Centralizers of differential operators of rank $h$

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MPIM 22-61
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Abstract

In the paper “Burchnall-Chaundy bundles” (Lecture Notes in Pure and Appl. Math., 200, Dekker, New York, 1998, pages 377–383) the second author conjectured that the centralizer of a differential operator $L = x^{-n}\delta(n-m)(\delta - 2m)...(\delta - m(n-1))$ where $\delta = x^{\frac{d}{dx}}$ is generated by operators $L$ and $B = x^{-m}\delta(n)(\delta - n)...(\delta - n(m-1))$ and therefore has rank equal to the greatest common divisor $h$ of $m$ and $n$. In this note we will show that this is indeed the case if the ground field $K$ has characteristic zero. Here we restrict ourselves to purely algebraic considerations; the reader interested in geometric aspects and historical background is advised to see the paper mentioned above and the paper ”Centralizers of rank one in the first Weyl algebra” by the first author (SIGMA, 17 (2021), Paper No. 052).

§1. Algebraic background: first Weyl algebra $A_1$ and its skew field of fractions $D_1$

The first Weyl algebra $A_1$ is an algebra over a field $K$ generated by two elements (denoted here by $x$ and $\partial$) which satisfy a relation $\partial x - x \partial = 1$.

When characteristic of $K$ is zero $A_1$ has a natural representation over the ring of polynomials $K[x]$ by operators of multiplication by $x$ and the derivative $\partial$ relative to $x$. Hence the elements of the Weyl algebra can be thought of as differential operators with polynomial coefficients. They can be written as ordinary polynomials: $a = \sum c_{i,j}x^i\partial^j$, $c_{i,j} \in K$ with ordinary addition but a more complicated multiplication.

*Unfortunately during the preparation of this note Emma Previato passed away on June 29, 2022. It is a sad duty of the first author to finish this project.
Algebra $A_1$ is rather small, its Gelfand-Kirillov dimension is 2, hence it is a two-sided Ore ring. Because of that it can be embedded in a skew field $D_1$. A detailed discussion of skew fields related to Weyl algebras and their skew fields of fraction, as well as a definition of Gelfand-Kirillov dimension can be found in a paper [GK].

In this note we will be interested in the subalgebra $B$ of $D_1$ generated by $\partial$ and $x^{-1}$. Both $A_1$ and $B$ are subalgebras of a larger algebra $D$ of differential operators: the elements of $D$ are $\sum_{i=0}^d a_i \partial^i$ where $a_i$ are differentiable functions of $x$.

If characteristic of $K$ is zero then the centralizer $C(a)$ of any element $a \in D \setminus K$ is a commutative subalgebra of $D_1$ of the transcendence degree one (therefore it is a maximal commutative subalgebra of $D_1$). This was established by Issai Schur who proved that this is the case in 1904 using pseudo-differential operators (see [S]) and much later by Harley Flanders (see [F]) and Shimshon Amitsur (see [A]) by purely algebraic means.

The research of commuting differential operators which was started by Georg Wallenberg in 1903 (see [W]), became quite popular about fifty years ago because of its connection to some important partial differential equations.

There is quite a few papers devoted to the centralizers of differential operators in various algebras of differential operators. In characterization of these centralizers an important role is played by a notion of rank. The rank of a centralizer $C(a)$ of a differential operator $a$ is the greatest common divisor of orders of all elements of this centralizer.

If an operator $a$ is given it is often not clear how to compute the rank of $C(a)$. In this note we are concerned with the centralizer of an operator $L = x^{-n} \delta(\delta - m)(\delta - 2m)...(\delta - m(n-1))$ where $\delta = x\partial$ is the Euler operator. It is rather easy to check that $B = x^{-m} \delta(\delta - n)(\delta - 2n)...(\delta - n(m-1))$ commutes with $L$. Since the order of $L$ is $n$ and the order of $B$ is $m$ the rank of $C(L)$ must divide $h = (m, n)$.

We prove in this note that the rank of $C(L)$ is indeed $(m, n)$, as it was conjectured in [P], where a connection between commutative subalgebras of rank $h$ and vector bundles (or coherent sheaves) of rank $h$ is explained.

