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PLUS CONSTRUCTIONS, PLETHYSM, AND UNIQUE FACTORIZATION CATEGORIES WITH APPLICATIONS TO GRAPHS AND OPERAD–LIKE THEORIES

RALPH M. KAUFMANN AND MICHAEL MONACO

To Yuri Ivanovich Manin on the occasion of his 85th birthday

Abstract. Plus constructions are at the nexus of relative bimodules, indexed enrichments, modules over functors and graphical calculus. We define several of these in different settings. The first takes a category to a monoidal category and the second is an endofunctor for monoidal categories. There are localized and unital versions of these constructions. These serve three main purposes. The first is to define a notion that generalizes modules over algebras to monoidal functors and modules, sometimes called algebras, over them. This is realized via indexed enrichment. The second is to provide a theory of twists, which is closely related, and the third is to classify relative bi–modules over a given groupoid. In this guise they classify bi-module monoids with respect to a plethysm products over a the homomorphisms of a category thought of as a bi–module of the underlying groupoid.

The presented plus constructions generalize the plus construction for Feynman categories and explain the appearance of the plethysm monoid definition for operads, properads and props. To this end we introduce a new notion of unique factorization category (UFC) and show that the plus construction of a hereditary UFC is a Feynman category. Just as finite sets are the fundamental indexing Feynman category cospans are the fundamental indexing category hereditary UFC.

We give a local presentation of these constructions as well as a global description of the morphisms and a graphical version using decorated groupoid colored graphs. We furthermore consider an enriched setting. The global presentation utilizes pasting diagram from 2–categories or equivalently double categories, which is of independent interest. The graphical description is a consequence of the local definition. In the special case of a UFC there is also a formalism with groupoid colored graphs. In both cases levels appear when one adds units, which clarifies their role in this type of construction.

Contents

1. Introduction

In this paper, we introduce several plus constructions. A plus construction essentially takes the morphisms of a category and turns them into the new objects. The morphisms are then generated by isomorphisms (in the arrow category), the composition maps and, in the monoidal case, also the monoidal product. These generators are subject to natural relations of associativity, equivariance with respect to isomorphisms and in the monoidal case an interchange relation.

The first construction takes a category $\mathcal{C}$ to a monoidal category $\mathcal{C}^{\boxtimes+}$. The second main construction is an endofunctor for monoidal categories $\mathcal{M}$. This comes in several flavors, the basic one being $\mathcal{M}^{nc+}$. Here nc stands for non–connected much like in the category co–representing lax monoidal functors via strong monoidal functors, see §A.1. The nomenclature non–connected stems from [KWZn15]. Important modifications are the unital or gcp (groupoid compatibly pointed) versions and the localization $\mathcal{M}^{+}$. These *unital plus constructions co–represent so–called indexed enrichments*, which is the first of three important uses. Co–representing means that the object are classified as functors out of the co–representing objects. Equivalently, these are pre–sheaves on the opposite category.

Indexed enrichment of a category $\mathcal{M}$ allows to define modules over algebras in a generalized fashion which includes algebras over operads and generalizations of this. The unital plus construction co-represents such enrichments and the modules over a functor $\mathcal{O}$ out of the unital plus construction can be defined as functors over the enriched $\mathcal{M}_{\mathcal{O}}$. The localized version $\mathcal{M}^{+}$ co–represents functors that are strong with respect to the original monoidal product. In this guise, localized unital, the plus constructions appeared in the opetopic constructions of [BD98]. Other constructions of this type can e.g. be found in [BM18], [BB17], and [Ber21]. The plus construction also play an important role in the comparison of Feynman categories and operadic categories [BKM22]. The second application is to define *twists* which are necessary for bar and cobar construction and appear in Koszul duality [KW21]. This facet appeared in the form of hyper modular operads in [GK98] and is used widely, [GK95], [MMS09], etc. see [KWZn15] and references therein.

In the localized case, these two aspects were united for Feynman categories in [KW17], with more details in [Kau21], where a plus construction for Feynman categories (FCs) yielding a new Feynman category was given. A Feynman category is a special type of monoidal category with a basis of objects and basic morphisms that co–represents operad-like theories. Thanks to a hereditary conditions there is a free–forget adjunction which allows to view these functors as algebras over a triple, see also [Get09]. This includes the usual generalizations of operads, modular operads, props, properads, and the whole zoo of them. One instructive example is that the Feynman category $\mathcal{F}^{\text{operads}}$ for operads is the plus construction of the Feynman category for finite sets and surjections, $\mathcal{F}^{\text{surj}}$. So that algebras over an operad $\mathcal{O}$ can be identified with functors out of $F_{\mathcal{O}}^{\text{operads}}$, see [Kau21] for details. One caveat remains, namely that not every Feynman category is the plus construction of another Feynman category. This motivated the study undertaken in this paper. In particular, we introduce the notion of a hereditary unique factorization category or UFC, which is a generalization of Feynman categories. The unique factorization is the correct requirement for the localized plus construction to yield a Feynman category. Indeed the localized plus construction of a hereditary UFC is a Feynman category.

The third use is to have a formulation of operads, and the like, as monoids. Whereas Feynman categories guarantee that the functors are algebras over the monad of a free–forget adjoint pair, the monoid description involves a version of the plethysm product. We decipher the relevant monoidal structure here as that of a *groupoid bi–module monoids* with respect to a relative tensor product given by a co–end aka. *plethysm*. The monoid description then involves such monoids over a given category whose morphisms are considered as a unital bi–module monoid over its isomorphism groupoid, see §2. There is a second monoidal structure if the underlying category is monoidal, that is we consider monoidal bi–module monoids. The non–unital versions of the plus construction co–represent the relative possibly non–unital bi–module monoids. In fact, this groupoid point of view is the basic philosophy underlying the constructions as we expound, see §2.1 and §A.4.

This paper generalizes all these constructions to arbitrary (monoidal) categories. It also introduces the non–localized versions and clarifies the role of units. We first give a generators and relations definition in §3 and prove that this is a categorically speaking good notion. This includes a localized version which yields the strong plus construction. A nice by–product of the localization is a new description of the hereditary condition in terms of the possibility of realizing the localized category through a right roof calculus. The unital version are defined in §4 and their co–representing properties are given there as well.

We give global descriptions by proving that the categories stemming from the plus construction are equivalent to a category obtained by decorating certain types of graphs generalizing string diagrams of 2–categories with decorations in §7. The exact results are summarize in §1.3. There is also an algebraic/logic version using valid formulas. This is done in a step–by–step approach providing intermediate results of independent interest, e.g. for defining "planar" or non–Sigma versions of the construction. This gives a relation to decomposable little 2–cubes as they appear in [Dun88, Bri01, BFSV03]. It also explains the exact role of *leveled* structures.

The plus construction has a particularly nice global description for unique factorization categories, which we introduce in §6. These generalize Feynman categories. Cospans are the fundamental example of a hereditary UFC and serve as their indexing category —just like the category of finite sets is the fundamental indexing category for Feynman categories, see §6.4. Roughly, Feynman categories are free on basic objects and the morphisms are generated by basic "many–to–one" morphisms. Cospans have also very recently been considered in [BH22] in a related situation.

A UFC, is a generalization in which the objects are still freely generated as well as the morphisms by basic morphisms, however, the hereditary condition is relaxed to allow for basic "many–to–many" maps which play the role of prime factors in a unique factorization. This generalization is not automatically compatible with the composition, but is guaranteed by an extra hereditary condition. Generalizing to "many–to–many", one looses the forget/free adjunction, but one still can recover many interesting features among them the plethysm description.

Hereditary UFCs also allow for a second graphical calculation of the plus construction using groupoid colored graphs. This generalizes the construction for FCs using groupoid colored trees of [Kau21], see §B.3. The graphical language used is the groupoid extension of the one used in [KW17], which is grounded in the graph category introduced in [BM08]. This is reviewed in the appendix B.3. In this graphical description as well as in the string picture, the role of the units in leveling the graphs becomes apparent. The non–decorated groupoid colored graphs are also at the heart of Koszul duality [KW21].

The plus constructions naturally produces Feynman categories. This explains why there is a Feynman category for props, as this FC is obtained via the plus construction for cospans. It also explains why props are a plethysm monoids. For the localized version the main result is that the $\mathcal{M}^+$ for a hereditary UFC yields Feynman category. This complements the previous result that the plus construction of Feynman category is again a Feynman category. For example, it explains why there is a Feynman category for properads: it is the localized plus construction for cospans. This is the plus construction of a hereditary UFC and not a Feynman category. Thus, "plus" is a kind of stabilization.

1.1. **Dedication.** It is a great pleasure and honor to dedicate this paper to Yuri Ivanovich Manin, whom we thank for his continued guidance and support. His example of regarding mathematical truths, formulating and sharing his insights have been a guiding light for mathematics. His unmistakeable style is an aspirational goal for the field. His character, vision and overarching influence in and outside of mathematics have been a constant inspiration. The current results as well as many others in the field have a direct line to the the themes and presentation of [Man99, Man19, Man18].

1.2. **Acknowledgments.** This has been a multi–year effort dating back to at least 2019. During this time RK acknowledges support from the MPIM in Bonn, the Czech Academy of Sciences, the CRM in Barcelona and the KMPB in Berlin and thanks the hosts, especially Yuri Manin, Martin Markl, Carles Casacuberta and Dirk Kreimer for the invitations and discussions. RK also acknowledges recent support from the Simons foundation. We especially wish to thank Clemens Berger for continued key discussions on the subject. RK would also like to thank Don Zagier and Alexei Davydov for related discussions.

1.3. **Main results.** We will briefly summarize the main theorems.

Notation 1.1. We will use the short hand $(\mathcal{C}, \otimes)$ to denote monoidal categories. $Iso(\mathcal{C})$ is the underlying groupoid of $\mathcal{C}$. It has the objects of $\mathcal{C}$ and only the isomorphisms as morphisms. For two functors $F : \mathcal{D} \to \mathcal{C}$ and $G : \mathcal{E} \to \mathcal{C}$, we denote the comma category by $(F \downarrow G)$, If F and G are clear, we also write $(\mathcal{D} \downarrow \mathcal{E})$. For instance $(\mathcal{C} \downarrow \mathcal{C})$ is the arrow category. The morphism in $Iso(\mathcal{C} \downarrow \mathcal{C})$ given by two isomorphisms σ, σ' will be denoted by $(\sigma \Downarrow \sigma')$. The action of such an element is given by $(\sigma \Downarrow \sigma')(\phi) = \sigma' \phi \sigma^{-1}$.

For a category $\mathcal{C}$, $\mathcal{C}^{\otimes}$ will denote the free (symmetric) monoidal category. Objects and morphisms are bracketed tuples of objects and morphisms. There is an equivalent bigger model parameterized by sets $\mathcal{C}^{\otimes \mathcal{S}et}$, see Appendix A.1.4 and a smaller model which is strict.

For a groupoid $\mathcal{G}$ a groupoid bimodule is a functor $\mathcal{G}^{op} \times \mathcal{G}$ to some target category $\mathcal{D}$. These have a monoidal structure $\otimes_{\mathcal{G}}$ given by a relative tensor product, see §A.3.

A *simple indexing* of a monoidal category $\mathcal{M}$ over another monoidal category $\mathcal{M}_0$ is a strong monoidal functor $I : \mathcal{M} \to \mathcal{M}_0$, an *indexing is surjective* on objects and a *strong indexing* is bijective on objects.

To state the results in a concise fashion results, we include optional adjectives by parentheses.

1.3.1. **Bi–module monoids, plus constructions co–representing them and algebras.** The main result of the section 2 is:

Theorem I. *Let $\mathcal{C}$ be a ((symmetric) monoidal) category, $\mathcal{G} = \mathrm{Iso}(\mathcal{C})$ and $\mathcal{E}$ be symmetric monoidal category. There are isomorphisms between the categories of*

(1) *(Unital) (counital) ((symmetric) monoidal) indexing data.*
(2) *(Unital) (counital) ((symmetric) monoidal) enrichment functors to* $\underline{\mathcal{E}}$.

In the counital case, a condition which is automatically satisfied if $\mathcal{E}$ *is Cartesian the above are equivalent to*

(3) *(Unital) ((symmetric) monoidal)* $\mathrm{Iso}(\mathcal{C})$ *bimodule monoids in* $\mathcal{E}$ *over* $\mathrm{Hom}_{\mathcal{C}}$.

In the unital and counital case the above are equivalent to

(4) $\mathcal{E}$ *enriched ((symmetric) monoidal) categories which are index enriched over* $\mathcal{C}$ *with a section on the isomorphisms.*

See Theorem 2.24 in the text. Thus, for a given unital counital $\mathrm{Iso}(\mathcal{C})$ bimodule ρ there is a category $\mathcal{C}(\rho)$ of the same type, which is index enriched over $\mathcal{C}$ with the indexed enrichment/data given by an enrichment functor $\mathcal{D}(\rho)$, and vice–versa.

We define plus constructions in the section 3, which allow us to corepresent these functors.

Theorem II. *Let* $\mathcal{C}$ *be a category, then there is a monoidal category* $\mathcal{C}^{\boxtimes+}$ *given in Definition 3.1, such that the category of counital bi–module monoids* ρ *over* $\mathrm{Hom}_{\mathcal{C}}$ *is equivalent to the category of counital strong monoidal functors, that is those with a given natural transformation to the trivial functor* $\mathcal{T}:\mathcal{C}^{\boxtimes+}\to\mathcal{E}$. *Furthermore, the isomorphism split bimodules correspond to the split functor, see §3.4.*

See Theorem 3.3 in the text. There is a symmetric version of $\mathcal{C}^{\boxtimes+}$. Note that for a Cartesian $\mathcal{E}$, there is always a natural transformation to the trivial functor as the unit of $\mathcal{E}$ is a final object.

Theorem III. *Let* $(\mathcal{M},\otimes)$ *be a (symmetric) monoidal category then there is a symmetric monoidal category* $\mathcal{M}^{nc+}$ *defined in Definition 3.4 such that (symmetric) strong monoidal counital functors* $[\mathcal{M}^{nc+},\mathcal{E}]_{\otimes}$ *is equivalent to the category of (symmetric) lax–monoidal* $\mathrm{Iso}(\mathcal{M})$*–bimodules monoids over* $\mathrm{Hom}_{\mathcal{M}}$. *The isomorphism split bimodules correspond to the split functor, see §3.4.*

There is also a symmetric monoidal category $\mathcal{M}^{+}$ *defined in Definition 3.9 such that* $[\mathcal{M}^{loc,+},\mathcal{E}]_{\otimes}$ *is equivalent to strong monoidal bimodule monoids over* $\mathrm{Hom}_{\mathcal{M}}$.

See Theorems 3.6 and 3.10.

If $\mathcal{M}$ satisfies more conditions given in Definition 3.13, then localization can described more explicitly by equivalent categories.

Theorem IV. *If* $\mathcal{M}$ *has factorizable isomorphisms then there is a category* $\mathcal{M}^{+}$ *equivalent to* $\mathcal{M}^{loc+}$, *whose underlying groupoid is* $\mathrm{Iso}(\mathcal{M}\downarrow\mathcal{M})$ *and whose generating morphisms are with additional generating morphisms the images of the composition morphisms* γ_{ϕ_1,ϕ_2}, *which are morphisms* $\bar{\gamma}_{\phi_1,\phi_2}:\phi_1\otimes\phi_2\to\phi_1\circ\phi_2$ *that satisfy equivariance with respect to isomorphisms and internal associativity, as given by the formulas (3.3), (3.4) and (3.5), where all occurrences of* $\boxtimes$ *are replaced by* $\otimes$.

If additionally $\mathcal{M}$ *is has common factorizations and is hereditary, then* $\mathcal{M}^{+}$ *is given by a right roof calculus.*

See Definition-Proposition 3.12 and Proposition 3.14. In particular, this recovers the definition of the plus construction in [KW17, Kau21].

To handle unital monoidal bi–module monoids, we introduce groupoid compatible pointed (gcp) and reduced gcp functors, in analogy to [GK98, KW17, Kau21], see Definitions 4.1 and

4.7. These again be co–represented. To make the statements more concise, we adopt the following notation. $\mathcal{P}l^+$ is any of the plus constructions $\mathcal{C}^{\boxtimes+}, \mathcal{M}^{nc+}, \mathcal{M}^{loc+}, \mathcal{M}^+$.

Theorem V. *A groupoid compatible pointing for a functor $\mathcal{D}$ is a strong monoidal unit for the $\mathcal{G} = \mathrm{Iso}(\mathcal{M})$ bimodule plethysm monoid ρ defined by it via (2.13).*

There are symmetric monoidal categories $\mathcal{P}l^{+gcp}$ and defined in Definitions 4.3 such that

1. *The category of strong monoidal functors $[\mathcal{C}^{\boxtimes+,gcp}, \mathcal{E}]_\otimes$ is equivalent to the category of unital* $\mathrm{Iso}(\mathcal{C})$ *bi–module monoids over* $\mathrm{Hom}_\mathcal{C}$.
2. *The category of strong monoidal functors $[\mathcal{M}^{nc+,gcp}, \mathcal{E}]_\otimes$ is equivalent to the category of unital lax–monoidal* $\mathrm{Iso}(\mathcal{M})$*–bimodules monoids over* $\mathrm{Hom}_\mathcal{C}$.
3. *The category of strong monoidal functors $[\mathcal{M}^{loc+,gcp}, \mathcal{E}]_\otimes$ is equivalent to the category of unital strong–monoidal* $\mathrm{Iso}(\mathcal{M})$*–bimodules monoids over* $\mathrm{Hom}_\mathcal{C}$.
4. *The category of strong monoidal functors $[\mathcal{M}^{+,gcp}, \mathcal{E}]_\otimes$ is equivalent to the category of unital strong–monoidal* $\mathrm{Iso}(\mathcal{M})$*–bimodules monoids over* $\mathrm{Hom}_\mathcal{C}$.

In the non–Cartesian case the statements are restricted to counital functors and bimodules, and the isomorphism split bimodules correspond to the split functors.

There are furthermore categories $\mathcal{P}l^{+hyp}$, see 4.8, which corepresent the reduced versions.

See Proposition 4.2, Theorem 4.6 and Proposition 4.9. The relevance of hyperfunctors is that they do not change the isomorphisms an can be used for twisting [GK98] and more generally [KW17, §4.1]

Theorem VI. *The plus construction $\mathcal{C}^{\boxtimes+}$ is a functor from categories to symmetric monoidal categories all other plus constructions are endofunctors of symmetric monoidal categories. In particular, these are good categorical notions meaning that (symmetric (monoidally)) equivalent categories yield (symmetric (monoidally)) equivalent plus constructions.*

See Theorem 5.5 and Corollary 5.6. Note it is possible to restrict to the non–symmetric case forgetting the symmetric structure.

Grace to the results above, one can define algebras over functors.

Definition. If a category $\mathcal{M} = \mathcal{P}l^+(\mathcal{N})$ for some category $\mathcal{N}$ of the same type, then an *algebras* over a (strong (symmetric) monoidal) functor $\mathcal{D} : \mathcal{M} \to \mathcal{E}$ is an algebra over the corresponding bimodule $\rho(\mathcal{D})$, or equivalently a functor out of $\mathcal{N}_\mathcal{D} = \mathcal{N}(\rho)$ which is indexed enriched category over $\mathcal{N}$.

This extends the definition of [KW17, Kau21] to arbitrary (symmetric (monoidal)) categories. Examples are modules over algebras, algebras over operads, and newly algebras over props and properads.

1.3.2. **Special categories and global presentations of the plus construction.** We introduce the new notion of a unique factorization categories (UFCs) see Definition 6.1 generalizing that of Feynman categories [KW17], see Lemma 6.12.

The main result is that

Theorem VII. *For any category $\mathcal{C}$ or monoidal category $\mathcal{M}$, $\mathcal{C}^{\boxtimes+}$ and $\mathcal{M}^{nc+}$ are cubical Feynman categories.*

If $\mathcal{M}$ is the underlying monoidal category of a hereditary UFC, then $\mathcal{M}^+$ is the underlying monoidal category of a cubical Feynman category.

The hyp versions are also cubical Feynman categories. The gpc versions are Feynman categories with a additional generators of degree -1 corresponding to the units.

See Theorem 6.42.

There is a useful criterion to check if a UFC is hereditary using cospans in Proposition 6.17.

More generally, there is a graphical description of the morphisms.

Theorem VIII.

(1) *The morphisms of $\mathcal{C}^{\boxtimes +}$ are forests of b/w bipartite rooted linear trees decorated with isomorphisms and morphisms with an enumeration of the vertices..*
(2) *If $\mathcal{M}$ has factorizable isomorphisms, the basic morphisms of $\mathcal{M}^{nc+}$ are composition graphs decorated with isomorphisms and morphisms. Where the decoration entails an enumeration of the vertices. (A composition graph is a planar b/w bipartite graph that has a special property of being decomposable.)*
(3) *The morphisms of $\mathcal{M}^+$ for a UFC are generated under composition by* connected *composition graphs decorated by morphisms and isomorphisms.*
(4) *If $\mathcal{M}$ is hereditary these are the basic morphisms of the Feynman category.*
(5) *If $\mathcal{M}$ is hereditary the the basic morphisms of the Feynman category $\mathcal{M}^+$ are classes of leveled composition diagrams which are decorated by morphisms and isomorphisms. The classes are with respect to re-levelling that is the introduction of units,*

This is contained in Proposition 7.18, Proposition 7.56, Theorem 7.64. Proposition 7.67 and Corollary 7.68, which provide more technical details.

There are also intermediate results yielding Feynman categories given by formulas and planar diagrams, see Corollary 7.50 and Proposition 7.55, which also link the theory to that of decomposable tight little 2-cubes.

1.3.3. **Graphical version.** For hereditary UFCs there is a second graphical interpretation that ties into the commonly used definitions of the plus construction using trees. In particular, using groupoid colored graphs allows us to identify the morphisms of the plus construction with decorations of a groupoid graphs, see §8.1. The underlying graphs are those that are used to define properads and props, see §B.3.2. In analogy to the formalism of groupoid trees in [Kau21] these are generalized to groupoid graphs. A reduction occurs in the discrete case where the decoration is a functor from the Feynman category of properads or props.

Theorem IX. *There are graphical constructions of $\mathcal{M}^{nc,+}$ and $\mathcal{M}^+$ and their gcp and hyp versions based on groupoid colored decorated graphs, which are equivalent to them.*

Here the decoration is the technical decoration of [KL17,BK17]. See Propositions 8.7, 8.11 and 8.15 for details.

There is also a direct conversion from decorated composition graphs to the groupoid colored graphs, see §8.5.

2. Categories as plethysm bimodules

2.1. **Philosophy for constructing categories.** The main philosophy is that categories $\mathcal{C}$ are constructed in a three–step process.

(1) Objects and their isomorphisms.
(2) A (set) of general morphisms and the action of the isomorphisms on them.
(3) Composition and units.

2.1.1. **Double category and category in groupoids.** There is a double categorical way to view the philosophy. Every category $\mathcal{C}$ defines a double category $\mathcal{D}^{\square}(\mathcal{C})$, where the horizontal and vertical morphisms are the morphisms of $\mathcal{C}$ and the two–morphisms are the commutative squares, see [Bén67]. The double category $\mathcal{D}(\mathcal{C})$ will be the sub–double category which has the restriction that the vertical morphisms are the isomorphisms. This means that the 2–morphisms are precisely the squares $(\sigma \overset{\phi}{\Downarrow} \sigma')$

$$
\begin{tikzcd}
X \arrow[r, "\phi"] \arrow[d, "\sigma"', "\simeq"] \arrow[dr, phantom, "\Downarrow (\sigma \overset{\phi}{\Downarrow} \sigma')"] & Y \arrow[d, "\sigma'", "\simeq"'] \\
X' \arrow[r, "\phi'"'] & Y'
\end{tikzcd}
\tag{2.1}
$$

Thinking of a double category as a category in categories, $\mathcal{D}(\mathcal{C})$ is a category in groupoids as all the vertical morphisms are isomorphisms.

In particular, the groupoid of objects is $\mathfrak{Obj} = \mathrm{Iso}(\mathcal{C})$. The groupoid of morphisms is $\mathfrak{Mor} = Iso(\mathcal{C} \downarrow \mathcal{C})$. The source and target functors sends $(\sigma \Downarrow \sigma')$ to σ and to σ' while the identity functor sends X to id_X and id_X to $(id_X \Downarrow id_X)$, see §A.4 and especially Example A.14 in the appendix for details. The composition in $\mathfrak{Mor}$ composes the morphisms as objects of $\mathfrak{Mor}$ and the morphisms of $\mathfrak{Mor}$ as $(\sigma \Downarrow \sigma') \circ (\sigma' \Downarrow \sigma'') = (\sigma \Downarrow \sigma'')$.

Remark 2.1. By construction, there is a unique filler in $\mathcal{D}(\mathcal{C})$ for any commutative outer square. This is the restriction of the this structure of $\mathcal{C}^{\square}$.

2.1.2. **Enrichment.** We can consider the case that $\mathcal{C}$ is enriched over $\mathcal{E}$, where $\mathcal{E}$ is closed symmetric monoidal. More generally functors may take values in any (symmetric) monoidal $\mathcal{E}$. There are two cases, $\mathcal{E}$ is Cartesian, which is a straightforward generalization, or $\mathcal{E}$ is not, e.g. $\mathcal{E}$ is linear. The latter needs some extra care. Following [KW17, Kau21] being a groupoid, in the non–Cartesian case means that $\mathcal{G} = \mathcal{G}' \odot \mathcal{E}$, that is a freely enriched groupoid; see [Kel82] for free enrichment.

To identify the groupoid as the isomorphisms $\mathcal{C}$ has to be *isomorphism split* this means that $\mathrm{Hom}_{\mathcal{C}}(X, Y) = I(X, Y) \odot \mathcal{E} \oplus \overline{Hom}(X, Y)$ where $I(X, Y)$ is a set of isomorphisms, $\odot\mathcal{E}$ again denotes the free enrichment, and $\overline{Hom}(X, Y)$ contains no isomorphisms. As a mnemonic, we often use $\oplus$ for the coproduct in a non–Cartesian setting, e.g. for an Abelian category like $\mathcal{V}ect_k$ or dg-$\mathcal{V}ect$. Note that if one does not insist on the identification of $\mathcal{G}$ with $\mathrm{Iso}(\mathcal{M})$, then there is no problem. We will not dwell too much on these details in the following, since the versed reader would know how to make the adjustments, while the uninitiated would risk being confused by overly detailed exposition. But, we point out extra assumptions and throughout in the text. We also refer to [Kau21] for examples.

ASSUMPTION: We will tacitly assume that the category $\mathcal{C}$ is isomorphism split if it is enriched and $\mathcal{G}$ is taken to be the isomorphisms. Also, all colimits will be indexed colimits, cf. [Kel82].

2.2. **Bimodule monoids.** The first interpretation of the philosophy is as bimodules. Step (1) is the specification of an underlying groupoid $\mathcal{G}$. We will denote the morphisms of $\mathcal{G}$ as σ.

In step (2), the putative morphisms are then given by a $\mathcal{G}$ bimodule $\rho \in [\mathcal{G}^{op} \times \mathcal{G}, \mathcal{S}et]$.

The action is given by $\phi' = (\sigma \Downarrow \sigma')(\phi) = \sigma' \circ \phi \circ \sigma^{-1}$ if $s(\phi) = s(\sigma), t(\phi) = s(\sigma')$. This fits into a diagram (2.1) where $(\sigma \Downarrow \sigma')(\phi)$ is the target of the morphisms $(\sigma \Downarrow \sigma')\phi$.

NB: The natural indexing for the two–cells is by $\mathcal{G}\times\mathcal{G}$, not $\mathcal{G}^{op}\times\mathcal{G}$, but using the inverse on the left yields an isomorphism ${}^{-1}\times id:\mathcal{G}\times\mathcal{G}\xrightarrow{\sim}\mathcal{G}^{op}\times\mathcal{G}$. We will use the notation $(\sigma\Downarrow\sigma')$ for elements of $\mathcal{G}\times\mathcal{G}$ and use (σ,σ') for those of $\mathcal{G}^{op}\times\mathcal{G}$ if necessary. Usually, we will use $(\sigma\Downarrow\sigma')$ and tacitly make the identification $(\sigma\Downarrow\sigma')\leftrightarrow(\sigma^{-1},\sigma')$. The standard actions are $(\sigma,\sigma')\phi=\sigma'\phi\sigma$ and$(\sigma\Downarrow\sigma')(\phi)=\sigma'\phi\sigma^{-1}=(\sigma^{-1},\sigma')$.

For step (3) one specifies the structure of a unital monoid for the plethysm product $\otimes_\mathcal{G}$ on $\mathcal{G}$–bimodules, see §A.3. The monoid structure is given by a natural transformation

$$\gamma:\rho\otimes_\mathcal{G}\rho\to\rho \tag{2.2}$$

which is strictly associative $\gamma(\gamma\otimes_\mathcal{G} id)=\gamma(id\otimes_\mathcal{G}\gamma)$. This will yield the compositions.

The unit for the plethysm product is $\mathrm{Hom}_\mathcal{G}(-,-):\mathcal{G}^{op}\times\mathcal{G}\to\mathcal{S}et$.

Definition 2.2. A *unit* unit for ρ is a natural transformation

$$u:\mathrm{Hom}_\mathcal{G}\to\rho \tag{2.3}$$

which is a strict unit $\gamma(u\otimes_\mathcal{G} id_\rho)=\rho=\gamma(id_\rho\otimes_\mathcal{G} u)$.

This will yield elements that will serve as identity maps: $id_X:=u(id_X)\in\rho(X,X)$ and the additional elements $u(\sigma)\in\rho(s(\sigma),t(\sigma))$. To have such putative identity maps, one ostensibly only needs a pointing of ρ, which a unit i of ρ considered via restriction as a $\mathcal{G}_{disc}^{op}\times\mathcal{G}_{disc}$ module. Here $\mathcal{G}_{disc}$ is the discrete subgroupoid which only retains the identity maps.

Lemma 2.3. *The data of a pointing is equivalent to that of a unit.*

Proof. The restriction of a unit is clearly a pointing. Given a pointing, set $u(id_X)=i(id_X)$. By the action there are elements $u_\sigma=(\sigma,id)u(id_X)$ and ${}_\sigma u=(id,\sigma)u(id_X)$. By equivariance and the fact that i is a unit, it follows that $\gamma(u_\sigma,\phi)=\sigma\phi$ and $\gamma(\phi,{}_\sigma u)=\phi\sigma$ from which it follows that $u_\sigma=({}_{\sigma^{-1}}u)^{-1}$. By outer equivariance of the composition it follows that $u_{\sigma^{-1}}=(u_\sigma)^{-1}$ and thus $u_\sigma={}_\sigma u$ and finally that $(\sigma\Downarrow\sigma')u_\tau=u_{(\sigma\Downarrow\sigma')(\tau)}$ establishing that u is a morphism of $\mathcal{G}^{op}\times\mathcal{G}$-modules. The fact that this is a unit is now straightforward. □

Definition 2.4. A unital bimodule monoid will be called *groupoid compatibly pointed (gcp)* if u is injective, that is $(\mathcal{G}^{op}\times\mathcal{G})(X,Y)\hookrightarrow\rho(X,Y)$ and *hyper* if additionally the image of u are the only invertible elements in the monoid ρ.

The terminology "hyper" goes back to [GK98], cf. [KW17,Kau21] for the expanded use in Feynman categories. The condition for hyper is what is needed for twists.

Example 2.5. Given a category $\mathcal{C}$, set $\mathcal{G}=\mathrm{Iso}(\mathcal{C})$ and let $\rho=Hom(-,-)$ be the restriction of the Hom bifunctor to $\mathcal{G}^{op}\times\mathcal{G}$. As composition is associative $(\phi\circ\sigma)\circ\psi=\phi\circ(\sigma\circ\phi)$, the composition $\circ:Hom_t\times_s Hom$ descends to a monoid $\gamma=\bar{\circ}$, see Example A.14. The unit is the inclusion $u:\mathrm{Iso}(X,Y)\hookrightarrow\mathrm{Hom}(X,Y)$ and the splitting $\mathrm{Hom}(X,Y)=\mathrm{Iso}(X,Y)\amalg\overline{\mathrm{Hom}}(X,Y)$ where $\overline{\mathrm{Hom}}(X,Y)$ contains no isomorphisms. This functor is gcp pointed and hyper.

Definition 2.6. Assuming a (symmetric) monoidal $\otimes$ structure on $\mathcal{G}$, a symmetric lax–monoidal structure for ρ is given by the data

(1) A lax–monoidal structure for ρ, i.e. is a natural transformation which is 2–cell for the following diagram

$$\begin{tikzcd}
(\mathcal{G}^{op}\times\mathcal{G})\times(\mathcal{G}^{op}\times\mathcal{G}) \arrow[r, "\otimes"] \arrow[d, "\rho\times\rho"'] \arrow[dr, phantom, "\overset{\mu}{\Rightarrow}"] & \mathcal{G}^{op}\times\mathcal{G} \arrow[d, "\rho"] \\
\mathcal{S}et\times\mathcal{S}et \arrow[r, "\times"] & \mathcal{S}et
\end{tikzcd} \tag{2.4}$$

with unit morphisms, associators and, in the symmetric case, commutators according to those of $\mathcal{G}$, see Example A.9. Explicitly the monoidal structure is given by morphisms

$$\mu : \rho(X,Y) \times \rho(X',Y') \to \rho(X \otimes X', Y \otimes Y') \tag{2.5}$$

which satisfy

$$\mu \circ ((\sigma_1 \Downarrow \sigma_1') \times (\sigma_2 \Downarrow \sigma_2')) = ((\sigma_1 \Downarrow \sigma_1') \otimes (\sigma_2 \Downarrow \sigma_2')) \circ \mu \tag{2.6}$$

In the unital case, this has to be compatible with the unit in the sense that $u_{id_{\mathbb{1}}} \in \rho(\mathbb{1},\mathbb{1})$ is the lax unit of ρ and $\mu \circ (u \otimes u) = u \circ \otimes$ or in elements

$$\mu(u_\sigma, u_{\sigma'}) = u_{\sigma \otimes \sigma'} \tag{2.7}$$

The monoidal structure is *strong* if the morphisms in (2.5) are isomorphisms and *strict* is they are equalities.

NB: $(\sigma_1 \Downarrow \sigma_1') \otimes (\sigma_2 \Downarrow \sigma_2') = ((\sigma_1 \otimes \sigma_2) \Downarrow (\sigma_1' \otimes \sigma_2'))$ by definition.

Example 2.7. If $\mathcal{M}$ is a monoidal category and $\rho = \mathrm{Hom}_{\mathcal{M}}(-,-)$ the monoidal structure $\otimes$ for $\mathrm{Iso}(\mathcal{M})$ is simply the restriction. The transformation μ is given by the monoidal structure: $\mu : \mathrm{Hom}(X,Y) \times \mathrm{Hom}(X',Y') \to \mathrm{Hom}(X \otimes X', Y \otimes Y')$, is given by $(\phi,\psi) \mapsto \phi \otimes \psi$. Note the compatibility (2.6) holds, since $(\sigma_1 \Downarrow \sigma_1') \otimes (\sigma_2 \Downarrow \sigma_2') = ((\sigma_1 \otimes \sigma_1') \Downarrow (\sigma_2 \otimes \sigma_2'))$ and as u is the inclusion $u_\sigma = \sigma$ and $\mu(\sigma \otimes \sigma') = \sigma \otimes \sigma'$, so that (2.7) holds tautologically.

Example 2.8. The trivial bi–module monoid is given by $\mathcal{T}(X,Y) = *$. The composition map being the unique possible map. This is the terminal bi–module monoid.

A unit for the functor $\mathcal{T}$ will have to send any $\sigma \in \mathcal{G}(X,Y)$ to the unique element $* \in \mathcal{T}(X,Y)$.

Example 2.9. Concretely, a particularly important groupoid is the groupoid $\mathcal{G} = \mathbb{S}$ whose objects are the natural numbers and whose morphisms are $Hom(n,n) = \mathbb{S}_n$ and $Hom(n,m) = \emptyset$ for $n \neq m$. This is the free symmetric monoidal category on the trivial category $\underline{*}$ which is the category with one object and only its identity as morphisms; that is $\mathbb{S} = \underline{*}^{\boxtimes}$. A functor to $\mathcal{T}$ will send $\sigma \in \mathbb{S}_n$ to id_n. Note that this is not an injection on the invertible elements nor is it a surjection onto the invertible elements as all the elements $\mathcal{T}(n,m)$ are invertible.

As it turns out, $\mathcal{T}$ is not the correct terminal object if one considers bi–modules with the condition that isomorphisms are conserved, see below.

2.2.1. **Categories from unital monoids.** A strict unital bi–module monoid defines a category $\mathcal{C}(\rho)$ with $\mathrm{Hom}_{\mathcal{C}(\rho)} = \rho(X,Y)$. The source and target maps are clear the units are given by $id_X = u(id_X)$ and the composition is specified by γ, that is $\gamma : \rho(X,Y) \times \rho(Y,Z) \to \rho(X,Z)$.

Remark 2.10.

(1) There is an additional action by $\mathcal{G}^{op} \times \mathcal{G}$ on the composition which amounts to the equivariance $\phi \circ \sigma\psi = \phi\sigma \circ \psi$. Here $\phi\sigma$ and $\sigma\psi$ are the right and left actions.

(2) The unit gives rise to elements $u_\sigma := u(\sigma) \in \rho(X,Y), \forall \sigma \in \mathcal{G}(X,Y)$ which satisfy

$$(\sigma \Downarrow \sigma')(u_\tau) = u_{(\sigma \Downarrow \sigma')(\tau)} \tag{2.8}$$

and

$$(\sigma \Downarrow id)(\phi) = \phi \circ u_{\sigma^{-1}} \quad (id \Downarrow \sigma')(\phi) = u_{\sigma'} \circ \phi \tag{2.9}$$

This implies that left and right multiplication by u_{id_X} act as identities and $u_\sigma \circ u_{\sigma'} = u_{\sigma\sigma'}$.

If ρ is a (symmetric) lax-monoidal unital monoid then $\mathcal{C}(\rho)$ is a (symmetric) monoidal category.

In this construction a further possible step according to the philosophy is the identification of $\mathcal{G}$ as isomorphisms or as *the* isomorphisms.

(1) An identification *as* isomorphisms is given by demanding that u be an inclusion, viz, $\mathrm{Hom}_{\mathcal{G}}(X,Y) \hookrightarrow \rho(X,Y)$. This condition allows one to identify $\mathcal{G}(X,Y) \subset \mathrm{Iso}(\mathcal{C})(X,Y)$. That is $u_\sigma = \sigma$.
(2) To force that $\mathcal{G} = \mathrm{Iso}(\mathcal{C}(\rho))$, u has to be a bijection with $\mathcal{G}$. This means there is a splitting
$$\rho(X,Y) = \mathcal{G}(X,Y) \amalg \bar{\rho}(X,Y) \tag{2.10}$$
where $\bar{\rho}(X,Y)$ contains no invertible elements.

Example 2.11. This allows us to recover the category $\mathcal{C}$ from its bi–module $\mathcal{C}(Hom_{\mathcal{C}}) = \mathcal{C}$.

$C(\mathcal{T})$ is the complete groupoid on $\mathrm{Obj}(\mathcal{G})$, viz. exactly one isomorphism between any two objects. This is also the groupoid associated to the complete graph. The unit functor is not an injection and the complement does contain isomorphisms.

2.2.2. **Enrichment.** Instead of being $\mathcal{S}et$ valued one can consider bimodules in a (symmetric monoidal) $\mathcal{E}$, viz. $\rho : \mathcal{G}^{op} \times \mathcal{G} \to \mathcal{E}$ where $\mathcal{E}$ is a monoidal category. In this case one replaces the $\times$ above by $\otimes_{\mathcal{E}}$.

If furthermore $\mathcal{G}$ is enriched over some $\mathcal{E}'$ and $\mathcal{E}$ is also enriched over $\mathcal{E}'$ (in most applications $\mathcal{E} = \mathcal{E}'$), then the bi–module ρ should be taken to be $\mathcal{E}'$ enriched. If $\mathcal{G}$ is $\mathcal{S}et$ groupoid, and $\mathcal{E}$ is closed symmetric monoidal then one can freely enrich over $\mathcal{E}$ to obtain bi–modules over $\mathcal{E}$.

A bi–module monoid is *isomorphism split* if $\rho(X,Y) = \iota(X,Y) \odot \mathcal{E} \oplus \bar{\rho}(X,Y)$ where $\iota(X,Y)$ are invertible elements and $\bar{\rho}(X,Y)$ contains no invertible elements of $\rho(X,Y)$. For a bi–module monoid in $\mathcal{E}$ to be gcp, we require that u is split injective, viz. $\rho(X,Y) = (\bigoplus_{\sigma \in (\mathcal{G}^{op} \times \mathcal{G})(X,Y)} \mathbb{1}_{\mathcal{E}}) \oplus \rho^c(X,Y)$ and for it to be hyper means that ρ^c contains no γ invertibles. In this case ρ is isomorphism split and $\rho^c = \bar{\rho}$.

Example 2.12. The trivial bi–module is given by $\mathcal{T}_{\mathcal{E}}(X,Y) = \mathbb{1}_{\mathcal{E}}$. This is terminal if $\mathcal{E}$ is Cartesian.

2.2.3. **Algebras.** An algebra for a bimodule monoid ρ is a left $\mathcal{G}$–module $\alpha \in [\mathcal{G}, \mathcal{E}]$ with a morphism, that is natural transformation of functors from $\mathcal{G}$ to $\mathcal{E}$:
$$a : \alpha \otimes_{\mathcal{G}} \rho \to \alpha \tag{2.11}$$
which is associative and unital in the usual fashion. This means that there is an equivariant action, viz. for every $\phi \in \rho(X,Y)$ there is a morphism $a(\phi) : \alpha(X) \to \alpha(Y)$. The natural transformations that intertwine the action imbue algebras with a categorical structure. If ρ has a unit then α is unital if $a(u(\sigma)) = \alpha(\sigma)$. If $\mathcal{G}$ and ρ are monoidal then requiring α to be of the same type defines (strong)(symmetric) monoidal algebras.

Lemma 2.13. *If ρ is unital, the category of unital algebras over ρ is equivalent to the category of functors $[C(\rho), \mathcal{E}]$. Similarly this is true for (strong)(symmetric) monoidal algebras and the selected type of functors from $C(\rho) \to \mathcal{E}$.*

Proof. This is a unwinding of definitions. Given α we define the following functor $A : \mathcal{C}(\rho) \to \mathcal{E}$. As $\mathcal{G}$ and $C(\rho)$ have the same objects set $A(X) = \alpha(X)$. A morphisms $\phi \in C(\rho)(X,Y)$

is by definition an element of $\rho(X,Y)$. Setting $A(\phi) := a(\phi) : A(X) \to A(Y)$ defines the functor on morphisms and finally $A(\phi\psi) = a(\phi\psi) = a(\gamma(\phi\otimes\psi)) = a(\phi)a(\psi)$ by associativity. The remaining statements are analogous. □

2.3. **Relative bimodules and Indexed enrichment.**

2.3.1. **Bimodules over** $\mathrm{Hom}_{\mathcal{C}}$**.** A central question is the construction is the enriching of a given category $\mathcal{C}$ respecting the isomorphisms. By Example 2.5, $\mathrm{Hom}_{\mathcal{C}}$ is a $\mathcal{G} = \mathrm{Iso}(\mathcal{C})$ bimodule monoid, and which in case that $\mathcal{C}$ is monoidal is also a monoidal monoid by Example 2.7. We say a bimodule ρ is *indexed* over $\mathcal{C}$ if it is in the slice category of $\mathrm{Hom}_{\mathcal{C}}$.

In the Cartesian case, we have the pull–backs $\mathcal{D}(\phi)$ defined by

$$\begin{tikzcd} \mathcal{D}(\phi) := \{\phi\}_i\times_b\rho(X,Y) \arrow[r] \arrow[d] & \rho(X,Y) \arrow[d, "b"] \\ \{\phi\} \arrow[r, "i"] & \mathrm{Hom}_{\mathcal{C}}(X,Y) \end{tikzcd} \tag{2.12}$$

and

$$\mathrm{Hom}_{\mathcal{C}(\rho)}(X,Y) = \rho(X,Y) = \coprod_{\phi\in Hom_{\mathcal{C}}(X,Y)} \mathcal{D}(\phi) \tag{2.13}$$

In the general enriched setting, as one is not guaranteed terminal objects and morphisms do not have to split, we assume that the coproducts exist in $\mathcal{E}$, the natural transformation is to $Hom_{\mathcal{C}\odot\mathcal{E}}$ where $\mathcal{C}\odot\mathcal{E}$ is the freely enriched category of $\mathcal{E}$ and respects the coproducts. This means that $\mathrm{Hom}_{\mathcal{C}\odot\mathcal{E}}(X,Y) = \bigoplus_{\phi\in \mathrm{Hom}_{\mathcal{C}}(X,Y)} \mathbb{1}_{\mathcal{E}}$, there is a decomposition (2.12) and

$$b = \bigoplus_{\phi\in Hom_{\mathcal{C}}(X,Y)} \epsilon_\phi \text{ with } \epsilon(\phi) : \mathcal{D}(\phi) \to \mathbb{1}_{\mathcal{E}} \tag{2.14}$$

The $\epsilon(\phi)$ are not extra data in the Cartesian case, but they are in the non–Cartesian case.

Example 2.14. In the Cartesian case every bimodule is in the slice category of $\mathcal{T}$. In the non–Cartesian case, the morphisms to $\mathcal{T}$ are given by $\epsilon : \rho(X,Y) \to \mathbb{1}$.

Definition 2.15. A possibly Cartesian enriched category $\hat{\mathcal{C}}$ is indexed enriched over a category $\mathcal{C}$ if there is a functor $b : \hat{\mathcal{C}} \to \mathcal{C}$, which is bijective on objects. For (symmetric) monoidal categories the functor needs to be (symmetric) monoidal.

A section on the isomorphisms is a functor $u : \mathrm{Iso}(C) \to b^{-1}(\mathrm{Iso}(\mathcal{C}))$. It is full if u is a bijection. Note, all isomorphisms of $\hat{\mathcal{C}}$ are mapped to isomorphisms of $\mathcal{C}$ and a section is always injective.

In the non–Cartesian case, we postulate that $Hom_{\hat{\mathcal{C}}}$ splits as (2.13) and the functor b splits according (2.14). A section on isomorphisms is then given by $u : \mathrm{Iso}(\mathcal{C})\odot\mathcal{E} \to \hat{\mathcal{C}}$ such that $bu = i\odot\mathcal{E}$ where $i\odot\mathcal{E}$ is the enriched inclusion $Iso(\mathcal{C})\odot\mathcal{E} \to \mathcal{C}\odot\mathcal{E}$ induced by the inclusion $i : \mathcal{G} \to \mathcal{C}$.

