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ON SIMPLE LEFT-SYMMETRIC ALGEBRAS

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ABSTRACT. We prove that the multiplication algebra $M(A)$ of any simple finite-dimensional left-symmetric nonassociative algebra A over a field of characteristic zero coincides with the right multiplication algebra $R(A)$. In particular, A does not contain any proper right ideal. These results immediately give a description of simple finite-dimensional Novikov algebras over an algebraically closed field of characteristic zero [29].

The structure of finite-dimensional simple left-symmetric nonassociative algebras from a very narrow class $\mathcal{A}$ of algebras with the identities $[[x,y],[z,t]] = [x,y]([z,t]u) = 0$ is studied in detail. We prove that every such algebra A admits a $\mathbb{Z}_2$-grading $A = A_0 \oplus A_1$ with an associative and commutative A_0. Simple algebras are described in the following cases: (1) A is four dimensional over an algebraically closed field of characteristic not 2, (2) A_0 is an algebra with the zero product, and (3) A_0 is simple; in the last two cases, the description is given in terms of root systems. A necessary and sufficient condition for A to be complete is given.

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1. **Introduction**

An algebra A over a field F is called *left-symmetric* (or *pre-Lie*) if it satisfies the identity

$$(xy)z - x(yz) = (yx)z - y(xz). \tag{1.1}$$

This means that the associator $(x, y, z) := (xy)z - x(yz)$ is symmetric with respect to two left arguments, i. e.,

$$(x, y, z) = (y, x, z). \tag{1.2}$$

Left-symmetric algebras arise in many different areas of mathematics and physics (for example, see [7]).

The variety of left-symmetric algebras is Lie-admissible, i. e., each left-symmetric algebra A with the operation $[x, y] := xy - yx$ is a Lie algebra. We denote this Lie algebra by $A^{(-)}$ and call it the *adjoint* Lie algebra of A.

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A linear basis for free left-symmetric algebras was given by D. Segal in 1994 [21]. The identities of left-symmetric algebras were studied by V. Filippov [10], and he proved that any left-nil left-symmetric algebra over a field of characteristic zero is left nilpotent. An analogue of the PBW basis Theorem for the universal (multiplicative) enveloping algebra of a right-symmetric algebra was given in [14]. The Freiheitssatz and the decidability of the word problem for one-relator right-symmetric algebras were proven in [15].

The left-symmetric Witt algebras $\mathcal{L}_n$ [25] are one of the most important series of infinite-dimensional simple left-symmetric algebras over fields of characteristic zero. These algebras are very convenient to describe some famous problems of affine algebraic geometry, including the Jacobian Conjecture, in purely ring theoretic terms [25]. Some results on the identities of the left-symmetric Witt algebras $\mathcal{L}_n$ are proven in [16].

The class of left-symmetric algebras is a wide extension of the class of associative algebras, and it contains the class of assosymmetric algebras, Novikov algebras, and $(-1,0)$-algebras. Recall that an *assosymmetric* algebra is a left-symmetric algebra, which is right-symmetric as well, i. e., it also satisfies the identity

$$(x, y, z) = (x, z, y).$$

In 1957 E. Kleinfeld [12] proved that if R is an assosymmetric ring of characteristic different from 2 and 3 and without zero-product ideals then R is associative. A *Novikov algebra* is a left-symmetric algebra with commuting right multiplications, i. e., the Novikov algebras satisfy the identity $(xy)z = (xz)y$ in addition to the left-symmetric identity (1.1). In 1987 E. Zelmanov [29] proved that any finite-dimensional simple Novikov algebra over an algebraically closed field of characteristic zero is one-dimensional. V. Filippov constructed a wide class of simple Novikov algebras of characteristic $p \geq 0$ [9]. J. Osborn [17, 18, 19] and X. Xu [27, 28] continued the study of simple finite-dimensional Novikov algebras over fields of positive characteristic and simple infinite-dimensional Novikov algebras over fields of characteristic zero. A complete classification of finite-dimensional simple Novikov algebras over algebraically closed fields of characteristic $p > 2$ is given in [27]. Some interesting results on the structure of nilpotent, solvable, and Lie solvable Novikov algebras were recently obtained in [22, 24, 26, 31, 30].

The class of $(-1,0)$-algebras is a part of the class of (γ,δ)-algebras introduced by A. Albert [1]. It is well known [13] that every simple finite-dimensional algebra of type $(-1,0)$ of characteristic not equal to 2 and 3 is associative.

In contrast to assosymmetric algebras, Novikov algebras, and $(-1,0)$-algebras, the class of simple (finite-dimensional) non-associative left-symmetric algebras is immense. For example, as it was shown in [20], starting from an arbitrary (finite-dimensional) nontrivial left-symmetric algebra A, one can construct a simple (finite-dimensional) left-symmetric algebra, which contains A as a subalgebra.

There exist infinitely many non-isomorphic simple left-symmetric structures on the Lie algebra gl_n [5]; they are classified in [5] as deformations of the associative matrix algebra structure. A classification of 2 and 3-dimensional simple left-symmetric algebras over $\mathbb{C}$ was obtained in [6]. Classification of 4-dimensional simple left-symmetric algebras are already quite complicated. However, it is feasible for complete left-symmetric algebras

[6]. Recall that a left-symmetric algebra A is called *complete* if the operator $Id + R(x)$ is bijective for all $x \in A$ (this condition arises naturally in the context of affine transformations).

It is well known that the adjoint Lie algebra of a left-symmetric algebra cannot be semisimple [4] and the adjoint Lie algebra of a simple left-symmetric algebra cannot be nilpotent [6]. There are many examples of simple left-symmetric algebras with solvable and reductive adjoint Lie algebras. The adjoint Lie algebra of a complete left-symmetric algebra is always solvable [3].

This paper is devoted to the study of simple finite-dimensional left-symmetric algebras over algebraically closed fields of characteristic zero. We prove that the multiplication algebra $M(A)$ of any simple finite-dimensional left-symmetric nonassociative algebra A over a field of characteristic zero coincides with the right multiplication algebra $R(A)$ and A is an irreducible $R(A)$-module. In particular, A does not contain any proper right ideal. Recall that a similar result holds for $(-1, 0)$ and $(1, 1)$-algebras (see [13]). Moreover, these results can be immediately applied to get the description of simple finite-dimensional Novikov algebras over an algebraically closed field of characteristic zero given in [29].

The remaining part of the paper is focused on the study of finite-dimensional simple left-symmetric nonassociative algebras from a very narrow variety $\mathfrak{M}$ of algebras with the identities $[[x, y], [z, t]] = [x, y]([z, t]u) = 0$. We establish that in some sense $\mathfrak{M}$ is the smallest reasonable variety of the left-symmetric algebras such that $\mathfrak{M}$ contains nontrivial finite-dimensional simple algebras. We show that even this smallest class contains a huge number of simple algebras. We prove that every simple finite-dimensional algebra $A \in \mathfrak{M}$ admits a $\mathbb{Z}_2$-grading $A = A_0 \oplus A_1$ with an associative and commutative A_0. Simple algebras are described in the following cases: (1) A is four dimensional, (2) A_0 is an algebra with the zero product, and (3) A_0 is simple; in the last two cases the description is given in terms of root systems. A necessary and sufficient condition for A to be complete is given.

The paper is organized as follows. In the preliminary Section 2 we give some constructions of ideals of left-symmetric algebras. In Section 3 we prove that the multiplication algebra of any simple finite-dimensional left-symmetric nonassociative algebra coincides with the right multiplication algebra and show that such an algebra is right simple. In Section 4 we define a very small variety of algebras $\mathfrak{M}$ such that $\mathfrak{M}$ contains simple finite-dimensional Lie-metabelian algebras, and we define the class of simple algebras $\mathcal{A}$ in $\mathfrak{M}$. In particular, every algebra A in $\mathcal{A}$ admits a $\mathbb{Z}_2$-grading $A = A_0 \oplus A_1$. In Section 5 we give a necessary and sufficient condition for $A \in \mathcal{A}$ to be complete. In Section 6 we study root decompositions for algebras in $\mathcal{A}$. In Section 7, using the obtained results, we give a complete description of simple four-dimensional algebras in $\mathcal{A}$. Section 8 is devoted to the study of algebras $A \in \mathcal{A}$ when either A_0 is an algebra with the zero product or A_0 is simple.

2. **Preliminaries**

Let A be an arbitrary left-symmetric algebra over a field F. Given $a \in A$, we define the operators $L_a : x \mapsto ax$ and $R_a : x \mapsto xa$ of the *left* and *right multiplication*, respectively. By (1.1) we get

$$[L_x, L_y] = L_{[x,y]} \tag{2.1}$$

and

$$[L_x, R_y] = R_{xy} - R_y R_x. \tag{2.2}$$

Let $\mathrm{End}(A)$ be the algebra of linear mappings of the vector space A. The subalgebra $M = M(A)$ of $\mathrm{End}(A)$ that is generated by the operators L_a and R_a, where $a \in A$, is called the *multiplication* algebra of A. The *left* multiplication algebra $L = L(A)$ and the *right* multiplication algebra $R = R(A)$ are some subalgebras of $M(A)$ generated by the operators L_a and R_a, respectively, where a ranges over A.

Lemma 2.1. *Let A be a left-symmetric algebra, and let $Ann_l(A) = \{x \in A : xA = 0\}$. Then $Ann_l(A)$ is an ideal of A.*

Proof. It suffices to prove that $Ann_l(A)$ is a left ideal of A. Take $x \in Ann_l(A)$ and $a \in A$. Then for every $b \in A$ we have

$$(bx)a = (b, x, a) = (x, b, a) = (xb)a - x(ba) = 0$$

by (1.2). Therefore, $bx \in Ann_l(A)$. Consequently, $Ann_l(A)$ is an ideal of A. □

Lemma 2.2. *$RL \subseteq LR + R$ and $LR + R$ is an ideal of M.*

Proof. Notice that every element of L is a linear combination of elements of the form

$$u = L_{x_1} \ldots L_{x_n}, n \geq 1,$$

and every element of R is a linear combination of elements of the form

$$v = R_{y_1} \ldots R_{y_m}, m \geq 1.$$

Using (2.2) we can represent the product vu as a linear combination of elements of the form

$$L_{a_1} \ldots L_{a_k} R_{b_1} \ldots R_{b_s}, s \geq 1.$$

Consequently, $RL \subseteq LR + R$ and $LR + R$ is an ideal of M. □

Lemma 2.3. *Let I be an ideal of R such that $[L_x, I] \subseteq I$ for all $x \in A$. Then $K = LI + I$ is an ideal of M.*

Proof. By Lemma 2.2,

$$RK = R(LI + I) \subseteq RLI + RI \subseteq LRI + RI + I \subseteq K,$$

since I is an ideal of R. Clearly, $LK \subseteq K$. Hence, K is a left ideal of M.