§2 Proof of the Theorem

Our goal is to prove the following
**Theorem.** The centralizer of the element

\[ L = x^{-n}\delta(\delta - m)(\delta - 2m)...(\delta - m(n - 1)) \in \mathcal{D} \]

belongs to the algebra \( \mathcal{B} = K[x^{-1}, \partial] \) and is generated by \( L \) and

\[ B = x^{-m}\delta(\delta - n)(\delta - 2n)...(\delta - n(m - 1)) \]

if characteristic of the ground field \( K \) is zero. \( \square \)

The proof is based on “computations”, so we start with an example to illustrate these computations and then give a proof of the Theorem.

2.1 The first interesting example is when \( n = 6, \ m = 9 \):

\[ L = x^{-6}\delta(\delta - 9)(\delta - 18)...(\delta - 45), \ B = x^{-9}\delta(\delta - 6)(\delta - 12)...(\delta - 48). \]

From \( \partial x - x\partial = 1 \) we can see that

\[ (\delta + i)x = (x\partial + i)x = x(x\partial + 1) + ix = x(\delta + i + 1), \]

\[ (\delta + i)x^{-1} = x^{-1}(\delta + i - 1), \ (\delta + i)x^j = x^j(\delta + i + j) \text{ and} \]

\[ (\delta + i)^{-1}x^{-1} = x^j(\delta + i + j)^{-1}. \]

Operators \( L, B \in \mathcal{D}_1 \). Hence

\[ D_1 \ni BL^{-1} = [x^{-9}\delta(\delta - 6)(\delta - 12)...(\delta - 48)][x^{-6}\delta(\delta - 9)(\delta - 18)...(\delta - 45)]^{-1} = x^{-9}\delta(\delta - 6)(\delta - 12)...(\delta - 48)\delta^{-1}(\delta - 9)^{-1}(\delta - 18)^{-1}... (\delta - 45)^{-1}x^6 = \]

\[ x^{-3}(\delta + 6)\delta(\delta - 6)...(\delta - 42)(\delta + 6)^{-1}(\delta - 3)^{-1}(\delta - 12)^{-1}...(\delta - 39)^{-1} = x^{-3}\delta(\delta - 6)(\delta - 21)(\delta - 36)(\delta - 42). \]

Denote this element of \( \mathcal{D}_1 \) by \( M \). It is easy to check that \( L = M^2, \ B = L^3. \)

Say, \( M^2 = x^{-3}\delta(\delta - 6)(\delta - 18)(\delta - 24)(\delta - 39)(\delta - 42) x^{-3}\delta(\delta - 6)(\delta - 18)(\delta - 24)(\delta - 36)(\delta - 42) = \]

\[ x^{-6}(\delta^{-3}\delta - 9)(\delta - 21)(\delta - 27)(\delta - 39)(\delta - 45) = \delta^{-1}(\delta - 18)(\delta - 42)(\delta - 39)(\delta - 42). \]

Though \( M \) is not a differential operator, its square and cube are. Of course because of that \( M^i \) is a differential operator if \( i > 1 \).

If the centralizer \( C(L) \neq K[L, B] \) then \( C(L) \ni N = \sum_{j=0}^{d} \nu_j(x)\partial^{d-j} \), a differential operator of order \( d \) not divisible by 3.

Consider an automorphism \( \alpha \) of \( D_1 \) given by \( \partial \to \lambda \partial, \ x \to \lambda^{-1}x \). Since \( \alpha(L) = \lambda^6L \) we can conclude that \( L = \sum_{i=0}^{6} \lambda_i x^{-i} \partial^{i-1} \) (where \( \lambda_0 = 1 \) and \( \lambda_i \in \mathbb{Z} \) because \( \partial x^i = x^i \partial + ix^{i-1} \)).

Since \( LN = NL \) we have an equality

\[ \partial^0\nu_0\partial^i + (\partial^1\nu_1\partial^{i-1} + \lambda_1 x^{-1} \partial \nu_0 \partial^0) + \cdots = \nu_0\partial^i\partial^0 + (\nu_1 \partial^{i-1} \partial^0 + \nu_0 \partial^d \lambda_1 x^{-1} \partial^5) + \cdots \]
Since $\partial^i f(x) = \sum_{j=0}^{i} (\nu_j f^{(j)}) \partial^{i-j}$ (can be easily checked by induction or recall the Leibniz law) we get the following restrictions on $\nu$:

$6\nu_0 \partial^{d+5} = 0$, $\nu_0 = 0$ and we can put $\nu_0 = 1$;

$(6\nu_1 + \lambda_1 dx^{-2}) \partial^{d+4} = 0$, $\nu_1 = -\frac{\lambda_1}{6} x^{-2}$ and $\nu_1 = \mu_{1,1} + \frac{\lambda_1}{6} x^{-1}$ where $\mu_{1,1} \in K$.