Proposition 2.16. *There is a bijection between unital ((symmetric) monoidal) bimodule monoids over* $\mathrm{Hom}_{\mathcal{C}}$ *and ((symmetric) monoidal) indexed enriched categories with a section on the isomorphisms.*

Proof. For the Cartesian case, given ρ over $\mathrm{Hom}_{\mathcal{C}}$ there is an induced functor $b : \mathcal{C}(\rho) \to \mathcal{C}$ as above. Let $\mathcal{D}(\phi)$ be the fibers over ϕ as in (2.12). The objects of $b^{-1}(\mathrm{Iso}(C))$ are those of $\mathcal{C}$ and the morphisms are $Hom_{b^{-1}(\mathcal{C})}(X,Y) = \coprod_{\sigma \in \mathrm{Iso}(X,Y)} \mathcal{D}(\sigma)$. Define the section by $u(\sigma) = u_\sigma \in \mathcal{D}(\sigma)$.

In the other direction, let $\rho(X,Y) = Hom_{\hat{\mathcal{C}}}(X,Y)$ and given b let the fibers over b be the $\mathcal{D}(\phi)$. The section u defines the bi–module structure via (2.9). It also yields the unit for the monoid structure using (2.13) and the embedding of $b^{-1}\,\mathrm{Iso}(\mathcal{C}) = \coprod_{\sigma \in \mathrm{Iso}(X,Y)} \mathcal{D}(\sigma)$.

In the non–Cartesian case the data of the $\mathcal{D}(\phi)$ and ϵ_ϕ are part of the data by assumption. The (symmetric) monoidal case is an analogous matching of data. $\square$

Example 2.17. Every category $C(\rho)$ is naturally index enriched over $\mathcal{C}(\mathcal{T})$ in the Cartesian case. In the non–Cartesian case such an indexing is given by a functor $\epsilon : Hom_{\mathcal{C}}(X,Y) = \rho(X,Y) \to \mathbb{1}$.

Remark 2.18 (Enriched $\mathcal{C}$)**.** The construction and the theorem generalize to the case that $\mathcal{C}$ is already enriched over $\mathcal{E}'$ and $\mathcal{E}$ is an enrichment category over $\mathcal{E}'$. The construction goes through *mutatis mutandis* in this case as well using indexed colimits, cf. [Kel82].

2.4. **Indexing data and indexed enrichment.** If $\mathcal{E}$ is Cartesian, the morphisms of the natural transformation $b : \rho \to \mathrm{Hom}_{\mathcal{C}}$ has fibers and the structures of the bimodule translate to the following data for $\mathcal{D}$, the totality of which is defined to be *an indexing data* $\mathcal{D}$.

(1) The objects $\mathcal{D}(\phi)$ of $\mathcal{E}$ for each $\phi \in \mathrm{Mor}(\mathcal{C})$ with the fiberwise action of $\mathcal{G}^{op} \times \mathcal{G}$:
$$(\sigma \overset{\phi}{\Downarrow} \sigma') : \mathcal{D}(\phi) \to \mathcal{D}(\sigma'\phi\sigma^{-1}) \tag{2.15}$$

(2) The fiberwise *associative* multiplication, i.e.
$$\gamma^{\mathcal{D}}_{\phi_1,\phi_0} : \mathcal{D}(\phi_1) \otimes_{\mathcal{E}} \mathcal{D}(\phi_0) \to \mathcal{D}(\phi_1 \circ \phi_0) \tag{2.16}$$
for any pair of composable morphisms compatible with the action:
$$\gamma^{\mathcal{D}} \circ (\sigma \Downarrow \sigma') \otimes (\sigma' \Downarrow \sigma'') = (\sigma' \Downarrow \sigma'') \circ \gamma^{\mathcal{D}} \tag{2.17}$$

In the category *unital* monoids, there indexing data furthermore includes

(3) Elements
$$\forall \sigma \in \mathrm{Iso}(\mathcal{C}) : u_\sigma : \mathbb{1}_{\mathcal{E}} \to \mathcal{D}(\sigma) \tag{2.18}$$
satisfying $(\sigma \Downarrow \sigma')(u_\tau) = u_{\sigma'\tau\sigma^{-1}}$, such that the $u_{id_{s(\phi)}}$ is a left identity for $\mathcal{D}(\phi)$ and $u_{id_{t(\phi)}}$ is a right identity and the following diagrams commute.
$$\begin{tikzcd} \mathcal{D}(\phi) \arrow[r, "\sim"] \arrow[d, "\mathcal{D}((\sigma \overset{\phi}{\Downarrow} \sigma'))"'] & \mathbb{1}_{\mathcal{E}} \otimes_{\mathcal{E}} \mathcal{D}(\phi) \otimes_{\mathcal{E}} \mathbb{1}_{\mathcal{E}} \arrow[d, "u_\sigma \otimes id \otimes u_{\sigma'}"] \\ \mathcal{D}((\sigma \Downarrow \sigma')(\phi)) & \mathcal{D}(\sigma) \otimes \mathcal{D}(\phi) \otimes \mathcal{D}(\sigma') \arrow[l, "\gamma(\gamma \otimes id)"] \end{tikzcd} \tag{2.19}$$

NB: In particular, $u_\sigma = \mathcal{D}((\mathrm{id}_{s(\sigma)} \Downarrow \sigma))u_{\mathrm{id}_{s(\sigma)}} = \mathcal{D}((\sigma^{-1} \Downarrow id_{t(\sigma)})u_{t(\sigma)})$.

For a *(symmetric) monoidal* ρ the indexing data also includes

(4)
$$\mu^{\mathcal{D}}_{\phi,\psi} : \mathcal{D}(\phi) \otimes \mathcal{D}(\psi) \to \mathcal{D}(\phi \otimes \psi) \tag{2.20}$$

(5)
$$\mathcal{D}((\sigma\Downarrow\sigma'))\otimes_{\mathcal{E}}\mathcal{D}((\tau\Downarrow\tau'))\to\mathcal{D}((\sigma\otimes\tau\Downarrow\sigma'\otimes\tau')) \tag{2.21}$$
compatible with $\mu^{\mathcal{D}}$.
$$\mu^{\mathcal{D}}\circ\mathcal{D}((\sigma\Downarrow\sigma'))\otimes\mathcal{D}((\tau\Downarrow\tau'))=\mathcal{D}((\sigma\otimes\tau\Downarrow\sigma'\otimes\tau'))\circ\mu^{\mathcal{D}} \tag{2.22}$$
In the unital case (symmetric) monoidal case the elements u_σ also have to satisfy the compatibility
$$\mu^{\mathcal{D}}_{\sigma,\sigma'}(u_\sigma\otimes u_{\sigma'})=u_{\sigma\otimes\sigma'} \tag{2.23}$$

Finally for a *non–Cartesian enrichment* the indexing data contains

(6) counit morphisms $\epsilon_\phi:\mathcal{D}(\phi)\to\mathbb{1}_{\mathcal{E}}$ which are functorial:
$$\epsilon_\phi(\hat\phi)\epsilon_\psi(\hat\psi)=\epsilon_{\phi\circ\psi}(\hat\phi\circ\hat\psi) \tag{2.24}$$
and are equivariant
$$\epsilon_{(\sigma\Downarrow\sigma')\phi}(\sigma\Downarrow\sigma')=(\sigma\Downarrow\sigma')\epsilon_\phi \tag{2.25}$$
and in the unital case unital
$$\epsilon_\sigma(u_\sigma)=1 \tag{2.26}$$
here 1 is the element $id_{\mathbb{1}_{\mathcal{E}}}:\mathbb{1}_{\mathcal{E}}\to\mathbb{1}_{\mathcal{E}}$.

Proposition 2.19. *There is a bijection between (unital) (counital) (symmetric (monoidal)) indexing data and (unital) (counital) (symmetric (monoidal)) bimodules.*

Definition-Proposition 2.20. *A set of unital indexing data defines a category $\mathcal{C}_{\mathcal{D}}$ with the same objects* $\mathrm{Obj}(\mathcal{C}_{\mathcal{D}}):=\mathrm{Obj}(\mathcal{C})$ *and*
$$\mathrm{Hom}_{\mathcal{C}_{\mathcal{D}}}(X,Y)=\coprod_{\phi\in Hom_{\mathcal{C}}(X,Y)}\mathcal{D}(\phi) \tag{2.27}$$
with the $\mathcal{D}(\phi)$ objects of an enrichment category $\mathcal{E}$, where we assume that the coproducts exist in E. The source and target maps are $s(\mathcal{D}(\phi))=s(\phi)$, $t(\mathcal{D}(\phi))=t(\phi)$. The composition and unit maps are given by the additional indexing data. If the data is counital there is a morphism $\epsilon:\mathcal{C}_{\mathcal{D}}\to\mathcal{C}$.

In this way there is a bijection of unital counital indexing data and indexed enrichment.

2.5. **Enrichment functors.** A succinct way to define this data is as coming from an enrichment functor. This is the point of view of [KW17,Kau21], which will now be generalized to ((symmetric) monoidal) categories. We give the definition and unravel the data. Let $\mathcal{C}$ be a category and $\mathcal{E}$ be an enrichment category. Consider $\mathcal{D}(\mathcal{C})$ as in §2.1.1. Let $\mathcal{D}(\mathcal{E})$ be the usual horizontal double category for a monoidal category $\mathcal{E}$. By this we mean that $\mathcal{D}(\mathcal{E})$ has one object, $\mathrm{Mor}_h(\mathcal{D}(\mathcal{E}))=Obj(\mathcal{E})$ with $\circ_h=\otimes$ as composition. $\mathrm{Mor}_v(\mathcal{D}(\mathcal{E}))$ is trivial and 2-$Mor(\mathcal{D}(\mathcal{E}))=Mor(\mathcal{E})$.

Definition 2.21. An *enrichment functor* is a horizontally lax functor and vertically strict functor of bi–categories $\mathcal{D}(\mathcal{C})\to\mathcal{D}(\mathcal{E})$. If $\mathcal{C}$ is monoidal, then $\mathcal{D}(\mathcal{C})$ is monoidal and a *monoidal* enrichment functor is a lax monoidal functor $\mathcal{D}(\mathcal{C})\to\mathcal{D}(\mathcal{E})$.

To give such a functor one needs to specify the following data:

(1) A functor $\mathcal{D}:Iso(\mathcal{C}\downarrow\mathcal{C})\to\mathcal{E}$ that is for all $\phi\in Mor(\mathcal{C})$, $\mathcal{D}(\phi)\in\mathcal{E}$ and an action $\mathcal{D}((\sigma\overset{\phi}{\Downarrow}\sigma')):\mathcal{D}(\phi)\to\mathcal{D}((\sigma\Downarrow\sigma')(\phi))$, such that $\mathcal{D}((id_{s(\phi)}\overset{\phi}{\Downarrow}id_{t(\phi)}))=id_{\mathcal{D}(\phi)}$.

(2) Morphisms for the lax–horizontal structure:

$$\gamma^{\mathcal{D}}_{\phi_1,\phi_0} : \mathcal{D}(\phi_1) \otimes_{\mathcal{E}} \mathcal{D}(\phi_0) \to \mathcal{D}(\phi_1 \circ \phi_0) \tag{2.28}$$

for any pair of composable morphisms compatible with the $\mathcal{D}((\sigma \Downarrow \sigma'))$:

$$\gamma^{\mathcal{D}} \circ \mathcal{D}((\sigma \Downarrow \sigma')) \otimes \mathcal{D}(\sigma' \Downarrow \sigma'') = \mathcal{D}(\sigma' \Downarrow \sigma'') \circ \gamma^{\mathcal{D}} \tag{2.29}$$

and units

$$u_{id_x} : \mathbb{1}_E \to \mathcal{D}(id_X) \tag{2.30}$$

which satisfy $\gamma(id \otimes u)(id \otimes l) = \gamma(u \otimes id)(r \otimes u) = id$ where l, r are the inverse unit constraints in $\mathcal{E}$ and the indices have been suppressed. The long form for the left unit is

$$\begin{tikzcd} \mathcal{D}(\phi) \arrow[r, "l=\lambda^{-1}_{\mathcal{D}(\phi)}"] \arrow[d, "id_{D(\phi)}"] & \mathbb{1}_{\mathcal{E}} \otimes \mathcal{D}(\phi) \arrow[d, "u_{id_{s(\phi)}} \times id_{\mathcal{D}(\phi)}"] \\ \mathcal{D}(\phi) & \mathcal{D}(id_{s(\phi)}) \otimes \mathcal{D}(\phi) \arrow[l, "\gamma"] \end{tikzcd} \tag{2.31}$$

(3) The u_{id_X} are compatible with the action of $\mathcal{D}((\sigma \Downarrow \sigma'))$. This means that there is a coherent set of elements $u_\sigma : \mathbb{1}_{\mathcal{E}} \to \mathcal{D}(\sigma)$ for all isomorphisms σ, satisfying $\mathcal{D}((\sigma \Downarrow \sigma'))(u_\tau) = u_{\sigma'\tau\sigma^{-1}}$, such that the following diagrams commute.

$$\begin{tikzcd} \mathcal{D}(\phi) \arrow[r, "\sim"] \arrow[d, "\mathcal{D}((\sigma \overset{\phi}{\Downarrow} \sigma'))=:"'] & \mathbb{1}_{\mathcal{E}} \otimes_{\mathcal{E}} \mathcal{D}(\phi) \otimes_{\mathcal{E}} \mathbb{1}_{\mathcal{E}} \arrow[d, "u_\sigma \otimes id \otimes u_{\sigma'}"] \\ \mathcal{D}((\sigma \Downarrow \sigma')(\phi)) & \mathcal{D}(\sigma) \otimes \mathcal{D}(\phi) \otimes \mathcal{D}(\sigma') \arrow[l, "\gamma(\gamma \otimes id)"] \end{tikzcd} \tag{2.32}$$

NB: In particular, $u_\sigma = \mathcal{D}((\mathrm{id}_{s(\sigma)} \Downarrow \sigma))u_{\mathrm{id}_{s(\sigma)}} = \mathcal{D}((\sigma^{-1} \Downarrow id_{t(\sigma)}))u_{id_{t(\sigma)}}$.

A lax (symmetric) monoidal enrichment functor also includes the data

(4)

$$\mu^{\mathcal{D}}_{\phi,\psi} : \mathcal{D}(\phi) \otimes \mathcal{D}(\psi) \to \mathcal{D}(\phi \otimes \psi) \tag{2.33}$$

(5)

$$\mathcal{D}((\sigma \Downarrow \sigma')) \otimes_{\mathcal{E}} \mathcal{D}((\tau \Downarrow \tau')) \to \mathcal{D}((\sigma \otimes \tau \Downarrow \sigma' \otimes \tau')) \tag{2.34}$$

compatible with $\mu^{\mathcal{D}}$.

$$\mu^{\mathcal{D}} \circ \mathcal{D}((\sigma \Downarrow \sigma')) \otimes \mathcal{D}((\tau \Downarrow \tau')) = \mathcal{D}((\sigma \otimes \tau \Downarrow \sigma' \otimes \tau')) \circ \mu^{\mathcal{D}} \tag{2.35}$$

The trivial enrichment functor $\mathcal{T}_{\mathcal{E}}$ is given by the $\mathcal{D}(\phi) = \mathbb{1}_{\mathcal{E}}$ and the unit squares.

A *counit* for an enrichment functor $\mathcal{D}$ is a strict double natural transformation $\epsilon : \mathcal{D} \to \mathcal{T}_{\mathcal{E}}$. This is given by the data $\epsilon_\phi : \mathcal{D}(\phi) \to \mathbb{1}$ that respects the other structures.

Let $\mathcal{D}(i) : \mathcal{D}(\mathrm{Iso}(\mathcal{C})) \to \mathcal{D}(\mathcal{C})$ be the natural inclusion. A *unit* for an enrichment functor is given by a strict double natural transformation $u : \mathcal{T}_{\mathcal{E}} \to \mathcal{D} \circ \mathcal{D}(i)$ of enrichment functors of $\mathrm{Iso}(\mathcal{C})$. the data are coherent morphisms $u(\sigma) : \mathbb{1} \to \mathcal{D}(\sigma)$

NB: If $\mathcal{D}$ is monoidal, then $\mathcal{C}_{\mathcal{D}}$ is monoidal using the $\mu_{\phi,\psi}$ with unitors, associators and, in the symmetric case, commutators given as in Example A.9.

By inspection of the lists, one arrives at the following proposition which identifies indexing data with enrichment functors.

Proposition 2.22. *The data of a (unital) counital ((symmetric) monoidal) enrichment functor is identical with a set of (unital) ((symmetric) monoidal) indexing data.* □

Remark 2.23 (Connection and holonomy)**.** The axioms are closely related to connections and holonomy in the sense of [BS76] of $\mathcal{C}^\square$. There the connection is given simply by $\bar{\phi} = \phi$ and the holonomy by the unique squares. There is a restriction of the structure fo $D(\mathcal{C})$. Preserving the structure in the double functor yields the unit and counit. Weakly preserving connection yields morphisms $u_\sigma : \mathcal{D}_v(\sigma) = \mathbb{1}_\mathcal{E} \to \mathcal{D}_h(\sigma)$ preserving the holonomy is equivalent to $\mathcal{D}((\sigma \Downarrow \sigma'))(u_\tau) = u_{(\sigma \Downarrow \sigma')(\tau)}$. The equation (2.32) is then the extension back to all morphisms, cf. [Kau21, Appendix C].

2.5.1. **Summary.** Summarizing the results of this section.

Theorem 2.24. *Let $\mathcal{C}$ be a ((symmetric) monoidal) category, $\mathcal{G} = \mathrm{Iso}(\mathcal{C})$ and $\mathcal{E}$ be symmetric monoidal category. There are isomorphisms between the categories of*

(1) *(Unital) (counital) ((symmetric) monoidal) indexing data.*
(2) *(Unital) (counital) ((symmetric) monoidal) enrichment functors to $\underline{\mathcal{E}}$.*

In the counital case, a condition which is automatically satisfied if $\mathcal{E}$ is Cartesian, the above are equivalent to

(3) *(Unital) ((symmetric) monoidal) $\mathrm{Iso}(\mathcal{C})$ bimodule monoids in $\mathcal{E}$ over $\mathrm{Hom}_\mathcal{C}$.*

In the unital and counital case the above are equivalent to

(4) *$\mathcal{E}$ enriched ((symmetric) monoidal) categories which are index enriched over $\mathcal{C}$ with a section on the isomorphisms.*

Proof. This follows from Propositions 2.16, 2.19 and 2.22 □

3. The plus constructions

The plus construction come comes in several flavors like the free monoidal construction and its nc–version for monoidal categories, see §A.1.3 and §A.1.5. These are the plus construction for a category 3.1, the nc–plus construction for a monoidal category, §3.2 and its strong version §3.3. It is possible to consider these constructions over enriched categories, but we will not spell out the details here.

3.1. The plus construction for a category: $\mathcal{C}^{\boxtimes+}$.

Definition 3.1. The underlying strict (symmetric) monoidal groupoid of $\mathcal{C}^{\boxtimes+}$ is the free groupoid

$$\mathrm{Iso}(\mathcal{C}\downarrow\mathcal{C})^{\boxtimes} = \mathrm{Iso}(\mathcal{C}^{\boxtimes}\downarrow C^{\boxtimes}) \tag{3.1}$$

This means that an object of $\mathcal{C}^{\boxtimes+}$ is a word $\Phi = \phi_1 \boxtimes \cdots \boxtimes \phi_n$ of morphisms $\phi_i \in \mathcal{C}$ and an isomorphisms is given by a word $(\sigma_1 \Downarrow \sigma'_1) \boxtimes \cdots \boxtimes (\sigma_n \Downarrow \sigma'_n)$.

To give the *general morphisms* of $\mathcal{C}^{\boxtimes+}$ and their composition, we adjoin generators to the (symmetric) monoidal category and mod out by relations.

Generators: For every composable pair (ϕ_1, ϕ_0), $X_0 \stackrel{\phi_0}{\to} X_1 \stackrel{\phi_1}{\to} X_2$, there is one generator

$$\gamma_{\phi_1,\phi_0} : \phi_1 \boxtimes \phi_0 \to \phi_1 \circ \phi_0 \tag{3.2}$$

NB: We will use the notation which is commensurate to the common composition convention, where the composable morphisms are in $\mathrm{Mor}\,{}_s\times_t\,\mathrm{Mor}$. In this way γ turns $\boxtimes$ into

$\circ$. Another equivalent option would be to use the nerve convention, that is the composable morphisms are in $\mathrm{Mor}\,{}_t\!\times_s \mathrm{Mor}$.

RELATIONS: The relations are the usual relations for a (symmetric) monoidal category, that is associativity, identities and interchange, and the following additional relations: equivariance with respect to isomorphisms (3.3), (3.4) and inner associativity (3.5).

(1) *Equivariance with Isomorphisms.*
   (a) *"Inner" equivariance* with respect to isomorphisms. For $\sigma : X_1 \to X_1'$. The following diagram commutes:

$$\begin{tikzcd} \phi_1 \boxtimes \phi_0 \arrow[r, "(\sigma\Downarrow id)\boxtimes(id\Downarrow\sigma)"] \arrow[d, "\gamma_{\phi_1,\phi_0}"'] & (\phi_1 \circ \sigma^{-1}) \boxtimes (\sigma \circ \phi_0) \arrow[d, "\gamma_{\phi_1\circ\sigma^{-1},\sigma\circ\phi_0}"] \\ \phi_1 \circ \phi_0 \arrow[r, equal] & \phi_1 \circ \phi_0 \end{tikzcd} \tag{3.3}$$

   (b) *"Outer" equivariance* with respect to isomorphisms on $\sigma : X_0 \to X_0'$ and $\sigma' : X_2 \to X_2'$.

$$\begin{tikzcd} \phi_1 \boxtimes \phi_0 \arrow[r, "(id\Downarrow\sigma')\boxtimes(\sigma\Downarrow id)"] \arrow[d, "\gamma_{\phi_1,\phi_0}"'] & (\sigma' \circ \phi_1) \boxtimes (\phi_0 \circ \sigma^{-1}) \arrow[d, "\gamma_{\sigma'\circ\phi_1,\phi_0\circ\sigma^{-1}}"] \\ \phi_1 \circ \phi_0 \arrow[r, "(\sigma\Downarrow\sigma')"'] & \sigma' \circ \phi_1 \circ \phi_0 \circ \sigma^{-1} \end{tikzcd} \tag{3.4}$$

(2) *Internal Associativity*: For triples of composable morphisms (ϕ_2, ϕ_1, ϕ_0), $X_0 \xrightarrow{\phi_0} X_1 \xrightarrow{\phi_1} X_2 \xrightarrow{\phi_2} X_3$ the following diagram commutes:

$$\begin{tikzcd} \phi_2 \boxtimes \phi_1 \boxtimes \phi_0 \arrow[r, "\gamma_{\phi_2,\phi_1}\boxtimes id_{\phi_0}"] \arrow[d, "\mathrm{id}_{\phi_2}\boxtimes\gamma_{\phi_1,\phi_0}"'] & (\phi_2 \circ \phi_1) \boxtimes \phi_0 \arrow[d, "\gamma_{\phi_2\circ\phi_1,\phi_0}"] \\ \phi_2 \boxtimes (\phi_1 \circ \phi_0) \arrow[r, "\gamma_{\phi_2,\phi_1\circ\phi_0}"'] & \phi_2 \circ \phi_1 \circ \phi_0 \end{tikzcd} \tag{3.5}$$

   where $id_\phi : \phi \to \phi$ are the unit morphisms.

As a short hand, we will use $\gamma_{\phi_0,\dots,\phi_n} : \phi_0 \boxtimes \cdots \boxtimes \phi_n \to \phi_0 \circ \cdots \circ \phi_n$ for the unique morphism resulting from any n–fold iteration of γ's. In particular the morphism in (3.5) is $\gamma_{\phi_0,\phi_1,\phi_2}$. We also use the convention that $\gamma_\phi := \mathrm{id}_\phi = (id_{s(\phi)} \overset{\phi}{\Downarrow} id_{t(\phi)})$.

Remark 3.2. The inner and outer compatibilities reflect the structure of being a monoid for the plethysm product in accordance with §2.2. They are also necessary to guarantee good properties with respect to equivalence of categories, see §5.

The internal associativity and interchange are certain strictness conditions. One could alternatively introduce 2–morphisms or ∞-structures.

Theorem 3.3. *For a Cartesian $\mathcal{E}$, the category of strong monoidal functors $[\mathcal{C}^{\boxtimes+}, \mathcal{E}]_\otimes$ is equivalent to the category of* $\mathrm{Iso}(\mathcal{C})$ *bi–module monoids ρ over* $\mathrm{Hom}_\mathcal{C}$*.*

In the non–Cartesian case the category of counital bi–module monoids ρ over $\mathrm{Hom}_\mathcal{C}$ *is equivalent to the category of counital strong monoidal functors, that is those with a given natural transformation to the trivial functor $\mathcal{T} : \mathcal{C}^{\boxtimes+} \to \mathcal{E}$. Furthermore, the isomorphism split bimodules correspond to the split functor, see §3.4.*

NB: Counits are automatic in the Cartesian case.

Proof. We will use enrichment data to show the equivalence. That a functor determines the enrichment data (1)-(2) of 2.4 and hence by Proposition 2.19 a monoidal bi–module ρ over $\mathrm{Hom}_{\mathcal{C}}$ is clear. In the other direction define the functor using the indexing data, first use the $\mathcal{D}(\phi) \in \mathcal{E}$ together with maps $\mathcal{D}((\sigma \Downarrow \sigma')) : \mathcal{D}(\phi) \to \mathcal{D}((\sigma \Downarrow \sigma')(\phi))$ to define a functor $\mathcal{D} \in [\mathrm{Iso}(\mathcal{C} \Downarrow \mathcal{C}), \mathcal{E}]$. This extends to a monoidal functor $\mathcal{D}^{\boxtimes} \in [\mathrm{Iso}(\mathcal{C} \Downarrow \mathcal{C})^{\boxtimes}, \mathcal{E}]_{\otimes}$. Extend $\mathcal{D}^{\boxtimes}$ to $\mathcal{D}^{\boxtimes +}$ by setting $\mathcal{D}^{\boxtimes +}(\gamma_{\phi_1,\phi_0}) := \gamma^{\mathcal{D}}_{\phi_1,\phi_0} : \mathcal{D}(\phi_1) \otimes_{\mathcal{E}} \mathcal{D}(\phi_0) \to \mathcal{D}(\phi_1 \circ \phi_0)$ for the generating morphisms. This is associative and the equivariance is guaranteed by (2.16) and hence defines a monoidal functor from $\mathcal{C}^{\boxtimes +}$ to $\mathcal{E}$. This is easily checked to be an equivalence using the coherence data (3) to define a natural transformation as in §A.1.5.

In the non–Cartesian case the data of the counits $\epsilon_\phi : \mathcal{D}(\phi) \to \mathbb{1}$ is exactly a natural transformation from $\mathcal{D}$ to $\mathcal{T}$. $\square$

3.2. A plus construction for (symmetric) monoidal categories $\mathcal{M}^{nc+}$.

Definition 3.4. Similar to $\mathcal{M}^{nc}$, see §A.1.5, the (symmetric) monoidal category $\mathcal{M}^{nc+}$ is obtained from $\mathcal{M}^{\boxtimes +}$ by additionally adjoining new generators and imposing new relations.
ADDITIONAL GENERATORS:

$$\mu_{\phi_0,\phi_1} : \phi_1 \boxtimes \phi_2 \to \phi_1 \otimes \phi_2 \tag{3.6}$$

ADDITIONAL RELATIONS:

(1) *Equivariance with respect to isomorphisms*

$$\begin{tikzcd}[column sep=huge]
\phi_1 \boxtimes \phi_2 \arrow[r, "(\sigma_1 \overset{\phi_1}{\Downarrow} \sigma_1')\boxtimes(\sigma_2 \overset{\phi_2}{\Downarrow} \sigma_2')"] \arrow[d, "\mu"] & \tilde{\phi_1} \boxtimes \tilde{\phi_2} \arrow[d, "\mu"] \\
\phi_1 \otimes \phi_2 \arrow[r, "(\sigma_1\otimes\sigma_2 \overset{\phi_1\otimes\phi_2}{\Downarrow} \sigma_1'\otimes\sigma_2')"] & \tilde{\phi_1} \otimes \tilde{\phi_2}
\end{tikzcd} \tag{3.7}$$

(2) *Internal Associativity for μ.* I.e. the following diagrams commute

$$\begin{tikzcd}
\phi_1 \boxtimes \phi_2 \boxtimes \phi_3 \arrow[r, "\mu_{\phi_1,\phi_2}\boxtimes id_{\phi_3}"] \arrow[d, "id\boxtimes\mu_{\phi_2,\phi_3}"] & \phi_1 \otimes \phi_2 \boxtimes \phi_3 \arrow[d, "\mu_{\phi_1\otimes\phi_2,\phi_3}"] \\
\phi_1 \boxtimes \phi_2 \boxtimes \phi_3 \arrow[r, "\mu_{\phi_1,\phi_2\otimes\phi_3}"] & \phi_1 \otimes \phi_2 \otimes \phi_3
\end{tikzcd} \tag{3.8}$$

(3) *Internal Interchange.* I.e. the following diagrams commute

$$\begin{tikzcd}
\phi_0 \boxtimes \psi_0 \boxtimes \phi_1 \boxtimes \psi_1 \arrow[r, "\tau_{23}"] \arrow[d, "(\mu\boxtimes\mu)"] & \phi_0 \boxtimes \phi_1 \boxtimes \psi_0 \boxtimes \psi_1 \arrow[r, "(\gamma_{\phi_0,\phi_1}\boxtimes\gamma_{\psi_0,\psi_1})"] & \phi_1 \circ \phi_0 \boxtimes \psi_0 \circ \psi_1 \arrow[d, "\mu"] \\
(\phi_0 \otimes \psi_0) \boxtimes (\phi_1 \otimes \psi_1) \arrow[r, "\gamma_{\phi_0\otimes\phi_1,\psi_0\otimes\psi_1}"] & (\phi_0 \otimes \psi_0) \circ (\phi_1 \otimes \psi_1) \arrow[r, equal] & (\phi_1 \circ \phi_0) \otimes (\psi_0 \circ \psi_1)
\end{tikzcd} \tag{3.9}$$

(4) *Compatibility with commutators*
When $\mathcal{M}$ is symmetric monoidal there is the additional quadratic relation.

$$\begin{tikzcd} \phi_1 \boxtimes \phi_2 \arrow[r, "\tau_{12}"] \arrow[d, "\mu_{\phi_1,\phi_2}"] & \phi_2 \boxtimes \phi_1 \arrow[d, "\mu_{\phi_2,\phi_1}"] \\ \phi_1 \otimes \phi_2 \arrow[r, "(C_{12} \Downarrow C_{12})"] & \phi_2 \otimes \phi_1 \end{tikzcd} \tag{3.10}$$

NB: The morphisms in a monoidal category always have a symmetric structure. The isomorphisms contain the associativity, unit and in the symmetric case commutativity constraints on the level of morphisms, see Example A.8.

Remark 3.5. There are several technical remarks about these constructions:

(1) A general morphism in $\mathcal{C}^{\boxtimes+}$ and $\mathcal{M}^{nc+}$ is obtained by concatenating and forming of tensor products of identities, generating morphisms, and isomorphisms, modulo the relations of a monoidal category — associativity, units, interchange — and modulo the additional relations above.
(2) There is a natural *degree of morphism* which is defined to be the number of γ's and μ's. This is well defined as all relations are homogeneous with respect to this degree. The degree is additive under tensor product and composition. The degree of isomorphisms including identities is 0. It is also the number of $\boxtimes$ changed to $\otimes$ or $\circ$.
(3) The unit, associators, and in the symmetric case, the commutators descend to the quotients. The unit is $id_{\mathbb{1}}$. In the strict case $\phi \otimes \mathrm{id}_{\mathbb{1}} = \phi$ as a morphism $X \otimes \mathbb{1} = X \to Y$. In the non–strict case, the unit constraints and associativity constraints descend. As do the commutativity constraints, see Example A.9.
(4) Associativity and, in the symmetric case, interchange hold automatically in the quotient and do not lead to additional identifications there.
   (a) Associativity:

$$\begin{aligned} & \gamma_{\phi_0\circ\phi_1\circ\phi_2,\phi_3} \circ (\gamma_{\phi_0\circ\phi_1,\phi_2} \boxtimes id) \circ (\gamma_{\phi_0,\phi_1} \boxtimes id \boxtimes id) \\ = \; & \gamma_{\phi_0\circ\phi_1\circ\phi_2,\phi_3} \circ (\gamma_{\phi_0,\phi_1,\phi_2} \boxtimes id) \\ = \; & \gamma_{\phi_0,\phi_1.\phi_2,\phi_3} \\ = \; & \gamma_{\phi_0\circ\phi_1,\phi_2,\phi_3} \circ (\gamma_{\phi_0,\phi_1} \boxtimes id \boxtimes id) \\ = \; & (\gamma_{\phi_0\circ\phi_1\circ\phi_2,\phi_3} \circ \gamma_{\phi_0\circ\phi_1,\phi_2} \boxtimes id) \circ (\gamma_{\phi_0,\phi_1} \boxtimes id \boxtimes id) \end{aligned} \tag{3.11}$$

   For the isomorphisms among themselves this is the fact that $\mathrm{Iso}(\mathcal{M} \downarrow \mathcal{M})$ is a subcategory. The compatibility equation together with associativity mean that there is an action of the isomorphisms on the γ's and hence associativity holds in $\mathcal{C}^{nc+}$. The computations for μ are similar and associativity holds automatically in $\mathcal{M}^{nc+}$ as well.
   (b) Interchange: The interchange relation holds automatically in $\mathcal{C}^{nc+}$ and $\mathcal{M}^{nc+}$ considered with product induced by $\boxtimes$. For two isomorphisms, this is clear. For the other combinations, this follows directly by calculation. As an example:

$$\begin{aligned} & (\gamma_{\phi_2,\phi_1\circ\phi_0} \circ (id_{\phi_2} \boxtimes \gamma_{\phi_1,\phi_0})) \boxtimes (\gamma_{\psi_2,\psi_1\circ\psi_0} \circ (id_{\psi_0} \boxtimes \gamma_{\psi_1,\psi_2})) \\ = \; & \gamma_{\phi_2,\phi_1,\phi_0} \boxtimes \gamma_{\psi_2,\psi_1,\psi_0} \\ = \; & (\gamma_{\phi_2,\phi_1\circ\phi_0} \boxtimes \gamma_{\psi_2,\psi_1\circ\psi_0}) \circ ((id_{\phi_2} \boxtimes \gamma_{\phi_1,\phi_0}) \boxtimes (id_{\psi_0} \boxtimes \gamma_{\psi_1,\psi_2})) \end{aligned} \tag{3.12}$$

Analogous equations hold for all similar expressions that is for all other composable iterations of γ's and isomorphisms. The same holds for morphisms including μ.

(5) One can pack the two compatibilities into one equation:

$$\begin{aligned} &\gamma_{\sigma'_Z\circ\phi\circ\sigma_Y^{-1},\tilde\sigma_Y\circ\sigma'_Y\circ\psi\circ\sigma_X^{-1}}\circ((\sigma_Y\Downarrow\sigma'_Z)\otimes(\sigma_X\Downarrow\tilde\sigma\circ\sigma'_Y))\\ =\;&(\sigma_X\Downarrow\sigma'_Z)\circ\gamma_{\phi\circ\sigma_Y^{-1}\circ\tilde\sigma_Y,\sigma'_Y\circ\psi}\circ((\tilde\sigma_Y^{-1}\circ\sigma_Y\Downarrow id)\otimes(id\Downarrow\sigma'_Y)) \end{aligned}\tag{3.13}$$

as a morphism: $\phi\otimes\psi\to\sigma_Z\circ\phi\circ\sigma_Y^{-1}\circ\tilde\sigma_Y\circ\sigma'_Y\circ\psi\circ\sigma_X^{-1}$.

We will use the following short hand notation for μ's. Given a source $\phi_1\boxtimes\ldots\boxtimes\phi_{n+1}$ for $I\subset\{1,\ldots,n\}$ we let μ_I be the map that converts the ith occurrence of $\boxtimes$ to $\otimes$ for all $i\in I$. Explicitly, if $I=\{i_1,\ldots,i_k\}$ with $i_1<\cdots<i_k$, then inductively set $\mu_I=\mu_{I\setminus\{i_k\}}\circ(id_{\phi_1}\otimes\cdots\otimes id_{\phi_{i_k}-1}\otimes\mu_{\phi_{i_k},\phi_{i_k+1}}\otimes\mathrm{id}_{\phi_{i_k+2}}\otimes\cdots\otimes id_{\phi_{n+1}})$. We also set $\mu_n=\mu_{\{1,\ldots,n\}}$.

Theorem 3.6. *The category of strong monoidal functors $[\mathcal{M}^{nc+},\mathcal{E}]_\otimes$ is equivalent to the category of lax–monoidal* $\mathrm{Iso}(\mathcal{M}\downarrow\mathcal{M})$*–bimodules monoids over* $\mathrm{Hom}_{\mathcal{M}}$.

In the non–Cartesian case the statement is restricted to counital functors and bimodules, and the isomorphism split bimodules correspond to the split functor, see §3.4.

Proof. This is similar to the proof of Theorem 3.3. The extra data needed for the lax monoidal structure are the morphisms $\mu^{\mathcal{D}}$ of (2.33). These are obtained via $\mu^{\mathcal{D}}_{\phi,\psi}:\mathcal{D}(\phi)\otimes_{\mathcal{E}}\mathcal{D}(\psi)\overset{\mathcal{D}_{\phi,\psi}}{\overset{\sim}{\to}}\mathcal{D}(\phi\boxtimes\psi)\overset{\mathcal{D}(\mu_{\phi,\psi})}{\to}\mathcal{D}(\phi\otimes\psi)$ and vice–versa, the $\mu^{\mathcal{D}}_{\phi,\psi}$ yield the $\mathcal{D}(\mu_{\phi,\psi})$ as the $\mathcal{D}_{\phi,\psi}$ are invertible. □

3.2.1. **Consolidated generators.** The following compositions of generators will be useful, see §7.7. Set

$$\begin{aligned} \overset{\sigma_0}{-}\phi_1\overset{\sigma_1}{-}\phi_2\overset{\sigma_2}{-}\cdots\overset{\sigma_{n-1}}{-}\phi_n\overset{\sigma_n}{-}:=\;&\gamma_{\sigma_0\circ\phi_1\circ\sigma_1,\ldots,\phi_{n-1}\circ\sigma_{n-1},\phi_n}\circ\\ &[(\mathrm{id}\Downarrow\sigma_0)(\phi_1)\otimes\cdots\otimes(id\Downarrow\sigma_{n-2})(\phi_{n-1})\otimes(\sigma_n^{-1}\Downarrow\sigma_{n-1})(\phi_n)] \end{aligned}\tag{3.14}$$

This is a morphism $\phi_1\boxtimes\cdots\boxtimes\phi_n\to\sigma_0\circ\phi_1\circ\sigma_1\circ\cdots\circ\sigma_{n-1}\circ\phi_n\circ\sigma_n$

Remark 3.7. Due to the equivariance under isomorphisms, there are several other ways to write this morphisms, by "distributing" the action of the isomorphisms, e.g.:

$$\begin{aligned} \overset{\sigma_0}{-}\phi_1\overset{\sigma_1}{-}\phi_2\overset{\sigma_2}{-}\cdots\overset{\sigma_{n-1}}{-}\phi_n\overset{\sigma_n}{-}=\;&(\sigma_n^{-1}\Downarrow\sigma_0)(\phi_1\circ\sigma_1\circ\cdots\circ\sigma_{n-1}\circ\phi_{n-1}\circ\phi_n)\\ &\circ\gamma_{\phi_1,\sigma_1\circ\phi_2.,\ldots,\sigma_{n-1}\circ\phi_n}\circ[(\mathrm{id}\Downarrow\mathrm{id})(\phi_1)\otimes(\mathrm{id}\Downarrow\sigma_1)(\phi_2)\otimes\cdots\otimes(id\Downarrow\sigma_{n-1})(\phi_n)] \end{aligned}\tag{3.15}$$

Lemma 3.8. *The following identities hold:*

(i) *Concatenation:*

$$\begin{aligned} &\overset{\sigma_0}{-}\phi_1\overset{\sigma_1}{-}\phi_2\overset{\sigma_2}{-}\cdots\overset{\sigma_{n-1}}{-}\phi_n\overset{\sigma_n}{-}\circ\overset{\sigma'_0}{-}\phi'_1\overset{\sigma'_1}{-}\phi'_2\overset{\sigma'_2}{-}\cdots\overset{\sigma'_{m-1}}{-}\phi'_m\overset{\sigma'_m}{-}=\\ &\qquad\overset{\sigma_0}{-}\phi_1\overset{\sigma_1}{-}\phi_2\overset{\sigma_2}{-}\cdots\overset{\sigma_{n-1}}{-}\phi_n\overset{\sigma'_0\circ\sigma_n}{-}\phi'_1\overset{\sigma'_1}{-}\phi'_2\overset{\sigma'_2}{-}\cdots\overset{\sigma'_{m-1}}{-}\phi'_m\overset{\sigma'_m}{-} \end{aligned}\tag{3.16}$$

(ii) *Action of isomorphisms:*

$$(\nu'\Downarrow\nu')\circ \overset{\sigma_0}{-}\phi_1\overset{\sigma_1}{-}\phi_2\overset{\sigma_2}{-}\cdots\overset{\sigma_{n-1}}{-}\phi_n\overset{\sigma_n}{-}\circ(\tau_1\Downarrow\tau_1')\boxtimes\cdots\boxtimes(\tau_n\Downarrow\tau_n') =$$
$$\overset{\nu\sigma_0}{-}\phi_1\overset{\sigma_1\tau_1'}{-}\phi_2\overset{\tau_1^{-1}\sigma_2\tau_2'}{-}\cdots\overset{\tau_{n-2}^{-1}\sigma_{n-1}\tau_{n-1}'}{-}\phi_n\overset{\sigma_n\nu^{-1}}{-} \quad (3.17)$$

(iii) *Interchange:*

$$\mu\circ\left[\overset{\sigma_0}{-}\phi_1\overset{\sigma_1}{-}\phi_2\overset{\sigma_2}{-}\cdots\overset{\sigma_{n-1}}{-}\phi_n\overset{\sigma_n}{-}\boxtimes\overset{\tau_0}{-}\psi_1\overset{\tau_1}{-}\psi_2\overset{\tau_2}{-}\cdots\overset{\tau_{n-1}}{-}\psi_n\overset{\tau_n}{-}\right]$$
$$=\overset{\sigma_0\otimes\tau_0}{-}\phi_1\otimes\psi_1\overset{\sigma_1\otimes\tau_2}{-}\phi_2\otimes\psi_2\overset{\sigma_2\otimes\tau_2}{-}\cdots\overset{\sigma_{n-1}\otimes\tau_{n-1}}{-}\phi_n\otimes\psi_n\overset{\sigma_n\otimes\tau_n}{-}\circ[\mu^{\boxtimes n}]\circ c^+_{n,n} \quad (3.18)$$

where $C^+_{n,n}$ is the (n,n)–shuffle shuffling in the ψ_i to the right of the ϕ_i.

Proof. The first statement is clear using the definition. The second is an application of (3.3) and (3.4). The last statement follows from (3.9). □

3.3. The strong plus construction for a (symmetric) monoidal category: $\mathcal{M}^+$.

Definition 3.9. Let $(\mathcal{M},\otimes)$ be a (symmetric) monoidal category. The *localized plus construction* $\mathcal{M}^{loc+}=\mathcal{M}^{nc+}[\mu^{-1}]$ is given by the localization with respect to the morphisms μ. In particular, this is given by inverting the morphisms μ in $\mathcal{M}^{nc+}$ fiberwise. That is adjoin generators $\mu^{-1}_{\phi;\phi_1,\phi_2}:\phi\to\phi_1\boxtimes\phi_2$ whenever ϕ is decomposable as $\phi=\phi_1\otimes\phi_2$ and mod out by the relations $\mu^{-1}\mu=\mu\mu^{-1}=id$.

A monoidal functor $\mathcal{D}\in[\mathcal{M}^{nc+},\mathcal{E}]_\otimes$ is called μ strong if the maps $\mathcal{D}(\mu_{\phi,\psi})$ are isomorphisms. By the universal property of localization, $[M^{loc+},\mathcal{E}]_\otimes$ is equivalent to the subcategory $[M^{nc+},\mathcal{E}]_{\otimes,\mu-strong}\subset[M^{nc+},\mathcal{E}]_\otimes$ of μ strong functors.

Theorem 3.10. *The category $[M^{loc,+},\mathcal{E}]_\otimes$ is equivalent to strong monoidal bimodule monoids over* $\mathrm{Hom}_\mathcal{M}$*. In the non–Cartesian case the statement is restricted to counital functors and bimodules, and the isomorphism split bimodules correspond to the split functor, see §3.4.*

Proof. Straightforward from Theorem 3.6 and the above. □

3.3.1. Reduced version of the strong plus construction: $\mathcal{M}^+$.

Definition 3.11. A monoidal category $\mathcal{M}$ has *factorizable isomorphisms* if for any given ϕ with given decomposition $\phi=\phi_1\otimes\phi_2$, for any 2-cell $(\sigma\overset{\phi}{\Downarrow}\sigma')$ there are 2-cells $(\sigma_1\overset{\phi_1}{\Downarrow}\sigma_1')$ and $(\sigma_2\overset{\phi_2}{\Downarrow}\sigma_2')$ such that $(\sigma\overset{\phi}{\Downarrow}\sigma')=(\sigma_1\otimes\sigma_2\overset{\phi_1\otimes\phi_2}{\Downarrow}\sigma_1'\otimes\sigma_2')$ is decomposable.

$$\begin{tikzcd}
\tilde\phi_1\boxtimes\tilde\phi_2 \arrow[d, dotted, "\mu"] & & \phi_1\boxtimes\phi_2 \arrow[ll, dotted, "(\sigma_1\overset{\phi_1}{\Downarrow}\sigma_1')\boxtimes(\sigma_2\overset{\phi_2}{\Downarrow}\sigma_2')"'] \arrow[d, "\mu"] \\
\tilde\phi_1\otimes\tilde\phi_2=\tilde\phi & & \phi=\phi_1\otimes\phi_2 \arrow[ll, "(\sigma\overset{\phi}{\Downarrow}\sigma')"]
\end{tikzcd} \quad (3.19)$$

In the symmetric monoidal case either $(\sigma\overset{\phi}{\Downarrow}\sigma')$ or $(C_{12}\Downarrow\mathcal{C}_{12})\circ(\sigma\overset{\phi}{\Downarrow}\sigma')$ has to be decomposable.

This allows for a characterization which is analogous to that of [KW17, Kau21].

Definition-Proposition 3.12. *Assume $\mathcal{M}$ has factorizable isomorphisms. Let $\mathcal{M}^+$ be the category whose underlying groupoid is $\mathrm{Iso}(\mathcal{M}\downarrow\mathcal{M})$ with additional generating morphisms the images of the γ's, which are morphisms $\bar\gamma_{\phi_0,\phi_1}:\phi_1\otimes\phi_2\to\phi_0\circ\phi_1$ that satisfy equivariance with respect to isomorphisms and internal associativity, as given by the formulas (3.3), (3.4) and (3.5), where all occurrences of $\boxtimes$ are replaced by $\otimes$. Then, $\mathcal{M}^+$ is symmetric monoidally equivalent to $\mathcal{M}^{loc+}$.*

Proof. Define a monoidal functor $F:(\mathcal{M}^{loc+},\boxtimes)\to(\mathcal{M}^+,\otimes)$ by $F(\phi_1\boxtimes\cdots\boxtimes\phi_n)=\phi_1\otimes\cdots\otimes\phi_n$, $F((\sigma\overset{\phi}{\Downarrow}\sigma'))=(\sigma\overset{\phi}{\Downarrow}\sigma')$, $F(\mu_{\phi_1,\phi_2})=id_{\phi_1\otimes\phi_2}$, $F(\gamma_{\phi_0,\phi_1})=\bar\gamma_{\phi_0,\phi_1}$. Similarly define the functor $G:\mathcal{M}^+\to\mathcal{M}^{loc+}$ to be the inclusion on the groupoid part and $G(\bar\gamma_{\phi_0,\phi_1})=\gamma_{\phi_0,\phi_1}\mu^{-1}_{\phi_0,\phi_1}$.