Show that K is a right ideal of M. For any $x \in A$ we get

$$IL_x \subseteq L_x I + [L_x, I] \subseteq L_x I + I.$$

Therefore,

$$KL_x \subseteq LIL_x + IL_x \subseteq LL_xI + LI + L_xI + I \subseteq K.$$

Clearly, $KR \subseteq K$, since I is an ideal of R. Hence, K is a right ideal of M. This proves that K is an ideal of M. □

Corollary 2.1. *If e is a central idempotent of R then $LRe + Re$ is an ideal of M and e is a central idempotent of M.*

Proof. We have

$$[L_x, e] = [L_x, e^2] = e[L_x, e] + [L_x, e]e = 2[L_x, e]e$$

for all $x \in A$, since $[L_x, e] \in R$ by (2.2). Consequently,

$$[L_x, e]e = 2[L_x, e]e^2 = 2[L_x, e]e.$$

Hence, $[L_x, e]e = 0$ and $[L_x, e] = 0$. Thus, e is a central idempotent of M. Moreover, Re is an ideal of R and

$$[L_x, Re] \subseteq [L_x, R]e + R[L_x, e] \subseteq Re.$$

Hence, $LRe + Re$ is an ideal of M. □

3. The multiplication algebra of a simple left-symmetric algebra

We may assume that A is a left M-module regarding the action $w \cdot a = w(a)$, where $w \in M, a \in A$. Similarly, we can consider A as a left R-module. Obviously, A is a faithful M-module and A is a faithful R-module.

Recall that an arbitrary algebra A is *simple* if A does not contain nontrivial ideals and $A^2 \neq 0$.

Now, let A be a simple finite-dimensional left-symmetric algebra over a field F. Then its multiplication algebra M is a matrix algebra over a skew-field. Hence, $M = LR + R$ by Lemma 2.2. Let e be the identity element of M and let $B = (id - e) \cdot A$. Obviously, $w \cdot B = 0$ for all $w \in M$. Consequently, B is an ideal of A and either $B = 0$ or $B = A$, since A is simple. If $B = A$ then we get $A^2 = 0$. Hence, $B = 0$, and A is a unitary M-module.

Let $C_M(R)$ be the *centralizer* of the subalgebra R in M, i.e.,

$$C_M(R) = \{x \in M : [x, a] = 0 \ \forall a \in R\}.$$

Lemma 3.1. *Let J be the Jacobson radical of R. Then the following assertions hold:*

(1) *if $[L_x, J] \subseteq J$ for all $x \in A$ then $J = 0$, R is a simple subalgebra of M, and R contains the identity element of M;*

(2) *if F is an algebraically closed field then $M \cong R \otimes C_M(R)$;*

(3) *if F is of characteristic zero then $[L_x, J] \subseteq J$ for every $x \in A$.*

Proof. Assume that $J \neq 0$. Then $LJ + J$ is a nonzero ideal of M by Lemma 2.3, since $[L_x, J] \subseteq J$ for all $x \in A$. Hence, $M = LJ + J$, since M is simple. The Jacobson radical J

of the finite-dimensional algebra R is nilpotent. Suppose that $J^n = 0$ and $J^{n-1} \neq 0$. Then $MJ^{n-1} \subseteq LJ^n + J^n = 0$. Consequently, $J^{n-1} = 0$. This contradiction implies $J = 0$.

Therefore,

$$R = R_1 \oplus \ldots \oplus R_k$$

is the direct sum of some simple algebras.

Let e be the identity element of R_1. Then e is a central idempotent of R. Set $K = LRe + Re$. By Corollary 2.1, K is an ideal of M and e is a central idempotent of M. Therefore, $M = LRe + Re$, since M is simple. Hence, e is the identity element of M, and $R = R_1$, i. e., R is simple.

Let F be an algebraically closed field. Then the center $Z(R)$ of R coincides with F. Therefore, $M \cong R \otimes C_M(R)$ by the coordinatization theorem [11].

Let F be a field of characteristic zero. By (2.2), $[L_x, R] \subseteq R$ for all $x \in A$, and $\mathrm{ad}(L_x) : R \to R$, which maps r into $[L_x, r]$, is a derivation of R. It is well known that the Jacobson radical is closed under derivations in characteristic zero [2] (see also [23]). Hence, $[L_x, J] \subseteq J$ for all $x \in A$. □

Notice that an arbitrary algebra satisfies the identity

$$a(b, c, d) - (ab, c, d) - (a, b, cd) + (a, bc, d) + (a, b, c)d = 0, \tag{3.1}$$

and every left-symmetric algebra satisfies the identity

$$(a, b, c) = [ab, c] - a[b, c] - [a, c]b. \tag{3.2}$$

Theorem 3.1. *Let A be a finite-dimensional simple left-symmetric algebra over an algebraically closed field F of characteristic zero. Then either A is associative or $R = M = M_n(F)$, where $n = \dim_F A$, and A is a simple R-module.*

Proof. By Lemma 3.1, A is a unitary R-module, and R is a simple finite-dimensional algebra. Therefore,

$$A = A_1 \oplus \ldots \oplus A_m$$

is the direct sum of some irreducible R-modules. Notice that A_i is a right ideal of A.

Assume that $m > 1$. If $i \neq j$ then

$$(A_i, A_j, A) = (A_j, A_i, A) \subseteq A_i \cap A_j = 0.$$

By (3.1),

$$A_i(A_j, A_j, A) \subseteq (A_iA_j, A_j, A) + (A_i, A_jA_j, A) + (A_i, A_j, A_jA) + (A_i, A_j, A_j)A \subseteq$$

$$(A_i, A_j, A) + (A_i, A_j, A)A = 0.$$

Therefore, $R_{(A_j,A_j,A)} \subseteq Ann_R(A_i)$. Since $Ann_R(A_i)$ is an ideal of R and R is simple by Lemma 3.1; therefore, either $Ann_R(A_i) = R$ or $Ann_R(A_i) = 0$. Clearly, $Ann_R(A_i) \neq R$. Hence, $Ann_R(A_i) = 0$ and $R_{(A_j,A_j,A)} = 0$, i. e., $A(A_j, A_j, A) = 0$ for all $j = 1, \ldots, m$. Consequently,

$$A(A, A, A) \subseteq \sum_{ij} A(A_i, A_j, A) = 0.$$

Applying (3.1) again, we get

$$(A, A, A)A \subseteq A(A, A, A) + (A, A, A) \subseteq (A, A, A).$$

Thus, (A, A, A) is an ideal of A. Therefore, either $(A, A, A) = 0$ or $(A, A, A) = A$.

If $A = (A, A, A)$ then we get $A^2 = A(A, A, A) = 0$. Consequently, $(A, A, A) = 0$, i. e., A is an associative algebra.

Hence, if A is not associative then $m = 1$. Consequently, A is an irreducible R-module. Let c be a nonzero element in $C_M(R)$. Note that $c \cdot A$ is an R-submodule of the R-module A. Since A is a faithful and irreducible R-module; therefore, $c \cdot A = A$. Consequently, $C_M(R)$ is a skew-field. Taking into account that $C_M(R)$ is finite-dimensional and F is an algebraically closed field we get $C_M(R) = F$. By Lemma 3.1 we obtain $R = M$. □

Corollary 3.1. *Every finite-dimensional simple left-symmetric algebra over an algebraically closed field of characteristic zero does not contain any nontrivial right ideal.*

Theorem 3.1 immediately implies Zel'manov's result [29] on finite-dimensional simple Novikov algebras of characteristic 0.

Corollary 3.2. [29] *Let N be a finite-dimensional simple Novikov algebra over a field F of characteristic zero. Then N is a field.*

Proof. By Lemma 3.1, the right multiplication algebra $R = R(N)$ is simple. This implies that R is a field, since R is commutative in the case of Novikov algebras.

Let $x \in N$. Then the map $w \in R \mapsto [L_x, w] \in R$ is a derivation of R. Let $w \in R$. Let $f(t) \in F[t]$ be a polynomial of minimal degree such that $f(w) = 0$. Then $f'(t) = \frac{df}{dt} \neq 0$ and $f'(w) \neq 0$. On the other hand, $0 = [L_x, f(w)] = f'(w)[L_x, w]$. Consequently, $[L_x, w] = 0$ for all $w \in R$. Hence, $R_{xy} - R_y R_x = [L_x, R_y] = 0$ for all $x, y \in N$ by (2.2). Therefore, $(z, x, y) = (R_y R_x - R_{xy})(z) = 0$, i. e., N is a simple associative algebra. Then N possesses a unity. Since $R_x R_y = R_y R_x$ for all $x, y \in N$; therefore, $xy = yx$. Thus, N is a field. □

4. The class $\mathcal{A}$ of simple Lie-metabelian algebras

Lemma 4.1. *Let A be a left-symmetric algebra over a field F. Then $I = [A, A] + [A, A]A$ is an ideal of A.*

Proof. By (3.2), we have

$$IA \subseteq [A, A]A + ([A, A]A)A \subseteq [A, A]A + ([A, A], A, A) \subseteq$$

$$[A, A]A + [[A, A]A, A] + [A, A][A, A] + [[A, A], A]A \subseteq [A, A] + [A, A]A \subseteq I.$$

Consequently, I is a right ideal of A. Since $AI \subseteq [A, I] + IA$, I is a left ideal of A. □

In this section, we always assume that A is a finite-dimensional simple left-symmetric nonassociative algebra over an algebraically closed field F of characteristic 0. Denote by $\mathfrak{g} = A^{(-)}$ the adjoint Lie algebra of A by $\mathfrak{g} = A^{(-)}$. It is well known [6] that $\mathfrak{g}$ cannot be nilpotent. But there exist many examples of simple algebras with solvable $\mathfrak{g}$ [6]. We also assume that $\mathfrak{g}$ is a solvable Lie algebra.

For a subspace V of A, we set

$$L_V = \{L_x : x \in V\}.$$

Lemma 4.2. *There exists a natural number n such that $L^n_{[A,A]} = 0$ and $L^{n-1}_{[A,A]} \neq 0$. Furthermore,*

$$A = \sum_{i=0}^{n-1} L^i_{[A,A]}[A, A].$$

Proof. By (2.1), L_A is a Lie subalgebra of $M = M(A)$ and the map $\mathfrak{g} \to L_A$ that is defined by $x \mapsto L_x$ is an epimorphism of Lie algebras. Consequently, L_A is solvable. By the Lie theorem [8], $[L_A, L_A] = L_{[A,A]}$ is nilpotent. Assume that $L^n_{[A,A]} = 0$ and $L^{n-1}_{[A,A]} \neq 0$ for some natural n.

We have $[A, A] \neq 0$, since A is nonassociative. By Lemma 4.1, we get

$$A = [A, A] + [A, A]A.$$

Therefore,

$$A \subseteq [A, A] + [A, A]([A, A] + [A, A]A) \subseteq [A, A] + L_{[A,A]}[A, A] + L_{[A,A]}L_{[A,A]}A.$$

Continuing this process we obtain

$$A = \sum_{i=0}^{n-1} L^i_{[A,A]}[A, A].$$

□

Corollary 4.1. *The algebra A cannot contain an identity element.*

Proof. By Lemma 4.2, we may assume that $L^n_{[A,A]} = 0$ and $L^{n-1}_{[A,A]} \neq 0$.