The first two coefficients of $N$ are polynomials in $x^{-1}$. We can use induction to prove that all coefficients of $N$ are polynomials in $x^{-1}$. Assume that coefficients $\nu_j$ of $N$ are polynomials in $x^{-1}$ if $j < k$. Recall that the commutator $[a, b]$ denotes $ab - ba$. Since $[L, N] = \sum_{i,j} [\lambda_i x^{-i} \partial^{d-i}, \nu_j(x) \partial^{d-j}]$ the coefficient with $\partial^{d+5-k}$ in $[L, N]$ is $6\nu'_k + \sum_{s=2}^{6} \nu_k^{(s)} + p_k$ where $p_k$ is a linear combination of coefficients with $\partial^{d+5-k}$ in commutators $[\lambda_i x^{-i} \partial^{d-i}, \nu_j(x) \partial^{d-j}]$ where $i > 0$ and $j < k$.

Because $[a, b c] = [a, b][c + b[a, c]$ a commutator $[x^{-i} \partial^{d-i}, \nu_j(x) \partial^{d-j}] = \nu_j(x)[x^{-i}, \partial^{d-j}] \partial^{d-i} x^{-i} \partial^{d-j} \nu_j(x)$ and the corresponding coefficients are linear combinations of $x^{-1}$ where $l > 1$. Thus $\nu'_k = x^{-2} f(x^{-1})$ where $f$ is a polynomial in $x^{-1}$ and an antiderivative of $x^{-2} f(x^{-1})$ is also a polynomial in $x^{-1}$. Therefore $N \in K[x^{-1}, \partial] \subset D_1$.

We can assume without loss of generality that $N = \sum_{i=0}^{d} \xi_j x^{-j} \partial^{d-j}$, $\xi_j \in K$. $N$ can be presented as a sum of a semi-invariants of the automorphism $\alpha$ each of which commutes with $L$. Hence $N = x^{-d} q(\delta)$ where $q$ is a polynomial.

Since $d$ is not divisible by 3 we can find $t_1$, $t_2$, $t_3 \in \mathbb{Z}$ for which $6t_1 + 9t_2 + dt_3 = 1$. Then $P = L^t_1 B^{t_2} N^{t_3} = x^{-1} r(\delta) \in D_1, r \in K(\delta)$.

Now, $MP^{-3} \in K(\delta)$ and commutes with $L$. Therefore $MP^{-3}$ is a constant which is equal to 1 since the leading coefficients of $M$ and $P$ are equal to 1.

We will prove that the rank of $C(L)$ is three and that $C(L) = K(L, B)$ if we show that equality $M = P^3$ is impossible.

If $M = P^3$ then $x^{-3} \frac{\delta(\delta-6)(\delta-18)(\delta-24)(\delta-36)(\delta-42)}{\delta-3} \frac{\delta-21}{\delta-39} = x^{-3} r(\delta) r(\delta-1) r(\delta-2)$ and $\frac{\delta(\delta-6)(\delta-18)(\delta-24)(\delta-36)(\delta-42)}{\delta-21} = r(\delta) r(\delta-1) r(\delta-2)$.

Assume that the ground filed $K$ is algebraically closed and present $r(\delta) = r_0(\delta) r_1(\delta)$ where all roots and poles of $r_0$ are integers and all roots and poles of $r_1$ are not integers. Then $r_1(\delta) r_1(\delta-1) r_1(\delta-2) = c \in K$, hence $r_1(\delta-1) r_1(\delta-2) r_1(\delta-3) = c$ and $r_1(\delta) = r_1(\delta-3)$. Since $r_1$ is a rational function this is possible only if $r_1$ is a constant.

Present now $r_0 = r_00 r_01 r_02$ where all roots and poles of $r_00$ are divisible by 3, all roots and poles of $r_01$ are $\equiv 1 \pmod{3}$, all roots and poles of $r_02$ are $\equiv 2 \pmod{3}$. 