Using that any object $\phi_1\boxtimes\cdots\boxtimes\phi_n$ is isomorphic to $\phi=\phi_1\otimes\cdots\otimes\phi_n$ and that the condition (3.19) identifies the isomorphisms of ϕ with those of any decomposition, it is straightforward to show that F and G witness the equivalence. □

3.3.2. **Roof calculus and hereditary condition.** The computation of a localization is greatly simplified in the presence of a roof calculus.

Definition 3.13. A monoidal category has *common factorizations of morphisms* if given the two solid arrows the dotted arrows exist.

$$\begin{tikzcd}
\phi_1^1\boxtimes\phi_1^2\boxtimes\phi_2^1\boxtimes\phi_2^2 \arrow[r, dotted, "(\mu\boxtimes\mu)\sigma_{23}"] \arrow[d, dotted, "\mu\boxtimes\mu"] & \psi_1\boxtimes\psi_2 \arrow[d, "\mu"] \\
\phi_1\boxtimes\phi_2 \arrow[r, "\mu"] & \phi_1\otimes\phi_2=\phi=\psi_1\otimes\psi_2
\end{tikzcd} \tag{3.20}$$

A monoidal category is called *hereditary* if for every pair of composable morphisms (ϕ_1,ϕ_2) and decomposition of $\psi=\phi_2\phi_1$ as $\psi=\psi_1\otimes\psi_2$ there are morphismsd $\phi_1^1,\phi_1^2,\phi_2^1,\phi_2^2$ such that $\phi_1=\phi_1^1\otimes\phi_1^2$, $\phi_2=\phi_2^1\otimes\phi_2^2$, $\psi_1=\phi_2^1\phi_1^1$ and $\psi_2=\phi_2^2\phi_1^2$

In other words given the two solid arrows the dotted arrows exist:

$$\begin{tikzcd}
\phi_1^1\boxtimes\phi_1^2\boxtimes\phi_2^1\boxtimes\phi_2^2 \arrow[r, dotted, "(\gamma\boxtimes\gamma)\sigma_{23}"] \arrow[d, dotted, "\mu\boxtimes\mu"] & \psi_1\boxtimes\psi_2 \arrow[d, "\mu"] \\
\phi_1\boxtimes\phi_2 \arrow[r, "\gamma"] & \phi_2\circ\phi_1=\phi=\psi_1\otimes\psi_2
\end{tikzcd} \tag{3.21}$$

We call $\mathcal{M}$ *fully* hereditary, if it has factorizable isomorphisms, has common factorization for morphisms and is hereditary.

Proposition 3.14. *If the $\mathcal{M}$ is fully hereditary, then the localization can be realized by a right roof calculus*

Proof. The sub–category generated by the μ_{ϕ_1,ϕ_2} is closed under composition and contains all identities. First $\mu\circ\sigma_{12}=\sigma_{12}\circ\mu$. Since the $(\sigma\Downarrow\sigma')$ are invertible, (3.19) is equivalent to the condition that the dotted arrows exist if the solid arrows are given in the following diagrams

$$\begin{tikzcd}
\phi_1\boxtimes\phi_2 \arrow[r, dotted, "(\sigma_1\overset{\phi_1}{\Downarrow}\sigma_1')\boxtimes(\sigma_2\overset{\phi_2}{\Downarrow}\sigma_2')"] \arrow[d, dotted, "\mu"] & \tilde\phi_1\boxtimes\tilde\phi_2 \arrow[d, "\mu"] \\
\phi_1\otimes\phi_2=\phi \arrow[r, "(\sigma\overset{\phi}{\Downarrow}\sigma')"] & \tilde\phi=\tilde\phi_1\otimes\tilde\phi_2
\end{tikzcd} \tag{3.22}$$

The equations (3.21), (3.20) and (3.22) are precisely the right Ore conditions. There are no non–trivial cancelability conditions to check, since —due to the nature of the relations— there are no non–identical parallel morphisms coequalized by a combination of μ's. □

Lemma 3.15. *Two roofs (μ_1, f_1) and (μ_2, f_2) are equivalent if and only if there is a roof (μ_l, μ_r) such that the diagram (3.23) commutes*

$$\begin{tikzcd}
 & & \Phi_a \arrow[dl, "\mu_l"'] \arrow[dr, "f=\mu_r"] & & \\
 & \Phi_v \arrow[dl, "\mu_1"'] \arrow[drrr, "f_1"] & & \Phi_w \arrow[dlll, "\mu_2"'] \arrow[dr, "f_2"] & \\
\Phi_s & & & & \Phi_t
\end{tikzcd} \tag{3.23}$$

Proof. By definition any roof (μ_L, f) could be used in (3.23), but since $\mu_1\mu_l = \mu_I$ for some index set I. Now, inspecting the relations we see that f has to be of the form μ_r with $\mu_2\mu_r = \mu_I$ as no γ and hence no isomorphism may appear. □

3.4. **Enriched case.** When enriching over an arbitrary monoidal category, the issue of splitting becomes relevant in the discussion of isomorphisms and units. The examples one should have in mind are the monoidal categories of R–bimodules and differential graded bi–modules.

Definition 3.16. We say a functor $\mathcal{D} : \mathcal{P}l^+ \to \mathcal{E}$ is *split*, if

(1) for every isomorphism object σ in the underlying category, there exist two objects $\mathcal{D}(\sigma)^\times$ and $\mathcal{D}^{red}(\sigma)$ such that there is a splitting
$$\mathcal{D}(\sigma) = \mathcal{D}_I(\sigma) \odot \mathcal{E} \oplus \mathcal{D}^{red}(\sigma) \tag{3.24}$$
(2) If $\phi \in \mathcal{D}(\sigma)$ has an inverse $\phi^{-1} \in \mathcal{D}(\sigma^{-1})$, then $\phi \in D_I(\sigma) \odot \mathcal{E}$.
(3) The $\mathcal{D}_I$ form a groupoid over the isomorphisms, viz. $\mathcal{D}_I(X,Y) = \amalg_{\sigma \in Iso(X,Y)} \amalg_{\sigma_i \in D_I(\sigma)} \sigma_i$ and there is are composition morphisms $\mathcal{D}_I(\sigma) \times \mathcal{D}_I(\sigma') \to D_I(\sigma\sigma')$ for composable σ, σ', such that fixing an element in $\mathcal{D}_I(\sigma)$ or an element in $\mathcal{D}_I(\sigma')$ the composition morphism is a bijection.

Example 3.17. In the case $\mathcal{E} = \mathcal{S}et$ the splitting is simply given by $\mathcal{D}(\sigma) = \mathcal{D}(\sigma)^\times \amalg \mathcal{D}(\sigma)^{red}$. Condition 3 does not give anything new. And similarly in any Cartesian category.

4. Unital versions and unital plethysm monoids

The above monoids obtained from the plus construction are not necessarily unital. However, one application for the plus construction is indexed enrichments, see §2.4, where one must work with unital structures. To force the existence of a monoid unit, certain extra structures are necessary. These are parallel to those introduced in [Kau21]. They are unital and groupoid compatible pointings (gcp) and the hyper version. Since there are several flavors of plus construction, but the pointings are analogous, we adopt the following notation. $\mathcal{P}l^+$ is any of the plus constructions $\mathcal{C}^{\boxtimes +}, \mathcal{M}^{nc+}, \mathcal{M}^{loc+}, \mathcal{M}^+$.

There are two natural ways to introduce units into the formalism: equipping a functor $\mathcal{D} : \mathcal{P}l^+ \to \mathcal{E}$ with a collection of pointings that satisfy certain properties or augmenting the category $\mathcal{P}l^+$ with extra morphisms. As expected, the two approaches yield the same result.

4.1. Groupoid compatible pointings.

Definition 4.1. A *groupoid compatible pointing* for a strong monoidal functor $\mathcal{D}:\mathcal{P}l^+\to\mathcal{E}$, is a choice of an element $u_\sigma:\mathbb{1}_\mathcal{E}\to\mathcal{D}(\sigma)$ for each isomorphism object σ in $\mathcal{C}$ respectively $\mathcal{M}$ which is compatible with the groupoid action and the composition data:

(1) Compatibility with the groupoid structure:
$$\mathcal{D}(\sigma\Downarrow\sigma')\circ u_\tau = u_{\sigma'\circ\tau\circ\sigma^{-1}} \tag{4.1}$$
(2) Compatibility with composition on the right:
$$\begin{tikzcd} \mathcal{D}(\phi_0)\boxtimes\mathcal{D}(\sigma_1) \arrow[r, "\mathcal{D}(\gamma)"] & \mathcal{D}(\phi_0\circ\sigma_1) \\ \mathcal{D}(\phi_0)\boxtimes\mathbb{1} \arrow[u, "id\boxtimes u_{\sigma_1}"] & \mathcal{D}(\phi_0) \arrow[l, "\simeq", "\text{unit constraints}"'] \arrow[u, "\simeq"', "\mathcal{D}(\sigma_1^{-1}\Downarrow id)"] \end{tikzcd} \tag{4.2}$$
(3) Compatibility with composition on the left:
$$\begin{tikzcd} \mathcal{D}(\sigma_0)\boxtimes\mathcal{D}(\phi_1) \arrow[r, "\mathcal{D}(\gamma)"] & \mathcal{D}(\sigma_0\circ\phi_1) \\ \mathbb{1}\boxtimes\mathcal{D}(\phi_1) \arrow[u, "u_{\sigma_0}\boxtimes id"] & \mathcal{D}(\phi_1) \arrow[l, "\simeq", "\text{unit constraint}"'] \arrow[u, "\simeq"', "\mathcal{D}(id\Downarrow\sigma_0)"] \end{tikzcd} \tag{4.3}$$
(4) In the case of $\mathcal{M}^{nc+}$ or $\mathcal{M}^+$ the following diagram has to commute
$$\begin{tikzcd} \mathbb{1}\otimes\mathbb{1} \arrow[r, "u_\sigma\otimes u_{\sigma'}"] & \mathcal{D}(\sigma)\otimes_\mathcal{E}\mathcal{D}(\sigma') \arrow[r, "\mathcal{D}_{\sigma,\sigma'}"] & \mathcal{D}(\sigma\boxtimes\sigma') \arrow[d, "\mathcal{D}(\mu_{\sigma,\sigma'})"] \\ \mathbb{1} \arrow[u, "\text{unit}"] \arrow[rr, "u_{\sigma\otimes\sigma'}"] & & \mathcal{D}(\sigma\otimes\sigma') \end{tikzcd} \tag{4.4}$$

In the non–Cartesian case, we also postulate that for $\epsilon:\mathcal{D}\to\mathcal{T}$
$$\epsilon(u_\sigma)=1 \tag{4.5}$$

A functor $\mathcal{D}$ and a choice of groupoid compatible pointing, is called *groupoid compatibly pointed (gcp)* functor. A natural transformation of *gcp* functors is a natural transformation which respects the pointing. That is if $N:\mathcal{D}\to\mathcal{D}'$ then $N_\sigma(u_\sigma)=u'_\sigma$.

NB: Alternatively, u can be thought of as a natural transformation of monoidal functors from $\mathcal{T}$ to $\mathcal{D}\circ i$ where $i:\mathrm{Iso}(\mathcal{M}^{nc+})\to\mathcal{M}^{nc+}$. Since $\mathrm{Iso}(\mathcal{M}^{nc+})=\mathrm{Iso}(\mathcal{M}\downarrow\mathcal{M})^{\boxtimes}$, $[\mathrm{Iso}(\mathcal{M}^{nc+},\mathcal{E}]_\otimes\simeq[\mathrm{Iso}(\mathcal{M}\downarrow\mathcal{M}),\mathcal{E}]$. This means that we extend the units by $u_{\sigma\boxtimes\sigma'}:=\mathcal{D}_{\sigma,\sigma}(u_\sigma\boxtimes u_{\sigma'})$. The situation for $\mathcal{C}^{\boxtimes+}$ and $\mathcal{M}^{loc+}$ is analogous. In the case of $\mathcal{M}^+$, this identification is already taken care of.

Proposition 4.2. *A groupoid compatible pointing for a functor $\mathcal{D}$ is a strong monoidal unit for the $\mathcal{G}=\mathrm{Iso}(\mathcal{M}\downarrow\mathcal{M})$ plethysm monoid ρ defined by it via* (2.13).

Proof. This is an unwrapping of definitions. A unit for a bimodule monoid is a natural transformation of bi–modules $u:Hom_\mathcal{G}\to\rho$. These are maps $u:\mathcal{G}(X,Y)=\mathrm{Iso}(X,Y)\to\rho(X,Y)=\amalg_{\phi\in Hom_{X,Y}}\mathcal{D}(\phi)$, which are groupoid compatible and satisfy $u\circ\mu=id=\mu\circ u$ where $\circ:\rho\otimes_\mathcal{G}\rho\to\rho$ is the monoid structure. From this it follows that $u(id_X)\subset\mathcal{D}(id_X)$ and hence $u(\sigma)\in\mathcal{D}(\sigma)$, that is a morphisms $\mathbb{1}_\mathcal{E}\to\mathcal{D}(\sigma)$, by the bimodule structure. The equations (4.2) and (4.3) are then the fact that this is indeed a unit. The final equation (4.4) is that the natural transformation is a natural transformation of strong monoidal functors and corresponds to (2.23). In the non–Cartesian case, the compatibilities (2.26) and (4.5) agree. Vice–versa it is clear that given the data this defines a unit for ρ using the same equations. □

Definition 4.3. Define the monoidal categories $\mathcal{P}l^{+,gcp}$ by first freely monoidally adjoining a morphism and then modding out by relations. The morphisms we adjoin are the morphism $i_\sigma : \mathbb{1} \to \sigma$ for each isomorphism σ in $\mathcal{C}$ respectively $\mathcal{M}$. Here the $\mathbb{1} = \emptyset$ is the unit with respect to the $\boxtimes$–product and freely monoidally means that the data of the i_σ is extended by

$$i_{\sigma_1 \boxtimes \cdots \boxtimes \sigma_n} := \mathbb{1} \underset{\text{unit constraints}}{\xrightarrow{\sim}} \mathbb{1} \boxtimes \cdots \boxtimes \mathbb{1} \xrightarrow{i_{\sigma_1} \boxtimes \cdots \boxtimes i_{\sigma_n}} \sigma_1 \boxtimes \cdots \boxtimes \sigma_n \tag{4.6}$$

This equation is not needed in the case of $\mathcal{M}^+$. The relations are:

$$(\sigma \Downarrow \sigma')(i_\tau) = i_{\sigma' \circ \tau \circ \sigma^{-1}} \tag{4.7}$$

implementing the compatibility with the groupoid structure. The compatibility with the generating morphisms is forced by modding out by the relations postulating that the following diagrams commute:

$$\begin{tikzcd} \phi_0 \boxtimes \sigma_1 \arrow[r, "\gamma"] & \phi_0 \circ \sigma_1 \\ \phi_0 \boxtimes \mathbb{1}_{\mathcal{M}^+} \arrow[u, "id_{\phi_0} \boxtimes i_{\sigma_1}"] & \phi_0 \arrow[l, "\sim"', "\text{unit constraint}"] \arrow[u, "\sim"', "(\sigma_1^{-1} \Downarrow id)"] \end{tikzcd} \tag{4.8}$$

where the right morphisms is given by the groupoid data.

$$\begin{tikzcd} \sigma_0 \boxtimes \phi_1 \arrow[r, "\gamma"] & \sigma_0 \circ \phi_1 \\ \mathbb{1}_{\mathcal{M}^+} \boxtimes \phi_1 \arrow[u, "i_{\sigma_0} \boxtimes id_{\phi_1}"] & \phi_1 \arrow[l, "\sim"', "\text{unit constraint}"] \arrow[u, "\sim"', "(id \Downarrow \sigma_0)"] \end{tikzcd} \tag{4.9}$$

For $\mathcal{M}^{nc+,gcp}$, $\mathcal{M}^{loc+,gcp}$ we mod out by the relation

$$\begin{tikzcd} \mathbb{1} \boxtimes \mathbb{1} \arrow[r, "i_\sigma \boxtimes i_{\sigma'}"] & \sigma \boxtimes \sigma'' \arrow[d, "\mu_{\sigma,\sigma'}"] \\ \mathbb{1} \arrow[u, "\text{unit}"] \arrow[r, "i_{\sigma \otimes \sigma'}"] & \sigma \otimes \sigma' \end{tikzcd} \tag{4.10}$$

and for $\mathcal{M}^{+,gcp}$ by

$$\begin{tikzcd} \mathbb{1} \otimes \mathbb{1} \arrow[r, "i_\sigma \otimes i_{\sigma'}"] & \sigma \otimes \sigma \\ \mathbb{1} \arrow[u, "\text{unit}"] \arrow[ur, "i_{\sigma \otimes \sigma'}"'] & \end{tikzcd} \tag{4.11}$$

Proposition 4.4. *Define the functor $GCP : \mathcal{P}l^+ \to \mathcal{P}l^{+,gcp}$ by inclusion. A strong monoidal functor $\mathcal{D} : \mathcal{P}l^+ \to \mathcal{E}$ with a groupoid compatible pointing $\{u_\sigma : \mathbb{1}_\mathcal{E} \to \mathcal{D}(\sigma)\}$ canonically factors through GCP:*

$$\begin{tikzcd} \mathcal{P}l^+ \arrow[rr, "\mathcal{D}"] \arrow[dr, "GCP"'] & & \mathcal{E} \\ & \mathcal{P}l^{+,gcp} \arrow[ur, dashed, "\mathcal{D}_{gcp}"'] & \end{tikzcd} \tag{4.12}$$

Proof. Define $\mathcal{D}_{gcp}$ as the extension of $\mathcal{D}$ that sends the generating morphism i_σ to the pointing $u_\sigma : \mathbb{1}_{\mathcal{E}} \to \mathcal{D}(\sigma)$. Then extend $\mathcal{D}_{gcp}$ to the rest of $\mathcal{P}l^{+,gcp}$ so that it is a strong monoidal functor. We must verify that this assignment respects the defining relations of $\mathcal{P}l^{+,gcp}$. The groupoid compatibility of the functor $\mathcal{D}_{gcp}$ just follows from the groupoid compatibility of the pointing. For compatibility with composition in $\mathcal{C}^{\boxtimes+,gcp}$ and $\mathcal{M}^{nc+,gcp}$ consider the following diagram:

$$\begin{tikzcd}
\mathcal{D}_{gcp}(\phi_0 \boxtimes \sigma_1) \arrow[r, "\simeq"] & \mathcal{D}(\phi_0) \boxtimes \mathcal{D}(\sigma_1) \arrow[r, "\mathcal{D}(\gamma)"] & \mathcal{D}(\phi_0 \circ \sigma_1) \\
\mathcal{D}_{gcp}(\phi_0 \boxtimes \mathbb{1}_{\mathcal{M}^+}) \arrow[u, "\mathcal{D}_{gcp}(id \boxtimes i_{\sigma_1})"] & \mathcal{D}(\phi_0) \boxtimes \mathbb{1} \arrow[l, "\simeq"'] \arrow[u, "id \boxtimes u_{\sigma_1}"'] & \mathcal{D}(\phi_0) \arrow[l, "\simeq", "\text{unit constraints}"'] \arrow[u, "\simeq", "\mathcal{D}(\sigma_1^{-1} \Downarrow id)"']
\end{tikzcd} \tag{4.13}$$

The left square commutes tautologically, because the left morphism was defined by extending generators. The right square commutes by groupoid compatibility of the pointing. Joining the two squares shows that $\mathcal{D}$ respects compatibility of composition on the right. Compatibility with composition on the left is similar. The compatibility for μ in $\mathcal{M}^{nc+}$ is similarly straightforward and the arguments descends to $\mathcal{M}^{loc+}$ and $\mathcal{M}^+$ by universality. □

For the non–Cartesian case, a splitting comes into play.

Definition 4.5. A functor $\mathcal{D} : \mathcal{P}l^{+,gcp} \to \mathcal{C}$ is *split* if the restriction to $\mathcal{P}l^+$ is split and if the morphism $\mathcal{D}(i_\sigma) : \mathbb{1} \to \mathcal{D}^\times(\sigma) \subset \mathcal{D}(\sigma)$ is split for each isomorphism object, that is $\mathcal{D}(\sigma)^\times = \mathbb{1} \amalg \bar{\mathcal{D}}(\sigma)^\times$ splits with the first component being $\mathcal{D}(i_\sigma)$.

Theorem 4.6.

(1) *The category of strong monoidal functors* $[\mathcal{C}^{\boxtimes+,gcp}, \mathcal{E}]_\otimes$ *is equivalent to the category of unital* $\mathrm{Iso}(\mathcal{C})$ *bi–module monoids over* $\mathrm{Hom}_{\mathcal{C}}$.
(2) *The category of strong monoidal functors* $[\mathcal{M}^{nc+,gcp}, \mathcal{E}]_\otimes$ *is equivalent to the category of unital lax–monoidal* $\mathrm{Iso}(\mathcal{M})$*–bimodules monoids over* $\mathrm{Hom}_{\mathcal{C}}$.
(3) *The category of strong monoidal functors* $[\mathcal{M}^{loc+,gcp}, \mathcal{E}]_\otimes$ *is equivalent to the category of unital strong–monoidal* $\mathrm{Iso}(\mathcal{M})$*–bimodules monoids over* $\mathrm{Hom}_{\mathcal{C}}$.
(4) *The category of strong monoidal functors* $[\mathcal{M}^{+,gcp}, \mathcal{E}]_\otimes$ *is equivalent to the category of unital strong–monoidal* $\mathrm{Iso}(\mathcal{M})$*–bimodules monoids over* $\mathrm{Hom}_{\mathcal{C}}$.

In the non–Cartesian case the statements are restricted to counital functors and bimodules, and the isomorphism split bimodules correspond to the split functors.

Proof. This follows from Theorems 3.3, Theorem 3.6 respectively Theorem 3.10 together with Proposition 4.2 and Proposition 4.4. □

4.2. **The reduced/hyper version.**

Definition 4.7. A gcp functor is $\mathcal{D}$ *reduced* if $u_\sigma : \mathbb{1}_{\mathcal{C}} \to \mathcal{D}(\sigma)$ is an isomorphism for all σ.

The corresponding universal quotient plus construction will be what we call the hyper construction, which is given by inverting the i_σ.

Definition 4.8. Let $\mathcal{P}l^{+,gcp}$ be any of the gcp plus constructions. Define $\mathcal{P}l^{hyp}$ that is $\mathcal{C}^{\boxtimes,hyp}, \mathcal{M}^{nc,hyp}, \mathcal{M}^{loc,hyp}$ respectively $\mathcal{M}^{hyp}$ by starting with the category $\mathcal{P}l^{+,gcp}$ and then formally adjoin a generator $r_\sigma : \sigma \to \mathbb{1}$ for each isomorphism and mod out by the relations $r_\sigma \circ i_\sigma = id_{\mathbb{1}}$ and $i_\sigma \circ r_\sigma = id_{\mathbb{1}}$. The groupoid compatibility of r_σ will follow from the groupoid compatibility of i_σ.

Proposition 4.9. *There is a natural functor*

$$Hyp : \mathcal{P}l^{+gcp} \to \mathcal{P}l^{hyp} \tag{4.14}$$

Moreover, any reduced gcp functor $\mathcal{D} : \mathcal{P}l^{+} \to \mathcal{C}$ factors through the composition $Hyp \circ GCP$:

$$\begin{tikzcd}
\mathcal{P}l^{+} \arrow[d, "GCP"'] \arrow[dr, "\mathcal{D}"] & \\
\mathcal{P}l^{+gcp} \arrow[r, "\mathcal{D}_{gcp}"] \arrow[d, "Hyp"'] & \mathcal{C} \\
\mathcal{P}l^{hyp} \arrow[ur, dashed, "\mathcal{D}_{hyp}"'] &
\end{tikzcd} \tag{4.15}$$

Proof. Like before, assign $\mathcal{D}_{hyp}(i_\sigma) = u_\sigma$. Using the reduced assumption, assign $\mathcal{D}_{hyp}(r_\sigma) = u_\sigma^{-1}$. The diagrams then commute.

In the non–Cartesian case the statements are restricted to counital functors and bimodules. And the isomorphism split bimodules correspond to the split functors, see §3.4 below. ☐

4.2.1. **Condensed version.** Lemma 4.10 below, tells us that the full subcategory of isomorphism objects is equivalent to the trivial category. Contracting the isomorphism objects to a single new object, we thus obtain an equivalent category. This allows us to recreate all of the essential properties of $\mathcal{P}l^{hyp}$ with a simplified presentation. Note that this is not a natural construction since it is only defined in the strict case. Rather, one should think of it as a technical tool.

Lemma 4.10. *In the category $\mathcal{P}l^{hyp}$, there is exactly one morphism between any two isomorphism objects.*

Proof. Let σ and σ' be a pair of isomorphism objects in $\mathcal{P}l^{hyp}$. The set $\mathrm{Hom}_{\mathcal{P}l^{+hyp}}(\sigma, \sigma')$ is non-empty because we always have at least $i_{\sigma'} \circ r_\sigma \in \mathrm{Hom}_{\mathcal{P}l^{+hyp}}(\sigma, \sigma')$. Through a series of reductions, we will narrow down the set $\mathrm{Hom}_{\mathcal{P}l^{+hyp}}(\sigma, \sigma')$ to show that this is the only element.

For our first reduction, we will show that the set $\mathrm{Hom}_{\mathcal{P}l^{+hyp}}(\sigma, \sigma')$ consists only of isomorphisms. It will suffice to check the γ-morphisms. Using compatibility of right multiplication, the generator $\gamma : \sigma_1 \otimes \sigma_0 \to \sigma_1 \circ \sigma_0$ can be rewritten as follows:

$$\begin{tikzcd}
\sigma_1 \otimes \sigma_0 \arrow[r, "\gamma"] \arrow[d, "id_{\sigma_1} \otimes r_{\sigma_0}"'] & \sigma_1 \circ \sigma_0 \\
\sigma_1 \otimes \mathbb{1} \arrow[r, "\text{unit constraints}"'] & \sigma_1 \arrow[u, "(\sigma_0 \Downarrow id)"']
\end{tikzcd} \tag{4.16}$$

For our second reduction, we will use the groupoid compatibility of r to rewrite any basic isomorphism in terms of r and i:

$$\begin{tikzcd}
\sigma \arrow[r, "(\tau_0 \Downarrow \tau_1)"] \arrow[dr, "r_\sigma"'] & \sigma' \arrow[r, "id_{\sigma'}"] \arrow[d, "r_{\sigma'}"] & \sigma' \\
& \mathbb{1} \arrow[ur, "i_{\sigma'}"'] &
\end{tikzcd} \tag{4.17}$$

We are now reduced to considering compositions of tensor products of i and r. Using the interchange relation, we can write these morphisms as compositions in i and r instead. We can reduce these compositions to the following:

$$\begin{array}{ll} \mathrm{Hom}_{\mathcal{P}l^{hyp}}(\sigma,\sigma') = \{i_{\sigma'} \circ r_\sigma\} & \mathrm{Hom}_{\mathcal{P}l^{hyp}}(\sigma,\mathbb{1}) = \{i_\sigma\} \\ \mathrm{Hom}_{\mathcal{P}l^{hyp}}(\mathbb{1},\sigma) = \{r_\sigma\} & \mathrm{Hom}_{\mathcal{P}l^{hyp}}(\mathbb{1},\mathbb{1}) = \{id_{\mathbb{1}}\} \end{array}$$

Hence there is exactly one morphism between any two isomorphism object. □

In the notation above, define the *condensed plus construction* $\mathcal{P}l^{hyp,cond}$ as the category obtained by identifying all the isomorphism objects $\mathcal{P}l^{hyp}$.

Definition 4.11. For a strict monoidal category $\mathcal{M}$, define

(1) The objects of $\mathcal{P}l^{hyp,cond}$ consist of $\mathbb{1}$ and all of the non–isomorphism objects of $\mathcal{P}l^+$
(2) For two non-isomorphism objects Φ and Ψ, the set $\mathrm{Hom}(\Phi,\Psi)$ is the same as $\mathcal{M}^+$. On the other hand, define $\mathrm{Hom}(\Phi,\mathbb{1})$ to be the set of all basic γ-morphisms $\gamma:\Phi\to\sigma$ whose codomain σ is some isomorphism object.

NB: We can identify the object $\star$ with $\mathbb{1}$.

Proposition 4.12. *This category is equivalent to* $\mathcal{P}l^{hyp}$

Proof. Straightforward. □

5. Equivalence principle for plus categories

The plus construction is an invariant (good) notion that respects the principle of equivalence: two monoidally equivalent monoidal categories $\mathcal{M}\simeq\mathcal{M}'$ have equivalent plus constructions, as is shown below. This allows us to replace $\mathcal{M}$ with a skeletal or strict model. The key input is equivariance which is essential for Lemma 5.4.

The idea is to show that there is also a plus construction for functors and natural isomorphisms. This will allow us to transfer the equivalence between $\mathcal{M}$ and $\mathcal{M}'$ to an equivalence between $\mathcal{M}^{nc+}$ and $(\mathcal{M}')^{nc+}$.

Definition 5.1. Suppose $F:\mathcal{M}\to\mathcal{M}'$ is a strong monoidal functor with structure morphisms $f_{XY}: F(Y)\otimes F(X)\to F(Y\otimes X)$. We define the strict monoidal functor $F^{+nc}:\mathcal{M}^{+nc}\to(\mathcal{M}')^{+nc}$ as follows:

Objects: as a strict monoidal functor, we necessarily have

$$F^{nc+}(\phi_1\boxtimes\dots\boxtimes\phi_n) = F(\phi_1)\boxtimes\dots\boxtimes F(\phi_n) \tag{5.1}$$

Basic isomorphisms: define $F^{nc+}(\sigma\Downarrow\sigma'):F(\Phi)\to F(\Psi)$ by

$$F^{nc+}(\sigma\Downarrow\sigma') = (F(\sigma)\Downarrow F(\sigma')) \tag{5.2}$$

γ-morphisms: for a composable pair of morphisms $\phi_1\in\mathrm{Hom}(Y,Z)$ and $\phi_0\in\mathrm{Hom}(X,Y)$ in $\mathcal{M}$, define $F^{nc+}(\gamma_{\phi_1,\phi_0}):F(\phi_1)\boxtimes F(\phi_0)\to F(\phi_1\circ\phi_0)$ by

$$F^{nc+}(\gamma_{\phi_1,\phi_0}) = \gamma_{F(\phi_1),F(\phi_0)} \tag{5.3}$$

μ-morphisms: for two morphisms $\phi_1\in\mathrm{Hom}(X_1,Y_1)$ and $\phi_2\in\mathrm{Hom}(X_2,Y_2)$ in $\mathcal{M}$, define $F^{nc+}(\mu_{\phi_1,\phi_2}):F(\phi_1)\boxtimes F(\phi_2)\to F(\phi_1\otimes\phi_2)$ to be the composition

$$F(\phi_1)\boxtimes F(\phi_2)\xrightarrow[\mu_{F(\phi_1),F(\phi_2)}]{} F(\phi_1)\otimes F(\phi_2)\xrightarrow[(f_{X_1,X_2}\Downarrow f_{Y_1,Y_2})]{} F(\phi_1\otimes\phi_0) \tag{5.4}$$

Lemma 5.2. *The correspondence* $F^{nc+}:\mathcal{M}^{nc+}\to(\mathcal{M}')^{nc+}$ *defined above is a strong monoidal functor.*

Proof. We just need to verify that commutative diagrams in $\mathcal{M}^{nc+}$ get sent to commutative diagrams in $(\mathcal{M}')^{nc+}$. The correspondence F^{nc+} sends the inner equivariance (3.3) diagrams in $\mathcal{M}^{nc+}$ to inner equivariance diagrams in $(\mathcal{M}')^{nc+}$ which commutes by definition. The same is true for the outer equivariance diagram (3.4) and the internal associativity diagram (3.5).

The equivariance of μ with respect to isomorphisms diagram gets sent to the following outer rectangle:

$$\begin{tikzcd}
F(\phi_1) \boxtimes F(\phi_2) \arrow[r, "\sim"] \arrow[d, "\mu"'] & F(\tilde{\phi}_1) \boxtimes F(\tilde{\phi}_2) \arrow[d, "\mu"] \\
F(\phi_1) \otimes F(\phi_2) \arrow[r, "\sim"] \arrow[d, "(f_{X_1,X_2} \Downarrow f_{Y_1,Y_2})"'] & F(\tilde{\phi}_1) \otimes F(\tilde{\phi}_2) \arrow[d, "(f_{\tilde{X}_1,\tilde{X}_2} \Downarrow f_{\tilde{Y}_1,\tilde{Y}_2})"] \\
F(\phi_1 \otimes \phi_2) \arrow[r, "\sim"] & F(\tilde{\phi}_1 \otimes \tilde{\phi}_2)
\end{tikzcd} \tag{5.5}$$

The top square commutes by equivariance with respect to isomorphisms in $(\mathcal{M}')^{nc+}$. The bottom square commutes by naturality of f.

The internal interchange diagrams in $\mathcal{M}^{nc+}$ gets mapped to the following diagram in $(\mathcal{M}')^{nc+}$:

$$\begin{tikzcd}
F\phi_0 \boxtimes F\phi_1 \boxtimes F\psi_0 \boxtimes F\psi_1 \arrow[rr, "\gamma\boxtimes\gamma"] \arrow[d, "(\mu\boxtimes\mu)\circ\sigma_{23}"'] & & F\phi_1 \circ F\phi_0 \boxtimes F\psi_1 \circ F\psi_0 \arrow[d, "\mu"] \\
F\phi_0 \otimes F\psi_0 \boxtimes F\phi_1 \otimes F\psi_1 \arrow[r, "\gamma"] \arrow[d, "(f\ldots\Downarrow f\ldots)\boxtimes(f\ldots\Downarrow f\ldots)"'] & (F\phi_0 \otimes F\phi_1) \circ (F\psi_0 \otimes F\psi_1) \arrow[r, equal] & (F\phi_1 \circ F\phi_0) \otimes (F\psi_1 \circ F\psi_0) \arrow[d, "(f\ldots\Downarrow f\ldots)"] \\
F(\phi_0 \otimes \phi_1) \boxtimes F(\psi_0 \otimes \psi_1) \arrow[r, "\gamma"'] & F(\phi_0 \otimes \phi_1) \circ F(\psi_0 \otimes \psi_1) \arrow[r, equal] & F(\phi_1 \circ \phi_0 \otimes \psi_1 \circ \psi_0)
\end{tikzcd} \tag{5.6}$$

The top commutes by internal interchange in $(\mathcal{M}')^{nc+}$ and the bottom commutes by equivariance of γ.

When it applies, the compatibility with commutators diagram gets sent to

$$\begin{tikzcd}
F\phi_1 \boxtimes F\phi_2 \arrow[r, "\tau_{12}"] \arrow[d, "\mu_{\phi_1,\phi_2}"'] & F\phi_2 \boxtimes F\phi_1 \arrow[d, "\mu_{\phi_2,\phi_1}"] \\
F\phi_1 \otimes F\phi_2 \arrow[r, "(FC_{12}\Downarrow FC_{12})"] \arrow[d, "f_{F\phi_1,F\phi_2}"'] & F\phi_2 \otimes F\phi_1 \arrow[d, "f_{F\phi_2,F\phi_1}"] \\
F(\phi_1 \otimes \phi_2) \arrow[r, "F(C_{12}\Downarrow C_{12})"] & F(\phi_2 \otimes \phi_1)
\end{tikzcd} \tag{5.7}$$

The top commutes by compatibility with commutators in $(\mathcal{M}')^{nc+}$ and the bottom commutes because the functor is symmetric monoidal. □

Lemma 5.3. *Given two strong monoidal functors F and G, we have*

$$(G \circ F)^{nc+} = G^{nc+} \circ F^{nc+} \tag{5.8}$$

Proof. The left and right hand sides affect basic isomorphisms in exactly the same way. The two sides also agree for γ-morphisms by functorality. We will check the μ-morphisms. To evaluate $G^{nc+}(F^{nc+}(\mu))$, first expand $F^{nc+}(\mu)$ and apply G^{nc+} to get the first column. Then expand $G^{nc+}(\mu_{F(\phi_1),F(\phi_2)})$ to get the second column.

$$\begin{tikzcd}
G(F\phi_1 \boxtimes F\phi_2) \arrow[r, equal] \arrow[dd, "G^{nc+}(\mu_{F(\phi_1),F(\phi_2)})"'] & (GF\phi_1) \boxtimes (GF\phi_2) \arrow[d, "\mu_{GF(\phi_1),GF(\phi_2)}"] \\
 & (GF\phi_1) \otimes (GF\phi_2) \arrow[d, "(g_{FX_1,FX_2} \Downarrow g_{FY_1,FY_2})"] \\
G(F\phi_1 \otimes F\phi_2) \arrow[r, equal] \arrow[d, "(Gf_{X_1,X_2} \Downarrow Gf_{Y_1,Y_2})"'] & G(F\phi_1 \otimes F\phi_2) \arrow[d, "(Gf_{X_1,X_2} \Downarrow Gf_{Y_1,Y_2})"] \\
GF(\phi_1 \otimes \phi_0) \arrow[r, equal] & GF(\phi_1 \otimes \phi_0)
\end{tikzcd} \tag{5.9}$$

The structure morphism for the monoidal functor $G \circ F$ is exactly the composition of the last two morphisms in the second column. Therefore $(G \circ F)^{nc+} = G^{nc+} \circ F^{nc+}$. $\square$

Lemma 5.4. *If $\alpha : F \Rightarrow G$ is a monoidal natural isomorphism, then $\{\alpha_\phi^{nc+}\}$ defined by $\alpha_\phi^{nc+} := (\alpha_{s(\phi)} \Downarrow \alpha_{t(\phi)})$ constitutes a monoidal natural isomorphism $\alpha^{nc+} : F^{nc+} \Rightarrow G^{nc+}$.*

Proof. Like before, it is enough to prove naturality for generators. Naturality of isomorphisms is a consequence of the naturality of α and functorality of F and G:

$$\begin{tikzcd}
F(X') \arrow[d, "\alpha_{X'}"] & F(X) \arrow[l, "F(\sigma)"'] \arrow[r, "F(\phi)"] \arrow[d, "\alpha_X"] & F(Y) \arrow[r, "F(\tau)"] \arrow[d, "\alpha_Y"] & F(Y') \arrow[d, "\alpha_{Y'}"] \\
G(X') & G(X) \arrow[l, "G(\sigma)"] \arrow[r, "G(\phi)"'] & G(Y) \arrow[r, "G(\tau)"'] & G(Y')
\end{tikzcd} \tag{5.10}$$

Naturality of γ-morphisms is a consequence of inner and outer equivariance.

$$\begin{tikzcd}
F(\phi_1) \boxtimes F(\phi_0) \arrow[r, "\gamma_{F(\phi_1),F(\phi_0)}"] \arrow[d, "\alpha^{nc+}_{\phi_1} \boxtimes \alpha^{nc+}_{\phi_0}"'] & F(\phi_1 \circ \phi_0) \arrow[d, "\alpha^{nc+}_{\phi_1 \circ \phi_0}"] \\
G(\phi_1) \boxtimes G(\phi_0) \arrow[r, "\gamma_{G(\phi_1),G(\phi_0)}"'] & G(\phi_1 \circ \phi_0)
\end{tikzcd} \tag{5.11}$$

To show naturality for the μ-morphisms, consider the following diagram with $F^{nc+}(\mu)$ on top and $G^{nc+}(\mu)$ on the bottom:

$$\begin{tikzcd}
F(\phi_1) \boxtimes F(\phi_2) \arrow[r, "\mu_{F(\phi_1),F(\phi_2)}"] \arrow[d, "\alpha^+_{\phi_1} \boxtimes \alpha^+_{\phi_2}"] & F(\phi_1) \otimes F(\phi_2) \arrow[r, "(f_{X_1,X_2} \Downarrow f_{Y_1,Y_2})"] \arrow[d, "\alpha^+_{\phi_1} \otimes \alpha^+_{\phi_2}"] & F(\phi_1 \otimes \phi_0) \arrow[d, "\alpha^+_{\phi_1 \otimes \phi_2}"] \\
G(\phi_1) \boxtimes G(\phi_2) \arrow[r, "\mu_{G(\phi_1),G(\phi_2)}"'] & G(\phi_1) \otimes G(\phi_2) \arrow[r, "(g_{X_1,X_2} \Downarrow g_{Y_1,Y_2})"'] & G(\phi_1 \otimes \phi_0)
\end{tikzcd} \tag{5.12}$$

The left square commutes by equivariance with respect to isomorphisms. The right square commutes because α is a monoidal natural transformation. $\square$

The constructions show that:

Theorem 5.5. *The plus construction is an endofunctor in the category of monoidal categories.* $\square$

Corollary 5.6. *If $\mathcal{M}$ and $\mathcal{M}'$ are monoidally equivalent, then their plus constructions are also monoidally equivalent:*

$$\mathcal{M}^{nc+} \cong (\mathcal{M}')^{nc+} \tag{5.13}$$

$$\mathcal{M}^{loc+} \cong (\mathcal{M}')^{loc+} \tag{5.14}$$

$$\mathcal{M}^{+} \cong (\mathcal{M}')^{+} \tag{5.15}$$

Moreover, if $\mathcal{C}$ and $\mathcal{C}$ are merely equivalent categories, then:

$$\mathcal{C}^{\boxtimes} \cong (\mathcal{C}')^{\boxtimes} \tag{5.16}$$

Proof. We start with (5.13). Suppose the functors $F : \mathcal{M} \to \mathcal{M}'$ and $G : \mathcal{M}' \to \mathcal{M}$ and the natural transformations $\alpha : GF \Rightarrow Id_{\mathcal{M}}$ and $\beta : FG \Rightarrow Id_{\mathcal{M}'}$ induce a monoidal equivalence. By our lemmas, we have functors $F^{nc+} : \mathcal{M}^{nc+} \to (\mathcal{M}')^{nc+}$ and $G^{nc+} : (\mathcal{M}')^{nc+} \to \mathcal{M}^{nc+}$ and the natural transformations $\alpha^{nc+} : G^{nc+}F^{nc+} \Rightarrow Id^{nc+}_{\mathcal{M}}$ and $\beta^{nc+} : F^{nc+}G^{nc+} \Rightarrow Id^{nc+}_{\mathcal{M}'}$. The only thing that remains is to show that $Id^{nc+}_{\mathcal{M}} = Id_{\mathcal{M}^{nc+}}$, but this is immediate.

Now consider (5.14). For a functor F, define $F^{+loc} : \mathcal{M}^{+loc} \to (\mathcal{M}')^{+loc}$ like we did with F^{nc+} but with the following addition:

μ^{-1}-MORPHISMS: for two morphisms $\phi_1 \in \mathrm{Hom}(X_1, Y_1)$ and $\phi_2 \in \mathrm{Hom}(X_2, Y_2)$ in $\mathcal{M}$, define $F^{+loc}(\mu^{-1}_{\phi_1,\phi_2}) : F(\phi_1 \otimes \phi_2) \to F(\phi_1) \boxtimes F(\phi_2)$ to be the composition

$$F(\phi_1) \boxtimes F(\phi_2) \xleftarrow[\mu^{-1}_{F(\phi_1),F(\phi_2)}]{} F(\phi_1) \otimes F(\phi_2) \xleftarrow[(f^{-1}_{X_1,X_2} \Downarrow f^{-1}_{Y_1,Y_2})]{} F(\phi_1 \otimes \phi_0) \tag{5.17}$$

For a natural isomorphism α, define $\alpha^{+loc}_{\phi} := (\alpha_{s(\phi)} \Downarrow \alpha_{t(\phi)})$, which is exactly the same as α^{nc+}_{ϕ}. The commutativity of the diagrams for μ-morphisms implies the commutativity of the diagrams for μ^{-1}-morphisms. Hence F^{+} and α^{+} are both well-defined. Therefore the argument for (5.13) carries over.

For (5.15), there is no exterior monoidal product, so we make the following modifications to the plus construction F^{+} of a functor:

OBJECTS: for any morphisms ϕ in $\mathcal{M}$, define

$$F^{+}(\phi) = F(\phi) \tag{5.18}$$

$\bar{\gamma}$-MORPHISMS: for two morphisms $\phi_1 \in \mathrm{Hom}(X_1, Y_1)$ and $\phi_2 \in \mathrm{Hom}(X_2, Y_2)$ in $\mathcal{M}$, define $F^{+}(\bar{\gamma}_{\phi_0,\phi_1}) : F(\phi_1 \otimes \phi_2) \to F(\phi_0 \circ \phi_1)$ to be the composition

$$F(\phi_1 \otimes \phi_2) \xrightarrow[(f^{-1}_{X_1,X_2} \Downarrow f^{-1}_{Y_1,Y_2})]{} F(\phi_1) \otimes F(\phi_0) \xrightarrow[\gamma]{} F(\phi_1) \circ F(\phi_0) =\!=\!= F(\phi_1 \circ \phi_0) \tag{5.19}$$

GENERAL MORPHISMS: Given two morphisms $\Gamma_1 : \Phi_1 \to \Psi_1$ and $\Gamma_2 : \Phi_2 \to \Psi_2$, define $F(\Gamma_1 \otimes \Gamma_2)$ inductively as the composition

$$F(\Phi_1 \otimes \Phi_2) \xrightarrow{\sim} F(\Phi_1) \otimes F(\Phi_2) \xrightarrow{F(\Gamma_1)\otimes F(\Gamma_2)} F(\Psi_1) \otimes F(\Psi_2) \xrightarrow{\sim} F(\Psi_1 \otimes \Psi_2) \tag{5.20}$$

Again, define $\alpha^{+}_{\phi} := (\alpha_{s(\phi)} \Downarrow \alpha_{t(\phi)})$. The commutative diagrams that we need to check now have additional morphisms involving f which "pack and unpack" the morphisms object. However, after eliminating compositions like (5.21), we can use essentially the same argument as we did for (5.13).

$$\begin{tikzcd} & F(\Phi_1)\otimes F(\Phi_2) \arrow[dr] & \\ F(\Phi_1\otimes\Phi_2) \arrow[ur] \arrow[rr, equal] & & F(\Phi_1\otimes\Phi_2) \end{tikzcd} \tag{5.21}$$

Finally, (5.16) follows by a simpler version of the argument for (5.13). □

Corollary 5.7. *Both the hyper and gcp constructions satisfy the principle of equivalence.*

Proof. Make the following modification for functors:

i-MORPHISMS: for any isomorphism σ in $\mathcal{M}$, define $F^{gcp+}(i_\sigma)$ and $F^{hyp}(i_\sigma)$ by

$$F(\mathbb{1}_{\mathcal{M}}) \xrightarrow{\sim} \mathbb{1}_{\mathcal{M}'} \xrightarrow{i_{F(\sigma)}} F(\sigma) \tag{5.22}$$

r-MORPHISMS: in the GCP case, define $F^{hyp}(r_\sigma)$ by

$$F(\mathbb{1}_{\mathcal{M}}) \xleftarrow[\sim]{} \mathbb{1}_{\mathcal{M}'} \xleftarrow[r_{F(\sigma)}]{} F(\sigma) \tag{5.23}$$

The rest of the argument is now routine. □

6. Special monoidal categories: UFCs and FCs

6.1. Essentially unique factorizations.

Definition 6.1. A (symmetric) monoidal category $(\mathcal{M},\otimes)$ has *essentially uniquely factorizable objects*, if there is a groupoid $\mathcal{V}$ of basic objects together with a functor $\imath:\mathcal{V}\to\mathcal{M}$, for which $\imath^{\boxtimes}$ induces an equivalence.

$$\imath^{\boxtimes}:\mathcal{V}^{\boxtimes}\xrightarrow{\sim}\mathrm{Iso}(\mathcal{M}) \tag{6.1}$$

A choice of such a pair $(\mathcal{V},\imath)$ will be called a *basis of objects* and its elements will be called *irreducibles* or *basic objects*.

