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AUTOMORPHISMS OF AFFINE VERONESE SURFACES

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Abstract. We prove that every derivation and every locally nilpotent derivation of the subalgebra $K[x^n, x^{n-1}y, \ldots, xy^{n-1}, y^n]$, where $n \geq 2$, of the polynomial algebra $K[x, y]$ in two variables over a field K of characteristic zero is induced by a derivation and a locally nilpotent derivation $K[x, y]$, respectively. Moreover, we prove that every automorphism of $K[x^n, x^{n-1}y, \ldots, xy^{n-1}, y^n]$ over an algebraically closed field K of characteristic zero is induced by an automorphism of $K[x, y]$. We also show that the group of automorphisms of $K[x^n, x^{n-1}y, \ldots, xy^{n-1}, y^n]$ admits an amalgamated free product structure.

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1. Introduction

Let K be an arbitrary field and let $\mathbb{A}^n$ and $\mathbb{P}^n$ be the affine and the projective n-space over K, respectively. The *Veronese map* of degree d is the map

$$\nu_d : \mathbb{P}^n \to \mathbb{P}^m$$

that sends $[x_0 : \ldots : x_n]$ to all $m+1$ possible monomials of total degree d, where

$$m = \binom{n+1}{d} - 1 = \binom{n+d}{d} - 1.$$

It is well known that the image of the Veronese map is a projective variety and is called the *Veronese variety* [3].

The *rational normal curve* $C \subset \mathbb{P}^d$ is a particular case of the Veronese variety and is defined to be the image of the map

$$\nu_d : \mathbb{P}^1 \to \mathbb{P}^d$$

given by

$$\nu_d : [x_o : x_1] \mapsto [x_0^d : x_0^{d-1}x_1 : \ldots : x_1^d] = [z_o : \ldots : z_d].$$

It is well known that C is the common zero locus of the polynomials

$$F_{i,j}(z_0, ..., z_n) = z_i z_j - z_{i-1}z_{j+1} \; for \; 1 \leq i \leq j \leq d-1. \tag{1}$$

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For $d = 2$ it is the plane conic $z_0z_2 = z_1^2$ and for $d = 3$ it is the twisted cubic [3].

Denote by $V_n \subset \mathbb{A}^{n+1}$ the common zero locus of the polynomials (1) in $\mathbb{A}^{n+1}$. The variety V_n will be called the *affine Veronese surface*. This paper is devoted to the study of the automorphism group of the affine Veronese surface V_n for all $n \geq 2$.

In 1990 L. Makar-Limanov described the generators of the automorpism groups of algebraic surfaces defined by an equation of the form $xy = P(z)$ over an algebraically closed field [8]. This result gives the generators of the automorphism group of V_2. The amalgamated free product structure of this group can deduced from Lamy's results [6, 7] on the structure of the group $\mathrm{Aut}_Q\,[\mathbb{C}^3]$ of polynomial automorphisms of $\mathbb{C}^3$ preserving the quadratic form $Q = xz + y^2$.

It is not difficult to show that the algebra of polynomial functions on V_n is isomorphic to the subalgebra $K[z_0^n, z_0^{n-1}z_1, \ldots, z_1^n]$ of $K[z_o, z_1]$. Thus the group of automorphisms of V_n is anti-isomorphic to the group of automorphisms of the algebra $K[z_0^n, z_0^{n-1}z_1, \ldots, z_1^n]$.

It is well known [4, 5] that all automorphisms of the polynomial algebra $K[x, y]$ in two variables x, y over a field K are tame. Moreover, the automorphism group $\mathrm{Aut}\,K[x, y]$ of this algebra admits an amalgamated free product structure [5, 12], i.e.,

$$\mathrm{Aut}\,K[x, y] = \mathrm{Aff}_2\,(K) *_C \mathrm{Tr}_2\,(K), \tag{2}$$

where $C = \mathrm{Aff}_2\,(K) \cap \mathrm{Tr}_2\,(K)$.

The well-known Nagata automorphism (see [9])

$$\sigma = (x + 2y(zx - y^2) + z(zx - y^2)^2, y + z(zx - y^2), z)$$

of the polynomial algebra $K[x, y, z]$ over a field K of characteristic 0 is proven to be non-tame [13].

