Max-Planck-Institut für Mathematik
Bonn

Horospherical geometry of relatively hyperbolic groups

by

Victor Gerasimov
Leonid Potyagailo

Max-Planck-Institut für Mathematik
Preprint Series 2010 (82)

Horospherical geometry of relatively hyperbolic groups

Victor Gerasimov
Leonid Potyagailo

Max-Planck-Institut für Mathematik
Vivatsgasse 7
53111 Bonn
Germany

Department of Mathematics
Federal University of Minas Gerais
Av. Antônio Carlos, 6627
Caixa Postal 702
CEP 30161-970 Belo Horizonte
Brazil

UFR de Mathématiques
Université de Lille 1
59655 Villeneuve d'Ascq cedex
France

MPIM 10-82

Horospherical geometry of relatively hyperbolic groups.

Victor Gerasimov and Leonid Potyagailo*

September 15, 2010

Abstract

We show that a relatively hyperbolic group G is finitely generated with respect to its parabolic subgroups. Using a system of Floyd metrics on the group completion we show that there is a system of "tight" curves satisfying the property of horospherical quasiconvexity. We then prove that the Floyd quasigeodesics are tight and so the parabolic subgroups of G are quasiconvex with respect to Floyd metrics.

1 Introduction.

Let T be a compact Hausdorff space (compactum) containing at least 3 points. The action of a discrete group G by homeomorphisms on T is called *convergence action* if the induced action on the space $\Theta^3 T$ of subsets of cardinality 3 is discontinuous. We say in this case that the action is *3-discontinuous*.

We assume throughout the paper that the induced action on the space $\Theta^2 T$ of subsets of cardinality 2 is cocompact. We say in this case that the action is 2-cocompact. An action is called *parabolic* if there is a unique fixed point, and *non-parabolic* in the opposite case.

Note that if G acts on T 3-discontinuously and 2-cocompactly then the action is *geometrically finite*, i.e. every point of T is either conical or bounded parabolic [Ge1]. The latter fact implies that G is relatively hyperbolic [Ya]. From the other hand every relatively hyperbolic group possesses a geometrically finite convergence action on a compact metrizable space X [Bo1]. Hence the space $\Theta^2 X/G$ is compact [Tu3]. These facts provide an equivalent "dynamical" definition of the relative hyperbolicity. We adopt it in this paper:

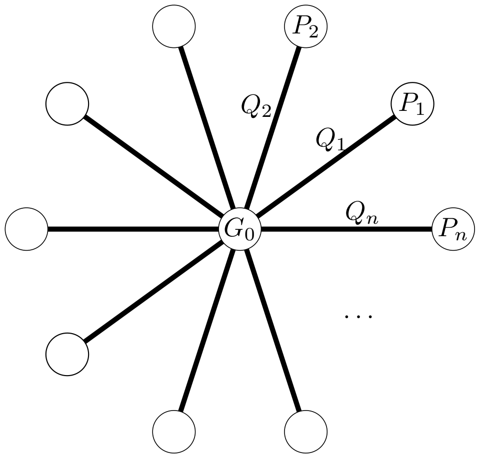
Convention. A group G is *relatively hyperbolic* if it admits a non-parabolic 3-discontinuous and 2-cocompact action on a compactum T.

Our first result shows that a infinitely generated relatively hyperbolic group can be "nicely" approximated by a finitely generated one.

**2010 Mathematics Subject Classification.* Primary 20F65, 20F67 - Secondary 30F40, 57M07, 22D05

Key words: Relatively hyperbolic groups, Relatively finitely generated groups, Floyd quasiconvexity.

Theorem A. *Let G be a relatively hyperbolic group with respect to a collection of parabolic subgroups $\{P_1, ..., P_k\}$. Then there exists a finitely generated subgroup G_0 of G which is relatively hyperbolic with respect to the collection $\{Q_i = P_i \cap G_0 \mid i = 1, .., k\}$ such that G is the fundamental group of the star graph*



whose central vertex group is G_0 and all other vertex groups are P_i $(i = 1, ..., k)$.

Furthermore for every finite subset $K \subset G$ the subgroup G_0 can be chosen to contain K.

If $\mathcal{H}$ be a set of subgroups of G, then G is called *finitely generated with respect to the system* $\mathcal{H}$ if G is generated by $S \cup \mathcal{H}$ where $S \subset G$ is a finite subset.

It follows immediately from Theorem A that if a group G acts 3-discontinuously and 2-cocompactly on a compactum T (i.e. the action $G \curvearrowright T$ is geometrically finite) then G is finitely generated with respect to the parabolic subgroups for this action (see Corollary 3.38). In particular if this action is without parabolic points then G is finitely generated.

The proof of Theorem A is based on the theory of topological entourages on the space $\mathbf{S}^2T$ of non-ordered pairs of points of T. In Section 3 using a discrete system A of entourages on T we construct a graph of entourages $\mathcal{G}$ on which G acts cocompactly. The subgroup G_0 will be chosen as the stabilizer of a connected connected component of a refined graph $\widetilde{\mathcal{G}}$ having the same set of vertices: $\widetilde{\mathcal{G}}^0 = \mathcal{G}^0 = A$. We will extensively use the notions of tubes and horospheres introduced in [Ge1] to prove the existence of the above splitting of G as a graph of groups.

In the rest of the paper we will consider finitely generated groups. Let Γ be an abstract locally finite, connected graph admitting a cocompact and discontinuous action of a finitely generated group G (e.g. its Cayley graph or the graph of entourages $\mathcal{G}$). According to W. Floyd by rescaling the graph distance d of Γ by a scalar function $f : \mathbb{N} \to \mathbb{R}_{\geq}0$ one obtains the Cauchy completion $\overline{\Gamma}_f$ of the metric space (Γ, δ_f) where δ_f is the rescaled metric. We call this space *Floyd completion* (see Section 4 below). The action of G extends continuously to $\overline{\Gamma}_f$. By [Ge2] there exists an equivariant continuous map F between the *Floyd boundary* $\partial_f\Gamma = \overline{\Gamma}_f \setminus \Gamma$ and the space T. The kernel of the map F was described in [GePo1, Thm A]. Namely if it is not a single point then it is equal to the topological boundary $\partial(\mathrm{Stab}_G p)$ of the stabilizer $\mathrm{Stab}_G p$ of a parabolic point $p \in T$. Let $\partial_f \mathrm{Stab}_G p$ denote the Floyd boundary of $\mathrm{Stab}_G p$ corresponding to a function f.

A subset X of Γ is called *Floyd (R-)quasiconvex* if every Floyd geodesic (with respect to the metric δ_f) with the endpoints in X belongs to R-neighborhood $N_r(X)$ for the graph metric d. In particular if the scalar function f is the identity the Floyd quasiconvexity coincides with the standard one. It is known (see e.g. [GePo1, Corollary 3.9]) that parabolic subgroups are quasiconvex with respect to d. Our next Theorem establishes the Floyd quasiconvexity of the parabolic subgroups.

Theorem C. *Let G be a finitely generated group acting 3-discontinuously and 2-cocompactly on a compactum T. Let Γ be a locally finite, connected graph admitting a cocompact discontinuous action of G. Then there exists a Floyd scaling function f, such that every parabolic subgroup H of G is Floyd quasiconvex for the Floyd metric δ_f.* $\square$

As a consequence of Theorem C we obtain the following Corollary which answers our question [GePo1, 1.1]:

Corollary 7.3 *For a scaling function f satisfying $(1-3)$ (see Section (7)) one has*

$$F^{-1}(p) = \partial_f(\mathrm{Stab}_G p), \qquad (*)$$

for every parabolic point $p \in T$. $\square$

We note that Corollary 7.3 provides a complete generalization of the Floyd theorem [F] to the case of relatively hyperbolic groups. The proof of Theorem C given in Section 7 is based on a description of a family of curves which are quasigeodesics locally and also in the horospherical neighborhoods. Their properties are given in the following Theorem (a more detailed formulation is given and proved in Section 6):

Theorem B. *For every tight curve γ in the graph of entourages $\mathcal{G}$ there exists a quasigeodesic $\alpha \subset A$ such that every non-horospherical vertex of γ belongs to a uniform neighborhood of α.*

The main step in proving Theorem C is to show that every Floyd quasigeodesic is tight. We notice that the graph of entourages $\mathcal{G}$ plays here a special role and in the proofs of Theorems B and C we deal mainly with it.
This is our second paper in a series of papers about relatively hyperbolic groups. Keeping the same definition of the relative hyperbolicity we apply here however different methods based on the theory of discrete systems of entourages not used in [GePo1].

Acknowledgements. During the work on this paper both authors were partially supported by the ANR grant BLAN $07-2183619$. We are grateful to the Max-Planck Institute für Mathematik in Bonn, where a part of the work was done. We also thank the Brasilian-French cooperation grant having supported our work.

2 Convergence Groups.

By *compactum* we mean a compact Hausdorff space. Let $\mathbf{S}^n T$ denote the quotient of the product space $\underbrace{T\times \ldots \times T}$ by the action of the permutation group on n symbols. The elements of $\mathbf{S}^n T$

are generalized non-ordered n-tuples (i.e. an element may belong to a tuple with some multiplicity). Let $\Theta^n T$ be the subset of $\mathbf{S}^n T$ whose elements are non-ordered n-tuples with all distinct components. Put $\mathbf{\Delta}^n T = \mathbf{S}^n T \setminus \Theta^n T$, the set $\mathbf{\Delta}^2 T$ is just the diagonal of T^2.

Convention. *If the opposite is not stated all group actions on compacta are assumed to have the convergence property.*

We refer to [Bo2], [GePo1], [GM], [Fr], [Tu2] where standard facts related to the convergence groups are stated. We recall below few facts really used in this paper.

The limit set $\Lambda(G)$ is the set of accumulation (limit) points of the G-orbit for the action of G on T. It is known that either $|\Lambda(G)| \in \{0, 1, 2\}$ in which case the action $G \curvearrowright T$ is called *elementary* or it is a perfect set and the action is not elementary [Tu2].

An elementary action of a group on T is called *parabolic* if there is unique fixed point called parabolic fixed point.

A limit point $x \in \Lambda(G)$ is called *conical* if there exists an infinite sequence $g_n \in G$ and distinct points $a, b \in T$ such that

$$\forall y \in T \setminus \{x\} \ : \ g_n(y) \to a \in T \ \wedge g_n(x) \to b.$$

A parabolic fixed point $p \in \Lambda(G)$ is called *bounded parabolic* if the quotient space $(\Lambda(G) \setminus \{p\})/\mathrm{Stab}_G p$ is compact.

A set M is called *G-finite* if M/G is a finite set.

An action of a group G on a compactum T is called *geometrically finite* if every limit point of T is either conical or bounded parabolic. As we have pointed out the relative hyperbolicity of G with respect to proper subgroups is equivalent to the existence of non-parabolic geometrically finite action.

Notation. From now on we fix the notation $\mathcal{P}$ for the set of parabolic points for the geometrically finite action $G \curvearrowright T$.

3 Visibility Geometry of Entourages.

3.1 Entourages, shadows, betweenness relation.

The following definition is motivated by [Bourb] and [W].

Definition 3.1. Let T be a compactum. Any neighborhood of the diagonal $\mathbf{\Delta}^2 T$ in $\mathbf{S}^2 T$ is called *entourage* of T. The set of all entourages of T is denoted by Ent T.

Convention. By definition an entourage consists of **non-ordered** pairs. However sometimes we identify an entourage $\mathbf{e} \in \mathrm{Ent}\ T$ with the **symmetric** neighborhood $\widetilde{\mathbf{e}}$ of the diagonal in $T \times T$.

We denote the entourages by bold small characters.

An entourage $\mathbf{e}$ determines a graph whose vertex set is T, and two vertices x, y are joined by an edge if and only if $\{x, y\} \in \mathbf{e}$. Denote by $\Delta_{\mathbf{e}}$ the corresponding graph distance which is the maximal distance function with the property $\{x, y\}{\in}\mathbf{e} \implies \mathbf{\Delta}_{\mathbf{e}}(x, y){\leqslant}1$. Note that $\Delta_{\mathbf{e}}(x, y) = \infty$

if and only if x and y belong to different connected components of the graph. A set $U \subset T$ is called **e**-small if its **e**-diameter is at most 1.

The set of all **e**-small sets is denoted by Small(**e**). For subsets $a, b \subset T$ we define $\Delta_{\mathbf{e}}(a, b) = \inf\{\Delta_{\mathbf{e}}(x, y) \mid x \in a,\ y \in b\}$. For a subset $a \subset T$ define its **e**-neighborhood $a\mathbf{e}$ as $\{x \in T \mid \Delta_{\mathbf{e}}(x, a) \leqslant 1\}$.

For a subset o of T its "*convex hull*" in $T \sqcup \mathsf{Ent}T$ is the set

$$\widetilde{o} = o \cup \{\mathbf{e} \in \mathsf{Ent}T : o' \in \mathsf{Small}(\mathbf{e})\}. \tag{\dagger}$$

We equip the space $T \sqcup \mathsf{Ent}T$ with the topology generated by the "*convex hulls*" of open subsets of T and the single-point subsets of Ent T. A set w in $T \sqcup \mathsf{Ent}T$ is declared *open* if for every point $t \in w \cap T$ there exists $o \in$ Open T such that $t \in o$ and $\widetilde{o} \subset w$. In particular Ent T is a discrete open subset and T is a closed subspace of $T \sqcup \mathsf{Ent}T$.

Remark. The definition of the topology on $T \sqcup$Ent T is motivated by the topology on $\mathbb{H}^n \cup \partial_\infty \mathbb{H}^n$ given by open sets $o \subset \partial_\infty \mathbb{H}^n$ and their convex hulls in $\mathbb{H}^n$. This analogy can be explained as follows. A bounded subset $B \subset \mathbb{H}^n$ defines an entourage $\mathbf{e}_B \in \mathrm{Ent}(\partial \mathbb{H}^n)$ in the following way: $\{x, y\} \in \mathbf{e}_B$ if and only if the geodesic $\gamma(x, y)$ with the endpoints x and y misses B. The entourage $\mathbf{e}_B$ is close to a point $a \in \partial \mathbb{H}^n$ if for an open neighborhood $o \subset \partial \mathbb{H}^n$ the convex hull $\widetilde{o}$ of o in $\mathbb{H}^n \cup \partial_\infty \mathbb{H}^n$ contains B (see Figure 1). $\square$

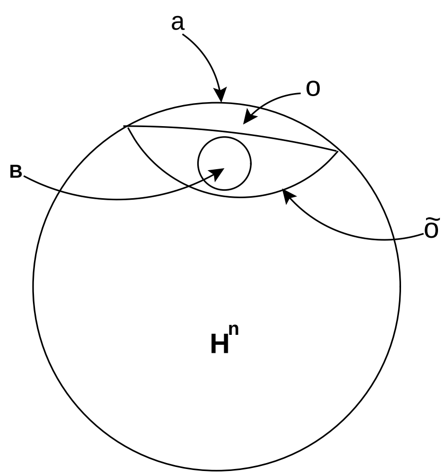


Figure 1: Bounded set in $\mathbb{H}^n$ and its visibility entourage.

Definition 3.2. [Ge1] Two entourages **a** and **b** are said to be *unlinked* if there exist $a \in \mathrm{Small}(\mathbf{a})$ and $b \in \mathrm{Small}(\mathbf{b})$ such that $T = a \cup b$. We denote this relation by $\mathbf{a} \bowtie \mathbf{b}$. In the opposite case we say that **a** and **b** are *linked*, and write $\mathbf{a} \# \mathbf{b}$. $\square$

Denote by La the set $\{\mathbf{b} \in \mathrm{Ent}\, T \mid \mathbf{a} \# \mathbf{b}\}$. It is enough for our purposes to consider only sufficiently small entourages implying the following.

Convention. All considered entourages are supposed to be self-linked :

$$\mathbf{a} \in \text{Ent } T \ : \ \mathbf{a}\#\mathbf{a}. \tag{1}$$

Definition 3.3. [Ge1] Let $\mathbf{a}$ and $\mathbf{b}$ be two unlinked entourages. We define the following "shadow" sets :

$$\text{Sh}_{\mathbf{a}}\mathbf{b} = \{a \in \text{Small}(\mathbf{a}) \mid a' \in \text{Small}(\mathbf{b})\},$$

and

$$\text{sh}_{\mathbf{a}}\mathbf{b} = \bigcap \text{Sh}_{\mathbf{a}}\mathbf{b} = (\bigcup \text{Sh}_{\mathbf{b}}\mathbf{a})'.$$

It is shown in [Ge1, Lemma S0] that if $\mathbf{a} \bowtie \mathbf{b}$ and $\text{diam}_{\mathbf{a}}T > 2$ then $\text{sh}_{\mathbf{a}}\mathbf{b} \neq \emptyset$; and if $\text{diam}_{\mathbf{a}}T > 4$ then $\text{sh}_{\mathbf{a}}\mathbf{b}$ has a nonempty interior.

Convention. We consider only the entourages $\mathbf{a}$ with $\mathsf{Diam}_{\mathbf{a}}T{>}4$. So every shadow has non-empty interior.

Remark. In the hyperbolic space $\mathbb{H}^n$ the shadows give rise to the sets at the sphere at infinity illustrated on the Figure 2.

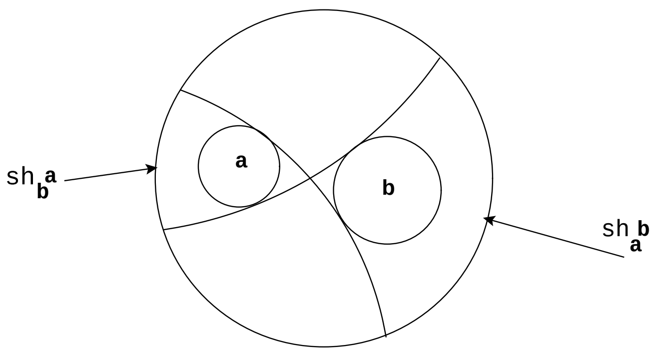


Figure 2: Shadows $\text{sh}_{\mathbf{b}}\mathbf{a}$ and $\text{sh}_{\mathbf{b}}\mathbf{a}$.

Definition 3.4. *(Betweenness relation). Let k be a positive integer.*

1) Suppose $\mathbf{a}, \mathbf{b}, \mathbf{c} \in \text{Ent } T$. We say that an entourage $\mathbf{b}$ lies *between* (or *k-between*) $\mathbf{a}$ and $\mathbf{c}$, and write $\mathbf{a} - \mathbf{b} - \mathbf{c}$ (k) (or simply $\mathbf{a} - \mathbf{b} - \mathbf{c}$), if $\mathbf{a} \bowtie \mathbf{b} \bowtie \mathbf{c}$ and $\Delta_{\mathbf{b}}(\text{sh}_{\mathbf{b}}\mathbf{a}, \text{sh}_{\mathbf{b}}\mathbf{c}) > k$.

2) Let $\mathbf{a}, \mathbf{b} {\in} \mathsf{Ent}T$ and let $p \in T$. We say that $\mathbf{b}$ lies *between* (or *k-between*) $\mathbf{a}$ and p if $\mathbf{a}{\bowtie}\mathbf{b}$ and $\Delta_{\mathbf{b}}(\mathsf{sh}_{\mathbf{b}}\mathbf{a}, b){>}k$ for some $\mathbf{b}$-small neighborhood b of p

   We write $\mathbf{a} - \mathbf{b} - p$ (k) (or simply $\mathbf{a} - \mathbf{b} - p$) in this case.

3) Let $\mathbf{b}\in\mathrm{Ent}T$ and let $p, q \in T$ be two distinct points. We say that $\mathbf{b}$ lies *between* (or *k-between*) p and q, and write $q - \mathbf{b} - p\ (k)$ (or simply $q - \mathbf{b} - p$), if $\Delta_{\mathbf{b}}(b_1, b_2) > k$ for some $\mathbf{b}$-small neighborhoods b_1 and b_2 of the points p and q respectively.

