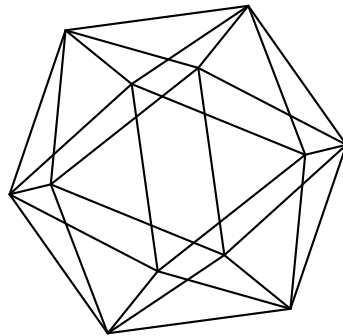


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HOMOTOPY COHERENT NERVES OF ENRICHED CATEGORIES

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ABSTRACT. We study several flavors of homotopy coherent nerves for enriched categories, in the form of right Quillen functors valued in simplicial objects. In particular, we extract explicit models for the (Segal) Reedy-injective fibrant replacement of the ordinary nerve of an enriched category. In the case of interest of categories enriched over complete Segal Θ_{n-1} -spaces, we also provide an explicit completion for its ordinary nerve. This is then used to obtain a direct Quillen equivalence between categories enriched over complete Segal Θ_{n-1} -spaces and complete Segal Θ_n -spaces.

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INTRODUCTION

I. Enriching over an ∞ -category. Many techniques in higher category theory involve the study of categories enriched in some sense over a monoidal ∞ -category. Often, the monoidal ∞ -categories are in fact cartesian closed ∞ -categories, namely they admit finite products and have internal homs. Examples of interest include enriching over the ∞ -category of spectra with the smash product, the cartesian closed ∞ -category of spaces, or the cartesian closed ∞ -category of $(n-1)$ -categories. Enriching over those in a suitable sense leads, respectively, to the notion of spectral ∞ -categories, ∞ -categories, and n -categories. Above all, our motivating example is the cartesian closed ∞ -category of $(\infty, n-1)$ -categories, for $n \geq 1$. The procedure of enriching over $(\infty, n-1)$ -categories defines (∞, n) -categories.

Traditionally, there are (at least) two ways to produce categories enriched in some sense in a cartesian closed ∞ -category \mathcal{M} .

- (1) One can first consider the ∞ -category $\mathcal{M}^{\Delta^{\text{op}}}$ of simplicial objects in \mathcal{M} , and localize to obtain the ∞ -category $\mathcal{M}_{\text{Seg}}^{\Delta^{\text{op}}}$ of Segal objects in \mathcal{M} . This would be considered a way to produce the ∞ -category of categories *internal* to \mathcal{M} . If then \mathcal{M} allows for a notion of completeness (including an appropriate constantness condition), one can further localize to obtain the ∞ -category $\mathcal{M}_{\text{Seg, comp}}^{\Delta^{\text{op}}}$ of complete Segal objects in \mathcal{M} .
- (2) If \mathcal{M} can be presented using a nice model category \mathcal{M} (meaning *excellent* in the sense of [Lur09, §A.3]), there is a model structure *M-Cat* on the category of categories strictly enriched over \mathcal{M} , in which weak equivalences are homotopically essentially surjective functors that are local weak

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equivalences. One can then consider its underlying ∞ -category $[\mathcal{M}\text{-Cat}]_\infty$. This ∞ -category ends up being equivalent to the ∞ -category $\mathcal{C}at_\infty^{\mathcal{M}}$ of ∞ -categories enriched over \mathcal{M} obtained through the machinery from [GH15], as shown in [Hau15].

The expectation is that the ∞ -categories $\mathcal{M}_{\text{Seg,comp}}^{\Delta^{\text{op}}}$ and $[\mathcal{M}\text{-Cat}]_\infty$ obtained by implementing the two viewpoints should be equivalent in cases of interest. Thanks to a zigzag of Quillen equivalences due to Bergner–Rezk [BR13, BR20], we know this is the case for our major application, namely when $\mathcal{M} := \mathcal{C}at_{(\infty, n-1)}$ is the ∞ -category of $(\infty, n-1)$ -categories, which can be presented by Rezk’s model category for complete Segal Θ_{n-1} -spaces [Rez10]. In this case, the ∞ -category $\mathcal{M}_{\text{Seg,comp}}^{\Delta^{\text{op}}}$ arises as the collection of (∞, n) -categories presented by complete Segal objects in a model of $(\infty, n-1)$ -categories, while the ∞ -category $[\mathcal{M}\text{-Cat}]_\infty$ arises as the collection of (∞, n) -categories presented by categories strictly enriched in a model of $(\infty, n-1)$ -categories. Such a comparison in a different framework is also treated in [GH15, §6].

In the scope of our research program, we are after an *explicit* equivalence between the homotopy theories of $\mathcal{M}_{\text{Seg,comp}}^{\Delta^{\text{op}}}$ and $[\mathcal{M}\text{-Cat}]_\infty$, even induced by a direct Quillen equivalence between the corresponding model structures $\mathcal{M}_{\text{Seg,comp}}^{\Delta^{\text{op}}}$ and $\mathcal{M}\text{-Cat}$, at least when \mathcal{M} is a cartesian closed model of $(\infty, n-1)$ -categories for some $n \geq 1$, e.g. the said model category on Θ_{n-1} -spaces by Rezk. We explain in Section IV how such a direct Quillen equivalence is important towards the goal of defining a consistent theory of (weighted) limits and colimits valued in an (∞, n) -category presented by different models.

In this paper, we construct a Quillen equivalence between categories enriched over complete Segal Θ_{n-1} -spaces and Rezk’s model structure for Segal complete Θ_n -spaces [Rez10] tailored to our scopes. Upon completion of this manuscript we learned that, using different techniques, Gindi constructed in [Gin21] a Quillen equivalence between a model category of enriched categories and a certain model structure on Θ_n -sets, which is based on work by Oury [Our10]. The two Quillen equivalences share similarities at the pointset level, and are possibly directly comparable. However, making a precise statement about their relationship at the homotopical level would at least require understanding how the model structure considered by Gindi compares with Ara’s model structure for n -quasi-categories [Ara14], which is to our knowledge an open problem. Currently, this has been addressed for $n = 2$ by Maehara [Mae20], and we did not investigate further the case of general n . More ideas on this discussion can be found at [Gin19].

II. The ordinary nerve of enriched categories. We now turn to describing how to relate $\mathcal{M}\text{-Cat}$ to $\mathcal{M}^{\Delta^{\text{op}}}$ and its localizations $\mathcal{M}_{\text{Seg}}^{\Delta^{\text{op}}}$ and $\mathcal{M}_{\text{Seg,comp}}^{\Delta^{\text{op}}}$. There is a naive way to try and encode explicitly the enriched categorical information into a simplicial direction via a nerve construction, which we recall in Section 1. Precisely, given an \mathcal{M} -enriched category \mathcal{C} , one can form an object NC of $\mathcal{M}^{\Delta^{\text{op}}}$ given in component 0 by the set $(NC)_0 := \text{Ob } \mathcal{C}$ of objects of \mathcal{C} and in component $m \geq 1$ by the object

$$(i) \quad (NC)_m := \coprod_{c_0, c_1, \dots, c_m \in \text{Ob } \mathcal{C}} \text{Hom}_{\mathcal{C}}(c_0, c_1) \times \text{Hom}_{\mathcal{C}}(c_1, c_2) \times \dots \times \text{Hom}_{\mathcal{C}}(c_{m-1}, c_m),$$

where $\text{Hom}_{\mathcal{C}}(-, -)$ denotes the hom object functor of \mathcal{C} , taking values in \mathcal{M} .

To see that this formula gives a well-defined functor of ∞ -categories, for convenience we focus on the situation where \mathcal{M} is presented by a cartesian closed localization $\mathcal{M} = s\text{Set}_S^{\Theta^{\text{op}}}$ of a model category of Θ -shaped space-valued diagrams at a set $S \subseteq s\text{Set}^{\Theta^{\text{op}}}$. There is evidence that this framework is not too restrictive¹, and it covers the majority of situations of interest for many purposes. For example, the ∞ -category $\mathcal{M} = \mathcal{C}at_{(\infty, n-1)}$ of $(\infty, n-1)$ -categories fits in this framework

¹Indeed, Dugger’s theorem [Dug01, Theorem 1.1] guarantees that any locally presentable ∞ -category can be presented by the left Bousfield localization of a model structures on simplicial presheaves valued in spaces. Also, Nikolaus–Sagave [NS17, Theorem 1.1] show that presentably symmetric monoidal ∞ -category is presented by a symmetric monoidal model category.

with Θ being Joyal's cell category Θ_{n-1} from [Joy97] and $\mathcal{M} = sSet_S^{\Theta_{n-1}^{\text{op}}}$ being Rezk's model structure for complete Segal Θ_{n-1} -spaces from [Rez10]. Alternatively, Θ could be taken to be the category $t\Delta$ and $\mathcal{M} = sSet_S^{t\Delta^{\text{op}}}$ the model structure for saturated $(n-1)$ -precomplicial spaces from [OR20].

