Max-Planck-Institut für Mathematik
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Leonid Makar-Limanov
Emma Previato

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Leonid Makar-Limanov
Emma Previato

Max-Planck-Institut für Mathematik
Vivatsgasse 7
53111 Bonn
Germany

Department of Mathematics
Wayne State University
Detroit, MI 48202
USA

Department of Mathematics
& Computer Science
The Weizmann Institute of Science
Rehovot 76100
Israel

Department of Mathematics and Statistics
Boston University
Boston, MA 02215-2411
USA

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L. Makar-Limanov, E.Previato*

Abstract

In the paper "Burchnall-Chaundy bundles" (Lecture Notes in Pure and Appl. Math., 200, Dekker, New York, 1998, pages 377–383) the second author conjectured that the centralizer of a differential operator $L = x^{-n}\delta(\delta - m)(\delta - 2m)...(\delta - m(n-1))$ where $\delta = x\frac{d}{dx}$ is generated by operators L and $B = x^{-m}\delta(\delta - n)(\delta - 2n)...(\delta - n(m-1))$ and therefore has rank equal to the greatest common divisor h of m and n. In this note we will show that this is indeed the case if the ground field K has characteristic zero. Here we restrict ourselves to purely algebraic considerations; the reader interested in geometric aspects and historical background is advised to see the paper mentioned above and the paper "Centralizers of rank one in the first Weyl algebra" by the first author (SIGMA, 17 (2021), Paper No. 052).

§1. **Algebraic background: first Weyl algebra A_1 and its skew field of fractions D_1**

The first Weyl algebra A_1 is an algebra over a field K generated by two elements (denoted here by x and ∂) which satisfy a relation $\partial x - x\partial = 1$.

When characteristic of K is zero A_1 has a natural representation over the ring of polynomials $K[x]$ by operators of multiplication by x and the derivative ∂ relative to x. Hence the elements of the Weyl algebra can be thought of as differential operators with polynomial coefficients. They can be written as ordinary polynomials: $a = \sum c_{i,j}x^i\partial^j,\ c_{i,j} \in K$ with ordinary addition but a more complicated multiplication.

*Unfortunately during the preparation of this note Emma Previato passed away on June 29, 2022. It is a sad duty of the first author to finish this project.

Algebra A_1 is rather small, its Gelfand-Kirillov dimension is 2, hence it is a two-sided Ore ring. Because of that it can be embedded in a skew field D_1. A detailed discussion of skew fields related to Weyl algebras and their skew fields of fraction, as well as a definition of Gelfand-Kirillov dimension can be found in a paper [GK].

In this note we will be interested in the subalgebra $\mathcal{B}$ of D_1 generated by ∂ and x^{-1}. Both A_1 and $\mathcal{B}$ are subalgebras of a larger algebra $\mathcal{D}$ of differential operators: the elements of $\mathcal{D}$ are $\sum_{i=0}^{d} a_i\partial^i$ where a_i are differentiable functions of x.

If characteristic of K is zero then the centralizer $C(a)$ of any element $a \in \mathcal{D} \setminus K$ is a commutative subalgebra of D_1 of the transcendence degree one (therefore it is a maximal commutative subalgebra of D_1). This was established by Issai Schur who proved that this is the case in 1904 using pseudo-differential operators (see [S]) and much later by Harley Flanders (see [F]) and Shimshon Amitsur (see [A]) by purely algebraic means.

The research of commuting differential operators which was started by Georg Wallenberg in 1903 (see [W]), became quite popular about fifty years ago because of its connection to some important partial differential equations.

There is quite a few papers devoted to the centralizers of differential operators in various algebras of differential operators. In characterization of these centralizers an important role is played by a notion of *rank*. The rank of a centralizer $C(a)$ of a differential operator a is the greatest common divisor of orders of all elements of this centralizer.

If an operator a is given it is often not clear how to compute the rank of $C(a)$. In this note we are concerned with the centralizer of an operator $L = x^{-n}\delta(\delta-m)(\delta-2m)...(\delta-m(n-1))$ where $\delta = x\partial$ is the Euler operator. It is rather easy to check that $B = x^{-m}\delta(\delta-n)(\delta-2n)...(\delta-n(m-1))$ commutes with L. Since the order of L is n and the order of B is m the rank of $C(L)$ must divide $h = (m, n)$.

We prove in this note that the rank of $C(L)$ is indeed (m, n), as it was conjectured in [P], where a connection between commutative subalgebras of rank h and vector bundles (or coherent sheaves) of rank h is explained.