Remarks 3.5. a) The betweenness relations 2) and 3) represent extension "by continuity" of the relation 1) between entourages to the points of T. Note that the middle object in the relation $\mathbf{a} - \mathbf{b} - \mathbf{c}$ is always an entourage.

b) Definition 3.4 in cases 2) and 3) differs from the corresponding definition in [Ge1] where the condition $\Delta_{\mathbf{b}}(\mathrm{sh}_{\mathbf{a}}\mathbf{b}, p) > k$ is stated instead of 2). The above betweenness definition is stronger than that of [Ge1] and so is easier to use. However both of them are quite close: the k-betweenness 2) implies k-betweenness of [Ge1] and by the triangle inequality $k+1$-betweenness of [Ge1] implies k-betweenness 2). We will use results of [Ge1] keeping in mind this relation.

Lemma 3.6. *(Continuity property). Suppose that* $\mathbf{a} - \mathbf{c} - p(k)$ $(k \in \mathbb{N})$ *where* $\mathbf{a} \in T \sqcup \mathrm{Ent}\ T$, $\mathbf{c} \in T$, $p \in T$. *Suppose that* $p \in T$ *is an accumulation point for an infinite subset* B *of* $\mathrm{Ent}\ T$. *Then there exists* $\mathbf{b} \in B$ *such that* $\mathbf{a} - \mathbf{c} - \mathbf{b}(k)$.

Proof: Let first $\mathbf{a} \in \mathrm{Ent}\ T$ be an entourage. Let c be an open $\mathbf{c}$-small set containing p such that $\Delta_{\mathbf{c}}(c, \mathrm{sh}_{\mathbf{c}}\mathbf{a}) > k$. For every $\mathbf{c}$-small neighborhood U_p of p there exists an entourage $\mathbf{b} \in U_p \cap B$. By definition of the topology of $T \sqcup \mathrm{Ent}\ T$ the complement U'_p is $\mathbf{b}$-small, i.e. $U'_p \subset b \in \mathrm{Small}(\mathbf{b})$. Then $U'_p \subset \bigcup \mathrm{Sh}_{\mathbf{b}}\mathbf{c}$, and $U_p \supset \mathrm{sh}_{\mathbf{c}}\mathbf{b} = (\bigcup \mathrm{Sh}_{\mathbf{b}}\mathbf{c})'$. Thus $\Delta_{\mathbf{c}}(\mathrm{sh}_{\mathbf{c}}\mathbf{a}, \mathrm{sh}_{\mathbf{c}}\mathbf{b}) > \Delta_{\mathbf{c}}(\mathrm{sh}_{\mathbf{c}}\mathbf{a}, c) > k$.

If now $a \in T$ then using a $\mathbf{c}$-small neighborhood U containing a, we obtain similarly $\Delta_{\mathbf{c}}(U, \mathrm{sh}_{\mathbf{c}}\mathbf{b}) > \Delta_{\mathbf{c}}(U, U_p) > k$. So we still have $a - \mathbf{c} - \mathbf{b}\ (k)$ for $\mathbf{b} \in B$. □

Definition 3.7. *(Tubes). [Ge1]* A sequence P of elements $\mathbf{a}_n$ of $T \sqcup \mathrm{Ent}\ T$ is called *k-tube* (or *tube*) if

$$\forall n :\ (\mathbf{a}_n \bowtie \mathbf{a}_{n+1}) \wedge (\mathbf{a}_{n-1} - \mathbf{a}_n - \mathbf{a}_{n+1}(k))$$

whenever $\mathbf{a}_{n\pm1}$ are defined. □

Lemma 3.8. 1) *(Ordering) For any three entourages at most one can be between the others.*

2) *(Convexity) If* $\mathbf{a} - \mathbf{b} - \mathbf{c}(3)$ *and* $\mathbf{a}, \mathbf{c} \in \mathrm{L}\mathbf{d}$ *then* $\mathbf{b} \in \mathrm{L}\mathbf{d}$.

Proof: 1) Indeed if not, we obtain $\mathbf{a} - \mathbf{b} - \mathbf{c}$ and $\mathbf{a} - \mathbf{c} - \mathbf{b}$ for some $\mathbf{a}, \mathbf{b}, \mathbf{c}$. The transitivity of the betweenness relation [Ge1] would imply $\mathbf{a} - \mathbf{b} - \mathbf{a}$ and so $\mathbf{a} \bowtie \mathbf{a}$ which is impossible by our convention (1).

2) Otherwise $\mathbf{b} \bowtie \mathbf{d}$ and we have $T = b \cup d = a \cup b_1 = c \cup b_2$ where $b_i \in \mathrm{Small}(\mathbf{b})$, $d \in \mathrm{Small}(\mathbf{d})$, $a \in \mathrm{Small}(\mathbf{a})$. It follows that $b \cap b_1 = \emptyset$ or $b \cap b_2 = \emptyset$ as otherwise we have $\Delta_{\mathbf{b}}(\mathrm{sh}_{\mathbf{b}}\mathbf{a}, \mathrm{sh}_{\mathbf{b}}\mathbf{c}) \leq 3$ which is impossible. If, for instance, $b \cap b_1 = \emptyset$ then $b_1 \subset d$ and $\mathbf{a} \bowtie \mathbf{d}$. A contradiction. □

3.2 Discrete sets of entourages. Horospheres.

Until the end of Section 3 we fix a 3-discontinuous 2-cocompact action $G \curvearrowright T$ of a group G on a compactum T.

Definition 3.9. A set A of entourages on T is called *discrete* if

$$\forall \mathbf{w} \in \text{Ent } T \ : \ |\{\mathbf{a} \in A \ : \ \mathbf{a}\#\mathbf{w}\}| < \infty. \tag{1}$$

By [Ge1, Proposition P] the set $\{g \in G \ : \ g\mathbf{a}\#\mathbf{w}\}$ is finite for each $\mathbf{w} \in \text{Ent } T$. This is called *Dynkin property* [Fu], [Ya1]. Hence every G-finite set is discrete for any $\mathbf{a} \in \text{Ent } T$.

Let $A \subset \text{Ent } T$ be a discrete set of entourages. Denote by $\widetilde{T}$ the subspace $T \sqcup A$ of $T \sqcup \text{Ent } T$. It is compact if and only if A is discrete [Ge1, Proposition D].

Definition 3.10. Let $\mathcal{G} = \mathcal{G}_A$ be the graph whose vertex set $\mathcal{G}^0$ is A and the edge set $\mathcal{G}^1$ is the set of pairs $\{\mathbf{a}, \mathbf{b}\}$ such that $\mathbf{a}\#\mathbf{b}$. Denote by d_A the corresponding graph distance.

Lemma 3.11. *The group G is finitely generated if and only if there exists a connected graph $\mathcal{G}_A$.*

Proof: Suppose first that G admits a finite set of generators S ($\text{id} \in S$). Since A is G-finite we have $A = \bigcup_{i=1}^{l} G(\mathbf{a}_i)$. Any entourage $\mathbf{a}_i$ contains a sub-entourage $\mathbf{a}'_i$ such that

$$\forall s \in S \ \ : \ \mathbf{a}'_i\#s\mathbf{a}'_j \ \ (i, j \in \{1, ..., l\}).$$

So up to choosing the entourages $\mathbf{a}_i$ $(i = 1, ..., l)$ to be sufficiently small we can assume that the above property is satisfied. Then all vertices in the set $\bigcup_i S\mathbf{a}_i$ are pairwise connected by edges. For any vertex $\mathbf{v} \in \mathcal{G}_A$ there exists $i \in \{1, ..., l\}$ and $g \in G$ such that $\mathbf{v} = g(\mathbf{a}_i)$ and $g = s_{i_1}s_{i_2}...s_{i_k}$ $(s_{i_j} \in S)$. Then $\mathcal{G}_A$ contains the edges $(s_{i_k}(\mathbf{a}_i), \mathbf{a}_i)$, $(s_{i_{k-1}}(\mathbf{a}_i), \mathbf{a}_i)$ and so $(s_{i_{k-1}}s_{i_k}(\mathbf{a}_i), s_{i_{k-1}}(\mathbf{a}_i))$. Hence it contains also a path between $\mathbf{a}_i$ and $s_{i_{k-1}}s_{i_k}(\mathbf{a}_i)$. Continuing in this way we obtain a path between $\mathbf{v}$ and $\mathbf{a}_i$.

Conversely suppose that $\mathcal{G}_A$ is connected. Let S be the set $\{s \in G \mid s\mathbf{a}_j\#\mathbf{a}_i\}$ where $A = \bigcup_{i=1}^{l} G\mathbf{a}_i$. By Dynkin property the set S is finite. For any $g \in G$ there is a path $l = \{\mathbf{a}_i, \mathbf{b}_2, ..., \mathbf{b}_{n-1}, \mathbf{a}\} \subset \mathcal{G}_A$ between the vertices $\mathbf{a} = g(\mathbf{a}_i)$ and $\mathbf{a}_i$. Then $\mathbf{b}_2\#\mathbf{a}_i$ so $\exists \ s_1 \in S \ :$ $\mathbf{b}_2 = s_1(\mathbf{a}_i)$. Thus $s_1^{-1}\mathbf{b}_3\#\mathbf{a}_i$ and $\exists \ s_2 \in S \ : \ \mathbf{b}_3 = s_1s_2\mathbf{a}_i$. Continuing in this way we obtain $\mathbf{a} = s_1s_2...s_n\mathbf{a}_i$. Then $g^{-1}(s_1s_2...s_n)$ fixes $\mathbf{a}_i$ and so belongs to S (by (1) of 3.1). The Lemma is proved. $\square$

It follows from Dynkin property and our convention (1) that the stabilizer of each edge and each vertex of $\mathcal{G}$ is finite. The action $G \curvearrowright T$ is 2-cocompact so by [Ge1, Proposition E] we can suppose that the set A is a single orbit $G(\mathbf{a}_0)$ $(\mathbf{a}_0 \in \text{Ent } T)$ having the following properties :

i) *m-separation property*:

$$\forall (p,q) \in \Theta^2 T \ \exists \ \mathbf{a} \in A \ : p - \mathbf{a} - q(m), \tag{2}$$

for a fixed $m \in \mathbb{N}$.

ii) *generating property:*

$$\forall \ \mathbf{u} \in \text{Ent } T \ \exists \ \mathbf{a}_i \in A \ (i = 1,..,l) : \ \mathbf{u} \supset \bigcap_{i=1}^{l} \mathbf{a}_i. \tag{3}$$

i.e. A generates Ent T as a filter.

Convention 3.12. From now on we fix an unlinked entourage $\mathbf{a}_0 \in$ Ent T (see (1)) of 3.1), and its orbit $A = G(\mathbf{a}_0)$ satisfying m-separating and generating properties. The value of m can be easily restored in each statement. Keeping in mind that this value might be needed to be increased further we just suppose that m is sufficiently large.

Furthermore if G is finitely generated we will always assume (by Lemma 3.11) that the graph $\mathcal{G}$ is connected.

Remarks. The graph $\mathcal{G}$ plays the role of the Cayley graph $\mathcal{C}a(G)$ if G is finitely generated, however it is always a locally finite graph. The space $\widetilde{T} = T \sqcup A$ is a compactification of $A = \mathcal{G}^0$ similar to the Floyd completion (see Section 4). Every action $G \curvearrowright T$ can be naturally extended to the space $\widetilde{T}$.

Lemma 3.13. *The space $\widetilde{T} = T \sqcup A$ is a compactum.*

Proof: The space T is Hausdorff. To prove that $\widetilde{T}$ is Hausdorff we will consider three different cases. Let first x, y be distinct points of T then there exist disjoint closed neighborhoods U_x and U_y in T. Their convex hulls $\widetilde{U}_x = U_x \cup \{\mathbf{e} \in A : \ U'_x \in \mathsf{Small}(\mathbf{e})\}$ and $\widetilde{U}_y = U_y \cup \{\mathbf{d} \in A : \ U'_y \in \mathsf{Small}(\mathbf{d})\}$ are neighborhoods of these points in the topology of $\widetilde{T}$ induced from $T \sqcup$Ent T (see (†) of 3.1). If $\mathbf{a} \in A \cap \widetilde{U}_x \cap \widetilde{U}_y$ then U'_x and $\widetilde{U}'_y$ are both $\mathbf{a}$-small. Since U_x and U_y are disjoint we have $U'_x \cup U'_y = T$ and so $\mathbf{a}\#\mathbf{a}$ contradicting our Convention 3.12. Hence $\widetilde{U}_x \cap \widetilde{U}_y = \emptyset$.

If now one of the points is an entourage $\mathbf{x} \in A$ and $y \in T$ then by the same reason any $\mathbf{x}$-small neighborhood of y in $\widetilde{T}$ cannot contain $\mathbf{x}$. Since every entourage is open in $\widetilde{T}$ we are done in this case too. If finally both points are entourages they coincide with their disjoint neighborhoods. So $\widetilde{T}$ is Hausdorff.

The compactness of $\widetilde{T}$ follows from [Ge1, Proposition D]. □

Proposition 3.14. *If a group G acts 3-discontinuously on a compactum T then the induced action on $\widetilde{T} = T \cup A$ is also 3-discontinuous.*

Remark. In [Ge2, Thm 5.1] it is proved that there is a unique topology on the compactified space $\widetilde{T}$ with respect to which the action is 3-discontinuous. The argument below provides a simple proof of this for the induced topology on $\widetilde{T} \subset T \sqcup \text{Ent } T$ introduced above.

Proof: For a subset $X \subset T$ denote by $\widetilde{X} = X \cup \{\mathbf{a} \in A \mid X' \in \text{Small}(\mathbf{a})\} \subset \widetilde{T}$ its convex hull in $\widetilde{T}$. In case if $X = \{\mathbf{a}\}$ where $\mathbf{a} \in A$ is an entourage we put $\widetilde{X} = \mathbf{a}$. For every $g \in G$ denote by $\widetilde{g}$ its natural extension to $\widetilde{T}$.

Every point $x \in \Theta^3\widetilde{T}$ admits a closed neighborhood which is a "cube" $\widetilde{K} = \widetilde{X} \times \widetilde{Y} \times \widetilde{Z}$ where X, Y and Z are either disjoint closed subsets of T or some of $\widetilde{X}, \widetilde{Y}, \widetilde{Z}$ are isolated entourages (in the latter case we call the corresponding cube *degenerate*). Every compact subset of $\Theta^3\widetilde{T}$ is a finite union of such cubes. So it is enough to prove that for two cubes $\widetilde{K}_i = \widetilde{X}_i \times \widetilde{Y}_i \times \widetilde{Z}_i \subset \Theta(\widetilde{T})$ $(i = 0, 1)$ the following set is finite:

$$S = \{g \in G \mid\ :\ \widetilde{g}\widetilde{X}_0 \cap \widetilde{X}_1 \neq \emptyset,\ \widetilde{g}\widetilde{Y}_0 \cap \widetilde{Y}_1 \neq \emptyset,\ \widetilde{g}\widetilde{Z}_0 \cap \widetilde{Z}_1 \neq \emptyset\}.$$

Suppose to the contrary that S is infinite. Since the action $G \curvearrowright T$ is 3-discontinuous every accumulation point of S with respect to Vietoris topology is a *cross* $< p, q >^{\times} = p \times T \sqcup T \times q$ [Ge1, Proposition P]. Consider now all possible cases.

Case 1. Both cubes are not degenerate, i.e. X_i, Y_i, Z_i $(i = 0, 1)$ are all closed disjoint subsets of T.

At least one of the "squares" $X_0 \times X_1,\ Y_0 \times Y_1$ or $Z_0 \times Z_1$ does not meet the cross. Indeed otherwise two of them intersect both either $p \times T$ or $T \times q$ which is impossible as X_i, Y_i and Z_i are pairwise disjoint for $i \in \{0, 1\}$.

Let us assume that e.g. $Z_0 \times Z_1 \cap < p, q >^{\times} = \emptyset$. Let $g \in S$ be a homeomorphism whose graph is contained in the neighborhood $T^2 \setminus Z_0 \times Z_1$ of $< p, q >^{\times}$. Then $gZ_0 \cap Z_1 = \emptyset$. However $\widetilde{g}\widetilde{Z}_0 \cap \widetilde{Z}_1 \neq \emptyset$. So there exists $\mathbf{a} \in \widetilde{Z}_0 \setminus Z_0$ such that $\widetilde{g}\mathbf{a} \in \widetilde{Z}_1$. By definition of the convex hull Z_0' and $(g^{-1}(Z_1))'$ are $\mathbf{a}$-small. Since $(g^{-1}Z_1)' \cup Z_0' = T$ we obtain that T is the union of two $\mathbf{a}$-small sets, so $\mathbf{a}\#\mathbf{a}$ contradicting our Convention 3.12.

Case 2. At least one of the cubes is degenerate.

Then some of the sets $\widetilde{X}_i, \widetilde{Y}_i, \widetilde{Z}_i$ are entourages. Note that since $g\widetilde{X}_0 \cap \widetilde{X}_1 \neq \emptyset$ for infinitely many $g \in S$, by Dynkin property $\widetilde{X}_0$ and $\widetilde{X}_1$ cannot be entourages simultaneously. The same is true for $\widetilde{Y}_i$ and Z_i $(i = 0, 1)$. So there could be at most 3 entourages among these 6 sets. We consider all the possibilities below.

Subcase 2.1. There is only one degenerate cube.

We can assume that $\widetilde{X}_0 = \mathbf{a}$ for some $\mathbf{a} \in A$. Then $\forall g \in S$ we have $g\mathbf{a} \in \widetilde{X}_1$. So $g^{-1}X_1'$ is $\mathbf{a}$-small. For a limit cross $< p, q >^{\times}$ for the set S and $\mathbf{a}$-small neighborhoods U_p and U_q of the points p and q respectively there exists $g \in S$ such that $gU_p' \subset U_q$ or $g^{-1}U_q' \subset U_p$. If now $U_q \cap X_1 = \emptyset$ then T would be the union of $\mathbf{a}$-small sets $g^{-1}X_1'$ and $g^{-1}U_q'$ contradicting the unlinkness condition $\mathbf{a}\#\mathbf{a}$. So for every $\mathbf{a}$-small neighborhood U_q of q we have $U_q \cap X_1 \neq \emptyset$. Since X_1 is closed it follows that $q \in X_1$.

At most one of the disjoint sets Y_0 or Z_0 can contain the other point p of the cross, let $p \notin Z_0$. Then for any neighborhood U_q and for infinitely many elements $g \in S$ we have $gZ_0 \subset U_q$. If $gZ_0 \cap Z_1 \neq \emptyset$ for infinitely many $g \in S$ then q is an accumulation point for Z_1, and since Z_1 is closed we obtain that $z \in Z_1 \cap X_1$ which is impossible. So for almost all $g \in S \ : \ gZ_0 \cap Z_1 = \emptyset$ and this situation has been excluded in Case 1.

Subcase 2.2. There are two degenerate cubes.

Note that they cannot belong to the same level, namely if $\widetilde{X}_0 = \mathbf{a} \in A$ and $\widetilde{Y}_0 = \mathbf{b} \in A$ then by the argument of Subcase 2.1 we must have $q \in Y_1 \cap X_1$ which is impossible.

So let $\widetilde{Y}_1 = \mathbf{b} \in A$. By the argument of Subcase 2.1 applied now to the inverse elements of S we obtain that $p \in Y_0$. Hence for almost all elements $g \in S$ we still have $gZ_0 \cap Z_1 = \emptyset$ which is impossible by Case 1.

Subcase 2.3. There are three degenerate cubes.

Then there are at least two of three entourages which are among of the sets of the same level: $\widetilde{X}_i, \widetilde{Y}_i, \widetilde{Z}_i$ ($i = 0$ or $i = 1$) which is impossible. So neither case can happen. The Proposition is proved. $\square$

Lemma 3.15. *Let B be an infinite subset of A and $C = N_d(B)$ where $N_d(B)$ is a d-neighborhood of B in $\widetilde{T}$. Then the topological boundaries of B and C coincide.*

In particular, if $(\mathbf{b}_n)_n$ and $(\mathbf{c}_n)_n$ are two sequences in A such that $d_A(\mathbf{b}_n, \mathbf{c}_n)$ is uniformly bounded. Then $(\mathbf{b}_n)_n$ converges to a point $p \in T$ if and only if $\mathbf{c}_n \to p$.