In this paper we show that over a field K of characteristic zero every derivation and every locally nilpotent derivation of the algebra $K[x^n, x^{n-1}y, ..., xy^{n-1}, y^n]$ is induced by a derivation and a locally nilpotent derivation of $K[x, y]$, respectively. Using the proof of the Rentchler's Theorem [10] on locally nilpotent derivations of $K[x, y]$ given in [2, Ch. 5], we prove that every automorphism of $K[x^n, x^{n-1}y, ..., xy^{n-1}, y^n]$ is induced by an automorphism of $K[x, y]$ if K is an algebraically closed field of characteristic zero. We also show that the amalgamated free product structure of the automorphism group of $K[x, y]$ induces an amalgamated free product structure on the automorphism group of $K[x^n, x^{n-1}y, ..., xy^{n-1}, y^n]$.

The paper is organized as follows. In Section 2 we recall some necessary results on the structure of the automorphism group of $K[x, y]$ from [1, 2]. Section 3 is devoted to lifting of derivations of $K[x^n, x^{n-1}y, ..., xy^{n-1}, y^n]$ to derivations of $K[x, y]$. In Section 4 we prove that so called n-derivations of $K[x, y]$ are triangulable. In Section 5 we prove that every automorphism of $K[x^n, x^{n-1}y, ..., xy^{n-1}, y^n]$ is induced by an automorphism of $K[x, y]$. The amalgamated free product of the automorphism group of $K[x^n, x^{n-1}y, ..., xy^{n-1}, y^n]$ is given in Section 6.

2. Automorphisms of $K[x, y]$

Let $K[x,y]$ be the polynomial algebra in the variables x, y over a field K and let $\operatorname{Aut} K[x,y]$ be the group of automorphisms of $K[x,y]$. Denote by $\phi = (f, g)$ the automorphism of $K[x,y]$ such that $\phi(x) = f$ and $\phi(y) = g$, where $f, g \in K[x,y]$. If $\phi = (f_1, g_1)$ and $\psi = (f_2, g_2)$ then the product in $\operatorname{Aut} K[x,y]$ is defined by

$$\phi \circ \psi = (f_2(f_1, g_1), g_2(f_1, g_1)).$$

An automorphism $\phi \in \operatorname{Aut} K[x,y]$ is called *elementary* if it has the form

$$\phi = (x, \alpha y + f(x))$$

or

$$\phi = (\alpha x + g(y), y),$$

where $f(x) \in K[x]$, $g(y) \in K[y]$, and $0 \neq \alpha \in K$. The subgroup of $\operatorname{Aut} K[x,y]$ generated by all elementary automorphisms is called the *tame subgroup*. Elements of this subgroup are called *tame automorphisms* of $K[x,y]$.

An automorphism $\phi \in \operatorname{Aut} K[x,y]$ is called *affine* if it has the form

$$\phi = (\alpha_1 x + \beta_1 y + \gamma_1, \alpha_2 x + \beta_2 y + \gamma_2)$$

where $\alpha_1\beta_2 \neq \beta_1\alpha_2$ and $\alpha_1, \alpha_2, \beta_1, \beta_2, \gamma_1, \gamma_2 \in K$. The subgroup $\operatorname{Aff}_2(K)$ of $\operatorname{Aut} K[x,y]$ generated by all affine automorphisms is called the *affine subgroup*. If $\gamma_1, \gamma_2 = 0$ then the affine automorphism ϕ is called *linear*. The subgroup $\operatorname{GL}_2(K)$ of $\operatorname{Aff}_2(K)$ generated by all linear automorphisms is called the *linear subgroup*.

An automorphism $\phi \in \operatorname{Aut} K[x,y]$ is called *triangular* if it has the form

$$\phi = (\alpha x + f(y), \beta y + \gamma), \tag{3}$$

where $0 \neq \alpha, \beta \in K$ and $f(y) \in K[y]$. The subgroup $\operatorname{Tr}_2(K)$ of $\operatorname{Aut} K[x,y]$ generated by all triangular automorphisms is called the *triangular subgroup*.

The well known Jung-van der Kulk Theorem [4, 5] says that all automorphisms of the polynomial algebra $K[x,y]$ in two variables x, y over a field K are tame. Moreover, van der Kulk and Shafarevich [5, 12] proved that the automorphism group $\operatorname{Aut} K[x,y]$ of this algebra admits an amalgamated free product structure, i.e.,

$$\operatorname{Aut} K[x,y] = \operatorname{Aff}_2(K) *_C \operatorname{Tr}_2(K),$$

where $C = \operatorname{Aff}_2(K) \cap \operatorname{Tr}_2(K)$.