In this context, one can easily obtain the following, that we state as Proposition 1.3.6.

Proposition. *With \mathcal{M} , Θ , and S as above, the nerve defines right adjoint functors of ∞ -categories*

$$N: [\mathcal{M}\text{-Cat}]_{\infty} \rightarrow \mathcal{M}^{\Delta^{\text{op}}} \quad \text{and} \quad N: [\mathcal{M}\text{-Cat}]_{\infty} \rightarrow \mathcal{M}_{\text{Seg}}^{\Delta^{\text{op}}},$$

that are induced by right Quillen functors

$$N: sSet_S^{\Theta^{\text{op}}}\text{-Cat} \rightarrow (sSet_S^{\Theta^{\text{op}}})_{\text{proj}}^{\Delta^{\text{op}}} \quad \text{and} \quad N: sSet_S^{\Theta^{\text{op}}}\text{-Cat} \rightarrow (sSet_S^{\Theta^{\text{op}}})_{\text{proj,Seg}}^{\Delta^{\text{op}}}.$$

However, this implementation has two issues.

- (1) The functor N is bad for computations in several regards, related to the fact that NC is essentially never injectively fibrant. Hence this requires to work with the projective model structure $\mathcal{M}_{\text{proj}}^{\Delta^{\text{op}}}$ to present the ∞ -category $\mathcal{M}^{\Delta^{\text{op}}}$, as opposed to the much more manageable injective model structure $\mathcal{M}_{\text{inj}}^{\Delta^{\text{op}}}$. To mention one consequent limitation, the formula for the left adjoint is completely unexplicit (at the level of model categories it involves an unknown projective cofibrant replacement).
- (2) When \mathcal{M} allows for a notion of completeness (e.g. when $\mathcal{M} = \text{Cat}_{(\infty, n-1)}$), the functor NC is generally not complete. Meaning, it does not define a functor valued in the ∞ -category $\mathcal{M}_{\text{Seg,comp}}^{\Delta^{\text{op}}}$.

In this paper, we propose two variants of the construction NC – given by the *homotopy coherent nerve* $\mathfrak{N}\mathcal{C}$ and the *complete homotopy coherent nerve* $\mathfrak{N}^c\mathcal{C}$ – that in cases of interest remedy the issues (1) and (2), respectively.

III. The (complete) homotopy coherent nerve of enriched categories. The main reason behind the technical issue (1) is caused by the fact that the formula (i) for $(NC)_m$ is in a sense too strict. To give a rough intuition, a point in $(NC)_2$ consists of a pair of composable morphisms in \mathcal{C} , as opposed to a pair of composable morphisms in \mathcal{C} with a specified choice of a (weak) composite.

Inspired by the classical homotopy coherent nerve by Cordier–Porter [CP86], we propose in Section 2 a homotopy coherent variant $\mathfrak{N}\mathcal{C}$ for NC , and prove in Theorems 2.3.1 and 2.3.7 the following.

Theorem A. *With \mathcal{M} , Θ and S as above, the homotopy coherent nerve defines right adjoint functors of ∞ -categories*

$$\mathfrak{N}: [\mathcal{M}\text{-Cat}]_{\infty} \rightarrow \mathcal{M}^{\Delta^{\text{op}}} \quad \text{and} \quad \mathfrak{N}: [\mathcal{M}\text{-Cat}]_{\infty} \rightarrow \mathcal{M}_{\text{Seg}}^{\Delta^{\text{op}}}$$

which are induced by right Quillen functors

$$\mathfrak{N}: sSet_S^{\Theta^{\text{op}}}\text{-Cat} \rightarrow (sSet_S^{\Theta^{\text{op}}})_{\text{inj}}^{\Delta^{\text{op}}} \quad \text{and} \quad \mathfrak{N}: sSet_S^{\Theta^{\text{op}}}\text{-Cat} \rightarrow (sSet_S^{\Theta^{\text{op}}})_{\text{Seg}}^{\Delta^{\text{op}}}.$$

When \mathcal{M} models certain collections of higher categorical structures, there is a notion of completeness for Segal objects, that encodes the fact that the objects form a space, which is determined by the higher structure. It then also makes sense to localize $\mathcal{M}_{\text{Seg}}^{\Delta^{\text{op}}}$ to obtain the ∞ -category $\mathcal{M}_{\text{Seg,comp}}^{\Delta^{\text{op}}}$ of complete Segal objects in \mathcal{M} .

When $\mathcal{M} = \text{Cat}_{(\infty, n-1)}$ is presented by Rezk's model structure $sSet_{(\infty, n-1)}^{\Theta_{n-1}^{\text{op}}}$, the homotopy coherent nerve $\mathfrak{N}\mathcal{C}$ is generally not complete. Indeed, the component $(\mathfrak{N}\mathcal{C})_0$ is just the set $\text{Ob}\mathcal{C}$ of objects of \mathcal{C} , rather than the underlying space of the (∞, n) -category \mathcal{C} . To correct this issue, we propose in Section 3 an explicit completion $\mathfrak{N}^c\mathcal{C}$ of $\mathfrak{N}\mathcal{C}$, and prove in Theorem 3.6.2 the following.

Theorem B. *For $n > 1$, the complete homotopy coherent nerve defines an equivalence of ∞ -categories*

$$\mathfrak{N}^c : [s\text{Set}_{(\infty, n-1)}^{\Theta_{n-1}^{\text{op}}}\text{-Cat}]_{\infty} \rightarrow (\text{Cat}_{(\infty, n-1)})_{\text{Seg, comp}}^{\Delta_{\text{op}}}$$

which is induced by a right Quillen equivalence

$$\mathfrak{N}^c : s\text{Set}_{(\infty, n-1)}^{\Theta_{n-1}^{\text{op}}}\text{-Cat} \rightarrow (s\text{Set}_{(\infty, n-1)}^{\Theta_{n-1}^{\text{op}}})_{\text{CCS}}^{\Delta_{\text{op}}}.$$

Our proof relies on the fact that n is at least 2. However, the analog statement does hold for $n = 1$, as we treat in Section 3.2. It can be deduced by combining the Quillen equivalence – given by the homotopy coherent nerve – between Kan-enriched categories and quasi-categories from [Joy07, Theorem 2.10] (or [Lur09, Theorem 2.2.5.1] and [DS11, Corollary 8.2]) with the Quillen equivalence between quasi-categories and complete Segal spaces from [JT07, Theorem 4.12].

The underlying ∞ -categories of $s\text{Set}_{(\infty, n-1)}^{\Theta_{n-1}^{\text{op}}}\text{-Cat}$ and $s\text{Set}_{(\infty, n)}^{\Theta_n^{\text{op}}}$ are known to be abstractly equivalent, as the model structures are connected by a zigzag of three Quillen equivalences. As a direct consequence of our result, however, we establish as Corollary 3.6.4 the following direct and explicit comparison.

Corollary. *For $n > 1$, there is a direct right Quillen equivalence*

$$d_* \circ \mathfrak{N}^c : s\text{Set}_{(\infty, n-1)}^{\Theta_{n-1}^{\text{op}}}\text{-Cat} \rightarrow s\text{Set}_{(\infty, n)}^{\Theta_n^{\text{op}}}.$$

For $n = 2$ one could achieve a similar comparison combining work by Gindi [Gin21] and Maehara [Mae20], but for general n we believe no such comparison exists in the literature.

IV. Limits in an (∞, n) -category across different models. The correct notion of a (weighted) limit for diagrams valued in an (∞, n) -category presented by an enriched category over a model of $(\infty, n-1)$ -categories is already established as part of a more general pattern, see [Shu06].

However, even for the case $n = 1$, it is notably harder to work with enriched categories rather than other models based on presheaves. In that case, instead of studying limits for diagrams valued in a category strictly enriched over spaces, one prefers to use the notion of limit for diagrams valued in the corresponding quasi-category, as defined by Joyal in [Joy02].