Proof: The second claim directly follows from the first one. So to prove the lemma we need only to show that every accumulation point of C is also an accumulation point of B. Suppose not and there exists a point $r \in \partial C \setminus \partial B$. Then for every neighborhood U_r of r there exists an infinite subset $C_0 \subset C$ such that $\forall \mathbf{c} \in C_0$ we have $\mathbf{c} \in U_r$ implying that $U'_r \subset c$ for some $c \in \mathrm{Small}(\mathbf{c})$.

Arguing by induction on d without loss of generality we may assume that $d = 1$. So $\forall \mathbf{c} \in C \ \exists \ \mathbf{b} \in B \ : \ \mathbf{c}\#\mathbf{b}$. Then there exists a subset $B_0 \subset B$ such that $d_A(B_0, C_0) \leq 1$. Since C_0 is infinite by discreteness of A the set B_0 is infinite too. Let $p \in T \setminus \{r\}$ be an accumulation point of B_0. Then for every neighborhood U_p of p there exists $\mathbf{b} \in B_0$ such that $U'_p \subset b$ for some $b \in \mathrm{Small}(\mathbf{b})$. Choosing U_p and U_r to be disjoint we obtain $b \cup c = T$ and so $\mathbf{b} \bowtie \mathbf{c}$. A contradiction. $\square$

Definition 3.16. *[Ge1] (Horospheres, Conical and Parabolic Points).* Let k be a fixed positive integer, and let A be the above discrete set of entourages.

1) We say that a point $p \in T$ and an entourage $\mathbf{e}$ are *neighbors* (with respect to A) and write $\mathbf{e} \underset{A,k}{\#} p$, if there is no $\mathbf{a} \in A$ such that $\mathbf{e} - \mathbf{a} - p(k)$.

2) The *horosphere* $T_{A,k}(p)$ (or $T_k(P)$ or $T(p)$) at the point $p \in T$ is the set

$$T_{A,k}(p) = \{\mathbf{e} \in A \mid \mathbf{e} \underset{A,k}{\#} p\}.$$

3) A point $x \in T$ is called (A, k)-*conical* (or just *conical*) if $T_{A,k}(x) = \emptyset$.

4) A point $p \in T$ is called (A, k)-*parabolic* (or just *parabolic*) if $T_{A,k}(p)$ is infinite.

It is shown in [Ge1] that the notions of (A, k)-conical and (A, k)-parabolic points for $k \geq 3$ (see Remark 3.5) are equivalent to the standard definitions (see Section 2) of conical and bounded parabolic points respectively.

Lemma 3.17. *[Ge1] If the action $G \curvearrowright T$ is 3-discontinuous and 2-cocompact then every limit point of this action is either conical or bounded parabolic. Furthermore the set of non-conical points is G-finite and for every parabolic point $p \in T$ the set $T(p)$ is $\mathrm{Stab}_G p$-finite.*

The next Lemma is proved in [Ge1, Lemma P2] for closed entourages. We prove it below in a general form.

Lemma 3.18. *For every $d > 0$ the parabolic point p is the unique accumulation point of the d-neighborhood $N_d(T_{A,k}(p))$ of the horosphere $T_{A,k}(p)$.*

Proof: By Lemma 3.15 it is enough to prove the statement for the horosphere $T_{A,k}(p)$. Suppose it admits two distinct accumulation points p and q. Since the set A is m-separating there exists $\mathbf{a} \in A$ such that $p - \mathbf{a} - q(k)$ for some $k \leq m$. Then by Lemma 3.6 there exists $\mathbf{b} \in T_{A,k}(p)$ such that $p - \mathbf{a} - \mathbf{b}(k)$ which is not possible. $\square$

We have the following transitivity property:

Lemma 3.19. *If $\mathbf{a}, \mathbf{b}, \mathbf{c} \in \mathrm{Ent}\ T$, $p \in T$ and $k \geq 3$. Then $\mathbf{a} - \mathbf{b} - p(k)$ and $\mathbf{b} - \mathbf{c} - p(k)$ imply $\mathbf{a} - \mathbf{c} - p(k)$.*

Proof: If $a \in \mathrm{Sh}_{\mathbf{a}}\mathbf{b}$ and $c \in \mathrm{Sh}_{\mathbf{c}}\mathbf{b}$, then the sets $b = a', b_1 = c'$ are $\mathbf{b}$-small and $a \cup b = b_1 \cup c = T$. There exists a $\mathbf{c}$-small neighborhood c_0 of p such that $\Delta_{\mathbf{c}}(c, c_0) \geq \Delta_{\mathbf{c}}(\mathrm{sh}_{\mathbf{c}}\mathbf{b}, c_0) - \Delta_{\mathbf{c}}(\mathrm{sh}_{\mathbf{c}}\mathbf{b}, c) > k - 1$. So $\Delta_{\mathbf{c}}(c, p) > k - 1 > 0$ and $p \in b_1$. Note that $b \cap b_1 = \emptyset$ since otherwise $\Delta_{\mathbf{b}}(b_1, \mathrm{sh}_{\mathbf{b}}\mathbf{a}) \leq \Delta_{\mathbf{b}}(b_1, b) + \Delta_{\mathbf{b}}(b, \mathrm{sh}_{\mathbf{b}}\mathbf{a}) \leq 3$ which is impossible as $\mathbf{a} - \mathbf{b} - p(k)$ and $k \geq 3$. Thus $b_1 \subset a$ and $a \cup c = T$. Since c was an arbitrary element of $\mathrm{Sh}_{\mathbf{c}}\mathbf{b}$, it follows that $\mathrm{Sh}_{\mathbf{c}}\mathbf{b} \subset \mathrm{Sh}_{\mathbf{c}}\mathbf{a}$ and $\mathrm{sh}_{\mathbf{c}}\mathbf{a} \subset \mathrm{sh}_{\mathbf{c}}\mathbf{b}$. Thus $\Delta_{\mathbf{c}}(\mathrm{sh}_{\mathbf{c}}\mathbf{a}, c_0) > k$. $\square$

The above notions allow us to introduce the following relation on the set $\mathrm{Ent}\ T$.

Definition 3.20. *(***Busemann order***)* For $\mathbf{a}, \mathbf{b} \in \mathrm{Ent}\ T$, and $p \in T$ we say that $\mathbf{a}$ and $\mathbf{b}$ are *Busemann ordered* with respect to p if

$$\text{either } \mathbf{a} = \mathbf{b}, \quad \text{or} \quad \mathbf{a} - \mathbf{b} - p(k).$$

We will denote this relation by $\mathbf{a} \geq_{p,k} \mathbf{b}$.

Lemma 3.19 implies that this relation is a partial order on $\mathrm{Ent}\ T$. Using Busemann order we can reformulate the above definitions of conical and parabolic points as follows.

Lemma 3.21. *A point $p \in T$ is A-conical if and only if its Busemann order has no minimal elements. A point p is A-parabolic if and only if its Busemann order has infinitely many minimal elements.*

3.3 Non-refinable tubes.

Lemma 3.22. *The set* $\Psi_k(\mathbf{a},\mathbf{b}) = \{\mathbf{c} \in A \ : \ \mathbf{a} - \mathbf{c} - \mathbf{b}(k)\}$ *is finite for any* $k \geq 1$.

Proof: Suppose that $\mathbf{a} - \mathbf{c} - \mathbf{b}(k)$ and let us prove that $\mathbf{c}\#(\mathbf{a} \cap \mathbf{b})$. If it is not true, then we have $\mathbf{c} \bowtie (\mathbf{a} \cap \mathbf{b})$, i.e. there exists $c \in \mathrm{Small}(\mathbf{c})$, $w \in \mathrm{Small}(\mathbf{a} \cap \mathbf{b})$ such that $c \cup w = T$. Thus $c \in \mathrm{Sh}_{\mathbf{c}}\mathbf{a} \cap \mathrm{Sh}_{\mathbf{c}}\mathbf{b}$ and $\mathrm{sh}_{\mathbf{c}}\mathbf{a} \subset c$, $\mathrm{sh}_{\mathbf{c}}\mathbf{b} \subset c$. Hence $\Delta_{\mathbf{c}}(\mathrm{sh}_{\mathbf{c}}\mathbf{a}, \mathrm{sh}_{\mathbf{c}}\mathbf{b}) \leq 1$ which is impossible. It follows that $\mathbf{c}\#(\mathbf{a} \cap \mathbf{b})$. The finiteness of $\Psi_k(\mathbf{a},\mathbf{b})$ now follows from the discreteness of A $\square$

Definition 3.23. (**Refinability**). A pair $\{\mathbf{a},\mathbf{b}\} \subset A$ is called *(k-)refinable* if $\Psi_k(\mathbf{a},\mathbf{b}) \neq \emptyset$, and *(k-)non-refinable* otherwise.

The Proposition 3.25 below guarantees the existence of a finite non-refinable tube between two given entourages in A. To prove it we need the following:

Lemma 3.24. *For every integer* $k \geq 2$, *every pair* $\{\mathbf{a},\mathbf{b}\} \subset A$ *is either* $k+1$*-nonrefinable or there exists* $\mathbf{c} \in \Psi_k(\mathbf{a},\mathbf{b})$ *such that the pair* $\{\mathbf{a},\mathbf{c}\}$ *is* $k+1$*-nonrefinable.*

Proof: Suppose this is not true and let a pair $\{\mathbf{a},\mathbf{b}\}$ be a counter-example. By Lemma 3.22 the set $\Psi_k(\mathbf{a},\mathbf{b})$ is finite so we can assume in addition that the number $|\Psi_k(\mathbf{a},\mathbf{b})|$ is the minimal one among all such counter-examples. So $\{\mathbf{a},\mathbf{b}\}$ is $k+1$-refinable. By our assumption there exists $\mathbf{c} \in \Psi_{k+1}(\mathbf{a},\mathbf{b})$ such that the pair $(\mathbf{a},\mathbf{c})$ is $k+1$-refinable too. We now claim that

$$\Psi_{k+1}(\mathbf{a},\mathbf{c}) \subset \Psi_{k+1}(\mathbf{a},\mathbf{b}) \ (k > 1). \tag{1}$$

Let $\mathbf{d} \in \Psi_{k+1}(\mathbf{a},\mathbf{c})$. By [Ge1, Lemma T2] we have $\mathbf{d} - \mathbf{c} - \mathbf{b}(k)$. Then $\mathrm{sh}_{\mathbf{d}}\mathbf{b} \subset \mathrm{sh}_{\mathbf{d}}\mathbf{c}$ [Ge2, Lemma B1]. Therefore $\Delta_{\mathbf{d}}(\mathrm{sh}_{\mathbf{d}}\mathbf{b}, \mathrm{sh}_{\mathbf{d}}\mathbf{a}) \geq \Delta_{\mathbf{d}}(\mathrm{sh}_{\mathbf{d}}\mathbf{c}, \mathrm{sh}_{\mathbf{d}}\mathbf{a})$. So $\mathbf{d} \in \Psi_{k+1}(\mathbf{a},\mathbf{b})$ and (1) follows.

As $\mathbf{c} \in \Psi_k(\mathbf{a},\mathbf{b}) \setminus \Psi_k(\mathbf{a},\mathbf{c})$ we obtain that $|\Psi_k(\mathbf{a},\mathbf{c})| < |\Psi_k(\mathbf{a},\mathbf{b})|$. Thus by the minimality of $(\mathbf{a},\mathbf{b})$ the pair $(\mathbf{a},\mathbf{c})$ cannot be a counter-example. Then $(\mathbf{a},\mathbf{d})$ is $(k+1)$-nonrefinable. Since $\mathbf{d} \in \Psi_{k+1}(\mathbf{a},\mathbf{b}) \subset \Psi_k(\mathbf{a},\mathbf{b})$ the pair $(\mathbf{a},\mathbf{b})$ cannot be a counter-example neither. A contradiction. $\square$

For a tube $P = \mathbf{a} - \mathbf{a}_1 - ... - \mathbf{a}_n - \mathbf{b}$ we denote by ∂P its boundary $\{\mathbf{a},\mathbf{b}\}$.

Proposition 3.25. *For every pair* $\{\mathbf{a},\mathbf{b}\} \subset A$ *and integer* $k \geq 2$ *there exists a finite* $k+2$*-nonrefinable k-tube* $P \subset A$ *such that* $\partial P = \{\mathbf{a},\mathbf{b}\}$.

Proof: Suppose this is not true. Let a pair $\{\mathbf{a},\mathbf{b}\}$ be a counter-example such that it has the minimal cardinality $|\Psi_k(\mathbf{a},\mathbf{b})|$ among all such pairs. Since $\{\mathbf{a},\mathbf{b}\}$ is $k+2$-refinable by the above Lemma there exists $\mathbf{c} \in \Psi_{k+1}(\mathbf{a},\mathbf{b})$ such that $\{\mathbf{a},\mathbf{c}\}$ is $k+2$-nonrefinable. Since the inclusion $\Psi_k(\mathbf{c},\mathbf{b}) \subset \Psi_k(\mathbf{a},\mathbf{b})$ is strict there exists a $k+2$-nonrefinable k-tube Q with $\partial Q = \{\mathbf{c},\mathbf{b}\}$. By the transitivity property [Ge1, Lemma T2] the set $R = \{\mathbf{a}\} \cup Q$ is a k-tube with the boundary $\{\mathbf{a},\mathbf{b}\}$. It is $k+2$-nonrefinable by construction. Thus the pair $\{\mathbf{a},\mathbf{b}\}$ is not a counterexample. We have a contradiction. $\square$

Definition 3.26. *[Ge1] (Horospherical projection). Let $p \in \mathcal{P}$ be a parabolic point and $T(p)$ be a horosphere at p. Define a projection map $\Pi_p : A \to T(p)$ (or $\Pi_{p,k}$) called horospherical projection as follows. If $\mathbf{a} \notin T_k(p)$ then $\Pi_p(\mathbf{a}) = \{\mathbf{p} \in T_k(p) \ : \ \mathbf{a} - \mathbf{p} - p(k)\}$; and if $\mathbf{a} \in T_k(p)$ then $\Pi_p(\mathbf{a}) = \mathbf{a}$.*

Proposition 3.27. *Let $\mathcal{P}$ denote the set of parabolic points for the action $G \curvearrowright T$. Then for any constants $k > 1$ and $d > 0$ the following sets are G-finite:*

1) $\forall \{\mathbf{a}, \mathbf{b}\} \subset A \ : \ \{\{\mathbf{c}, \mathbf{d}\} \mid \mathbf{c} \in \Pi_p(g\mathbf{a}), \ \mathbf{d} \in \Pi_p(g\mathbf{b}), \ p \in \mathcal{P}, g \in G\}$

2) $\mathcal{A}_1 = \{(\mathbf{a}, \mathbf{b}) \mid \Psi_k(\mathbf{a}, \mathbf{b}) = \emptyset, \ \{\mathbf{a}, \mathbf{b}\} \not\subset T_{A,k}(p), \ p \in \mathcal{P}\}$.

3) *a)* $\{\{p, q\} \subset \ \mathcal{P} \mid N_d(T_{A,k}(p)) \cap N_d(T_{A,K}(q) \neq \emptyset\}$, *and*
   b) $\{N_d(T_{A,k}(p)) \cap N_d(T_{A,k}(q)) \mid \{p, q\} \subset \mathcal{P}\}$.

Proof: 1) Suppose to the contrary that the set 1) is infinite. Assume first that $\mathbf{a} \neq \mathbf{b}$. Then there exist an infinite sequence of elements $g_n \in G$, distinct entourages $\{\mathbf{c}_n, \mathbf{d}_n\} \subset A$ such that

$$g_n\mathbf{a} - \mathbf{c}_n - p_n(k) \text{ and } g_n\mathbf{b} - \mathbf{d}_n - p_n(k), \ \mathbf{c}_n \in T_{A,k}(p_n), \ \mathbf{d}_n \in T_{A,k}(p_n), \ p_n \in \mathcal{P} \qquad (2).$$

Since the set $\mathcal{P}$ is G-finite (Lemma 3.17) up to choosing an infinite subsets we can fix the parabolic point $p_n = p$. Since the stabilizer $\mathrm{Stab}_G p$ acts cofinitely on $T_{A,k}(p)$ (Lemma 3.17) we can also fix $\mathbf{c}_n = \mathbf{c} \in T_{A,k}(p)$, and assume that $\mathbf{d}_n = h_n(\mathbf{d}), \ \mathbf{d} \in T_{A,k}(p), h_n \in \mathrm{Stab}_G p$. So (2) gives

$$g_n\mathbf{a} - \mathbf{c} - p(k), \quad g_n\mathbf{b} - \mathbf{d}_n - p(k), \ \mathbf{c} \in T_{A,k}(p), \ \mathbf{d}_n \in T_{A,k}(p), \ p \in \mathcal{P} \qquad (2').$$

The following Lemma implies that p is a limit point of $\{g_n\mathbf{b}\}_n$.

Lemma 3.28. *If $\mathbf{b}_n - \mathbf{d}_n - p(k)$ $(k > 1)$, $\mathbf{d}_n \in T_{A,k}(p)$ and $\lim\limits_{n\to\infty} \mathbf{d}_n = p$ then $\lim\limits_{n\to\infty} \mathbf{b}_n = p$.*

Proof: Suppose by contradiction that there exists an accumulation point $q \in T$ of the set $\{\mathbf{b}_n\}_n$ distinct from p. Let U_q be a closed neighborhood of q not containing p. Then U'_q is $\mathbf{b}_{m_n}$-small for infinitely many $\{\mathbf{b}_{m_n}\}_n$. Since $\lim\limits_{n\to\infty} \mathbf{d}_n = p$ and U'_q is a neighborhood of p its complement U_q is $\mathbf{d}_{m_n}$-small for some m_n. Then $U_q \in \mathrm{Sh}_{\mathbf{d}_{m_n}} \mathbf{b}_{m_n}$. By assumption there exists a neighborhood U_p of p such that $\Delta_{\mathbf{d}_{m_n}}(\mathrm{sh}_{\mathbf{d}_{m_n}} \mathbf{b}_{m_n}, U_p) > k$. Thus $\Delta_{\mathbf{d}_{m_n}}(U_q, U_p) > k - 1$. So $q - \mathbf{d}_{m_n} - p(k-1)$ for an infinite subset $\{\mathbf{d}_{m_n}\}_n$. Therefore

$$h_{m_n}^{-1}(q) - \mathbf{d} - p(k-1) \qquad (3).$$

Since the action of $\mathrm{Stab}_p G$ on $T \setminus p$ is discontinuous and cocompact we have $\lim\limits_{n\to\infty} h_{m_n}^{-1}(q) = p$. From the other hand (3) implies that $\Delta_{\mathbf{d}}(h_{m_n}^{-1}(p), \widetilde{U}_q) > k - 1 > 0$ for some neighborhood $\widetilde{U}_q$ of q. So $\forall m_n \ h_{m_n}^{-1}(p) \notin \widetilde{U}_q$. A contradiction. $\square$

It follows from (2') that there exist $g_n\mathbf{a}$-small set a_n and $\mathbf{c}$-small set c_n such that $a_n \cup c_n = T$ and $\Delta_{\mathbf{d}_n}(c_n, U_p) > k$ for some neighborhood U_p of p. Then $U_p \subset a_n$ and U_p is $g_n\mathbf{a}$-small too.