Let e be the identity element of A. Then

$$L^{n-1}_{[A,A]}[A, A] \subseteq L^{n-1}_{[A,A]}L_{[A,A]}(e) = L^n_{[A,A]}(e) = 0.$$

Then, by Lemma 4.1, we get

$$L^{n-1}_{[A,A]}A = L^{n-1}_{[A,A]}[A, A] = 0.$$

Hence, $L^{n-1}_{[A,A]} = 0$, which is a contradiction. □

Corollary 4.2. *The space $[A, A]$ is left nilpotent but not nilpotent.*

Proof. By Lemma 4.2, $[A, A]$ is left nilpotent. Suppose that $[A, A]^k = 0$ and $[A, A]^{k-1} \neq 0$. Lemma 4.1 implies that $[A, A]^{k-1}$ is contained in the left annihilator of A and Lemma 2.1 gives that $[A, A]^{k-1} = 0$. □

Taking into account these results we define a reasonable minimal class of simple finite-dimensional left-symmetric algebras with solvable adjoint Lie algebras such that it contains a nonassociative algebra. Let $\mathcal{A}$ be the class of all simple finite-dimensional left-symmetric nonassociative algebras over an algebraically closed field F of characteristic 0 satisfying the identities

$$[[x, y], [z, t]] = 0 \tag{4.1}$$

and

$$[x, y]([z, t]u) = 0. \tag{4.2}$$

Thus, if $A \in \mathcal{A}$ then $\mathfrak{g} = A^{(-)}$ is a metabelian Lie algebra by (4.1). The metabelian Lie algebras form a minimal solvable variety of Lie algebras that is not nilpotent. Note that (4.2) is equivalent to $[x, y]([z, t][u, v]) = 0$ for $A \in \mathcal{A}$, and it can be rewritten also as $L^2_{[A,A]} = 0$.

Proposition 4.1. *Let $A \in \mathcal{A}$. Set $A_0 = [A, A]^2$, $A_1 = [A, A]$. Then the following assertions hold*:

(i) $A = A_0 \oplus A_1$ is a $\mathbb{Z}_2$-graded algebra;

(ii) $A_1A_0 = 0$, $[A_1, A_1] = 0$, $A_0 = A_1^2$, and $A_1 = A_0A_1$;

(iii) A_0 is an associative commutative algebra and A is an associative right A_0-module;

(iv)

$$a(xy) = (ax)y + x(ay) \tag{4.3}$$

for all $a \in A_0$ and $x, y \in A_1$.

Proof. We have $L^2_{[A,A]} = 0$ by (4.2). Then Lemma 4.2 implies that $A = A_0 + A_1$. We have $A_1^2 = [A, A][A, A] = A_0$. We get $A_1A_0 = 0$ by (4.2) and $[A_1, A_1] = 0$ by (4.1). Obviously,

$$A_0A_1 \subseteq [A_0, A_1] + A_1A_0 \subseteq [A_0, A_1] \subseteq [A, A] = A_1.$$

By (2.1), we obtain

$$A_0^2 \subseteq A_0(A_1A_1) \subseteq A_1(A_0A_1) + [A_0, A_1]A_1 \subseteq A_1A_1 = A_0.$$

Consequently, A_0 is a subalgebra of A.

Set $I = A_0 \cap A_1$. Then $IA_0 \subseteq A_0A_0 \subseteq A_0$, and $IA_0 \subseteq A_1A_0 \subseteq A_1$. Therefore, $IA_0 \subseteq I$. Similarly, $IA_1 \subseteq I$. Consequently, I is a right ideal of A. Analogously, I is a left ideal of A. Since A is simple, either $A = I$ or $I = 0$.

If $A = I$ then $A = A_0 = A_1$. Since $A_1A_0 = 0$, $A^2 = A_1A_0 = 0$. Therefore, $A_0 \cap A_1 = I = 0$. Thus, $A = A_0 \oplus A_1$ is a $\mathbb{Z}_2$-graded algebra.

Since $A^2 = A$ and $A_1A_0 = 0$, we have $A = A_0^2 + A_1^2 + A_0A_1$. Consequently, $A_1 = A_0A_1$.

Take arbitrary $a, b \in A_0$. Then $[a, b] \in A_0 \cap A_1 = 0$. Hence, A_0 is a commutative algebra, whence A_0 is associative by (3.2).

Since $A_1A_0 = 0$, we get

$$(A, A_0, A_0) = (A_0, A_0, A_0) + (A_1, A_0, A_0) = 0.$$

Thus, A is an associative A_0-module.

Now, let $a \in A_0$ and $x, y \in A_1$. Then

$$a(xy) = x(ay) + (ax)y$$

by (1.2), since $xa = 0$. □

5. **A bilinear form and complete left-symmetric algebras**

Let A be a finite-dimensional left-symmetric algebra. Consider the symmetric bilinear form

$$f(x,y) = \mathrm{tr}(R_x R_y)$$

on A. By (2.2), we have

$$\mathrm{tr}(R_{xy}) = \mathrm{tr}([L_x, R_y] + R_y R_x) = \mathrm{tr}(R_y R_x) = \mathrm{tr}(R_x R_y).$$

Therefore, $\mathrm{tr}(R_{xy}) = \mathrm{tr}(R_{yx})$. Consequently, $\mathrm{tr}(R_{[x,y]}) = 0$ for all $x, y \in A$.

Lemma 5.1. *For all $a, b, c \in A$ we have*

$$f([a,b],c) = f(a,bc) - f(b,ac).$$

Proof. By definition, $f(ab,c) = \mathrm{tr}(R_{(ab)c})$. By (3.2),

$$\begin{aligned} f(ab,c) &= \mathrm{tr}(R_{(ab)c}) = \mathrm{tr}(R_{a(bc)}) + \mathrm{tr}(R_{[ab,c]}) - \mathrm{tr}(R_{a[b,c]}) - \mathrm{tr}(R_{[a,c]b}) \\ &= f(a,bc) - f(a,[b,c]) - f([a,c],b) = f(a,cb) - f([a,c],b). \end{aligned}$$

Consequently, $f([a,c],b) = f(a,cb) - f(c,ab)$. □

Let $T(A) = \{x \in A : \mathrm{tr}(R_x) = 0\}$. The largest left ideal of A which is contained in $T(A)$ is called the *radical* of A, and it is denoted by $\mathrm{rad}\,(A)$. A left-symmetric algebra A is called *complete* if $A = \mathrm{rad}\,(A)$.

Lemma 5.2. *Let $A \in \mathcal{A}$ and let $A = A_0 \oplus A_1$ be its $\mathbb{Z}_2$-grading from Proposition 4.1 (i). If A_0 is nilpotent then the form f is degenerate on A, i. e., $f(A,A) = 0$.*

Proof. Let $a \in A_0$. We have $R_a^n(A) \subseteq R_{a^n}(A)$, since A is an associative right A_0-module by Proposition 4.1 (*iii*). Consequently, R_a is nilpotent, since $a \in A_0$ is nilpotent. Hence, $\mathrm{tr}(R_a) = 0$. Consequently, for all $a, b \in A_0$ we have $f(a,b) = tr(R_a R_b) = tr(R_{ab}) = 0$. Thus, $f(A_0, A_0) = 0$.

Let $a, b \in A_0$ and $x \in A_1$. By Lemma 5.1 and Proposition 4.1 (*ii*), we get

$$f(ax,b) = f([a,x],b) = f(a,xb) - f(x,ab) = -f(x,ab).$$

It means that

$$f(L^n_{A_0} A_1, A_0) \subseteq f(A_1, A_0^{n+1})$$

for all $n \geq 0$. Since $A_1 = L^n_{A_0} A_1$ by Proposition 4.1(*ii*) and A_0 is nilpotent; therefore, $f(A_1, A_0) = 0$.

If $x, y \in A_1$ then

$$f(x,y) = \mathrm{tr}(R_x R_y) = \mathrm{tr}(R_{xy}) = 0,$$

since $xy \in A_0$. Consequently, $f(A_1, A_1) = 0$. Thus, f is degenerate on A. □

Theorem 5.1. *Let $A \in \mathcal{A}$ and let $A = A_0 \oplus A_1$ be the $\mathbb{Z}_2$-grading of A from Proposition 4.1 (i). Then A is complete if and only if A_0 is nilpotent.*

Proof. Assume that A is complete. Then by [6, Lemma 1.1], A is right nil, i. e., R_x is nilpotent for every $x \in A$. Therefore, A_0 is an associative and commutative finite-dimensional nil algebra over a field of characteristic zero. Consequently, A_0 is nilpotent.

If A_0 is nilpotent then $f(A, A) = 0$ by Lemma 5.2. Hence, $\mathrm{tr}(R_x R_y) = 0$ for all $x, y \in A$. Then

$$\mathrm{tr}(R_A) = \mathrm{tr}(R_{A^2}) = \mathrm{tr}(R_A R_A) = 0,$$

since $\mathrm{tr}(R_{xy}) = \mathrm{tr}(R_x R_y)$ and A is simple. Therefore, $T(A) = A$ and A is complete. □

6. **The root decomposition**

Now, let F be an algebraically closed field, and let $A = A_0 + A_1$ be a simple $\mathbb{Z}_2$-graded finite-dimensional left-symmetric algebra such that A_0 is an associative commutative algebra, $A_0 = A_1^2$, $A_1 = A_0 A_1$, $[A_1, A_1] = 0$, and $A_1 A_0 = 0$. Notice that by (1.2) we have

$$[L_x, L_y] = L_{[x,y]}.$$

The algebra A_0 acts on the vector space A_i by the left multiplication operators, where $i = 0, 1$. Notice that for $a, b \in A_0$ the left multiplication operators L_a and L_b are commuting. Denote by A_0^* the dual space for A_0. Take $a \in A_0$, $\alpha \in A_0^*$, and $i = 0, 1$. Then

$$A_i(\alpha) = \{v \in A_i : (L_a - \alpha(a)id)^n(v) = 0,\ n \in \mathbb{N}\}$$

are the *root subspaces* and $\alpha \in A_0^*$ are the *roots* provided that $A_i(\alpha) \neq 0$. Let Φ_i be the system of roots of the algebra A_0 on the vector space A_i, where $i = 0, 1$, i. e., $\Phi_i = \{\alpha \in A_0^* : A_i(\alpha) \neq 0\}$. Since L_a and L_b are the commuting operators; therefore,

$$A_i = \bigoplus_{\alpha \in \Phi_i} A_i(\alpha)$$

is the *root decomposition* of A_i with respect to A_0, where $i = 0, 1$. Clearly, $A_0 A_1(\alpha) \subseteq A_1(\alpha)$ for all $\alpha \in \Phi_1$. Then we have the following

Lemma 6.1. *Given $\alpha \in \Phi_0$, there exist $\beta, \gamma \in \Phi_1$ such that $\alpha = \beta + \gamma$. Moreover,*

$$A_0(\alpha) \ = \sum_{\alpha=\beta+\gamma,\ \beta,\gamma\in\Phi_1} A_1(\beta)A_1(\gamma),$$

$A_1(0) = 0$, and $A_1(\beta)A_1(\gamma)$ is an ideal of A_0. If $\alpha, \beta \in \Phi_0$ and $\alpha \neq \beta$ then $A_0(\alpha)A_0(\beta) = 0$.