A (symmetric) monoidal category $\mathcal{M}$ has *essentially uniquely factorizable morphisms* if there is a groupoid $\mathcal{P}$ and a functor $\jmath:\mathcal{P}\to\mathrm{Iso}(\mathcal{M}\downarrow\mathcal{M})$ for which $\jmath^{\boxtimes}$ is an equivalence:

$$\jmath^{\boxtimes}:\mathcal{P}^{\boxtimes}\simeq\mathrm{Iso}(\mathcal{M}\downarrow\mathcal{M}) \tag{6.2}$$

A choice of such a pair $(\mathcal{P},\jmath)$ will be called a *basis of morphisms* and its elements will be called *irreducibles* or *basic morphisms*.

A compatibility between a choice of basic objects and a choice of basic morphisms is a choice of functor $\imath_{\mathcal{P}}:\mathcal{P}\to(\mathcal{V}^{\boxtimes}\downarrow\mathcal{V}^{\boxtimes})$, such that $\jmath=(\imath^{\boxtimes},id,\imath^{\boxtimes})\circ\imath_{\mathcal{P}}$. This means that the following diagram commutes:

$$\begin{tikzcd} \mathcal{P} \arrow[r] \arrow[d, "\imath_{\mathcal{P}}"'] \arrow[rr, bend left, "\jmath"] & \mathcal{P}^{\boxtimes} \arrow[r, "\jmath^{\boxtimes}"] \arrow[d, "\imath_{\mathcal{P}}^{\boxtimes}"'] & \mathrm{Iso}(\mathcal{M}\downarrow\mathcal{M}) \arrow[d, equal] \\ \mathrm{Iso}(\mathcal{V}^{\boxtimes}\downarrow\mathcal{V}^{\boxtimes}) \arrow[r] \arrow[rr, bend right, "(\imath^{\boxtimes},id,\imath^{\boxtimes})"'] & \mathrm{Iso}(\mathcal{V}^{\boxtimes}\downarrow\mathcal{V}^{\boxtimes})^{\boxtimes} \arrow[r, "(\imath^{\boxtimes},id,\imath^{\boxtimes})^{\boxtimes}"'] & \mathrm{Iso}(\mathcal{M}\downarrow\mathcal{M}) \end{tikzcd} \tag{6.3}$$

A compatible choice of basis is a choice of basic objects, basic morphisms and a compatibility.

Remark 6.2. For the objects $\mathcal{V}^{\boxtimes}$ can be the non–symmetric or symmetric version, cf. [Kau18] and §A.1.3. The morphisms in a monoidal category are always symmetric monoidal, either lying in set, or in a symmetric monoidal enrichment category. This is due to the interchange relation (A.1) in which the second and third component are switched. Thus there is no natural non–symmetric version of unique factorization of morphisms.

Definition 6.3. If a category has essentially uniquely factorizable objects, then each object has a degree $|X|$ which is the length of any isomorphic object in $\mathcal{V}^{\boxtimes}$. This gives is a natural *bi–degree* or *type* of a morphism $type(\phi) = (|s(\phi)|, |t(\phi)|)$ and a *degree*, the length decrease, $|\phi| = |s(\phi)| - |t(\phi)|$, both are invariant under isomorphism. Both are additive under $\otimes$ and the latter is additive under $\circ$ as well.

If $\mathcal{M}$ has essentially uniquely factorizable morphisms there is a natural *degree* for morphisms $depth(\phi)$ which is the length of a monoidal decomposition in $\mathcal{P}^{\boxtimes}$ isomorphic to ϕ. This is well defined additive under $\otimes$.

Notation 6.4. To simplify the notation, we will often work in $\mathcal{V}^{\otimes Set}$ and $\mathcal{P}^{\otimes Set}$, see §A.1.4, fix a presentation and choose pseudo–inverses $\tilde{\imath}$ and $\tilde{\jmath}$. Thus $\tilde{\imath}(X) = \bigotimes_{s\in S} *_s$ for a collection of objects $*_s$ in $\mathcal{V}^S$ and $X \simeq \bigotimes_{s\in S} \imath(*_S)$ in a chosen fashion $(\sigma \Downarrow \sigma')$. As a short hand for these choice we will simply write $X \simeq \bigotimes_{s\in S} *_s$. Any isomorphism of objects $\sigma : X = \bigotimes_{s\in S} *_S \xrightarrow{\sim} \tilde{X} = \bigotimes_{t\in T} *_T$ is then given by the image of a bijection $\bar{\sigma} : S \leftrightarrow T$ and fixed isomorphisms $\sigma_s : *_s \to *_{\bar{\sigma}(t)}$. We will simply write $S \leftrightarrow T$ for such a map. We define the index of a such a tensor product as $\mathrm{idx}(\bigotimes_{s\in S} *_s) = S$ and $\mathrm{idx}(X) = \mathrm{idx}(\tilde{\imath}(X))$.

Remark 6.5.

(1) All choices of groupoid of basic objects are equivalent to the essential image of any choice. If not specified, we use this as the set of basic objects. Similarly, for a groupoid of basic morphisms.

(2) The condition (6.1) means that up to isomorphism any object X is decomposable into basic objects $X \simeq \bigotimes_{v\in V} \imath(*_v)$. Moreover, the only isomorphisms are given by wreath products of isomorphism groups, that is isomorphisms on the $*_v$ and bijection of indexing sets, inducing permutations of the isomorphic factors. In particular, any isomorphism $\sigma : X \to X'$ decomposes as

$$\sigma \simeq P \circ \bigotimes_{i=1}^{n} \sigma_i = \bigotimes_{i=1}^{n} \sigma_{P(i)} \circ P \tag{6.4}$$

where $\sigma_i : *_i \to *'_i$ for chosen decompositions $X \simeq \bigotimes_{i=i}^{n} \imath(*_i), X' \simeq \bigotimes_{i=1}^{n} \imath(*'_i)$.

(3) The factorization condition for morphisms (6.5) means that any $\phi \in \mathcal{M}(X, Y)$ can, up to isomorphism $(\sigma \Downarrow \sigma')$, be decomposed as

$$\begin{tikzcd} X \arrow[r, "\phi"] \arrow[d, "\sigma"', "\simeq"] & Y \arrow[d, "\simeq"', "\sigma'"] \\ \bigotimes_{v\in V} X_v \arrow[r, "\bigotimes_{v\in V} \phi_v"] & \bigotimes_{v\in V} Y_v \end{tikzcd} \tag{6.5}$$

with $\phi_v \in \jmath(\mathcal{P})$ and this decomposition is essentially unique, in the sense that it is unique up to isomorphisms —in the arrow category— on the factors ϕ_v and permutations of these factors. We will write $\phi \simeq \bigotimes_{v\in V} \phi_v$ for a diagram of type (6.5).

(4) A compatibility allows one to think of the objects of $\mathcal{P}$ as multi–to–multi morphisms with a given decomposition of the source and target. If $\jmath(\phi) = (X, \jmath(\phi), Y)$ then $\imath_{\mathcal{P}}(\phi) = (\bigotimes_{v\in V} X_v, i_{\mathcal{P}}(\phi), \bigotimes_{w\in W} X_w)$ with $X = \bigotimes_{v\in V} \imath(X_v)$, $Y = \bigotimes_{w\in W} \imath(X_w)$ and $i_{\mathcal{P}}(\phi) = j(\phi)$ by compatibility.
  (a) In particular, a choice of compatible bases allows us to fix the vertical isomorphisms in (6.5) in such a way, that if $\tilde{\imath}(X) = \bigotimes_{s\in S} *_s$ and $\tilde{\imath}(Y) = \bigotimes_{t\in T} *'_t$ are realized by σ and σ', that is $\mathrm{idx}(X) = S$ and $\mathrm{idx}(Y) = T$ then there is a V–partition of S and a V–partition of T such that $\tilde{\imath}(X_v) = \bigotimes_{s\in S_v} *_s$ and $\tilde{\imath}(Y_v) = \bigotimes_{v\in T_v} *'_v$. This defines a V–partition of S and T and hence a pair of maps $s_V : S \to V$ and $t_V : T \to V$.
  (b) Thus a compatible choice of basis yields a pair of maps which form a co-span: $\mathrm{idx}(\phi) : (S \xrightarrow{s_V} V \xleftarrow{t_V} T)$. It turns out that idx being a functor is equivalent to the condition that $\mathcal{M}$ is part of a hereditary UFC, see §6.3 below.

Lemma 6.6. *A category with essentially unique factorization of objects has factorizable isomorphism and has common factorizations of morphisms.*

Proof. Straightforward from Remark 6.5 (2). Given two factorizations of a morphisms $\phi = \phi_1 \otimes \phi_2 = \psi_1 \otimes \psi_2$, decomposing ϕ into irreducibles yields the desired decomposition. $\square$

Corollary 6.7. *A category with essentially unique factorization of objects that is hereditary is fully hereditary.* $\square$

Proposition 6.8. *If $\mathcal{M}$ has essentially uniquely factorizable morphisms, then it has essentially uniquely factorizable objects in particular, it has factorizable isomorphism and common factorizations of morphisms.*

Concretely, if $\mathcal{P} \subset Iso(\mathcal{M} \downarrow \mathcal{M})$ is a maximal set of basic morphisms, which can be obtained after replacing some choice of basic morphisms with its essential image, $\mathcal{V}$ can be chosen to be the full subgroupoid of $\mathrm{Iso}(\mathcal{M})$ *whose objects are those X which $id_X \in \mathcal{P}$. The inclusion $\imath_{\mathcal{P}}$ which is given by the source and target is a compatibility.*

Proof. Let $\mathcal{V}$ be the full subgroupoid defined above. We have to show that this is equivalent to $\mathrm{Iso}(\mathcal{M})$. First, we will show that any object X of $\mathcal{M}$ factorizes essentially uniquely. By the assumption for any object X of $\mathcal{M}$, id_X factorizes essentially uniquely as $\bigotimes_{v\in V} \phi_v$, so that $depth(id_X) = |V|$. Let $X_v = s(\phi_v)$, then X factorizes as $X \simeq \bigotimes_{v\in V} X_v$ and $id_X \simeq \bigotimes_{v\in V} id_{X_v}$ is another decomposition of the morphism id_X. Hence $depth(id_{X_v}) = 1$ and there are irreducible and unique up to isomorphisms and permutations of the identities. This in turn means that the decomposition into the X_v is essentially unique up to isomorphisms and permutations and that the X_v are irreducible. Indeed, if $X_v \simeq X'_v \otimes X''_v$ then $id_{X_v} \simeq id_{X'_v} \otimes id_{X''_v}$ and one of the $X'', X' = \mathbb{1}$ as otherwise, the identity morphisms of the other would not be in $\mathcal{P}^{\otimes}$.

Second, we have to show that $\mathcal{V}^{\otimes} \simeq Iso(\mathcal{M})$ also on the level of morphisms. Consider an isomorphism $\sigma : X \xrightarrow{\sim} Y$. Let $\mathrm{id}_Y \simeq \bigotimes_{v\in V} id_{Y_v}$ be a decomposition of id_Y into irreducibles, then $(\sigma \Downarrow id_Y) : \sigma \to \bigotimes_{v\in V} id_{Y_v}$ is an isomorphism in $\mathrm{Iso}(\mathcal{M}\downarrow\mathcal{M})$, see (6.6).

$$\begin{tikzcd}
X \arrow[r, "\sigma"] \arrow[d, "\sigma"'] & Y \arrow[r, "\simeq"] \arrow[d, "id_Y"] & \bigotimes_{v\in V} Y_v \arrow[d, "\bigotimes_v id_{Y_v}"] \\
Y \arrow[r, "id_Y"'] & Y \arrow[r, "\simeq"'] & \bigotimes_{v\in V} Y_v
\end{tikzcd} \tag{6.6}$$

Thus any isomorphism in $Obj(\mathcal{P}^{\otimes}) = \mathrm{Iso}(\mathcal{M} \downarrow \mathcal{M})$ is isomorphic to tensor products of depth one isomorphisms which are between objects of $\mathcal{V}$. Any σ thus decomposes essentially uniquely as $\bigotimes_{v\in V} \sigma_v$ with $\sigma_v \in \mathcal{V}(X_v, Y_v)$. In fact essentially uniquely as identities of irreducible objects.

That the inclusion is a compatibility is clear. $\square$

Note that by the proof of the proposition above, there is always a compatible choice of basis and, up to equivalence, we can always assume that the choices are compatible. It is, however, useful to have presentations given by choices. "Good" (vs. evil) categorical notions, that is those up to equivalence are of course independent of such a choice, but concrete applications and constructions warrant them.

Definition 6.9. A *unique factorization category* (UFC) is a symmetric monoidal category with uniquely factorizable morphisms, whose slice categories are essentially small, together with a choice of basis, viz. a triple $(\mathcal{M}, \mathcal{P}, \jmath)$.

A morphism of UFCs is a pair of a functor $b : \mathcal{P} \to \mathcal{P}'$ and a strong monoidal functor $f : \mathcal{M} \to \mathcal{M}'$ that commute with the functors $\jmath^{\boxtimes}$ and $\jmath'^{\boxtimes}$.

We will call a UFC *strict* if $\jmath^{\boxtimes}$ is the identity.

The new terminology allows us to succinctly reword the definition of a Feynman category [KW17].

Definition 6.10. A Feynman category is a symmetric monoidal category $\mathcal{F}$ whose slice categories are all essentially small, with a choice of basic objects $(\mathcal{V}, \imath)$ such that $\mathrm{Iso}(\mathcal{F} \downarrow \mathcal{V}), \jmath = (id_{\mathcal{F}}, id_{\mathcal{F}}, \imath)$ is a compatible choice of basic morphisms $\mathcal{P}$ making $\mathcal{F}$ into a UFC.

Note for a Feynman category all the basic morphisms are necessarily of type $(n, 1)$; that is multi–to–one.

Definition 6.11. A presentation of a UFC is category with essentially factorizable morphisms together with a choice of basic objects, and a compatibility. This is thus a tuple $(\mathcal{V}, \mathcal{P}, \mathcal{M}, \imath, \jmath, \imath_{\mathcal{P}})$. We call a presentation strict, if the equivalences $\imath^{\otimes}, \jmath^{\otimes}$ are identities.

A morphism between UFCs with a presentation $\mathfrak{f} : (\mathcal{M}, \mathcal{V}, \mathcal{P}, \imath, \jmath) \to (\mathcal{M}', \mathcal{V}', \mathcal{P}', \imath', \jmath')$ is a triple functors $\mathfrak{f} = (v, p, f)$, with f (symmetric) monoidal, which fit to the following commutative diagrams.

$$\begin{tikzcd} \mathcal{V} \arrow[r, "\imath"] \arrow[d, "v"'] & \mathcal{M} \arrow[d, "f"] \\ \mathcal{V}' \arrow[r, "\imath'"] & \mathcal{M}' \end{tikzcd} \quad \begin{tikzcd} \mathcal{P} \arrow[r, "\jmath"] \arrow[d, "p"'] & \mathrm{Iso}(\mathcal{M}\downarrow\mathcal{M}) \arrow[d, "{(f,f,f)}"] \\ \mathcal{P}' \arrow[r, "\jmath'"] & \mathrm{Iso}(\mathcal{M}'\downarrow\mathcal{M}') \end{tikzcd} \quad \begin{tikzcd} \mathcal{P} \arrow[r, "p"] \arrow[d, "\imath_{\mathcal{P}}"'] & \mathcal{P}' \arrow[d, "\imath_{\mathcal{P}'}"] \\ \mathrm{Iso}(\mathcal{V}^{\otimes}\downarrow\mathcal{V}^{\otimes}) \arrow[r, "{(v^{\otimes},f,v^{\otimes})}"] & \mathrm{Iso}((\mathcal{V}')^{\otimes}\downarrow(\mathcal{V}')^{\otimes}) \end{tikzcd} \tag{6.7}$$

In the case of strict hereditary UFCs, it suffices to specify f and for Feynman categories it suffices to specify (v, f), cf. [KW17, §1].

A morphism of presentations of UFCs categories is a (strong/strict) *indexing* if f is a (strong/strict) indexing.

This generalizes the notions of [Kau21, Definition 2.1.1].

Lemma 6.12. *Feynman categories correspond presentations of UFCs all of whose basic morphisms satisfy* $\mathcal{P} = \mathrm{Iso}(\mathcal{V}^{\otimes} \downarrow \mathcal{V})$, *and* $\imath_P$ *is the image of the canonical map* $\mathrm{Iso}(\mathcal{V}^{\otimes} \downarrow \mathcal{V})^{\otimes} \to \mathrm{Iso}(\mathcal{V}^{\otimes} \downarrow \mathcal{V}^{\otimes})$ *sending a formal product of morphisms to the monoidal product of morphisms.*

Proof. This is straightforward. Given a Feynman Category, $(\mathcal{V}, \imath, \mathcal{F})$, we complete to a UFC by the given data. If $\mathcal{P} = Iso(\mathcal{V}^{\otimes} \downarrow \mathcal{V})$ then restricting the data yields a FC. $\square$

6.2. **Hereditary UFCs.** UFCs, unlike Feynman categories, are not automatically hereditary.

Definition-Proposition 6.13. *A basis for morphisms $(\mathcal{P}, \jmath)$ is hereditary if for every pair of composable morphisms (ϕ_0, ϕ_1), with $\phi_1\phi_0 = \phi$, and decomposition into irreducible morphisms*

$$\phi_0 \simeq \bigotimes_{v\in V} \phi_{0,v}, \quad \phi_1 \simeq \bigotimes_{w\in W} \phi_{1,w}, \ \text{and } \phi = \bigotimes_{u\in U} \phi_u \tag{6.8}$$

there exists a partition of $V \amalg W = \amalg_{u\in U} P_u$ indexed by U, such that for each $u \in U$ there is a decomposition pair $(\phi_{0,u}, \phi_{1,u})$ of the ϕ_u, viz. $\phi_{1,u}\phi_{0,u} = \phi_u$, such that

$$\phi_{0,u} \simeq \bigotimes_{v\in P_u\cap V} \phi_{0,v} \ \text{and } \phi_{1,u} \simeq \bigotimes_{w\in P_u\cap W} \phi_{1,w} \tag{6.9}$$

A UFC is hereditary UFC *if its basis is hereditary. The underlying category $\mathcal{M}$ is hereditary if and only if it has a hereditary basis.*

Proof. Straightforward. $\square$

NB: Without the hereditary conditions there is still a decomposition into morphisms ϕ_u of ϕ, called connected components, which, however, need not be irreducible, see (6.11) and §6.3.

Proposition 6.14. *A Feynman category $(\mathcal{F}, \mathcal{P}, \jmath)$ is a hereditary UFC, since the hereditary condition will hold automatically, if the morphisms are all multi–to–one, see Corollary 6.18.*

Proof. This follows from Corollary 6.18 below. $\square$

6.3. **Connected components.** To get a useful criterion when hereditary UFCs are hereditary, we will show that any composable pair gives rise to a diagram (6.11) with the a maximal choice of U, but with the ϕ_u not necessarily in $\mathcal{P}$. This generalizes the construction in $\mathcal{C}ospan$ and uses the connected components of $\mathcal{S}pan$, see §6.4.

Proposition 6.15. *For $\mathcal{C}$ with uniquely factorizable objects. Given any two decompositions of an object X into objects $X \simeq \bigotimes_{s\in S} X_s$ and $X \simeq \bigotimes_{t\in T} X_t$ for any decomposition X into basic objects $X \simeq \bigotimes_{v\in V} *_v$ there is a span $S \stackrel{l}{\leftarrow} V \stackrel{r}{\to} T$, whose connected components yield isomorphisms: $\bigotimes_{s\in S_c} X_s \simeq \bigotimes_{v\in V_c} *_v \simeq \bigotimes_{t\in T_c} X_t$.*

Proof. Decomposing as above, by essentially unique factorization there are maps $V \to S$ and $V \to T$ such that $X_s \simeq \bigotimes_{v\in l^{-1}(s)} *_v, X_t \simeq \bigotimes_{v\in r^{-1}(t)} *_v$. Considering the connected components of the span as in Remark 6.26, we see that the restriction $S_c \leftarrow V_c \to T_c$ yields the isomorphisms $\bigotimes_{s\in S_c} X_s \simeq \bigotimes_{v\in V_c} *_v \simeq \bigotimes_{t\in T_c} X_t$. $\square$

Corollary 6.16. *Given two composable morphisms, $X_0 \stackrel{\phi_0}{\to} X_1 \stackrel{\phi_1}{\to} X_2$, choosing a presentation and pseudo-inverses as above yields the diagram*

$$\begin{tikzcd}[column sep=small]
& X_1 \arrow[dl, "\simeq"'] \arrow[d, "\eta", "\simeq"'] \arrow[dr, "\simeq"] & \\
\bigotimes_{v\in V} X_{1,v} \arrow[dr, "\mu'", "\simeq"'] & \bigotimes_{s\in S} \imath(*_s) \arrow[l, "\simeq"'] \arrow[r, "\simeq"] \arrow[d, "\simeq"] & \bigotimes_{w\in W} X_{1,w} \arrow[dl, "\simeq"] \\
& \bigotimes_{u\in U} X_{1,u} &
\end{tikzcd} \tag{6.10}$$

where $\phi : X_0 \to X_1$ *is decomposed as* $\phi = \bigotimes_{v\in V} \jmath(\phi_v)$ *inducing decompositions of* X *into the* $X_{0,v} = s(\jmath\phi_v)$ *and* $X_{1,v} = t(\jmath\phi_v)$. *The middle product is the chosen decomposition of* X_1, *the* $X_{1,v}$ *and* $X_{1,w}$ *come from decomposing* $\phi_0 \simeq \bigotimes_{v\in V} \phi_{0,v}$ *and* $\phi_1 = \bigotimes_{w\in W} \phi_{1,w}$ *and* U *is the push–out of the span* $V \leftarrow S \to W$ *which defines the* $X_{1,u}$ *via Lemma 6.15.*

Proposition 6.17. *Given a pair of composable morphisms* (ϕ_0, ϕ_1) *in a UFC* $(\mathcal{M}, \mathcal{P}, \jmath)$ *together with decompositions* $\phi_0 \simeq \bigotimes_{v\in V} \phi_{0,v}$ *and* $\phi_1 = \bigotimes_{w\in W} \phi_{1,w}$, *there exists a partition of* $V \amalg W = \amalg_{u\in U} P_u$ *indexed by* U, *such that for each* $u \in U$ *there is a composable pair* $(\phi_{0,u}, \phi_{1,u})$ *with* $\phi_{1,u} \circ \phi_{0,u} = \phi_u$ *and* $\phi_{0,u} \simeq \bigotimes_{v\in P_u} \phi_{0,v}, \phi_{1,u} \simeq \bigotimes_{v\in P_u} \phi_{1,v}$. *These fit into the diagram* (6.11).

The ϕ_u *are essentially unique and will be called the* connected components *of the decomposition. A UFC is hereditary precisely if all the connected components are irreducible.*

Proof. To do the computation, fix compatible bases and pseudo–inverse functors $\tilde{\imath}, \tilde{\jmath}$, defining $(\sigma \Downarrow \sigma')$ for ϕ_0 and $(\nu \Downarrow \nu')$ for ϕ_1 in (6.11). This also fixes $\tilde{\imath}(X_1) \simeq \bigotimes_{s\in S} *_s$ then we obtain a decomposition that defines a "middle square" via the push–out of $V \leftarrow S \to W$ according to the Corollary above. This defines the remaining morphisms in (6.11), namely the composable pairs of morphisms $(\phi_{0,u}, \phi_{1,u}), u \in U$ and their compositions $\phi_u = \phi_{1,u} \circ \phi_{0,u}$.

By definition in a hereditary UFC these are irreducible, and vice–versa that the ϕ_u are irreducible is the condition for a hereditary UFC. ☐

We will call the essentially unique ϕ_u *the connected components* of the composition.

Corollary 6.18. *If for a UFC the elements of* $\mathcal{P}$ *are all of type* $(n, 1)$ *the UFC is a hereditary UFC.*

Proof. As the $\phi_{0,v}$ are of type $(n_v, 1)$ the left arrow in the span indexing the diagram is a bijection that is $V \leftrightarrow S \to W$. The pushout are thus just the fibers of r and are in bijection with W. Thus the ϕ_u are of type $(n_u, 1)$ and hence irreducible since the type is additive. ☐

6.4. **Examples.** Feynman categories are examples of UFCs. We review the paradigmatic example of the Feynman category of finite sets and then give the paradigmatic example of a hereditary UFC which is $\mathcal{C}ospan$. We also give an example of a UFC that is not hereditary.

6.4.1. **The FC of finite Sets.** Let $\mathcal{F}in\mathcal{S}et$ be the category of finite sets with the monoidal product of disjoint union $\amalg$. Let $\underline{*}$ be the category with one object $*$ and its identity id_*. We will denote the unique map $T \to \{*\}$ by π_T. Then $\mathcal{F}in\mathcal{S}et$ is a Feynman category by $\mathcal{F} = \mathcal{F}in\mathcal{S}et$, $\mathcal{V} = \underline{*}$, and $\imath(*) = \{1\}$. Note that $\mathcal{V}^{\otimes}$ has the groupoid $\mathbb{S}$ as its strict version and skeleton.

$$\begin{tikzcd}
X_0 \arrow[r, "\simeq", "\sigma"'] \arrow[dd, "\phi_0"'] \arrow[rr, bend left, "\chi", "\simeq"'] \arrow[dddd, bend right=60, "\phi"'] & \bigotimes_{v\in V} X_{0,v} \arrow[r, "\simeq", "\mu"'] \arrow[d, "\bigotimes_{v\in V}\phi_{0,v}"'] & \bigotimes_{u\in U}\bigotimes_{v\in P_u} X_v \arrow[dd, "\bigotimes_{u\in U}\phi_{0,u}"'] \arrow[dddd, bend left=60, "\bigotimes_{u\in U}\phi_u"] \\
 & \bigotimes_{v\in V} X_{1,v} \arrow[dr, "\simeq", "\mu'"'] & \\
X_1 \arrow[ur, "\simeq", "\sigma'"'] \arrow[dr, "\nu", "\simeq"'] \arrow[dd, "\phi_1"'] & & \bigotimes_{u\in U} X_{1,u} \arrow[dd, "\bigotimes_{u\in U}\phi_{1,u}"'] \\
 & \bigotimes_{w\in W} X_{1,w} \arrow[ur, "\tau", "\simeq"'] \arrow[d, "\bigotimes_{w\in W}\phi_{1,w}"'] & \\
X_2 \arrow[r, "\nu'", "\simeq"'] \arrow[rr, bend right, "\simeq", "\chi'"'] & \bigotimes_{w\in W} X_{2,w} \arrow[r, "\tau'", "\simeq"'] & \bigotimes_{u\in U} X_{2,u}
\end{tikzcd} \tag{6.11}$$

FIGURE 1. The diagram for the condition of hereditary and the definition of connected components

The condition on morphisms is guaranteed by the fact that maps of sets have fibers

$$\begin{tikzcd}
S \arrow[r, "f"] \arrow[d, "\simeq"'] & T \arrow[d, "\simeq"] \\
\coprod_{i=1}^{|T|} f^{-1}(i) \arrow[r, "\amalg \pi_{\{t\}} f|_{f^{-1}(i)}"'] & \coprod_{i=1}^{|T|} \imath(*) = \underline{|T|}
\end{tikzcd} \tag{6.12}$$

where we chose a representative for the disjoint union. The fibers are the decomposition into irreducibles. The objects in $\mathcal{P} = \mathrm{Iso}(\mathcal{M}\downarrow\mathcal{V})$ are the surjections $\pi_S : S \to \{*\}$ if $S \neq \emptyset$ and the empty map $i_\emptyset : \emptyset \to \{*\}$ if the source is empty. We set $\mathfrak{FinSet} = (\underline{*}, \mathcal{F}in\mathcal{S}et, \imath)$.

Remark 6.19.

(1) Working with the skeleton $sk(\mathcal{F}in\mathcal{S}et)$ has "same" objects as $\mathbb{S}$ under the identification $n \leftrightarrow \underline{n} = \{1, \dots, n\}$, where the equivalence class of a finite set T is $\underline{|T|}$ in the skeleton. Under the above identification $\mathbb{S} = \mathrm{Iso}(sk(\mathcal{F}in\mathcal{S}et)) = sk(\mathrm{Iso}(\mathcal{F}in\mathcal{S}et))$. In the skeleton, the monoidal structure is then given by addition or $\underline{n} \amalg \underline{m} = \underline{n+m}$ and $\underline{0} = \emptyset$ is the monoidal unit. This identifies $sk(\underline{*}^{\otimes}) = \mathbb{S}$ and establishes the equivalence.

(2) Working with the unbiased strict monoidal category $\underline{*}^{\otimes \mathcal{S}et} = \mathrm{Iso}(\mathcal{F}in\mathcal{S}et)$.

(3) The restrictions to injections, FI, and surjections, FS, are also Feynman categories, [KW17, Kau21].
(4) There is a non–Sigma version given by finite ordered sets, [Kau21] for details, and the respective subcategories OI and OS.
(5) A graphical representation for the surjections, see e.g. [Kau18] is given by depicting a morphism as a collection of rooted corollas. In particular the basic morphism $\pi : S \to \{t\}$ is depicted by the corolla with root t and leaves S. The composition is given by grafting corollas to a level forests and then contracting the edges, see [Kau04]. The connected components of a pair of composable morphisms, discussed in the next paragraph, are the 2-level trees of the 2-level forest.

 In the ordered case, these corollas and forests are planar, cf. [Kau21].

6.4.2. **The hereditary UFC of Cospans.** The analogous hereditary UFC to $\mathcal{F}in\mathcal{S}et$ for Feynman categories is the category of cospans of finite sets $\mathcal{C}ospan$. The objects are finite sets and morphisms are isomorphism classes of cospans. A cospan is a diagram

$$\begin{tikzcd} & V & \\ S \arrow[ur, "l"] & & T \arrow[ul, "r"'] \end{tikzcd} \tag{6.13}$$

a short hand notation is (l, r). An isomorphism of cospans is given by an isomorphism in the middle

$$\begin{tikzcd} & V \arrow[dd, "\sigma", "\simeq"'] & \\ S \arrow[ur, "l"] \arrow[dr, "l'"'] & & T \arrow[ul, "r"'] \arrow[dl, "r'"] \\ & V' & \end{tikzcd} \tag{6.14}$$

That is $(l, r) \simeq (\sigma \circ l, \sigma \circ r)$ for an isomorphism $\sigma : V \to V'$. Let $[(l, r)]$ denote the isomorphism class containing (l, r).

The composition of two classes cospans is given by push–out:

$$\begin{tikzcd} & & U & & \\ & V \arrow[ur, dashed, "f'"] & & W \arrow[ul, dashed, "g'"'] & \\ S \arrow[ur, "f_1"] & & C \arrow[ul, "g_1"'] \arrow[ur, "f_2"] & & T \arrow[ul, "g_2"'] \end{tikzcd} \tag{6.15}$$

Note this is well defined and associative on isomorphism classes, due to the universal property of cospans. The units are the cospans $[(id_S, id_S)]$.

The *isomorphisms* in the category $\mathcal{C}ospan$ are the morphisms $[(\sigma, \tau)]$ where σ and τ are bijections.

This category is *monoidal* with respect to the disjoint union $\amalg$: $[(l, r)] \amalg [(l', r')] = [(l \amalg l', r \amalg r')]$. The unit $\mathbb{1}_{\mathcal{C}ospan}$ is the class of the empty cospan: $\emptyset \to \emptyset \leftarrow \emptyset$.

Remark 6.20. If one does not pass to isomorphism classes, one ends up with a 2–category, see [Bén67]. Here a 2–morphism between two cospans (l, r) and (l', r') with the same source and target is given by any morphisms $m : V \to V'$ such that $l' = lm, r' = rm$. Such morphisms, more precisely surjections m, are natural when considering mergers for directed graphs, see below.

We will call a cospan *connected* if $|V| = 1$. This notion is well defined under isomorphisms of cospans and such a map is given by the class $[(\pi_S, \pi_T)]$, where for any set $X : \pi_X : S \to \{*\}$ is the unique map to a final object. A *singleton map* is a map in which $|S| = |T| = |V| = 1$. The following is straightforward:

Lemma 6.21.

(1) *Any morphisms $\phi \in Hom_{\mathcal{C}ospan}(S,T)$ is isomorphic to a disjoint union of connected morphisms. In particular:*

$$(S \xrightarrow{l} V \xleftarrow{r} T) \simeq \amalg_{v \in V}(\pi_{l^{-1}(v)}, \pi_{r^{-1}(v)}) \tag{6.16}$$

This decomposition is unique up to unique isomorphism given by simultaneous permutation of fibers and isomorphisms fixing the fibers of l and r.

(2) $Aut([(\pi_S, \pi_T)]) = \mathrm{Aut}(S) \times \mathrm{Aut}(T)$.

(3) *The isomorphisms in the category $\mathcal{C}ospan$ are given by disjoint unions of singleton maps.*

□

Remark 6.22.

(1) A cospan is the same as a V-partition of $S \amalg T$ given by $l \amalg r$. This induces an equivalence relation on $S \amalg T$ whose classes are in 1–1 correspondence with the union of the images of the maps l and r. The elements not in this image are empty classes that cannot be recovered from the equivalence relation alone.

(2) The push–out U is the relative coproduct $U = V\,{}_{g_1}\!\amalg_{f_2} W = (V \amalg W)/\sim$ where $\sim$ is the equivalence relation given by $g_1(c) \sim f_2(c)$ for $c \in C$. The resulting equivalence relation on $S \amalg T$ is given by the image of $f_1 \circ f' \amalg g_1 \circ g'$.

(3) An isomorphism class of cospans still has a residual action of $(Aut(S) \times Aut(T))/\mathrm{Aut}(V)$ on $Hom(S,V) \times Hom(T,V)$ where $Aut(V)$ acts diagonally permuting the fibers of l and r. This identifies $Aut([(l,r)])$ as the classes of those elements of $Aut(S) \times Aut(T)$ which simultaneously preserve the fibers of l and r over each v.

(4) There is a left and a right injection of $\mathcal{F}in\mathcal{S}et$ to $\mathcal{C}ospan$ given by identity on objects and by sending f to (f, id) or (id, f).

(5) Note that $[(\sigma, \tau)] = [(\tau^{-1}\sigma, id)]$ and $[(\sigma, id_T)] = [(id_S, \sigma^{-1})]$ where $\sigma : S \xrightarrow{\sim} T$. Thus one can identify $\mathrm{Iso}(\mathcal{C}ospan) \simeq \mathrm{Iso}(\mathcal{F}in\mathcal{S}et)$ by the identity on objects and sending an isomorphism σ to $[(\sigma, id)]$.

Proposition 6.23. *$\mathcal{C}ospan$ is part of a hereditary UFC with $\mathcal{V} = \{1\}$ and $\mathcal{P}$ being the full subgroupoid $\mathcal{C}td$ of $\mathrm{Iso}(\mathcal{C}ospan \downarrow \mathcal{C}ospan)$ whose objects are connected cospans, with $\jmath$ the inclusion and $\jmath^{\otimes}$ the identity.*

Proof. The fact that $\mathcal{C}ospan$ is a UFC with basis $\mathcal{C}td$ is clear from Lemma 6.21. To show the hereditary condition note that if $\phi_0 = [(f_1, g_1)]$ and $\phi_1 = [(f_2, g_2)]$ as in (6.15) then the decomposition of $\phi := \phi_1\phi_0 = \amalg_{u \in U}\phi_u$ is given by $\phi_u = [(f'f_1|_{(f'f_1)^{-1}(u)}, g'g_2|_{g'g_2^{-1}(u)})]$. Letting $P_u = f'^{-1}(u) \amalg g'^{-1}(u)$ gives the partition of $V \amalg W$ and each ϕ_u decomposes into $\phi_{0,u} = \amalg_{v \in f'^{-1}(u)}\phi_{0,v}$ and $\phi_{1,u} = \amalg_{w \in g'^{-1}(u)}\phi_{0,w}$ which constitute the desired decomposition. □

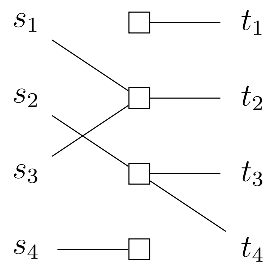


FIGURE 2. An example of a morphism $S \to T$ in $\mathcal{C}ospan$ for $S = \{s_1, s_2, s_3, s_4\}$ and $T = \{t_1, t_2, t_3, t_4\}$.

A graphical representation of a morphism is given in Figure 2. The morphisms in Figure 2 has four connected components.

The boxes represent the subsets of the partition and a edge represents membership. In this graphical representation, the composition is obtained by collecting the boxes and middle dots that are connected into one "big box" and retaining the box as a new vertex erasing the inside. An example is given in Figure 3.

Remark 6.24. There are several important interpretations of this structure.

(1) *Graph interpretation.* For this, one interprets V as a set of vertices of an aggregate of corollas, S as the set of input flags and T as the set of output flags. The map $\partial = l \amalg r : S \amalg T \to V$ is the attaching maps of the flags. If V is a singleton set then the irreducible $S \to \{*\} \leftarrow T$ is a directed corolla. The composition matches input and output flags and then contracts them, generalizing the usual corolla interpretation of the associative and commutative operad, see [Kau04].
(2) *PROP interpretation.* This composition is not by chance reminiscent of the composition in PROPs. It corresponds the terminal functor $\mathcal{T}$ for the Feynman category for PROPs $\mathfrak{F}_{PROP}$, see [KW17, §2], with values in $\mathcal{S}et$.
(3) *Surface and cobordism interpretation.* In this interpretation, consider the cospan $S \to \{*\} \leftarrow T$ to be a surface with S–labeled input boundaries and T–labeled output boundaries. Upon gluing the cobordisms one stabilizes by removing all extra handles. This corresponds to the contraction above and corresponds to the stabilized arc structures of [Kau09]. To recover the "lost genus" one can use a push–forward as in [BK17, BK22].

6.4.3. **Skeletal versions.** There is a skeletal version of $\mathcal{C}ospan$ whose objects are the sets $\underline{n}, n \in \mathbb{N}_0$ and the morphisms are the cospans for these objects. The monoidal structure is then given by $+$, where one identifies $\underline{n} \amalg \underline{m} \leftrightarrow \underline{n+m}$, by identifying $\underline{n}$ with the first n elements and $\underline{m}$ with the last m elements. This is strictly associative and unital. The commutativity constraints are explicitly given as follows:

$$B_{n,m} : \underline{n} + \underline{m} \to \underline{m} + \underline{n} \tag{6.17}$$

By definition, this is a partition of $\underline{n+m} \amalg \underline{n+m}$. Define the partition $B_{n,m}$ by the following pairing:

- If i on the left is such that $i \leq n$, pair it with $i + m$ on the right.
- If i on the left is such that $i > n$, pair it with $i - n$ on the right.

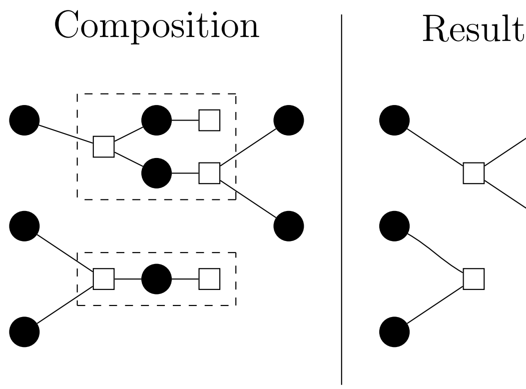
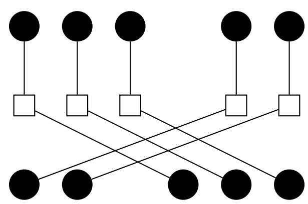


FIGURE 3. A composition of two morphisms.

Graphically, the monoidal product is obtained by juxtaposing two graphs and relabeling. The commutativity constraint is a block transposition graph. For example, $B_{3,2}$ would correspond to the following graph:



(6.18)

6.4.4. **Spans.** Just like cospans, one can consider equivalence classes of spans $(l,r) = S \leftarrow V \to T$ by reversing the arrows. The composition is then by pull–back. The disjoint union again gives a monoidal structure. A span is connected if the pushout $S \amalg_V T$ has a single element. Let $\mathcal{C}td_S$ be the subgroupoid of connected spans.

Proposition 6.25. *$\mathcal{S}pan$ is a hereditary UFC with basis $\mathcal{C}td_S$.*

Proof. Let $C = S \amalg_V T$ and $\pi : S \amalg T \to S \amalg_V T$ be the projection. For $c \in C$ let $S_c = \pi^{-1}(c) \cap S$ and $T_c = \pi^{-1}(C) \cap T$ be the preimages, and set $V_c = l^{-1}(S_c)$ then $r(V_c) = T_c$ and the restriction (l_c, r_c) is well defined. First assume that l, r are surjective. Then the V_c are non–empty and $(l,r) = \amalg_{c \in C}(l_c, r_c)$. If the maps are not surjective decompose $S = im(l) \amalg S_0$ and $T = im(r) \amalg T_0$. Then $C = im(l) \amalg_V im(r) \amalg S_0, \amalg T_0$ and $(l,r) = \amalg_{c \in C}(l_c, r_c)$, were for $s \in S_0 : l_s : \{s\} \leftarrow \emptyset \to \emptyset$ and similarly for $t \in T_0$. As is easily checked, this decomposition is unique up to unique isomorphisms and hence $\mathcal{S}pan$ is a UFC with the basis $\mathcal{C}td_S$. For the hereditary condition note that the connected components of the composition naturally decompose into connected components of the two constituent morphisms by the universal properties of push–forwards. □

Remark 6.26. From the proof we see that if both maps are surjective, then we can think of both S and T giving partitions of V via their fibers, and the connected components give the greatest common partition of $V = \amalg_{c \in C} V_c$.

6.4.5. **A UFC that is not a hereditary UFC.** For a finite set X a *tie* is given by a pair $\mathcal{T} = (X_0, \{X_1, , \dots, X_k\})$ in which $X_0 \subseteq X$ is a possibly empty subset, and $\{X_1, , \dots, X_k\}$ is a partition of $X \setminus X_0$ into nonempty subsets. X_0 is the subset of *untied* elements and each

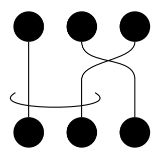
$X_i, i > 0$ is a tied subset. Note that a single element x that can be untied $x \in X_0$, a part of a tied subset, or tied by itself $U_i = \{x\}$ for some i. Let $T(X)$ be the set of ties for X.

Consider the category $\mathcal{T}ies$ whose objects are finite sets and whose morphisms will be *isomorphisms* of finite sets together with a choice of tie. Let $\mathrm{Hom}(X, Y) = \mathrm{Iso}(X, Y) \times T(Y)$. Note that under an isomorphisms canonically $T(X) \simeq T(Y)$. Now define the composition of $(\sigma, \mathcal{T}) \in \mathrm{Iso}(X, Y) \times T(Y)$ and $(\sigma', \mathcal{T}') \in \mathrm{Iso}(Y, Z) \times T(Z)$ to be $(\sigma' \circ \sigma, \mathcal{T}' \circ \mathcal{T})$ where

$$\mathcal{T}' \circ \mathcal{T} = (\sigma(U_0) \cap V_0, \{\sigma(U_i) \cap V_j : (i, j) \neq (0, 0)\} \setminus \{\emptyset\})$$

This formula says that the resulting partition is the least common refinement of the partition for the tied elements with the untied elements acting as identity on the ties. In particular, the identity of X in this category is $(\mathrm{id}_X, (X, \emptyset))$.

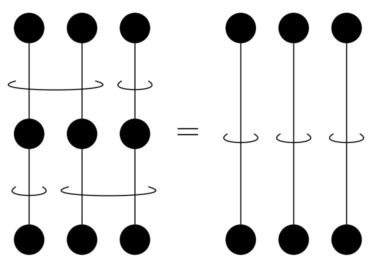
Pictorially one can write a morphisms as a line diagrams for a bijection with at most one tie around each strands, where we think of $T(Y)$ as bands tied at the bottom.



(6.19)

The strand on the right is untied.

Composition amounts to joining the strands and retying the ties if they overlap:



(6.20)

The category $\mathcal{T}ies$ is monoidal. The monoidal product on objects will be disjoint union $X \otimes Y = X \amalg Y$.

To give the structure on morphisms, define a product $\otimes : T(X) \times T(Y) \to T(X \amalg Y)$ by

$$(X_0, \{X_i\}_{i=1}^n) \otimes (Y_0, \{\mathcal{Y}_j\}_{j=1}^m) = (\iota_X X_0 \cup \iota_Y Y_0, \{\iota_X X_i\}_{i=1}^n \cup \{\iota_Y Y_j\}_{j=1}^m) \tag{6.21}$$

Where $\iota_X : X \to X \amalg Y$ and $\iota_Y : Y \to X \amalg Y$ are the natural inclusion maps. Then, $(\sigma, \mathcal{T}) \otimes (\sigma', \mathcal{T}') = (\sigma \amalg \sigma', \mathcal{T} \otimes \mathcal{T}')$.

Proposition 6.27. *The category $\mathcal{T}ies$ together with subset of morphisms $\mathcal{P}$ for which the tie is either $(\{x\}, \emptyset)$ for a singleton set, or $(\emptyset, \{X\})$ for any set is a UFC, but not a hereditary UFC.*

Proof. Note that the isomorphisms of $\mathcal{T}ies$ are of the form $(\sigma, (X, \emptyset))$ and in the arrow category $(\sigma, \mathcal{T})$ is equivalent to $(id_X, \mathbb{T}) = \bigotimes_{x \in X_0}(id_{\{x\}}, (\{x\}, \emptyset)) \otimes \bigotimes_{i=1}^k (id_{X_i}, (\emptyset, \{X_i\}))$, if $\mathbb{T} = (X_0, \{X_1, \ldots, X_k\})$ then $(id_X, \mathbb{T})$. This is unique up to unique isomorphisms proving $\mathcal{T}ies$ is a UFC.