From the other hand by Proposition 3.14 we have that $G \curvearrowright \widetilde{T}$ is a convergence action. Then by [GePo1, Lemma 5.1] for every pair of distinct non-conical points $\{x, y\} \subset \widetilde{T}$ the accumulation points of the orbit $G(x, y)$ belong to the diagonal $\Delta^2\widetilde{T}$. By Lemma 3.28 $\lim\limits_{n\to\infty} g_n(\mathbf{b}) = p$ so $\lim\limits_{n\to\infty} g_n(\mathbf{a}) = p$. Hence U'_p is $g_n\mathbf{a}$-small and U_p is $g_n\mathbf{a}$-small for some $n \in \mathbb{N}$. This is impossible by our Convention (1) of 3.1. Claim 1) is proved. $\square$

2) Suppose that $\{(\mathbf{a}_i, \mathbf{b}_i) \in A \times A \mid i \in I\}$ is an infinite set such that for every $i \in I$ there is no $\mathbf{c}_i \in A$ such that $\mathbf{a}_i - \mathbf{c}_i - \mathbf{b}_i(k)$. The set A is G-finite so we can fix $\mathbf{a} = \mathbf{a}_i$ and assume that $\mathbf{b}_i = g_i(\mathbf{b})\ :\ g_i \in G$. Since the space $\widetilde{T}$ is compact, the set $\{\mathbf{b}_i\}_{i\in I}$ admits an accumulation point p which is a limit point for the geometrically finite action $G \curvearrowright \widetilde{T}$. By Lemma 3.17 p is either k-conical or k-parabolic point for any $k > 1$. Consider these two cases separately.

Let first, p be a k-conical point. Then there exists $\mathbf{c} \in A$ such that $\mathbf{a} - \mathbf{c} - p(k)$. By Lemma 3.6 we have $\mathbf{a} - \mathbf{c} - \mathbf{b}_i(k)$ $(i \in I)$ contradicting the k-non-refinability of the pair $\{\mathbf{a}, \mathbf{b}_i\}$.

Let now suppose that p is k-parabolic. We will now show that for almost all $i \in I$ the entourages $\mathbf{a}$ and $\mathbf{b}_i$ belong to the same horosphere $T_{A,k}(p)$. We claim first that $\mathbf{a} \in T_{A,k}(p)$. Indeed if not, then there exists $\mathbf{c} \in A$ such that $\mathbf{a} - \mathbf{c} - p(k)$ contradicting by the same argument the k-non-refinability of the pair $\{\mathbf{a}, \mathbf{b}_i\}$ $(i \in I)$. So $\mathbf{a} \in T_{A,k}(p)$.

Suppose by contradiction that there exist $\mathbf{b}_i \notin T_{A,k}(p)$ for infinitely many $i \in I$. Then there exist $\mathbf{c}_i \in T_{A,k}(p)$ such that

$$\mathbf{b}_i - \mathbf{c}_i - p(k). \tag{$*$}$$

We first note that in (*) we cannot have the same entourage $\mathbf{c}_0$ for infinitely many different $\mathbf{b}_i$. Indeed if not, then from (*) we have $\Delta_{\mathbf{c}_0}(\mathrm{sh}_{\mathbf{c}_0}\mathbf{b}_i, c_0) > k$ $(i \in I)$ for a $\mathbf{c}_0$-small set c_0 containing p. By Lemma 3.18 p is an accumulation point for the set $\{\mathbf{b}_i\}_{i\in I}$ so c'_0 is $\mathbf{b}_i$-small for infinitely many $i \in I$. Thus $c_0 \supset \mathrm{sh}_{\mathbf{c}_0}\mathbf{b}_i$, and $\Delta_{\mathbf{c}_0}(c_0, \mathrm{sh}_{\mathbf{c}_0}\mathbf{b}_i) \leq 1$ which is impossible. So we can assume that $\mathbf{c}_i$ are all distinct. By Lemma 3.17 the quotient $T_{A,k}(p)/\mathrm{Stab_Gp}$ is finite, so there exists $h_i \in \mathrm{Stab_Gp}$ such that we have $h_i(\mathbf{c}_i) = \mathbf{c} \in T_{A,k}(p)$ for infinitely many $i \in I$. Hence $h_i(\mathbf{b}_i) - \mathbf{c} - p(k)$. Since $\mathbf{a} \in T_{A,k}(p)$ by Lemma 3.18 p is an accumulation point for the set $\{h_i(\mathbf{a})\}_{i\in I}$. Then by Lemma 3.6 we obtain $h_i(\mathbf{b}_i) - \mathbf{c} - h_i(\mathbf{a})(k)$ and so $\mathbf{b}_i - \mathbf{c}_i - \mathbf{a}(k)$ which is impossible.

So $\mathbf{b}_i \in T_{A,k}(p)$ for almost all $i \in I$. This shows that the set A_1 is G-finite. Part 2) is proved.$\square$

3) We omit the index k below. Suppose that the first set is infinite. Then there exists an infinite set of G-non-equivalent pairs of parabolic points $(p_i, q_i) \in \mathcal{P}^2$ for which $N_d(T(p_i)) \cap N_d(T(q_i)) \neq \emptyset$ $(i \in I)$. Since the action of G on Θ^2T is cocompact there exist $g_i \in G$ such that the pair $(g_i(p_i), g_i(q_i))$ belong to a compact subset of Θ^2T. So without lost of generality we may assume that the sets $\{p_i\}_{i\in I}$ and $\{q_i\}_{i\in I}$ admits two distinct accumulation points p and q. It follows from [Ge1, Lemma P3] that there cannot exist an entourage belonging to the intersection of infinitely many distinct horospheres. So there is an infinite sequence of entourages $\mathbf{b}_i \in N_d(T(p_i)) \cap N_d(T(q_i))$ $(i \in I)$. The set $\{\mathbf{b}_i\}_{i\in I}$ admits an accumulation point $x \in T$. Let $(\mathbf{c}_i)_i \subset T(p_i)$ and $(\mathbf{d}_i)_i \subset T(q_i)$ be two subsets for which $d_A(\mathbf{b}_i, \mathbf{c}_i)$ and $d_A(\mathbf{b}_i, \mathbf{d}_i)$ are bounded by the constant d. Thus $d_A(\mathbf{c}_i, \mathbf{d}_i) \leq 2d$ and by Lemma 3.15 we have $p = q = x$. A contradiction.

The G-finiteness of the second set directly follows from the last argument too. Indeed if $|N_d(T(p)) \cap N_d(T(q))| = \infty$ we must have $p = q$. The Proposition is proved. $\square$

Corollary 3.29. *Suppose that G is a finitely generated group acting 3-discontinuously and 2-cocompactly on a compactum T. Then there exists a constant $C > 0$ such that the d_A-diameter of each of the sets 1), 2) and 3b) of Proposition 3.27 is bounded by C.*

Proof: The group G acts isometrically on the graph $(\mathcal{G}, d_A)$ and the set A is G-finite. So the only thing which is left to prove is that in the case 3b) we have $\mathrm{diam}(\mathrm{N_d(T(q))\cap N_d(T(p))}) \leq \mathrm{C}$ for all parabolic points p and q and every $d > 0$. Since the set of parabolic point is G-finite (Lemma 3.17) so we may suppose that $p_n = p$. Then by 3.27.3b we have $\forall q \in \mathcal{P}\ : \mathrm{diam}(\mathrm{N_d(T(p))\cap N_d(T(q))}) <$ C for some constant C. Then the same is true for any pair of parabolic points. The Corollary is proved. □

From Proposition 3.27.2 we immediately obtain:

Corollary 3.30. *Let $G \curvearrowright T$ be a 3-discontinuous and 2-cocompact action satisfying the above conditions. Then if for a fixed $\mathbf{a} \in A$ and infinitely many $\mathbf{b}_n \in A$ the pairs $(\mathbf{a}, \mathbf{b}_n)$ are all non-refinable then for all but finitely many n one has $\mathbf{b}_n \in T(p)$.* □

We will now obtain few more finiteness properties characterizing the horospherical projection $\Pi_p : A \to T_{A,k}(p)(p \in \mathcal{P})$.

Definition 3.31. *For a fixed $k > 5$ a visibility neighborhood of the point $\mathbf{p} \in \Pi_p(\mathbf{a}) \subset T_{A,k}(p)$ from the point $\mathbf{a} \in A$ is the following set*

$$\mathcal{N}(\mathbf{a}, \mathbf{p}, p) \ = \ \{\mathbf{x} \in T_{A,k}(p) \mid \mathbf{a} - \mathbf{p} - p(k) \ \wedge \ \neg\ \mathbf{a} - \mathbf{p} - \mathbf{x}(k-1)\},$$

where $\neg$ denotes the opposite logical statement.

The following Proposition establishes the G-finiteness properties of two more sets (by continuing the notations of 3.27):

Proposition 3.32. *For every $k > 1$ the following sets are G-finite:*

1) $\mathcal{A}_2 = \{(\mathbf{x}, \mathbf{p}) \in T_k^2(p) | \ \mathbf{x} \in \mathcal{N}(\mathbf{a}, \mathbf{p}, p),\ \mathbf{a} \in A,\ p \in \mathcal{P}\}$.

2) $\mathcal{A}_3 = \{\Pi_p(T_k(q)) \mid \{p, q\} \subset \mathcal{P}\}$.

Proof: 1) Suppose by contradiction that it is not true and $\mathcal{A}_2$ is not G-finite for some $k > 1$. Since A is one G-orbit up to taking an infinite subset of $\mathcal{A}_2$ we can fix the entourage $\mathbf{p}$. By Proposition 3.27.3 (or by [Ge1, Lemma P3]) $\mathbf{p}$ can belong to at most finitely many different horospheres. So up to a passing to a new infinite subset we can fix the parabolic point $p \in \mathcal{P}$.

If first the set of entourages $\{\mathbf{a} \mid (\mathbf{x}, \Pi_p(\mathbf{a})) \in \mathcal{A}_2\}$ is finite, up to choosing a new infinite subset of $\mathcal{A}_2$ we have $\mathbf{a} - \mathbf{p} - p(k)$ and $\neg\ \mathbf{a} - \mathbf{p} - \mathbf{x}(k-1)$ for a fixed $\mathbf{a}$. Then the set of the first coordinates $\{\mathbf{x} \mid (\mathbf{x}, \cdot) \in \mathcal{A}_2\} \subset T_p$ is infinite and by Lemma 3.18 its accumulation point is p. Then by Lemma 3.6 there exists $\mathbf{x}$ in this set such that $\mathbf{a} - \mathbf{p} - \mathbf{x}(k)$. A contradiction.

If now the set $\{\mathbf{a} \mid (\mathbf{x}, \Pi_p(\mathbf{a})) \in \mathcal{A}_2\}$ is infinite let $q \in T$ be an accumulation point of it. Taking a $\mathbf{p}$-small neighborhood U_q of q we obtain that U'_q is $\mathbf{a}$-small for some $\mathbf{a}$. So $U_q \supset \mathrm{sh}_{\mathbf{p}}\mathbf{a}$. Since $\mathbf{a}-\mathbf{p}-p(k)$, so $\Delta_{\mathbf{p}}(U_q, U_p) > k-1$ for some $\mathbf{p}$-small neighborhood U_p of p. It yields $q-\mathbf{p}-p(k-1)$.

For $\mathbf{x} \in U_p$ we have $\Delta(\mathrm{sh}_{\mathbf{p}}\mathbf{a}, \mathrm{sh}_{\mathbf{p}}\mathbf{x}) \geq \Delta_{\mathbf{p}}(U_p, U_q) > k-1$. Therefore $\mathbf{a} - \mathbf{p} - \mathbf{x}(k-1)$. Again a contradiction. $\square$

2) Suppose not. Since the set of parabolic points $\mathcal{P}$ is G-finite we can fix the point $p \in \mathcal{P}$. Using the action of $\mathrm{Stab}_G p$ on $T_k(p)$ we can also assume that there is a fixed entourage $\mathbf{c} \in T(p)$ such that for every $q \in \mathcal{P}\ :\ \mathbf{c} \in \Pi_p(T_q)$. So there exists an infinite set I such that for all $i \in I$ we have

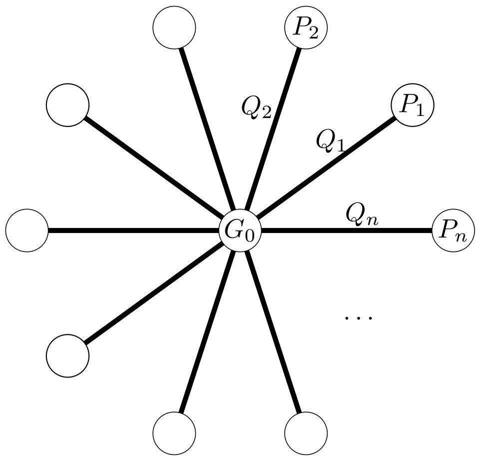
$$\mathbf{b}_i - \mathbf{d}_i - p(k),\ \mathbf{a}_i - \mathbf{c} - p(k), \{\mathbf{a}_i, \mathbf{b}_i\} \subset T(q_i),\ \mathbf{d}_i \in \Pi_p(T_{q_i}),\ q_i \in \mathcal{P}.$$

Since p is the unique accumulation point of $T(p)$, up to passing to an infinite subsequence of I, we may assume that $\lim\limits_{i\to\infty} \mathbf{d}_i = p$. Then by Lemma 3.28 we have $\lim\limits_{i\to\infty} \mathbf{b}_i = p$. Let $q \in T$ be an accumulation point of the set $\{q_i\}_{i\in I}$. We claim that $q = p$. Indeed if not then there exists an entourage $\mathbf{a} \in A$ such that $q - \mathbf{a} - p(k+1)$. Hence for infinitely many $i \in I$ we have $q - \mathbf{a} - \mathbf{b}_i(k)$ (Lemma 3.6). Thus there exists a neighborhood U_q of q such that $\Delta_{\mathbf{a}}(U_q, \mathrm{sh}_{\mathbf{a}}\mathbf{b}_i) > k$. So for some $i \in I$ we have $q_i \in U_q$ and hence $q_i - \mathbf{a} - \mathbf{b}_i(k)$. The latter one is impossible since $\mathbf{b}_i \in T_k(q_i)$. By the same argument since $\lim\limits_{i\to\infty} q_i = p$ we also have $\lim\limits_{i\to\infty} \mathbf{a}_i = p$. This is a contradiction as $\mathbf{a}_i - \mathbf{c} - p(k)$. The Proposition is proved. $\square$

3.4 Exhaustion by finitely generated subgroups.

The aim of this subsection is the following.

Theorem A. *Let G be a relatively hyperbolic group with respect to a collection of parabolic subgroups $\{P_1, ..., P_k\}$. Then there exists a finitely generated subgroup G_0 of G which is relatively hyperbolic with respect to the collection $\{Q_i = P_i \cap G_0 \mid i = 1, .., k\}$ such that G is the fundamental group of the star graph*



whose central vertex group is G_0 and all other vertex groups are P_i ($i = 1, ..., k$).

Furthermore for every finite subset $K \subset G$ the subgroup G_0 can be chosen to contain K.

Proof: Recall that $A = G(\mathbf{a}_0)$ ($\mathbf{a}_0 \in \text{Ent } T$) is a discrete orbit of entourages forming the vertex set of the graph $\mathcal{G}$ satisfying our Convention 3.12. Without lost of generality we can assume that the group G is not finitely generated and $\mathbf{a}_0 \in K$. So the graph $\mathcal{G}$ is not connected (see Lemma 3.11). The distance $d_A(\mathbf{x}, \mathbf{y})$ is a pseudo-distance being infinity if and only if $\mathbf{x}$ and $\mathbf{y}$ belong to different connected components of $\mathcal{G}$. By Lemmas 3.17 and 3.18 the set $\mathcal{P}$ of parabolic points for the action $G \curvearrowright T$ is G-finite; and for every $p \in \mathcal{P}$ the stabilizer $H_p = \mathrm{Stab_G p}$ acts cofinitely on its horosphere $T(p)$.

Let $\mathcal{A}_i$ $(i = 1, 2, 3) \subset A^2$ be the G-finite sets introduced in Propositions 3.27.2 and 3.32.

We now construct a new graph $\widetilde{\mathcal{G}}$ whose set of vertices is A and the set of edges is given by the pairs of entourages belonging to the following sets:

a) the finite set K^2 and the set of all its horospherical projections $\{\Pi_p(K^2) \mid p \in \mathcal{P}\}$;

b) the set $\mathcal{A}_1$ and the set of all its horospherical projections $\{\Pi_p(\mathcal{A}_1) \mid p \in \mathcal{P}\}$;

c) the set $\mathcal{A}_2$;

d) the set $\mathcal{A}_3$.

All these sets are G-finite. Indeed the set $\mathcal{A}_1$ is G-finite by Proposition 3.27.2. So the sets a) and b) being projections of finitely many G-orbits of pairs are G-finite by Proposition 3.27.1. The sets $\mathcal{A}_2$ and $\mathcal{A}_3$ are G-finite by Proposition 3.32.

Lemma 3.33. *There exists a finitely generated subgroup G_0 of G containing any finite subset $K \subset G$ and which is relatively hyperbolic with respect to $Q_i = P_i \cap G_0$ $(i = 1, ..., k)$.*

Proof: Let $\mathcal{G}_0$ be the connected component of $\widetilde{\mathcal{G}}$ containing K. Set $G_0 = \mathrm{Stab}_G \mathcal{G}_0$, $\widetilde{A} = \widetilde{\mathcal{G}}^0$ and $A_0 = \mathcal{G}_0^0$. By Lemma 3.11 the group G_0 is finitely generated. We are left to prove that G_0 is relatively hyperbolic with respect to the subgroups $\{Q_i\}_{i=1}^k$.

Let T_0 be a subset of T which is the limit set of G_0. We will first show that the action $G_0 \curvearrowright T_0$ is 2-cocompact. By [Ge1, Prop. E] the 2-cocompactness is equivalent to the k-separation property:

$$\forall p, q \in T_0 \ : \ p \neq q \ \exists \ \mathbf{b} \in A_0 \ : p - \mathbf{b} - q(k), \tag{1}$$

for some $k > 0$. Since the action of G on T is 2-cocompact, (1) is true for some $\mathbf{b} \in \widetilde{A}$. If $\mathbf{b} \in A_0$ we are done, so suppose that $\mathbf{b} \notin A_0$. Then for two $\mathbf{b}$-small neighborhoods U_p and U_q of the accumulation points p and q of A_0, we find entourages $\mathbf{a}, \mathbf{c} \in A_0$ such that U'_p is $\mathbf{a}$-small and U'_q is $\mathbf{c}$-small. So $U_p \supset \mathrm{sh}_{\mathbf{b}}\mathbf{a}$ and $U_q \supset \mathrm{sh}_{\mathbf{b}}\mathbf{c}$. Hence

$$\mathbf{a} - \mathbf{b} - \mathbf{c}(k). \tag{2}$$

By Proposition 3.25 up to refining the pair $\{\mathbf{a}, \mathbf{b}\}$ we can suppose that the pair $\{\mathbf{a}, \mathbf{b}\}$ is $k+2$-nonrefinable. Since $\mathbf{b} \notin A_0$, by operation b) above the pair $\{\mathbf{a}, \mathbf{b}\}$ must belong to an horosphere $T(r)$ $(r \in \mathcal{P})$. As $\{\mathbf{a}, \mathbf{c}\} \subset A_0$ and $\mathcal{G}_0$ is connected there exists a path $\gamma = \gamma(\mathbf{a}, \mathbf{c}) \subset \mathcal{G}_0$. Let $\mathbf{e} = \Pi_p(\mathbf{c})$. Note that for every edge $l \in \mathcal{G}_0^1$ we have $\Pi_p(l) \in \mathcal{G}_0^1$. Indeed if l joins two vertices of A_0 then by the operations a), b) and d) all their horospherical projections are joined by edges too. So $\Pi_p(\mathcal{G}_0) \subset \mathcal{G}_0$. Since $\{\mathbf{a}, \mathbf{e}\} \subset T(r) \cap \Pi_p(\gamma)$ we have $\mathbf{e} \in A_0$.

Operation c) then implies that $\mathbf{b} \notin \mathcal{N}(\mathbf{c}, \mathbf{e}, r)$. By Definition 3.31 we have

$$\mathbf{b} - \mathbf{e} - \mathbf{c}(k-1) \qquad (3).$$

So $\mathrm{sh}_{\mathbf{b}}\mathbf{c} \subset \mathrm{sh}_{\mathbf{b}}\mathbf{e}$ and (2) yields $\mathbf{a}-\mathbf{b}-\mathbf{e}(k-1)$. Thus $\mathrm{sh}_{\mathbf{e}}\mathbf{a} \subset \mathrm{sh}_{\mathbf{e}}\mathbf{b}$ and by (3) we have $\mathbf{a}-\mathbf{e}-\mathbf{c}(k-1)$ with $\mathbf{e} \in A_0$. We have proved that the action $G_0 \curvearrowright T_0$ is $(k-1)$-separating and so is 2-cocompact [Ge1, Prop. E].