§2 **Proof of the Theorem**

Our goal is to prove the following

Theorem. The centralizer of the element

$$L = x^{-n}\delta(\delta - m)(\delta - 2m)...(\delta - m(n-1)) \in \mathcal{D}$$

belongs to the algebra $\mathcal{B} = K[x^{-1}, \partial]$ and is generated by L and

$$B = x^{-m}\delta(\delta - n)(\delta - 2n)...(\delta - n(m-1))$$

if characteristic of the ground field K is zero. $\Box$

The proof is based on "computations", so we start with an example to illustrate these computations and then give a proof of the Theorem.

2.1 The first interesting example is when $n = 6,\ m = 9:$
$L = x^{-6}\delta(\delta - 9)(\delta - 18)...(\delta - 45),\ B = x^{-9}\delta(\delta - 6)(\delta - 12)...(\delta - 48).$

From $\partial x - x\partial = 1$ we can see that
$(\delta + i)x = (x\partial + i)x = x(x\partial + 1) + ix = x(\delta + i + 1),$
$(\delta + i)x^{-1} = x^{-1}(\delta + i - 1),\ (\delta + i)x^j = x^j(\delta + i + j)$ and
$(\delta + i)^{-1}x^j = x^j(\delta + i + j)^{-1}.$

Operators $L,\ B \in D_1$. Hence
$D_1 \ni BL^{-1} = [x^{-9}\delta(\delta-6)(\delta-12)...(\delta-48)][x^{-6}\delta(\delta-9)(\delta-18)...(\delta-45)]^{-1} =$
$x^{-9}\delta(\delta - 6)(\delta - 12)...(\delta - 48)\delta^{-1}(\delta - 9)^{-1}(\delta - 18)^{-1}...(\delta - 45)^{-1}x^6 =$
$x^{-3}(\delta + 6)\delta(\delta - 6)...(\delta - 42)(\delta + 6)^{-1}(\delta - 3)^{-1}(\delta - 12)^{-1}...(\delta - 39)^{-1} =$
$x^{-3}\frac{\delta(\delta-6)}{\delta-3}\frac{(\delta-18)(\delta-24)}{\delta-21}\frac{(\delta-36)(\delta-42)}{\delta-39}.$

Denote this element of D_1 by M. It is easy to check that $L = M^2,\ B = L^3$.
Say, $M^2 = x^{-3}\frac{\delta(\delta-6)}{\delta-3}\frac{(\delta-18)(\delta-24)}{\delta-21}\frac{(\delta-36)(\delta-42)}{\delta-39}x^{-3}\frac{\delta(\delta-6)}{\delta-3}\frac{(\delta-18)(\delta-24)}{\delta-21}\frac{(\delta-36)(\delta-42)}{\delta-39} =$
$x^{-6}\frac{(\delta-3)(\delta-9)}{\delta-6}\frac{(\delta-21)(\delta-27)}{\delta-24}\frac{(\delta-39)(\delta-45)}{\delta-42}\frac{\delta(\delta-6)}{\delta-3}\frac{(\delta-18)(\delta-24)}{\delta-21}\frac{(\delta-36)(\delta-42)}{\delta-39} =$
$x^{-6}\delta(\delta - 9)(\delta - 18)(\delta - 27)(\delta - 36)(\delta - 45).$

Though M is not a differential operator, its square and cube are. Of course because of that M^i is a differential operator if $i > 1$.

If the centralizer $C(L) \neq K[L, B]$ then $C(L) \ni N = \sum_{j=0}^{d}\nu_j(x)\partial^{d-j}$, a differential operator of order d not divisible by 3.

Consider an automorphism α of D_1 given by $\partial \to \lambda\partial,\ x \to \lambda^{-1}x$. Since $\alpha(L) = \lambda^6 L$ we can conclude that $L = \sum_{i=0}^{6}\lambda_i x^{-i}\partial^{6-i}$ (where $\lambda_0 = 1$ and $\lambda_i \in \mathbb{Z}$ because $\partial x^i = x^i\partial + ix^{i-1}$).

Since $LN = NL$ we have an equality
$\partial^6\nu_0\partial^d + (\partial^6\nu_1\partial^{d-1} + \lambda_1 x^{-1}\partial^5\nu_0\partial^d) + \cdots = \nu_0\partial^d\partial^6 + (\nu_1\partial^{d-1}\partial^6 + \nu_0\partial^d\lambda_1 x^{-1}\partial^5) +$

. . . .