We fix a grading

$$K[x,y] = K[x,y]_0 \oplus K[x,y]_1 \oplus K[x,y]_2 \oplus \ldots \oplus K[x,y]_{n-1}$$

of the polynomial algebra $K[x,y]$, where $K[x,y]_i$ is the linear span of all homogeneous monomials of degree $i + ns$, $i = 0, 1, \ldots, n-1$, and s is an arbitrary nonnegative integer. This is a $\mathbb{Z}_n$-grading of $K[x,y]$, i.e.,

$$K[x,y]_i K[x,y]_j \subseteq K[x,y]_{i+j},$$

where $i, j \in \mathbb{Z}_n = \mathbb{Z}/n\mathbb{Z}$.

An automorphism $\phi \in \operatorname{Aut} K[x,y]$ will be called an *n-automorphism* if $\phi(x), \phi(y) \in K[x,y]_1$. Obviosly every n-automorphism induces an automorphism of the algebra $K[x^n, x^{n-1}y, \ldots, xy^{n-1}, y^n]$. An n-automorphism will be called a *tame n-automorphism* if it is a product of elementary n-automorphisms.

Set $K[x,y]_1 \cap K[y] = K[y]_1$. An automorphism $\phi \in \operatorname{Aut} K[x,y]$ is called *n-triangular* if it has the form

$$\phi = (\alpha x + f(y), \beta y),$$

where $0 \neq \alpha, \beta \in K$ and $f(y) \in K[y]_1$.

3. Derivations of $K[x^n, x^{n-1}y, ..., xy^{n-1}, y^n]$

Let K be an arbitrary field of characteristic zero. Let A be any algebra over K. A derivation D of A is called *locally nilpotent* if for every $a \in A$ there exists a positive integer $n = n(a)$ such that $D^n(a) = 0$.

If D is a locally nilpotent derivation of A then

$$\exp D = \sum_{p \geq 0} \frac{1}{p!} D^p$$

is an automorpism of A and is called an *exponential* automorphism.

Moreover, if D is any derivation of A then

$$\exp TD = \sum_{i=0}^{\infty} \frac{1}{i!} D^i T^i$$

is an automorpism of the formal power series algebra $A[[T]]$. If D is locally nilpotent then $\exp TD$ is an automorphism of $A[T]$

A derivation D of $K[x,y]$ will be called an *n-derivation* if $D(x), D(y) \in K[x,y]_1$. Obviously, every n-derivation of $K[x,y]$ induces a derivation of $K[x^n, x^{n-1}y, ..., xy^{n-1}, y^n]$. The reverse is also true.

Lemma 1. *Every derivation of $K[x^n, x^{n-1}y, ..., xy^{n-1}, y^n]$ can be uniquely extended to an n-derivation of $K[x,y]$.*

Proof. Let D be a derivation of $K[x^n, x^{n-1}y, ..., xy^{n-1}, y^n]$. Denote by T the unique extension of D [14, p. 120] to a derivation of the field of fractions $K(x^n, x^{n-1}y, ..., xy^{n-1}, y^n)$ of $K[x^n, x^{n-1}y, ..., xy^{n-1}, y^n]$. Obviously, the field extension

$$K(x^n, x^{n-1}y, ..., xy^{n-1}, y^n) \subseteq K(x,y)$$

is algebraic. This extension is separable since K is a field of characteristic zero. By Corollaries 2 and 2' in [14, pages 124–125], every derivation of the field $K(x^n, x^{n-1}y, ..., xy^{n-1}, y^n)$ can be uniquely extended to a derivation of $K(x,y)$. Let S be the unique extension of T to a derivation of $K(x,y)$. Suppose that

$$S(x) = \frac{f_1}{g_1},\ S(y) = \frac{f_2}{g_2}, \tag{4}$$

where $f_1, f_2 \in K[x,y]$, $0 \neq g_1, g_2 \in K[x,y]$, and the pairs f_1, g_1 and f_2, g_2 are relatively prime. We have

$$D(x^{n-i}y^i) = S(x^{n-i}y^i) = (n-i)x^{n-i-1}y^i \frac{f_1}{g_1} y_1 + i x^{n-i} y^{i-1} \frac{f_2}{g_2}$$

for all $0 \leq i \leq n$.

Since $D(x^n), D(x^{n-1}y), \dots, D(xy^{n-1}), D(y^n) \in A$ it follows that

$$g_1g_2|(n-i)x^{n-(i+1)}y^if_1g_2 + ix^{n-i}y^{i-1}f_2g_1$$

for all $0 \le i \le n$. Consequently,

$$g_1|(n-i)x^{n-(i+1)}y^i$$

and

$$g_2|ix^{n-i}y^{i-1}$$

for all $0 \le i \le n$.