In this context, the existence of the direct Quillen equivalence between the two models of $(\infty, 1)$ -categories given by the homotopy coherent nerve $\overline{\mathfrak{N}} : s\text{Set}_{(\infty, 0)}\text{-Cat} \rightarrow \text{Set}_{(\infty, 1)}^{\Delta_{\text{op}}}$ by Cordier–Porter [CP86] was heavily exploited in [RV20, Rov21] to show that, given a (fibrant) $s\text{Set}_{(\infty, 0)}$ -enriched category \mathcal{C} , the limit of a diagram $X \rightarrow \overline{\mathfrak{N}}\mathcal{C}$ agrees with the limit of the corresponding diagram $\overline{\mathcal{C}}X \rightarrow \mathcal{C}$. The complete homotopy coherent nerve $\mathfrak{N}^c : s\text{Set}_{(\infty, 0)}\text{-Cat} \rightarrow s\text{Set}_{(\infty, 1)}^{\Delta_{\text{op}}}$ could also be used to obtain a further comparison with the notion of limits for diagrams valued in an $(\infty, 1)$ -category presented by a complete Segal space – as studied in [Ras17].

In a similar vein, in our research program, we plan to define the notion of (weighted) limit for diagrams valued in (∞, n) -categories presented by a complete Segal object in complete Segal Θ_{n-1} -spaces. Then we will deduce definitions for diagrams valued in an (∞, n) -category presented by a complete Segal Θ_n -space or an n -fold complete Segal space, and show this is done consistently with the enriched approach using the explicit Quillen equivalence constructed in this paper.

More precisely, given a (fibrant) $s\text{Set}_{(\infty, n-1)}^{\Theta_{n-1}^{\text{op}}}$ -enriched category \mathcal{C} , we can use the explicit Quillen equivalences from Section III to represent the same (∞, n) -category as the complete Segal object in Θ_{n-1} -spaces $\mathfrak{N}^c\mathcal{C}$, or as the complete Segal Θ_n -space $d_*\mathfrak{N}^c\mathcal{C}$. Furthermore, every diagram $X \rightarrow d_*\mathfrak{N}^c\mathcal{C}$ can be represented as a diagram $d^*X \rightarrow \mathfrak{N}^c\mathcal{C}$, or as a diagram $\mathcal{C}^c d^*X \rightarrow \mathcal{C}$. We plan to show that the limits of all these different representations of the same diagram across different models agree, providing a consistent theory of limits for diagrams valued in an (∞, n) -category.

The approach using the model of complete Segal objects in complete Segal Θ_{n-1} -spaces is particularly useful to formulate a notion of (weighted) limits, since it is directly contained in a model of “internal” categories to $(\infty, n-1)$ -categories, namely, Segal objects in complete Segal

Θ_{n-1} -spaces. This is already apparent when treating the strict case [GP99, cM20, cM22], namely when studying limits for diagrams valued in a 2-category. In that context, it is crucial to embed the theory of 2-categories into that of internal categories to categories, i.e., double categories, where more constructions become available. The upshot is that limits are for many purposes better encoded using internal categorical structures rather than directly using (∞, n) -categories.

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1. THE STRICT NERVE AND ITS PROPERTIES

In this section, we let Θ be a small category and S be a set of maps in $sSet^{\Theta^{\text{op}}}$ such that the model structure $sSet_S^{\Theta^{\text{op}}}$, obtained as the left Bousfield localization at S of the injective model structure $(sSet_{(\infty, 0)}^{\Theta^{\text{op}}})_{\text{inj}}^{\Theta^{\text{op}}}$ on the category of Θ -presheaves valued in the Kan-Quillen model structure $sSet_{(\infty, 0)}$, is cartesian closed.

We first recall in Section 1.1 the strict $(m+1)$ -point suspension construction for $m \geq 0$ $\Sigma_m: sSet^{\Theta^{\text{op}}} \rightarrow sSet^{\Theta^{\text{op}}}\text{-Cat}$, and use it in Section 1.2 to recall the strict nerve construction $N: sSet^{\Theta^{\text{op}}}\text{-Cat} \rightarrow sSet^{\Theta^{\text{op}} \times \Delta^{\text{op}}}$.

We then study in Section 1.3 the homotopical properties and limitations of the strict nerve. Roughly speaking, we show that N is well-behaved when endowing its target category with model structures based on the projective model structure, and not with the injective model structure.

1.1. The strict suspension construction. Many of the $sSet^{\Theta^{\text{op}}}$ -enriched categories that feature in this paper have the following property, so we introduce a terminology that streamlines the exposition. We refer to an object of $sSet^{\Theta^{\text{op}}}$ as a Θ -space.

Definition 1.1.1. A $sSet^{\Theta^{\text{op}}}$ -enriched category \mathcal{C} is *directed* if

- its set of objects $\text{Ob } \mathcal{C}$ is $\{0, 1, \dots, m\}$, for some $m \geq 0$,
- for $0 \leq j \leq i \leq m$, the hom Θ -space $\text{Hom}_{\mathcal{C}}(i, j)$ is given by

$$\text{Hom}_{\mathcal{C}}(i, j) = \begin{cases} \emptyset & \text{if } j < i \\ \Delta[0] & \text{if } j = i. \end{cases}$$

In particular, compositions in a directed $sSet^{\Theta^{\text{op}}}$ -enriched category \mathcal{C} involving the above hom Θ -spaces are uniquely determined. Moreover, the value of a $sSet^{\Theta^{\text{op}}}$ -enriched functor from a directed $sSet^{\Theta^{\text{op}}}$ -enriched category is also uniquely determined on these hom Θ -spaces.

Definition 1.1.2. Let $m \geq 0$. For $X \in sSet^{\Theta^{\text{op}}}$, the $(m+1)$ -point *strict suspension* of X is the directed $sSet^{\Theta^{\text{op}}}$ -enriched category $\Sigma_m X$ such that

- its set of objects $\text{Ob}(\Sigma_m X)$ is $\{0, 1, \dots, m\}$,
- for $0 \leq i < j \leq m$, the hom Θ -space is $\text{Hom}_{\Sigma_m X}(i, j) := X^{\times(j-i)}$,
- for $0 \leq i < j < k \leq m$, the composition map is given by

$$\begin{array}{ccc} \text{Hom}_{\Sigma_m X}(i, j) \times \text{Hom}_{\Sigma_m X}(j, k) & = & X^{\times(j-i)} \times X^{\times(k-j)} \\ \circ_{i,j,k} \downarrow & & \downarrow \cong \\ \text{Hom}_{\Sigma_m X}(i, k) & = & X^{\times(k-i)}. \end{array}$$

The construction extends canonically to a functor

$$\Sigma_m : sSet^{\Theta^{\text{op}}} \rightarrow sSet^{\Theta^{\text{op}}}\text{-Cat}.$$

The following propositions record some elementary properties of this construction, and are of straightforward verification.

Proposition 1.1.3. *The assignment $([m], X) \mapsto \Sigma_m X$ defines a functor*

$$\Sigma_{\bullet} : \Delta^{\text{op}} \times sSet^{\Theta^{\text{op}}} \rightarrow sSet^{\Theta^{\text{op}}}\text{-Cat}.$$

Notice that the notation is set up so that $\Sigma_0 X$ is the terminal category $[0]$, and $\Sigma_1 X$ coincides with the “usual” suspension traditionally denoted ΣX , which fits into the adjunction

$$\{0,1\}/sSet^{\Theta^{\text{op}}}\text{-Cat} \begin{array}{c} \xleftarrow{\Sigma} \\ \perp \\ \xrightarrow{\text{Hom}} \end{array} sSet^{\Theta^{\text{op}}},$$

where $\{0,1\}/sSet^{\Theta^{\text{op}}}\text{-Cat}$ denotes the category of bi-pointed $sSet^{\Theta^{\text{op}}}$ -enriched categories. For $m > 1$, the iterated suspension $\Sigma_m X$ is built out of ΣX and $[0]$ as follows.

Proposition 1.1.4. *For $m \geq 1$ and $X \in sSet^{\Theta^{\text{op}}}$, there are natural isomorphisms in $sSet^{\Theta^{\text{op}}}\text{-Cat}$*

$$\Sigma_m X \cong \Sigma X \amalg_{[0]} \Sigma_{m-1} X \cong \Sigma X \amalg_{[0]} \dots \amalg_{[0]} \Sigma X,$$

where the right-hand side is the colimit of m copies of ΣX under their target or source.