Proof. Take $a \in A_0$, $x, y \in A_1$, and $\beta, \gamma \in \Phi_1$. Then by (4.3) we get

$$(L_a - (\beta + \gamma)(a)id)^n(xy) = \sum_{i=0}^{n} C_i^n (L_a - \beta(a)id)^i(x)(L_a - \gamma(a)id)^{n-i}(y),$$

where C_i^n are the binomial coefficients. Consequently, $A_1(\beta)A_1(\gamma) \subseteq A_0(\beta + \gamma)$.

Since $A_0 = A_1^2$, we have

$$A_0 = \sum_{\beta,\gamma\in\Phi_1} A_1(\beta)A_1(\gamma).$$

Hence, there are $\beta, \gamma \in \Phi_1$ such that $A_1(\beta)A_1(\gamma) \neq 0$. Therefore, $\beta + \gamma \in \Phi_0$ and

$$A_0 = \bigoplus_{\alpha \in \Phi_0} \Big(\sum_{\substack{\beta,\gamma \in \Phi_1 \\ \beta+\gamma=\alpha}} A_1(\beta)A_1(\gamma) \Big).$$

Since $A_0 = \bigoplus_{\alpha \in \Phi_0} A_0(\alpha)$; therefore, for every $\alpha \in \Phi_0$ we have

$$A_0(\alpha) = \sum_{\beta,\gamma \in \Phi_1,\, \beta+\gamma=\alpha} A_1(\beta)A_1(\gamma).$$

By (4.3), we get $A_0(A_1(\beta)A_1(\gamma)) \subseteq (A_0A_1(\beta))A_1(\gamma) + A_1(\beta)(A_0A_1(\gamma)) \subseteq A_1(\beta)A_1(\gamma)$. Consequently, $A_1(\beta)A_1(\gamma)$ is an ideal of A_0. Clearly, $A_0(\alpha)A_0(\beta) = 0$ for distinct $\alpha, \beta \in \Phi_0$.

Prove that $A_1(0) = 0$. Notice that every operator of left multiplication L_a, where $a \in A_0$, acts nilpotently on $A_1(0)$. Since A_0 is finite-dimensional and L_a are pairwise commuting; therefore, there exists $n \in \mathbb{N}$ such that $L_{a_1} \dots L_{a_n} A_1(0) = 0$ for all $a_1, \dots, a_n \in A_0$. By Proposition 4.1, we have $A_1 = A_0A_1$. Consequently, $A_1(0) = A_0A_1(0)$. Therefore, $A_1(0) = \underbrace{A_0(\dots(A_0}_{n} A_1(0)\dots) = 0$. □

Lemma 6.2. *Let A_0 be a nilpotent algebra. Then*

$$A_0 = \sum_{\alpha \in \Phi_1} A_1(-\alpha)A_1(\alpha).$$

Moreover, $A_1(\alpha)A_1(\beta) = 0$ for all $\alpha, \beta \in \Phi_1$ such that $\beta \neq -\alpha$. Furthermore, $-\alpha \in \Phi_1$ for every $\alpha \in \Phi_1$.

Proof. Since A_0 is nilpotent, $\Phi_0 = 0$. Take $\alpha \in \Phi_0$. Then, by Lemma 6.1, there are $\beta, \gamma \in \Phi_1$ such that $\alpha = \beta + \gamma$. Therefore, $\beta + \gamma = 0$. Consequently,

$$A_0 = \bigoplus_{\alpha \in \Phi_1} A_1(-\alpha)A_1(\alpha).$$

Let $\alpha, \beta \in \Phi_1$ and $\beta \neq -\alpha$. Then $A_1(\alpha)A_1(\beta) \subseteq A_0(\alpha+\beta) = 0$. Assume that $-\alpha \notin \Phi_1$. Then $A_1(\alpha)A_1(\beta) = 0$ for all $\beta \in \Phi_1$. Since $A_0A_1(\alpha) \subseteq A_1(\alpha)$ and $A_1(\alpha)A_0 = 0$; therefore, $A_1(\alpha)$ is an ideal of A. Consequently, $A_1(\alpha) = 0$. Therefore, $-\alpha \in \Phi_1$ for all $\alpha \in \Phi_1$. □

7. The four-dimensional Lie-solvable left-symmetric algebras in $\mathcal{A}$

In this section, we describe the four-dimensional simple $\mathbb{Z}_2$-graded left-symmetric algebras $A = A_0 + A_1$ over an algebraically closed field F of characteristic not 2 such that A_0 is an associative commutative algebra, $A_0 = A_1^2$, $A_1 = A_0A_1$, $[A_1, A_1] = 0$, and $A_1A_0 = 0$.

In what follows, $\langle \Upsilon \rangle_F$ is used for the linear span of a set Υ over a field F, where we omit F if the field is clear from the context.

Lemma 7.1. *The algebra A_0 is not nilpotent.*

Proof. Assume that A_0 is nilpotent. Then, $A_0 = \sum_{\alpha\in\Phi_1} A_1(-\alpha)A_1(\alpha)$ by Lemma 6.2, and $A_1 = \sum_{\alpha\in\Phi_1} A_1(\alpha)$. By Lemma 6.1, $\alpha \neq 0$ for all $\alpha \in \Phi_1$. Since $\dim A = 4$; therefore, $\Phi_1 = \{\alpha, -\alpha\}$. By Lemma 6.2, $\dim A_1 = 3$. Since A_0 is nilpotent, $A_0^2 = 0$.

Let e_2, e_3, e_4 be a basis for A_1. We may suppose that $A_1(\alpha) = \langle e_2, e_3\rangle$, $A_1(-\alpha) = \langle e_4\rangle$. By Lemma 6.2, $A_1(\alpha)^2 = 0$. Then for all nonzero $x \in A_1(\alpha)$ we have $xe_4 \neq 0$, since otherwise $xA = 0$. Hence, $x \in Ann_l(A)$; a contradiction by Lemma 2.1. Consequently, $e_2e_4 \neq 0$ and $e_3e_4 \neq 0$. Then $e_3e_4 = \beta e_2e_4$ for some $\beta \in F$, and $(e_3 - \beta e_2)e_4 = 0$. It means that if $\Phi_1 = \{\alpha, -\alpha\}$ then A_0 is not nilpotent. □

In what follows, we assume that A_0 is not nilpotent.

Lemma 7.2. *Let* $\Phi_1 = \{\alpha\}$. *Then* $\dim A_0 = 1$.

Proof. Since $\Phi_1 = \{\alpha\}$, $\Phi_0 = \{2\alpha\}$. Assume that $\dim A_1 = 2$. Let x, y be a basis for A_1 such that

$$ax = \alpha(a)x, ay = \alpha(a)y + \beta(a)x,$$

where $a \in A_0, \beta \in A_0^*$. Then $\dim A_0 = 2$ and $A_0 = \langle x^2, xy, y^2\rangle$. By (4.3), $ax^2 = 2\alpha(a)x^2$ for all $a \in A_0$. Hence, $\langle x^2\rangle$ is an ideal of A_0.

Assume that $x^2 = 0$. Then, by (4.3), for all $a \in A_0$ we get

$$a(xy) = (ax)y + x(ay) = 2\alpha(a)xy + \beta(a)x^2 = 2\alpha(a)xy.$$

Therefore, $\langle xy\rangle$ is an ideal of A_0. Since

$$(xy)y^2 = 2\alpha(xy)y^2 + 2\beta(xy)xy;$$

therefore, $\alpha(xy)y^2 \in \langle xy\rangle$. Since $\dim A_0 = 2$, $\alpha(xy) = 0$ and $(xy)^2 = 2\alpha(xy)xy = 0$. From here we conclude that $\langle xy, x\rangle$ is an ideal of A, since $(xy)y = \beta(xy)x$. Thus, $x^2 \neq 0$.

Now, let $x^2 \neq 0$. By (4.3),

$$a(xy) = 2\alpha(a)xy + \beta(a)x^2, ay^2 = 2\alpha(a)y^2 + 2\beta(a)xy$$

for all $a \in A_0$. Since $\langle x^2\rangle$ is an ideal of A_0; therefore, $\alpha(x^2)xy \in \langle x^2\rangle$ and $\alpha(x^2)^2y^2 \in \langle x^2\rangle$. Consequently, $\alpha(x^2) = 0$, since otherwise $\dim A_0 = 1$.

From here we get $x^2(xy) = \beta(x^2)x^2$. On the other hand, $x^2(xy) = 2\alpha(xy)x^2$. Then $\beta(x^2) = 2\alpha(xy)$. Similarly,

$$x^2y^2 = 2\beta(x^2)xy = 2\alpha(y^2)x^2.$$

If $\beta(x^2) = 0$ then $\alpha(xy) = 0$ and $\alpha(y^2) = 0$. Consequently, $\alpha(a) = 0$ for all $a \in A_0$. In this case A_0 is nilpotent. Hence, $\beta(x^2) \neq 0$, and $\alpha(xy) \neq 0$. Since $\beta(x^2)xy = \alpha(y^2)x^2$; therefore, $xy \in \langle x^2\rangle$ and $\alpha(xy) = 0$, a contradiction. □

Example 7.1. *Let* $\Phi_1 = \{\alpha\}$ *and* $A_0 = \langle e_1\rangle$. *Assume that* L_{e_1} *is a semisimple operator on* A_1. *Then the vector space* A_1 *possesses a basis* e_2, e_3, e_4 *such that the algebra* A *has the following multiplication table*

$$e_1^2 = 2e_1, e_1e_2 = e_2, e_1e_3 = e_3, e_1e_4 = e_4, e_2^2 = e_3^2 = e_4^2 = e_1, \tag{7.1}$$

and all other products are zero.

Proof. Since L_{e_1} is semisimple; therefore, for some basis x, y, z of A_1 we have

$$e_1x = \alpha x, e_1y = \alpha y, e_1z = \alpha z.$$

Since $\Phi_0 = \{2\alpha\}$ and A_0 is not nilpotent, $\alpha \neq 0$. Hence, we may assume that $\alpha = 1$. Since $e_1^2 = 2\alpha e_1$, $e_1^2 = 2e_1$.

Suppose that $x^2 \neq 0$. Then we may assume that $x^2 = e_1$ and $xy = \beta x^2$, where $\beta \in F$. Therefore, $x(y - \beta x) = 0$. Hence, we may assume that $xy = 0$. Similarly, $xz = 0$.