By [Ge1, Main Thm] every point of T_0 is either conical or parabolic for the action of G_0 on T_0. Let $p \in T_0$ be a parabolic point. We claim that it is also parabolic for the action of G on T. Indeed if not then $\forall \mathbf{b} \in T(p) \cap A_0$ $\exists$ $\mathbf{c} \in \widetilde{A} \setminus A_0$ such that $\mathbf{b} - \mathbf{c} - p(2)$. Then by Lemma below we can suppose up to refining the couple $(\mathbf{b}, \mathbf{c})$ that it is not k-refinable ($k > 3$). By our assumption p is a conical point for the action $G \curvearrowright T$ and $\mathbf{c} \notin T(p)$. Then by operation b) above the vertices $\mathbf{c}$ and $\mathbf{b}$ are joined by an edge in $\widetilde{\mathcal{G}}$. So $\mathbf{c} \in A_0$ which is not possible as $\mathbf{b} \in T(p) \cap A_0$.. This is a contradiction and p is a parabolic point for the action of G on T. We have $\mathrm{Stab}_{G_0} p = \mathrm{Stab}_G p \cap A_0$. Lemma 3.33 is proved modulo the following Lemma.

Sublemma 3.34. *If* $\mathbf{b} - \mathbf{c} - p(2)$ *and* $\mathbf{b} - \mathbf{c}_1 - \mathbf{c}(k)$ *then* $\mathbf{b} - \mathbf{c}_1 - p(k-1)$ $(k > 3)$.

Proof: Let us first show that $\mathbf{c}_1-\mathbf{c}-p(1)$. Indeed the second assumption implies that $\mathrm{sh}_{\mathbf{c}}\mathbf{c}_1 \supset \mathrm{sh}_{\mathbf{c}}\mathbf{b}$. So for a $\mathbf{c}$-small neighborhood U_p of p using the first assumption we have

$$\Delta_{\mathbf{c}}(\mathrm{sh}_{\mathbf{c}}\mathbf{c}_1, U_p) > \Delta_{\mathbf{c}}(\mathrm{sh}_{\mathbf{c}}\mathbf{b}, U_p) - \Delta_{\mathbf{c}}(\mathrm{sh}_{\mathbf{c}}\mathbf{b}, \mathrm{sh}_{\mathbf{c}}\mathbf{c}_1) > 2 - 1 = 1.$$

Hence $\Delta_{\mathbf{c}_1}(\mathrm{sh}_{\mathbf{c}_1}\mathbf{b}, U_p) > \Delta_{\mathbf{c}_1}(\mathrm{sh}_{\mathbf{c}_1}\mathbf{b}, \mathrm{sh}_{\mathbf{c}_1}\mathbf{c}) - \Delta_{\mathbf{c}_1}(\mathrm{sh}_{\mathbf{c}_1}\mathbf{c}, U_p) > k-1$. The Lemma and the Proposition are proved. $\square$

The following Lemma finishes the proof of the Theorem.

Lemma 3.35. *The action $G \curvearrowright \widetilde{\mathcal{G}}$ induces an action on a bipartite simplicial tree $\mathcal{T}$ such that the graph $X = \mathcal{T}/G$ satisfies Theorem A.*

Proof: Using the graph $\widetilde{\mathcal{G}}$ we construct the tree $\mathcal{T}$ to have vertices belonging to two subsets $\mathcal{C}$ and $\mathcal{H}$. The elements of $\mathcal{C}$ are components of $\mathcal{G}$ and the elements of $\mathcal{H}$ are the horospheres of T. We call them *non-horospherical* and *horospherical* respectively. Two vertices C and H of $\mathcal{T}$ are joined by an edge if and only if $C \in \mathcal{C}$, $H \in \mathcal{H}$, and $C \cap H \neq \emptyset$.

Let us first show that $\mathcal{T}$ is connected. Indeed by construction every horospherical vertex is joined with a non-horospherical one. So it is enough to prove that every two non-horospherical vertices can be joined by a path. Let C_i $(i = 1, 2)$ be the corresponding connected components of $\widetilde{\mathcal{G}}$ and let us fix two entourages $\mathbf{a} \in C_1^0$ and $\mathbf{b} \in C_2^0$. Let $P = \mathbf{a} - \mathbf{b}_1 - ... - \mathbf{b}_n - \mathbf{b} \subset A$ be a non-refinable tube between them. By operation b) above every non-refinable pair $(\mathbf{b}_i, \mathbf{b}_{i+1})$ either belongs to an horosphere $T(p)$ or corresponds to an edge in the graph $\widetilde{\mathcal{G}}$. In the latter case it stays in the same component of $\widetilde{\mathcal{G}}$. In the former case the horosphere T_p corresponds to a single vertex in the graph $\mathcal{T}$. So the tube P produces a path in $\mathcal{T}$ between the corresponding vertices. Thus $\mathcal{T}$ is connected.

Let us now show that $\mathcal{T}$ is a tree. Suppose not and it contains a simple loop α. Since the vertices of two types alternate on α we can fix a horospherical vertex H having two non-horospherical neighboring vertices C_1 and C_2. Let α_1 be a subpath of α containing H and $\alpha_2 =$

$\overline{\alpha \setminus \alpha_1}$. As we have seen above α_2 corresponds to an alternating sequence of components of $\widetilde{\mathcal{G}}$ and horospheres. So we can choose a sequence of tubes $P_i \subset C_i$ where each C_i is a component of $\widetilde{\mathcal{G}}$ corresponding to a non-horospherical vertex of α_2. The tube P_i connects two entourages from C_i each belonging to horospheres $T(q_i)$ and $T(q_i')$ intersecting C_i. Note that these horospheres differ from the initial horosphere $T(p)$ corresponding to h as α is a simple loop. By operations b) and d) above it follows that $\bigcup\limits_i \Pi_p(P_i \cup T(q_i) \cup T(q_i'))$ is a connected path on T_p. It implies that the vertices C_1 and C_2 correspond to the same connected component of $\widetilde{\mathcal{G}}$ which is impossible. So $\mathcal{T}$ is a tree.

Since G acts transitively on A and so on $\mathcal{C}$, there is one non-horospherical vertex $v_0 = \mathcal{C}/G$ in the graph $X = \mathcal{T}/G$. The set of horospheres on T is G-finite (Lemma 3.17) so X contains n vertices of non horospherical type each representing the G-orbit of an horosphere $T(p)$ ($p \in \mathcal{P}$). Every one of them is connected with v_0 by a unique edge. The vertex group of v_0 is G_0 as every element in $\mathcal{C}$ is stabilized by a subgroup conjugate to G_0. Similarly every vertex group of horospherical type is P_i and the edge groups are $Q_i = P_i \cap G_0$ ($i = 1, ..., n$). The Theorem is proved. □

Theorem A admits several immediate corollaries describing different type of finiteness properties of relatively hyperbolic groups.

Corollary 3.36. *Let G be a relatively hyperbolic group with respect to the system P_j ($j = 1, .., n$). Then there exists an exhaustion $G = \bigcup\limits_{i \in I} G_i$ where G_i is a finitely generated relatively hyperbolic group with respect to $P_j \cap G_i$ ($j = 1, ..., n$).* □

Definition 3.37. *A group G is called finitely generated with respect to subgroups H_i ($i \in I$) if it is generated by a finite set S and the subgroups H_i.*

Corollary 3.38. *Let G a group acting 3-discontinuously and 2-cocompactly on a compactum T. Then G is finitely generated with respect to the stabilizers of the parabolic points.* □

Corollary 3.39. *A group G acting 3-discontinuously and 2-cocompactly on a compactum T without parabolic points is finitely generated.* □

Remark. If in particular G acts 3-discontinuously and 3-cocompactly on T without isolated points then every point of T is conical [GePo1, Appendix]. So by Corollary 3.39 G is finitely generated in this case. By a direct argument one can now deduce that G is word-hyperbolic [GePo1, Appendix]. This provides a new proof of a theorem due to B. Bowditch [Bo3]. □

4 Floyd metrics and shortcut metrics.

From now on we will assume that G is a finitely generated group acting 3-discontinuously and 2-cocompactly on a compactum T. Let us first recall few standard definitions concerning Floyd compactification (see [F], [Ka], [Tu1], [Ge2] for more details).

We will deal with abstract graphs even without assuming any group action. In particular it can be the Cayley graph of a finitely generated group.

Let Γ be a locally finite connected graph. For a finite path $\alpha : I \to \Gamma$ ($I \subset \mathbb{Z}$) we define its *length* to be $|I| - 1$. We denote by $d(,)$ the canonical shortest path distance function on Γ, and by $B(v, R)$ the ball at a vertex $v \in \Gamma^0$ of radius R.

Let $f : \mathbb{N} \to \mathbb{R}_{>0}$ be a function satisfying the following conditions :

$$\exists\, \lambda > 0\ \forall n \in \mathbb{N}\ :\ 1 < \frac{f(n)}{f(n+1)} < \lambda \tag{1}$$

$$\sum_{n \in \mathbb{N}} f(n) < +\infty. \tag{2}$$

Define the Floyd length $L_{f,v}(\alpha)$ of a path $\alpha = \alpha(a, b) \subset \Gamma$ with respect to a vertex v as follows:

$$L_{f,v}(\alpha) = \sum_i f(d(v, \{x_i, x_{i+1}\})). \tag{$*$}$$

where $\alpha^0 = \{x_i\}_i$ is the set of vertices of α (we assume $f(0) := f(1)$ to make it well-defined).

The Floyd metric $\delta_{f,v}$ is defined to be the corresponding shortest path metric:

$$\delta_{f,v}(a, b) = \inf_\alpha L_{f,v}(\alpha), \tag{$**$}$$

where the infimum is taken over all paths between the vertices a and b in Γ. We denote by $\overline{\Gamma}_f$ be the Cauchy completion of the metric space $(\Gamma, \delta_{f,v})$ and call it *Floyd completion*. Let

$$\partial_f \Gamma = \overline{\Gamma}_f \setminus \Gamma$$

be its boundary, called *Floyd boundary*.

If Γ is a Cayley graph $\mathcal{C}a(G, S)$ of a group G with respect to a finite generating system S we denote by $\overline{G}_f$ and by $\partial_f G$ its Floyd completion and Floyd boundary respectively. Then the condition (1) above implies that the G-action extends to its Floyd completion $\overline{G}_f$ by homeomorphisms [Ka]. Therefore in this case for any $g \in G$ the Floyd metric δ_g is the g-shift of δ_1:

$$\delta_g(x, y) = \delta_1(g^{-1}x, g^{-1}y), \quad x, y \in \overline{G}_f, g \in G,$$

where 1 is the neutral element of G. Every two metrics δ_{g_1} and δ_{g_2} are bilipshitz equivalent with a Lipshitz constant depending on $d(g_1, g_2)$.

Recall also that a *quasi-isometric map* (or *c-quasi-isometric map*) $\varphi : X \to Y$ between two metric spaces X and Y is a correspondence such that :

$$\frac{1}{c}d_X(x,y) - c < d_Y(\varphi(x), \varphi(y)) \leq c d_X(x,y) + c,$$

where c is a uniform constant and $d_X,\ d_Y$ denote the metrics of X and Y respectively.

If in addition $d_X(\mathrm{id}, \psi \circ \varphi) \leq$ const for a converse quasi-isometric map $\psi : Y \to X$ we say that φ is a quasi-isometry between X and Y.

A c-quasi-isometric map $\varphi : I \to X$ is called *c-quasigeodesic* if I is a convex subset of $\mathbb{Z}$ or $\mathbb{R}$. A quasigeodesic path $\gamma : I \to \Gamma$ defined on a half-infinite subset I of $\mathbb{Z}$ is called *(quasi-)geodesic ray*; a (quasi-)geodesic path defined on the whole $\mathbb{Z}$ is called *(quasi-)geodesic line.*

The following Lemma will be often used.

Lemma 4.1. (Karlsson Lemma) *For every $\varepsilon > 0$ and every $c > 0$, there exists a finite set D such that δ_v-length of every c-quasigeodesic $\gamma \subset \Gamma$ that does not meet D is less than ε.* $\square$

Remark. A. Karlsson [Ka] proved it for geodesics in the Cayley graphs of finitely generated groups. The proof of [Ka] does not use the group action and is also valid for quasigeodesics.

Consider now a set S of paths in the graph Γ with unbounded length. Every path $\alpha : I{=}[0, n] \to \Gamma$ started at a point $a = \alpha(0) \in \Gamma$ can be considered as an element of the product $\prod_{i \in I} B(a, i)$. By the compactness of the latter space in the Tikhonov topology every infinite sequence $(\alpha_n)_n \subset S$ possesses a "limit path" $\delta : [0, +\infty) \to \Gamma$ whose initial segments are initial segments of α_n.

The following Lemma illustrates the properties of limits of infinite quasigeodesics of Γ.

Lemma 4.2. *[GePo1] Let Γ be a locally finite connected graph. Then the following statements are true:*

1) *Every infinite ray $r : [0, +\infty[\to \Gamma$ converges to a point at the boundary: $\lim\limits_{n\to\infty} r(n) = p \in \partial_f \Gamma$.*

2) *For every point $p {\in} \partial_f \Gamma$ and every $a \in \Gamma$ there exists a geodesic ray joining a and p.*

3) *Every two distinct points in $\partial_f \Gamma$ can be joined by a geodesic line.*

$\square$

Let Γ be a locally finite, connected graph on which a finitely generated group G acts cocompactly (e.g. its Cayley graph). Besides the Floyd metrics the Floyd completion $\overline{\Gamma}_f$ possesses a set of *shortcut* pseudometrics which can be introduced as follows (see also [Ge2], [GePo1]). Let ω be a closed G-invariant equivalence relation on $\overline{\Gamma}_f$. Then there is an induced G-action on the compactum quotient space $\overline{\Gamma}_f/\omega$. A shortcut pseudometric $\overline{\delta}_g$ is the maximal element in the set of symmetric functions $\varrho : \overline{\Gamma}_f {\times} \overline{\Gamma}_f \to \mathbb{R}_{\geqslant 0}$ that vanish on ω and satisfy the triangle inequality, and the inequality $\varrho {\leqslant} \delta_g$.

For $p, q {\in} \overline{\Gamma}_f$ the value $\overline{\delta}_g(p, q)$ is the infimum of the finite sums $\sum\limits_{i=1}^{n} \delta_g(p_i, q_i)$ such that $p{=}p_1$, $q{=}q_n$ and $\langle q_i, p_{i+1} \rangle {\in} \omega$ $(i{=}1, \ldots, n{-}1)$ [BBI, pp 77]. Obviously, the shortcut pseudometric $\overline{\delta}_g$ is the g-shift of $\overline{\delta}_1$. The metrics $\overline{\delta}_{g_1}$, $\overline{\delta}_{g_2}$ are bilipschitz equivalent for the same constant as for δ_{g_1}, δ_{g_2}.

The pseudometric $\overline{\delta}_g$ is constant on $\omega-$equivalent pairs of points of $\partial_f\Gamma$, so it induces a pseudometric on the quotient space $\overline{\Gamma}_f/\omega$. We denote this induced pseudometric by the same symbol $\overline{\delta}_g$.

Let $\mathcal{G}$ be the graph given by the discrete system $A = G(\mathbf{a}_0)$ ($\mathbf{a}_0 \in \text{Ent } T$) of entourages (see Definition A and Convention 3.12). The graph $\mathcal{G}$ is locally finite, G-finite and connected (Lemma 3.11). The correspondence $g \in G \to g(\mathbf{a}_0) \in A$ gives rise to an equivariant c-quasi-isometry $\varphi : \mathcal{C}a(G, S) \to \mathcal{G}$. Let f and g be scaling functions satisfying (1-2) and the condition:

$$\frac{g(n)}{f(cn)} < D \ (n \in \mathbb{N})$$

for a constant $D > 0$. Then by [GePo1, Lemma 2.5] the map φ extends to a G-equivariant Lipshitz map between the Floyd completions $\overline{G}_f$ and $\overline{\mathcal{G}}_g$ of these graphs. We denote this map by the same letter φ. The following Lemma is a direct consequence of the main result of [Ge2]:

Lemma 4.3. *(*Floyd map) Let G be a finitely generated group acting 3-discontinuously and 2-cocompactly on a compactum T. Then there exist $\mu \in]0, 1[$ and a continuous G-equivariant map $F : \overline{G}_f \to \widetilde{T}$ for the scaling function $f(n) = \mu^n$.

Furthermore for every vertex $v \in \mathcal{C}a(G, S)$ the quantity $\overline{\delta}_{\mathbf{v}}((F(x), F(y))$ is a metric on $\widetilde{T}$ where $x, y \in \overline{G}_f$ and $\mathbf{v} = \varphi(v) = F(v)$.

Proof: It follows from [Ge2] that there exists $\lambda \in]0, 1[$ and a continuous G-equivariant map $\mathcal{F} : \overline{\mathcal{G}}_g \to \widetilde{T} = A \sqcup T$ where $g(n) = \lambda^n$. Furthermore by [Ge2] the map $\mathcal{F}$ transfers every shortcut pseudometric on $\overline{\mathcal{G}}_g$ to a shortcut metric on $\widetilde{T}$.

Let $\varphi : \overline{G}_f \to \overline{\mathcal{G}}_g$ be the G-equivariant Lipshitz map described above where $f(n) = \mu^n$ and $\mu = \lambda^{1/c}$. Set $F = \mathcal{F} \circ \varphi$. The kernel of F is the closed G-invariant equivalence relation on $\overline{G}_f$ such that $\overline{\delta}_{\mathbf{v}}(F(x), F(y)) = 0$ implies $F(x) = F(y)$ $(x, y \in \overline{G}_f)$.
So the map F transfers the pseudometric $\overline{\delta}_v$ on $\overline{G}_f$ to a metric on $\widetilde{T}$ as follows :

$$\overline{\delta}_{\mathbf{v}}(F(x), F(y)) = \overline{\delta}_v(x, y), \text{ where } \mathbf{v} = F(v),\ v \in \mathcal{C}a(G, S).$$

$\square$

Remarks 4.4. *1) We will call the obtained metric $\overline{\delta}_{\mathbf{v}}$ ($\mathbf{v} = F(v) \in A$) on $\widetilde{T}$ shortcut (Floyd) metric.*

2) Lemma 4.3 is in particularly true for any polynomial scalar function f. Moreover one has $f = g$ as $f(cn)/f(n) = \text{const}$ in this case.

3) Since $\overline{\delta}_g \leq \delta_g$ the Karlsson Lemma 4.1 is also true when one replaces the Floyd δ_v-length by the shortcut $\overline{\delta}_g$-length.

5 Horospheres and tubes.

Let a finitely generated group G act 3-discontinuously and 2-cocompactly on a compactum T. Let $\mathcal{G}$ denote the connected graph of entourages introduced in Section 3 and satisfying Convention 3.12. We will use the graph distance d_A on $\mathcal{G}$ as well as shortcut metrics $\overline{\delta}_{\mathbf{v}}$ ($\mathbf{v} \in \mathcal{G}$) on the compactified space $\widetilde{T} = T \cup A$ where $A = \mathcal{G}^0$.

We obtain in this Section several properties of tubes and horospheres which will be used later on.