This means that $g_1|x^{n-1}$ and $g_1|y^{n-1}$ and, consequently, we may assume that $g_1 = 1$. Similarly, $g_2|y^{n-1}$ and $g_2|x^{n-1}$ give that $g_2 = 1$. Obviously, $f_1, f_2 \in K[x,y]_1$. □

For any derivation D of $K[x^n, x^{n-1}y, ..., xy^{n-1}, y^n]$ denote by $\widetilde{D}$ its unique extension to a derivation of $K[x,y]$ determined by Lemma 1. Obviously D is locally nilpotent if $\widetilde{D}$ is locally nilpotent. The reverse statement is also true.

Lemma 2. *If D is a locally nilpotent derivation of $K[x^n, x^{n-1}y, ..., xy^{n-1}, y^n]$ then $\widetilde{D}$ is a locally nilpotent n-derivation of $K[x,y]$.*

Proof. Suppose that D is a locally nilpotent derivation of $K[x^n, x^{n-1}y, ..., xy^{n-1}, y^n]$. Then $\exp TD$ is an automorphism of $K[x^n, x^{n-1}y, ..., xy^{n-1}, y^n][T]$. Recall that $\exp T\widetilde{D}$ is an automorphism of $K[x,y][[T]]$. We have

$$\exp TD(x^n) = \exp T\widetilde{D}(x^n) = \exp T\widetilde{D}(x)^n.$$

This implies that $\exp T\widetilde{D}(x) \in K[x,y][T]$ since $\exp TD(x^n) \in K[x,y][T]$. Similarly, $\exp T\widetilde{D}(y) \in K[x,y][T]$. This means that there exist natural numbers m and n such that $\widetilde{D}^m(x) = 0$ and $\widetilde{D}^n(y) = 0$. Therefore $\widetilde{D}$ is locally nilpotent. □

4. Triangulation of locally nilpotent n-derivations

A derivation D of $K[x,y]$ is called *triangular* if

$$D(x) = f(y) \in K[y], \quad D(y) = \alpha \in K.$$

A derivation D of $K[x,y]$ is called *triangulable* if there exists an automorphism $\alpha \in \operatorname{Aut} K[x,y]$ such that $\alpha^{-1}D\alpha$ is triangular.

Every triangular derivation, and hence every triangulable derivation, is locally nilpotent. In 1968 R. Rentschler [10] proved that every locally nilpotent derivation of the polynomial algebra $K[x,y]$ over a field of characteristic zero is triangulable. In this section we adopt the proof of this result given in [2, Ch. 5] to prove that every locally nilpotent n-derivation of $K[x,y]$ is triangulable by a tame n-automorphism.

First recall some necessary definitions from [2].

Let $0 \neq w = (w_1, w_2) \in \mathbb{Z}^2$. Then w-degree of the monomial $x^{a_1}y^{a_2}$ is defined by $w(x^{a_1}y^{a_2}) = a_1w_1 + a_2w_2$. This degree function leads to the w-grading

$$K[x,y] = \sum_d W_d$$

of $K[x,y]$, where W_d is the span of all monomials of w-degree d.

Let $T := cx^{a_1}y^{a_2}\partial_i$ be a monomial derivation of $K[x,y]$, where $i = 1, 2$. Set $(s,t) = (a_1, a_2) - e_i$, where e_i is the i-th vector of the standard basis of $\mathbb{K}^2$. Then

$$T(x^{m_1}y^{m_2}) \in Kx^{m_1+s}y^{m_2+t}$$

for all m_1, m_2. We call (s,t) the *strength* of T.

Every derivation D is a linear combination of monomial derivations. Set

$$\operatorname{supp} D = \{(s,t) \in \mathbb{Z}^2 \mid D \textit{ contains a term of strength} (s,t)\}.$$

Let us denote by $D(s,t)$ the sum of all terms in D of strength (s,t) and set

$$D_p = \sum_{sw_1+tw_2=p} D(s,t).$$

Obviously,

$$D = \sum_p D_p$$

and this decomposition is called the *w-homogeneous* decomposition of D. If p is maximal with $D_p \neq 0$ then p is called the *w-degree of* D and is denoted by *wdeg* D. When $w = (1,1)$ p is called the *degree of* D and is denoted by *deg* D.

It is easy to check [2] that $D_pW_d \subset W_{p+d}$ for all $p, d \in \mathbb{Z}$.