Let $y^2 \neq 0$. Then $yz = \beta y^2$, where $\beta \in F$. Hence, we may suppose that $xy = xz = yz = 0$. In this case, $z^2 \neq 0$, since otherwise $\langle z \rangle$ is an ideal of A. Since $y^2 \neq 0$; therefore, $y^2 = \beta x^2$, $\beta \in F$, and $\beta \neq 0$. Hence, we may assume that $y^2 = x^2$. Similarly, $z^2 = x^2$. Finally, in the case under consideration we arrive at the multiplication table (7.1).

Let $y^2 = z^2 = 0$. If $yz = 0$ then $\langle y \rangle$ is an ideal of A. Therefore, $yz \neq 0$ and $(\frac{y+z}{2})^2 = \frac{yz}{2} \neq 0$. Since $x \cdot \frac{y+z}{2} = 0$; therefore, replacing y by $\frac{y+z}{2}$ we arrive to the case considered above.

Let $x^2 = 0, y^2 = 0, z^2 = 0$. Then either $xy \neq 0$ or $xz \neq 0$, since otherwise $\langle x \rangle$ is an ideal of A. Repeating the previous argument, we get the required basis for A with the multiplication table (7.1). □

In [6], the left-symmetric simple four-dimensional algebras $I_4^d(\alpha, \beta, \gamma)$ were introduced, where $\alpha, \beta, \gamma \in F$. The algebra of Example 7.1 is $I_4^d(0, 0, 0)$.

Lemma 7.3. *Let $\Phi_1 = \{\alpha\}$ and $A_0 = \langle e_1 \rangle$. The case of non-semisimple L_{e_1} with a minimal polynomial of degree two is impossible.*

Proof. Assume that L_{e_1} is not semisimple and its minimal polynomial is of degree two. Then A_1 possesses a basis x, y, z such that

$$e_1x = \alpha x,\ e_1y = \alpha y,\ e_1z = \alpha z + y.$$

Let $y^2 \neq 0$. Then $yz = \beta y^2$, where $\beta \in F$. Therefore, $y(z - \beta y) = 0$. Moreover,

$$e_1(z - \beta y) = \alpha(z - \beta y) + y.$$

Hence, we may replace z by $z - \beta y$. Consequently, we may suppose that $yz = 0$. By Proposition 4.1,

$$0 = e_1(yz) = (e_1y)z + y(e_1z) = \alpha yz + y(\alpha z + y) = y^2,$$

which is a contradiction. Hence, $y^2 = 0$.

Let $y^2 = 0$. Then either $xy \neq 0$ or $yz \neq 0$, since otherwise $\langle y \rangle$ is an ideal of A.

Let $xy \neq 0$ and $xz = \beta xy$, where $\beta \in F$. Then $x(z - \beta y) = 0$. Put $u = z - \beta x$. Then $xu = 0$. Moreover,

$$e_1u = e_1(z - \beta x) = \alpha(z - \beta x) + y = \alpha u + y.$$

By Proposition 4.1,

$$0 = e_1(xu) = (e_1x)u + x(e_1u) = \alpha xu + x(\alpha u + y) = xy,$$

a contradiction. Therefore, $xy = 0$. Consequently, $yz \neq 0$.

Let $yz \neq 0$. Then $z^2 = \beta yz$, where $\beta \in F$. Therefore, $(z - \frac{\beta}{2}y)^2 = 0$. Put $u = z - \frac{\beta}{2}y$. Then $u^2 = 0$, $yu = yz$, and

$$e_1 u = e_1(z - \frac{\beta}{2}y) = \alpha(z - \frac{\beta}{2}y) + y = \alpha u + y.$$

By Proposition 4.1,

$$0 = e_1 u^2 = 2(e_1 u)u = 2(\alpha u + y)u = yu = yz,$$

which is a contradiction. □

Example 7.2. *Let $\Phi_1 = \{\alpha\}$, $A_0 = \langle e_1 \rangle$, and let the minimal polynomial for L_{e_1} be of degree 3. Then A_1 possesses a basis e_2, e_3, e_4 such that A has the following multiplication table*

$$\begin{aligned} e_1^2 &= 2e_1,\ e_1e_2 = e_3 + e_2,\ e_1e_3 = e_4 + e_3,\ e_1e_4 = e_4, \\ e_2^2 &= \beta e_1,\ e_3^2 = -e_1,\ e_2e_4 = e_4e_2 = e_1,\ \beta \in F, \end{aligned} \tag{7.2}$$

and all other products are zero.

Proof. By the hypothesis, A_1 possesses a basis x, y, z such that

$$e_1x = \alpha x + y, e_1y = \alpha y + z, e_1z = \alpha z.$$

The root α is nonzero. Therefore, we may assume that $\alpha = 1$. Since $\Phi_0 = \{2\alpha\}$, $e_1^2 = 2e_1$.

Let $z^2 \neq 0$. Then $e_1z^2 = 2z^2$ and $yz = \beta z^2$, where $\beta \in F$. By Proposition 4.1,

$$e_1(yz) = (e_1y)z + y(e_1z) = (y + z)z + yz = 2yz + z^2 = 2\beta z^2 + z^2 = 2\beta z^2,$$

whence $z^2 = 0$.

Let $yz \neq 0$. Then $e_1(yz) = 2yz$ and $y^2 = \beta yz$, where $\beta \in F$. By Proposition 4.1,

$$e_1y^2 = 2(e_1y)y = 2(y + z)y = 2y^2 + 2yz = 2\beta yz,$$

whence $yz = 0$. Thus, $z^2 = yz = 0$. Therefore, $xz \neq 0$, since otherwise $\langle z \rangle$ is an ideal of A. We may assume that $xz = e_1$.

Since $x^2 = \beta e_1$ with $\beta \in F$; therefore, by Proposition 4.1 we have

$$e_1x^2 = 2(e_1x)x = 2x^2 + 2xy = 2\beta e_1,$$

whence $xy = 0$. Then,

$$0 = e_1(xy) = (e_1x)y + x(e_1y) = 2xy + y^2 + xz = y^2 + e_1$$

by Proposition 4.1. Therefore, $y^2 = -e_1$. Consequently, we arrive at (7.2). □

Let F be a field of characteristic not 2, and let $\alpha \in F$. Consider a new basis

$$\begin{pmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & i & \alpha & -\frac{1+\alpha^2+\beta}{2}i \\ 0 & -1 & \alpha i & -\frac{1-\alpha^2-\beta}{2} \\ 0 & 0 & i & \alpha \end{pmatrix} \begin{pmatrix} e_1 \\ e_2 \\ e_3 \\ e_4 \end{pmatrix}$$

for A from Example 7.2. Then A has the product of the algebra $I_4^d(0, 1, i)$ with respect to the basis f_1, f_2, f_3, f_4.

Lemma 7.4. *The case $\Phi_1 = \{\alpha, \beta\}$ is impossible.*

Proof. Let $\Phi_1 = \{\alpha, \beta\}$. Assume that $\dim A_1 = 2$. Then A_1 has a basis x, y such that

$$ax = \alpha(a)x, ay = \beta(a)y$$

for all $a \in A_0$. Then $A_0 = \langle x^2, xy, y^2\rangle$. Clearly, $x^2 \neq 0$ or $y^2 = 0$, since $\dim A_0 = 2$.

We may suppose that $x^2 \neq 0$. Let $y^2 \neq 0$. Then $\Phi_0 = \{2\alpha, 2\beta\}$ by Lemma 6.1. Therefore, $A_0(2\alpha) = \langle x^2\rangle, A_0(2\beta) = \langle y^2\rangle$, and $A_0(\alpha+\beta) = 0$, i.e., $xy = 0$. We also have $A_0(2\alpha)A_0(2\beta) = 0$ by Lemma 6.1. Moreover, $\alpha(y^2) = \beta(x^2) = 0$. Then $\langle x^2, x\rangle$ is an ideal of A. Consequently, $y^2 = 0$.

Let $y^2 = 0$. Then $xy \neq 0$. Hence, $\Phi_0 = \{2\alpha, \alpha+\beta\}$, $A_0(2\alpha)A_0(\alpha+\beta) = 0$, and $\alpha(xy) = 0$, since $xy \in A_0(\alpha+\beta)$. Therefore, $\langle xy, y\rangle$ is an ideal of A. Consequently, $\dim A_1 \neq 2$.

Thus, $\dim A_1 = 3$, and A_1 has a basis x, y, z. Let $A_1(\alpha) = \langle x\rangle$ and $A_1(\beta) = \langle y, z\rangle$.

Assume that $x^2 \neq 0$. Then $\Phi_0 = \{2\alpha\}$. Therefore,

$$A_1(\beta)A_1 \subseteq A_1(\beta)(A_1(\beta) + A_1(\alpha)) \subseteq A_0(2\beta) + A_0(\alpha+\beta) = 0.$$

Consequently, $A_1(\beta)A = 0$, i.e., $A_1(\beta) \subseteq Ann_l(A)$. Since A is simple, $Ann_l(A) = 0$ by Lemma 2.1. Hence, $A_1(\alpha)^2 = 0$.

Let $A_1(\beta)^2 \neq 0$. Then $\Phi_0 = \{2\beta\}$. Therefore,

$$A_1(\alpha)A_1 \subseteq A_1(\alpha)(A_1(\beta) + A_1(\alpha)) \subseteq A_0(\alpha+\beta) + A_0(2\alpha) = 0.$$

Consequently, $A_1(\alpha)A = 0$. Hence, $A_1(\beta)^2 = 0$.

Then $A_1(\alpha)A_1(\beta) \neq 0$. Therefore, $\Phi_0 = \{\alpha+\beta\}$. Let $xy \neq 0$. Then $xz = \gamma xy$, where $\gamma \in F$. From here we get $x(z - \gamma y) = 0$. Hence, $(z - \gamma y) \in Ann_l(A)$, a contradiction. □

Example 7.3. *Let $\Phi_1 = \{\alpha, \beta, \gamma\}$, and let all the roots be different. Let $A_0 = \langle e_1\rangle$. Then A_1 possesses a basis e_2, e_3, e_4 such that A has the following multiplication table*

$$e_1^2 = 2e_1, e_1e_2 = e_2, e_1e_3 = \beta e_3, e_1e_4 = (2-\beta)e_4, \tag{7.3}$$
$$e_2^2 = e_1, e_3e_4 = e_4e_3 = e_1,$$

where $\beta \in F$, $\beta \neq 0, 1, 2$, and all other products are zero.

Proof. Let $\Phi_1 = \{\alpha, \beta, \gamma\}$, and let all the roots be different. Then $A_0 = \langle e_1\rangle$. Choose a basis x, y, z for A_1 such that

$$e_1x = \alpha x, e_1y = \beta y, e_1z = \gamma z.$$

By Lemma 6.1 we have $\alpha, \beta, \gamma \neq 0$.