The fact that this is not a hereditary UFC is proven by the example (6.20). The decomposition of the composition of the two morphisms into indecomposables is $(id_{\{x\}}, (\emptyset, \{x\})^{\otimes 3})$, while the decomposition of the constituent morphisms each have two factors and the composable decomposition of these involve both factors, so that there is no decomposition into three tensor factors as required in (6.9). $\square$

6.5. **Indexing and connected components of multiple composition.** The decomposition of a morphisms for a UFC is governed by $\mathcal{C}ospan$ in a functorial fashion.

Lemma 6.28. *For a UFC with a presentation there is the following factorization of morphisms of type* (n,m).

$$Hom(\bigotimes_{s\in S} *_s, \bigotimes_{t\in T} *'_t) \simeq \bigoplus_k \bigotimes_{S\xrightarrow{l}\underline{k}\xleftarrow{r}T} \bigotimes_{1\leq i\leq k} \mathcal{P}(\bigotimes_{s\in l^{-1}(i)} *_s, \bigotimes_{t\in r^{-1}(i)} *'_t) \tag{6.22}$$

where the product is over all skeletal cospans, we used the notation $\mathcal{P}(X,Y)$ *to be the morphisms in* $\mathcal{P}$ *from* X *to* Y *by using the compatibility.*

Remark 6.29. Note that if the index set is empty then the product is equal to $\mathbb{1}$. The morphisms $R = \mathcal{P}(\mathbb{1},\mathbb{1})$ play the role of a ground monoid. Thus in application either on uses the condition that $\mathcal{P}$ is reduced, viz. $\mathcal{P}(\mathbb{1},\mathbb{1}) = \mathbb{1}$, or one extends coefficients to R. Another way to handle these factors is to use a power series variable q to count them. This is common in quantum deformation situations.

Proof. Straightforward using a skeletal version of the indexing of the tensor products. □

Remark 6.30. This generalizes the formulas for a Feynman category from maps of finite sets to correspondences as for Feynman categories $k = |T|$ in which case (6.22) specializes to

$$Hom(\bigotimes_{s\in S} *_s, \bigotimes_{t\in T} *'_t) \simeq \bigotimes_{f:S\twoheadrightarrow T} \bigotimes_{t\in T} Hom(\bigotimes_{s\in f^{-1}(t)} *_s, *'_t) \tag{6.23}$$

cf. [KW17] and the reformulation [BKW18]. The reason being that the connected components are of type $(n_j, 1)$ where in particular $n_j = |f^{-1}(j)|$.

More generally, a set indexed presentation defines morphism of Feynman categories $\mathfrak{index}_{\tilde{\imath}} : \mathfrak{F} \to \mathfrak{Fin}\mathfrak{S}et$: On $\mathcal{V}$, this is the only possible functor $v : \mathcal{V} \to \underline{*}$. On objects of $\mathcal{F}$ the functor is $f(X) = \mathrm{idx}(\tilde{\imath}(X))$, where idx is given in (A.5). On the morphisms of f the functor is defined by the formula (6.23), given by applying $\tilde{j}$ to the morphisms and picking the factor corresponding to $\mathcal{F}$ in which it lies.

Note that even in the skeletal case, $S = \underline{n}$ and $T = \underline{m}$ the fibers are unordered and the monoidal products are most naturally indexed by sets.

Proposition 6.31. *If* $\mathcal{M}$ *has essentially uniquely factorizable objects and is hereditary then a presentation defines a functor* $\mathrm{idx} : \mathcal{M} \to \mathcal{C}ospan$ *which extends to a functor hereditary UFCs* $\mathfrak{index} : \mathfrak{M} \to \mathfrak{C}ospan$.

Proof. Given $\mathcal{M}$ fix a compatible basis $\tilde{\imath}, \tilde{\jmath}$. The functor $f : \mathcal{M} \to \mathcal{C}ospan$ is given by $\mathrm{idx}(X) = \mathrm{idx}(\tilde{\imath}(X))$ on objects and by $f(\phi) = [(\pi_{\mathrm{idx}(\tilde{\imath}(s(\phi)))}, \pi_{\mathrm{idx}(\tilde{\imath}(t(\phi)))})]$ on morphisms coming from applying the index functor to given by $\tilde{\jmath}(\phi) = \bigotimes_{v\in V} \phi_v$. This is clearly strong monoidal and $f(\mathbb{1}_{\mathcal{M}}) = \mathbb{1}_{\mathcal{C}ospan}$. For the units $f(id_{\imath(\bigotimes_{s\in S}(*_s))}) = (S \overset{id}{\to} S \overset{id}{\leftarrow} S) = id_S \in \mathcal{C}ospan(S,S)$. Thus f is a strong monoidal functor, if the equation $f(\phi_0 \circ \phi_1) = f(\phi_0) \circ f(\phi_1)$ holds. This is the case as the connected components are indexed by the composition of the cospans and the vertex U is the index set of the irreducibles of the decomposition. The functor f automatically extends to an indexing of $\mathfrak{M}$ by $\mathfrak{C}ospan$ as follows: The restriction to $\mathcal{P}$ takes values in $\mathcal{C}td$ as $|V| = 1$ for irreducibles. On $\mathcal{V}$ this restricts to $v : \mathcal{V} \to \underline{*}$ the trivial functor. □

By abuse of notation, the functor f will be called idx as well.

Definition 6.32. If $\mathcal{M}$ is not hereditary, then the connected components of a composable pair (ϕ_0, ϕ_1) give a new cospan on the level of indexing of morphisms. Let $V_0 = \mathrm{idx}(\phi_0), V_1 = \mathrm{idx}(\phi_1)$, $W = \mathrm{idx}(\phi_1 \circ \phi_0)$ and U be the the push-out defining the connected components. Then this defines a co-span *the index of* (ϕ_0, ϕ_1) *denoted by* $\mathrm{idx}(\phi_0, \phi_1)$

$$\begin{tikzcd} & U & \\ V \arrow[ur] & & W \arrow[ul] \\ & S \arrow[ul] \arrow[ur] & \end{tikzcd} \tag{6.24}$$

We define *depth*$(\phi_0, \phi_1) := |U|$.

Similarly, given composable morphisms $(\phi_0, \dots, \phi_{n-1})$: $X_0 \xrightarrow{\phi_0} X_1 \xrightarrow{\phi_1} X_2 \to \cdots \to X_{n-1} \xrightarrow{\phi_{n-1}} X_n$, fixing bases $X_i \simeq \bigotimes_{s \in S_i} *_s$ and $\phi_i \simeq \bigotimes_{v \in V_i} \phi_v$, then using indexing $\mathfrak{index}$ we get the first two rows of a push–out diagram

$$\begin{tikzcd} & V_0 & & V_1 & \cdots & V_{n-1} & \\ S_0 \arrow[ur] & & S_1 \arrow[ul] \arrow[ur] & & S_2 \arrow[ul] \arrow[ur, dashed] & \cdots & S_n \arrow[ul] \end{tikzcd} \tag{6.25}$$

We will call the isomorphism class of the full diagram $\mathrm{idx}(\phi_0, \dots, \phi_n)$. Let U be the full–pushout, that is the combination of all the equivalence relations, then by iterating the construction above, we obtain composable morphisms $(\phi_{u,0}, \dots, \phi_{u,n-1})$ with $\phi_i \simeq \bigotimes_{u \in U} \phi_{u,i}$ and if $\phi = \phi_{n-1} \circ \cdots \circ \phi_0$ and $\phi_u = \phi_{n-1,u} \circ \cdots \circ \phi_{u,0}$, then $\phi_i \simeq \bigotimes_{u \in U} \phi_{u,i}$.

We call the ϕ_u the *connected components of the composition* and $(\phi_{u,0}, \dots, \phi_{u,n-1})$ the *connected components of the composable functions* $(\phi_0, \dots, \phi_n)$. A sequence $(\phi_0, \dots, \phi_n)$ is called *connected*, if it only has only one connected component. This is equivalent to the statement that $\mathrm{idx}(\phi_0 \circ \cdots \circ \phi_n)$ is connected, that is $|U| = 1$. More generally we define *depth*$(\phi_0, \dots, \phi_n) = |U|$. Note that all the morphisms are uniquely fixed by choosing bases. Different choices of bases, however, only change these morphisms by isomorphisms, so that the condition that the connected components are irreducible is independent of basis.

Proposition 6.33. *A UFC is hereditary if and only if the connected components ϕ_u of any set of composable morphisms $(\phi_0, \dots, \phi_{n-1})$ are irreducible.*

Proof. If the connected components are irreducible for any composable sequence, then they are for pairs. The reverse direction can be done by induction. For two morphisms this follows from the same proposition. If the statement is true for composable morphisms $(\phi_0, \dots, \phi_m)$, $2 \le m \le n$, consider a composable sequence $(\phi_0, \dots, \phi_n)$, this gives rise to the pair $(\phi_0, \phi_1' = \phi_n \circ \cdots \circ \phi_1)$. Now the connected components $\phi'_{1,u'}$ of ϕ'_1, with $u' \in U'$ which is the pushout of the $S_i, V_i, i = 1, \dots, n$ are irreducible and are thus part of a diagram (6.11) which in turn shows that the connected components corresponding to the final push–out U, that is the ϕ_u are irreducible. □

By a similar argument:

Lemma 6.34. *A sequence $(\phi_0, \dots, \phi_n)$ is connected if and only if all $0 \le i_1 \le \dots, \le i_k \le n$: $(\phi_0 \circ \cdots \circ \phi_{i_1}, \phi_{i_1+1} \circ \cdots \circ \phi_{i_2}, \cdots, \phi_{i_k+1} \circ \cdots \circ \phi_n)$ are connected.* □

Remark 6.35. Alternatively, this argument can be done by using the functor $\mathfrak{index}$. In $\mathcal{C}ospan$, the corresponding push–out diagram must have at least one one–point set on the level 2 and above. If there is such an object, it propagates, since it is not possible to have the push–out $\{*\} \leftarrow \emptyset \rightarrow \{*\}$ after the first level. The only co-span whose middle set is the empty set is the unit. (This is analogous to the fact that S^0 is not connected, but all higher spheres are.) This means that when composing sequences, composing with something connected makes the result connected. It is possible that composing two non–connected sequences, the result is connected. This mirrors the connectivity of cobordisms, see Remark 6.24 (3).

6.6. **Some structural results.** In the case of a UFC, there is more structure which allows us to reduce the number of generators, and hence data for the plus constructions. The index is a tool for these considerations. We will choose a presentation. The results are easily checked to be independent of this choice.

6.6.1. **Connected Generators.**

Lemma 6.36. *For a UFC $\mathfrak{M}$ the γ_{ϕ_1,ϕ_0}, with $\phi_0, \phi_1 \in \jmath^{\otimes}(\mathcal{P}^{\otimes})$ together with the μ_{ϕ_1,ϕ_2} where $\phi_1, \phi_2 \in \jmath^{\otimes}(\mathcal{P}^{\otimes})$ and the isomorphisms $(\sigma \overset{\phi}{\Downarrow} \sigma')$ with $\phi \in \jmath^{\otimes}(\mathcal{P}^{\otimes})$ form a set of essential generators under concatenation and monoidal product of $\mathcal{M}^{nc+}$.*

Here essential means that they generate together with the isomorphisms given by $\jmath^{\otimes}$, so in particular these generate an equivalent subcategory. Thus, in the following, we will assume strictness without loss of generality.

Proof. We first show that the γ_{ϕ_0,ϕ_1} $\phi_0, \phi_1 \in \jmath^{\otimes}(\mathcal{P}^{\otimes})$ generate together with the isomorphisms given by $\jmath^{\otimes}$. Given a composable pair (ϕ_0, ϕ_1) there are isomorphisms $(\sigma \Downarrow \sigma')$ and $(\tau \Downarrow \tau')$, such that $(\sigma \Downarrow \sigma')(\phi_0) = \phi_0'' \in \mathcal{P}^{\otimes}$ and $(\tau \Downarrow \tau')(\phi_1) = \phi_1' \in \mathcal{P}$. These fit into a diagram

$$\begin{tikzcd}
X_0 \arrow[rr, "\phi_0"] \arrow[d, "\sigma"'] & & X_1 \arrow[rr, "\phi_1"] \arrow[dl, "\sigma'"'] \arrow[dr, "\tau"] & & X_2 \arrow[d, "\tau'"] \\
X_0' \arrow[r, "\phi_0'"'] \arrow[rrr, dashed, bend right, "\phi_0''"'] & X_1'' \arrow[rr, "\tau(\sigma')^{-1}"] \arrow[rrr, bend right, "\phi_1'"'] & & X_1' \arrow[r, "\phi_1''"'] & X_2'
\end{tikzcd} \tag{6.26}$$

Setting $\phi_1' = (id \Downarrow \tau \circ (\sigma')^{-1})(\phi'') \in \mathcal{P}$, we have $\phi_1' = \phi_1''\tau(\sigma'^{-1}) = \tau'\phi_1\tau^{-1}\tau\sigma'^{-1} = (\sigma' \Downarrow \tau')(\phi_1)$, we find that

$$\begin{aligned}
\gamma_{\phi_1,\phi_0} &= (\sigma \Downarrow \tau')^{-1} \circ \gamma_{\tau'\circ\phi_1,\phi_0\circ\sigma^{-1}} \circ ((id \Downarrow \tau') \otimes (\sigma \otimes id)) \\
&= (\sigma \Downarrow \tau')^{-1} \circ \gamma_{\tau'\circ\phi_1\circ(\sigma')^{-1},\sigma'\circ\phi_0\circ s^{-1}} \circ (\sigma' \Downarrow id) \otimes (id \Downarrow \sigma') \circ ((id \Downarrow \tau') \otimes (\sigma \otimes id)) \\
&= (\sigma \Downarrow \tau')^{-1} \circ \gamma_{\phi_1',\phi_0'} \circ ((\sigma' \Downarrow \tau') \otimes (\sigma \Downarrow \sigma'))
\end{aligned} \tag{6.27}$$

The essential reductions for the other morphisms is analogous. □

Lemma 6.37. *For a UFC $\mathfrak{M}$ for the essential generators of $\mathcal{M}^{nc+}$*

(1) *The μ_{ϕ_1,ϕ_2} with $\phi_1, \phi_1 \in \mathcal{P}$ suffice.*
(2) *The γ_{ϕ_0,ϕ_1} with* connected *pairs $(\phi_0, \phi_1) \in \mathcal{P}^{\times 2}$ suffice. Note, if the underlying $\mathcal{M}$ is hereditary then these are the γ's whose output is in $\mathcal{P}$.*

(3) *The isomorphisms in* $\mathrm{Iso}(\mathcal{M}\downarrow\mathcal{M})$ *are generated by the* $(\sigma\overset{\phi}{\Downarrow}\sigma')$ *with* $\phi\in\mathcal{P}$ *and permutations. That is are given by are of the form* $(\sigma\overset{\phi}{\Downarrow}\sigma') = (P\Downarrow P')((\sigma_1\overset{\phi_1}{\Downarrow}\sigma'_1)\otimes\cdots\otimes(\sigma_n\overset{\phi_n}{\Downarrow}\sigma'_n))$ *where each* $\sigma_i=\sigma_i^1\otimes\cdots\otimes\sigma_i^{n_i}, \sigma'_i=\sigma_i'^1\otimes\cdots\otimes s_i'^{m_i}$.

Proof. The statement about the reduction of the generators μ_{ϕ_1,ϕ_2} to those with irreducible ϕ_1,ϕ_2 follows from the condition (3.8).

The reduction of the generators γ_{ϕ_1,ϕ_2} follows from Proposition 6.17.

The reduction of the isomorphisms follows from the factorizability of isomorphisms in a UFC. First we can factor into $(P\Downarrow P)((\sigma_1\overset{\phi_1}{\Downarrow}\sigma'_1)\otimes\cdots\otimes(\sigma_n\overset{\phi_n}{\Downarrow}\sigma'_n))$ using (6.4) and (3.19). Then applying (3.19) to the σ_i,σ'_i and pulling out the permutations yields the result. □

Proof. For the first statement, by Remark 7.15, we already have that the $\gamma_{\phi_n,\dots,\phi_0}$ are generators. Using $\mathrm{idx}(\phi_0,\dots,\phi_n)$ we see that the connected $\gamma_{\phi_n,\dots,\phi_0}$ are precisely the irreducibles. □

6.6.2. **Consolidated generators in UFCs.** Consider a hereditary UFC $\mathfrak{M}$. Set

$$\begin{aligned}\mathrm{idx}\left(\overset{\sigma_0}{-}\phi_1\overset{\sigma_1}{-}\phi_2\overset{\sigma_2}{-}\cdots\overset{\sigma_{n-1}}{-}\phi_n\overset{\sigma_n}{-}\right) &= \mathrm{idx}(\sigma_0\phi_0,\sigma_1\phi_1,\dots,\sigma_{n-1}\phi_{n-1}\sigma_n)\\ \mathit{depth}\left(\overset{\sigma_0}{-}\phi_1\overset{\sigma_1}{-}\phi_2\overset{\sigma_2}{-}\cdots\overset{\sigma_{n-1}}{-}\phi_n\overset{\sigma_n}{-}\right) &= \mathit{depth}(\sigma_0\phi_0,\sigma_1\phi_1,\dots,\sigma_{n-1}\phi_{n-1}\sigma_n)\end{aligned}$$

The morphism $\overset{\sigma_0}{-}\phi_1\overset{\sigma_1}{-}\phi_2\overset{\sigma_2}{-}\cdots\overset{\sigma_{n-1}}{-}\phi_n\overset{\sigma_n}{-})$ is called if *connected* if $\mathit{depth}(\overset{\sigma_0}{-}\phi_1\overset{\sigma_1}{-}\phi_2\overset{\sigma_2}{-}\cdots\overset{\sigma_{n-1}}{-}\phi_n\overset{\sigma_n}{-})=1$.

Remark 6.38. Note that due to the definition of push-outs, up to isomorphisms, we are free to "rebracket" the isomorphisms in the arguments, that is $(\phi_0\circ\sigma_0,\dots)$ or $(\dots,\phi_i\circ\sigma_{i-1},\dots)$.

Proposition 6.39. *If* $\mathfrak{M}$ *is a hereditary UFC then the target of* $\overset{\sigma_0}{-}\phi_1\overset{\sigma_1}{-}\phi_2\overset{\sigma_2}{-}\cdots\overset{\sigma_{n-1}}{-}\phi_n\overset{\sigma_n}{-}$ *is irreducible if and only if it is connected.*

Proof. If $\gamma=\overset{\sigma_0}{-}\phi_1\overset{\sigma_1}{-}\phi_2\overset{\sigma_2}{-}\cdots\overset{\sigma_{n-1}}{-}\phi_n\overset{\sigma_n}{-}$ is connected, assume that it is the product of at least two generators, then by (3.21) the morphisms can be iteratively split to $\mu\circ(\gamma\simeq\gamma'\boxtimes\gamma'')$, then as *depth* is additive under μ, $1=\mathit{depth}(\gamma'\otimes\gamma'')=\mathit{depth}(\gamma')+\mathit{depth}(\gamma'')=\mathit{depth}(\gamma)$ and thus either $\mathit{depth}(\gamma')=0$ or $\mathit{depth}(\gamma'')=0$. Let's assume $\mathit{depth}(\gamma')=0$, then this means that the whole diagram of push–outs (6.25) only has entries $\emptyset$, which in turn means that $\gamma'\simeq id_{\mathbb{1}}$; and, hence γ is irreducible. Note as depth is independent under isomorphisms this also holds for any essential decomposition.

Vice–versa, assume that $\overset{\sigma_0}{-}\phi_1\overset{\sigma_1}{-}\phi_2\overset{\sigma_2}{-}\cdots\overset{\sigma_{n-1}}{-}\phi_n\overset{\sigma_n}{-}$ is not connected, then $|U|\geq 2$ and tracing back the pre–images in $\mathrm{idx}(\overset{\sigma_0}{-}\phi_1\overset{\sigma_1}{-}\phi_2\overset{\sigma_2}{-}\cdots\overset{\sigma_{n-1}}{-}\phi_n\overset{\sigma_n}{-})$ one obtains a monoidal decomposition into connected components with more than one generator and hence it is reducible. □

Remark 6.40. If $\mathcal{M}$ is just a UFC *depth* is defined and irreducible implies connected, but not vice–versa.

Corollary 6.41. *For a hereditary UFC any composite generator can be decomposed as*

$$\mu^k\circ[\overset{\sigma_0^1}{-}\phi_1^1\overset{\sigma_1^1}{-}\phi_2^1\overset{\sigma_2^1}{-}\cdots\overset{\sigma_{n_1-1}^1}{-}\phi_{n_1}^1\overset{\sigma_{n_1}^1}{-}\boxtimes\cdots\boxtimes\overset{\sigma_0^k}{-}\phi_1^k\overset{\sigma_1^k}{-}\phi_2^k\overset{\sigma_2^k}{-}\cdots\overset{\sigma_{n_k-1}^k}{-}\phi_{n_k}^k\overset{\sigma_{n_k}^k}{-}]\circ P \tag{6.28}$$

where now the $\overset{\sigma_0^j}{-}\ \phi_1^j \overset{\sigma_1^j}{-}\ \phi_2^j \overset{\sigma_2^j}{-} \cdots \overset{\sigma_{n_1-1}^j}{-}\ \phi_{n_1}^j \overset{\sigma_{n_1}^j}{-}$ *are connected and* P *is the permutation rearranging the factors.*

6.6.3. **Degrees.** The isomorphisms $(\sigma \Downarrow \sigma')$ preserve the single degree and the type of a morphism. The maps γ_{ϕ_1,ϕ_0} preserve the single degree but are non–decreasing in both entries of the bidegree. $||(\phi_1 \otimes \phi_0)|| = (|s(\phi_1)|+|s(\phi_0)|, |t(\phi_1)|+|t(\phi_0)|) \geq (|s(\phi_1 \circ \phi_0)|, |t(\phi_1 \circ \phi_0)|) = (|s(\phi_0)|, |t(\phi_1)|)$. Equality means that $|s(\phi_1)| = |t(\phi_0)| = 0$ which means that $s(\phi_1) = t(\phi_0) = \mathbb{1}$.

6.7. **UFCs, FCs and the plus construction - Results.** The plus construction is a source for Feynman categories. This is proven in the next section through a detailed analysis which yields standard forms for morphisms.

Theorem 6.42.

(1) $\mathcal{C}^{\boxtimes +}$ *is a cubical Feynman category with* $\mathcal{V}^+ = \mathrm{Mor}(\mathcal{C})$ *and one type of degree* 1 *generators.*
(2) $\mathcal{M}^{nc+}$ *is a cubical Feynman category with* $\mathcal{V}^+ = \mathcal{P}^{\boxtimes} \simeq \mathrm{Mor}(\mathcal{M})$ *and two types of degree* 1 *generators.*
(3) *For a hereditary UFC* $\mathcal{M}^+$ *is a cubical Feynman category with* $\mathcal{V}^+ = \mathcal{P}$ *and one type of degree* 1 *generators.*

The hyp versions are also cubical Feynman categories. The gcp versions are Feynman categories with additional generators of degree -1 *corresponding to the units.*

The generators are the γ_{ϕ_1,ϕ_0}, γ_{ϕ_1,ϕ_0} and μ_{ϕ_1,ϕ_2}, and $\bar{\gamma}_{\phi_1,\phi_0}$ respectively.

Proof. The first statement is contained in Propositions 7.18 and 7.19. The second statement is Proposition 7.56, and the final statement is in Proposition 7.67. □

Theorem 6.43.

(1) *In* $\mathcal{M}^{nc+,gcp}$ *the basic morphisms are*

$$\mu^k \circ (\overset{\sigma_0^1}{-}\ \phi_1^1 \overset{\sigma_1^1}{-}\ \phi_2^1 \overset{\sigma_2^1}{-} \cdots \overset{\sigma_{n_1-1}^1}{-}\ \phi_{n_1}^1 \overset{\sigma_{n_1}^1}{-} \boxtimes \cdots \boxtimes \overset{\sigma_0^k}{-}\ \phi_1^k \overset{\sigma_1^k}{-}\ \phi_2^k \overset{\sigma_2^k}{-} \cdots \overset{\sigma_{n_k-1}^k}{-}\ \phi_{n_k}^k \overset{\sigma_{n_k}^k}{-})$$

with connected $\overset{\sigma_0^j}{-}\ \phi_1^j \overset{\sigma_1^j}{-}\ \phi_2^j \overset{\sigma_2^j}{-} \cdots \overset{\sigma_{n-1}^j}{-}\ \phi_n^j \overset{\sigma_n^j}{-}$.

(2) *For a hereditary UFC, the basic morphisms of the category* $\mathcal{M}^{+,gcp}$ *are the connected morphisms* $\overset{\sigma_0}{-}\ \phi_1 \overset{\sigma_1}{-}\ \phi_2 \overset{\sigma_2}{-} \cdots \overset{\sigma_{n-1}}{-}\ \phi_n \overset{\sigma_n}{-}$.

Proof. See Propositions 7.66 and 7.67. □

7. Decompositions and standard forms for morphisms in plus constructions

The plus catogories $\mathcal{P}l^+$, true to the general philosophy, are generated by isomorphisms and the new generators. We now formalize this.

7.1. **Monoidal categories generated by subcategories.** In the general situation of two monoidally generating subcategories there are several structural results. For instance, one can bring any morphism into a standard form as a word alternating in morphisms from the two subcategories. Reducing this from further leads to a set of free (symmetric) monoidal generators in the case of a hereditary UFC. At each step there is a presentation which is of independent interest.

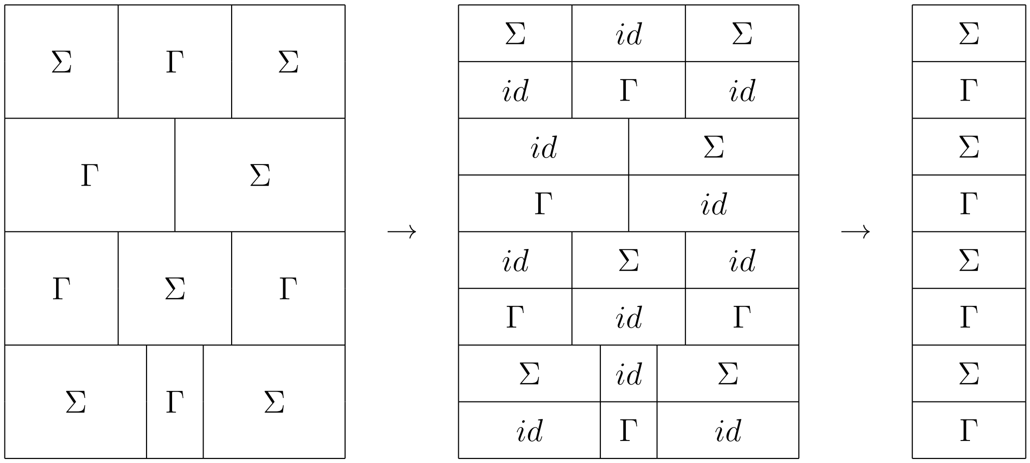


FIGURE 4. Schematic of compositions. First tensor along rows then compose the rows in the column direction. Note the target of a row being the source of a column allows for different tensor decompositions, which in general can be of different length. The interstices are to mark the relative "cuts" of these tensors. In a hereditary UFC the tensor lengths agree —per unique decomposition.

7.1.1. **Standard form for two (crossed) generating subcategories.**

Proposition 7.1. *If $\mathcal{C}$ is generated by two monoidal subcategories $I(\mathcal{C})$ and $\mathrm{Plr}(\mathcal{C})$, then any morphism in $\mathcal{C}$ can be written as $\Gamma_n \circ \Sigma_n \circ \cdots \circ \Gamma_1 \circ \Sigma_1$ with the $\Sigma_i \in I(\mathcal{C})$ and the $\Gamma_i \in \mathrm{Plr}(\mathcal{C})$. In particular, if $\mathcal{C}$ is* $(\mathrm{Plr}(\mathcal{C}), \mathrm{I}(\mathcal{C}))$-crossed*, then any morphism is of the form $\Gamma\Sigma$.*

Proof. Any morphisms in $\mathcal{C}$ can be written as a concatenation of tensor products by using the interchange equation (A.1) after equating the tensor lengths by adding identities if necessary, viz. using $(f \circ g) \otimes h = (f \circ g) \otimes (h \circ id) = (f \otimes h) \circ (g \otimes id)$ and its symmetric partner iteratively.

A convenient way to encode this is in a type of "brick wall" picture. Where the rows correspond to tensoring and the columns to concatenation, see Figure 4. Note the horizontal seams go through, by the preparation step above, while the vertical ones can be interrupted. The entries Σ are for elements of $\mathrm{Iso}(\mathcal{M}^+)$ and Γ for an element in $\mathrm{Plr}(\mathcal{M}^+)$.

Due to the interchange relation,

$$\begin{aligned} \Gamma \otimes \Sigma &= (\Gamma \otimes id) \circ (id \otimes \Sigma) \\ \Sigma \otimes \Gamma &= (id \otimes \Gamma) \circ (\Sigma \otimes id) \end{aligned} \tag{7.1}$$

one can "pull apart" the rows into only Γ or Σ by "pulling up" the factors of Σ, starting with a row of Γ tensoring together the rows gives the desired form, see Figure 4.

The expression for the Γ_i follows from the fact that the γ_{ϕ_0,ϕ_1} generate (by definition $\gamma_{\phi_0} = id_{\phi_0}$) and Remark 7.15. The expression for the Σ_i follows from the fact that $\mathrm{Iso}(\mathcal{M}^+) = \mathrm{Iso}(\mathcal{M}\downarrow\mathcal{M})$ is a (symmetric) monoidal subcategory. $\square$

Remark 7.2.

(1) These standard forms are not unique, as there are relations. For instance, there can be identity rows.

(2) The monoidal structure in terms of the brick wall structure on these generators is given by placing the brick walls next to each other. If they do not have the same height, then one inserts rows of identities to make them have equal height. This creates a vertical seam. Vice-versa, clearly any such diagram with a vertical seam is a product of the two diagrams to the left and right of the seam.

(3) We will use the convenient and suggestive notation
$$\Gamma_n \overset{\Sigma_n}{-} \Gamma_{n-1} \overset{\Sigma_{n-1}}{-} \cdots \overset{\Sigma_2}{-} \Gamma_1 \overset{\Sigma_1}{-} := \Gamma_n \circ \Sigma_n \circ \cdots \circ \Gamma_1 \circ \Sigma_1 \tag{7.2}$$
For later purposes it is convenient to think of this as a column as in Figure 4.

(4) The composition of the generators is given by concatenating the diagrams or columns.
$$\begin{aligned}&\left(\Gamma_n \overset{\Sigma_n}{-} \Gamma_{n-1} \overset{\Sigma_{n-1}}{-} \cdots \overset{\Sigma_2}{-} \Gamma_1 \overset{\Sigma_1}{-}\right) \circ \left(\Gamma'_m \overset{\Sigma'_m}{-} \Gamma'_{m-1} \overset{\Sigma'_{m-1}}{-} \cdots \overset{\Sigma'_2}{-} \Gamma'_1 \overset{\Sigma'_1}{-}\right)\\ &= \Gamma_n \overset{\Sigma_n}{-} \Gamma_{n-1} \overset{\Sigma_{n-1}}{-} \cdots \overset{\Sigma_2}{-} \Gamma_1 \overset{\Sigma_1}{-} \Gamma'_m \overset{\Sigma'_m}{-} \Gamma'_{m-1} \overset{\Sigma'_{m-1}}{-} \cdots \overset{\Sigma'_2}{-} \Gamma'_1 \overset{\Sigma'_1}{-}\end{aligned} \tag{7.3}$$
Note some of the Γ_i or Σ_i can be identities and a standard form for an identity is $id \overset{id}{-}$.

(5) The monoidal structure on these generators is given by placing the brick wall diagrams/consolidated columns next to each other, and including them into a bigger diagram, where one adds extra identities if the diagrams are not the same length. Note that there is not a unique representative for this. In particular: if the two generators have the same length, the product is given by
$$\begin{aligned}&\left(\Gamma_n \overset{\Sigma_n}{-} \Gamma_{n-1} \overset{\Sigma_{n-1}}{-} \cdots \overset{\Sigma_2}{-} \Gamma_1 \overset{\Sigma_1}{-}\right) \otimes \left(\Gamma'_n \overset{\Sigma'_n}{-} \Gamma'_{n-1} \overset{\Sigma'_{n-1}}{-} \cdots \overset{\Sigma'_2}{-} \Gamma'_1 \overset{\Sigma'_1}{-}\right)\\ =\ & \Gamma_n \otimes \Gamma'_n \overset{\Sigma_n \otimes \Sigma'_n}{-} \Gamma_{n-1} \otimes \Gamma'_{n-1} \overset{\Sigma_{n-1} \otimes \Sigma'_{n-1}}{-} \cdots \overset{\Sigma_2 \otimes \Sigma'_2}{-} \Gamma_1 \otimes \Gamma'_1 \overset{\Sigma_1 \otimes \Sigma'_1}{-}\\ =\ & \Gamma''_n \overset{\Sigma''_n}{-} \Gamma''_{n-1} \overset{\Sigma''_{n-1}}{-} \cdots \overset{\Sigma''_2}{-} \Gamma''_1 \overset{\Sigma''_1}{-}\end{aligned} \tag{7.4}$$
On the other hand, given generators $\Gamma_n \overset{\Sigma_n}{-} \Gamma_{n-1} \overset{\Sigma_{n-1}}{-} \cdots \overset{\Sigma_2}{-} \Gamma_1 \overset{\Sigma_1}{-}$ and $\Gamma'_m \overset{\Sigma'_m}{-} \Gamma'_{m-1} \overset{\Sigma'_{m-1}}{-} \cdots \overset{\Sigma'_2}{-} \Gamma'_1 \overset{\Sigma'_1}{-}$ with say $n > m$ formally add $n-m$ identity rows/entries $\Gamma_m \overset{\Sigma_m}{-} \Gamma_{m-1} \overset{\Sigma_{m-1}}{-} \cdots \overset{\Sigma_2}{-} \Gamma_1 \overset{\Sigma_1}{-} id \cdots \overset{id}{-} id$ to get length n and use the formula above. Note that there is an ambiguity where to add the identities as all these formal expressions actually represent the same morphism. The well-definedness is guaranteed by the interchange equation.

The following are immediate.

Lemma 7.3. *A morphisms* $\Gamma_n \overset{\Sigma_n}{-} \Gamma_{n-1} \overset{\Sigma_{n-1}}{-} \cdots \overset{\Sigma_2}{-} \Gamma_1 \overset{\Sigma_1}{-} := \Gamma_n \circ \Sigma_n \circ \cdots \circ \Gamma_1 \circ \Sigma_1$ *is* $\boxtimes$ *reducible if and only if it has a representative brick–wall representative with a seam.* □

Corollary 7.4. *Any morphisms in* $\mathcal{P}l^+$ *has a decomposition* $\Gamma_n \overset{\Sigma_n}{-} \Gamma_{n-1} \overset{\Sigma_{n-1}}{-} \cdots \overset{\Sigma_2}{-} \Gamma_1 \overset{\Sigma_1}{-} := \Gamma_n \circ \Sigma_n \circ \cdots \circ \Gamma_1 \circ \Sigma_1$ *with* $\Gamma_i \in \mathrm{Plr}(\mathcal{P}l^+)$ *and* $\Sigma \in \mathrm{Iso}(\mathcal{P}l^+)$*.*

Such a morphism is $\boxtimes$ *reducible if and only if it has a representative brick–wall representative with a seam.* □

Definition 7.5. Given two monoidal subcategories $I(\mathcal{C})$ and $\mathrm{Plr}(\mathcal{C})$, which generate $\mathcal{C}$, we call $\mathcal{C}$ $(\mathrm{Plr}(\mathcal{C}), \mathrm{I}(\mathcal{C}))$-*crossed* if for composable pair (Σ, Γ), $\Sigma \in I(\mathcal{C})$ and $\Gamma \in \mathrm{Plr}(\mathcal{C})$ there are $\Sigma' \in I(\mathcal{C})$ and $\Gamma' \in \mathrm{Plr}(\mathcal{C})$ such that $\Sigma\Gamma = \Gamma'\Sigma'$.

$$\begin{tikzcd} \Phi' \arrow[r, "\Gamma"] \arrow[d, dashed, "\Sigma'"] & \Psi \arrow[d, "\Sigma"] \\ \Phi \arrow[r, dashed, "\Gamma'"] & \Psi' \end{tikzcd} \tag{7.5}$$

From Proposition 7.1:

Corollary 7.6. *If $\mathcal{C}$ is* $(\mathrm{Plr}(\mathcal{C}), \mathrm{I}(\mathcal{C}))$-crossed*, then any morphism can be written as $\Gamma\Sigma$ with $\Gamma \in \mathrm{Plr}(\mathcal{C})$ and $\Sigma \in I(\mathcal{C})$.* □

Both $\mathrm{Plr}(\mathcal{C}^{\boxtimes+})$ and $\mathrm{Iso}(\mathcal{C}^{nc+})$ naturally embed into $\mathrm{Iso}(\mathcal{C}^{nc+})$. These generate the category in a specific fashion if certain conditions are met. The following definition is inspired by [FL91], see also [KW17, §5.2].

Definition 7.7.

(1) The subcategory $\mathrm{Plr}(\mathcal{C}^{\boxtimes+}) \subset \mathcal{C}^{\boxtimes+}$ is the wide monoidal subcategory which only has the morphisms γ.
(2) The subcategory $\mathrm{Plr}(\mathcal{M}^{\mathrm{nc}+}) \subset \mathcal{M}^{\mathrm{nc}+}$ is the wide monoidal subcategory which only has the morphisms γ and μ.
(3) The subcategory $\mathrm{Plr}(\mathcal{M}^{\mathrm{loc}+}) \subset \mathcal{M}^{\mathrm{loc}+}$ is the wide monoidal subcategory which contains the morphisms γ, μ and μ^{-1}. We furthermore let $\overline{\mathrm{Plr}}(\mathcal{M}^{loc,+})$ be the subcategory which is the image of $\mathrm{Plr}(\mathcal{M}{+}^{\mathrm{nc}+})$. We let $\mathcal{L}$ be the wide category generated by the μ^{-1}.
(4) Finally, $\mathrm{Plr}(\mathcal{M}^{+}) \subset \mathcal{M}^{+}$ is the wide monoidal subcategory which only has the morphisms $\bar{\gamma}$.

With the exception of $\mathcal{M}^{loc+}$, set $\mathcal{I}(Pl^{+}) = \mathrm{Iso}(\mathcal{P}l^{+})$. For $\mathcal{M}^{loc+}$, let $\mathcal{I}$ be the image of $\mathrm{Iso}(\mathcal{M}^{nc+})$, viz. it does not contain the morphisms μ or μ^{-1}.

Remark 7.8. It is clear that the monoidal subcategories $\mathrm{Plr}(\mathcal{P}\mathrm{l}^{+})$ and $\mathcal{I}(\mathcal{P}l^{+})$ generate. Note that $\mathrm{Plr}(\mathcal{P}\mathrm{l}^{+})$ is not symmetric monoidal. In the symmetric monoidal category all the commutators are in $\mathcal{I}(\mathcal{P}l^{+})$.

Proposition 7.9.

(1) *The category $\mathcal{C}^{\boxtimes+}$ is $(\mathrm{Plr}(\mathcal{C}^{\boxtimes+}), \mathrm{Iso}(\mathcal{C}^{\boxtimes+}))$–crossed.*
(2) *If a monoidal category $\mathcal{M}$ has factorizable isomorphisms then*
   (a) *$\mathcal{M}^{nc+}$ is $(\mathrm{Plr}(\mathcal{M}^{\mathrm{nc}+}), \mathrm{Iso}(\mathcal{M}^{\mathrm{nc}+}))$–crossed,*
   (b) *$\mathcal{M}^{loc+}$ is $(\mathrm{Plr}(\mathcal{M}^{\mathrm{loc}+}_{\mathrm{red}}), \mathcal{I}(\mathcal{M}^{\mathrm{loc}+}))$–crossed, and*
   (c) *$\mathcal{M}^{+}$ is $(\mathrm{Plr}(\mathcal{M}^{+}), \mathcal{I}(\mathcal{M}^{+}))$–crossed.*

Proof. The statement (1) follows from (3.4) and the (2) (a) subsequently from (3.19). For $\mathcal{M}^{+}$ note that the equation (3.7) crosses the other way– $\Sigma M = M'\Sigma'$ which upon inversion crosses the correct way $M^{-1}\Sigma = \Sigma'(M^{-1})'$, proving (b). This descends to $\mathcal{M}^{+}$, whence (c). □

Corollary 7.10.

(1) *Any morphism in $\mathcal{C}^{\boxtimes+}$ can be written as $\Gamma\Sigma$ with $\Gamma \in \mathrm{Plr}(\mathcal{C}^{\boxtimes+}), \Sigma \in \mathrm{Iso}(\mathcal{C}^{\boxtimes+})$.*

(2) *If $\mathcal{M}$ has factorizable isomorphisms then*
   (a) *Any morphism in $\mathcal{M}^{nc+}$ can be written as $\Gamma\Sigma$ with $\Gamma \in \mathrm{Plr}(\mathcal{M}^{\mathrm{nc}+}), \Sigma \in \mathrm{Iso}(\mathcal{M}^{\mathrm{nc}+})$.*
   (b) *Any morphism in $\mathcal{M}^{loc+}$ can be written as $\Gamma\Sigma$ with $\Gamma \in \mathrm{Plr}(\mathcal{M}^{\mathrm{loc}+}), \Sigma \in \mathcal{I}(\mathcal{M}^{\mathrm{loc}+})$. If in addition $\mathcal{M}$ is hereditary then any morphism can be written as $\Gamma\Sigma M^{-1}$ with $\Gamma \in \overline{\mathrm{Plr}}(\mathcal{M}^{loc+}), \Sigma \in \mathcal{I}(\mathcal{M}^{loc+})$ and $M \in \mathcal{L}(\mathcal{M}^{loc+})$.*
   (c) *Any morphism in $\mathcal{M}^{+}$ can be written as $\Gamma\Sigma$ with $\Gamma \in \mathrm{Plr}(\mathcal{M}^{+}_{\mathrm{red}}), \Sigma \in \mathrm{Iso}(\mathcal{M}^{+}_{\mathrm{red}}) = \mathrm{Iso}(\mathcal{M}\downarrow\mathcal{M})$.*

Proof. This follows from Corollary 7.6 with the addition of Proposition 7.9 in the hereditary case. □

7.1.2. **External and internal Isomorphism.** In the symmetric case, $\mathcal{I}(\mathcal{P}l^{+})$ is generated by the $(\sigma \Downarrow \sigma')$ and the commutators $\tau_{12}^{\boxtimes} : \phi_1 \boxtimes \phi_2 \xrightarrow{\sim} \phi_2 \boxtimes \phi_1$. In fact by definition they are the wreath product of the two. In particular, $\tau_{12}^{\boxtimes}[(\sigma_1 \Downarrow \sigma_1')\boxtimes(\sigma_2 \Downarrow \sigma_2')]\tau_{12}^{\boxtimes} = (\sigma_2 \Downarrow \sigma_2')\boxtimes(\sigma_1 \Downarrow \sigma_1')$. Define the following subcategories of $\mathrm{Iso}(\mathcal{M}^{nc+})$: Σ_{ext} is generated by the $\tau_{ii+1}^{\boxtimes}$ and Σ_{int} is generated by the $(\sigma \Downarrow \sigma')$.

Proposition 7.11. *$\mathcal{I}(\mathcal{P}l^{+})$ is both $(\Sigma_{ext}, \Sigma_{int})$-crossed and $(\Sigma_{int}, \Sigma_{ext})$–crossed. Hence, any isomorphism can be written as $P\boldsymbol{\sigma}$ or $\boldsymbol{\sigma}P$ where $P \in \Sigma_{ext}$ and $\boldsymbol{\sigma} \in \Sigma_{int}$. Letting $\mathcal{I}_{ext}$ and $\mathcal{I}_{int}$ be the corresponding subcategories in $\mathcal{I}(\mathcal{P}l^{+})$, this decomposition refines all the decomposition in Corollary 7.10.*

Proof. Immediate from Corollary 7.6 and the above. □

Definition 7.12. We define $\mathrm{Plr}(\mathcal{P}\mathrm{l}^{+})\Sigma_{int/ext}$ to be the subcategory generated by $\mathrm{Plr}(\mathcal{P}\mathrm{l}^{+})$ and $\mathcal{I}_{int}(\mathcal{P}l^{+})$ respectively $\mathcal{I}_{int}(\mathcal{P}l^{+})$ of $\mathcal{P}l^{+}$.

Corollary 7.13. *Any morphism in $\mathrm{Plr}(\mathcal{P}\mathrm{l}^{+})\Sigma_{\mathrm{ext/int}}$ can be written as $\Gamma\Sigma$ with $\Gamma \in \mathrm{Plr}(\mathcal{P}\mathrm{l}^{+})$ and $\Sigma \in \mathcal{I}_{ext}(\mathcal{P}l^{+})$ respectively $\Sigma \in \mathcal{I}_{int}(\mathcal{P}l^{+})$.*

Proof. The isomorphism in $\Sigma\mathrm{Plr}(\mathcal{P}\mathrm{l}^{+})$ are always factorizable. □

7.2. **The structure of $\mathcal{C}^{\boxtimes+}$.** Using the decomposition, we will put together the category by building it up from the three subcategories.

7.2.1. **The structure of $\mathrm{Plr}(\mathcal{C}^{\boxtimes+})$.** For $\mathrm{Plr}(\mathcal{C}^{\boxtimes+})$ removing the associativity brackets leaves the monoidal generators $\gamma_{\phi_1,\dots,\phi_n}$. These compose as follows. Given composable tuples $(\phi_1^i, \dots, \phi_{n_i}^i)$ for $i = 1, \dots, k$ set $\psi_i = \phi_1^i \circ \dots \circ \phi_{n_i}^i$ then

$$\gamma_{\psi_1,\dots,\psi_k} \circ [\gamma_{\phi_1^1,\dots,\phi_{n_1}^1} \boxtimes \dots \boxtimes \gamma_{\phi^k,\dots,\phi_{n_k}^k}] = \gamma_{\phi_1^1,\dots,\phi_{n_1}^1,\dots,\phi_1^k,\dots,\phi_{n_k}^k} \tag{7.6}$$

If one thinks of the generators as directed, or equivalently rooted, linear graphs, whose vertices are labelled by the ϕ_i, then this composition is the typical behavior of graph insertion, see e.g. [KW17] for a concrete definition. The monoidal product $\boxtimes$ becomes disjoint union in this interpretation.

Proposition 7.14. *The generators above freely generate $\mathrm{Plr}(\mathcal{C}^{\boxtimes+})$ under the monoidal product $\boxtimes$ and $\mathrm{Plr}(\mathcal{C}^{\boxtimes+})$ with the basis of objects $\mathcal{V} = \mathrm{Mor}(M)$ as a discrete groupoid is a non–Sigma Feynman category.*

Proof. These morphisms clearly generate and are closed under composition. There are no relation between the different generators. The associativity has been incorporated and all other relations involve the isomorphisms, see §3.1. The statement about being a Feynman category follows readily. □

Equivalently, the generators form a colored non–Sigma operad, cf [KW17, 1.11.2].