Proposition 1. *Let D be a locally nilpotent n-derivation of $K[x,y]$. Then there exists a tame n-automorphism α of $K[x,y]$ and $f(y) \in K[y]_1$ such that $\alpha^{-1}D\alpha = f(y)\partial_x$.*

Proof. Let D be a locally nilpotent n-derivation of $K[x,y]$. According to Corollary 5.1.16 in [2, p. 91], the following three cases are possible:

(*i*) $D = f(y)\partial_x$, for some $f(y) \in K[y]$;

(*ii*) $D = f(x)\partial_y$, for some $f(x) \in K[x]$;

(*iii*) there exist $s_0, t_0 \geq 0$ such that $(s_0, -1)$ and $(-1, t_0)$ belong to *supp* D and, furthermore, $\operatorname{supp} D$ is contained in the triangle with vertices $(s_0, -1)$, $(-1, -1)$, $(-1, t_0)$.

Case (*i*). If $D = f(y)\partial_x$ with $f(y) \in K[y]_1$ then set $\alpha = \mathrm{id}$. Obviously, the identity automorphism is an n-automorphism.

Case (*ii*). If $D = f(x)\partial_y$ with $f(x) \in K[x]_1$ then set $\alpha = (y, x)$. Obviously α is a n-automorphism of $K[x,y]$ and $\alpha^{-1}D\alpha = f(y)\partial_x$ with $f(y) \in K[y]_1$.

Case (*iii*). Suppose that we have $s_0, t_0 \geq 0$ such that $(s_0, -1), (-1, t_0) \in \operatorname{supp} D$. This implies that D contains differential monomials of the form $x^{s_0}\partial_y$ and $y^{t_0}\partial_x$ with nonzero coefficients. Hence $s_0 = 1 + nk$, $t_0 = 1 + nl$, $k, l \in \mathbb{Z}$ since $x^{s_0}, y^{t_0} \in K[x,y]_1$.

Let L be the line passing through the points $(1+nk, -1)$ and $(-1, 1+nl)$. The defining equation of L is

$$(nl+2)x + (nk+2)y = n^2kl + nk + nl = nM.$$

Set $w = (nl+2, nk+2)$ and $p = n^2kl + nk + nl$. Obviously $\operatorname{wdeg} D = p$ and D_p is the highest homogeneous part of D with respect to the w-degree. It is well known that the highest homogeneous part of a locally nilpotent derivation is locally nilpotent (see, for example [2, p. 90]). Consequently, D_p is a locally nilpotent n-derivation.

We can write $D_p = gD_1$, where $D_1 = a\partial_x + b\partial_y$ with $gcd(a,b) = 1$. By Corollary 1.3.34 in [2, p. 29], D_1 is locally nilpotent and $D_1(g) = 0$. By Proposition 1.3.46 in [2, p. 31], D_1 has a slice in $K[x,y]$, i.e., there exists $s \in K[x,y]$ such that $D_1(s) = 1$. This implies that $a(0,0) \neq 0$ or $b(0,0) \neq 0$. Assume that $a(0,0) \neq 0$. This means that D_1 has a term of the form $c\partial_x$, where $c \in K^*$. Since $(1+nk, -1) \in \operatorname{supp} D_p$ and $D_p = gD_1$ it follows that D_1 also has a term of the form $dx^r\partial_y$ with $r \geq 0$ and $d \in K^*$. Moreover, g and D_1 are w-homogeneous since D_p is w-homogeneous. Therefore $\operatorname{supp} D_1$ is on the line passing through the points $(-1,0)$ and $(r,-1)$. Notice that this line does not contain any other points with integer coordinates. Hence $D_1 = c\partial_x + dx^r\partial_y$. Since D_p is an n-derivation it follows that $g \in K[x,y]_1$ and $n|r$.

We have $g \in \operatorname{Ker} D_1 = K[y - \frac{d}{(r+1)c}x^{r+1}]$ since $D_1(g) = 0$. Consequently, $g = a(y - \frac{d}{(r+1)c}x^{r+1})^N$ for some $a \in K^*$ and $N \in \mathbb{N}$ since g is w-homogeneous. So

$$D_p = a(y - \frac{d}{(r+1)c}x^{r+1})^N(c\partial_x + dx^r\partial_y),$$

where $a, c, d \in K^*$, $r \geq 0$, and $N \in \mathbb{N}$. Obviously, $t_0 = N$ and $s_0 = (r+1)N + r$.