Remark 1.1.5. For $m \geq 1$ there is an adjunction

$$\{0,1,\dots,m\}/sSet^{\Theta^{\text{op}}}\text{-Cat} \begin{array}{c} \xleftarrow{\Sigma_m} \\ \perp \\ \xrightarrow{\quad} \end{array} sSet^{\Theta^{\text{op}}},$$

where $\{0,1,\dots,m\}/sSet^{\Theta^{\text{op}}}\text{-Cat}$ denotes the category of $sSet^{\Theta^{\text{op}}}$ -enriched categories pointed on $m+1$ objects. In particular, the functor $\Sigma_m : sSet^{\Theta^{\text{op}}} \rightarrow \{0,1,\dots,m\}/sSet^{\Theta^{\text{op}}}\text{-Cat}$ preserves colimits.

1.2. The strict nerve. We record the following notations, used throughout the whole paper.

Notation 1.2.1. We write:

- $F[m] \in \text{Set}^{\Delta^{\text{op}}}$ for the representable at $m \geq 0$,
- $\Theta[\theta] \in \text{Set}^{\Theta^{\text{op}}}$ for the representable at $\theta \in \Theta$,
- $\Delta[k] \in sSet$ for the representable at $k \geq 0$,
- $\Theta[\theta] \times \Delta[k] \in sSet^{\Theta^{\text{op}}}$ for the representable at $(\theta, [k]) \in \Theta \times \Delta$,
- $F[m] \times \Theta[\theta] \in \text{Set}^{\Theta^{\text{op}} \times \Delta^{\text{op}}}$ for the representable at $([m], \theta) \in \Delta \times \Theta$,
- $F[m] \times \Theta[\theta] \times \Delta[k] \in sSet^{\Theta^{\text{op}} \times \Delta^{\text{op}}}$ for the representable at $([m], \theta, [k]) \in \Delta \times \Theta \times \Delta$.

All categories $\text{Set}^{\Delta^{\text{op}}}$, $\text{Set}^{\Theta^{\text{op}}}$, $sSet$, $sSet^{\Theta^{\text{op}}}$, $\text{Set}^{\Theta^{\text{op}} \times \Delta^{\text{op}}}$ are naturally included into $sSet^{\Theta^{\text{op}} \times \Delta^{\text{op}}}$, and we regard all the above as objects of it without further specification.

By taking the left Kan extension of the assignment $\Delta \times \Theta \times \Delta \rightarrow sSet^{\Theta^{\text{op}}}\text{-Cat}$ given by

$$([m], \theta, [k]) \mapsto \Sigma_m(\Theta[\theta] \times \Delta[k]),$$

we obtain an adjunction

$$sSet^{\Theta^{\text{op}}}\text{-Cat} \begin{array}{c} \xleftarrow{c} \\ \perp \\ \xrightarrow{N} \end{array} sSet^{\Theta^{\text{op}} \times \Delta^{\text{op}}}.$$

The right adjoint in this pair is what is traditionally considered to be the strict nerve of a $sSet^{\Theta^{\text{op}}}$ -enriched category, which we now spell out.

Definition 1.2.2. For a $sSet^{\Theta\text{op}}$ -enriched category \mathcal{C} , the *strict nerve* NC is the $(\Delta \times \Theta)$ -space given at $m \geq 0$, $\theta \in \Theta$, and $k \geq 0$ by the set

$$(NC)_{m,\theta,k} := sSet^{\Theta\text{op} \times \Delta\text{op}}(\Sigma_m(\Theta[\theta] \times \Delta[k]), \mathcal{C}).$$

Alternatively, NC is given at $m = 0$ by $(NC)_0 = \text{Ob } \mathcal{C}$ – the set of objects of \mathcal{C} seen as an object in $sSet^{\Theta\text{op}}$ – and at $m \geq 1$ by the object in $sSet^{\Theta\text{op}}$

$$\begin{aligned} (NC)_m &\cong \coprod_{c_0, c_1, \dots, c_m \in \mathcal{C}} \text{Hom}_{\mathcal{C}}(c_0, c_1) \times \text{Hom}_{\mathcal{C}}(c_1, c_2) \times \dots \times \text{Hom}_{\mathcal{C}}(c_{m-1}, c_m) \\ &\cong \text{Mor } \mathcal{C} \times_{\text{Ob } \mathcal{C}} \text{Mor } \mathcal{C} \times_{\text{Ob } \mathcal{C}} \dots \times_{\text{Ob } \mathcal{C}} \text{Mor } \mathcal{C} \end{aligned}$$

with the convention that $\text{Mor } \mathcal{C}$ is the object of $sSet^{\Theta\text{op}}$ given by

$$\text{Mor } \mathcal{C} := \coprod_{c_0, c_1 \in \mathcal{C}} \text{Hom}_{\mathcal{C}}(c_0, c_1).$$

Remark 1.2.3. Given a $sSet^{\Theta\text{op}}$ -enriched category \mathcal{C} and $m \geq 1$, by Proposition 1.1.4 the Segal map is an isomorphism in $sSet^{\Theta\text{op}}$

$$(NC)_m \cong (NC)_1 \times_{(NC)_0} \dots \times_{(NC)_0} (NC)_1.$$

1.3. Homotopical properties of the strict nerve. We now introduce a homotopical framework, and study the homotopical properties of the strict nerve construction $N: sSet^{\Theta\text{op}}\text{-Cat} \rightarrow sSet^{\Theta\text{op}}$ and its limitations.

Let $(sSet_{(\infty,0)}^{\Theta\text{op}})_{\text{inj}}^{\Theta\text{op}}$ denote the injective model structure on the category of Θ -presheaves valued in $sSet_{(\infty,0)}$. Given a set S of maps in $sSet^{\Theta\text{op}}$, let $sSet_S^{\Theta\text{op}}$ denote the left Bousfield localization of $(sSet_{(\infty,0)}^{\Theta\text{op}})_{\text{inj}}^{\Theta\text{op}}$ at the set S , and assume furthermore that the set S is such that $sSet_S^{\Theta\text{op}}$ is cartesian closed. This is enough to guarantee that the model structure $sSet_S^{\Theta\text{op}}$ is excellent in the sense of [Lur09, Definition A.3.2.16]. Consequently, the category $sSet^{\Theta\text{op}}\text{-Cat}$ supports the model structure $sSet_S^{\Theta\text{op}}\text{-Cat}$ from [Lur09, Proposition A.3.2.4, Theorem A.3.2.24], whose main features we now recall.

In the model structure $sSet_S^{\Theta\text{op}}\text{-Cat}$, a $sSet^{\Theta\text{op}}$ -enriched category \mathcal{C} is *fibrant* if, for every pair of objects $c, c' \in \text{Ob } \mathcal{C}$, the Θ -space $\text{Hom}_{\mathcal{C}}(c, c')$ is fibrant in $sSet_S^{\Theta\text{op}}$, and a $sSet^{\Theta\text{op}}$ -enriched functor $F: \mathcal{C} \rightarrow \mathcal{D}$ is:

- a *weak equivalence* if the induced functor $\text{Ho } F: \text{Ho } \mathcal{C} \rightarrow \text{Ho } \mathcal{D}$ between homotopy categories is essentially surjective on objects, and for every pair of objects $c, c' \in \text{Ob } \mathcal{C}$ the induced map

$$F_{c,c'}: \text{Hom}_{\mathcal{C}}(c, c') \rightarrow \text{Hom}_{\mathcal{D}}(Fc, Fc')$$

is a weak equivalence in $sSet_S^{\Theta\text{op}}$,

- a *fibration between fibrant objects* if the induced functor $\text{Ho } F: \text{Ho } \mathcal{C} \rightarrow \text{Ho } \mathcal{D}$ between homotopy categories is an isofibration of categories, and for every pair of objects $c, c' \in \text{Ob } \mathcal{C}$ the induced map

$$F_{c,c'}: \text{Hom}_{\mathcal{C}}(c, c') \rightarrow \text{Hom}_{\mathcal{D}}(Fc, Fc')$$

is a fibration in $sSet_S^{\Theta\text{op}}$,

- a *trivial fibration* if it is surjective on objects, and for every pair of objects $c, c' \in \text{Ob } \mathcal{C}$ the induced map

$$F_{c,c'}: \text{Hom}_{\mathcal{C}}(c, c') \rightarrow \text{Hom}_{\mathcal{D}}(Fc, Fc')$$

is a trivial fibration in $sSet_S^{\Theta\text{op}}$.

We do not recall the description of the homotopy category construction $\text{Ho } \mathcal{C}$, since we will not make an explicit use of it. For the purpose of this paper, all one needs to know is that if $F: \mathcal{C} \rightarrow \mathcal{D}$ is bijective on objects, then $\text{Ho } F: \text{Ho } \mathcal{C} \rightarrow \text{Ho } \mathcal{D}$ is essentially surjective on objects.