Proof. Base of induction: $\alpha_{00}(\delta)r_{01}(\delta)r_{02}(\delta)r_{00}(\delta - 1)r_{01}(\delta - 2)\alpha_{00}(\delta - 1)r_{02}(\delta - 2)r_{01}(\delta - 2)r_{02}(\delta - 2)\alpha_{00}(\delta - 3)r_{01}(\delta - 3)r_{02}(\delta - 3)\alpha_{00}(\delta - 3) = c_0\alpha_{00}(\delta)r_{01}(\delta - 2)r_{02}(\delta - 1),\ c_0 \in K.$

and $\frac{\alpha_{00}(\delta - 1)r_{01}(\delta)r_{02}(\delta - 2)}{\alpha_{00}(\delta - 2)r_{01}(\delta - 1)r_{02}(\delta)} = c_1 \in K,\ r_{00}(\delta - 2)r_{01}(\delta - 1)r_{02}(\delta) = c_2 \in K,\ c_0c_1c_2 = 1.$

But this is impossible since $3 = \deg_\delta(r_{00}(\delta)r_{01}(\delta - 2)r_{02}(\delta - 1)) = \deg_\delta(r_{00}(\delta - 1)r_{01}(\delta)r_{02}(\delta - 1)) = 0.$ Hence $C(L) \subseteq K[M]$ and has rank 3.

2.2 We are ready to consider the general case

$L = x^{-n}\prod_{i=0}^{n-1}(\delta - im),\ B = x^{-m}\prod_{j=0}^{m-1}(\delta - jn);$

$n = n_1h,\ m = m_1h,\ (n_1, m_1) = 1.$

For the reader’s convenience the proof is split into seven Lemmas (some of which are certainly not new).

**Lemma 1.** $L^m = B^n.$

Proof. $L^m = x^{-mn}\prod_{j=0}^{m-1}(\prod_{i=0}^{n-1}(\delta - im - jn))$ and $B^n = x^{-mn}\prod_{i=0}^{n-1}(\prod_{j=0}^{m-1}(\delta - jn - im))$ since $(\delta + k)x^j = x^j(\delta + k + l).$ □

Remark. Hence $[L, B] = 0.$ Indeed $L^m$ commutes with $L$ and $B$ and by Schur’s theorem $[L, B] = 0.$ □

**Lemma 2.** $L = \sum_{i=0}^{n}\lambda_ix^{-i}\partial_0^{-i}$ where $\lambda_0 = 1.$

Proof. Since $\alpha(L) = \lambda^nL$ where $\alpha$ is an automorphism of $D_1[t, t^{-1}]$ given by $\alpha(\partial) = t\partial,\ \alpha(x) = t^{-1}x,\ \alpha(t) = t$ ($t$ is a central variable), we can conclude that $L = \sum_{i=0}^{n}\lambda_ix^{-i}\partial_0^{-i}$ where $\lambda_0 = 1$ and $\lambda_i \in \mathbb{Z}$ because $\partial x^j = x^j\partial + ix^{-i-1}.$ □

**Lemma 3** (Leibniz law). $[\partial^i, f] = \sum_{j=1}^{i} (i_j^j) f^{(j)}\partial^{i-j}.$

Proof. Base of induction: $[\partial, f] = f'.$ Induction step: $[\partial^{i+1}, f] = [\partial, f]\partial + \partial[f^{(i)}] = f^{(i)}\partial + \partial\sum_{j=1}^{i} (i_j^j) f^{(j)}\partial^{i-j} = \sum_{j=1}^{i+1} (i_{j+1}^j) f^{(j)}\partial^{i+1-j}.$ □

Remark. If $f \in K[\alpha^{-1}]$ and $i > 0$ then $[\partial^i, f] \in x^{-2}B.$ □
Lemma 4. If $E \in \mathcal{B}$ and is a monic differential operator of positive order then $C(E) \subseteq \mathcal{B}$.

Proof. $E = \partial^e + \sum_{i=1}^{e} \epsilon_i \partial^{e-i}, e > 0$ where $\epsilon_i \in K[x^{-1}]$. If $[E,N] = 0$ and $N = \sum_{j=0}^{d} \nu_j(x) \partial^{d-j}$ where $\nu_j$ are differentiable functions of $x$ then $\nu_0 = 0$ and we can assume that $\nu_0 = 1$. Assume further that $\nu_j \in K[x^{-1}]$ for $j < k$. Since $[E, \sum_{j=0}^{k-1} \nu_j(x) \partial^{d-j}] = \sum_{i=0,j=0}^{e,j=k-1} [\epsilon_i \partial^{e-i}, \nu_j(x) \partial^{d-j}] = \sum_{i=0,j=0}^{e,j=k-1} \nu_j(x)[\epsilon_i, \partial^{d-j}] \partial^{e-i} + \sum_{i=0,j=0}^{e,j=k-1} \epsilon_i [\partial^{e-i}, \nu_j(x)] \partial^{d-j} \in x^{-2} B$ and the coefficient with $\partial^{e+d-k-1}$ of $[E,N]$ is $ev_k$ plus the coefficient with $\partial^{e+d-k-1}$ of $[E, \sum_{j=1}^{k-1} \nu_j(x) \partial^{d-j}]$ we can conclude that $\nu_k \in K[x^{-1}]$. □