Let α be the automorphism given by

$$\alpha(x) = x,\ \alpha(y) = y - \frac{d}{(r+1)c}x^{r+1}.$$

This is an elementary n-automorphism since $n|r$. Direct calculations give that

$$\alpha^{-1}D_1\alpha = c\partial_x$$

and

$$\alpha^{-1}D_p\alpha = acy^{t_0}\partial_x.$$

Since α is w-homogeneous, $\alpha^{-1}D_p\alpha$ is the highest w-homogeneous part of $\alpha^{-1}D\alpha$. Thus α turns all points of $\operatorname{supp} D_p$ to one point $(-1, t_0)$. Consequently, $s_0(D) < s_0(\alpha^{-1}D\alpha)$. Leading an induction on $s_0(D) + t_0(D)$ we can conclude that the statement of the proposition is true. $\square$

5. Automorphisms of $K[x^n, x^{n-1}y, ..., xy^{n-1}, y^n]$

As we noticed above, every n-automorphism of $K[x,y]$ induces an automorphism of $K[x^n, x^{n-1}y, ..., xy^{n-1}, y^n]$. In this section we prove the reverse of this statement.

Theorem 1. *Every automorphism of $K[x^n, x^{n-1}y, ..., xy^{n-1}, y^n]$ over an algebraically closed field K of characteristic zero is induced by an n-automorphism of $K[x,y]$.*

Proof. Consider the derivation $D = y\partial_x$ of $K[x,y]$. Let $\overline{D}$ be the derivation of $K[x^n, x^{n-1}y, ..., xy^{n-1}, y^n]$ induced by D.

Let α be an arbitrary automorphism of $K[x^n, x^{n-1}y, ..., xy^{n-1}, y^n]$. Set $T = \alpha\overline{D}\alpha^{-1}$. This derivation is locally nilpotent since D is locally nilpotent. Let $\widetilde{T}$ be the extension of T to a derivation of $K[x, y]$ that uniquely defined by Lemma 1. By Lemma 2, $\widetilde{T}$ is a locally nilpotent n-derivation of $K[x, y]$. By Proposition 1, there exists an n-tame automorphism β of $K[x, y]$ such that $S = \beta^{-1}\widetilde{T}\beta$ is a triangular n-derivation of $K[x, y]$. Let

$$S = \beta^{-1}\widetilde{T}\beta = g(y)\partial_x,$$

where $g(y) \in K[y]_1$. We get

$$S(f) = g(y)\frac{\partial f}{\partial x}, \quad f \in K[x, y].$$

Let $\overline{\beta}$ be the automorphism of $K[x^n, x^{n-1}y, ..., xy^{n-1}, y^n]$ induced by β. Then S induces the derivation $\overline{S} = \overline{\beta}^{-1}T\overline{\beta} = \overline{\beta}^{-1}\alpha\overline{D}\alpha^{-1}\overline{\beta}$ of $K[x^n, x^{n-1}y, ..., xy^{n-1}, y^n]$.

Let $\phi = \overline{\beta}^{-1}\alpha$. Assume that $\phi(x^{n-i}y^i) = f_i$, where $0 \leq i \leq n$. Applying the equation $\phi\overline{D} = \overline{S}\phi$ to $x^{n-i}y^i$ for all i, we get

$$(n-i)f_{i+1} = g(y)\frac{\partial f_i}{\partial x},$$

i.e.,

$$0 = g(y)\frac{\partial f_n}{\partial x}, f_n = g(y)\frac{\partial f_{n-1}}{\partial x}, \ldots, (n-1)f_2 = g(y)\frac{\partial f_1}{\partial x}, nf_1 = g(y)\frac{\partial f_0}{\partial x}.$$

These equalities immediately give that

$$\deg_x f_n = 0, \deg_x f_{n-1} = 1, \ldots, \deg_x f_{n-i} = i, \ldots, \deg_x f_0 = n.$$

In particular, $f_n \in K[y]$.

We have

$$\frac{f_0}{f_1} = \frac{f_1}{f_2} = \ldots = \frac{f_{n-1}}{f_n} \tag{5}$$

since the generators $x^n, x^{n-1}y, ..., xy^{n-1}, y^n$ of $K[x^n, x^{n-1}y, ..., xy^{n-1}, y^n]$ satisfy the relations

$$\frac{x^n}{x^{n-1}y} = \frac{x^{n-1}y}{x^{n-2}y^2} = \ldots = \frac{xy^{n-1}}{y^n} = \frac{x}{y}.$$

Let $\frac{f_0}{f_1} = \frac{p}{q}$, where $p, q \in K[x, y]$ are relatively prime. Then $\frac{f_0}{f_n} = \frac{p^n}{q^n}$ by (5). Since p^n and q^n are relatively prime it follows that $f_0 = p^n u$ and $f_n = q^n u$ for some $u \in K[x, y]$. Moreover, (5) implies that $f_i = p^{n-i}q^i u$ for all i. From this we get

$$K[x^n, x^{n-1}y, ..., xy^{n-1}, y^n] \subseteq K + (u),$$

where (u) is the ideal of $K[x, y]$ generated by u. This is possible if the leading word of u divides all of the words $x^n, x^{n-1}y, ..., xy^{n-1}, y^n$. Consequently, $u \in K^*$. Over an algebraically closed field we can write $u = v^n$ for some $v \in K^*$. Replacing vp with vp and vq with q, we may assume that $u = 1$ and $f_i = p^{n-i}q^i$ for all i.