Lemma 5.1. *For any integer $k > 1$ there exists a constant $\nu > 0$ such that*

$$\forall\ \mathbf{a}, \mathbf{c} \in \widetilde{T} = T \sqcup A,\ \forall \mathbf{b} \in A\ :\ \mathbf{a} - \mathbf{b} - \mathbf{c}(k) \text{ then } \overline{\delta}_{\mathbf{b}}(\mathbf{a}, \mathbf{c}) \geq \nu.$$

Proof: For a fixed entourage $\mathbf{b} \in A$ let $C_{\mathbf{b},k}$ denote the closure of the set $\{\{\mathbf{a}, \mathbf{c}\} \in \widetilde{T} \times \widetilde{T}\ :\ \mathbf{a} - \mathbf{b} - \mathbf{c}(k)\}$ in $\widetilde{T}$. We first claim that the set $C_{\mathbf{b},k}$ does not intersect the diagonal of $\widetilde{T} \times \widetilde{T}$. Suppose not and $(p, p) \in C_{\mathbf{b},k} \cap \Delta^2\widetilde{T}$. Then there exist two infinite sequences $(\mathbf{a}_n)_n$ and $(\mathbf{c}_n)_n$ in $C_{\mathbf{b},k}$ converging to p. By discreteness of A we may suppose that $p \in T$. By Lemma 3.6 we have $\mathbf{a}_n - \mathbf{b} - \mathbf{c}_n(k)$. Let U be a $\mathbf{b}$-small neighborhood of p. Then U' is $\mathbf{a}_n$-small and $\mathbf{c}_n$-small simultaneously for $n > n_0$. Hence $\mathrm{sh}_{\mathbf{b}}\mathbf{a}_n \cup \mathrm{sh}_{\mathbf{b}}\mathbf{c}_n \subset U$, and so $\Delta_{\mathbf{b}}(\mathrm{sh}_{\mathbf{b}}\mathbf{a}_n, \mathrm{sh}_{\mathbf{b}}\mathbf{c}_n) \leq 1$ which is impossible. It follows that $C_{\mathbf{b},k} \cap \Delta^2\widetilde{T} = \emptyset$.

Since $C_{\mathbf{b},k}$ is a closed subset of $\widetilde{T} \times \widetilde{T}$, and $\overline{\delta}_{\mathbf{b}}$ is a metric on $\widetilde{T}$, there exists a constant $\nu > 0$ such that $\overline{\delta}_{\mathbf{b}}(\mathbf{a}, \mathbf{c}) \geq \nu$ on $C_{\mathbf{b},k}$. Thus our statement holds for the set $C_{\mathbf{b},k}$ of entourages separated by the fixed entourage $\mathbf{b}$.

We have $A = G(\mathbf{a}_0)$. If now $\mathbf{a} - \mathbf{b} - \mathbf{c}(k)$ then $\exists\ g \in G\ :\ \mathbf{b} = g(\mathbf{a}_0)$. Thus $\overline{\delta}_{\mathbf{b}}(\mathbf{a}, \mathbf{c}) = \overline{\delta}_{\mathbf{a}_0}(g^{-1}(\mathbf{a}), g^{-1}(\mathbf{c})) \geq \nu$. Lemma 5.1 is proved. $\square$

The following Lemmas give a local description of C-quasigeodesics around tubes and horospheres.

Lemma 5.2. *There exists a constant $D > 0$ such that for every C-quasigeodesic $\gamma = \gamma(\mathbf{a}, \mathbf{c})$ in $\mathcal{G}$ with endpoints $\mathbf{a}, \mathbf{c}$ we have :*

$$\forall \mathbf{b} \in \Psi_k(\mathbf{a}, \mathbf{c})\ :\ d_A(\mathbf{b}, \gamma) \leq D \qquad (1).$$

Proof: By Lemma 5.1 we have $\overline{\delta}_{\mathbf{b}}(\mathbf{a}, \mathbf{c}) \geq \nu$, and so the Floyd length $L_{f,\mathbf{b}}(\gamma)$ of γ is at least ν. By Karlsson's Lemma 4.1 there exists a constant $D > 0$ such that $\gamma \cap B(\mathbf{b}, D) \neq \emptyset$ for the d_A-ball $B(\mathbf{b}, D)$ in $\mathcal{G}$ centered at $\mathbf{b}$ with the radius D. The Lemma is proved. $\square$

Lemma 5.3. *The following statements are true :*

1) *For any $C > 0$ and $E > 0$ there exists $L > 0$ such that for any parabolic point $p \in T$ and any C-quasigeodesic $\gamma : [0, 1] \to \mathcal{G}$ one has*

$$d_A(\gamma(1), T(p)) \leq E \implies d_A(\gamma, \Pi_p(\gamma(0))) \leq L \qquad (2).$$

2) *There exists a constant $D > 0$ such that for any parabolic point $p \in T$ and any C-quasigeodesic $\gamma : [0, \infty[\to \mathcal{G}$ one has*

$$\lim_{n \to \infty} \gamma(n) = p \implies d_A(\gamma, \Pi_p(\gamma(0))) \leq D. \tag{3}$$

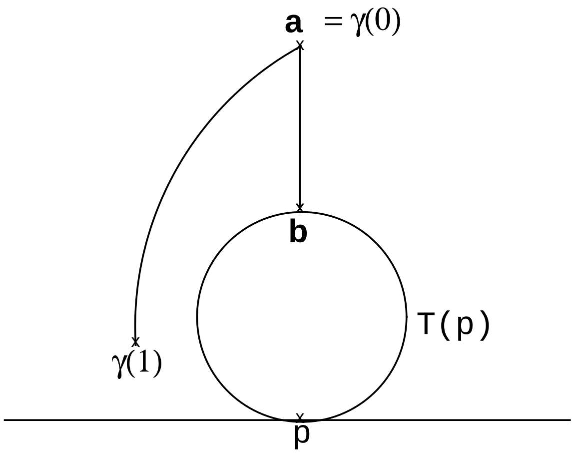


Figure 3: Quasigeodesics around horospheres.

Proof: 1) Suppose not, then there exist constants C and E such that for any n there exist a parabolic point p_n and a C-quasigeodesic $\gamma_n : [0, 1] \to \mathcal{G}$ such that $d_A(\gamma_n(1), T(p_n)) \leq E$ and $d_A(\gamma_n, \Pi_{p_n}(\gamma_n(0))) > n$ for all $n \in \mathbb{N}$. By Lemma 3.17 there are at most finitely many G-non-equivalent parabolic points. So we may assume that $p = p_n$ and let $\mathbf{b}_n \in \Pi_p(\gamma_n(0))$. By Lemma 3.17 the group $\mathrm{Stab}_G p$ acts cofinitely on $T(p)$ so we may also suppose that $\mathbf{b} = \mathbf{b}_n$.

Since $d_A(\gamma_n(1), T(p)) \leq E$ by Lemmas 3.18 and 4.2 we have up to a subsequence $\gamma_n(1) \to p$. Denote $\mathbf{a}_n = \gamma_n(0)$ and $\mathbf{c}_n = \gamma_n(1)$. By our assumption $\mathbf{a}_n \notin T(p)$ so $\mathbf{a}_n - \mathbf{b} - p$. By Lemma 3.6 we have $\mathbf{a}_n - \mathbf{b} - \mathbf{c}_n (n > n_0)$. Thus Lemma 5.2 implies that $d_A(\mathbf{b}, \gamma_n) \leq D$ which is a contradiction. The statement 1) is proved.

2) We have $\lim\limits_{n \to \infty} (\gamma(n) = \mathbf{c}_n) = p$ and without lost of generality we can suppose that $\mathbf{a} = \gamma(0) \notin T(p)$. Then arguing similarly we obtain $\mathbf{a} - \mathbf{b} - \mathbf{c}_n$ $(n > n_0)$ where $\mathbf{b} = \Pi_p(\mathbf{a})$. From Lemma 5.2 we have $d_A(\mathbf{b}, \gamma) \leq D$. $\square$

The following Lemma is a generalization of the previous one to geodesics with variable endpoints:

Lemma 5.4. *The following statements are true :*

1) *For any $C > 0$ and $E > 0$ there exists $M > 0$ such that for any parabolic point $p \in T$ and any C-quasigeodesic $\gamma : [-1, 1] \to \mathcal{G}$ one has*

$$d_A(\{\gamma(-1), \gamma(1)\}, T(p)) \leq E \implies d_A(\gamma(0), \Pi_p(\gamma(0))) \leq M \tag{2'}$$

2) *There exists a constant $D > 0$ such that for any parabolic point $p \in T$ and any C-quasigeodesic $\gamma : [-\infty, +\infty[\to \mathcal{G}$ one has*

$$\lim_{n\to\pm\infty} \gamma(n) = p \implies d_A(\gamma(0), \Pi_p(\gamma(0))) \le D \qquad (3').$$

Proof: 1) As before using the finiteness of G-non-equivalent parabolic points, we fix a parabolic point p. Then we apply the previous Lemma to C-quasigeodesics $\gamma_- = \gamma([-1, 0])$, and $\gamma_+ = \gamma([0, 1])$. If $\mathbf{a} = \gamma(0) \notin T(p)$ and $\mathbf{b} = \Pi_p(\mathbf{a})$ then by 5.3.1 we have $d_A(\gamma_\pm, \mathbf{b}) \le L$. Let $\mathbf{z} \in \gamma_+$ and $\mathbf{y} \in \gamma_-$ be the points realizing these distances. Since there is a path from $\mathbf{z}$ to $\mathbf{y}$ through $\mathbf{b}$ of length $2L$, the length $l(\gamma(\mathbf{z}, \mathbf{y})))$ of the C-quasigeodesic $\gamma(\mathbf{z}, \mathbf{y})$ between $\mathbf{z}$ and $\mathbf{y}$ is at most $2L(C + 1)$. So at least for one of these entourages, e.g. $\mathbf{z}$, we have $l(\gamma(\mathbf{a}, \mathbf{z})) \le L(C + 1)$. By the triangle inequality we obtain $d_A(\mathbf{a}, \mathbf{b}) \le M = L \cdot (C + 2)$.

The same argument and 5.3.2 imply the statement 2). □

The graph $\mathcal{G}$ is quasi-isometric to the Cayley graph of the group G. So the statement 5.4.1 immediately implies the quasiconvexity of parabolic subgroups of G (see also [Ge1]):

Corollary 5.5. *Suppose G acts 3-discontinuously and 2-cocompactly T. Then every parabolic subgroup of G is quasiconvex.*

Remark. The above Lemmas 5.3 and 5.4 are close to some Lemmas contained in our work [GePo1] where the horospheres were defined without using entourages. We need the above results in terms of entourages to apply them in the further argument where the language of entourages is crucial.

Definition 5.6. Let $\gamma \subset \widetilde{T}$ be a C-quasigeodesic. We call an entourage $\mathbf{v} \in \gamma$ *d-horospherical* if there exist parts $[\mathbf{v}, \mathbf{c}]$ and $[\mathbf{a}, \mathbf{v}]$ of γ of length greater than a constant e and which are contained in a d-neighborhood $N_d(T(p))$ of a horosphere $T(p)$.

The entourage $\mathbf{v} \in \gamma$ is called *non-horospherical* in the opposite case.

Remark. The only restriction on e which will be used in future is that $e > 2d$ for the constant d from Proposition 3.27.3b. Then the parabolic point p with respect to which the (non)-horosphericity is considered is unique.

Lemma 5.7. *Let $\gamma = \gamma(\mathbf{a}, \mathbf{c})$ be a c-quasigeodesic. Suppose that $P = P(\mathbf{a}, \mathbf{c})$ is a non-refinable tube having the same ending vertices $\mathbf{a}$ and $\mathbf{c}$ as γ. For every sufficiently large $d > 0$ there exists a constant $E > 0$ such that $d_A(\mathbf{g}, P) \le E$ for every d-non-horospherical point $\mathbf{g} \in \gamma$.*

Proof: Note that the non-refinable tube $P(\mathbf{a}, \mathbf{c})$ exists by Proposition 3.25. By Lemma 5.2 there exists $D > 0$ such that for every $\mathbf{p}_i \in P$ we have $d_A(\mathbf{p}_i, \gamma) \le D$ $(i = 1, ..., m)$. So let us fix a non-horospherical entourage $\mathbf{g} \in \gamma$, and let $\mathbf{g}_i \in \gamma$ be such that $d_A(\mathbf{p}_i, \gamma) = d_A(\mathbf{p}_i, \mathbf{g}_i)$ $(i = 0, ..., m)$. Let us also assume that $\mathbf{g} \in [\mathbf{g}_i, \mathbf{g}_{i+1}]$ where $[\mathbf{g}_i, \mathbf{g}_{i+1}]$ denotes the part of γ between $\mathbf{g}_i$ and $\mathbf{g}_{i+1}$ and $|\cdot|$ its length.

By Corollary 3.29 there exists a constant $C > 0$ such that if $d_A(\mathbf{p}_i, \mathbf{p}_{i+1}) > C$ then the pair $\{\mathbf{p}_i, \mathbf{p}_{i+1}\}$ is contained in a horosphere $T(p)$. In this case $\{\mathbf{g}_i, \mathbf{g}_{i+1}\} \subset N_D(T(p))$ and by Lemma 5.4 we have that $[\mathbf{g}_i, \mathbf{g}_{i+1}] \subset N_L(T(p))$ for some constant $L > 0$.

Let d be any number bigger than L. If g is d-non-horospherical then there exists a constant l_0 such that either $d_A(\mathbf{g}, \mathbf{g}_i)$ or $d_A(\mathbf{g}, \mathbf{g}_{i+1})$ is less than l_0. Thus $d_A(\mathbf{g}, P) \leq l_0 + d$.

If now $d_A(\mathbf{p}_i, \mathbf{p}_{i+1}) \leq C$ then $|[\mathbf{g}_i, \mathbf{g}]| \leq c(C + 2D + c)$. So $d_A(\mathbf{g}_i, P) \leq d_A(\mathbf{g}_i, \mathbf{g}) + D \leq c(C + 2D + c) + D$.

Put $E = \max\{l_0 + d, c(C + 2D + c) + D\}$. The Lemma is proved. $\square$

Remark. The constants d and l_0 depend on the constants D, C and $L = L(D)$ given respectively by the statements 5.2, 3.25 and 5.5.

6 Tight curves in $\mathcal{G}$.

Let a finitely generated group G act 3-discontinuously and 2-cocompactly on a compactum T. For a parabolic point p and a constant $d > 0$ we denote by $N_d(T(p))$ a d-neighborhood of the horosphere $T(p)$ in the metric d_A (see Section 3.2). The notation diam$(\cdot)$ is used for the diameter of a set with respect to the distance d_A. We denote by $c^{-1}(n)$ the linear function $\dfrac{n}{c} - c$ for some constant $c > 0$.

Definition 6.1. For positive integers l, c and d, a curve $\gamma : I \to \mathcal{G}$ is called (l, d, c)-*tight* (or just *tight* when the values of l, d and c are fixed) if for every $J \subset I$ the following conditions hold:

1. $|J| \leq l \implies \gamma|_J$ is a c-quasigeodesic.

2. $\mathrm{diam}(\gamma(J) \cap N_d(T(p))) > l \implies \mathrm{diam}(\gamma(\partial J)) > c^{-1}(l).$ $\square$

The rest of the Section is devoted to the proof of the following Theorem describing the non-horospherical points (see Definition 5.6) of tight curves.

Theorem B. *For every $c > 0$ there exist positive constants l_0, w_0, d, c' such that for all $l \geq l_0$ and every (l, d, c)-tight curve $\gamma \subset \mathcal{G}$ there exists a c'-quasigeodesic $\alpha \subset A$ such that every d-non-horospherical vertex of γ belongs to the w_0-neighborhood $N_{w_0}(\alpha)$ of α.*

The following three lemmas are close to the results of the previous Section. We use below the notation $\mathrm{diam}_{\overline{\delta}_{\mathbf{v}}}$ for the diameter of a set with respect to the shortcut metric $\overline{\delta}_{\mathbf{v}}$ ($\mathbf{v} \in A$).

Lemma 6.2. *There exist positive constants ρ and d such that for every c-quasigeodesic $\gamma : I \to \mathcal{G}$ of non-zero length and a d-non-horospherical point $\gamma(0) \in \mathcal{G}$ one has:*

$$\mathrm{diam}_{\overline{\delta}_{\gamma(0)}}(\gamma(\partial I)) > \rho.$$

Proof: Let us first prove that there exists a constant $r > 0$ such that

$$d_A(\gamma(0), \gamma(\partial I)) > r \implies \overline{\delta}_{\gamma(0)}(\gamma(\partial I)) > \rho \qquad (*).$$

Suppose by contradiction that it is not true. Then for every $d > 0$ there exists a sequence of quasigeodesics γ_n such that $d_A(\gamma_n(0), \gamma_n(\partial I)) \to +\infty$ and $\overline{\delta}_{\gamma_n(0)}(\gamma(\partial I)) \to 0$ where $\gamma_n(0)$ is a d-non-horospherical point of γ_n.

Up to choosing a subsequence we may suppose that the sequence $(\gamma_n)_n$ converges in the Tikhonov topology to a c-quasigeodesic $\gamma : \mathbb{Z} \to \mathcal{G}$ such that $\lim\limits_{n\to\pm\infty} \gamma(n) = p \in T$. Then γ is a horocycle at p and by [GePo1, Lemma 3.6] the point p is parabolic. By Lemma 5.4.2 for every $i \in \mathbb{Z}$ the distance $d_A(\gamma_n(i), T(p))$ is uniformly bounded by a constant $D > 0$. So the points $\gamma_n(0)$ are D-horospherical for sufficiently large n. The obtained contradiction proves (*).

We are left now with the case when $d_A(\gamma(0), \gamma(\partial I)) \leq r$ where the constant r satisfies (*). Suppose first that the distance between $\gamma(0)$ and both endpoints of $\gamma(\partial I)$ is less than r. By translating $\gamma(0)$ to a fixed basepoint $\mathbf{v} \in A$ we obtain that γ is contained in a finite ball $B(\mathbf{v}, r + c(r))$. Then the $\overline{\delta}_{\mathbf{v}}$-length of γ is uniformly bounded from below. If the distance between $\gamma(0)$ and only one of the endpoints is bigger than r then the $\overline{\delta}$-length of γ is still bounded from below.

Denoting by ρ the minimum among all of these constants we obtain the Lemma. $\square$

Remark. Above we have used Lemma 3.6 from [GePo1] stated there for the Cayley graphs. Since our graph $\mathcal{G}$ is quasi-isometric to the Cayley graph this result can be applied.

Recall that $A = G(\mathbf{a}_0)$ is the vertex set of the graph $\mathcal{G}$. Using a "refining" procedure we will now introduce a new graph $\mathcal{G}^*$ whose vertex set A^* satisfies some additional conditions.

From now on we fix the constant ρ coming from Lemma 6.2 and an integer $k > 5$ which will be used in the betweenness relation below. Let δ be a number such that

$$0 < \delta < \frac{\rho}{k+2}. \tag{**}$$

<u>*Definition of the set A^**</u> : For every $\mathbf{v} \in A$ denote by $\mathbf{v}^*$ the entourage $\{\{x, y\} \in \mathbf{S}^2 T \ : \ \overline{\delta}_{\mathbf{v}}(x, y) < \delta\}$.

It follows from the following Lemma that the compactifying topology on T coming from the graphs A^* and A is the same.

Lemma 6.3. If $\mathbf{a}_n \to p \in T$ then $\mathbf{a}_n^* \to p$.

Proof: Suppose it is not true and $\mathbf{a}_n^* \not\to p$. Then there exists a neighborhood U_p of the point p such that $U'(p)$ is not $\mathbf{a}_n^*$-small for $n > n_0$. So $\exists\ \mathbf{x}_n,\ \mathbf{y}_n \in U'_p \ : \ \overline{\delta}_{\mathbf{a}_n^*}(\mathbf{x}_n, \mathbf{y}_n) > \delta$. It follows that up to subsequences we have $\mathbf{x}_n \to x \in T,\ \mathbf{y}_n \to y \in T\ (n \to \infty)$ and $x \neq y \neq p \neq x$. Let U_x and U_y be closed neighborhoods of x and y such that $U_p \cap U_x \cap U_y = \emptyset$.