Let $x^2 = y^2 = z^2 = 0$. Then either $xy \neq 0$ or $xz \neq 0$, since otherwise $xA = 0$, which is a contradiction by Lemma 2.1. We may assume that $xy \neq 0$. Hence, $xz = \delta xy$, where $\delta \in F$. From here we get $x(z - \delta y) = 0$. If $yz = 0$ then $y(z - \delta y) = z(z - \delta y) = 0$. Therefore, $(z - \delta y)A = 0$; a contradiction by Lemma 2.1. Consequently, $yz \neq 0$. Similarly, $xz \neq 0$. Then $\Phi_0 = \{\alpha+\beta\} = \{\beta+\gamma\} = \{\alpha+\gamma\}$; a contradiction, since all the roots are different. Therefore, the case $x^2 = y^2 = z^2 = 0$ is impossible.

Let $x^2 \neq 0$. Then $\Phi_0 = \{2\alpha\}$ and $e_1^2 = 2\alpha e_1$. Therefore, $y^2 = xy = xz = z^2 = 0$, since all the roots are different. Moreover, we may assume that $x^2 = e_1$. In this case, $yz = \delta e_1$, where $\delta \in F$ and $\delta \neq 0$, since otherwise $\langle y \rangle$ is an ideal of A. Therefore, we may suppose that $yz = e_1$. Then $2\alpha = \beta + \gamma$, since $yz \in A_0(\beta + \gamma)$. Since $\alpha \neq 0$; therefore, we may assume that $\alpha = 1$. Hence, $2 = \beta + \gamma$. Put $e_2 = x, e_3 = y, e_4 = z$. Then we arrive at (7.3). □

Let F be a field of characteristic not 2. Consider a new basis

$$\begin{pmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -i & \frac{i}{2} \\ 0 & 0 & 1 & \frac{1}{2} \end{pmatrix} \begin{pmatrix} e_1 \\ e_2 \\ e_3 \\ e_4 \end{pmatrix}$$

for A from Example 7.3. Then A has the product of $I_4^d(0, 0, i(1-\beta))$ with respect to the basis f_1, f_2, f_3, f_4.

Finally, we collect the results obtained in this section in the following

Theorem 7.1. *Let A be a four-dimensional algebra in $\mathcal{A}$ over an algebraically closed field of characteristic not 2. Then A is isomorphic to one of the algebras* (7.1) – (7.3).

8. On algebras in $\mathcal{A}$, whose even part is either simple or zero-product

8.1. **Simplicity conditions.** Let $A = A_0 \oplus A_1$ be a Z_2-graded algebra such that $A_1 A_0 = 0$, A_0 and A_1 are commutative, and A_0 is associative. Denote the class of such algebras by $\mathcal{B}$. The following lemma is immediate.

Lemma 8.1. *Let $A = A_0 \oplus A_1 \in \mathcal{B}$. Then A is left-symmetric if and only if*

$$a(xy) = (ax)y + x(ay), \; a(bx) = b(ax)$$

hold for all $a, b \in A_0$ and $x, y \in A_1$.

Lemma 8.2. *Let $A = A_0 \oplus A_1 \in \mathcal{B}$ be left-symmetric. Then A is simple if and only if $A_0 = A_1^2$, $A_1 = A_0 A_1$, A_0 lacks proper ideals I such that $(IA_1)A_1 \subseteq I$, and A_1 lacks proper ideals of A.*

Proof. Let A be simple. Since $A^2 \trianglelefteq A$ and $A^2 = A_0^2 + A_0 A_1 + A_1^2$, we have $A_1 = A_0 A_1$. If $A_1^2 \neq A_0$ then $A_1^2 + A_1$ is a proper right ideal of A, which is impossible. Let I be an ideal of A_0 such that $(IA_1)A_1 \subseteq I$. It is easy to see that $I + IA_1$ is a right ideal of A. Therefore, A_0 lacks proper ideals I such that $(IA_1)A_1 \subseteq I$. Obviously, A_1 lacks proper ideals of A.

Conversely, assume that I is an ideal of A. Let I_k be the projection of I on A_k, $k = 1, 2$. Then $I \subseteq I_0 + I_1$, and $I_0 A_0 \subseteq I_0$, whence I_0 is an ideal of A_0. Since $(I_0 A_1)A_1 \subseteq I_0$; therefore, either $I_0 = 0$ or $I_0 = A_0$. If $I_0 = 0$ then I_1 is an ideal of A, whence $I_1 = 0$. If $I_0 = A_0$ then $A_1 = A_0 A_1 \subseteq I_1$, whence $I_1 = A_1$. Now, since $A_1 A_1 = A_0$ and $A_1(I_0 + I_1) \subseteq I$, we have $A_0 \subseteq I$, whence $I = A$. □

Put $A_0(\alpha) := A_1(\alpha)A_1(-\alpha)$ for every $\alpha \in \Phi_1$. We say that $A_1(\alpha)$ is *nondegenerate* if for every $x_\alpha \in A_1(\alpha)$ there is $x_{-\alpha} \in A_1(-\alpha)$ such that $x^0_\alpha := x_\alpha x_{-\alpha} \neq 0$. Put $A_0(\alpha_1, \ldots, \alpha_s) := \sum_{i=1}^s A_0(\alpha_i)$. We say that Φ_1 is *nondegenerate* if $A_1(\alpha)$ is nondegenerate for every $\alpha \in \Phi_1$, and Φ_1 possesses a *chain property* provided that it is nondegenerate and for every $\alpha_1 \in \Phi_1$ there is a chain of roots $\alpha_2, \ldots, \alpha_k \in \Phi_1$ such that $\alpha_{s+1}(A_0(\alpha_1, \ldots, \alpha_s)) \neq 0$ for all $s = 1, \ldots, k-1$ and $A_0(\alpha_1, \ldots, \alpha_k) = A_0$. The number n of linearly independent roots of Φ_1 is the *rank* of Φ_1. The chain $\alpha_1, \ldots, \alpha_s$ is a *CP-system* or an α_1-*system*, and s is its *length*. Denote by $\mathcal{C}$ the class of left-symmetric algebras in $\mathcal{B}$ such that $A_0 = A_1^2$ and $A_1 = A_0A_1$.

Proposition 8.1. *Let $A = A_0 \oplus A_1$ be a left-symmetric algebra in $\mathcal{C}$ with a nilpotent subalgebra A_0 of dimension n such that the action of A_0 on A_1 is diagonalizable. Then A is simple if and only if Φ_1 is a root system of rank n with the chain property.*

Proof. Let A be simple. If there are no n linearly independent roots in Φ_1 then $I := \cap Ker\, \alpha_i \neq 0$. Since $a(x_\alpha \cdot x_{-\alpha}) = 0$ for every $a \in I$ by (4.3) and $A_0 = \sum_{\alpha\in\Phi_1} A_0(\alpha)$; therefore, $aA_0 = 0$ and $I \trianglelefteq A$. Take $\alpha_1 \in \Phi_1$. Then $A_1(\alpha_1)$ is nondegenerate, since otherwise if $xA_1(-\alpha_1) = 0$ for some $x \in A_1(\alpha_1)$ then $\langle x\rangle$ is a right ideal of A, which is impossible.

Take $x \in A_0(\alpha_1)$. Then for every $\alpha \in \Phi_1$ we have $(x \cdot x_\alpha)x_{-\alpha} = \alpha(x)x^0_\alpha$. If $\alpha(x) = 0$ for all $\alpha \in \Phi_1\backslash\{\alpha_1\}$ then we may apply Lemma 8.2 to $I_0 = A_0(\alpha_1)$. Thus, either $A_0(\alpha_1) = A_0$ or there is $\alpha_2 \in \Phi_1\backslash\{\alpha_1\}$ such that $\alpha_2(x) \neq 0$ and $\alpha_2(A_0(\alpha_1)) \neq 0$. Continuing this process we arrive at the assertion of the lemma.

Conversely, assume that Φ_1 is a root system of rank n with a chain property. Consider a nonzero ideal I of A. If $y = a + \sum_{\gamma\in\Phi_1} x_\gamma \in I$ with some $x_\alpha \neq 0$ then there is $h \in A_0$ such that $\alpha(h) \neq 0$, whence $hy = ha + \sum_{\gamma\in\Phi_1} \gamma(h)x_\gamma \in I,\ yh = ah = ha \in I$. Therefore, we may assume that $x^0_\alpha \in I$. If $y = a \in I$ then $\alpha(a) \neq 0$ for some $\alpha \in \Phi_1$, whence $x_\alpha \in I$ and $x^0_\alpha \in I$. Thus, we may suppose that $x^0_{\alpha_1} \in I$ for some $\alpha_1 \in \Phi_1$. Take α_2 such that $\alpha_2(x^0_{\alpha_1}) \neq 0$. Then $x^0_{\alpha_1}x_{\alpha_2} = \alpha_2(x^0_{\alpha_1})x_{\alpha_2} \in I$, whence $A_0(\alpha_2) \subseteq I$. From here we may assume initially that $A_0(\alpha_1) \subseteq I$. Continuing this process we arrive at the assertion of the lemma. □

In the case of an arbitrary even part, we can prove an analogous statement, modifying the definition of $A_0(\alpha)$ by

$$A_0(\alpha, \beta) = A_1(\alpha)A_1(\beta).$$

We say that $A_1(\alpha)$ is *nondegenerate* provided that for every $x_\alpha \in A_1(\alpha)$ there is $x_\beta \in A_1(\beta)$ such that $x_\alpha x_\beta \neq 0$ (β is a *companion* for α). Put $A_0(\alpha_1, \beta_1, \ldots, \alpha_s, \beta_s) := \sum_{i=1}^s A_0(\alpha_i, \beta_i)$. We say that Φ_1 possesses a *chain property* provided that it is nondegenerate and for every pair $\alpha_1, \beta_1 \in \Phi_1$ there is a chain of roots $\alpha_2, \beta_2, \ldots, \alpha_k, \beta_k \in \Phi_1$ such that $\alpha_{s+1}(A_0(\alpha_1, \beta_1, \ldots, \alpha_s, \beta_s)) \neq 0$ for all $s = 1, \ldots, k-1$ and $A_0(\alpha_1, \beta_1, \ldots, \alpha_k, \beta_k) = A_0$, where β_i is a companion for α_i.

Lemma 8.3. *Let $A = A_0 \oplus A_1$ be a left-symmetric algebra in $\mathcal{C}$ with A_0 of dimension n such that the action of A_0 on A_1 is diagonalizable. Then A is simple if and only if Φ_1 is a root system of rank n with the chain property.*

Proof repeats one of Proposition 8.1. $\square$

The following lemma and the examples below show the immensity of the class of algebras, satisfying the hypothesis of Proposition 8.1.