Remark 7.15. In $\mathcal{M}^{nc+}$ these generators satisfy internal interchange for generators of the same "length" n generalizing (3.9), and for different length they satisfy equations of the type

$$\begin{aligned}
&\mu_{\phi_0\circ\phi_1\circ\phi_2,\psi_0\circ\psi_1}\circ(\gamma_{\phi_0,\phi_1,\phi_2}\boxtimes\gamma_{\psi_0,\psi_1})\\
&=(\gamma_{\phi_0\otimes\psi_0,(\phi_1\circ\phi_2)\otimes\psi_1})\circ(\mu_{\phi_0,\psi_0}\boxtimes\mu_{\phi_1\circ\phi_2,\psi_1})\circ\tau_{23}\circ((\mathrm{id}_{\phi_0}\boxtimes\gamma_{\phi_1,\phi_2})\boxtimes(\mathrm{id}_{\psi_0}\boxtimes\mathrm{id}_{\psi_1})) \qquad (7.7)\\
&=(\gamma_{(\phi_0\circ\phi_1)\otimes\psi_0,\phi_1\circ\phi_2\otimes\psi_1})\circ(\mu_{\phi_0\circ\phi_1,\psi_0}\boxtimes\mu_{\phi_2,\psi_1})\circ\tau_{23}\circ((\gamma_{\phi_0,\phi_1}\boxtimes id_{\phi_2})\boxtimes(\mathrm{id}_{\psi_0}\boxtimes\mathrm{id}_{\psi_1}))
\end{aligned}$$

We will address these relations below.

7.2.2. **Adding external isomorphisms.** Including the external symmetries $\mathrm{Plr}(\mathcal{C}^{\boxtimes+})\Sigma_{\mathrm{ext}}$ amounts to a decoration of linear graphs.

Proposition 7.16. *The monoidal generators of morphisms of* $\mathrm{Plr}(\mathcal{C}^{\boxtimes+})\Sigma_{\mathrm{ext}}$ *can be written as pairs* $(\gamma_{(\phi_n,\dots,\phi_1)},\sigma)$ *with* $\sigma\in\mathbb{S}_n$. *The composition is given by the wreath product. These can be alternatively thought of as a decorated linear rooted tree whose vertices are labelled by morphisms of* $\mathcal{C}$ *and an enumeration of the vertices.* $\mathrm{Plr}(\mathcal{C}^{\boxtimes+})\Sigma_{\mathrm{ext}}$ *is a Feynman category with discrete* $\mathcal{V}=\mathrm{Mor}(\mathcal{C})$.

Proof. The first statement follows from Corollary 7.13. For the second, we interpret the morphism

$$\phi_1\boxtimes\cdots\boxtimes\phi_n\xrightarrow{\sigma}\phi_{\sigma^{-1}(1)}\boxtimes\cdots\boxtimes\phi_{\sigma^{-1}(n)}\xrightarrow{\gamma_{\phi_{\sigma^{-1}(1)},\dots,\phi_{\sigma^{-1}(n)}}}\phi \qquad (7.8)$$

as a linear rooted tree whose vertices decorated by the ϕ_i. This fixes the map γ. The enumeration of the vertices gives the source, namely the $\boxtimes$ product of the morphisms decorating the vertices in that order.

The statement about being an FC follows readily. □

Alternatively, the morphisms of $\mathrm{Plr}(\mathcal{C}^{\boxtimes+})\Sigma_{\mathrm{ext}}$ form a colored symmetric operad.

7.2.3. **Adding internal isomorphisms.** Set

$$\stackrel{\sigma_0}{-}\phi_1\stackrel{\sigma_1}{-}\phi_2\stackrel{\sigma_2}{-}\cdots\stackrel{\sigma_{n-1}}{-}\phi_n\stackrel{\sigma_n}{-}:=\gamma_{\sigma_0\circ\phi_1\circ\sigma_1,\dots,\phi_{n-1}\circ\sigma_{n-1},\phi_n}\circ\\ [(\mathrm{id}\Downarrow\sigma_0)(\phi_1)\otimes\cdots\otimes(id\Downarrow\sigma_{n-2})(\phi_{n-1})\otimes(\sigma_n^{-1}\Downarrow\sigma_{n-1})(\phi_n)] \qquad (7.9)$$

Graphically these generators are linear rooted b/w biptartite trees with a black root, whose vertices are decorated by the ϕ_i and the edges by the σ_i. The enumeration left to right correspond to the enumeration starting at the root, due to the function notation for composition. Alternatively, these can be seen as b/w bipartite linear rooted trees, where the black vertices are decorated by elements σ_i and white vertices are decorated by elements ϕ_i.

Proposition 7.17. *The morphisms of* $\mathrm{Plr}(\mathcal{C}^{\boxtimes+})\Sigma_{\mathrm{int}}$ *are monoidally freely generated by the* $\stackrel{\sigma_0}{-}\phi_1\stackrel{\sigma_1}{-}\phi_2\stackrel{\sigma_2}{-}\cdots\stackrel{\sigma_{n-1}}{-}\phi_n\stackrel{\sigma_n}{-}$. $\mathrm{Plr}(\mathcal{C}^{\boxtimes+})\Sigma_{\mathrm{int}}$ *is a non–Sigma FC with groupoid* $\mathcal{V}=Iso(\mathcal{C}\downarrow\mathcal{C})$.

Proof. By Corollary 7.13 and Proposition 7.16 all the morphisms are of the type $\gamma_{\phi_1\dots\phi_n}\boldsymbol{\sigma}$. If $\tilde\phi_i=(\sigma_i\stackrel{\sigma_i'}{\Downarrow}\phi_i)$ and $\boldsymbol{\sigma}=(\sigma_1\stackrel{\sigma_n'}{\Downarrow}\phi_i)\boxtimes\cdots\boxtimes(\sigma_n\stackrel{\sigma_n'}{\Downarrow}\phi_n)$ and then by inner equivariance (3.3) then

$$\gamma_{\tilde\phi_n,\dots,\tilde\phi_1}\boldsymbol{\sigma}=\stackrel{\sigma_1'}{-}\phi_1\stackrel{\sigma_1^{-1}\sigma_2'}{-}\phi_2\stackrel{\sigma_2^{-1}\sigma_2'}{-}\dots\stackrel{\sigma_{n-1}^{-1}\sigma_n}{-}\phi_n\stackrel{\sigma_n^{-1}}{-} \qquad (7.10)$$

Thus the purported elements generate. It is easily seen that they are independent as only the equation (3.3) has been taken care of. The statement about being an FC follows readily. □

This means that the generators form a groupoid colored non–Sigma operad if suitably defined in the enriched case.

7.2.4. **General morphisms.** Adding general isomorphisms, we can write any morphisms as $\Gamma\boldsymbol{\sigma}P$ or $\Gamma P\boldsymbol{\sigma}$ with $\Gamma \in \mathrm{Plr}, \mathrm{P} \in \mathcal{I}_{\mathrm{ext}}, \sigma \in \mathcal{I}_{\mathrm{int}}$. It follows that

Proposition 7.18. *The morphisms of $\mathcal{C}^{\boxtimes+}$ are freely monoidally generated by linear rooted trees together with a labelling of the vertices by morphisms of $\mathcal{C}$ and edges by isomorphisms of $\mathcal{C}$ and an enumeration of the vertices. $\mathcal{C}^{\boxtimes+}$ is a FC with groupoid $\mathcal{V} = Iso(\mathcal{M}\downarrow\mathcal{M})$.*

Proof. Parallel to the results above. □

This means that the generators form a groupoid colored operad if suitably defined, see [KW17, §1.11.2], in the enriched case.

Moreover, the morphisms are generated, cf. [KW17, §5], γ_{ϕ_1,ϕ_0} the relation (3.5) is quadratic. Declaring the degree of γ's to be 1 and that of $((\sigma \Downarrow \sigma'))$ yields a proper degree function [KW17, Definition 7.2.1]. This is also the natural degree, see 6.3.

Proposition 7.19. *With the proper degree function above, $\mathcal{C}^{\boxtimes+}$ and its hyp version are cubical Feynman category as defined in [KW17, Definition 7.2.2]. The gcp version is a Feynman category with additional generators of type $(0,1)$ with degree -1.*

Proof. The fact that $\mathcal{C}^{\boxtimes+}$ is cubical is a straightforward check from the definitions as the relations are quadratic. The unit maps are indeed $(0,1)$ maps and have degree -1. There are no new monoidal relations.

In the hyp version these maps eliminates become isomorphisms of degree 0 and the relations for non–isomorphism are quadratic. □

Remark 7.20. These constructions can be seen as a categorification of the plus construction on the trivial category which yields the FC for monoids, cf. [Kau21, Proposition 3.20, Proposition 3.34].

7.3. **Formulas, cells and graphs.** To give the morphisms in $\mathcal{M}^{nc+}$ we define several formalisms.

7.3.1. **Formulas.** A *fully bracketed irreducible pre–formula* is a formal expression of two formal binary operations $\circ$ and $\otimes$. For instance $(- \circ (- \circ -)) \otimes (- \circ -)$. The arity of a formula is the number of $-$. This is the number of operations minus one.

A fully bracketed pre–formula gives rise to a flow chart and vice–versa. This is a planar planted binary tree with black vertices for the binary operations $\otimes$ and a white vertex for the operation $\circ$. The level of nesting of brackets is the distance to the root vertex plus one, if the outside parenthesis are of level 1.

To model associative binary operations, we use reduced pre–formulas. That is use $(- \circ - \circ -)$ to represent both $((- \circ -) \circ -)$ and $(- \circ (- \circ -))$ and likewise form $\otimes$, for instance $(- \circ - \circ -) \otimes (- \circ -)$. Replacing a nested expression of brackets of the same type by just one bracket defines reduced pre–formulas.

In the flow chart this becomes particularly transparent and one obtains a planar planted b/w bi–partite tree: Call an edge black (resp. white) if it is between two black (respectively white) vertices, the edges between a black and white vertex will be called mixed. The associativity equation acts as usual by edge collapses and expansions of black and white edges, see e.g. [Kau04, Kau07b]. Contracting all black and white edges, one is left with only mixed edges, that is a black and white bipartite tree.

Remark 7.21. Irreducible pre–formulas naturally form a non–Sigma operad by substitution. As flow charts this is gluing the leaves/inputs to the root flag/output.

A *pre–formula* (fully bracketed or reduced) is a formal conjunction of fully bracketed irreducible pre–formulas by a associative formal binary operator $\boxtimes$ —for instance

$$f = ((- \circ (- \circ -)) \otimes (- \circ -)) \boxtimes (- \circ -) \boxtimes (- \otimes -) \tag{7.11}$$

in the fully bracketed case. The arity is the sum of arities. As flow charts pre–formulas are planar, viz. ordered, forests of trees of the given type

Definition-Proposition 7.22.

(1) *Consider the category $\mathcal{F}_{b/w-bin-tree}$ whose objects are the natural numbers and whose morphisms $Hom(n,m)$ are ordered forests with of b/w binary planar planted trees with m trees having a total of n leaves. With one basic object 1 and the basic $(n,1)$ morphisms being b/w binary planar planted trees with 1 leaf this is a non–Sigma Feynman category.*

Isomorphically, the set of basic $(m,1)$ morphisms are the n–ary irreducible fully bracketed pre-formulas.

(2) *Similarly taking the same objects, but planar planted bi–partied trees as basic $(n,1)$ morphisms, general morphisms being ordered forests, defines an FC.*

Isomorphically, we can use reduced formulas as morphisms with the irreducible reduced ones being basic. Call this category $\mathcal{F}_{form}$ and the Feynman category given by the presentation $\mathfrak{F}_{form}$.

(3) *The morphisms are generated, cf. [KW17, §5], by γ and μ in $Hom(2,1)$ and in the fully bracketed case, they generate freely, and in the reduced case they have quadratic relations and the resulting Feynman category is cubical, [KW17, Definition 7.2.1].*

Proof. For the cubical structure one needs to specify a proper degree function, which is given by the natural degree $(n-1)$ for basic morphisms of type $(n,1)$. Then everything is a straightforward check of the axioms. □

Remark 7.23.

(1) From the general theory [KW17, GCKT20], or by direct check, irreducible pre–formulas form a non–connected non–Sigma operad by substitution. The cubical structure entails that this is quadratic, [KW21].

(2) The appearance of b/w bipartite trees suggests a connection to little 2-cubes by [Kau07b, Luc16, Bri01]. This is made precise in §7.4.3 below.

7.3.2. **Valid formulas.** Let S be a set with two possibly colored associative binary operations $\circ$ and $\otimes$ which satisfy the interchange equation (A.1). Here colored means that there are source and target maps and they have to coincide, cf. e.g. [KY21]. An example anticipating the next section is furnished by the 2–morphisms of a double category.

Consider the free associative monoid $S^{\boxtimes}$ on S. The population of an n–ary formula (fully bracketed or reduced) f by elements $\phi_1,\dots,\phi_n$ of S the formal substitution of the ϕ_i into the i–th slot of the formula.

Definition 7.24. A valid formula (fully bracketed or reduced) is a population of a pre–formula whose evaluation is possible. We will denote this evaluation by $eval(f)(\phi_1,\dots,\phi_n)$ and call it the target of $f(\phi_1,\dots,\phi_n)$. The source of $f(\phi_1,\dots,\phi_n)$ is defined to be the expression $\phi_1 \boxtimes \cdots \boxtimes \phi_n$.

The morphism corresponding to the formulas $\phi_1 \circ \phi_2$ will be called $\gamma_{\phi_1,\phi_2} : \phi_1 \boxtimes \phi_2 \to \phi_1 \circ \phi_2$, and the morphism corresponding to the formulas $\phi_1 \otimes \phi_2$ will be called $\mu_{\phi_1,\phi_2} : \phi_1 \boxtimes \phi_2 \to \phi_1 \otimes \phi_2$. These two morphisms generate under composition. They satisfy associativity relations and the interchange relation.

Example 7.25. The expression

$$f(\phi_1, \dots, \phi_8) = \phi_1 \circ (\phi_2 \otimes (\phi_3 \circ \phi_4)) \circ (\phi_5 \otimes \phi_6) \boxtimes (\phi_7 \circ \phi_8) \tag{7.12}$$

is a fully bracketed valid formula given by a population of (7.11) if the source and target maps align properly for $\phi_1 \circ (\phi_2 \otimes (\phi_3 \circ \phi_4)) \circ (\phi_5 \otimes \phi_6)$ and for $\phi_7 \circ \phi_8$. The formula (7.12) then specifies a morphism from $\phi_1 \boxtimes \cdots \boxtimes \phi_6 \to \psi_1 \boxtimes \psi_2$ where $\psi_1 = \phi_1 \circ (\phi_2 \otimes (\phi_3 \circ \phi_4)) \circ (\phi_5 \otimes \phi_6)$ and $\psi_2 = \phi_7 \circ \phi_8$. The morphism can be read off as

$$[\gamma_{\phi_1,\phi_2 \otimes (\phi_3 \circ \phi_4),\phi_5 \otimes \phi_6)} \circ (id_{\phi_1} \boxtimes \mu_{\phi_2,\phi_3 \circ \phi_4} \boxtimes \mu_{\phi_5,\phi_6}) \circ (id_{\phi_1} \boxtimes id_{\phi_2} \boxtimes \gamma_{\phi_3,\phi_4} \boxtimes id_{\phi_5} \boxtimes id_{\phi_6}] \boxtimes \gamma_{\phi_7,\phi_8} \tag{7.13}$$

The formula $(\phi_1 \boxtimes \phi_2) \circ \phi_3$ is never valid as there no element of S ϕ_3 whose source is given by $t(\phi_1) \boxtimes t(\phi_2)$. The underlying pre–formula $(- \boxtimes -) \otimes -$ is also not valid.

For the flow chart/tree picture a population corresponds to a decoration of the inputs/leaves by the ϕ_i in their order and the decoration of the output/root by the target, see Figure??. We call the decoration valid if the corresponding formula is valid. This entails that each edge is naturally labeled by the output of the compositions above it. The following is readily checked.

Definition-Proposition 7.26.

(1) *Valid formulas, both fully bracketed and reduced, form a non-Sigma Feynman category whose basis of objects are given by S whose basis of morphisms is given by valid populated irreducible formulas with source and target as defined above.*

(2) *If $\mathcal{F}$ is the underlying category, the morphisms are generated by $\gamma_{\phi_1,\phi_0} \in \mathcal{F}(\phi_1, \phi_0; \phi_1 \circ \phi_0)$, for all pairs where the compositions are defined, and $\mu_{\psi_1,\psi_2} \in \mathcal{F}rm(\psi_1, \psi_2; \psi_1 \otimes \psi_2)$.*

In the fully bracketed case, they generate freely, and in the reduced case they have quadratic relations and the resulting Feynman category is cubical.

We denote the monoidal category of reduced valid formulas by $\mathcal{F}rm(S)$ and the corresponding Feynman category by $\mathfrak{F}_{form}(S)$.

Proof. This is just a rephrasing of structures. Being a non–Sigma Feynman category is checked readily $\mathcal{V} = S$ as a discrete category and the underlying category $\mathcal{F}rm$ of $\mathfrak{F}$ has objects $S^{\boxtimes}$. $\mathcal{P}$ is freely monoidally generated by the $(n, 1)$ morphisms $\mathcal{F}rm(\phi_1, \dots, \phi_n; \phi)$. The statement about generation follows from the definition. The proper degree function, is defined to be the natural degree $(n - m)$ for morphisms of type (n, m). □

Remark 7.27.

(1) The morphism can be read off from a valid formula as follows: Given a valid formula, the target of the corresponding morphism is the value of the formula and the source is obtained by replacing all occurrences of $\circ$ and $\otimes$ by $\boxtimes$. The morphism expressed in generators is the combination of γ's an μ's that changes the respective occurrences of $\boxtimes$ to $\circ$ and $\otimes$ whose iteration is determined by the nesting.

(2) The FCs above as colored structures are obtained from their uncolored counterparts by a decoration and restriction, cf. [KW17, §2.5] and more generally [KL17, 6.1.4]. The decoration yields a Feynman category of populated formulas and the restriction restricts to the valid ones.
(3) The irreducible reduced formulas form non–Sigma S-colored operad $\mathcal{F}rm$ freely generated by two generators. Composition is given by substitution: For example $(\phi_1 \circ \phi_2) \boxtimes \phi_3$ corresponds to $\mu_{\phi_1 \circ \phi_2, \phi_3} \circ_1 \gamma_{\phi_1,\phi_2}$. This colored operad is again quadratic.

7.4. **Cells and diagrams.** A particularly helpful way to think about fully bracketed valid formulas is a diagram or pasting scheme in a 2–category $\mathcal{D}$ with a composition ordering. The two operations are horizontal $\circ_h$ and vertical composition $\circ_v$. Equivalence classes defined by the associativity relations for the decomposition correspond to compatible enumerations of the cells. There are again three levels: unpopulated diagrams, populated diagrams and valid diagrams. In the application, $\mathcal{D} = \underline{\mathcal{M}}$, the 2-category with one object, 1-morphisms given by objects of $\mathcal{M}$ and two–morphisms by morphisms of $\mathcal{M}$. The vertical composition is composition of morphisms and the horizontal is tensor product. String diagrams provide a graphical version.

Notation 7.28. With a view towards the application to the 2–category $\underline{\mathcal{M}}$, we will write $\circ := \circ_v$ for the vertical composition and $\otimes := \circ_h$. The maps for these compositions of 2–cells will be called γ and μ.

Considering the associated double category leads to decomposable tight square arrangements considered up to isotopies called basic decomposable box diagrams. These basic box diagrams can be generalized to account for different sides of the interchange equation.

There is a dual b/w bipartite graph for a basic box diagram and a b/w bipartite suspension graph. The former is related to string diagrams, while the latter contains finer information about interchanges.

7.4.1. **Pasting diagrams.** Starting from the 2–categorical view, unpopulated diagrams can be seen as abstract diagrams in the categorical sense, where the index 2-category does not need 0 or 1 cell labels or enumeration, as the 0 and 1–cells then need to have compatible labels and the 0-cells so this information can be omitted from the enumeration, but we do label and enumerate the 2–cells as $1, \dots, n$. These enumeration may be compatible or not. Given a diagram in a double category $\mathcal{D}$ a decomposition into elementary composition steps involving only one horizontal or vertical composition defines a full enumeration of the cells. An enumeration of cells is *compatible* if and only if it is the image of the morphism.

A 2-functor $D \to \mathcal{D}$ of such a diagram into a 2–category is directly a population where sources and targets match. To obtain a formula one has a 2–cell ϕ_i for each i, and uses the following algorithm: If two consecutively numbered cells i, $i+1$ are composable, horizontally or vertically

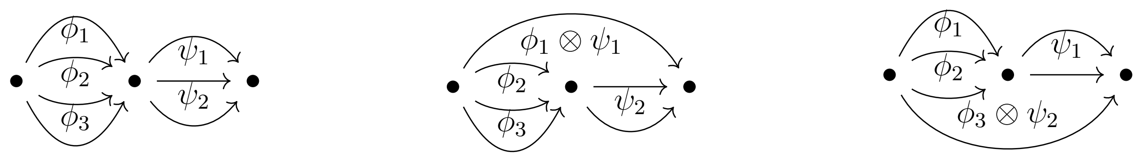
(1) compose them, this is the new decoration, and replace the number by i and renumber j to $j-1$ for $j > i$.
(2) record this in the formula as the morphism $\mu_{\phi_i,\phi_{i+1}}$ for horizontal composition respectively $\gamma_{\phi_i,\phi_{i+1}}$ for vertical composition.

repeat as long as possible.

If there is only one 2–cell labelled by one morphism, left, the enumeration is *compatible* and the diagram is called *valid*.

This can be seen as map from the free monoid on 2–morphisms to 2–morphisms. The composition is the target of the formula and the source is $\phi_1 \boxtimes \cdots \boxtimes \phi_n$, where $\boxtimes$ is the free product. Ordered collections of such diagrams then provide maps from the free product to the free product.

Example 7.29. The formula $(\phi_3 \circ \phi_2 \circ \phi_1) \boxtimes (\psi_2 \circ \psi_1)$ corresponding to the morphism given by $\mu_{\phi_3\circ\phi_2\circ\phi_1,\psi_2\circ\psi_1} \circ (\gamma_{\phi_0,\phi_1,\phi_2)} \boxtimes \gamma_{\psi_2,\psi_1}) : \phi_3 \boxtimes \phi_2 \boxtimes \phi_1 \boxtimes \psi_2 \boxtimes \psi_1 \to (\phi_3 \circ \phi_2 \circ \phi_1) \boxtimes (\psi_2 \circ \psi_1)$ is given in Figure 5 (A) with the decomposition given by first doing the vertical composition and then the horizontal decomposition. If the first step in the composition is given by (B) and then the standard decomposition the formula reads $((\phi_3 \circ \phi_2) \otimes \psi_2) \circ (\phi_1 \otimes \psi_1)$. This has a different source and enumeration, namely $\phi_3 \boxtimes \phi_2 \boxtimes \psi_2 \boxtimes \phi_1 \boxtimes \psi_1$. The formula obtained by evaluating a different first horizontal composition as in (C) and then using the standard decomposition is $(\phi_3 \otimes \psi_2) \circ ((\phi_2 \circ \phi_1) \otimes \psi_1)$ which again has a different source.



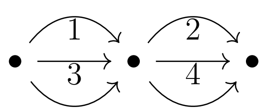
(A) (B) (C)

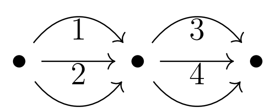
FIGURE 5. A diagram (A) and two different first compositions (B) and (C).

Proposition 7.30. *The compatible enumeration of the morphism, as cells in the diagram or letters, is a complete invariant of the associativity relations. Thus, compatibly enumerated populated valid diagrams are in bijection with reduced valid formulas.*

Proof. The fact that it is an invariant of the associativity transformations is straightforward. The fact that it is complete follows from the fact that the only other relation possibly resulting in different (de)compositions is the interchange relation (3.9). This however changes enumeration of the order and hence the source thereby changing the morphism. $\square$

Definition 7.31 (Standard numbering)**.** Given a non–enumerated diagram, there is a standard way to give a decomposition by prioritizing the vertical compositions over the horizontal ones. For this first perform all possible vertical compositions, then all possible horizontal ones, continue in this manner. This gives a particular formula and enumeration which we call the *standard* enumeration. An irreducible formula is in *standard* form or a *standard formula*, if its cell diagram has a standard enumeration.

An example is given in Figure 7. In general there may be more that one compatible enumeration. This is due to the interchange relation. In particular (A.1) corresponds to the two enumerations of cells 

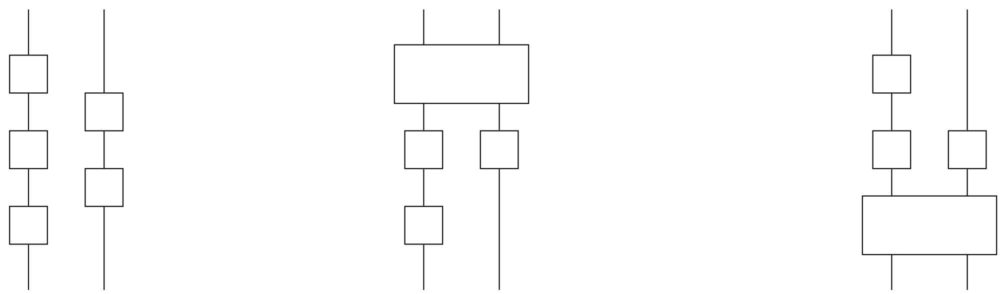
respectively 

see also Figure 9. This means that the l.h.s. of (A.1) is in standard form, while the r.h.s. is not.

Lemma 7.32. *All compatible enumerations are obtained from the standard enumeration by permutations stemming from the interchange relation.*

Proof. This is clear, since the interchange relation is the only relation other than associativity between the operations of horizontal and vertical composition. □

7.4.2. **String diagrams.** It is well known that the diagrams of a 2–category can be equivalently represented by string diagrams. These are the dual graphs. Given a diagram there is a vertex for each 2-cell and an edge for each 1-cell connecting the two vertices of the 2-cells it bounds with the inherited labelling. The outer edges, those that are on the boundary of only one 2-cell are leaves or tails. Each edge/tail is directed induced by the source to target maps. The monoidal product is formal disjoint union and composition is by insertion into vertices. Note, the string diagrams need not be connected, which is why the monoidal product is formal.



(A) (B) (C)

FIGURE 6. The string diagrams for the pasting diagrams of Figure 5.

Definition 7.33. We call a diagram horizontally or —in the case of $\underline{M}$— ⊗-decomposable if its string diagram is disconnected. The horizontal irreducible, respectively ⊗ irreducible components are consequently the connected components of the string diagram.

Remark 7.34. In terms of the original diagram as a graph this means that removing a vertex the and irreducible diagram is still connected. Dually one can think or a reducible diagram as merged from irreducible ones by merging vertices.

For the diagrams of Figure 5 (A) is horizontally irreducible while (B) and (C) are horizontally reducible. This can be read off from Figure 6.

Remark 7.35. Another related graphical realization can be found in [JS91].

7.4.3. **Box diagrams and their graphs.** Reinterpreting the 2–category as a double category, where the vertical morphisms is $id_* = \mathbb{1}$ for the only vertical morphisms one obtains box diagram of tight decomposable rectangles, by expanding the nodes to vertical lines with segments according to the 2–cells. This is closely related to the considerations of [Dun88,Bri01,BFSV03] in that there is a non–Sigma operad structure which is a quotient of the suboperad of decomposable cubes of tight little 2–cubes, see Remark 7.38. "Tight" means that the little squares fill out the entire cube and "decomposable" means precisely that the configuration is in the image of the two basic operations, see below. The quotient is by isotopies of moving line segments. Conversely, given such a diagram, collapsing the horizontal line segments yields a two–cell diagram, see Figure 7. We will now make this precise.

FIGURE 7. The box and 2–cell realization of the formula $f = (\phi_6 \circ (\phi_3 \otimes (\phi_5 \circ \phi_4))) \circ (\phi_1 \otimes \phi_2)$ in the standard enumeration. The diagram is horizontally irreducible and has only one compatible enumeration

Definition 7.36. A *basic box diagram* is a class of decomposable arrangements of boxes (little squares) obtained iteratively by starting with a unit squares and subdividing horizontally or vertically modulo isotopies on the vertical and horizontal line segments.

The horizontal line segments can be moved, but are not allowed to cross each other. One may also not break triple crossing points or "T" junctions but one *is* allowed to move the horizontal line segments to possibly align in a four point crossing and to break such a crossing, compare Figure 8.

The vertical line segments may be moved, but are not allowed to not cross each other *or* any of the intersection points, see Figure 8. Given a box diagram one obtains a cell diagram by shrinking all horizontal lines, see Figure 7 for an example.

A *generalized box diagram* is again a class of box diagrams up to isomorphism, but where now the horizontal isotopies are not allowed to break four point or cross, "+", intersections or to create them. Such a diagram is called *generic* if it has only "T" intersections.

A *population* or coloring is given by labelling the boxes with 2–morphisms of $\mathcal{D}$ —that is morphisms of $\mathcal{M}$ if $\mathcal{D} = \underline{\mathcal{M}}$. A population is *valid* if the compositions line up correctly. That is for each full line segment the tensor product of the target boundaries of the two cells bordering this line segment from above agrees with the tensor product of the sources of the 2–cells bordering this line segment from below.

For example for the population to be valid in Figure 7 the following equality must hold $t(\phi_1) \otimes t(\phi_2) = s(\phi_3) \otimes s(\phi_4)$.

As box diagrams the generators μ_{ϕ_1,ϕ_2} and γ_{ϕ_1,ϕ_2} are a box with a vertical line receptively a box with a horizontal line, and the following valid decorations:

$$\begin{array}{|c|c|}\hline \phi_1 & \phi_2 \\ \hline\end{array} \quad \text{and} \quad \begin{array}{|c|}\hline \phi_1 \\ \hline \phi_2 \\ \hline\end{array} \tag{7.14}$$

Or just unlabelled boxes in the case of unpopulated diagrams.

Being decomposable means tautologically that it stems from an iteration as above. The recursive recognition definition is that there is either a horizontal of a vertical line which cuts the diagram into two nonempty parts each of which has the same property.

The box diagrams that are obtained in this way are decomposable by definition and valid formulas are obtained by this iterated substitution. The ambiguity in bracketing operations

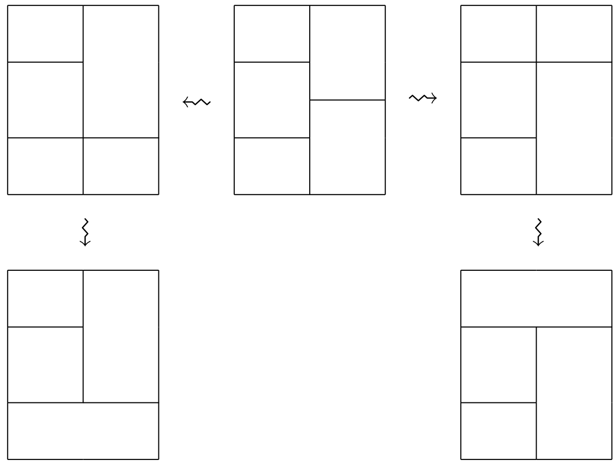


FIGURE 8. Different drawings of boxes that are obtained by moving the horizontal line segments. In the top row moving the line segments into different non–generic positions is an equality in basic diagrams. The diagrams are different as generalized box diagrams. If the line segments are aligned, one can perform horizontal compositions changing the diagram as indicated. The diagram in the bottom row are neither equivalent as basic nor as generalized box diagrams. The steps correspond to the (de)compositions in Figures 5 and 6.

leading to multiple parallel horizontal or vertical line segments in the choice is taken care of by associativity of composition respectively the strict monoidal structure. The basic box diagrams also already incorporate the interchange relation, while the generalized box diagram encodes the different sides differently — see Figure 9.

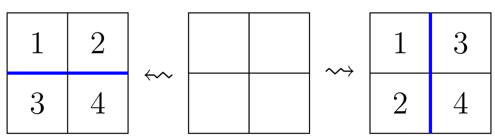


FIGURE 9. Different enumerations corresponding to the different decomposition related by the interchange relation. The basic non–enumerated box diagrams agree while the generalized box diagrams differ as on the left the horizontal line segments may not be moved separately in the generalized case, but in both cases, they may be on the right.

Lemma 7.37. *Substitution into the labelled basic/generalized box diagrams form a non–Sigma colored operad structure and hence the morphisms of a non–Sigma FC. There is a morphism of such operads, functor of Feynman categories, from generalized box diagrams to basic box diagrams by classes under the coarser equivalence.*

Proof. Straightforward. ∎

A substitution by composition with a generator (7.14) in the box picture divides one rectangle with a horizontal of a vertical line.

Remark 7.38. Enumerated unlabelled basic/generalized box diagrams form an operad under substitution. Decomposable tight squares form a suboperad of the decomposable squares of [Dun88, Bri01] and there is a morphism operad to the enumerated basic/box configurations, by taking the isotopy classes.

Proposition 7.39.

(1) *Contracting horizontal line segments provides a surjective morphism of the colored non–Sigma operad of basic/generalized box diagrams (with a valid population) to 2–cell diagrams (with a valid population). In particular, the boxes and 2–cells are in bijection.*
(2) *This morphism is an isomorphism for basic box diagrams.*
(3) *The morphism from generalized box digrams to 2–cells factors through the morphisms from generalized to basic box diagrams.*
(4) *The fiber over a fixed 2–cell diagram are the possible decompositions of it.*

Proof. All but the last statement are straightforward. For the fiber, we see that these are the generalized box diagrams that map to the same basic box diagram. These are enumerated by a choice of matching movable horizontal line segments with neighboring movable line segments. Matching or not matching corresponds to the choice of a side of a use of an interchange relation. This is the only relation leading to different compositions, and thus the last statement follows. $\square$

An enumeration of the boxes of a basic is box diagram is called compatible/standard if the enumeration on the 2-cell side is compatible/standard.

Example 7.40. The box diagram and 2–cell diagram for the morphism

$$\gamma_{\phi_6,\phi_1\otimes\phi_2,\phi_3\otimes\phi_5\circ\phi_4}\circ(id_{\phi_6}\otimes\gamma_{\phi_1\otimes\phi_2,\phi_3\otimes\phi_5\circ\phi_4})\circ(id_{\phi_6}\otimes id_{\phi_5}\otimes\gamma(\phi_4,\phi_5)\otimes id_{\phi_2}\otimes id_{\phi_1}):\phi\to\tilde{\phi} \quad (7.15)$$

where $\phi=\phi_6\boxtimes\phi_3\boxtimes\phi_5\boxtimes\phi_4\boxtimes\phi_1\boxtimes\phi_2$ which corresponds to the formula $f=\phi_6\circ(\phi_3\otimes(\phi_5\circ\phi_4))\circ(\phi_1\otimes\phi_2)$ are given in Figure 7.

Corollary 7.41. *Fully bracketed formulas are equivalent to*

(1) *enumerated basic box diagrams*
(2) *generalized box diagrams (with standard enumeration)*

Proof. Fixing an iteration, a fully bracketed formula is equivalent to specifying a compatible enumeration for the corresponding 2–cell diagram by Proposition 7.30. This is the same as a compatibly enumerated basic box diagram by Proposition 7.39. Hence the first claim follows. The last claim follows from Proposition 7.39. $\square$

Remark 7.42. The following algorithm produces a generalized box diagram from a pre–formula. Produce the corresponding diagram by drawing horizontal and vertical line segments. Then takes its class as a generalized box diagram. Given such a diagram or suspension graph, inversely use the algorithm to remove full horizontal or vertical line segment and note the operations as $\circ$ or $\otimes$ in the pre–formula. This process will terminate in just one box by decomposablility.

7.4.4. **Composition and suspension graphs.**

Definition 7.43. The *planar b/w graph* of a basic decomposable box diagram is the planar directed bipartite b/w graph, whose black vertices are the horizontal line segments, whose white vertices are the boxes and there is an edge if a the horizontal sub–line–segment is the boundary of the box. The edges are directed from top to bottom.

The *suspension graph* of generalized decomposable box diagram is the planar directed bipartite b/w graph whose white vertices are the boxes, whose black vertices are the full line segments. There is an edge if the full line segment has a sub–line segment which is part of the box and the direction is down.

If the box diagram is decorated, there is an induced decoration on the graph and a direction from source to target. For instance the cell arrangement in Figures 5 and 7 (B) translates to the planar b/w graph and the suspension b/w graph in Figure 10 both orientated downward.

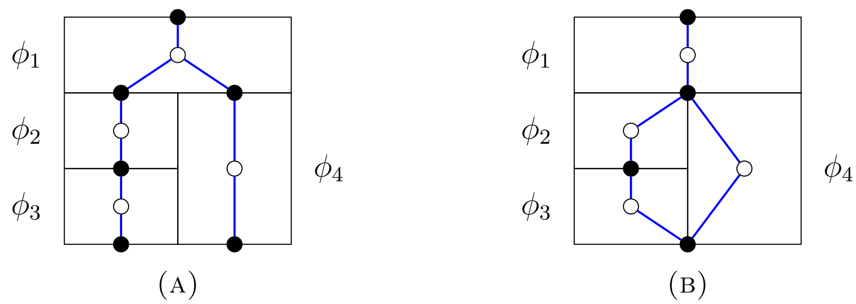


FIGURE 10. A box diagram with its the dual b/w string diagram and dual supspension graph.

A more complicated diagram given by the Example of Figure 7 is in Figure 11.

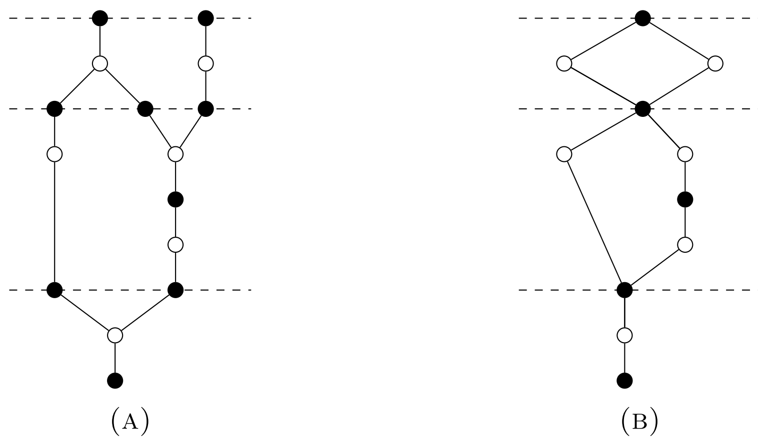


FIGURE 11. The planar directed b/w bipartite graph and suspension graph corresponding to the example in Figure 7. The orientation is downward. There is a merging/splitting of vertices when going from one diagram to the other.

Remark 7.44. The removal of line segments that determine the decomposition translate as follows:

(1) In the planar black and white graph, bivalent black vertices are those that border bivalent white vertices correspond to freely movable line segments. These are the ones than can be removed in a (de)composition.

(2) If line segments are moved to "+" crossings the effect on the graphs is a merging of two black vertices
(3) In the b/w graph the black vertices are decorated by the source and target objects.
(4) in the suspension graph the black vertices are the tensor product of the objects belonging to the sub–line–segments of the full line segment.

The following Remark translates the algorithm of §7.4.1.

Remark 7.45.

(1) In the b/w graph the removal of a movable horizontal line segment in the decomposable box diagram corresponds to the deletion of the corresponding bivalent black vertex and the contraction of the edge between the two neighboring white vertices. If there is a labelling, the new label is the composition.

    The removal of a horizontal line segment corresponds to the merging of two parallel neighboring black-white-black strands where the white vertex is bivalent. If there is a labelling, upon merging labels are tensored.
(2) In the suspension graph, a horizontal line piece can be removed in the generalized box diagram, if it belongs to a bivalent black vertex. Then the procedure is the same as above.

    A removal of a vertical line piece is possible if two white vertices are suspended from the same two black vertices and are neighbors. The removal is then the merging of these two white vertices.

The graph, thus captures the different sides of the interchange equation. In the basic picture, one has to align the line segments horizontally and then compose to do the μ operations first, compare Figure 6. The choice of the prep-step is recorded by the suspension graph.

The basic b/w bipartite graph *b-w-b* is the graph with a black input and output and one white vertex.

Definition-Proposition 7.46. *A directed planar b/w bipartite graph is the dual of a basic diagram, if it has black input and output vertices and the algorithm above terminates with b-w-b. We will call these* composition graphs*.*

A directed b/w bipartite graph is the dual of a generalized box diagram, if it has black input and output vertices the algorithm above terminates with b-w-b. We will call these suspension graphs*.*

Proof. That the images are of this type is clear. Conversely building up the diagram by running the algorithm backwards produces the box diagram. □

The algorithm allows us to read off the bracketed formula by recording the operations as $\circ$ or $\otimes$ in the pre–formula.

Corollary 7.47. *The set of pre–formulas is bijective with respect to the decomposable suspension graphs. This yields an isomorphism of non–Sigma FCs.*

Remark 7.48. Substitution for the diagrams leads to substitution for the b/w graphs. For this a white vertex is replaced by the b/w graph, minus the input and output black vertices. In suspension graph, these vertices are merged with the corresponding black vertex above and vertex below the white vertex.

In the basic b/w graph, the gluing splits these vertices according to the labelling. This depends on the labelling and the combinatorics of the possibilities is, not by chance, that appearing in the compositions of the discretezation of the arc operad [KLP03, Kau07a, Kau08].

7.5. **The structure of $\mathcal{M}^{nc+}$.** Since equivalent categories yield equivalent plus constructions with given equivalences, in the following, we *assume that $\mathcal{M}$ is strict (symmetric) monoidal.*

7.5.1. **The non–Sigma FC $\mathrm{Plr}(\mathcal{M}^{\mathrm{nc+}})$.** Let $S = \mathrm{Mor}(\mathcal{M})$ for the monoidal category $(\mathcal{M}, \otimes)$. This has the partially defined binary operation of composition $\circ$ and that of tensor product $\otimes$. The free monoid $S^{\boxtimes}$ can be identified with the morphisms of the non–symmetric free monoidal category $\mathrm{Mor}(\mathcal{M}^{\neg\Sigma\boxtimes})$. A formula is valid if all the compositions $\circ$ can be performed, that is sources and targets line up correctly. The basic discrete groupoid of objects is $\mathcal{V} = (\mathcal{M}{\downarrow}\mathcal{M})^{disc}$.

Proposition 7.49. *The category $\mathrm{Plr}(\mathcal{M}^{\mathrm{nc+}})$ can be identified with the category $\mathcal{F}rm(S)$ for $S = \mathrm{Mor}(\mathcal{M})$ and is part of the corresponding Feynman category $\mathfrak{F}_{form}(S)$.*

Proof. By definition $\mathrm{Plr}(\mathcal{M}^{\mathrm{nc+}})$ is generated by $\gamma_{\phi_0,\phi_1}, \mu_{\phi_1,\phi_2}$ and the identities under monoidal product and composition modulo inner associativity and the interchange relation. In particular, any morphism $\phi \to \tilde{\phi}$ is given by a sequence of tensor products of these operations. Such a sequence is in bijective correspondence with a fully bracketed formula. Such a formula is not a unique morphism, but subject to associativity according to (3.5) and (3.8). The result follows from the fact that the relations on both sides are identified by the bijection, and any class of valid formulas corresponds to a reduced formula. $\square$

NB: Note that the interchange relation (3.9) is not yet part of the relations as it contains a commutation isomorphism, see also Remark 7.15.

Corollary 7.50. *The morphisms of $\mathrm{Plr}(\mathcal{M}^{\mathrm{nc+}})$ and the corresponding non–Sigma Feynman category are isomorphically given by the following non–Sigma colored operads with monoidal structure being ordered disjoint union.*

1. *Planar planted b/w bipartite trees with a valid decorations by morphisms of $\mathcal{M}$ on the leaves, where gluing decorated b/w bipartite trees at leaves gives the operad structure and hence the compositions*
2. *Compatibly enumerated 2–cell diagrams with a valid population and substitution. Or equivalently, string diagrams with a valid population and compatible enumeration and substitution.*
3. *Compatibly enumerated basic box diagrams with a valid population and substitution. Or dually, composition graphs with a valid decoration and a compatible enumeration. The operad structure is substitution of this type of graph, see Remark 7.48.*
4. *Generalized box diagrams with a valid population and substitution. Or dually, suspension graphs with a valid decoration. The operad structure is substitution of this type of graph, see Remark 7.48.*

Proof. This follows from regarding a formula as a flow chart, Proposition 7.30, Proposition 7.39, Corollary 7.41 and Definition–Proposition 7.46. $\square$

7.5.2. **Symmetric version** $\mathrm{Plr}(\mathcal{M}^{\mathrm{nc}+})\Sigma_{\mathrm{ext}}$**.** Freely adding the commutators imbues the non–Sigma structures with a free $\mathbb{S}$ action. Quotienting out by the interchange relation defines the symmetric versions of the results above. In this process, entire orbits are identified. This works for the 2-cell diagrams and equivalently for the basic or generalized box diagrams, as well as their dual graphs.

Proposition 7.51. *Basic box diagrams with standard enumerations of 2-cells generate the morphisms of* $\mathrm{Plr}(\mathcal{M}^{\mathrm{nc}+})\Sigma_{\mathrm{ext}}$ *symmetrically monoidally that is under the free symmetric product* $\boxtimes$. $\mathrm{Plr}(\mathcal{M}^{\mathrm{nc}+})\Sigma_{\mathrm{ext}}$ *is part of a Feynman category with* $\mathcal{V} = (\mathcal{M} \downarrow \mathcal{M})$ *and* $\mathcal{P}$ *given by the set of basic box diagrams with standard enumeration.*

Isomorphically the $(n, 1)$ *morphisms together with their composition structure are*

(1) *Composition graphs with labelled and enumerated white vertices.*
(2) *Pairs* (f, σ) *of irreducible valid standard formula of arity* n *and a permutation* $\mathbb{S}_n$ *of arity with* $\mathbb{S}_n$ *acting on* σ *on the right.*
(3) *2–cell diagrams in* $\underline{\mathcal{M}}$ *with a decoration of cells by elements of* $\mathcal{M}$ *and an arbitrary enumeration of the cells. Or, equivalently the associated string diagrams with a decoration and an arbitrary enumeration.*

Proof. The proof is in two steps. First adding arbitrary permutations, we obtain pairs (b, σ) of a box diagram with n boxes in standard enumeration and an element $\sigma \in \mathbb{S}_n$ where $\mathbb{S}_n$ acts on the right. This maps to morphisms in $\mathrm{Plr}(\mathcal{M}^{\mathrm{nc}+})\Sigma_{ext}$ by $\Gamma(b) \circ \sigma$ where $\Gamma(f = b)$ is the morphism in $\mathrm{Plr}(\mathcal{M}^{\mathrm{nc}+})$ defined by b. By Corollary 7.13 any morphism is of this type and the map is surjective. Note that morphisms σ permutes the source $\phi_1 \boxtimes \cdots \boxtimes \phi_n$ to $\phi_{\sigma(1)} \boxtimes \cdots \boxtimes \phi_{\sigma(n)}$ and there is only one box diagram in standard enumeration in the orbit. This is due to the fact that any use of the interchange relation (A.1) changes the numbering by a transposition. This is particularly transparent in the b/w tree picture. Due to the same fact any other decomposition is also in the orbit as any formula having the same diagram, but with possibly different compatible enumeration, can be obtained by a repeated application of the interchange relation in the 2–category. By (3.9) this corresponds to the pre–composition by a permutation σ. The element (b, σ) is mapped to $\Gamma(b)\sigma$ and if $b = b'\sigma'$ is the permutation to a standard form b then $\Gamma(b)\sigma = \Gamma(b')\sigma'\sigma$ by (3.9). On the other hand, if b is in standard form $\Gamma(b)\sigma = \Gamma(b')\sigma'$ means that $\sigma = \sigma'$ and $b = b'$ there is only one standard form in each orbit. Thus, a full orbit of the standard enumeration is given by an arbitrary enumeration. The rest of the statements again follow from Propositions 7.30 and 7.39, Corollary 7.41, and Definition–Proposition 7.46. $\square$

Remark 7.52.