Remark. $N$ can be represented as a sum of semi-invariants of the automorphism $\alpha$. Hence $C(L)$ has a linear basis consisting of operators $x^{-i}q(\delta)$ where $q(\delta) \in K[\delta]$, $\deg(q) = i$, $q$ is a monic polynomial. □

Lemma 5. Skew field $D_1$ contains such an element $M$ that $L = M^{n_1}, B = M^{m_1}$.

Proof. We can find integers $t_1, t_2$ such that $t_1 n_1 + t_2 m_1 = 1$ since $(n_1, m_1) = 1$. Therefore $M = L^{t_1} B^{t_2} \in D_1$. Since $M = x^{-h} s(\delta), s(\delta) \in K(\delta)$ elements $LM^{-n_1}, BM^{-m_1} \in K(\delta)$ and commute with $L$. Hence $LM^{-n_1}, BM^{-m_1} \in K$. Even more, $LM^{-n_1} = BM^{-m_1} = 1$ since $L$ and $B$ are monic operators. □

Remark. All roots and poles of $s(\delta)$ are integers divisible by $h$ and $\deg(s) = h$. □

Lemma 6. If $C(L) \ni N$, a differential operator of order $d$ then $D_1$ contains an element $P = x^{-(h,d)} r(\delta), r(\delta) \in K(\delta)$, such that $M = P^{(h,d)}$.

Proof. We can present $N$ as a sum of semi-invariants of $\alpha$ all of which commute with $L$ and replace it by a semi-invariant of order $d$. Thus we may assume that $N = x^{-d} q(\delta)$ where $q$ is a monic polynomial of degree $d$. We can find integers $t_3, t_4$ such that $t_3 h + t_4 d = (h,d)$. Then $P = M^{t_3} N^{t_4}$ is the element of $D_1$ we are looking for. □

Lemma 7. If $C(L) \ni N$, a differential operator of order $d$ then $(h,d) = h$.

Proof. If $(h,d) \neq h$ then $M = P^k, k > 1$ and $h = kh_1$. Therefore $s(\delta) = \prod_{i=0}^{k-1} r(\delta - ih_1)$. Assume that the ground filed $K$ is algebraically closed. Then we can write $r(\delta) = r_0(\delta) r_1(\delta)$ where all roots and poles of $r_0$ are integers divisible by $h_1$ and all roots and poles of $r_1$ are not integers divisible by $h_1$. By Remark to Lemma 5 all roots and poles of $s(\delta)$ are divisible by $h$ and
hence by $h_1$. Because of that $\prod_{i=0}^{k-1} r_1(\delta - ih_1) = c \in K$, $\prod_{i=1}^{k} r(\delta - ih_1) = c$ and $r_1(\delta) = r_1(\delta - h)$. Since $r_1$ is a rational function this is possible only if $r_1$ is a constant and $s(\delta) = c^{-1} \prod_{j=0}^{k-1} r_0(\delta - ih_1)$

On the other hand $r_0 = \prod_{j=0}^{k-1} r_{0j}$ where all roots and poles of $r_{0j}$ are $\equiv -jh_1 \pmod{h}$. Hence $s(\delta) = c_0 \prod_{j=0}^{k-1} r_{0j}(\delta - jh_1)$ and, say, $r_{00}(\delta + h_1 - h) \prod_{j=1}^{k-1} r_{0j}(\delta - jh_1 + h_1) = c_1 \in K$. But then $\deg(s) = 0$, i.e. $h = 0$ which is absurd (see Remark to Lemma 5).

A proof of the Theorem is done. Lemma 4 establishes that $C(L) \in \mathcal{B}$ and Lemma 7 shows that the order $d$ of an operator $N \in C(L)$ must be divisible by $h$. Therefore the rank of $C(L)$ is $h$.

Acknowledgments.

The first author is grateful to the Max-Planck-Institut für Mathemtik in Bonn where he is presently a visitor. He was also supported by a FAPESP grant awarded by the State of Sao Paulo, Brazil.

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