The model structure $sSet_S^{\Theta\text{op}}\text{-Cat}$ is designed so that the following holds. Here $\{0,1\}/sSet_S^{\Theta\text{op}}\text{-Cat}$ denotes the slice model structure, in which cofibrations, fibrations, and weak equivalences are created by the forgetful functor to $sSet_S^{\Theta\text{op}}\text{-Cat}$.

Proposition 1.3.1. *The adjunction*

$$\{0,1\}/sSet_S^{\Theta\text{op}}\text{-Cat} \begin{array}{c} \xleftarrow{\Sigma} \\ \xrightarrow[\text{Hom}]{\perp} \end{array} sSet_S^{\Theta\text{op}}$$

is a Quillen pair.

Proof. This follows directly from the local properties of trivial fibrations and fibrations between fibrant objects, using the fact that trivial cofibrations are determined by their lifting properties against fibrations between fibrant objects by [Joy08, Lemma E.2.13]. \square

Let $(sSet_S^{\Theta\text{op}})_{\text{inj}}^{\Delta\text{op}}$ and $(sSet_S^{\Theta\text{op}})_{\text{proj}}^{\Delta\text{op}}$ denote the injective and projective model structures on the category $(sSet_S^{\Theta\text{op}})^{\Delta\text{op}} \cong sSet^{\Theta\text{op} \times \Delta\text{op}}$ of simplicial objects in $sSet_S^{\Theta\text{op}}$.

Notation 1.3.2. Let $A \rightarrow B$ and $X \rightarrow Y$ be two maps in a presheaf category. We denote by $(A \rightarrow B) \widehat{\times} (X \rightarrow Y)$ the pushout-product map

$$(A \rightarrow B) \widehat{\times} (X \rightarrow Y) := (A \times Y \amalg_{A \times X} B \times X \rightarrow B \times Y).$$

Remark 1.3.3. We recall sets of generating (trivial) cofibrations for the injective and projective model structures $(sSet_S^{\Theta\text{op}})_{\text{inj}}^{\Delta\text{op}}$ and $(sSet_S^{\Theta\text{op}})_{\text{proj}}^{\Delta\text{op}}$.

- By [Hir03, Definition 11.5.33], a set of generating (trivial) cofibrations for the projective model structure $(sSet_S^{\Theta\text{op}})_{\text{proj}}^{\Delta\text{op}}$ is given by

$$\{F[m] \times (X \rightarrow Y) \mid m \geq 0, X \rightarrow Y \in \mathcal{J}\},$$

where \mathcal{J} is a set of generating (trivial) cofibrations in $sSet_S^{\Theta\text{op}}$.

- As a consequence of [Hir03, Corollary 15.7.2], the injective model structure $(sSet_S^{\Theta\text{op}})_{\text{inj}}^{\Delta\text{op}}$ coincides with the Reedy model structure $(sSet_S^{\Theta\text{op}})_{\text{Reedy}}^{\Delta\text{op}}$, so by [Hir03, Theorem 15.6.27] a set of generating (trivial) cofibrations for $(sSet_S^{\Theta\text{op}})_{\text{inj}}^{\Delta\text{op}}$ is given by

$$\{(\partial F[m] \rightarrow F[m]) \widehat{\times} (X \rightarrow Y) \mid m \geq 0, X \rightarrow Y \in \mathcal{J}\},$$

where $\partial F[m] \rightarrow F[m]$ is the boundary inclusion and \mathcal{J} is a set of generating (trivial) cofibrations in $sSet_S^{\Theta\text{op}}$.

1.3.4. Denote by $(sSet_S^{\Theta\text{op}})_{\text{Seg}}^{\Delta\text{op}}$ and $(sSet_S^{\Theta\text{op}})_{\text{proj,Seg}}^{\Delta\text{op}}$ the left Bousfield localizations of the model structures $(sSet_S^{\Theta\text{op}})_{\text{inj}}^{\Delta\text{op}}$ and $(sSet_S^{\Theta\text{op}})_{\text{proj}}^{\Delta\text{op}}$ at the set of *Segal maps*

$$\text{Seg}_{\Theta} = \{(F[1] \amalg_{F[0]} \dots \amalg_{F[0]} F[1] \rightarrow F[m]) \times \Theta[\theta] \mid m \geq 1, \theta \in \Theta\},$$

where $F[1] \amalg_{F[0]} \dots \amalg_{F[0]} F[1] \rightarrow F[m]$ is the spine inclusion.

The fibrant objects of the model structure $(sSet_S^{\Theta\text{op}})_{\text{proj,Seg}}^{\Delta\text{op}}$ are those $(\Delta \times \Theta)$ -spaces X such that X_m is fibrant in $sSet_S^{\Theta\text{op}}$, for all $m \geq 0$, and the Segal map

$$X_m \rightarrow X_1 \times_{X_0}^h \dots \times_{X_0}^h X_1$$

is a weak equivalence in $sSet_S^{\Theta\text{op}}$, for all $m \geq 1$. In comparison, the fibrant objects of the model structure $(sSet_S^{\Theta\text{op}})_{\text{inj}}^{\Delta\text{op}}$ are the *Segal objects* in S -local Θ -spaces, as defined below.

Definition 1.3.5. An object $X \in sSet^{\Theta\text{op} \times \Delta\text{op}}$ is a *Segal object in S -local Θ -spaces* if X is fibrant in $(sSet_S^{\Theta\text{op}})_{\text{inj}}^{\Delta\text{op}}$, and the Segal map

$$X_m \rightarrow X_1 \times_{X_0}^{(h)} \dots \times_{X_0}^{(h)} X_1$$

is a weak equivalence in $sSet_S^{\Theta^{op}}$, for all $m \geq 1$.

Note that the fibrancy of X in $(sSet_S^{\Theta^{op}})_{inj}^{\Delta^{op}}$ implies that X_m is fibrant in $sSet_S^{\Theta^{op}}$, for all $m \geq 0$, and that the pullbacks appearing in the Segal maps are in fact homotopy pullbacks.

The following can be deduced using the explicit description of the generating (trivial) cofibrations of $(sSet_S^{\Theta^{op}})_{proj}^{\Delta^{op}}$ and [Hir03, Theorem 3.3.20(1)(a)], and we omit the proof.

Proposition 1.3.6. *The adjunctions*

$$sSet_S^{\Theta^{op}}\text{-Cat} \xleftarrow[\perp]{c} (sSet_S^{\Theta^{op}})_{proj}^{\Delta^{op}} \quad \text{and} \quad sSet_S^{\Theta^{op}}\text{-Cat} \xleftarrow[\perp]{c} (sSet_S^{\Theta^{op}})_{proj,Seg}^{\Delta^{op}}$$

are Quillen pairs.

However, the analog statement fails when replacing the projective with the injective model structure.

Remark 1.3.7. The adjunction

$$sSet_S^{\Theta^{op}}\text{-Cat} \xleftarrow[\perp]{c} (sSet_S^{\Theta^{op}})_{inj}^{\Delta^{op}}$$

is *not* a Quillen pair. Indeed, given a fibrant $sSet_S^{\Theta^{op}}$ -enriched category \mathcal{C} , Example 1.3.8 shows that the nerve $N\mathcal{C}$ is generally not fibrant in $(sSet_S^{\Theta^{op}})_{inj}^{\Delta^{op}}$. As a consequence, the adjunction

$$sSet_S^{\Theta^{op}}\text{-Cat} \xleftarrow[\perp]{c} (sSet_S^{\Theta^{op}})_{Seg}^{\Delta^{op}}$$

is also not a Quillen pair.

Example 1.3.8. Let X be a fibrant object in $sSet_S^{\Theta^{op}}$ that is non-discrete (meaning that it is not in the image of $Set^{\Theta^{op}} \hookrightarrow sSet^{\Theta^{op}}$). The $sSet^{\Theta^{op}}$ -enriched category ΣX is by construction fibrant in $sSet_S^{\Theta^{op}}\text{-Cat}$, however its strict nerve $N\Sigma X$ is not fibrant in $(sSet_S^{\Theta^{op}})_{inj}^{\Delta^{op}}$. To see this, we first observe that the model structure $(sSet_S^{\Theta^{op}})_{inj}^{\Delta^{op}}$ is enriched over $sSet^{\Theta^{op}}$ (see e.g. [Mos19, Theorem 5.4]), and we denote by $\text{Hom}_{sSet^{\Theta^{op}} \times \Delta^{op}}(-, -)$ its hom Θ -space functor. Now, the map $\partial F[2] \rightarrow F[2]$ is a cofibration in $(sSet_S^{\Theta^{op}})_{inj}^{\Delta^{op}}$, but the map

$$\text{Hom}_{sSet^{\Theta^{op}} \times \Delta^{op}}(F[2], N\Sigma X) \rightarrow \text{Hom}_{sSet^{\Theta^{op}} \times \Delta^{op}}(\partial F[2], N\Sigma X),$$

is isomorphic to the map

$$\Delta[0] \amalg X \amalg X \amalg \Delta[0] \rightarrow \Delta[0] \amalg (X \times X) \amalg (X \times X) \amalg \Delta[0],$$

induced by the diagonal map of the non-discrete Θ -space X . It is therefore not a fibration in $(sSet_{(\infty,0)}^{\Theta^{op}})_{inj}^{\Theta^{op}}$ and hence also not a fibration in $sSet_S^{\Theta^{op}}$.