Lemma 8.4. *Let $A = A_0 \oplus A_1 \in \mathcal{A}$, and let A_0 be zero-product. Assume that A_0 acts diagonally on A_1, $dim\, A_0 = n$, and $\Phi_1 = \pm\{\alpha_1, \dots, \alpha_n\}$ consists of $2n$ roots, where $\alpha_1, \dots, \alpha_n$ are linearly independent. Let $dim\, A_0(\alpha) := 1$ for all $\alpha \in \Phi_1$. Then $dim\, A_1(\alpha_i) = dim\, A_1(-\alpha_i) = k_i \in \mathbb{N}$, $A_1(\alpha_i) := \left\langle x^{(j)}_{\alpha_i} : j = 1, \dots, k_i \right\rangle$ for every $i = 1, \dots, n$, and*

$$A = A_0 \oplus \sum_{\alpha_i \in \Phi_1} A_1(\alpha_i)$$

with the following nonzero products

$$x^{(j)}_{\alpha_i} x^{(j)}_{-\alpha_i} = x^{(j)}_{-\alpha_i} x^{(j)}_{\alpha_i} = a_i \in A_0,\ a x^{(j)}_{\alpha_i} = \alpha_i(a) x^{(j)}_{\alpha_i}$$

for all $a \in A_0$, $\alpha_i \in \Phi_1, j = 1, \dots, k_i$. In particular, $dim\, A = n + 2\sum_{i=1}^{n} k_i \geq 3n$.

Proof. By Proposition 8.1, the rank of Φ_1 is n, and $A_1(\alpha_i)$ is nondegenerate for every $i = 1, \dots, n$. Consider $A_1(\alpha_1)$. Assume that $dim\, A_1(\alpha_1) \neq dim\, A_1(-\alpha_1)$. Without loss of generality, we may suppose that $dim\, A_1(\alpha_1) = k + 1$, $dim\, A_1(-\alpha_1) = k$. Let $A_1(\alpha_1) = \langle x_1, \dots, x_{k+1} \rangle$, $A_1(-\alpha_1) = \langle y_1, \dots, y_k \rangle$. Changing a base if needed, it is easy to see that we may assume $x_i \cdot y_j = \delta_{ij} a_1$ for all $i, j = 1, \dots, k$, where δ_{ij} is Kronecker's delta. Let $x_{k+1} \cdot y_i = \gamma_i a_1$ for some $\gamma_i \in F$ and for all $i = 1, \dots, k$. Then $x := x_{k+1} - \sum_{i=1}^{k} \gamma_i x_i$ satisfies $x \cdot A_1(-\alpha_1) = 0$, whence $\langle x \rangle$ is a right ideal of A. Therefore, $dim\, A_1(\alpha_1) = k_1 = dim\, A_1(-\alpha_1)$, and the product between $A_1(\alpha_1)$ and $A_1(-\alpha_1)$ satisfies the mentioned relations. $\square$

8.2. **Examples of CP-systems.** Note that the union of some systems with the chain property is a system with the chain property. A CP-system $\alpha := \{\alpha_1, \dots, \alpha_m\}$ is *minimal* if $\{\alpha_1, \dots, \alpha_m\} \setminus \{\alpha_i\}$ is not a CP-system for every $i = 1, \dots, m$, α is *invariant* if $\pm\alpha$ is a system with the chain property, and α is a *base* if it is minimal and invariant. Clearly, every system with the chain property contains a base. Obviously, if a nondegenerate system of roots Γ contains a base then Γ is a system with the chain property.

In what follows, $\{\delta_i\}$ is a dual basis for $\{e_i\}$.

Example 8.1. *Consider a cyclic system*: $\alpha_i(A_0(\alpha_{i-1})) \neq 0$, $\alpha_1(A_0(\alpha_m)) \neq 0$, $i = 2, \dots, m$. *Write explicitly a minimal invariant CP-system of rank n, which is cyclic*:

$A_0(\delta_i)$	e_1	e_2	e_3	$\dots$	e_n
δ_i	δ_n	δ_1	δ_2	$\dots$	δ_{n-1}

The importance of cyclic systems is obvious. Every nondegenerate root system, which contains a cyclic subsystem of rank n, is a system with the chain property. It is easy to show, for example, that every system with the chain property of rank 2 contains a cyclic subsystem of rank 2. Notice that it is easy to construct CP-systems with the root spaces

$A_1(\alpha)$ and $A_1(-\alpha)$ of distinct dimensions. Also, one may construct a base of rank n and length greater than n.

Example 8.2. *Give an example of a minimal invariant CP-system of rank n and length $n+1$ with n linearly independent roots $\alpha_i,\ i=1,\dots,n$:*

$A_0(\alpha_i)$	e_1	e_1	e_2	$\dots$	e_{n-1}	e_n
α_i	α_1	α_2	α_3	$\dots$	α_n	$2\alpha_n$

$\alpha_i = \delta_1 + \ldots + \delta_i$. *Note that this system is embedded into a cyclic system or it may be rewritten as a cyclic system*: $2\alpha_n, \alpha_n, \dots, \alpha_1$.

The following lemma is obvious.

Lemma 8.5. *Let A be a left-symmetric algebra in $\mathcal{C}$ with a nilpotent subalgebra A_0 of dimension n and a nondegenerate root system Φ_1. Fix a set Γ of n linearly independent roots in Φ_1. Then Φ_1 is a system with the chain property if and only if for every $\gamma \in \Gamma$ there is a γ-system in Φ_1.*

Note that the condition of diagonality of the action of A_0 is essential for existence of a system of rank n if $dim\, A_0 = n$. Show, for example, existence of algebras in $\mathcal{A}$ with a zero-product even part A_0 of an arbitrary dimension and of rank 1 (in this case the action of A_0 is not diagonal).

Let $A_\alpha = \langle u_1, \dots, u_k \rangle$, $A_{-\alpha} = \langle v_1, \dots, v_s \rangle$. Let $A := A_0 \oplus A_\alpha \oplus A_{-\alpha}$ and the action of A_0 be the following

$$au_i = \alpha(a)u_i + u_{i+1},\ au_k = \alpha(a)u_k,\ av_i = -\alpha(a)v_i + v_{i+1},\ av_s = -\alpha(a)v_s.$$

It is easy to see that $b(av) = a(bv)$ for all $a, b \in A_0,\ v \in A_1$. Then from $0 = a(u_i v_s) = (\alpha(a)u_i + u_{i+1})v_s - \alpha(a)u_i v_s$ we get $u_{i+1}v_s = 0$ for all $i \neq k$. Analogously, $v_{i+1}u_k = 0$ for all $i \neq s$. From

$$0 = a(u_i v_j) = (\alpha(a)u_i + u_{i+1})v_j + u_i(-\alpha(a)v_j + v_{j+1})$$

we obtain $u_{i+1}v_j + u_i v_{j+1} = 0$ for all $i \neq k,\ j \neq s$. Thus, $u_k v_1 \equiv_2 u_{k-s+1}v_s = 0$ if $k > s$, and A is not simple. Further, assume that $k = s$. From the obtained equalities we also see that $A_0 \neq A_\alpha A_{-\alpha}$ if $k < n$. Thus, we assume that $k = n$, $A_0 = A_\alpha A_{-\alpha} = \langle v_1 u_i : i = 1, \dots, n \rangle$. Finally, we have to require $\alpha(v_1 u_n) \neq 0$ for the simplicity. Now, we may apply Lemma 8.2 to A in order to prove the simplicity of A. Thus, we have proved the following

Lemma 8.6. *Let $A = A_0 \oplus A_\alpha \oplus A_{-\alpha}$ be as above with $dim\, A_\alpha = dim\, A_{-\alpha} = dim\, A_0 = n$, and $\alpha(v_1 u_n) \neq 0$. Then $A \in \mathcal{A}$.*

8.3. **On algebras in $\mathcal{A}$ with a simple even part.** In this subsection we assume A_0 to be simple, whence $dim\, A_0 = 1$ and A_0 coincides with the main field F. In what follows, for simplicity we assume F to be algebraically closed. First, we suppose that A_0 acts diagonally on A_1. We say that $A_1(\alpha)$ and $A_1(1-\alpha)$ are *dual* provided that $A_1(\alpha) := \left\langle x_\alpha^{(1)}, \dots, x_\alpha^{(k)} \right\rangle$, $A_1(1-\alpha) := \left\langle x_{1-\alpha}^{(1)}, \dots, x_{1-\alpha}^{(k)} \right\rangle$, and only the following products $x_\alpha^{(i)} \cdot x_{1-\alpha}^{(i)} = 1 = x_{1-\alpha}^{(i)} \cdot x_\alpha^{(i)}$ are nonzero for all $i = 1, \dots, k$.

Lemma 8.7. *Let $A = A_0 \oplus A_1 \in \mathcal{A}$, let A_0 be simple, and let A_0 act diagonally on A_1. Then one of the following cases holds*

$$\begin{aligned} A &= F \oplus \sum_{\alpha \neq \frac{1}{2}} (A_1(\alpha) \oplus A_1(1-\alpha)), \\ A &= F \oplus A_1(\tfrac{1}{2}) \oplus \sum_{\alpha \neq \frac{1}{2}} (A_1(\alpha) \oplus A_1(1-\alpha)), \end{aligned}$$

where $dim\, A_1(\alpha) = dim\, A_1(1-\alpha)$ for every $\alpha \in \Phi_1$, and $A_1(\alpha)$ and $A_1(1-\alpha)$ are dual. Conversely, every such algebra belongs to $\mathcal{A}$.

Proof. Notice that the left-symmetry of A follows from Lemma 8.1 and the fact that the action of A_0 is diagonal. Under hypothesis of the lemma, A_0 possesses the unique root $\mathbf{1}$. For every root α on A_1 there is a unique root β on A_1 such that $\alpha + \beta = \mathbf{1}$ and $A_1(\alpha)A_1(\beta) = F$. Thus, in this case we arrive at the algebra structure from the assertion of the lemma. In this case $dim\, A_1(\alpha) = dim\, A_1(1-\alpha)$ and the dual bases for $A_1(\alpha)$ and $A_1(1-\alpha)$ may be chosen as in Lemma 8.4. The converse statement follows immediately from Lemmas 8.1 and 8.2. $\square$

Let $A = A_0 \oplus A_1 := A^{\alpha}_{1,n} \in \mathcal{A}$, let $A_0 = \langle e \rangle$ be simple, and let A_0 act on $A_1 = \langle x_1, \ldots, x_n \rangle$ as follows:

$$e \cdot x_i = \alpha x_i + x_{i+1},\ i = 1, \ldots, n-1,\ e \cdot x_n = \alpha x_n.$$

We say that x_n is a *minimal* vector and x_1 is a *maximal* vector for $A^{\alpha}_{1,n}$. Denote $A^{\frac{1}{2}}_{1,n}$ by $A_{1,n}$. In what follows, we say that an algebra $A \in \mathcal{C}$ is *degenerate* if its odd part A_1 contains a degenerate root subspace.

Lemma 8.8. *The algebra $A_{1,n}$ has the following product*:

$$\begin{cases} x_i \cdot x_j = 0 & \text{if } i+j > n+1, \\ x_i \cdot x_j = 0 & \text{if } i-j \equiv 1 \pmod 2, \\ x_i \cdot x_j = (-1)^{\frac{(i-j)}{2}} x^2_{\frac{(i+j)}{2}} & \text{otherwise.} \end{cases}$$

In the case $n = 2k$, the algebra $A_{1,2k}$ is degenerate. In the case $n = 2k+1$, the algebra $A_{1,2k+1}$ is nondegenerate if and only if $x^2_{k+1} \neq 0$.