Since $\partial^i f(x) = \sum_{j=0}^{i} \binom{i}{j} f^{(j)} \partial^{i-j}$ (can be easily checked by induction or recall the Leibniz law) we get the following restrictions on ν_i:
$6\nu_0' \partial^{d+5} = 0,\ \nu_0' = 0$ and we can put $\nu_0 = 1$;
$(6\nu_1' + \lambda_1 d x^{-2}) \partial^{d+4} = 0,\ \nu_1' = -\frac{\lambda_1 d}{6} x^{-2}$ and $\nu_1 = \mu_{1,1} + \frac{\lambda_1 d}{6} x^{-1}$ where $\mu_{1,1} \in K$.

The first two coefficients of N are polynomials in x^{-1}. We can use induction to prove that all coefficients of N are polynomials in x^{-1}. Assume that coefficients ν_j of N are polynomials in x^{-1} if $j < k$. Recall that the commutator $[a,b]$ denotes $ab - ba$. Since $[L,N] = \sum_{i,j} [\lambda_i x^{-i} \partial^{6-i}, \nu_j(x) \partial^{d-j}]$ the coefficient with ∂^{d+5-k} in $[L,N]$ is $6\nu_k' + \sum_{s=2}^{6} \binom{6}{s} \nu_{k-s+1}^{(s)} + p_k$ where p_k is a linear combination of coefficients with ∂^{d+5-k} in commutators $[\lambda_i x^{-i} \partial^{6-i}, \nu_j(x) \partial^{d-j}]$ where $i > 0$ and $j < k$.

Because $[a, bc] = [a,b]c + b[a,c]$ a commutator $[x^{-i} \partial^{6-i}, \nu_j(x) \partial^{d-j}] = \nu_j(x)[x^{-i}, \partial^{d-j}] \partial^{6-i} + x^{-i} [\partial^{6-i}, \nu_j(x)] \partial^{d-j}$ and the corresponding coefficients are linear combinations of x^{-l} where $l > 1$. Thus $\nu_k' = x^{-2} f(x^{-1})$ where f is a polynomial in x^{-1} and an antiderivative of $x^{-2} f(x^{-1})$ is also a polynomial in x^{-1}. Therefore $N \in K[x^{-1}, \partial] \subset D_1$.

We can assume without loss of generality that $N = \sum_{i=0}^{d} \xi_j x^{-j} \partial^{d-j},\ \xi_j \in K:$ N can be presented as a sum of a semi-invariants of the automorphism α each of which commutes with L. Hence $N = x^{-d} q(\delta)$ where q is a polynomial.

Since d is not divisible by 3 we can find $t_1,\ t_2,\ t_3 \in \mathbb{Z}$ for which $6t_1 + 9t_2 + dt_3 = 1$. Then $P = L^{t_1} B^{t_2} N^{t_3} = x^{-1} r(\delta) \in D_1,\ r \in K(\delta)$.

Now, $MP^{-3} \in K(\delta)$ and commutes with L. Therefore MP^{-3} is a constant which is equal to 1 since the leading coefficients of M and P are equal to 1.

We will prove that the rank of $C(L)$ is three and that $C(L) = K(L,B)$ if we show that equality $M = P^3$ is impossible.

If $M = P^3$ then $x^{-3} \frac{\delta(\delta-6)}{\delta-3} \frac{(\delta-18)(\delta-24)}{\delta-21} \frac{(\delta-36)(\delta-42)}{\delta-39} = x^{-3} r(\delta) r(\delta-1) r(\delta-2)$ and $\frac{\delta(\delta-6)}{\delta-3} \frac{(\delta-18)(\delta-24)}{\delta-21} \frac{(\delta-36)(\delta-42)}{\delta-39} = r(\delta) r(\delta-1) r(\delta-2)$.

Assume that the ground filed K is algebraically closed and present $r(\delta) = r_0(\delta) r_1(\delta)$ where all roots and poles of r_0 are integers and all roots and poles of r_1 are not integers. Then $r_1(\delta) r_1(\delta-1) r_1(\delta-2) = c \in K$, hence $r_1(\delta-1) r_1(\delta-2) r_1(\delta-3) = c$ and $r_1(\delta) = r_1(\delta-3)$. Since r_1 is a rational function this is possible only if r_1 is a constant.

Present now $r_0 = r_{00} r_{01} r_{02}$ where all roots and poles of r_{00} are divisible by 3, all roots and poles of r_{01} are $\equiv 1 \pmod 3$, all roots and poles of r_{02} are are $\equiv 2 \pmod 3$.