In the next section, we propose a variant of N that remedies the issues presented in Remark 1.3.7.

2. THE HOMOTOPY COHERENT NERVE AND ITS PROPERTIES

Like in the previous section, we let Θ be a small category and S be a set of maps in $sSet^{\Theta^{op}}$ such that the model structure $sSet_S^{\Theta^{op}}$ – obtained as a left Bousfield localization of the injective model structure $(sSet_{(\infty,0)}^{\Theta^{op}})_{inj}^{\Theta^{op}}$ at the set S – is cartesian closed.

We define in Section 2.1 what we call the homotopy coherent $(m+1)$ -point suspension construction $\mathfrak{S}_m: sSet^{\Theta^{op}} \rightarrow sSet^{\Theta^{op}}\text{-Cat}$ for $m \geq 0$, and use it in Section 2.2 to define the homotopy coherent nerve construction $\mathfrak{N}: sSet^{\Theta^{op}}\text{-Cat} \rightarrow sSet^{\Theta^{op} \times \Delta^{op}}$.

We then study in Section 2.3 the homotopical properties of the homotopy coherent nerve – describing in which sense it is better behaved than the strict nerve N – but also its limitations. Roughly speaking, we show that \mathfrak{N} is well-behaved when endowing its target category with model

structures (based on the injective model structure) for Segal objects, but not for *complete* Segal objects. We also show that \mathfrak{NC} provides an injective (and Segal) fibrant replacement of \mathcal{NC} for a fibrant $sSet_S^{\Theta^{\text{op}}}$ -enriched category \mathcal{C} .

2.1. The homotopy coherent suspension construction. The following is designed to be a homotopical version of the strict suspension construction from Section 1.1.

Definition 2.1.1. Let $m \geq 0$. Given $X \in sSet^{\Theta^{\text{op}}}$, the $(m+1)$ -point homotopy coherent suspension of X is the directed $sSet^{\Theta^{\text{op}}}$ -enriched category $\mathfrak{S}_m X$ such that

- its set of objects $\text{Ob}(\mathfrak{S}_m X)$ is $\{0, 1, \dots, m\}$,
- for $0 \leq i < j \leq m$, the hom Θ -space is $\text{Hom}_{\mathfrak{S}_m X}(i, j) := \Delta[1]^{\times(j-i-1)} \times X^{\times(j-i)}$,
- for $0 \leq i < j < k \leq m$, the composition map is given by

$$\begin{array}{ccc} \text{Hom}_{\mathfrak{S}_m X}(i, j) \times \text{Hom}_{\mathfrak{S}_m X}(j, k) & = & \Delta[1]^{\times(j-i-1)} \times \Delta[1]^{\times(k-j-1)} \times X^{\times(j-i)} \times X^{\times(k-j)} \\ \circ_{i,j,k} \downarrow & & \downarrow \text{id}^{\times(j-i-1)} \times \langle 1 \rangle \times \text{id}^{\times(k-j-1)} \times \cong \\ \text{Hom}_{\mathfrak{S}_m X}(i, k) & = & \Delta[1]^{\times(k-i-1)} \times X^{\times(k-i)}. \end{array}$$

The construction extends canonically to a functor

$$\mathfrak{S}_m : sSet^{\Theta^{\text{op}}} \rightarrow sSet^{\Theta^{\text{op}}}\text{-Cat}.$$

The following proposition records some elementary properties of this construction, and are of straightforward verification.

Proposition 2.1.2. *The assignment $([m], X) \mapsto \mathfrak{S}_m X$ defines a functor*

$$\mathfrak{S}_\bullet : \Delta^{\text{op}} \times sSet^{\Theta^{\text{op}}} \rightarrow sSet^{\Theta^{\text{op}}}\text{-Cat}.$$

Notice that we again have that $\mathfrak{S}_0 X$ is the terminal category $[0]$, and that $\mathfrak{S}_1 X$ coincides with the usual suspension ΣX .

Remark 2.1.3. For $m \geq 1$ there is an adjunction

$$\{0, 1, \dots, m\} / sSet^{\Theta^{\text{op}}}\text{-Cat} \begin{array}{c} \xleftarrow{\mathfrak{S}_m} \\ \perp \\ \xrightarrow{\quad} \end{array} sSet^{\Theta^{\text{op}}},$$

where $\{0, 1, \dots, m\} / sSet^{\Theta^{\text{op}}}\text{-Cat}$ denotes the category of $sSet^{\Theta^{\text{op}}}$ -enriched categories pointed on $m+1$ objects. In particular, the functor $\mathfrak{S}_m : sSet^{\Theta^{\text{op}}} \rightarrow \{0, 1, \dots, m\} / sSet^{\Theta^{\text{op}}}\text{-Cat}$ preserves colimits.

We understand several relations between Σ_m and \mathfrak{S}_m .

Remark 2.1.4. For $m \geq 0$ and $X \in sSet^{\Theta^{\text{op}}}$, there is a natural isomorphism in $sSet^{\Theta^{\text{op}}}\text{-Cat}$

$$\mathfrak{S}_m X \cong \Sigma_m X \times_{\Sigma_m \Delta[0]} \mathfrak{S}_m \Delta[0].$$

In particular, there is an induced map $\mathfrak{S}_m X \rightarrow \Sigma_m X$ given by one of the projections.

This comparison map is in fact a weak equivalence in $sSet_S^{\Theta^{\text{op}}}\text{-Cat}$. Indeed, for a fixed m it admits a section (that is not natural in m) – described in the following proposition – which we show is a weak equivalence in $sSet_S^{\Theta^{\text{op}}}\text{-Cat}$.

Proposition 2.1.5. *For $m \geq 0$ and $X \in sSet^{\Theta^{\text{op}}}$, there is a weak equivalence in $sSet_S^{\Theta^{\text{op}}}\text{-Cat}$*

$$\Sigma_m X \xrightarrow{\simeq} \mathfrak{S}_m X.$$

Proof. Let $m \geq 0$. We define a $sSet^{\Theta^{\text{op}}}$ -enriched functor $\Sigma_m X \rightarrow \mathfrak{S}_m X$ between directed $sSet^{\Theta^{\text{op}}}$ -enriched categories as follows:

- on objects, it is the identity at $\{0, 1, \dots, m\}$,
- for $0 \leq i < j \leq m$, it is given by the map

$$\begin{array}{ccc} \mathrm{Hom}_{\Sigma_m X}(i, j) = X^{\times(j-i)} & & \\ \downarrow & & \downarrow \langle 1 \rangle^{\times(j-i-1)} \times \mathrm{id}_X^{\times(j-i)} \\ \mathrm{Hom}_{\mathfrak{S}_m X}(i, j) = \Delta[1]^{\times(j-i-1)} \times X^{\times(j-i)} & & \end{array}$$

It is straightforward to see that this data is compatible with compositions. The $s\mathrm{Set}^{\Theta^{\mathrm{op}}}$ -enriched functor is by construction a weak equivalence in $s\mathrm{Set}_S^{\Theta^{\mathrm{op}}}\text{-Cat}$ since $\Delta[1]$ is contractible in $s\mathrm{Set}_{(\infty, 0)}$, and hence also in $s\mathrm{Set}_S^{\Theta^{\mathrm{op}}}$. \square

2.2. The homotopy coherent nerve. By taking the left Kan extension along the assignment $\Delta \times \Theta \times \Delta \rightarrow s\mathrm{Set}^{\Theta^{\mathrm{op}}}\text{-Cat}$ given by

$$([m], \theta, [k]) \mapsto \mathfrak{S}_m(\Theta[\theta] \times \Delta[k]),$$

we obtain an adjunction

$$s\mathrm{Set}^{\Theta^{\mathrm{op}}}\text{-Cat} \xleftarrow[\mathfrak{N}]{\mathfrak{C}} s\mathrm{Set}^{\Theta^{\mathrm{op}} \times \Delta^{\mathrm{op}}}.$$

Spelling out the description of the right adjoint, we obtain the following.