(1) The $(n, 1)$ morphisms, again by general theory, form a symmetric colored operad. The relation to the operad of decomposable little 2–cubes is apparent in this interpretation.
(2) The permutation can be viewed as part of the source map, so that morphisms are diagrams together with a source map. This is parallel to the fact [KW17] that morphisms in the graphical Feynman categories are given by their ghost graph together with extra structures defining the source and target maps.

7.5.3. **Adding internal isomorphisms.** The generators will be classes of generalized box diagrams (or equivalently suspension graphs) decorated by morphisms and isomorphisms.

Definition 7.53. A *valid decoration by morphisms and isomorphisms*

(1) for a *generalized* decomposable box diagrams is a decoration of the boxes by morphisms and the *full line* segment by isomorphisms. These have to be compatible in the way that the compositions line up correctly. That is the source of the isomorphism is the tensor product the target boundaries of the two cells bordering this line segment from above and the target of the isomorphism is the tensor product of the sources of the 2–cells bordering this line segment from below. For instance in the example in Figure 7 the isomorphism in the second line from the top will be from $t(\phi_1)\otimes t(\phi_2)\to s(\phi_3)\otimes s(\phi_4)$. The top and bottom lines only have constraints on the target and the source of the isomorphisms respectively.

    This translates to the *suspension graph* as black vertices being decorated by isomorphisms and white vertices by morphisms in a compatible fashion.

(2) for *basic decomposable box diagram* is a decoration of all cells by morphisms of $\mathcal{M}$ and all horizontal line segments by isomorphisms. This has to be compatible, so that the tensor product of the isomorphisms of the sub–segments satisfy the conditions above.

    This translates to the *dual b/w string diagram* as black vertices being decorated by isomorphisms and white vertices by morphisms in a compatible fashion, and for a *2–cell diagram* as decoration of a the 2-cells by morphisms and of the 1-cells by isomorphisms, such that the decoration obtained by censuring the decorations of the 1–cells that form the boundary of a 2–cell the decoration is a valid decoration as above.

Example 7.54. E.g. in Figure 7 The isomorphism decorating the second full interval will be decorated by an isomorphism $\tau(\phi)\otimes t(\phi_2)\to s(\phi_3)\otimes s(\phi_4)$

Substituting a morphisms $(\sigma\Downarrow\sigma')(\psi)$ into ϕ_4 would yield the composition with the isomorphism $id\otimes\sigma^{-1}$ on the top of the box and $id\otimes\sigma$ on the bottom if the outside isomorphism of the box being substituted is $(\sigma\Downarrow\sigma')$.

Proposition 7.55. $\mathrm{Plr}(\mathcal{M}^{\mathrm{nc}+}\Sigma_{int})$ *is a non–Sigma Feynman category with* $\mathcal{V}=Iso(\mathcal{M}\downarrow\mathcal{M})$ *The basic morphisms of* $\mathrm{Plr}(\mathcal{M}^{\mathrm{nc}+}\Sigma_{int})$ *can be described by the groupoid colored non–Sigma operad whose generators are generalized box diagrams (or equivalently suspension graphs) with a valid decoration by morphisms and isomorphisms.*

Proof. Analogously to Proposition 7.17, it follows from (3.3) and (3.7) that the generators are the suspension graphs decorated as stated. Substituting these generators, composes outer isomorphisms which can be pulled through by (3.4), where for longer intervals, these are extended by identities as tensor factors on the subintervals that are not part of the box that is being substituted. The upshot is all the isomorphisms can be associated to the full line segments and this is stable under composition.

□

7.5.4. **General case.** Note since there is no assumption of factorizable isomorphisms the possible isomorphisms on the different sides of the interchange relation (3.9) are different and choosing non–factorizable isomorphisms will not yield allow for an interchange relation, an interchange is only possible if the isomorphisms factor.

To get a concise answer, we assume that $\mathcal{M}$ has factorizable isomorphisms

Proposition 7.56. *If $\mathcal{M}$ has factorizable isomorphisms, then $\mathcal{M}^{nc+}$ is a FC whose basic groupoid of objects is $\mathcal{V} = Iso(\mathcal{M} \downarrow \mathcal{M})$. The basic morphisms of $\mathcal{M}^{nc+}$ are equivalently given*

(1) *Box diagrams with a valid decoration by morphisms and isomorphisms and an enumeration of their cells.*
(2) *Composition graphs with a valid decoration by morphisms and isomorphisms and an enumeration of their white vertices.*

Proof. If $\mathcal{M}$ has factorizable isomorphisms then any morphism can be factored as $\Gamma\Sigma_{ext}\Sigma_{int}$ by Corollary 7.6 and Proposition 7.11. The morphisms $\Gamma\Sigma_{ext}$ are given by box diagrams with a valid decoration by morphisms and an enumeration of the two cells. The decoration is by the morphisms Σ_{int} applied to the source. As in the proof of Proposition 7.55 the operation of Σ_{int} can be represented by decorating the boundaries of the boxes compatibly with isomorphisms. The description of graphs again follows from Definition–Proposition 7.46 □

Corollary 7.57. *If $\mathcal{M}$ has factorizable isomorphisms, then every morphism $\mathbf{\Gamma}$ splits as $\mu_n(\mathbf{\Gamma}_1 \boxtimes \cdots \boxtimes \mathbf{\Gamma}_n)P$, if the box diagram has $n-1$ full vertical lines (top to bottom), equivalently n is the number of connected components of the composition graph, $\mathbf{\Gamma}_i$ are the morphisms corresponding to the connected components and P is a permutation.*

Proof. This is clear since the algorithm determining the morphisms will perform the multiplication followed by the last, outer, isomorphism, whose permutation can be pulled to second last place by the factorizations of isomorphisms. In formulas: $\mathbf{\Gamma} = \Sigma\mu(\mathbf{\Gamma}'_1 \boxtimes \cdots \boxtimes \mathbf{\Gamma}'_n) = \mu P\Sigma'(\mathbf{\Gamma}_1 \boxtimes \cdots \boxtimes \mathbf{\Gamma}_n) = \mu P(\Sigma_1\mathbf{\Gamma}'_1 \boxtimes \cdots \boxtimes \Sigma_n\mathbf{\Gamma}'_n) = \mu(\mathbf{\Gamma}_{P(1)} \boxtimes \cdots \boxtimes \mathbf{\Gamma}_{P(n)})P$ where P is the permutation of the factors determined by $\Sigma = P(\Sigma_1 \otimes \cdots \otimes \Sigma_n)P$. □

Vice–versa:

Proposition 7.58. *If $\mathcal{M}$ is fully hereditary, then if the target ϕ of a basic morphism $\Gamma : \Phi \to \phi$ is $\otimes$ decomposable, that is the morphisms $\phi_1 \otimes \cdots \otimes \phi_n$, the so is $\Gamma = \Gamma_1 \otimes \cdots \otimes \Gamma_n$. That is given the solid arrows, the dotted arrows exist*

$$\begin{tikzcd}
\Phi_1 \boxtimes \cdots \boxtimes \Phi_n \arrow[r, dotted, "\mu^{\boxtimes n}"] \arrow[d, dotted, "(\Gamma_1 \boxtimes \cdots \boxtimes \Gamma_n)P"] & \Phi \arrow[d, "\Gamma"] \\
\phi_1 \boxtimes \cdots \boxtimes \phi_n \arrow[r, "\mu"] & \phi_1 \otimes \cdots \otimes \phi_n
\end{tikzcd} \tag{7.16}$$

In particular for a hereditary UFC, each basic morphisms $\mathbf{\Gamma}$ splits into n irreducible factors if its target is of length n.

In terms of the decorated box diagrams diagrams this means that they have at least $n-1$ full vertical lines and for the composition graphs this means that they have can be split into a disjoint union of at least n graphs.

Proof. This follows from the fact that by the conditions the pullback of Φ along μ exists, for all generators.

□

Remark 7.59. If there are factorizable isomorphisms then the interchange relations are compatible with the action of the isomorphisms and the decoration of the generalized diagram can be transferred to the basic diagram in whose class it lies. This effectively divides out all the interchange relations.

In the general case, this procedure yields an equivalence relation, dividing out the possible interchange relation after applying isomorphisms —namely those where the isomorphisms factors. In the general case, the morphisms of $\mathcal{M}^{nc+}$ are classes of diagrams where two diagrams are equivalent if there is a compatibly enumerated basic diagram decorated by morphisms and isomorphisms, which maps to both of the generalized decorated diagrams. For example as in Figure 8.

7.6. Localization and $\mathcal{M}^+$.

Proposition 7.60. *If $\mathcal{M}$ is fully hereditary, then morphisms $Hom_{\mathcal{M}^+}(\phi, \psi)$ are equivalence classes of roofs given by pairs of a decomposition $\phi = \phi_1 \otimes \cdots \otimes \phi_n$ and a morphisms $\mathbf{\Gamma}$. The latter can be taken to be a box diagram with a valid decoration by the morphism $\phi_1, \dots, \phi_n$ and a choice of isomorphisms such that the target is ψ. Or, equivalently, a composition graph with this decoration.*

More generally, if only the isomorphisms are factorizable, the above morphisms generate under the monoidal product and composition.

Proof. By Proposition 3.14 if $\mathcal{M}$ is fully hereditary then the morphisms of $\mathcal{M}^+$ are pairs $(\mu_n, \mathbf{\Gamma})$ where $\mathbf{\Gamma} \in \mathcal{M}^{nc+}$ where $\mu_n : s(\mathbf{\Gamma}) = \phi_1 \boxtimes \cdots \boxtimes \phi_n \to \phi_1 \otimes \cdots \otimes \phi_n$ modulo the equivalence relation on roofs given in Lemma 3.15. Using Proposition 7.56 the first result follows. In the more general case, a morphism in $\mathcal{M}^+$ is a composition that is zig-zag of roofs by definition. □

We can say more for hereditary UFCs. Recall that we fixed $\mathcal{M}$ to be strict that is $Iso(\mathcal{M} \downarrow \mathcal{M}) = \mathcal{P}^{\otimes}$. Hence every $\phi = \phi_1 \otimes \cdots \otimes \phi_n$ uniquely. If one does not assume strictness, all arguments go through up to equivalence by using a presentation. The results will be independent of the presentation. The argument is parallel to Lemma 6.36.

Proposition 7.61. *For a hereditary UFC $\mathcal{M}$ every morphisms in $\mathcal{M}^+$ has an essentially unique representative, $(\mu_n, \mu_m(\mathbf{\Gamma}_1 \boxtimes \cdots \boxtimes \mathbf{\Gamma}_m)P)$ where the source of μ_n is $\phi_1 \boxtimes \cdots \boxtimes \phi_n$, its target is $\phi_1 \otimes \cdots \otimes \phi_n$ with each ϕ_i irreducible, and the $\mathbf{\Gamma}_i$ are $\otimes$–irreducible. This is unique if $\mathcal{M}$ is a strict UFC.*

Proof. Given a morphisms (μ_1, f_1) with $\mu_1 : (\phi_1^1 \boxtimes \cdots \boxtimes \phi_1^{m_1}) \boxtimes \cdots \boxtimes (\phi_n^1 \boxtimes \cdots \boxtimes \phi_n^{m_n}) \to \phi_1 \boxtimes \cdots \boxtimes \phi_n$ in the obvious indexing, decompose ϕ_j^i into irreducibles, let μ_n be the total map to the source and let μ_l be the map that factors the total map as $\mu_n = \mu_1 \mu_l$. The roof (μ_l, id) then gives an equivalence to (μ, f_2) with $f_2 = f_1 \mu_l$ and may assume that μ_n is maximal and thus unique, as any other roof used for an equivalence must have an identity on the left, cf. Lemma 3.15.

Finally, splitting the target $\psi = \psi_1 \otimes \cdots \otimes \psi_m$ into irreducibles and applying Propositions 7.58 and yields the result. □

Using the functor idx, we also have a criterion for a morphisms $\mathbf{\Gamma}$ to be $\otimes$–irreducible. Recall that for a UFC $\mathcal{M}$ after a choice of presentation there is an indexing functor idx : $\mathcal{M} \to \mathcal{C}ospan$, see §2.4. By the functionality of the plus construction there is a graph idx$(\mathbf{\Gamma})$ obtained functorially from $\mathbf{\Gamma}$. This is given by redecorating the underlying box diagram by idx(ϕ_i) and idx(σ_j) for the decoration by morphisms and isomorphisms. In analogy to Proposition 6.39 we have:

Lemma 7.62. *For a hereditary UFC, the underlying graph of a composition graph is the same graph obtained as the image of the morphisms induced by the functor* idx *and hence connectedness can be checked on the level of source and target objects.* □

Proposition 7.63. *For a UFC $\mathfrak{M}$, a morphisms $\mathbf{\Gamma}$ of $\mathcal{M}^{nc+}$ is $\otimes$–irreducible if the composition graph is connected.*

Proof. Assume that $\mathbf{\Gamma}$ is isomorphic to the $\otimes$–product of at least two generators $\Gamma = \mu \circ (\mathbf{\Gamma}' \boxtimes \mathbf{\Gamma}'')$, then $\mathrm{idx}(\mathbf{\Gamma}) = \mu(\mathrm{idx}(\mathbf{\Gamma}) \boxtimes \mathrm{idx}(\mathbf{\Gamma}'))$ whose corresponding graph has at least two components. Vice–versa, assume that $\mathrm{idx}(\mathbf{\Gamma}) = \mu(\mathrm{idx}(\mathbf{\Gamma}) \boxtimes \mathrm{idx}(\mathbf{\Gamma}')$ is reducible. Then tracing through the diagram $\mathbf{\Gamma}$, we see that the pre–images must also be disconnected, as they are simply decorations of the diagrams. □

In summary:

Theorem 7.64. *If $\mathcal{M}$ is a hereditary UFC, then the category $\mathcal{M}^+$ is equivalent to the Feynman category whose underlying groupoid is given by $\mathrm{Iso}(\mathcal{M}\downarrow\mathcal{M})$ and whose basic morphisms out of a given ϕ are given by a decomposition of $\phi \simeq \phi_1 \otimes \cdots \otimes \phi_n$, and a connected composition graph validly decorated by these morphisms and isomorphisms.*

If $\mathcal{M}$ is a not necessarily hereditary UFC these morphisms still generate under $\otimes$ and $\circ$.

Proof. The fact that the morphisms are described by decorated connected composition graph follows from Proposition 7.61 together with Proposition 7.63. This claim about being a FC is then clear as any morphisms of $\mathcal{M}^+$ by Proposition 7.61 is given by a not–necessarily connected decorated composition graph which is the disjoint union of its connected components and the ordered disjoint union acts freely, that is without relations, since this is already true on the underlying graphs.

The last statement is simply that any morphism is represented by a concatenation of roofs. □

Note that the permutation is incorporated into the isomorphism.

Remark 7.65. The exact conditions for a UFC to have a concise description of $\mathcal{M}^+$ will be treated in the future. We expect that the counter example 6.4.5 will play an important role.

7.7. **The gcp and hyp versions.** The gcp and hyp versions can also be described in this formalism. Since the unit was strict adding morphisms using i_{id_X} we have vertical identities at our disposal and like in the proof of Proposition 7.1 create a "brick wall" diagram and compose horizontally. In the graph picture this corresponds to leveling the graph, cf. [Kau21, Appendix B.1.4] for the corresponding leveling of trees. We will use labels i_σ as a morphism decoration to indicate the presence of such a map. E.g. in the box diagram for $\gamma_{\phi,\sigma} \circ (id_\phi \boxtimes i_\sigma)$ will have comprised of two vertically stacked boxes the labels ϕ_1 and i_σ. In particular, as $\gamma_{\phi, id_{s(\phi)}} \circ (id_\phi \boxtimes i_{id_{s(\phi)}}) = id_\phi$ we can introduce horizontal lines into each box and decorate the new boxes with $i_{s(\phi)}$ or $i_{t(\phi)}$. This means that the $\mathbf{\Gamma}$ up to permutations and precomposition with μs can be taken to be the $\overset{\sigma_0}{-} \phi_1 \overset{\sigma_1}{-} \phi_2 \overset{\sigma_2}{-} \cdots \overset{\sigma_{n-1}}{-} \phi_n \overset{\sigma_n}{-}$.

Proposition 7.66. *For $\mathcal{M}$ with factorizable isomorphisms $\mathcal{M}^{nc+,gcp}$ a basic morphism can be written as*

$$\mathbf{\Gamma} = \overset{\sigma_0}{-} \phi_1 \overset{\sigma_1}{-} \phi_2 \overset{\sigma_2}{-} \cdots \overset{\sigma_{n-1}}{-} \phi_n \overset{\sigma_n}{-} \circ [\mu^{m_1} \boxtimes \cdots \boxtimes \mu^{m_n}] \circ P \circ (i \boxtimes id^{\boxtimes}) \tag{7.17}$$

where i is an inclusion of identity maps i_{id_X} applied to added factors of $\mathbb{1}$. That is a basic morphism is specified by the data of a composite generator $\stackrel{\sigma_0}{-}\phi_1\stackrel{\sigma_1}{-}\phi_2\stackrel{\sigma_2}{-}\cdots\stackrel{\sigma_{n-1}}{-}\phi_n\stackrel{\sigma_n}{-}$ *together with decompositions $\phi_l=\phi_l^1\otimes\cdots\otimes\phi_l^{n_1}$ and a permutation P, where the ϕ_l may contain factors of i_{id_X} and P is a permutation rearranging the ϕ_l^j.*

The only relations are moving the insertions of the i_{id_X} which is a choice and may change the level of the l of the ϕ_l^j.

For a fully hereditary $\mathcal{M}$, in particular if $\mathcal{M}$ is part of a hereditary UFC, composite generators $\stackrel{\sigma_0}{-}\phi_1\stackrel{\sigma_1}{-}\phi_2\stackrel{\sigma_2}{-}\cdots\stackrel{\sigma_{n-1}}{-}\phi_n\stackrel{\sigma_n}{-}$ *can be further split according to (6.28).*

NB: The ϕ_l^j decorations do not have any factors of i_σ for $\sigma\neq id_X$.

Proof. Given a box diagram, using the unit constraints and the morphisms $i_{t(\phi_i)}$ or $i_{s(\phi_i)}$ we can insert horizontal lines into each box so to make the number of horizontal lines in any two adjacent boxes match. This means that the diagram can be put into a position where it has full horizontal lines (left to right). Using the interchange relation this means that we can first compose the horizontal and then the vertical compositions. The horizontal compositions at each level are the μ^{n_l} these yield the morphisms ϕ_i and the isomorphisms σ_i.

Since except of the units, there are no labels i_σ there are no relations stemming from the equations (4.8) and (4.9), except for the identities. Thus the statement follows from the definition of $\mathcal{M}^{nc+,gcp}$ together with Proposition 7.56.

The last statement is an application of the hereditary condition. □

Proposition 7.67. *For a hereditary UFC, $(\mathcal{M}^{+,gcp},\otimes)$ is part of a Feynman category whose basic objects are $\mathcal{P}$ and whose basic morphisms of $\mathcal{M}^{+,gcp}$ are given by pairs* $(\mu_m,(\stackrel{\sigma_0}{-}\phi_1\stackrel{\sigma_1}{-}\phi_2\stackrel{\sigma_2}{-}\cdots\stackrel{\sigma_{n-1}}{-}\phi_n\stackrel{\sigma_n}{-})P$ *with connected* $\stackrel{\sigma_0}{-}\phi_1\stackrel{\sigma_1}{-}\phi_2\stackrel{\sigma_2}{-}\cdots\stackrel{\sigma_{n-1}}{-}\phi_n\stackrel{\sigma_n}{-}$ *where $\phi_l=\phi_1^l\otimes\cdots\otimes\phi_{m_l}^l$ with $\phi_i^l\in\mathcal{P}$ or $\phi_i^l=i_{id_X}$, the source of μ_m is $\phi_1^1\boxtimes\cdots\boxtimes\phi_{m_n}^n$ and its target is $\phi_1^n\boxtimes\cdots\boxtimes\phi_{n_m}^m$.*

Proof. By Theorem 7.64 the basic morphisms are given by $(\mu_m,\mathbf{\Gamma}[\mu^{m_1},\dots\mu^{m_n}]P)$ with connected $\mathbf{\Gamma}$. Using the i_{id_X}, we can level $\mathbf{\Gamma}=\stackrel{\sigma_0}{-}\phi_1\stackrel{\sigma_1}{-}\phi_2\stackrel{\sigma_2}{-}\cdots\stackrel{\sigma_{n-1}}{-}\phi_n\stackrel{\sigma_n}{-}$. □

Corollary 7.68. *For a hereditary the basic morphisms of $\mathcal{M}^+$ are represented by connected* leveled *composition graphs decorated by isomorphisms and morphism in $\mathcal{P}$.*

Remark 7.69.

(1) The decoration includes an enumeration of the labels. This enumeration specifies the source of the morphisms: Identifying the enumeration $n_1^1,\dots,n_1^{m_1},\dots,n_1^m,\dots,n_{m_k}^m$ with $1,\dots,m=\sum_l m_l$, the enumeration is an element of $\mathbb{S}_m$, where $m=\sum_{j=1}^k m_j$. This also fixes the source as $\phi=\phi_1\otimes\cdots\otimes\phi_m$.
(2) Two representations yield the same morphism if they represent levelings, by insertion of identities, of the same underlying non–leveled graph. These are the only relations. There is a standard leveling, by inserting all identities at the top.

Proposition 7.70. *For a hereditary UFC the morphisms of $\mathcal{M}^{nc+,hyp}$ and $\mathcal{M}^{+,hyp}$ are the graphs as specified in Proposition 7.66 with the additional restriction that non of the labels $\phi_j^l\in\mathrm{Iso}(\mathcal{M})$.*

Proof. Any occurrence of such a label can be replaced by transforming into into an isomorphisms $(id\Downarrow\sigma)$ or $(\sigma\Downarrow id)$ and hence removed from a standard form. □

Remark 7.71. This leveling is necessary for these morphisms to lead to an indexing. As the corresponding bi–module has to be unital. A morphism with m levels in $\mathcal{M}^{+,gcp}$ will correspond to an element in the m–th level of the nerve. The hyp construction guarantees that the isomorphisms are not counted, just like in the depth. This corresponds to taking the groupoid nerve, as is appropriate according to the general philosophy.

8. Graphical plus construction

In the case where $\mathcal{M}$ is a strict hereditary UFC or Feynman category, the plus construction has a graphical description in more familiar terms. The formalism of graphs we use is that of [BM08, KW17], which is reviewed in Appendix B.3 for convenience. We will then build on this by introducing groupoid-colored graphs expanding on [Kau21, Appendix B]. This will allow us to describe the plus construction of a strict UFC $\mathfrak{M}$ via $Iso(M)$-colored graphs whose vertices are decorated by decorated $\mathcal{P}$. The graphical gcp and hyp constructions will be variations on this idea.

8.1. **The category of groupoid-colored graphs.** For groupoid-colored graphs, it is important to keep track of edge orientations. Recall that an edge is a 2-element set $e = \{f, i(f)\}$. A *directed edge* or *oriented edge* is a choice of order on this set. It will be denoted by $\vec{e} = (f, i(f))$. Note that each edge gives rise to two oriented edges. Given an oriented edge $\vec{e}$, we denote the edge with the opposite orientation by $\overleftarrow{e}$.

A *groupoid-colored graph* for a groupoid $\mathcal{V}$ is a triple $(\Gamma, clr_\Gamma, \boldsymbol{\sigma}_\Gamma)$. The data Γ is an ordinary graph. The data $clr : F \to Obj(\mathcal{V})$ is a function called the *coloring*. The data $\boldsymbol{\sigma}_\Gamma$ assigns to each directed edge $\vec{e} = (f, i(f))$ an isomorphism:

$$\sigma(f, i(f)) : clr(f) \xrightarrow{\sim} clr(i(f)) \tag{8.1}$$

Moreover, we require this to be compatible with reorientation: $\sigma(\vec{e}) = \sigma^{-1}(\overleftarrow{e})$.

A morphism of groupoid-colored graphs $\phi : (\Gamma, clr_\Gamma, \boldsymbol{\sigma}_\Gamma) \to (\Gamma, clr_{\Gamma'}, \boldsymbol{\sigma}_{\Gamma'})$ is a triple $(\phi, \boldsymbol{\sigma}_\phi, \boldsymbol{\tau})$. Here, $\phi : \Gamma \to \Gamma'$ is an ordinary graph morphism. The data $\boldsymbol{\sigma}_\phi$ is a collection of isomorphisms, one for each directed ghost edge $(f, i_\phi(f))$, $\sigma((f, i_\phi(f))) : clr(f) \xrightarrow{\sim} clr(i_\phi(f))$ — again satisfying $\sigma(\vec{e}) = \sigma^{-1}(\overleftarrow{e})$ —, and $\boldsymbol{\tau}$ is a collection of isomorphisms, one for each $f' \in F'$, $\tau(f) : clr(f') \xrightarrow{\sim} clr(\phi^F(f'))$. We call these *flag recolorings*. Again, we obtain a category $\mathcal{V}$-$\mathcal{G}raphs$ of $\mathcal{V}$–colored graphs.

Example 8.1. Let $\mathcal{V}$ be a discrete category, that is $\mathcal{V}$ only has identity morphisms, and let V be the underlying set of objects. Then V-$\mathcal{G}raphs = \mathcal{V}$-$\mathcal{G}raphs$.

Proposition 8.2. *Any functor $F : \mathcal{V} \to \mathcal{V}'$ of groupoids induces a functor*

$$\nu : \mathcal{V}\text{-}\mathcal{G}raphs \to \mathcal{V}'\text{-}\mathcal{G}raphs \tag{8.2}$$

given by $(\Gamma, \nu \circ clr, \nu \circ \boldsymbol{\sigma})$.

Proof. Straightforward. ☐

Example 8.3. In general, there is no functor between the category of graphs colored by the small groupoid $\mathcal{V}$ and the category of graphs colored by the set $\mathrm{Obj}(\mathcal{V})$. However, there is a natural functor $\mathcal{V} \to \mathcal{V}_{iso}$ which sends an object to the isomorphism class of that object. This gives the natural functor: $\mathcal{V}$-$\mathcal{G}raphs \to \mathcal{V}_{iso}$-$\mathcal{G}raphs$.

Example 8.4. The unique functor from the groupoid $\mathcal{V}$ to the trivial category yields a forgetful functor $\mathcal{V}$-$\mathcal{G}raphs \to \mathcal{G}raphs$

8.2. **Plus construction as a graph category.** We will give a version of the plus construction as decoration of a graph. As in [KW17] we will be concerned with the wide subcategories generated by aggregates of corollas. That is $\mathcal{V}$-$\mathcal{A}gg$. In particular, see appendix 5.

Definition 8.5. Define $\mathcal{V}$-$\mathcal{F}^{\text{nu-prop}}$ to be the wide subcategory of directed $\mathcal{V}$-colored aggregates such that such that the morphisms are generated by isomorphisms and morphisms with full directed ghost graphs for basic morphisms. Similarly define $\mathcal{V}$-$\mathcal{F}^{\text{nu-properad}}$ to be the wide subcategory of directed $\mathcal{V}$-colored aggregates such that such that the morphisms are generated by isomorphism and morphisms with full connected directed ghost graphs for basic morphisms.

Proposition 8.6. *For a strict hereditary UFC $\mathcal{M}$ such that* $\mathrm{Iso}(\mathcal{M}) = \mathcal{V}^{\otimes}$, *there is a canonical strong monoidal functor*

$$\mathcal{O}_{\mathcal{M}} : \mathcal{V}\text{-}\mathcal{F}^{\text{nu-properad}} \to \mathcal{S}et \tag{8.3}$$

Proof. Let $\mathcal{V}$-$\mathcal{P}in$ be the subcategory of $\mathcal{V}$-$\mathcal{F}^{\text{properad}}$ where the objects are monoidal products of corollas and the morphisms are isomorphisms.

For each product of groupoid-colored corollas, define the following set

$$Word\left(\coprod_{v\in V} *_v\right) = \prod_{v\in V} Indec\left(\bigotimes_{s\in F_{in}(v)} clr(s), \bigotimes_{t\in F_{out}(v)} clr(t)\right) \tag{8.4}$$

Where $F_{in}(v)$ and $F_{out}(v)$ are the set of in and out flags of the corolla $*_v$. The morphisms of $\mathcal{V}$-$\mathcal{P}in$ act on $Word$ in the following way:

Bijections of flags: For a bijection of flags $b : * \to *'$, let $b_{in}(v) : F'_{in}(v) \to F_{in}(v)$ and $b_{out}(v) : F'_{out}(v) \to F_{out}(v)$ be the induced bijections on in-flags and out-flags. These induce commutativity constraints $C_{in} : \bigotimes_{s'\in F'_{in}} clr(s') \to \bigotimes_{s\in F_{in}} clr(s)$ and $C_{out} : \bigotimes_{t'\in F'_{out}} clr(s') \to \bigotimes_{t\in F_{out}} clr(s)$. Define $\mathcal{O}_{\mathcal{M}}(b) : \mathcal{O}_{\mathcal{M}}(*_0) \to \mathcal{O}_{\mathcal{M}}(*_1)$ to be the following set map:

$$\begin{array}{c} Indec\left(\bigotimes_{s\in F_{in}} clr(s), \bigotimes_{t\in F_{out}} clr(t)\right) \\ \Big\downarrow{\scriptstyle (C_{in}^{-1} \Downarrow C_{out}^{-1})} \\ Indec\left(\bigotimes_{s'\in F'_{in}} clr(s'), \bigotimes_{t'\in F'_{out}} clr(t')\right) \end{array} \tag{8.5}$$

Recolorings: For a label-recoloring $r : * \to \star$, let clr_* denote the color function for $*$ and let $clr_\star$ denote the color function for $\star$. The graph morphism gives basic isomorphisms $\sigma_i : clr_\star(s_i) \to clr_*(s_i)$ for each index i. Likewise, we have isomorphisms $\tau_i : clr_\star(t_j) \to clr_*(t_j)$ for the out-flags. Define $\mathcal{O}_{\mathcal{M}}(r) : \mathcal{O}_{\mathcal{M}}(*) \to \mathcal{O}_{\mathcal{M}}(\star)$ to be the following set map:

$$\begin{array}{c} Indec\left(\bigotimes_i clr_*(s_i), \bigotimes_j clr_*(t_j)\right) \\ \Big\downarrow{\scriptstyle (\bigotimes_i \sigma_1^{-1} \Downarrow \bigotimes_j \tau_j^{-1})} \\ Indec\left(\bigotimes_i clr_\star(s_i), \bigotimes_j clr_\star(t_j)\right) \end{array} \tag{8.6}$$

Bijections of indices: Straightforward, commutativity constraints $\coprod_{v\in V}\to\coprod_{v'\in V'}$ become commutativity constraints $\prod_{v\in V}\to\prod_{v'\in V'}$

For an aggregate Γ with flags F and vertices V, define

$$\mathcal{O}_{\mathcal{M}}(\Gamma) := \int^{\amalg_v *_v\in\mathcal{V}\text{-}\mathcal{P}in} \mathrm{Iso}(\amalg_v *_v,\Gamma)\times Word(\otimes_v *_v) \tag{8.7}$$

using a coend.

The composition $\mathrm{Iso}(\otimes_v *_v,\Gamma)\times\mathrm{Iso}(\Gamma,\Gamma')\to\mathrm{Iso}(\otimes_v *_v,\Gamma')$ determines what the functor $\mathcal{O}_{\mathcal{M}}$ does to isomorphisms.

2-level morphisms For a two–level graph morphism $(\coprod_i *_i)\amalg(\coprod_j *'_j)\to\star$ we have:

$$\mathcal{O}_{\mathcal{M}}\left((\coprod_i *_i)\amalg(\coprod_j *'_j)\right)=\mathcal{O}_{\mathcal{M}}(\coprod_i *_i)\times\mathcal{O}_{\mathcal{M}}(\coprod_j *'_j) \tag{8.8}$$

Hence an element on the left is a pair of morphisms (ϕ,ϕ') written below where the objects (X_i), (Y_j), (Y'_j), and (Z_k) are all basic:

$$\phi\in\mathcal{O}_{\mathcal{M}}(\coprod_i *_i)\subseteq\mathrm{Hom}(X_1\otimes\cdots\otimes X_m, Y_1\otimes\cdots\otimes Y_n)$$
$$\phi'\in\mathcal{O}_{\mathcal{M}}(\coprod_j *'_j)\subseteq\mathrm{Hom}(Y'_1\otimes\cdots\otimes Y'_n, Z_1\otimes\cdots\otimes Z_p)$$

The graph morphism induces a bijection between the out-tails of the aggregate $\coprod_i *_i$ and the in-tails of the aggregate $\coprod_j *'_j$. This gives a commutativity constraint. Moreover, each ghost edge of the morphism corresponds to an isomorphism. All together, this defines an isomorphism $\sigma\in\mathrm{Hom}(Y_1\otimes\cdots\otimes Y_n, Y'_1\otimes\cdots\otimes Y'_n)$. Now define the set map

$$\mathcal{O}_{\mathcal{M}}\left((\coprod_i *_i)\amalg(\coprod_j *'_j)\right)\to\mathcal{O}_{\mathcal{M}}(\star)$$
$$\phi\otimes\phi'\mapsto\phi'\circ\sigma\circ\phi$$

From the definition, this is clearly strong monoidal, so it is an op. □

Proposition 8.7. *Given a strict hereditary UFC $\mathcal{M}$ such that* $\mathrm{Iso}(\mathcal{M})=\mathcal{V}^{\otimes}$*, the* graphical plus construction *is defined by the following decoration:*

$$\mathcal{M}^{G+} := \mathcal{V}\text{-}\mathcal{F}^{\text{properad}}_{dec\mathcal{O}_{\mathcal{M}}} \tag{8.9}$$

Moreover, there is a monoidal equivalence between $\mathcal{M}^+$ and $\mathcal{M}^{G+}$.

Here $\mathcal{F}_{dec\mathcal{O}}$ is the Grothendieck construction for Feynman categories, see [KL17]. The objects are pairs $(X, a_x\in O(X))$ and morphisms are $\phi: X\to Y$ for which $\mathcal{O}(\phi)(a_X)=a_Y$

Proof. As short hand, let $(\Gamma;\Sigma,(\phi_v)_V)$ denote the graph Γ decorated with the class of $(\Sigma,(\phi_v)_V)\in\mathcal{O}_{\mathcal{M}}(\Gamma)$. By the definition of a UFC, each morphism Φ in $\mathcal{M}^+$ can be written as $\Phi=(\sigma\Downarrow\sigma')(\phi_1\otimes\cdots\otimes\phi_n)$ where each ϕ_i is a basic morphisms in $\mathcal{M}^+$. Define a strong monoidal functor $Graph:\mathcal{M}^+\to\mathcal{M}^{G+}$ by

$$Graph(\Phi)=(\amalg_{i=1}^n *_i;(\sigma\Downarrow\sigma'),(\phi)_{i=1}^n) \tag{8.10}$$

Where $*_i$ is the $\mathcal{V}$–colored corolla whose in-flags are colored by the source of ϕ_i and the out-flags are colored by the target of ϕ_i. This is well defined because any other factorization belongs to the same class in $\mathcal{O}_{\mathcal{M}}(\Gamma)$.

The definition on morphisms is straightforward. The *Graph* functor respects inner equivariance, outer equivariance and internal associativity because composition in the category $\mathcal{M}$ is associative. Internal interchange follows from $\mathcal{O}_{\mathcal{M}}$ being a strong monoidal functor.

For the other direction, first pick a functor $Card : \mathcal{F}in\mathcal{S}et \to \mathcal{F}in\mathcal{S}et$ that sends every finite set X to $\{1, \ldots, |X|\}$ and a natural isomorphism $\kappa : \mathrm{id} \Rightarrow Card$. Now, define a monoidal functor $Mor : \mathcal{M}^{G+} \to \mathcal{M}^{+}$ by

Objects: define

$$Mor(\Gamma; \Sigma, (\phi_v)_V) = \Sigma\left(\phi_{\kappa_V^{-1}(1)} \otimes \ldots \otimes \phi_{\kappa_V^{-1}(|V|)}\right) \tag{8.11}$$

Vertex bijections: If B is a graph morphism which is exclusively a bijection of vertices, then $Word(B)$ is the morphism

$$(\Gamma; \Sigma, (\phi_v)_V) \to (B(\Gamma); B \circ \Sigma, (\phi_v)_V) \tag{8.12}$$

Flag isomorphisms: reverse the *Graph* functor.

2–level–contractions: reverse the *Graph* functor.

We can always arrange $\kappa_{\underline{n}} : \underline{n} \to \underline{n}$ to be the identity. In that case, the composition $Mor \circ Graph$ is the identity functor. On the other hand, $Graph(Mor(\Gamma; \Sigma, (\phi_v)_V)) = (\Gamma; \Sigma, (\phi_{\kappa_V^{-1}(i)})_{i=1}^{|V|})$. Let K_V be the vertex bijection induced by $\kappa_V : V \to Card(V)$. This gives a morphism of decorated graphs:

$$(\Gamma; \Sigma, (\phi_v)_V) \to (\Gamma; \Sigma \circ K_V^{-1}, (\phi_{\kappa_V^{-1}(i)})_{i=1}^{|V|}) \tag{8.13}$$

This is a natural isomorphism $\mathrm{id}_{\mathcal{M}^{G+}} \cong Graph \circ Mor$. ☐

Corollary 8.8. *$\mathcal{C}ospan^{+}$ is equivalent to $\mathfrak{F}^{\text{nu-properad}}$*

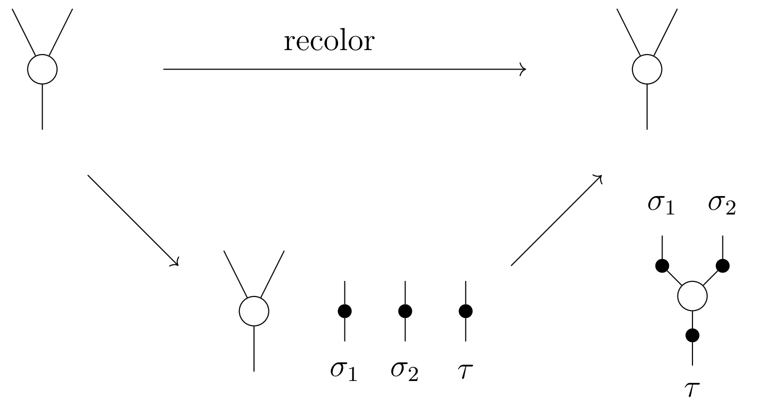
Proof. Let $\mathcal{C}$ denote the skeletal cospan category described in §6.4.3. Since the plus construction respects the principle of equivalence, $\mathcal{C}ospan^{+} \simeq \mathcal{C}^{+}$. From the last result, we know $\mathcal{C}^{+} \simeq \mathcal{C}^{G+}$. The *Indec* sets for $\mathcal{C}$ are always a singleton, so the objects of $\mathcal{C}^{G+}$ are all aggregates and the morphisms are those of $\mathfrak{F}^{\text{properad}}$. ☐

8.3. **Graphical gcp construction.** Define $|^{X}_{Y}$ to be a corolla with one in-tail labeled by X and one out-tail labeled by Y. Define $\mathcal{V}$-$\mathcal{F}^{\text{properad,gcp}}$ by adding morphisms $i_\sigma : \varnothing \to |^{X}_{Y}$ for each basic isomorphism $\sigma : X \to Y$ in $\mathcal{V}$. The presence of these morphisms will be encoded by bivalent vertices $|\!\!\bullet\sigma$. We then quotient by the following relations:

(1) Groupoid compatibility: For isomorphisms $\sigma : X \to Y$, $\tau : X \to X'$ and $\tau' : Y \to Y'$, the following diagram commutes

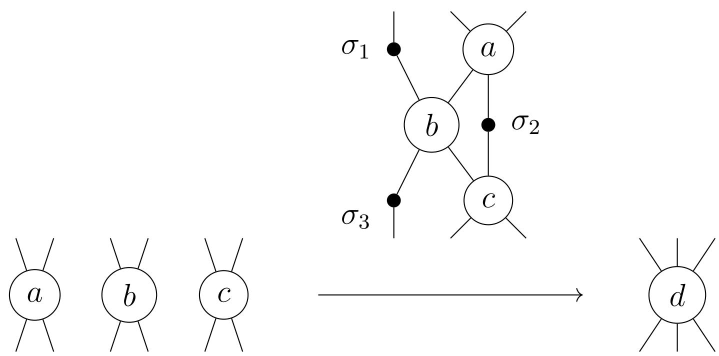
$$\begin{tikzcd} \varnothing \arrow[rr, "i_{\tau^{-1}\sigma\tau'}"] \arrow[dr, "i_\sigma"'] & & |^{X'}_{Y'} \\ & |^{X}_{Y} \arrow[ur, "(\tau\Downarrow\tau')"'] & \end{tikzcd} \tag{8.14}$$

(2) Compatibility with recoloring:



(8.15)

Example 8.9. A typical morphism of $\mathcal{V}$-$\mathcal{F}^{\text{properad,gcp}}$ is given below. The black dots indicate an application of $i_\sigma : \varnothing \to |_Y^X$.



(8.16)

Formally, the ghost graph is still a level graph. However, the added morphisms allow us to insert "scaffolding" into the ghost graph yielding different ways to combine the corollas.

Proposition 8.10. *There is a canonical extension of $\mathcal{O}_\mathcal{M}$:*

$$\mathcal{O}_\mathcal{M}^{gcp} : \mathcal{V}\text{-}\mathcal{P}rop^{gcp} \to \mathcal{S}et \tag{8.17}$$

Proof. Send $i_\sigma : \varnothing \to |_{t(\sigma)}^{s(\sigma)}$ to the pointing $\{\mathrm{id}_\mathbb{1}\} \to \mathcal{O}_\mathcal{M}(|_{t(\sigma)}^{s(\sigma)})$ which selects the element σ. It is clear that this respects the relations imposed on the pointings. Hence this defines a monoidal functor. $\square$

Proposition 8.11. *For a strict hereditary UFC $\mathcal{M}$, define the* graphical gcp construction *as the following decoration:*

$$\mathcal{M}^{G+gcp} := (\mathcal{V}\text{-}\mathcal{F}^{\text{properads,gcp}})_{dec\mathcal{O}_\mathcal{M}^{gcp}} \tag{8.18}$$

There is a monoidal equivalence between $\mathcal{M}^{+gcp}$ and $\mathcal{M}^{G+gcp}$.

Proof. This is a routine modification of the proof of Theorem 8.7. $\square$

Example 8.12. $\mathcal{C}ospan^{+,gcp} = \mathfrak{F}^{\text{properad}}$.

8.4. **Graphical Hyper construction.** Define $\mathcal{V}$-$\mathcal{F}^{\text{properad,hyp}}$ from $\mathcal{V}$-$\mathcal{F}^{\text{properad,gcp}}$ by adding morphisms $r_\sigma : |_{t(\sigma)}^{s(\sigma)} \to \varnothing$ for each basic morphism σ in $\mathcal{V}$ and quotient by the relation $r_\sigma \circ i_\sigma = id_\varnothing$.

Proposition 8.13. *There is a canonical extension of $\mathcal{O}_\mathcal{M}$:*

$$\mathcal{O}^{Hyp}_\mathcal{M} : \mathcal{V}\text{-}\mathcal{P}rop^{Hyp} \to \mathcal{S}et \tag{8.19}$$

Proof. The only option is to send $r_\sigma : |^{s(\sigma)}_{t(\sigma} \to \varnothing$ to the unique map $\mathcal{O}_\mathcal{M}(|^{s(\sigma)}_{t(\sigma)}) \to \{\mathrm{id}_1\}$. This clearly respects the required relations. ☐

Proposition 8.14. *For a strict hereditary UFC $\mathcal{M}$, define the following decoration:*

$$\mathcal{M}^{G+\,Hyp} := (\mathcal{V}\text{-}\mathcal{F}^{properads,hyp})_{dec\mathcal{O}^{Hyp}_\mathcal{M}} \tag{8.20}$$

Moreover, there is a monoidal equivalence between $\mathcal{M}^{Hyp}$ and $\mathcal{M}^{G+\,Hyp}$.

Proof. This is another routine modification of the proof of Theorem 8.7. ☐

8.4.1. **Graphical nc + construction.** One can extend the functors $\mathcal{O}_\mathcal{M}$ and $\mathcal{O}^{gcp}_\mathcal{M}$ above to non–connected graphs by assigning the merger μ where the labels are on the vertices are tensored to together to the extra morphisms μ_{ϕ_1,ϕ_2}.

This leads to a graphical nc plus construction and its gcp and hyp versions. $\mathcal{M}^{G,nc+} := \mathcal{V}\text{-}\mathfrak{F}^{\text{prop}}$ and $\mathcal{M}^{G,nc,gcp/hyp} := \mathcal{V}\text{-}\mathcal{F}^{\text{props,gcp/hyp}}$.

Proposition 8.15. *For a hereditary UFC, there are monoidal equivalences between $\mathcal{M}^{G,nc+}$ and $\mathcal{M}^{nc+}$ as well as between $\mathcal{M}^{G,nc,gpc/hyp}$ and $\mathcal{M}^{nc.gpc/hyp}$.*

Proof. This is again a variation of the construction above. ☐

Example 8.16. $\mathcal{C}ospan^{nc,+} = (\mathfrak{F}^{\text{nu-prop}})^{nc}$ and $\mathcal{C}ospan^{nc+,gcp} = (\mathfrak{F}^{\text{prop}})^{nc}$ which can be merged to $\mathfrak{F}^{\text{nu-prop}}$ and $\mathfrak{F}^{\text{prop}}$, cf. [KW17, Example 3.2.4].