Then $\frac{\delta(\delta-6)}{\delta-3}\frac{(\delta-18)(\delta-24)}{\delta-21}\frac{(\delta-36)(\delta-42)}{\delta-39} =$
$r_{00}(\delta)r_{01}(\delta)r_{02}(\delta)r_{00}(\delta-1)r_{01}(\delta-1)r_{02}(\delta-1)r_{00}(\delta-2)r_{01}(\delta-2)r_{02}(\delta-2)$
and $\frac{\delta(\delta-6)}{\delta-3}\frac{(\delta-18)(\delta-24)}{\delta-21}\frac{(\delta-36)(\delta-42)}{\delta-39} = c_0 r_{00}(\delta)r_{01}(\delta-2)r_{02}(\delta-1),\ c_0 \in K.$
$r_{00}(\delta-1)r_{01}(\delta)r_{02}(\delta-2) = c_1 \in K,\ r_{00}(\delta-2)r_{01}(\delta-1)r_{02}(\delta) = c_2 \in K,\ c_0c_1c_2 = 1.$

But this is impossible since
$3 = \deg_\delta(r_{00}(\delta)r_{01}(\delta-2)r_{02}(\delta-1)) = \deg_\delta(r_{00}(\delta-1)r_{01}(\delta)r_{02}(\delta-1)) = 0.$
Hence $C(L) \subset K[M]$ and has rank 3.

2.2 We are ready to consider the general case

$$L = x^{-n}\prod_{i=0}^{n-1}(\delta - im),\ B = x^{-m}\prod_{j=0}^{m-1}(\delta - jn);$$

$n = n_1h,\ m = m_1h,\ (n_1, m_1) = 1.$

For the reader's convenience the proof is split into seven Lemmas (some of which are certainly not new).

Lemma 1. $L^m = B^n$.
Proof. $L^m = x^{-mn}\prod_{j=0}^{m-1}(\prod_{i=0}^{n-1}(\delta - im - jn))$ and $B^n = x^{-mn}\prod_{i=0}^{n-1}(\prod_{j=0}^{m-1}(\delta - jn - im))$ since $(\delta + k)x^l = x^l(\delta + k + l)$. □
Remark. Hence $[L, B] = 0$. Indeed L^m commutes with L and B and by Schur's theorem $[L, B] = 0$. □

Lemma 2. $L = \sum_{i=0}^{n}\lambda_i x^{-i}\partial^{n-i}$ where $\lambda_0 = 1$.
Proof. Since $\alpha(L) = \lambda^n L$ where α is an automorphism of $D_1[t, t^{-1}]$ given by $\alpha(\partial) = t\partial,\ \alpha(x) = t^{-1}x,\ \alpha(t) = t$ (t is a central variable), we can conclude that $L = \sum_{i=0}^{n}\lambda_i x^{-i}\partial^{n-i}$ where $\lambda_0 = 1$ and $\lambda_i \in \mathbb{Z}$ because $\partial x^i = x^i\partial + ix^{i-1}$. □

Lemma 3 (Leibniz law). $[\partial^i, f] = \sum_{j=1}^{i}\binom{i}{j}f^{(j)}\partial^{i-j}$.
Proof. Base of induction: $[\partial, f] = f'$. Induction step: $[\partial^{i+1}, f] = [\partial, f]\partial^i + \partial[\partial^i, f] = f'\partial^i + \partial\sum_{j=1}^{i}\binom{i}{j}f^{(j)}\partial^{i-j} = \binom{i}{0}f'\partial^i + \sum_{j=1}^{i}\binom{i}{j}(f^{(j)}\partial + f^{(j+1)})\partial^{i-j} = \sum_{j=1}^{i+1}\binom{i+1}{j}f^{(j)}\partial^{i+1-j}$. □
Remark. If $f \in K[x^{-1}]$ and $i > 0$ then $[\partial^i, f] \in x^{-2}B$. □