Definition 2.2.1. For a $s\mathrm{Set}^{\Theta^{\mathrm{op}}}$ -enriched category \mathcal{C} , the *homotopy coherent nerve* $\mathfrak{N}\mathcal{C}$ is the $(\Delta \times \Theta)$ -space given at $m \geq 0$, $\theta \in \Theta$, and $k \geq 0$ by the set

$$(\mathfrak{N}\mathcal{C})_{m, \theta, k} := s\mathrm{Set}^{\Theta^{\mathrm{op}} \times \Delta^{\mathrm{op}}}(\mathfrak{S}_m(\Theta[\theta] \times \Delta[k]), \mathcal{C}).$$

In particular, we describe $(\mathfrak{N}\mathcal{C})_m$ explicitly for low values of m . For $m = 0, 1$ we have

$$(\mathfrak{N}\mathcal{C})_0 = \mathrm{Ob}\mathcal{C} \quad \text{and} \quad (\mathfrak{N}\mathcal{C})_1 = \mathrm{Mor}\mathcal{C},$$

and for $m = 2$ we have

$$(\mathfrak{N}\mathcal{C})_2 \cong \coprod_{c_0, c_1, c_2 \in \mathcal{C}} \mathrm{Hom}_{\mathcal{C}}(c_0, c_1, c_2),$$

where $\mathrm{Hom}_{\mathcal{C}}(c_0, c_1, c_2)$ is given by the following pullback in $s\mathrm{Set}^{\Theta^{\mathrm{op}}}$.

$$\begin{array}{ccc} \mathrm{Hom}_{\mathcal{C}}(c_0, c_1, c_2) & \longrightarrow & \mathrm{Hom}_{\mathcal{C}}(c_0, c_2)^{\Delta[1]} \\ \downarrow \lrcorner & & \downarrow (d^0)^* \\ \mathrm{Hom}_{\mathcal{C}}(c_0, c_1) \times \mathrm{Hom}_{\mathcal{C}}(c_1, c_2) & \xrightarrow{\circ_{c_0, c_1, c_2}} & \mathrm{Hom}_{\mathcal{C}}(c_0, c_2) \end{array}$$

2.3. Homotopical properties of the homotopy coherent nerve. We can now see that the homotopy coherent nerve construction $\mathfrak{N}: s\mathrm{Set}^{\Theta^{\mathrm{op}}}\text{-Cat} \rightarrow s\mathrm{Set}^{\Theta^{\mathrm{op}}}$ has some of the homotopical properties that the strict nerve construction N did not have (see Remark 1.3.7).

Theorem 2.3.1. *The adjunction*

$$s\mathrm{Set}_S^{\Theta^{\mathrm{op}}}\text{-Cat} \xleftarrow[\mathfrak{N}]{\mathfrak{C}} (s\mathrm{Set}_S^{\Theta^{\mathrm{op}}})_{\mathrm{inj}}^{\Delta^{\mathrm{op}}}$$

is a Quillen pair.

The proof of the proposition relies on several lemmas.

Notation 2.3.2. For $X \in s\mathrm{Set}^{\Theta^{\mathrm{op}}}$, we denote by $\partial\mathfrak{S}_m X$ the $s\mathrm{Set}^{\Theta^{\mathrm{op}}}$ -enriched category given by the canonical coequalizer in $s\mathrm{Set}^{\Theta^{\mathrm{op}}}\text{-Cat}$

$$\partial\mathfrak{S}_m X := \mathrm{coeq}\left(\coprod_{0 \leq i < j \leq m} \mathfrak{S}_{m-2} X \rightrightarrows \coprod_{0 \leq i \leq m} \mathfrak{S}_{m-1} X \right).$$

In fact $\partial\mathfrak{S}_m X$ can be computed as the directed $sSet^{\Theta^{\text{op}}}$ -enriched category such that

- its set of objects $\text{Ob}(\partial\mathfrak{S}_m X)$ is $\{0, 1, \dots, m\}$,
- for $0 \leq i < j \leq m$, its hom Θ -space $\text{Hom}_{\partial\mathfrak{S}_m X}(i, j)$ is given by

$$\text{Hom}_{\partial\mathfrak{S}_m X}(i, j) = \begin{cases} \Delta[1]^{\times(j-i-1)} \times X^{\times(j-i)} & \text{if } 0 < j - i < m \\ \partial(\Delta[1]^{\times(m-1)}) \times X^{\times m} & \text{if } i = 0, j = m, \end{cases}$$

where $\partial(\Delta[1]^{\times(m-1)})$ is the boundary of the $(m-1)$ -dimensional cube $\Delta[1]^{\times(m-1)}$ (see [Lur09, §2.2.5] and [RV20, Notation 5.1.6]),

- composition maps are induced by those of $\mathfrak{S}_m X$.

It comes with a natural inclusion $d_X: \partial\mathfrak{S}_m X \rightarrow \mathfrak{S}_m X$. This construction extends to a functor

$$\partial\mathfrak{S}_m: sSet \rightarrow sSet^{\Theta^{\text{op}}}\text{-Cat}.$$

The following lemma describes the pushout $\partial\mathfrak{S}_m Y \amalg_{\partial\mathfrak{S}_m X} \mathfrak{S}_m X$.

Lemma 2.3.3. *Let $f: X \rightarrow Y$ be a monomorphism in $sSet^{\Theta^{\text{op}}}$, and consider the following pushout in $sSet^{\Theta^{\text{op}}}\text{-Cat}$.*

$$\begin{array}{ccc} \partial\mathfrak{S}_m X & \xrightarrow{d_X} & \mathfrak{S}_m X \\ \partial\mathfrak{S}_m f \downarrow & & \downarrow \Gamma \\ \partial\mathfrak{S}_m Y & \longrightarrow & \partial\mathfrak{S}_m Y \amalg_{\partial\mathfrak{S}_m X} \mathfrak{S}_m X \end{array}$$

Then $\partial\mathfrak{S}_m Y \amalg_{\partial\mathfrak{S}_m X} \mathfrak{S}_m X$ is the directed $sSet^{\Theta^{\text{op}}}$ -enriched category such that

- its set of objects $\text{Ob}(\partial\mathfrak{S}_m Y \amalg_{\partial\mathfrak{S}_m X} \mathfrak{S}_m X)$ is $\{0, 1, \dots, m\}$,
- for $0 \leq i < j \leq m$, its hom Θ -space $\text{Hom}_{\partial\mathfrak{S}_m Y \amalg_{\partial\mathfrak{S}_m X} \mathfrak{S}_m X}(i, j)$ is given by

$$\text{Hom}_{\partial\mathfrak{S}_m Y \amalg_{\partial\mathfrak{S}_m X} \mathfrak{S}_m X}(i, j) = \begin{cases} \text{Hom}_{\partial\mathfrak{S}_m Y}(i, j) = \Delta[1]^{\times(j-i-1)} \times Y^{\times(j-i)} & \text{if } 0 < j - i < m \\ \text{Hom}_{\partial\mathfrak{S}_m Y \amalg_{\partial\mathfrak{S}_m X} \mathfrak{S}_m X}(0, m) & \text{if } i = 0, j = m, \end{cases}$$

where $\text{Hom}_{\partial\mathfrak{S}_m Y \amalg_{\partial\mathfrak{S}_m X} \mathfrak{S}_m X}(0, m)$ is given by the following pushout in $sSet^{\Theta^{\text{op}}}$,

$$\begin{array}{ccc} \partial(\Delta[1]^{\times(m-1)}) \times X^{\times m} & \xrightarrow{\iota \times \text{id}_X^{\times m}} & \Delta[1]^{\times(m-1)} \times X^{\times m} \\ \text{id} \times f^{\times m} \downarrow & & \downarrow \varphi_X(f) \\ \partial(\Delta[1]^{\times(m-1)}) \times Y^{\times m} & \xrightarrow{\varphi_Y} & \text{Hom}_{\partial\mathfrak{S}_m Y \amalg_{\partial\mathfrak{S}_m X} \mathfrak{S}_m X}(0, m) \end{array}$$

- composition maps are induced by those of $\partial\mathfrak{S}_m Y$.