8.4.2. **Summary.**

Theorem 8.17. *There are graphical constructions of $\mathcal{M}^{nc,+}$ and $\mathcal{M}^+$ and their gcp and hyp versions based on groupoid colored decorated graphs, which are equivalent to them.*

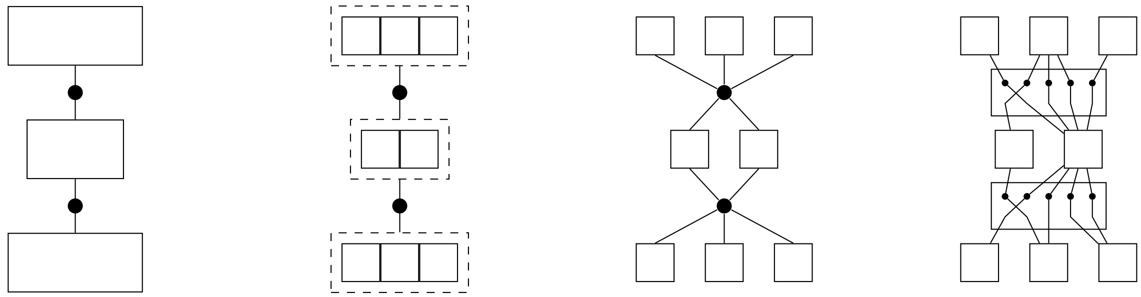
8.5. **Direct conversion.** As one can imagine, there is also a direct construction using the composition graphs. The graphical procedure is as follows: Given a composition graph,

(1) split each edge labeled by $X = X_1 \otimes \cdots \otimes X_k$ where the X_i are irreducible into k edges labeled by the X_i.
(2) split each white vertex labeled by $\phi = \phi_1 \otimes \cdots \otimes \phi_n$, with $\phi_i \in \mathcal{P}$ into n vertices labelled by ϕ_i where the flags are assigned to the vertices according to the sources and targets of the ϕ_i, using that $s(\phi) = s(\phi_1) \otimes \cdots \otimes s(\phi_n)$ an likewise for the target.
(3) Split the black labelled with an isomorphisms $\sigma = \sigma_1 \otimes \cdots \otimes s_n$ in a similar fashion where the $\sigma_i : X_i \to Y_i$ where the $X_i, Y_i \in \mathcal{V}$.

The result is a labelled graph of the type of of the graphical plus constructions, see Figure 12.

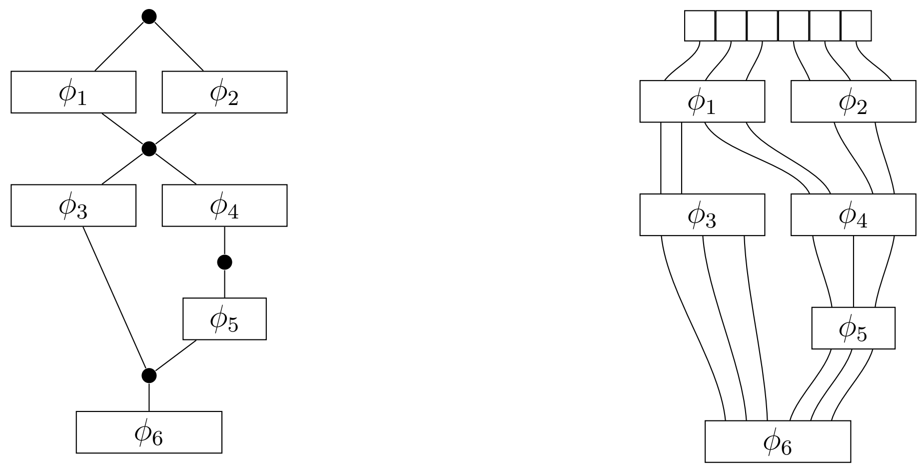
In this interpretation the γ_{ϕ_1,ϕ_2} which correspond to removing horizontal line segments correspond to 2-level contractions.

8.5.1. **Graphical planar version.** Note that the same procedure already works more generally for the suspension graphs, yielding a graphical description in that case as well, see Figure 13.



(A) (B) (C) (D)

FIGURE 12. An example of converting a composition graph into a graph appearing in the graphical plus construction. Here for instructive purposes, we fist split the white vertices splitting the output edges as well, then split the black vertices along with the edges.



(A) (B)

FIGURE 13. Graphical planar version

APPENDIX A. GLOSSARY AND NOTATION

A.1. Monoidal categories and functors.

A.1.1. **Monoidal categories and functors.** We first recall a few definitions and results in a abbreviated fashion referring to the literature (e.g. [Kas95, ML98]) for the full details. Additionally, we also fix notation that will be used in the following.

A *symmetric monoidal category* $(\mathcal{C}, \otimes)$ is a category $\mathcal{C}$ equipped with a functor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ called the *monoidal product*. In particular, this means that for any pair of objects X and Y, there is a new object $X \otimes Y$. Likewise, for any pair of morphisms $\phi : X \to Y$ and $\psi : X' \to Y'$, there is a new morphism $\phi \otimes \psi : X \otimes X' \to Y \otimes Y'$. Moreover, the following properties are satisfied:

(1) There exists a *unit object* $1_{\mathcal{C}}$ together with isomorphisms $\lambda : X \otimes 1_{\mathcal{C}} \xrightarrow{\sim} X$ and $\rho : 1_{\mathcal{C}} \otimes X \xrightarrow{\sim} X$ called *unit constraints*.

(2) There exists a collection of isomorphism $A_{XYX} : (X \otimes Y) \otimes Z \xrightarrow{\sim} X \otimes (Y \otimes Z)$ called *associativity constraints or associators.*
(3) There exists a collection of isomorphism $C_{XY} : X \otimes Y \xrightarrow{\sim} Y \otimes X$ called *commutativity constraints or commutators.*

This data satisfies several conditions: The triangle identity for the units, the pentagon identity for the associators and the hexagon identity for the commutators and in the symmetric case the interchange relation.

$$(\phi \otimes \psi) \circ (\phi' \otimes \psi') = (\phi \circ \phi') \otimes (\psi \circ \psi') \circ C_{s(\psi)s(\phi')} \tag{A.1}$$

and we suppressed the possible associators. Here s is the source map, t denotes the target map.

As a short hand, we will write $(\mathcal{C}, \otimes)$ to indicate that $\mathcal{C}$ is monoidal, instead of listing the full structures. Another short hand is σ_{ii+1} to indicate that the i-th and the $i+1$st position are interchanged.

These categories are called *strict* if the associativity and unit constrants in the data are identities. Using MacLane's coherence, one can show that every monoidal category is equivalent to a strict one. In the following, we will make this assumption in the calculations avoiding an extra layer of complication that can be handled in a standard fashion.

We will call an object or a morphism *reducible* if it is isomorphic to a tensor product of two or more objects or morphisms which are not invertible up to isomorphism with respect to the tensor product. Otherwise it is *irreducible*. Isomorphism for morphisms is meant in the comma category, see below.

A *lax-monoidal functor* f between monoidal categories $\mathcal{C}$ and $\mathcal{E}$ is a functor f together with morphisms $f_{X,Y} : f(X) \otimes f(Y) \to f(X \otimes Y)$ and $f_{\mathbb{1}} : \mathbb{1}_{\mathcal{D}} \to f(\mathbb{1}_{\mathcal{C}})$ satisfying compatibilities called coherences. An op-lax functor has morphisms $f_{X,Y} : f(X \otimes Y) \to f(X) \otimes f(Y)$ and $f^{\mathbb{1}} : f(\mathbb{1}_{\mathcal{C}}) \to \mathbb{1}_{\mathcal{E}}$. Such a functor is called *strong* if the morphisms are isomorphisms. It is *strict*, if the morphisms are identities. We sometimes omit the designation lax. A lax-monoidal functor F between symmetric monoidal categories is *symmetric* if it respects the commutativity constraints, that is satisfies the natural equation $f(c_{X,Y}) \circ f_{X,Y} = f_{Y.X} \circ c_{f(X),f(Y)}$.

We will denote the strong/strict/lax/op-lax monoidal functors between two monoidal categories $\mathcal{M}$ and $\mathcal{M}'$ by $[\mathcal{M}, \mathcal{M}']_{\otimes}$, $[\mathcal{M}, \mathcal{M}']_{strict-\otimes}$ $[\mathcal{M}, \mathcal{M}']_{lax-\otimes}$, respectively $[\mathcal{M}, \mathcal{M}']_{op-\otimes}$.

The *trivial* strong monoidal functor $\mathcal{T} : \mathcal{M} \to \mathcal{M}'$ is defined by $\mathcal{T}(X) = \mathbb{1}_{\mathcal{M}'}$ and $\mathcal{T}(\phi) = \mathrm{id}_{\mathbb{1}_{\mathcal{M}'}}$ with the isomorphisms $f_{X,Y}$ given by the unit constraint of $\mathcal{M}'$: $\mathcal{T}(X) \otimes \mathcal{T}(Y) = \mathbb{1}_{\mathcal{M}'} \otimes \mathbb{1} \to \mathbb{1}_{\mathcal{M}'}$.

Note that one can always restrict to the (essential) image of the functor to make it (essentially) surjective.[1] An indexing is called strong, if it is bijective on objects and surjective on morphisms. It is called strict if it induces an equivalence of $\mathrm{Iso}(\mathcal{C})$ with $\mathrm{Iso}(\mathcal{D})$.

A.1.2. **Ground ring.** Note that $Hom(\mathbb{1})$ plays the role of a ground monoid for the homomorphisms. Whose multiplication is given by the unit constraints. E.g. using the left unit constraint:

$$\mathrm{Hom}(\mathbb{1}, \mathbb{1}) \times \mathrm{Hom}(X, Y) \xrightarrow{\otimes} \mathrm{Hom}(\mathbb{1} \otimes X, \mathbb{1} \otimes Y) \xrightarrow{(\lambda^{*-1}, \lambda_*)} \mathrm{Hom}(X, Y) \tag{A.2}$$

This also holds in the enriched case.

[1] In [Kau21, KW17] the surjectivity was demanded for an indexing.

A.1.3. **Free monoidal categories.** Just like there is a free monoid, there are *free monoidal category* and *free symmetric monoidal categories*. We denote this by $\mathcal{C}^{\boxtimes}$, see [Kau18] for more details and examples. Passing to strict versions, a representation is given by a category whose objects are words (tuples) in the objects of C and the morphisms are words in morphisms. The empty word is the unit: $\mathbb{1}_{\mathcal{C}^{\boxtimes}} = \emptyset$. In the symmetric case, there are also permutations of the letters as extra morphisms, which act in a wreath product fashion.

More generally, a free (symmetric) monoidal category on $\mathcal{C}$ is given by a (symmetric) monoidal category $\mathcal{C}^{\boxtimes}$ and an inclusion $j : \mathcal{C} \to \mathcal{C}^{\boxtimes}$ satisfying the universal property that for any functor $f : \mathcal{C} \to \mathcal{E}$ where $(\mathcal{E}, \otimes)$ a monoidal category, there is a strict (symmetric) monoidal functor $f^{\boxtimes} : \mathcal{C}^{\boxtimes} \to \mathcal{E}$ with $f = f^{\boxtimes} \circ i$ that is unique up to unique isomorphism.

$$\begin{tikzcd} \mathcal{C} \arrow[rr, "f"] \arrow[dr, "i"'] & & \mathcal{E} \\ & \mathcal{C}^{\boxtimes} \arrow[ur, "f^{\boxtimes}"'] & \end{tikzcd} \tag{A.3}$$

The universal property for a free symmetric monoidal is analogous.

NB: A free monoidal category has a natural *length* for objects and for morphisms given by the length of the word, i.e. the number of tensor factors, e.g. if $X = X_1 \boxtimes \cdots \boxtimes X_n$ then $|X| = n$.

Example A.1. For the trivial category $\underline{*}$ with one object and its identity. The free strict monoidal category is the discrete category $\mathbf{N}$ which has objects the natural numbers and only identity morphisms and monoidal structure given by $+$. Here n stands for $*^{\boxtimes n}$. The free strict symmetric monoidal category on $\underline{*}$ is the category with objects $\mathbb{N}_0$, addition as monoidal structure and $Hom(n, n) = \mathbb{S}_n$ the symmetric group and $Hom(n, m) = \emptyset$. The category $\mathbf{N}$ is initial for strict monoidal functors and $\mathbb{S}$ is initial for strict symmetric monoidal functors, as these are fixed by their value on 1 which is necessarily $\mathbb{1}$.

Remark A.2. The strict symmetric monoidal product $\mathcal{C}^{\boxtimes}$ is a type of wreath product or crossed product of the non–symmetric free strict monoidal $\mathcal{C}^{\neg\Sigma\boxtimes}$ and $\mathbb{S}$ in the following sense. Any $\Phi \in Mor(\mathcal{C}^{\boxtimes})$ can be written

$$\Phi = P \circ \bigotimes_{i=1}^{n} \phi_i = \bigotimes_{i=1}^{n} \phi_{P(i)} \circ P \tag{A.4}$$

where the $\phi_i \in Mor(\mathcal{C})$, P is a permutation, and the two parts are unique. This can be seen as a crossed product via the star condition of [FL91].

Example A.3. If $\mathcal{M}$ is a monoidal category, by the universal property of $\mathcal{M}^{\boxtimes}$ the identity functor $id_{\mathcal{M}} : \mathcal{M} \to \mathcal{M}$ defines the functor $\mu = id_{\mathcal{M}}^{\boxtimes} : \mathcal{M}^{\boxtimes} \to \mathcal{M}$. It is given by sending $\boxtimes$ to $\otimes$. I.e. $\mu(X \boxtimes Y) = X \otimes Y$ and $\mu(\phi \boxtimes \psi) = \phi \otimes \psi$.

More generally, if $\mathcal{M}$ is monoidal and f is a strict monoidal functor, then $f^{\otimes}$ factors through μ, i.e. $f^{\otimes} = f \circ \mu$. If f is lax or op-lax then there is a natural transformation from one side to the other, and if it is strong, then the two functors in the equation are equivalent.

Proposition A.4. *There are equivalences of categories* $[\mathcal{C}^{\boxtimes}, \mathcal{E}]_{\otimes} \simeq [\mathcal{C}^{\boxtimes}, \mathcal{E}]_{strict-\otimes} \simeq [\mathcal{C}, \mathcal{E}]$.

Proof. Given a strict or lax functor the restriction to $\mathcal{C}$ is a functor. Vice–versa, given a functor $f : \mathcal{C} \to \mathcal{E}$ extend it to a monoidal functor by $f(X \boxtimes Y) := f(X) \otimes_{\mathcal{E}} f(Y)$ with $f(\emptyset) = \mathbb{1}_E$ using induction on length. This yields a strict functor. This is isomorphic to a given lax functor f by the coherence data. Again using induction on the length, the coherences, $\mathbb{1} \xrightarrow{\sim} f(\emptyset)$ $f_{X,Y} : f(X) \otimes f(Y) \xrightarrow{\sim} f(X \boxtimes Y)$ give the needed natural isomorphisms. $\square$

A.1.4. **Unbiased version.** There is a larger version of the free monoidal category, which is unbiased $\mathcal{C}^{\otimes \mathcal{S}et}$ as explained in [Del90], which is equivalent to the free monoidal category. The unbiased strict symmetric monoidal product contains objects given by S-collections of objects for each finite set S: $\bigotimes_{s \in S} X_s$. The morphisms $\bigotimes_{s \in S} X_s \to \bigotimes_{t \in T} Y_t$ are given by bijections $f : S \to T$ and morphisms $\phi_s : X_s \to Y_{f(s)}$, in particular there are isomorphisms rearranging the factors.

For many purposes it will be useful to work with this larger version. This is natural if there are natrual indexing sets that do not have a canonical order. A typical example being the flags at a vertex in a graph or the boundary components of a surface.

The functor $\mathcal{V}^{\otimes} \to \mathcal{V}^{\otimes \mathcal{S}et}$ given sending $X_1 \otimes \cdots \otimes X_n \to \bigotimes_{i \in \underline{n}} X_i$ induces an equivalence. We will tacitly choose a quasi–inverse once and for all.

The unbiased tensor product has an indexing functor to $\mathrm{Iso}(\mathcal{F}in\mathcal{S}et)$. It sends an object to its indexing set:

$$\mathrm{idx}(\bigotimes_{s \in S} X_s) = S \tag{A.5}$$

and a morphism $\bigotimes_{s \in S} X_s \to Y_t$ to the underlying bijection $f : S \to T$.

The unbracketed expressions have strict associators otherwise one has to additionally introduce brackets for the expressions. For the non–symmetric version, one utilizes ordered finite sets with order preserving maps. An order is most conveniently given by a bijection of $S \leftrightarrow \underline{|S|}$.

A.1.5. **NC category or strings.** Given a monoidal category $(\mathcal{M}, \otimes)$ there is a non–connected (nc) construction, aka. strings, [Bau81, KWZn15, KW17]. This is the category obtained by adjoining the the data of the functor μ, viz. the morphisms $\mu_{X,Y} : X \boxtimes Y \to X \otimes Y$ and the morphism $\epsilon : \mathbb{1}_{\boxtimes} \to \mathbb{1}_{\otimes}$.

Proposition A.5. *There is an equivalence of categories between* $[\mathcal{M}, M]_{lax-\otimes}$ *and* $[\mathcal{M}^{nc}, M]_{\otimes}$.

Proof. Straightforward, see e.g. [Kau21, §3.2.1] for details. $\square$

A.1.6. **Comma categories.** Given two functors $f : \mathcal{C}' \to \mathcal{C}$ and $g : \mathcal{C}'' \to \mathcal{C}$, we denote the comma category by $(f \downarrow g)$, The objects are triples (X, Y, ϕ) with $X \in Obj(\mathcal{C}'), Y \in Obj(\mathcal{C}'')$ and $\phi : f(X) \to g(Y)$. Morphisms from (X, Y, ϕ) to (X', Y', ϕ') are pairs $(\psi : X \to X', \psi' : Y \to Y')$ such that $g(\psi') \circ \phi = \phi' \circ f(\psi)$. If the functors are clear from the context we simply write $(\mathcal{C}' \downarrow \mathcal{C}'')$. E.g. $(\mathcal{C} \downarrow \mathcal{C})$ is $(id_{\mathcal{C}} \downarrow id_{\mathcal{C}})$, where $id_{\mathcal{C}} : \mathcal{C} \to \mathcal{C}$ is the identity functor. Comma categories are functorial. Given a commutative diagram

$$\begin{tikzcd}
\mathcal{C}' \arrow[r, "f"] \arrow[d, "f'"] & \mathcal{C} \arrow[d, "h"] & \mathcal{C}'' \arrow[l, "g"'] \arrow[d, "g'"] \\
\hat{\mathcal{C}}' \arrow[r, "\hat{f}"'] & \hat{\mathcal{C}} & \hat{\mathcal{C}}'' \arrow[l, "\hat{g}"]
\end{tikzcd} \tag{A.6}$$

then there is a functor $(f', h, g') : (f \downarrow g) \to (\hat{f} \downarrow \hat{g})$. On objects it is simply $(f', h, g')(X, \phi, Y) = (f'(X), h(\phi), g'(Y))$,

NB: Comma categories for strong (symmetric) monoidal functors yield (symmetric) monoidal categories; one actually only needs that f is lax and g is colax.

Example A.6. For a category $\mathcal{C}$, the category of arrows is the comma category $(id_{\mathcal{C}} \downarrow \mathrm{id}_{\mathcal{C}})$ which will also be denoted by $(\mathcal{C}\downarrow\mathcal{C})$. The objects are the morphisms of $\mathcal{C}$ and the morphisms in the arrow category from ϕ to ϕ' in $(\mathcal{C}\downarrow\mathcal{C})$ are given by a commutative diagrams:

$$\begin{tikzcd} X \arrow[r, "\phi"] \arrow[d, "l"'] & Y \arrow[d, "r"] \\ X' \arrow[r, "\phi'"] & Y' \end{tikzcd} \quad \text{and are denoted by } (l \Downarrow r) : \phi \to \phi' \tag{A.7}$$

The composition is given by vertical composition of diagrams. $(l \Downarrow r) \circ (l' \Downarrow r') = (l \circ l' \Downarrow r \circ r')$.

For a monoidal category $\mathcal{C} = \mathcal{M}$, the category $(\mathcal{M}\downarrow\mathcal{M})$ is a symmetric monoidal category. Note that even for a monoidal category, the morphisms form a symmetric monoidal category. This is what is used in the interchange law.

On the objects of $(\mathcal{M}\downarrow\mathcal{M})$, the monoidal structure is given by the underlying monoidal structure in $\mathcal{M}$, that is $\otimes(\phi, \psi) = \phi \otimes \psi$. On morphisms the monoidal product is given by

$$(l \Downarrow r) \otimes (l' \Downarrow r') = (l \otimes l' \Downarrow r \otimes r') : \phi \otimes \psi \to \phi' \otimes \psi' \tag{A.8}$$

The monoidal unit is $\mathbb{1}^+ = id_{\mathbb{1}}$ and $id_{\mathbb{1}^+} = (id_{\mathbb{1}} \Downarrow \mathrm{id}_{\mathbb{1}})$. The associativity constraints are given by $A^+_{\phi_0\phi_1\phi_2} = (A_{s(\phi_0)s(\phi_1)s(\phi_2)} \Downarrow A_{t(\phi_0)t(\phi_1)t(\phi_2)})$ and the unit constraints are $\lambda^+_\phi = (\lambda_{s(\phi)} \Downarrow \lambda_{t(\phi)})$ and $\rho^+_\phi = (\rho_{s(\phi)} \Downarrow \rho_{t(\phi)})$. The commutators are given by $C^+_{\phi,\psi}(\phi \otimes \psi) = \psi \otimes \phi$.

Note that if $\mathcal{C}$ is symmetric monoidal, the commutators satisfy

$$C^+_{\phi,\psi} = (C_{t(\phi),t(\psi)} \Downarrow C_{s(\phi),s(\psi)}) \tag{A.9}$$

or in short hand notation $C^+_{12} = (C_{12} \Downarrow C_{12})$, that is $C^+_{12}(\phi \otimes \psi) = C_{12} \circ (\phi \otimes \psi) \circ C_{12} = \psi \otimes \phi$ by the naturality of the commutativity constraints.

Definition A.7. A category $\mathcal{C}$ is called *essentially slice small* if or all objects X of $\mathcal{C}$, the slice category $(\mathcal{C} \downarrow X)$ is essentially small.

A.2. **Groupoids.** A *groupoid* is a category whose morphisms are all isomorphisms. Every category $\mathcal{C}$ has and underlying groupoid $\mathrm{Iso}(\mathcal{C})$ given by restricting the morphisms to only the isomorphisms. If $\mathcal{V}$ is a groupoid then $\mathrm{Iso}(\mathcal{V}) = \mathcal{V}$ and $\mathcal{V}^\otimes = \mathrm{Iso}(\mathcal{V}^\otimes)$ is a groupoid.

Example A.8 (The groupoid of isomorphisms $\mathrm{Iso}(\mathcal{C} \downarrow \mathcal{C})$)**.** The morphisms of $Iso(\mathcal{C} \downarrow \mathcal{C})$ are given by commutative diagrams as in (A.7).

As morphisms $(\sigma \overset{\phi}{\Downarrow} \sigma') : \phi \to \phi' = \sigma' \circ \sigma \circ \sigma^{-1}$. If the context is clear, we will omit ϕ. Note that by the above if $\mathcal{C}$ is (symmetric) monoidal, so is $\mathrm{Iso}(\mathcal{C} \downarrow \mathcal{C})$.

Example A.9 (Associator, unitor and commutator groupoids)**.** In a monoidal category $\mathcal{C}$ there is the subgroupoid $\mathcal{A}ss(\mathcal{C})$ of $\mathrm{Iso}(\mathcal{C})$ generated by all the associators, the unitor sub–groupoid $\mathcal{U}n(\mathcal{C})$ and the subgroupoid generated by both of them $\mathcal{UA}ss(\mathcal{C})$. Likewise in a symmetric monoidal category, there is the subgroupoid $\mathcal{C}om(\mathcal{C})$ generated by $\mathcal{UA}ss(\mathcal{C})$ and all commutators.

In particular for the free symmetric strict monoidal category on any category with only one element $\underline{M}$. We have that $\mathcal{C}om(\underline{M}^\otimes) \simeq \mathbb{S}$ which has objects $n \in \mathbb{N}_0$ corresponding to the $*^{\otimes n}$ and $\mathrm{Aut}(n) = \mathbb{S}_n$ with all other $\mathrm{Hom}(n, m) = \emptyset$ if $n \neq m$.

A.2.1. **Groupoid actions/Modules.** A left action of a groupoid $\mathcal{G}$ is given by a functor $\rho : \mathcal{G} \to \mathcal{C}$ and a right action by a functor $\rho : \mathcal{G}^{op} \to \mathcal{C}$.

This yields an action on objects of $\mathcal{C}$ by morphisms of $\mathcal{G}$ via $\rho : \mathrm{Mor}(\mathcal{G}) \to \mathrm{Mor}(\mathcal{C})$: $\sigma \mapsto \rho(\sigma) : \rho(X) \to \rho(Y)$.

Example A.10. The action of a group G on a set X is given by a functor $\rho : \underline{G} \to \mathcal{S}et$ with $\rho(*) = X$ and $\rho(g) : X \to X$ a set theoretic automorphism (bijection).

If $\mathcal{G}$ is a goupoid, so is $\mathcal{G}^e = \mathcal{G}^{op} \times \mathcal{G}$. A functor $F : \mathcal{G}^e \to \mathcal{C}$ is a left and a right action of $\mathcal{G}$ which commute. Such functors are called $\mathcal{G}$ *bi–modules*.

Example A.11. For the groupoid $Iso(\mathcal{C})$ of a category $\mathrm{Iso}(\mathcal{C})^e$ can be identified with $Iso(\mathcal{C}\downarrow \mathcal{C})$ by identifying (σ, σ') with $(\sigma \Downarrow \sigma')$. The Hom-sets of a category form an $Iso(\mathcal{C})$ bi–module under the action $(\sigma \Downarrow \sigma')$. Explicitly on objects $\mathcal{H}om_{\mathcal{C}}(X,Y) = Hom_{\mathcal{C}}(X,Y)$ and on morphisms $\mathcal{H}om_{\mathcal{C}}(\sigma, \sigma') = (\sigma \Downarrow \sigma') : Hom(s(\sigma), s(\sigma')) \to Hom(t(\sigma), t(\sigma'))$.

If $\mathcal{C}$ is (symmetric) monoidal then by functoriality the actions as functors extend to strong monoidal actions of $\mathcal{G}^{\otimes}$ and $\mathcal{G}^{\otimes} \times (\mathcal{G}^{op})^{\otimes} = (\mathcal{G} \times \mathcal{G}^{op})^{\otimes} = \mathcal{G}^{\otimes} \times (\mathcal{G}^{\otimes})^{op}$. In case that $\mathcal{G}$ and $\mathcal{C}$ are (symmetric) monidal then we also require that ρ is a strong (symmetric) monoidal functor.

Lemma A.12. *The right and left actions commute. Using the notation $\rho(\sigma, \sigma') = (\sigma \Downarrow \sigma')$ the following equation holds:*

$$(\sigma \Downarrow \sigma') = (\sigma \Downarrow id) \circ (id \Downarrow \sigma') = (\mathrm{id} \Downarrow \sigma') \circ (\sigma \Downarrow id) \tag{A.10}$$

If the action is strict monidal, then (A.8) holds.

Proof. Straightforward calculations. □

A.3. **Plethysm product.** There is a natural product, the *plethysm* product for $\mathcal{G}$–bimodules given by the relative monidal product: Given two bimodules F and G set

$$F \otimes_{\mathcal{G}} G = \int^{Y \in \mathcal{G}} F(Y, -) \otimes G(-, Y) \tag{A.11}$$

This coend formula can be expressed via a colimit or coequalizer and implents the quotient under the equivalence relation $(\phi, \psi) \sim (\lambda(\sigma)\phi, \psi\rho(\sigma))$ where λ, ρ are the left and right actions, see [ML98, IX.6].

The groupoid $\mathcal{G}$ as a bi–module with left and right action given by $((\sigma \Downarrow \sigma'))(\tau) = \sigma; \tau\sigma^{-1}$ is a uni.t

In other words, $\mathcal{G}^e$ functors form a monoidal category with the plethysm, aka., relative tensor product.

Proposition A.13. *For a category $\mathcal{C}$ and the functor $Hom : Iso(\mathcal{C})^e \to \mathcal{S}et$ the composition factors through the plethysm product.*

$$\begin{tikzcd}
Hom(-,-)_t \times_s Hom(-,-) \arrow[r, "\circ"] \arrow[d] & Hom(-,-) \\
Hom(-,-) \otimes_{Iso(\mathcal{C})^e} Hom(-,-) \arrow[ur, "\bar{\circ}"] &
\end{tikzcd} \tag{A.12}$$

and hence Hom is an monoid for the plethysm product, which is unital for the unit $\eta : I \to Hom$ given by $\eta_{X,X} = id_x$ viewed as a morphism $\{*\} \to Hom(X, X)$.

Proof. This amounts to the fact that $(\phi_0 \circ \sigma) \circ (\sigma^{-1} \circ \phi_1) = \phi_0 \circ \phi_1$ and the properties of the identities. $\square$

The generalization of this is the theory of indexed enrichment.

NB: The horizontal arrow can also be understood as stemming from the inclusion of $I \to \mathrm{Iso}(\mathcal{C})$ by the universal properties of co-ends.

A.4. **Categories in groupoids.**

A.4.1. **Internal categories.** A category can be given by the following data. Classes of objects and morphisms, together with the source, target and identity maps s, t, id with $s(\mathrm{id}(X)) = t(id(X)) = X$ and an associative unital composition $\circ$:

$$\begin{array}{c} \mathrm{Obj}(\mathcal{C}) \\ s\!\uparrow \downarrow id \uparrow t \\ \mathrm{Mor}(\mathcal{C}) \end{array} \qquad \circ : \mathrm{Mor}(\mathcal{C})_s \times_t \mathrm{Mor}(\mathcal{C}) \to \mathrm{Mor}(\mathcal{C}) \tag{A.13}$$

This formulation is what can be used to define internal categories. A category internal to a category $\mathcal{E}$ is a pair if objects $(\mathrm{Obj}(\mathcal{C}), \mathrm{Mor}(\mathcal{C}))$ of $\mathcal{E}$ together with morphisms $s, t, id, \circ$ in $\mathcal{E}$ as above. This requires that the relative product exists.

For example: a category internal to $\mathcal{S}et$ is a small category. A category in $\mathcal{C}at$ is a double category. We will be interested in categories internal to groupoids. A category $\mathfrak{C}$ internal to groupoids is given by a groupoid of objects $\mathfrak{Obj}(\mathfrak{C})$ a goupoid of morphisms $\mathfrak{Mor}(\mathfrak{C})$, source and target functors $\mathfrak{s}, \mathfrak{t} : \mathfrak{Obj} \to \mathfrak{Mor}$ a section $\mathfrak{id} : \mathfrak{Mor} \to \mathfrak{Obj}$ of both source and target: $\mathfrak{s} \circ \mathfrak{id} = \mathfrak{t} \circ \mathfrak{id} = \mathrm{id}_{\mathfrak{Obj}}$, that is the identify functor of $\mathfrak{Obj}$, and a composition functor $\mathfrak{Mor}_{\mathfrak{s}} \times_{\mathfrak{t}} \mathfrak{Mor} \to \mathfrak{Mor}$.

Example A.14 (Category as category in groupoids)**.** Any category $\mathcal{C}$ gives rise to a category in groupoids, $\mathcal{C}^{\circlearrowleft \mathrm{Iso}(\mathcal{C})}$ with $\mathfrak{Obj} = \mathrm{Iso}(\mathcal{C})$ and $\mathfrak{Mor} = Iso(\mathcal{C} \downarrow \mathcal{C})$ the groupoid with objects $\mathrm{Mor}(\mathcal{C})$ and morphisms $(\sigma \Downarrow \sigma')(\phi)$ which compose as $(\sigma \Downarrow \sigma') \circ (\tau \Downarrow \tau') = (\sigma \circ \tau \Downarrow \sigma' \circ \tau')$. The source and target functors are s, t on the object level of $\mathfrak{Mor}$ and $\mathfrak{s}((\sigma \Downarrow \sigma')) = \sigma$, $\mathfrak{t}((\sigma \Downarrow \sigma')) = \sigma'$ on the morphism level. The identity section is $\mathfrak{id}(X) = X$ and $\mathfrak{id}(\sigma) = (\sigma \Downarrow \sigma)$. All the conditions for the functors are easily checked. As for the composition, this is the functor $\bar{\circ}$ given by plethysm. On the object level of the fiber product $\mathfrak{Mor}_{\mathfrak{s}} \times_{\mathfrak{t}} \mathfrak{Mor} \to \mathfrak{Mor}$: $\bar{\circ}(\phi, \psi) = \phi \circ \psi$ for composable morphisms. On morphisms $\bar{\circ}((\sigma_Y \Downarrow \sigma'_Z)(\phi), (\sigma_X \Downarrow \sigma_Y)(\psi)) = (\sigma_X \Downarrow \sigma_Z)(\phi \circ \psi)$ which is well definied by Proposition A.13.

Example A.15 ($\mathcal{C}$ with an action as a category internal to groupoids)**.** A groupoid action $\rho : \mathcal{G} \to \mathcal{C}$, whose image is a wide subcategory $im(\rho) \subset \mathrm{Iso}(\mathcal{C})$ defines a category in groupoids $\mathcal{C}^{\circlearrowleft \rho}$ whose groupoid of objects is $im(\rho)$ and whose goupoid of morphisms is the wide subgroupoid $\mathfrak{Mor}(\mathcal{C}^{\circlearrowleft \rho}) \subset \mathfrak{Mor}(\mathfrak{C})$ whose morphisms are $im(\rho)^{op} \times im(\rho) \subset \mathcal{G}^{op} \times \mathcal{G}$. This is a subcategory of $\mathrm{Iso}(\mathcal{C})$.

A.4.2. **Functors of categories in groupoids.** A functor $\mathfrak{F} : \mathcal{C} \to \mathcal{D}$ in a category of groupoids is a pair of functors $F_O : \mathfrak{Obj}(\mathcal{C}) \to \mathfrak{Obj}(\mathcal{D})$ and $F_M : \mathfrak{Mor}(\mathcal{C}) \to \mathfrak{Mor}(\mathcal{D})$ satisfying the obvious compatibilities given by upgrading the usual diagram to diagrams of functors between groupoids. Usually, we consider strict functors which means that we are working in the 1-category of categories. If one wishes to consider lax functors, one can work in the 2-category of categories. The functors are then strong if the natural transformations are isomorphisms.

A.4.3. **Monoidal categories in groupoids.** A monoidal category in groupoids is given similar data as before (see §A.1), but where all the morphisms are now functors. E.g. $\otimes : \mathfrak{C}\times\mathfrak{C} \to \mathfrak{C}$ which is a monoidal structure on the categories $\mathfrak{Obj}(\mathfrak{C})$ and $\mathfrak{Mor}(\mathcal{C})$ compatible with $\mathfrak{s}, \mathfrak{t}$ and $\mathfrak{id}$. The unit is then a functor $\mathfrak{u} : \mathfrak{Triv} \to \mathfrak{C}$, where $\mathfrak{Triv}$ is the trivial category in groupoids, it has the morphisms and object groupoids $\underline{*}$ with the identity functor $id_{\underline{*}}$ for source, target and identity and the only possible composition.

Appendix B. Graphs and Feynman categories

Although the notion of graphs we will use is available in several places [BM08, KW17, Kau21], we will recall the basic structures here, as we will generalize them.

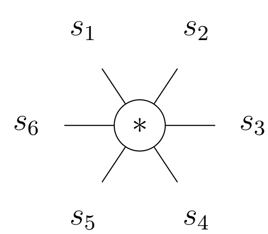
Definition B.1. A *graph* is a tuple $\Gamma = (V_\Gamma, F_\Gamma, \partial_\Gamma, i_\Gamma)$, where:

1. V_Γ is a set whose elements are called *vertices*.
2. F_Γ is a set whose elements are called *flags*
3. $\partial : F \to V$ is a map.
4. $i_\Gamma : F \to F$ is an involution so that $i_\Gamma^2 = id$.

We will adopt the standard graph nomenclature. We say the vertex v is *incident* to the flag f if $\partial(f) = v$. An *edge* $e = \{f, i(f)\}$ is an length 2 orbit of i_Γ. The two flags belonging to an edge are called *half-edges*. We use E_Γ to denote the set of edges. On the other hand, a *tail* is an orbits of length 1. The set of tails is denoted T_Γ. We will drop the subscript Γ if it is clear from the context.

A *loop edge* is an edge $e = \{f_1, f_2\}$ with $\partial(f_1) = \partial(f_2)$. A graph is a tree, if it is contractible. A disjoint union of trees is a forest. A tree with a distinguished tail f_0 is called a *rooted tree*. The *root vertex* of such a rooted tree is the vertex $r_0 = \partial(f_0)$. A graph is called finite if both V and F are finite.

Example B.2 (Corollas)**.** An example of a graph is $*_S := (\{*\}, S, \partial, id)$, where ∂ is the only possible map $S \to \{*\}$. These graphs will be called corollas.



(B.1)

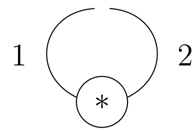
B.1. **Graph morphisms.**

B.1.1. **Definition.** A *morphism of graphs* $\phi : \Gamma \to \Gamma'$ is a triple (ϕ_V, ϕ^F, i_ϕ), where

1. $\phi_V : V_\Gamma \twoheadrightarrow V_{\Gamma'}$ is a surjection.
2. $\phi^F : \mathcal{F}_{\Gamma'} \hookrightarrow \mathcal{F}_\Gamma$ is an injection.
3. i_ϕ is a *fixed point free* involution of $F_\Gamma \setminus \phi^F(F_{\Gamma'})$.

There are several compatibility conditions which are technical to state and put the Appendix. It is possible to glue together two tails to a new edge, to contract an edge and to merge two or more vertices. In fact, one can show that all morphisms factor into a composition of isomorphisms, edge contractions and edge gluings.

Example B.3 (Simple loop)**.** To illustrate the advantages of using this setup, consider the simple loop graph given by $l = (\{*\}, \{1, 2\}, \partial, i)$ with the involution defined as $i(1) = 2$ and $i(2) = 1$.



(B.2)

Then $Aut(l) = \mathbb{Z}/2\mathbb{Z}$, where the non–trivial map is $\phi = (id, \tau_{1,2}, i_\emptyset)$, where τ_{12} is the involution that exchanges 1 and 2, and $i_\emptyset : \emptyset \to \emptyset$.

B.1.2. **Ghost edges.** The set of orbits of i_ϕ, is denoted by $E_{ghost}(\phi)$. All of these orbits are necessarily of order 2, we will call these orbits the *ghost edges*. The terminology comes from that fact that if flags disappear, i.e. are not in the image of ϕ^F, then they must disappear pairwise with the pairing given by i_ϕ. This happens, when one glues together two tails to an edge and then subsequently contracts the edge. In the composed morphism, two of the flags vanished, but not without a trace. This trace is precisely given by the ghost edge. The purpose of i_ϕ is to keep track of this and the conditions force that this is the "only way" that flags can disappear.

We defined the *underlying or ghost graph* of a morphism to be $\mathbb{\Gamma}(\phi) = (V_\Gamma, F_\Gamma, \partial_\Gamma, \hat{\imath}_\phi)$, where $\hat{\imath}_\phi$ is the trivial extension of ϕ to all of F. That is, if $f \notin \phi^F(F')$ then $\hat{\imath}_\phi(f) = f$ and if $f \in \phi^F(F')$ then $\hat{\imath}_\phi(f) = i_\phi(f)$. If the target of ϕ only has one vertex, then $\mathbb{\Gamma}(\phi)$ fixes the isomorphism class of ϕ, see [KW17, §2.1], that is $\phi \simeq \phi'$ if an only if $\mathbb{\Gamma}(\phi) \simeq \mathbb{\Gamma}(\phi')$.

We compose two graphs by combining the ghost edges of both graphs. More formally, define the composition of two graph morphisms $\phi : \Gamma \to \Gamma'$ and $\psi : \Gamma' \to \Gamma''$ as:

(1) $(\psi \circ \phi)_V$ is the composition $V_\Gamma \twoheadrightarrow V_{\Gamma'} \twoheadrightarrow V_{\Gamma''}$.
(2) $(\psi \circ \phi)^F$ is the composition $F_{\Gamma''} \hookrightarrow F_{\Gamma'} \hookrightarrow F_\Gamma$.
(3) $\imath_{\psi\circ\phi}$ is the appropriate restriction to the following involution:

$$\hat{\imath}_{\psi\circ\phi}(f) := \begin{cases} \phi^F(\hat{\imath}_\psi(f')) & \exists f' \in F' \text{ such that } f = \phi^F(f') \\ \hat{\imath}_\phi(f) & \text{otherwise} \end{cases} \tag{B.3}$$

B.2. **The Borisov-Manin category of graphs and subcategories $\mathcal{C}rl$ and $\mathcal{A}gg$.** Together, graphs and graph morphisms form a category. This can be made into a symmetric monoidal category by the disjoint union of graphs:

$$(V, F, \partial, i) \amalg (V', F', \partial', i') := (V \amalg V', F \amalg F', \partial \amalg \partial', i \amalg i') \tag{B.4}$$

The unit is given by the empty graph $\Gamma_\emptyset = \{\emptyset, \emptyset, \mathrm{id}_\emptyset, \mathrm{id}_\emptyset\}$.

Example B.4. An aggregate of corollas is a graph $\Gamma = (V, F, \partial, id)$ which is a disjoint union of corollas $\Gamma = \amalg_{v\in V} v_{F_v}$.

We let $\mathcal{C}rl$ be the subcategory of the graphs $*_S$ with finite S. There is an isomorphism of categories $\mathrm{Iso}(\mathcal{F}in\mathcal{S}et) \to \mathcal{C}rl$ given by $S \to *_S$. The skeleton of $\mathcal{C}rl$ is isomorphic to the groupoid $\mathbb{S}$ whose elements are natural numbers $n \in \mathbb{N}_0$ representing the set $\bar{n} = \{1, \dots, n\}$ with morphisms only being automorphisms and $\mathrm{Aut}(n, n) = \mathbb{S}_n$ is the symmetric group.

We let $\mathcal{A}gg$ be the full sub-category of finite Aggregates. As any corolla is an aggregate, $i : \mathcal{C}rl \to \mathcal{A}gg$ is a subcategory.

This inclusion yields a Feynman category $\mathfrak{F}^{graph} = (\mathcal{C}rl, \mathcal{A}gg, i)$, see [KW17, Proposition 2.1.2].

The subcategory $\mathcal{A}gg^{\text{ctd}}$ of $\mathcal{A}gg$ is defined to be the subcategory whose basic morphisms have connected graphs. This means that the ghost graph ϕ_v^{-1} only has one component, see e.g. [BK22, §1].

Then $\mathfrak{F}^{\text{ctd}}(\mathcal{C}rl, \mathcal{A}gg^{\text{ctd}}, i)$ is a Feyman category [KW17, §2.3.2]

B.3. **The category of oriented graphs.**

B.3.1. **Terminology.** A *directed* or *oriented edge* is a choice of order on this set. An oriented edge will be denoted by $e_{or} = (f, i(f))$, where the subscript maybe dropped if the orientation is clear from the context. Note that each edge gives rise to two oriented edges. Given an oriented edge $e_{or} = (f_1, f_2)$, we denote by $\bar{e}_{or} = (f_2, f_1)$ the edge with the opposite orientation.

A rooted tree is a tree τ together with a marked tail flag $r \in T_\tau$. The set $L_\tau = T_\tau \setminus \{r\}$ will be called the tails. A graph is linear if it is a tree and only has vertices of valence 2. A linear directed graph is a rooted linear graph.

B.3.2. **Directed graphs and their Feynman categories.** A directed graph is graph Γ together with a function $io : F \to \{-1, 1\}$, , such that if $f \neq i(f)$, $io(f) = -i(f)$. This means that all edges are naturally oriented as $(f, i(f))$ with $io(f) = -1$. Here -1 stands for "out" and $+1$ for "in".

For a morphism of directed graphs, we demand that $i_\phi(f) = -i_\phi(f)$. This yields the category of directed graphs.

A corolla is rooted, if it is directed and has exactly one output root.

A rooted tree is naturally oriented by demanding that the root flag r is "out"; $io(r) = -1$.

Using directions defines the Feynman categories [KW17, §2.1.4] $(\mathcal{C}rl^{\text{dir}}, \mathcal{A}gg^{\text{dir}}, i^{\text{dir}})$ of oriented corollas and aggregates and $(\mathcal{C}rl^{\text{dir,ctd}}, \mathcal{A}gg^{\text{dir,ctd}}, i^{\text{dir,ctd}})$ of oriented corollas and connected oriented aggregates. We will denote the corolla with input flags i and output flags j as $*_{i,o}$

B.3.3. **Non–unital props and properads.** A directed graph is *full* if at every vertex either no output flag is a part of an edge or none of them are and either no input flag is part of an edge or no input flag is part of an edge. A two–level contraction is a morphisms of corollas is a basic morphisms of corollas whose ghost graph is full and has two levels. That is it is isomorphic to a map $\phi : a_{i_1,o_1} \amalg a_{i_2,o_2} \to *_{i_1,o_2}$ where a_{i_1,o_1}, a_{i_2,o_2} are two aggregates and the ghost graph is connected and $\imath$ is a bijection between o_1 an i_2

Restricting to the sub–Feynman category generated by these basic morphisms one obtains. $\mathfrak{F}^{\text{nu-prop}} = (\mathcal{C}rl^{\text{dir}}, \mathcal{A}gg^{\text{dir,2-level}}, i)$ and $\mathfrak{F}^{\text{nu-properad}} = (\mathcal{C}rl^{\text{dir}}, \mathcal{A}gg^{\text{dir,2-level,ctd}}, i)$ where in the latter case the ghost graphs of the basic morphisms are also connected. The names come from the fact that the categories of strong monoidal functors out of them are non–unital props and non–unital properads.

Remark B.5. The generators of the generators of $\mathcal{F}^{\text{nu-properads}}$ are two-level-edge contractions. Similarly $\mathcal{F}^{\text{nu-prop}}$ is generated by 2–level contractions and mergers.

B.3.4. **Props and properads.** To add units, one proceeds with adding special black bivalent vertices expressing the presence of units, cf. [KW17, §2.8.2], see also [KW17, §2.2.1] for operads and [Kau21, B.1.4] for the plus construction in Feynman categories. These encode morphisms $u : \mathbb{1} \to id_{*_{1,1}}$ where $*_{1,1}$ is the corolla with vertex $*$, one input and one output. To implement the unit properties these graphs are taken modulo the relation of removing black vertices. The classes are then simply the graphs without special vertices and on extra class, that of [$\bullet\!\!|$] which can be expressed as the loose edge | following [Mar08]. This is the special generator u.

This construction yields the Feyman categories $\mathfrak{F}^{\text{prop}}$ and $\mathfrak{F}^{\text{properads}}$.

B.4. **Graphs with colors.**

B.4.1. **Set-colored graphs.** Colored graphs are a straightforward decoration of graphs, see [KW17, KL17]. For a set C, define a C–colored graph to be a graph Γ together with a morphism $clr : F \to C$ called a *coloring* such that $clr(f) = clr(i(f))$. In other words, the two flags of an edge have the same color.

A morphism of C–colored graphs is a morphism of the underlying graphs whose ghost graph is a C–colored graph. The disjoint union is defined as it was before. Hence, these assemble into a monoidal category C-$\mathcal{G}raphs$ of C–colored graphs.

Example B.6. Let $C = \{*\}$ be a singleton set. Then the category of C–colored graphs is essentially the same as the category of graphs.

Proposition B.7. *Any set map $f : C \to D$ induces a functor*

$$f : C\text{-}\mathcal{G}raphs \to D\text{-}\mathcal{G}raphs \tag{B.5}$$

Corollary B.8. *There is a natural forgetful functor C-$\mathcal{G}raphs \to \mathcal{G}raphs$ induced by the unique surjection $C \twoheadrightarrow \{*\}$.*

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