Let $H(U_{x,y}) \subset \mathcal{G}$ denote the set of geodesics whose endpoints are situated in $U_{x,y} = U_x \cup U_y$. By [GePo1, Main Lemma] $\overline{H(U_{x,y})} \cap T = \overline{U_{x,y}} \cap T$ where $\overline{U}$ means the closure of U in $\widetilde{T} = A \sqcup \mathrm{Ent}\, T$. It follows that the geodesics $\gamma_n(\mathbf{x}_n, \mathbf{y}_n) \subset \mathcal{G}$ between $\mathbf{x}_n$ and $\mathbf{y}_n$ do not intersect a neighborhood $V_p \subset U_p$ of p $(n > n_0)$. Since $\mathbf{a}_n \to p$ we have $d_A(\mathbf{a}_n, \gamma_n) \to \infty$. By Karlsson Lemma 4.1 (see also Remark 4.4.3) we obtain that $\overline{\delta}_{\mathbf{a}_n}(\mathbf{x}_n, \mathbf{y}_n) < \delta$ $(n > n_0)$ which is a contradiction. $\square$

The need of the graph A^* is explained by the following:

Lemma 6.4. *There exist constants $d, w > 0$ such that for every quasigeodesic $\gamma : I \to \mathcal{G}$ containing three vertices $\mathbf{a}, \mathbf{b}, \mathbf{c} \in A$ the following is true:*

$$\mathbf{b} \text{ is } d-\text{non}-\text{horospherical} \ \wedge\ d_A(\mathbf{b}, \{\mathbf{a}, \mathbf{c}\}) > w \implies \mathbf{a}^* - \mathbf{b}^* - \mathbf{c}^*(k).$$

Proof: Suppose not and there are sequences $\mathbf{a}_n, \mathbf{c}_n$ and $\mathbf{b}_n$ such that $\mathbf{b}_n$ is d-non-horospherical, $d_A(\mathbf{b}_n, \{\mathbf{a}_n, \mathbf{c}_n\}) \to \infty$ and $\mathbf{a}_n^* - \mathbf{b}_n^* - \mathbf{c}_n^*(k)$ is not true. Since A is G-finite we can suppose that $\mathbf{b}_n = \mathbf{b}$. Up to a subsequence we have $\mathbf{a}_n \to p,\ \mathbf{c}_n \to q$. Let $\gamma_n = \gamma_n(\mathbf{a}_n, \mathbf{c}_n) \subset \mathcal{G}$ be the corresponding geodesics. Since $\mathbf{b}$ is non-horospherical we have by Lemma 6.2 that $\overline{\delta}_{\mathbf{b}}(p, q) > \rho$, hence $p \neq q$.

Let U_p and U_q be disjoint $\mathbf{b}^*$-small neighborhoods of p and q respectively. By Lemma 6.3 we also have $\mathbf{a}_n^* \to p$ and $\mathbf{c}_n^* \to q$. So U'_p and U'_q are $\mathbf{a}_n^*$-small and $\mathbf{c}_n^*$-small respectively $(n > n_0)$. Hence $U_p \supset \mathrm{sh}_{\mathbf{b}^*}\mathbf{a}_n^*$ and $U_q \supset \mathrm{sh}_{\mathbf{b}^*}\mathbf{c}_n^*$. It yields $\overline{\delta}_{\mathbf{b}}(U, V) > \rho - 2\delta$. Then (**) implies that $\overline{\delta}_{\mathbf{b}}(U_p, U_q) \geq \rho - 2\delta > k \cdot \delta$. Hence $\Delta_{\mathbf{b}^*}(\mathrm{sh}_{\mathbf{b}^*}\mathbf{a}_n^*, \mathrm{sh}_{\mathbf{b}^*}\mathbf{c}_n^*) > k$. It follows that $\mathbf{a}_n^* - \mathbf{b}^* - \mathbf{c}_n^*(k)$ which is a contradiction. $\square$

Lemma 6.5. *For every $d > 0$ there exists a constant l_0 such that for every parabolic point p, and all entourages $\mathbf{b}, \mathbf{c}, \mathbf{d} \in N_d(T(p))$, and $\mathbf{a} \in A$ one has*

$$\forall\ l > l_0 \quad : \ d_A(\mathbf{b}, \mathbf{c}) > l \ \wedge\ d_A(\mathbf{b}, \mathbf{d}) > l \ \wedge\ \mathbf{a}^* - \mathbf{b}^* - \mathbf{c}^*\ (k) \implies \mathbf{a}^* - \mathbf{b}^* - \mathbf{d}^*\ (k-2) \qquad (1)$$

Proof: Since by Lemma 3.17 the set of parabolic points is G-finite it is enough to prove the statement for a fixed parabolic point $p \in T$. By Lemma 3.18 the parabolic point p is the unique limit point of $N_d(T(p))$. By definition of the topology on $T \sqcup \mathsf{Ent}T$ for sufficiently large l_0 our assumption implies that the entourages $\mathbf{c}$ and $\mathbf{d}$ are sufficiently close to p. By Lemma 6.3 the entourages $\mathbf{c}^*$ and $\mathbf{d}^*$ are also close to p. So for every $\mathbf{b}^*$-small neighborhood U_p of p its complement U'_p is $\mathbf{c}^*$-small and $\mathbf{d}^*$-small for $l > l_0$. Then $\mathrm{sh}_{\mathbf{b}^*}\mathbf{c}^* \subset U_p$ and $\mathrm{sh}_{\mathbf{b}^*}\mathbf{d}^* \subset U_p$. Therefore $\Delta_{\mathbf{b}^*}(\mathrm{sh}_{\mathbf{b}^*}\mathbf{c}^*, \mathrm{sh}_{\mathbf{b}^*}\mathbf{d}^*) \leq \Delta_{\mathbf{b}^*}(\mathrm{sh}_{\mathbf{b}^*}\mathbf{c}^*, p) + \Delta_{\mathbf{b}^*}(p, \mathrm{sh}_{\mathbf{b}^*}\mathbf{d}^*) \leq 2$. We obtain $\Delta_{\mathbf{b}^*}(\mathrm{sh}_{\mathbf{b}^*}\mathbf{d}^*, \mathrm{sh}_{\mathbf{b}^*}\mathbf{a}^*) \geq \Delta_{\mathbf{b}^*}(\mathrm{sh}_{\mathbf{b}^*}\mathbf{a}^*, \mathrm{sh}_{\mathbf{b}^*}\mathbf{c}^*) - \Delta_{\mathbf{b}^*}(\mathrm{sh}_{\mathbf{b}^*}\mathbf{c}^*, \mathrm{sh}_{\mathbf{b}^*}\mathbf{d}^*) > k - 2$. $\square$

Remark 6.6. *(about the constants).* We assume that the tightness constant l is much bigger than the constants $R,, e$ (see 5.6) and w (see 6.4). We will also suppose that the chosen constants satisfy the following relations:

$$l_0 > 4w,\ w > e > 2 \cdot \mathrm{diam}(N_d(T(p)) \cap N_d(T(q))),$$

where p and q are parabolic points. The finiteness of the last diameter comes from Corollary 3.29.

Proof of Theorem B. We fix the constant d coming from Lemma 6.2 (the term "(non)-horosphericity" means d-(non)-horosphericity" below). The proof will proceed in the following way: we first choose d-non-horospherical points $\mathbf{v}_n$ of the curve γ which gives by Lemma 6.4 an auxiliary tube $P^* = \{\mathbf{v}_n^*\}$ in the graph A^*. There is a quasi-geodesic $\alpha^* \subset A^*$ whose non-horospherical points are in a bounded distance from P^*. Since the map $\varphi : \mathbf{v} \to \mathbf{v}^*$ is a quasi-isometry between the G-finite graphs $\mathcal{G}$ and $\mathcal{G}^*$ it will gives us a quasi-geodesic $\alpha \subset A$ satisfying the statement of the Theorem.

To construct the tube P^* we proceed inductively by choosing vertices of γ as follows. Let $\gamma(0)$ be the first non-horospherical point on γ, then we put $\mathbf{v}_0^* = \gamma^*(0)$. Suppose that a point $\mathbf{v}_n^* = \gamma^*(n)$ is already chosen. Then choose $i_{n+1} \geq i_n + w$ such that $\gamma(i_{n+1})$ is the first non-horospherical point on γ after $\gamma(i_n + w)$. We set $\mathbf{v}_{n+1}^* = \gamma^*(i_{n+1})$. The following Proposition shows that for every n each three chosen neighboring vertices form a tube $\mathbf{v}_{n-1}^* - \mathbf{v}_n^* - \mathbf{v}_{n+1}^*$ $(k-4)$ for the integer k fixed above. Then all the constructed vertices will give a tube $P^* = \mathbf{v}_0^* - \mathbf{v}_1^* - ... - \mathbf{v}_m^*$ $(k-4)$.

Proposition 6.7. *For every $n \in \mathbb{N}$ one has $\mathbf{v}_{n-1}^* - \mathbf{v}_n^* - \mathbf{v}_{n+1}^*$ $(k-4)$.*

Proof of the Proposition. There are four different cases depending on the lengths $|\gamma|_{[i_n, i_{n+1}]}| = i_{n+1} - i_n$ of the parts of γ $(n \in \mathbb{N})$.

Case 1. $i_n - i_{n-1} \leq l/2 \wedge\ i_{n+1} - i_n \leq l/2$,

By definition of a tight curve the points $\gamma(i_{n-1}), \gamma(i_n), \gamma(i_{n+1})$ belong to a c-quasigeodesic part of γ so the result follows from Lemma 6.4.

Case 2. $i_{n+1} - i_{n-1} > l$.

There are three subcases.

Subcase 2.1. $i_n - i_{n-1} \leq l/2\ \wedge\ i_{n+1} - i_n > l/2$,

Since $\gamma(i_{n+1})$ is the first non-horospherical point on γ after $\gamma(i_n + w)$ and $w < l/2$ the point $\gamma(i_n + w)$ is horospherical. Then there exists a unique horosphere $T(p)$ such that $d_A(\gamma(i_n + w), T(p)) \leq d$. As $\gamma|_{[i_n, i_n+w]}$ is a c-quasigeodesic we have

$$d_A(\gamma(i_n), T(p)) < c(w) + d, \text{ where } c(w) = cw + c. \qquad (**)$$

Furthermore Lemma 6.4 yields:

$$\gamma^*(i_{n-1}) - \gamma^*(i_n) - \gamma^*(i_{n-1} + l)\ (k). \qquad (2)$$

Since $i_{n+1} - i_{n-1} > l$ the point $\gamma(i_{n-1} + l)$ is also horospherical. Thus $\gamma(i_{n-1} + l) \in]\gamma(i_n + w), \gamma(i_{n+1})]$. Since the curve $\gamma|_{[i_n, i_n+l]}$ is still c-quasigeodesic we have

$$d_A(\gamma(i_n), \gamma(i_{n-1} + l)) > \frac{i_{n-1} + l - i_n}{c} - c \geq \frac{l}{2c} - c > \frac{l}{4c}, \qquad (3)$$

where we assume that $l > l_0 > 4c^2$ for the constant l_0 from Lemma 6.5.

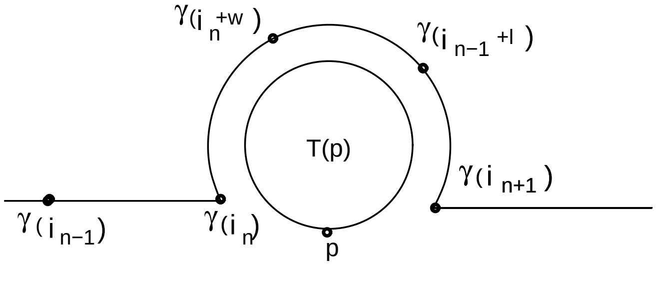


Figure 3: Tight curves around horospheres.

By construction we can also suppose that $\gamma(i_{n+1}) \in N_d(T(p))$ as otherwise there is another non-horospherical point on γ after $\gamma(i_n + w)$ preceding $\gamma(i_{n+1})$. So by (**) $\{\gamma(i_n), \gamma(i_{n+1})\} \subset N_{d_1}(T(p))$, where $d_1 = c(w) + d$.

If, first, $i_{n+1} - i_n \leq l$ then $\gamma|_{[i_n, i_{n+1}]}$ is a c-quasigeodesic, and $d_A(\gamma(i_n), \gamma(i_{n+1})) > l/2c - c > l/4c$. Hence by the choice of l_0 (see Remark 6.6) and all $l > l_0$ it follows from (2), (3) and Lemma 6.5 that

$$\gamma^*(i_{n-1}) - \gamma^*(i_n) - \gamma^*(i_{n+1}) \ (k-2). \tag{4}$$

If now $i_{n+1} - i_n > l$ then by 6.1.2 (applying to $N_{d_1}(p)$) we have $d_A(\gamma(i_n), \gamma(i_{n+1})) > c^{-1}(l)$ and again (4) follows from (2), (3) and Lemma 6.5. □

Subcase 2.2. $i_n - i_{n-1} \geq l/2 \ \wedge \ i_{n+1} - i_n \leq l/2$,

The argument is similar to that of Subcase 2.1 but it works in the opposite direction. We have the tube $\gamma^*(i_{n+1}) - \gamma^*(i_n) - \gamma^*(i_{n+1} - l)$ (k). As above if $i_n - i_{n-1} \leq l$ then the curve $\gamma|_{[i_{n-1}, i_n]}$ is c-quasigeodesic and so its diameter is greater than $l/4c$. If not then using the tightness we obtain that $d_A(\gamma(i_{n-1}), \gamma(i_n)) > c^{-1}(l)$ and (4) follows from the same argument as in the Subcase 2.1.

Subcase 2.2. $i_n - i_{n-1} \geq l/2 \ \wedge \ i_{n+1} - i_n \geq l/2$,

In this case we have that $\gamma(i_n) \in N_d(T(p))$ and $\gamma(i_{n+1}) \in N_d(T(q))$ where p and q are distinct parabolic points. The points $\gamma(i_n - l/4)$ and $\gamma(i_n + l/4)$ preceding respectively $\gamma(i_n)$ and $\gamma(i_{n+1})$ are both non-horospherical. This is true as $w < l/4$ and $\gamma(i_n)$ and $\gamma(i_{n+1})$ are the first non-horospherical points after $\gamma(i_{n-1})$ and $\gamma(i_n)$ respectively. Since $\gamma|_{[i_n - l/4, i_n + l/4]}$ is a quasigeodesic by Lemma 6.4 we have

$$\gamma^*(i_n - l/4) - \gamma^*(i_n) - \gamma^*(i_n + l/4) \ (k). \tag{4}$$

We may now assume that $d_A(\gamma(i_{n-1}), \gamma(i_n))$ and $d_A(\gamma(i_n), \gamma(i_{n+1}))$ are both greater than $l/4c$. Indeed if $i_n - i_{n-1} > l$ then by (l, c)-tightness we have $d_A(\gamma(i_{n-1}), \gamma(i_n)) > c^{-l}(l) > l/4c$. If $i_n - i_{n-1} \leq l$ then $\gamma|_{[i_{n-1}, i_n]}$ is c-quasigeodesic, and as above $d_A(\gamma(i_{n-1}), \gamma(i_n)) > l/4c$. In the same way we obtain $d_A(\gamma(i_n), \gamma(i_{n+1})) > l/4c$.

Applying now Lemma 6.5 to (4) two times for $l > 4c \cdot l_0$ we obtain

$$\gamma^*(i_{n-1}) - \gamma^*(i_n) - \gamma^*(i_{n+1}) \ (k-4).$$

The Proposition is proved. □

We continue the proof of Theorem B. By Proposition 6.7 the curve γ admits a set of non-horospherical points $\mathbf{v}_n = \gamma(i_n)$ such that $\mathbf{v}_n^* = \varphi(\gamma(i_n))$ is a vertex of the tube P^*. Let $\mathbf{u} = \gamma(i)$ be a non-horospherical point of γ which does not belong to the set $\{\mathbf{v}_n\}_n$. Then by construction $i_n \leq i < i_n + w$ for some $i_n \in \{0, ..., m\}$. Since $w < l$ the curve $\gamma|_{[i_n, i_n + w]}$ is a c-quasigeodesic so $d_A(\mathbf{v}_n, \mathbf{u}) \leq cw + c$. The map $\varphi : \mathbf{u} \in A \to \mathbf{u}^* \in A^*$ is a quasi-isometry so $d_{A^*}(\mathbf{u}^*, \mathbf{v}_n^*) \leq w_1$ for some uniform constant $w_1 > 0$. Let α^* be a geodesic in the graph $\mathcal{G}^*$ with the same endpoints as P^*. Then by Lemma 5.2 (applied to the graph $\mathcal{G}^*$) there is a constant $D^* > 0$ such that $\forall \ \mathbf{v}^* \in P^* \ : \ d_{A^*}(\alpha^*, \mathbf{v}^*) \leq D^*$. So for every non-horospherical point $\mathbf{u} \in \gamma$ we have $d_{A^*}(\mathbf{u}^*, \alpha^*) \leq w_1 + D^*$. The map $\varphi^{-1} : \mathbf{u}^* \to \mathbf{u}$ is a quasi-isometry too. Hence $\alpha = \varphi^{-1}(\alpha^*)$ is a c'-quasi-geodesic in $\mathcal{G}$. It follows that $d_A(\mathbf{u}, \alpha) \leq w_0$ for some positive constants c' and w_0. Theorem B is proved. $\Box$.

7 Floyd quasiconvexity of parabolic subgroups.

We suppose as before that a finitely generated group G acts 3-discontinuously and 2-cocompactly on a compactum T. Let Γ be a locally finite, connected graph on which a finitely generated group G acts discontinuously and cofinitely (e.g. its Cayley graph). We denote by $d(,)$ the graph distance of Γ. Let $f : \mathbb{N} \to \mathbb{R}_{>0}$ be a scaling function esatisfying the following conditions (1-2) (see Section 4):

$$\exists\, \lambda > 0\ \forall n \in \mathbb{N}\ :\ 1 < \tfrac{f(n)}{f(n+1)} < \lambda \tag{1}$$

$$\sum_{n \in \mathbb{N}} f(n) < +\infty. \tag{2}$$

To precise that f satisfies (1) with respect to some $\lambda \in]1, \infty[$ we will say that the function f is λ-*slow*. Denote by δ_f the corresponding Floyd metric on Γ with respect to a fixed vertex $v \in \Gamma^0$.

By a standard argument based on Arzela-Ascoli theorem it follows that the Floyd completion $\overline{\Gamma}_f$ of any locally finite graph G is a geodesic (strictly intrinsic) space (see e.g. [BBI, Theorem 2.5.14]. We call Floyd geodesic (or δ_f-geodesic) a geodesic in the space $\overline{\Gamma}_f$ with respect to the Floyd δ_f-metric.

Definition 7.1. A subgroup H of G is called *Floyd quasiconvex* if there exists a constant $R > 0$ such that every Floyd geodesic $\gamma = \gamma(h_1, h_2) \subset \Gamma$ in the metric δ_f with the endpoints $h_i \in H$ belongs to R neighborhood $N_R(H)$ of H for the graph metric:

$$\forall x \in \gamma\ :\ d(x, H) < R.$$

By Corollary 5.5 every parabolic subgroup of G is quasiconvex with respect to the word metric (see also [Ge1]). The aim of this Section is to prove the following Theorem stating the Floyd quasiconvexity of parabolic subgroups.

We will need (if the opposite is not stated) that the λ-slow Floyd function satisfies in addition the following assumption:

$$\frac{f(n)}{f(2n)} \leq \kappa, \tag{3}$$

for some constant $\kappa > 0$.

Theorem C. Let G be a finitely generated group acting 3-discontinuously and 2-cocompactly on a compactum T. Let Γ be a locally finite, connected graph admitting a cocompact discontinuous action of G. Then there exists a constant $\lambda_0 \in]1, \infty[$ such that for all $\lambda \in]1, \lambda_0[$ and a λ-slow

Floyd scaling function f satisfying (1-3), every parabolic subgroup H of G is Floyd quasiconvex for the Floyd metric δ_f. $\square$

Remarks. 1) Theorem C is valid for any polynomial function $f(n) = n^{-k}$ as the conditions (1-3) are satisfied for any fixed $\lambda > 1$ and $n > n_0$..