Proof. Since $ea = a$ for every $a \in A_0$; therefore, by (4.3) for all $i, j \neq n$ we have

$$x_i \cdot x_j = e \cdot (x_i \cdot x_j) = (\frac{1}{2}x_i + x_{i+1})x_j + x_i(\frac{1}{2}x_j + x_{j+1}),$$

$$x_i \cdot x_{j+1} + x_{i+1} \cdot x_j = 0, \tag{8.1}$$

$$x_i \cdot x_n = e \cdot (x_i \cdot x_n) = (\frac{1}{2}x_i + x_{i+1})x_n + \frac{1}{2}x_i x_n),$$

$$x_{i+1} \cdot x_n = 0, \tag{8.2}$$

whence $x_i x_{i+1} = 0$ for all $i \neq n$ and $x_i \cdot x_j = 0$ if $i + j > n + 1$. Now, applying (8.1) we see that $x_i \cdot x_j = 0$ if $i - j \equiv 1 \pmod 2$, and $x_i \cdot x_j = (-1)^{\frac{(i-j)}{2}} x^2_{\frac{(i+j)}{2}}$ otherwise.

In the case $n = 2k$ the algebra $A_{1,2k}$ is degenerate, since $x_i \cdot x_n = 0$ for all i. Show that in the case $n = 2k+1$ the algebra $A_{1,2k+1}$ is nondegenerate if and only if $x_{k+1}^2 \neq 0$. Indeed, if $A_{1,2k+1}$ is nondegenerate then $x_1 x_n \equiv_2 x_{k+1}^2 \neq 0$. Conversely, if $(\sum_{i=1}^n \alpha_i x_i)x = 0$ for all x then we obtain $\alpha_1, \dots, \alpha_n = 0$ putting consequentially $x = x_n, x_{n-1}, \dots, x_1$. □

Thus, to define the product in $A_{1,n}$ we have to put $x_i^2 = \beta_i e$ for some $\beta_i \in F$ and for all $i = 1, \dots, [\frac{n+1}{2}]$.

Assume that $A_{1,n}^\alpha$ and $A_{1,m}^\beta$ possess a common even part $A_0 = \langle e \rangle$, $A_{1,n}^\alpha = \langle e, x_1, \dots, x_n \rangle$, and $A_{1,m}^\beta = \langle e, y_1, \dots, y_m \rangle$.

Lemma 8.9. *Let $\alpha, \beta \neq \frac{1}{2}$. The algebra $A_{1,n}^\alpha + A_{1,m}^\beta$ has nonzero product of odd elements only in the case $\alpha + \beta = 1$. The product in $A_{1,n}^\alpha + A_{1,m}^\beta$ is such that*

$$y_{j+1} \cdot x_i + y_j \cdot x_{i+1} = 0, \tag{8.3}$$
$$y_{j+1} \cdot x_n = 0,\ y_m \cdot x_{i+1} = 0 \tag{8.4}$$

for all $j \neq m, i \neq n$. In particular, $A_{1,n}^\alpha + A_{1,m}^\beta$ is nondegenerate if and only if $n = m$ and $x_n y_1 \neq 0$.

Proof. Obviously, $\alpha + \beta = 1$. Since $ea = a$ for every $a \in A_0$; therefore, by (4.3) for all $i \neq n, j \neq m$ we have

$$\begin{aligned}
&x_i \cdot y_j = e \cdot (x_i \cdot y_j) = (\alpha x_i + x_{i+1}) y_j + x_i(\beta y_j + y_{j+1}),\\
&x_i \cdot y_{j+1} + x_{i+1} \cdot y_j = 0,\\
&x_i \cdot y_m = e \cdot (x_i \cdot y_m) = (\alpha x_i + x_{i+1}) y_m + \beta x_i y_m,\\
&x_n \cdot y_j = e \cdot (x_n \cdot y_j) = \alpha x_n y_j + x_n(\beta y_j + y_{j+1}),\\
&x_{i+1} \cdot y_m = 0,\ x_n \cdot y_{j+1} = 0.
\end{aligned}$$

Prove the non-degeneracy assertion. If $n > m$ then

$$y_1 x_n \equiv_2 \dots \equiv_2 y_m x_{n-m+1} = 0,$$

whence $x_n A_{1,m}^\beta = 0$. Thus, $m = n$ and $x_n y_1 \neq 0$. □

Proposition 8.2. *Let $A = A_0 \oplus A_1$ be a nondegenerate finite-dimensional left-symmetric algebra in $\mathcal{C}$ with the simple even part A_0 acting non-diagonally on A_1. Then*

$$A = \sum_{i \in I} A_{1,m_i}^{\alpha_i},$$

where the product is coordinated by the equalities (8.1) – (8.4). Let $e_1, \dots, e_n$ be some linearly independent set of minimal vectors of all $A_{1,k}^\alpha$ for every fixed $\alpha \in \Phi_1$ and $k \in \mathbb{N}$, and let $f_1, \dots, f_n$ be the corresponding set of maximal vectors in $A_{1,k}^{1-\alpha}$. Let $e_i \cdot f_j = \gamma_{ij} e$. The algebra A is simple if and only if the matrix $\Gamma_k(\alpha) := (\gamma_{ij})$ is nondegenerate for all such k and α.

Proof. We need to prove only the simplicity condition, which is equivalent to the non-degeneracy condition. Note that $xA = 0$ implies $(ex)A = 0$ by Lemma 2.1. Thus, if $x = \sum x_p \in Ann_l(A)$ then we may assume that every x_p is a minimal vector for some fixed root α. Furthermore, we may assume that x_p has a fixed length k, since only the minimal vectors of length k may give nonzero products with the corresponding maximal vectors of length k. Thus, $\sum_{i=1}^{n} \alpha_i e_i \in Ann_l(A)$, i. e., $(\sum_{i=1}^{n} \alpha_i e_i) f_j = 0$ for all j. Considering these equalities as a linear system with respect to α_i, we see that this system possesses a nontrivial solution if and only if $\Gamma_k(\alpha) = (\gamma_{ij})$ is degenerate. $\square$

Remark 8.1. *A similar assertion may be stated and proved for the algebras in $\mathcal{A}$ with a zero-product even part A_0 acting non-diagonally on A_1. In this case we have to modify the condition on $\Gamma_k(\alpha) = (e_i \cdot f_j) \in M_k(A_0)$, considering $\Gamma_k(\alpha)$ as a linear operator from F_k to A_0 with the usual right action. Thus, we have to require the non-degeneracy of this operator. Also, some non-degeneracy conditions for the set of roots should be required.*

8.4. **On algebras in $\mathcal{A}$ with an arbitrary even part.** In this subsection we firstly give an easy example of a simple left-symmetric algebra A in $\mathcal{A}$ such that its even part A_0 is the direct sum of a simple subalgebra S and a zero-product ideal N, i. e., $A_0 = S \oplus N$, and the action of N on A_1 is not diagonal. To this end we put

$$S = \langle e \rangle,\ N = \langle a \rangle,\ A_1 = V_{\alpha_1} \oplus V_{\alpha_2},\ V_{\alpha_1} = \{v_1, v_2\},\ V_{\alpha_2} = \{u_1, u_2\},$$

and define nonzero product on $A = A_0 \oplus A_1$ by the table

$ev_i = \alpha v_i$	$eu_i = (1-\alpha)u_i$	$v_1 u_1 = \beta e + \gamma a$
$av_1 = pv_1 + v_2$	$au_1 = -pu_1 + u_2$	$v_1 u_2 = \delta a$
$av_2 = pv_2$	$au_2 = -pu_2$	$v_2 u_1 = (\beta - \delta)a$

where $\alpha, \beta, \gamma, \delta, p \in F$, $\alpha \neq \frac{1}{2},\ p, \beta, \delta \neq 0,\ \beta \neq \delta$. We see that

$\alpha_1(e) = \alpha$	$\alpha_2(e) = 1 - \alpha$
$\alpha_1(a) = p$	$\alpha_2(a) = -p$

Applying Lemmas 8.1 and 8.2, we infer that A is a simple left-symmetric algebra.

Proposition 8.3. *Let k be the maximal order of Jordan blocks for L_a on A_1, where a ranges over A_0. Assume that A_0 is nilpotent. Then $L_a^{2k-1} = 0$ on A_0 for every $a \in A_0$, A_0 is a nil-algebra of index $\leq 2k$, and A_0 is nilpotent of index $\leq 4k^2$. In particular, if A_0 acts on A_1 diagonally then A_0 is zero-product. If $A_0 = S \oplus N$, where S is a semisimple subalgebra and N is a nilpotent ideal, then the nilpotency index of N is bounded by $4k^2$.*

Proof. Take $a \in A_0,\ \alpha \in \Phi_1$. Without loss of generality, we may assume that $A_\alpha = \langle u_1, \ldots, u_k \rangle$ and $A_{-\alpha} = \langle v_1, \ldots, v_s \rangle$ are some Jordan blocks with respect to L_a, $s \leq k$. Then from

$$\begin{aligned} a(u_i v_j) &= (\alpha(a)u_i + u_{i+1})v_j + u_i(-\alpha(a)v_j + v_{j+1}),\ i \neq k,\ j \neq s, \\ a(u_k v_s) &= (\alpha(a)u_k)v_s - \alpha(a)u_k v_s = 0 \end{aligned}$$

we have $u_1v_1 \overset{L_a}{\mapsto} u_2v_1 + u_1v_2 \overset{L_a}{\mapsto} u_3v_1 + 2u_2v_2 + u_1v_3 \overset{L_a}{\mapsto} \ldots 0$, and $L_a^{2k-1} = 0$ on A_0. In particular, if A_0 acts diagonally then $k = 1$ and $L_a = 0$ on A_0, i. e., A_0 is zero-product. By Razmyslov's theorem, A_0 is nilpotent of index $\leq 4k^2$.

In the case when $A_0 = S \oplus N$, where S is a semisimple subalgebra and N is a nilpotent ideal, we proceed analogously. Take some Jordan blocks $U = \langle u_1, \ldots, u_k\rangle \subseteq A_\alpha$ and $V = \langle v_1, \ldots, v_s\rangle \subseteq A_\beta$ of A_1. Then from

$$\begin{aligned} a(u_iv_j) &= (\alpha(a)u_i + u_{i+1})v_j + u_i(\beta(a)v_j + v_{j+1}),\ i \neq k,\ j \neq s, \\ a(u_kv_s) &= (\alpha + \beta)(a)u_kv_s \end{aligned}$$

for every $a \in N$ we have $(\alpha + \beta)(a) = 0$, since N is nilpotent. Proceeding by analogy with the previous case, we arrive at the required assertion. □

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