Lemma 4. If $E \in \mathcal{B}$ and is a monic differential operator of positive order then $C(E) \subset \mathcal{B}$.
Proof. $E = \partial^e + \sum_{i=1}^{e} \epsilon_i \partial^{e-i},\ e > 0$ where $\epsilon_i \in K[x^{-1}]$. If $[E, N] = 0$ and $N = \sum_{j=0}^{d} \nu_j(x)\partial^{d-j}$ where ν_j are differentiable functions of x then $\nu_0' = 0$ and we can assume that $\nu_0 = 1$. Assume further that $\nu_j \in K[x^{-1}]$ for $j < k$. Since $[E, \sum_{j=0}^{k-1} \nu_j(x)\partial^{d-j}] = \sum_{i=0,j=0}^{i=e,j=k-1} [\epsilon_i \partial^{e-i}, \nu_j(x)\partial^{d-j}] = \sum_{i=0,j=0}^{i=e,j=k-1} \nu_j(x)[\epsilon_i, \partial^{d-j}]\partial^{e-i} + \sum_{i=0,j=0}^{i=e,j=k-1} \epsilon_i[\partial^{e-i}, \nu_j(x)]\partial^{d-j} \in x^{-2}B$ and the coefficient with $\partial^{e+d-k-1}$ of $[E, N]$ is $e\nu_k'$ plus the coefficient with $\partial^{e+d-k-1}$ of $[E, \sum_{j=0}^{k-1} \nu_j(x)\partial^{d-j}]$ we can conclude that $\nu_k \in K[x^{-1}]$. □
Remark. N can be represented as a sum of semi-invariants of the automorphism α. Hence $C(L)$ has a linear basis consisting of operators $x^{-i}q(\delta)$ where $q(\delta) \in K[\delta],\ \deg(q) = i,\ q$ is a monic polynomial. □

Lemma 5. Skew field D_1 contains such an element M that $L = M^{n_1},\ B = M^{m_1}$.
Proof. We can find integers $t_1,\ t_2$ such that $t_1n_1 + t_2m_1 = 1$ since $(n_1, m_1) = 1$. Therefore $M = L^{t_1}B^{t_2} \in D_1$. Since $M = x^{-h}s(\delta),\ s(\delta) \in K(\delta)$ elements $LM^{-n_1},\ BM^{-m_1} \in K(\delta)$ and commute with L. Hence $LM^{-n_1},\ BM^{-m_1} \in K$. Even more, $LM^{-n_1} = BM^{-m_1} = 1$ since L and B are monic operators. □
Remark. All roots and poles of $s(\delta)$ are integers divisible by h and $\deg(s) = h$. □.

Lemma 6. If $C(L) \ni N$, a differential operator of order d then D_1 contains an element $P = x^{-(h,d)}r(\delta),\ r(\delta) \in K(\delta)$, such that $M = P^{\frac{h}{(h,d)}}$.
Proof. We can present N as a sum of semi-invariants of α all of which commute with L and replace it by a semi-invariant of order d. Thus we may assume that $N = x^{-d}q(\delta)$ where q is a monic polynomial of degree d. We can find integers $t_3,\ t_4$ such that $t_3h + t_4d = (h, d)$. Then $P = M^{t_3}N^{t_4}$ is the element of D_1 we are looking for. □

Lemma 7. If $C(L) \ni N$, a differential operator of order d then $(h, d) = h$.
Proof. If $(h, d) \neq h$ then $M = P^k,\ k > 1$ and $h = kh_1$. Therefore $s(\delta) = \prod_{i=0}^{k-1} r(\delta - ih_1)$. Assume that the ground filed K is algebraically closed. Then we can write $r(\delta) = r_0(\delta)r_1(\delta)$ where all roots and poles of r_0 are integers divisible by h_1 and all roots and poles of r_1 are not integers divisible by h_1. By Remark to Lemma 5 all roots and poles of $s(\delta)$ are divisible by h and

hence by h_1. Because of that $\prod_{i=0}^{k-1} r_1(\delta - ih_1) = c \in K$, $\prod_{i=1}^{k} r(\delta - ih_1) = c$ and $r_1(\delta) = r_1(\delta - h)$. Since r_1 is a rational function this is possible only if r_1 is a constant and $s(\delta) = c^{-1} \prod_{i=0}^{k-1} r_0(\delta - ih_1)$

On the other hand $r_0 = \prod_{j=0}^{k-1} r_{0j}$ where all roots and poles of r_{0j} are $\equiv -jh_1 \pmod h$. Hence $s(\delta) = c_0 \prod_{j=0}^{k-1} r_{0j}(\delta - jh_1)$ and, say, $r_{00}(\delta + h_1 - h) \prod_{j=1}^{k-1} r_{0j}(\delta - jh_1 + h_1) = c_1 \in K$. But then $\deg(s) = 0$, i.e. $h = 0$ which is absurd (see Remark to Lemma 5). $\square$

A proof of the Theorem is done. Lemma 4 establishes that $C(L) \in \mathcal{B}$ and Lemma 7 shows that the order d of an operator $N \in C(L)$ must be divisible by h. Therefore the rank of $C(L)$ is h.

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Department of Mathematics, Wayne State University, Detroit, MI 48202, USA;
Department of Mathematics & Computer Science, the Weizmann Institute of Science, Rehovot 76100, Israel.

E-mail address: lml@wayne.edu

Department of Mathematics and Statistics, Boston University, Boston, MA 02215-2411, USA.