2) The condition (3) above will be used in the proof only in order to pass from the Floyd completion of the graph Γ to that of $\mathcal{G}$. So Theorem C is valid for the graph $\mathcal{G}$ also for the exponential function $f(n) = \lambda^{-n}$ if $\lambda \in]1, \lambda_0[$ for some $\lambda_0 > 1$. $\square$

Since every parabolic subgroup H is quasiconvex in G there exists a quasi-isometric map φ of the group H into the graph Γ.

Corollary 7.2. *Let p be a parabolic point for the action of G on T and $H = \mathrm{Stab}_G p$ be its stabilizer. Then φ extends injectively to the Floyd boundaries:*

$$\varphi \ : \ \overline{H}_f \to \overline{\Gamma}_f, \tag{4}$$

where f is a λ-slow function satisfying (1-3).

$\square$

It follows from [GePo1, Thm A] that if the kernel of the equivariant Floyd map $F : \partial_f G \to T$ is not a single point then it is equal to the topological boundary $\partial(\mathrm{Stab}_G p)$ of the stabilizer $\mathrm{Stab}_G p$ of a parabolic point $p \in T$. We denote by $\partial_f \mathrm{Stab}_G p$ the Floyd boundary of $\mathrm{Stab}_G p$ corresponding to a function f. By Corollary 7.2 we have that $\varphi|_{\partial_f H}$ is a homeomorphism. So the following is immediate.

Corollary 7.3. *For a scaling function f satisfying $(1-3)$ one has*

$$F^{-1}(p) = \partial_f(\mathrm{Stab}_G p), \tag{5}$$

for every parabolic point $p \in T$.

$\square$

Corollary 7.3 answers positively our question [GePo1, 1.1] and provides complete generalization of the theorem of Floyd [F] for the relatively hyperbolic groups.

Let us first show that Theorem C implies Corollary 7.2.

Proof of Corollary 7.2. We suppose that $H \subset \Gamma^0$ and $\varphi : H \hookrightarrow \Gamma^0$ is the identity map inducing the quasi-isometry between the word metrics. Let $d'(,)$ and $d(,)$ be the graph distances of H and Γ respectively. We also denote by $\delta_{f,H}$ and $\delta_{f,G}$ the corresponding Floyd distances with respect to a fixed basepoint $v \in H$. Since f satisfies (3) by [GePo1, Lemma 2.5] the map φ extends to a Lipschitz map (denoted by the same letter) $\varphi : \overline{H}_f \to \overline{\Gamma}_f$ between the Floyd completions of H and Γ.

Let $x, y \in H \subset \Gamma$ be two distinct points. Connect them by a Floyd geodesic $\omega = \omega(x, y)$ in the graph Γ. Then $\delta_{f,G}(x, y) = L_f(\omega) = \sum_{i=1}^{l} f(d(v, \{x_i, x_{i+1}\}))$. Denote by $x'_i \in H$ one of the closest vertices to x_i in H $(i = 1, ..., l)$. By Theorem C there exists a constant $R > 0$ such that $d(x_i, x'_i) \leq R$. Thus $d(x'_i, x'_{i+1}) \leq 2R + 1$. So for any vertex x'_{ij} on a Floyd geodesic in H between x'_i and x'_{i+1} we obtain $d(v, \{x_i, x_{i+1}\}) \leq (3R + 1) + d(v, x'_{ij})$. Since φ is quasi-isometric we have $d(v, x'_{ij}) \leq \alpha d'(v, x'_{ij}) + \beta$ for some constants α and β. Let $\omega' = \omega'(x, y) \subset H$ be a curve between x and y obtained by connecting the vertices x'_i and x'_{i+1} by geodesics segments in H containing x'_{ij}. We have $(2R+1)f(d(v, \{x_i, x_{i+1}\})) \geq \sum_j f(\alpha d'(v, x'_{ij}) + m_1)$, where $m_1 = \beta + 3R + 1$. The conditions (1) and (3) yield

$$L_{f,G}(\omega) \geq \frac{L_{f,H}(\omega')}{(2R+1)\lambda^{m_1}\kappa^{k_1}},$$

where $k_1 = \min\{k \ : \ 2^k > \alpha\}$. Therefore

$$\forall x, y \in H \quad \delta_{f,G}(x, y) \geq \frac{1}{(2R+1)\lambda^{m_1}\kappa^{k_1}} \cdot \delta_{f,H}(x, y). \tag{6}$$

By continuity the inequality (6) remains valid for every pair of distinct points $x, y \in \overline{H}_f$. So the map $\varphi : \overline{H}_f \to \overline{\Gamma}_f$ is injective. The Lemma is proved. □

Proof of Theorem C. We start with two Lemmas.

Lemma 7.4. *For every $r > 0$ there exists $\lambda_0 > 1$ such that $\forall \lambda \in]1, \lambda_0[$ and every λ-slow function f the condition $d(x, y) \leq r$ $(x, y \in \Gamma^0)$ implies that every Floyd δ_f-geodesic $\gamma = \gamma(x, y)$ is a geodesic in Γ.*

Remark. A similar statement for δ-hyperbolic spaces is proved in [Gr, Lemma 7.2.1]

Proof: Let $v \in \Gamma^0$ be a basepoint and let $R = d(v, \{x, y\})$. Denote by $\omega = \omega(x, y)$ the word-geodesic between x and y for which $|\omega| = r$. We have $L_f(\omega) = \sum_{i=1}^{l} f(d(v, \{x_i, x_{i+1}\})) \leq rf(|R-r|)$.

Suppose by contradiction that γ is not word geodesic and so $|\gamma| \geq r + 1$. Hence $L_f(\gamma) \geq (r + 1)f(R + r + 1)$. So

$$\frac{f(R + r + 1)}{f(|R - r|)} \leq \frac{L_f(\gamma)}{(r + 1)f(|R - r|)} \leq \frac{L_f(\omega)}{(r + 1)f(|R - r|)} \leq \frac{r}{r + 1}.$$

Since f is λ-slow we have $\dfrac{f(r + R + 1)}{f(|R - r|)} \geq \dfrac{1}{\lambda^{2r+1}}$. Thus

$$\frac{1}{\lambda^{2r+1}} \leq \frac{r}{r + 1}. \tag{7}$$

Then there exists $\lambda_0 > 1$ such that for $\lambda \in]1, \lambda_0[$ the inequality (7) is not true for a fixed $r > 0$. So for such λ_0 we have a contradiction. The Lemma is proved. □

Remark. Obviously if r is not fixed and tends to infinity the above constant λ_0 does not exist.

The group G acts discontinuously and cofinitely on the graph Γ and on the graph $\mathcal{G}$ of entourages. So there exists a c-quasi-isometry $\psi : \Gamma \to \mathcal{G}$. We will denote by $\mathbf{x}$ the vertex $\psi(x) \in A = \mathcal{G}^0$ where $x \in \Gamma$.

Lemma 7.5. *Let $H \subset \Gamma^0$ be the stabilizer of a parabolic point $p \in T$. Then for every $h \in H$ the entourage $\mathbf{h} = \psi(h)$ belongs to a d-uniform neighborhood $N_d(T(p)) \subset \mathcal{G}$ of the horosphere $T(p) \subset \mathcal{G}$.*

Proof: For a scaling function f satisfying (1-3) the map ψ admits a bilipschitz map between the Floyd completions $\overline{\Gamma}_f \to \overline{\mathcal{G}}_f$ [GePo1, Lemma 2.5]. By [Ge2] there exist Floyd maps $\overline{\Gamma}_f \to T \cup \Gamma$ and $\overline{\mathcal{G}}_f \to T \cup \mathcal{G}$. So the map ψ induces a continuous mapping (denoted by the same letter) $\psi : T \cup \Gamma \to T \cup \mathcal{G}$ whose restriction on Γ is the initial quasi-isometry.

For any $h \in H \subset \Gamma^0$ there exists a horocycle at p passing through $\mathbf{h}$, i.e. a bi-infinite geodesic $\alpha : \mathbb{Z} \to \Gamma$ such that $h = \alpha(0)$ and $\lim\limits_{n \to \pm\infty} \alpha(n) = p$. Then the quasigeodesic $\beta = \psi(\alpha)$ is a horocycle in the graph $\mathcal{G}$ at the point $\psi(p)$. By [GePo1, Lemma 3.6] the point $\psi(p)$ is parabolic. So by Lemma 5.4 we have $\beta(\mathbb{Z}) \subset N_d(T(\psi(p)))$ for some uniform $d > 0$. $\square$

Lemma 7.6. *For every $l > 0$ there exists $\lambda_0 > 1$ such that for any $\lambda \in]1, \lambda_0[$ and λ-slow function f satisfying (1-3) one has: if $\gamma \subset \Gamma$ is δ_f-geodesic then the curve $\psi(\gamma) \subset \mathcal{G}$ is (l, c)-tight quasigeodesic, where c is the quasi-isometry constant of ψ.*

Proof: For a fixed $l > 0$ by Lemma 7.4 (applied to $r = l$) there exists $\lambda_0 > 1$ such that for any $\lambda \in]1, \lambda_0[$ and any λ-slow function f, every part of δ_f-geodesic of length less than l is geodesic in Γ. Then $\beta = \psi(\gamma)$ is c-quasigeodesic on every interval of length at most l. So the first condition of Definition 6.1.1 is satisfied for $\beta \subset \mathcal{G}$.

To prove the second condition 6.1.2 assume that

$$|\beta(J) \cap N_d(T(p))| > l \qquad (8),$$

where $p \in T$ is a parabolic point and $|\cdot|$ stands for the length of a curve. Using the map ψ and its quasi-isometric inverse ψ^{-1}, by Lemma 7.5 we have that $p = \psi(p_0)$ where the point $p_0 \in T$ is also parabolic. Furthermore $\psi(H = \mathrm{Stab}_G p_0) \subset N_d(T(p))$.

If first $|\gamma(J) \cap H| \leq l$ then by Lemma 7.4 $\gamma|_J$ is geodesic in Γ. So $\beta|_J$ is c-quasigeodesic in $\mathcal{G}$. It follows from (8) that $\mathrm{diam}(\partial(\beta(J))) > \mathrm{c}^{-1}(\mathrm{l}) = \mathrm{l}/\mathrm{c} - \mathrm{c}$

If now $|\gamma(J) \cap H| > l$ then again by Lemma 7.4 we must have $|\partial\gamma(J)| > l$, as otherwise $\gamma|_J$ is geodesic and $|\gamma(J) \cap H| \leq l$ what is impossible. So we have $|\partial\beta(J))| > c^{-1}(l)$ since ψ is a c-quasi-isometry. The Lemma is proved. $\square$

Proof of Theorem C. The group G acts 3-discontinuously and 2-cocompactly on a compactum T. Let Γ be a locally finite, connected graph admitting cocompact discontinuous action of G.

Let l_0 and λ_0 be the constants given by Theorem B and Lemma 7.6. Let f be a λ-slow function for $\lambda \in]1, \lambda_0[$. Suppose that $\gamma = \gamma(h_1, h_2) \subset \Gamma$ is a δ_f-geodesic between two elements h_1 and h_2 in the parabolic subgroup H. Then by Lemma 7.6 the curve $\beta = \psi(\gamma)$ is (l, c)-tight in $\mathcal{G}$.

A segment of a curve $\beta \subset \mathcal{G}$ having the extremities at points $\mathbf{h}_i \in \mathcal{G}$ $(i = 1, 2)$ we denote by $\beta[\mathbf{h}_1, \mathbf{h}_2]$. Using the quasi-isometric inverse ψ^{-1} we conclude that Theorem C follows from the following.

Proposition 7.7. *For every $c > 0$ there exist positive constants s, d and l_0 such that for all $l > l_0$ every (l, c)-tight curve $\beta(\mathbf{h}_1, \mathbf{h}_2) \subset \mathcal{G}$ with $\mathbf{h}_i \in N_d(T(p))$ $(i = 1, 2)$ is situated in $N_s(T(p))$.*

Proof of the Proposition. Suppose that β is a (l, c)-tight curve where $l > l_0$ and the constants l_0 and c are given by Theorem B. Then there exists a c'-quasigeodesic $\alpha \subset \mathcal{G}$ such that every non-horospherical point $\mathbf{v}$ of β belongs to the w_0-neighborhood $N_{w_0}(\alpha)$ with respect to the distance d_A. By Lemma 5.4.1 we have $\forall i \in I \ : \ d_A(\alpha(i), T(p)) \leq \text{const}$. Thus there exists a uniform constant $R > 0$ such that for any non-horospherical point $\mathbf{z} \in \beta$ we have $d_A(\mathbf{z}, T(p)) \leq R$.

Let now $\beta[\mathbf{x}, \mathbf{y}]$ be a horospherical part of β lying in the d-neighborhood $N_d(T(q))$ of another parabolic point q. Up to increasing the above part of β we can suppose that both extremal points $\mathbf{x}$ and $\mathbf{y}$ are non-horospherical. So we have $d_A(\mathbf{x}, T(p)) \leq R$ and $d_A(\mathbf{y}, T(p)) \leq R$. Let $\mathbf{x}_1$ and $\mathbf{y}_1$ be points on $T(p)$ realizing the minimal distance from the points $\mathbf{x}$ and $\mathbf{y}$ to $T(p)$ respectively. Denote by $\alpha_1 = [\mathbf{x}, \mathbf{x}_1]$ and $\alpha_2 = [\mathbf{y}, \mathbf{y}_1]$ the corresponding geodesics (see Figure below).

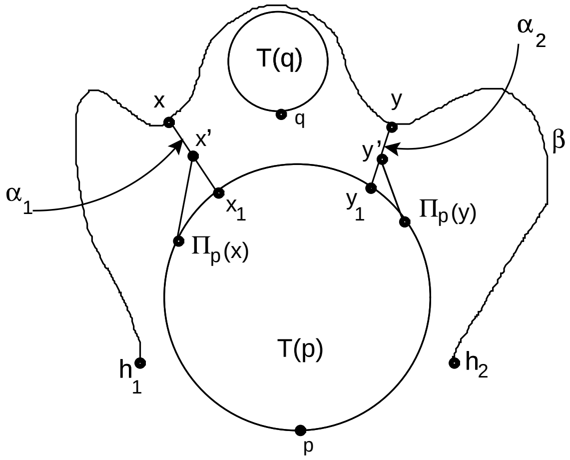


Figure 4: Tight curve β.

Let $\Pi_p(\mathbf{x})$ and $\Pi_p(\mathbf{y})$ be the projections of $\mathbf{x}$ and $\mathbf{y}$ on $T(p)$. By Lemma 5.3.1 we have $d_A(\alpha_1, \Pi_p\mathbf{x}) = d_A(\mathbf{x}', \Pi_p(\mathbf{x})) \leq L$ for some constant L depending only on R, where $\mathbf{x}' \in \alpha_1$. Hence $d_A(\mathbf{x}, \Pi_p(\mathbf{x})) \leq R + L$ and similarly $d_A(\mathbf{y}, \Pi_p(\mathbf{y}) \leq R + L$. By Proposition 3.32.2 the set $\Pi_p(T_q)$ is finite and so is $\Pi_p(N_d(T(q)))$. So there exists a constant $C > 0$ such that $d_A(\Pi_p\mathbf{x}, \Pi_p\mathbf{y}) \leq C$. Therefore $d_A(\mathbf{x}, \mathbf{y}) \leq C + 2R + 2L$. The above constants C, R and L are all uniform so we can choose the parameter l from Theorem B satisfying $l > \max(l_0, C + 2R + 2L)$. Then the segment $\beta[\mathbf{x}, \mathbf{y}]$ is c-quasigeodesic whose length is bounded by $c(C+2R+2L)+c$. Hence $\beta[\mathbf{x}, \mathbf{y}] \subset N_s(T(p))$ where $s = R + c(C + 2R + 2L) + c$. Theorem C is proved. $\square$

References

[BM] A. Beardon, B. Maskit, *Limit sets of Kleinian groups and finite sided fundamental polyhedra*, Acta Math. **132**(1974) 1–12.

[Bo1] B. H. Bowditch, *Relatively hyperbolic groups,* Preprint 1997.

[Bo2] B. H. Bowditch, *Convergence groups and configuration spaces*, in "Group theory down under" (ed. J.Cossey, C.F.Miller, W.D.Neumann, M.Shapiro), de Gruyter (1999) 23–54.

[Bo3] B. H. Bowditch, *A topological characterisation of hyperbolic groups*, J. Amer. Math. Soc. 11 (1998), no. 3, 643–667.

[BH] M. Bridson, A. Haefliger, *Metric spaces of non-positive curvature*, Grundlehren der Mathematischen Wissenschaften, 319. Springer-Verlag, Berlin, 1999.

[Bourb] N. Bourbaki, Topologie Générale Hermann, Paris, 1965.

[BBI] D. Burago, Y. Burago, S. Ivanov, A Course in Metric Geometry Graduate Studies in Math, AMS, Vol 13, 2001.

[Fa] B. Farb, *Relatively hyperbolic groups,* **GAFA**, , 8(5), 1998, 810-840.

[F] W. J. Floyd, *Group completions and limit sets of Kleinian groups*, Inventiones Math. **57** (1980), 205–218.

[Fr] E. M. Freden, *Properties of convergence groups and spaces*, Conformal Geometry and Dynamics, **1** (1997) 13–23.

[Fu] H. Furstenberg. *Poisson boundaries and envelopes of discrete groups,* Bull. Amer. Math. Soc. **73** (1967) 350–356.

[Ge1] V. Gerasimov, *Expansive Convergence Groups are Relatively Hyperbolic*, preprint, to appear in **GAFA**.

[Ge2] V. Gerasimov, *Floyd Theorem for Geometrically Finite Convergence Groups*, preprint.

[GePo1] V. Gerasimov, L. Potyagailo *Quasi-isometries and Floyd boundaries of relatively hyperbolic groups*, preprint.

[GM] F. W. Gehring and G. J. Martin, *Discrete quasiconformal groups I.* Proc. London Math. Soc. **55** (1987) 331–358.

[Gr] M. Gromov, *Hyperbolic groups*, in: "Essays in Group Theory" (ed. S. M. Gersten) M.S.R.I. Publications No. 8, Springer-Verlag (1987) 75–263.

[Gr1] M. Gromov, *Asymptotic Invariants of Infinite Groups*, "Geometric Group Theory II" LMS Lecture notes **182**, Cambridge University Press (1993)

[Hr] G. Hruska, *Relative hyperbolicity and relative quasiconvexity for countable groups,* Preprint 2008.

[Ka] A. Karlsson *Free subgroups of groups with non-trivial Floyd boundary*, Comm. Algebra, 31, (2003), 5361–5376.

[My] P. J. Myrberg, *Untersuchungen ueber die automorphen Funktionen beliebiger vieler Variabelen*, Acta Math. **46** (1925).

[Os] D. Osin, *Relatively hyperbolic groups: intrinsic geometry, algebraic properties and algorithmic problems*, , Mem. AMS **179** (2006) no. 843 vi+100pp.

[Tu1] P. Tukia, *A remark on the paper by Floyd*, Holomorphic functions and moduli, vol.II(Berkeley CA,1986),165–172, MSRI Publ.11, Springer New York 1988.

[Tu2] P. Tukia, *Convergence groups and Gromov's metric hyperbolic spaces* : New Zealand J. Math. **23** (1994) 157–187.

[Tu3] P. Tukia, *Conical limit points and uniform convergence groups*, J. Reine. Angew. Math. **501** (1998) 71–98.

[W] A. Weil, Sur les espace à structure uniforme et sur la topologie générale, Paris, 1937.

[Ya] A. Yaman, *A topological characterisation of relatively hyperbolic groups*, J. reine ang. Math. **566** (2004), 41–89.

[Ya1] A. Yaman*, A short note on Dynkin groups and convergence groups,* Groups Geom. Dyn. **1**, no.2 (2007), 205–208.

Victor Gerasimov : Department of Mathematics, Federal University of Minas Gerais,
Av. Antônio Carlos, 6627 - Caixa Postal 702 - CEP 30161-970 Belo Horizonte, Brazil;
email: victor@mat.ufmg.br

Leonid Potyagailo: UFR de Mathématiques, Université de Lille 1,
59655 Villeneuve d'Ascq cedex, France;
email: potyag@math.univ-lille1.fr