We have $q \in K[y]$ since $f_n = q^n \in K[y]$. We also have $\deg_x(p) = 1$ since $p^n = f_0$ and $\deg_x(f_0) = n$. Set $p = xa(y) + b(y)$. We get

$$K[x^n, x^{n-1}y, ..., xy^{n-1}, y^n] \subseteq K[f_n] + (p) \subseteq K[y] + (p),$$

where (p) is the ideal of $K[x,y]$ generated by p. Consequently,

$$x^n = (xa(y) + b(y))h + f(y).$$

Introducing a monomial order with prioritized x, we get that it is possible only if $a(y) = a \in K^*$. Consequently, $p = ax + b(y)$. Now it is easy to check that $p^n \in K[x^n, x^{n-1}y, ..., xy^{n-1}, y^n]$ implies that $p \in K[x,y]_1$. Set $\gamma = (ax+b(y), y)$. Then γ is an elementary n-automorphism of $K[x,y]$. Set $\psi = \overline{\gamma}^{-1}\phi$. Then $\psi(x^{n-i}y^i) = x^{n-i}q^i$ for all i. We have

$$K[x^n, x^{n-1}y, ..., xy^{n-1}, y^n] \subseteq K[q^n] + (x),$$

where (x) is the ideal of $K[x,y]$ generated by x. It is possible only if $q^n = cy^n$ for some $c \in K^*$. Consequently, $q = ey$ for some $e \in K^*$ since K is algebraically closed.

Let $\delta = (x, ey)$. Then $\overline{\delta}^{-1}\psi = \mathrm{id}$, i.e., $\overline{\delta}^{-1}\overline{\gamma}^{-1}\overline{\beta}^{-1}\alpha = \mathrm{id}$. Consequently, $\alpha = \overline{\beta}\overline{\gamma}\overline{\delta} = \overline{\beta\gamma\delta}$ is induced by a tame n-automorphism of $K[x,y]$. □

6. Amalgamated free product srtucture of Aut A

Let G_n be the group of all n-automorphisms of the polynomial algebra $K[x,y]$.

Lemma 3. *The subgroup G_n of* Aut $K[x,y]$ *is generated by all linear automorphisms and all automorphisms of the type $(x - \alpha y^m, y)$, where $m = 1 + ns$ is a positive integer and $\alpha \in K$.*

Proof. For any $f \in K[x,y]$ denote by $\bar{f}$ its highest homogeneous part with respect the standard degree function deg. Let $\phi = (f,g)$ be a n-automorphism of the algebra $K[x,y]$ and suppose that deg $f = k$ and deg $g = l$. If $k + l = 2$ then ϕ is a linear automorphism.

Suppose that $k+l \geq 3$. It is well known that $k|l$ or $l|k$ (see, for example [1, 2]). Assume that $l|k$. In this case we have $\bar{f} = \alpha\bar{g}^m$ for some $\alpha \in K^*$ and $m \in \mathbb{N}$. Since $\bar{f}, \bar{g} \in K[x,y]_1$ it follows that $m = 1+ns$ for some $s \geq 0$. In fact, let $\deg(\bar{f}) = 1+np$ and $\deg(\bar{g}) = 1+nq$. Then

$$1 + np = m(1 + nq).$$

Consequently, $m - 1 = np - mnq = ns$.

Therefore $(x - \alpha y^m, y)$ is an elementary n-automorphism of $K[x,y]$. We have

$$(f,g) \circ (x - \alpha y^m, y) = (f - \alpha g^m, g) = (f', g),$$

where $\deg(f') < \deg(f)$. Leading an induction on $k+l$ we may assume that (f', g) satisfies the statement of the lemma. Then (f,g) also satisfies the statement of the lemma. □

Let T_n be the group of all n-triangular automorphisms of the polynomial algebra $K[x,y]$.

Corollary 1. *$G_n = \mathrm{GL}_2(K) *_B T_n$, where $B = \mathrm{GL}_2(K) \cap T_n$.*

Proof. Lemma 3 says that G_n is generated by GL_2 and T_n. Consider (2). We have $\mathrm{GL}_2 \subseteq \mathrm{Aff}_2$, $T_n \subseteq \mathrm{Tr}_2(K)$, and $B \subseteq C$. This means that every decomposition of an element of G_n in the form

$$g_1 g_2 \cdots g_k,$$

where $g_i \in \mathrm{GL}_2 \cup T_n$ for all i and g_i and g_{i+1} do not belong together to GL_2 or T_n for all $i < k$, determined by the amalgated free product structure (2). Consequently,

$$G_n = \mathrm{GL}_2\,(K) *_B T_n \subseteq \mathrm{Aff}_2\,(K) *_C \mathrm{Tr}_2\,(K). \quad \square$$

Corollary 2. *Let* $E = \{\epsilon \mathrm{id} | \epsilon^n = 1, \epsilon \in K\}$. *Then*

$$\mathrm{Aut}\, K[x^n, x^{n-1}y, ..., xy^{n-1}, y^n] \cong G_n/E.$$

Proof. Consider the homomorphism

$$\psi : G_n \to \mathrm{Aut}\, K[x^n, x^{n-1}y, ..., xy^{n-1}, y^n] \tag{6}$$

defined by $\psi(\alpha) = \overline{\alpha}$, where $\overline{\alpha}$ is the automorphism of $K[x^n, x^{n-1}y, ..., xy^{n-1}, y^n]$ induced by the n-automorphism α of $K[x, y]$.

By Theorem 1, ψ is an epimorphism. Let $\alpha \in \mathrm{Ker}\, \psi$. Then

$$\alpha(x)^{n-i}\alpha(y)^i = x^{n-i}y^i$$

for all $0 \leq i \leq n$. This implies that $\alpha(x) = \epsilon x, \alpha(y) = \epsilon y$ for some nth root of unity $\epsilon \in K$. Consequently, $\alpha \in E$. Obviously, $E \subseteq \mathrm{Ker}\, \psi$. $\square$

Let

$$\overline{\mathrm{GL}_2\,(K)} = \mathrm{GL}_2\,(K)/E, \overline{T_n} = T_n/E, \overline{B} = B/E.$$

Theorem 2. $\mathrm{Aut}\, K[x^n, x^{n-1}y, ..., xy^{n-1}, y^n] \cong \overline{\mathrm{GL}_2\,(K)} *_{\overline{B}} \overline{T_n}$.

Proof. By Corollaries 1 and 2, the group $\mathrm{Aut}\, K[x^n, x^{n-1}y, ..., xy^{n-1}, y^n]$ is generated by $\overline{\mathrm{GL}_2\,(K)}$ and $\overline{T_n}$.

Let G be any group and $\psi_1 : \overline{\mathrm{GL}_2\,(K)} \to G$ and $\psi_2 : \overline{T_n} \to G$ be any homomorphisms with $\psi_1|_{\overline{B}} = \psi_1|_{\overline{B}}$.

Let $\alpha : \mathrm{GL}_2\,(K) \to \overline{\mathrm{GL}_2\,(K)}$ and $\beta : T_n \to \overline{T_n}$ be natural homomorphisms. Set $\phi_1 = \psi_1\alpha : \mathrm{GL}_2\,(K) \to G$ and $\phi_2 = \psi_2\beta : \mathrm{Tr}_n \to G$. Obviously, $\phi_1|_B = \phi_2|_B$. By the universal property of the amalgamated free products of groups [11, Ch. 1], there exists a unique homomorphism $\phi : \mathrm{GL}_2\,(K) *_B T_n \to G$ such that $\phi|_{\mathrm{GL}_2\,(K)} = \phi_1$, $\phi|_{T_n} = \phi_2$. Since $E \subseteq \mathrm{Ker}(\phi)$, ϕ induces the homomorphism $\overline{\phi} : (\mathrm{GL}_2\,(K) *_B T_n)/E \to G$. Obviously, $\overline{\phi}|_{\overline{\mathrm{GL}_2\,(K)}} = \psi_1$ and $\overline{\phi}|_{\overline{T_n}} = \psi_2$. By the definition of the amalgamated free product [11], we get

$$\mathrm{GL}_2\,(K) *_B T_n)/E \cong \overline{\mathrm{GL}_2\,(K)} *_{\overline{C_n}} \overline{\mathrm{Tr}_n}.$$

Corollary 1 finishes the proof of the theorem. $\square$

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