Proof. Let \mathcal{P} be the $sSet^{\Theta^{\text{op}}}$ -enriched category whose objects, hom Θ -spaces, and structure is as described in the statement, and we show that \mathcal{P} satisfies the universal property of the desired pushout. For this, we show that there is a unique $sSet^{\Theta^{\text{op}}}$ -enriched functor $L: \mathcal{P} \rightarrow \mathcal{C}$ making the following diagram commute.

$$\begin{array}{ccc} \partial\mathfrak{S}_m X & \xrightarrow{d_X} & \mathfrak{S}_m X \\ \partial\mathfrak{S}_m f \downarrow & & \downarrow G \\ \partial\mathfrak{S}_m Y & \xrightarrow{F} & \mathcal{P} \\ & \searrow H & \downarrow L \\ & & \mathcal{C} \end{array} \quad \begin{array}{c} \curvearrowright K \\ \downarrow \\ \curvearrowright \end{array}$$

For $0 \leq i \leq m$, we set $L(i) := H(i) = K(i)$, for $0 < j - i < m$, we set

$$L_{i,j} := H_{i,j}: \text{Hom}_{\mathcal{P}}(i, j) = \text{Hom}_{\partial \mathfrak{S}_m Y}(i, j) \rightarrow \text{Hom}_{\mathcal{C}}(L(i), L(j)),$$

and we set $L_{0,m}$ to be the unique map given by the universal property of the pushout, as in the following diagram.

$$\begin{array}{ccc} \text{Hom}_{\partial \mathfrak{S}_m X}(0, m) = \partial(\Delta[1]^{\times(m-1)}) \times X^{\times m} & \xrightarrow{\iota \times \text{id}_X^{\times m}} & \Delta[1]^{\times(m-1)} \times X^{\times m} = \text{Hom}_{\mathfrak{S}_m X}(0, m) \\ \text{id} \times f^{\times m} \downarrow & & \downarrow \varphi_X(f) \\ \text{Hom}_{\partial \mathfrak{S}_m Y}(0, m) = \partial(\Delta[1]^{\times(m-1)}) \times Y^{\times m} & \xrightarrow{\varphi_Y} & \text{Hom}_{\mathcal{P}}(0, m) \\ & \searrow H_{0,m} & \downarrow \lrcorner \\ & & \text{Hom}_{\mathcal{C}}(L(0), L(m)) \end{array}$$

$\begin{array}{c} \text{Hom}_{\mathfrak{S}_m X}(0, m) \\ \downarrow K_{0,m} \\ \text{Hom}_{\mathcal{C}}(L(0), L(m)) \end{array}$

The maps $L_{i,j}, L_{j,k}, L_{i,k}$ are compatible with composition for all $0 \leq i < j < k \leq m$ with $k - i < m$, since the corresponding maps of H do. Finally, the maps $L_{0,i}, L_{i,m}, L_{0,m}$ are also compatible with composition for all $0 \leq i \leq m$ by compatibility of K and H on $\partial \mathfrak{S}_m X$. Moreover, we have that L is the unique $s\text{Set}^{\Theta^{\text{op}}}$ -enriched functor with the desired properties. This shows that \mathcal{P} is the pushout $\mathcal{P} \cong \partial \mathfrak{S}_m Y \amalg_{\partial \mathfrak{S}_m X} \mathfrak{S}_m X$, and concludes the proof. \square

Lemma 2.3.4. *Let \mathcal{P} and \mathcal{Q} be directed $s\text{Set}^{\Theta^{\text{op}}}$ -enriched categories such that*

- *they have the same set of objects $\text{Ob } \mathcal{P} = \{0, 1, \dots, m\} = \text{Ob } \mathcal{Q}$,*
- *for $0 < j - i < m$, they have the same hom Θ -spaces $\text{Hom}_{\mathcal{P}}(i, j) = \text{Hom}_{\mathcal{Q}}(i, j)$.*

Let $F: \mathcal{P} \rightarrow \mathcal{Q}$ be a $s\text{Set}^{\Theta^{\text{op}}}$ -enriched functor such that

- *on objects, it is the identity at $\{0, 1, \dots, m\}$,*
- *for all $0 < j - i < m$, the map $F_{i,j}$ on hom Θ -spaces is the identity at $\text{Hom}_{\mathcal{P}}(i, j) = \text{Hom}_{\mathcal{Q}}(i, j)$.*

Then the following is a pushout in $s\text{Set}^{\Theta^{\text{op}}}$ -Cat.

$$\begin{array}{ccc} \Sigma \text{Hom}_{\mathcal{P}}(0, m) & \xrightarrow{\iota_{0,m}} & \mathcal{P} \\ \Sigma F_{0,m} \downarrow & & \downarrow F \\ \Sigma \text{Hom}_{\mathcal{Q}}(0, m) & \xrightarrow{\iota_{0,m}} & \mathcal{Q} \end{array}$$

Moreover, if $F_{0,m}$ is a (trivial) cofibration in $s\text{Set}_S^{\Theta^{\text{op}}}$, then $F: \mathcal{P} \rightarrow \mathcal{Q}$ is a (trivial) cofibration in $s\text{Set}_S^{\Theta^{\text{op}}}$ -Cat.

Proof. In order to show that \mathcal{Q} satisfies the universal property of the desired pushout, we show that there is a unique $s\text{Set}^{\Theta^{\text{op}}}$ -enriched functor $H: \mathcal{Q} \rightarrow \mathcal{C}$ making the following diagram commute.

$$\begin{array}{ccc} \Sigma \text{Hom}_{\mathcal{P}}(0, m) & \xrightarrow{\iota_{0,m}} & \mathcal{P} \\ \Sigma F_{0,m} \downarrow & & \downarrow F \\ \Sigma \text{Hom}_{\mathcal{Q}}(0, m) & \xrightarrow{\iota_{0,m}} & \mathcal{Q} \end{array}$$

$\begin{array}{c} \mathcal{P} \\ \downarrow F \\ \mathcal{Q} \\ \downarrow H \\ \mathcal{C} \end{array}$

g

For $0 \leq i \leq m$, we set $H(i) := G(i)$, for $0 < j - i < m$, we set

$$H_{i,j} := G_{i,j}: \text{Hom}_{\mathcal{Q}}(i, j) = \text{Hom}_{\mathcal{P}}(i, j) \rightarrow \text{Hom}_{\mathcal{C}}(G(i), G(j)),$$

and we set

$$H_{0,m} := g: \text{Hom}_{\mathcal{Q}}(0, m) \rightarrow \text{Hom}_{\mathcal{C}}(G(0), G(m)).$$

The maps $H_{i,j}, H_{j,k}, H_{i,k}$ are compatible with composition for all $0 \leq i < j < k \leq m$ with $k - i < m$ since the corresponding maps of G do. It remains to show that $H_{0,i}, H_{i,m}, H_{0,m}$ are compatible with composition for all $0 \leq i \leq m$. For $0 \leq i \leq m$ we have that the following diagram commutes,

$$\begin{array}{ccc} \text{Hom}_{\mathcal{Q}}(0, i) \times \text{Hom}_{\mathcal{Q}}(i, m) & \xrightarrow{\circ_{0,i,m}} & \text{Hom}_{\mathcal{Q}}(0, m) \\ \parallel & & \nearrow F_{0,m} \\ \text{Hom}_{\mathcal{P}}(0, i) \times \text{Hom}_{\mathcal{P}}(i, m) & \xrightarrow{\circ_{0,i,m}} & \text{Hom}_{\mathcal{P}}(0, m) \\ \downarrow H_{0,i} \times H_{i,m} = G_{0,i} \times G_{i,m} & & \searrow G_{0,m} \\ \text{Hom}_{\mathcal{C}}(G(0), G(i)) \times \text{Hom}_{\mathcal{C}}(G(i), G(m)) & \xrightarrow{\circ_{G(0),G(i),G(m)}} & \text{Hom}_{\mathcal{C}}(G(0), G(m)) \end{array} \quad \begin{array}{c} \downarrow g = H_{0,m} \\ \downarrow \end{array}$$

where the top rectangle commutes by compatibility of F with composition, the bottom one by compatibility of G with composition, and the right-hand triangle since $G \circ \iota_{0,m} = g \circ \Sigma F_{0,m}$. This shows that $H_{0,i}, H_{i,m}, H_{0,m}$ are compatible with composition for all $0 \leq i \leq m$. Moreover, we have that H is the unique $s\text{Set}^{\Theta^{\text{op}}}$ -enriched functor with the desired properties. This shows that \mathcal{Q} is the pushout

$$\mathcal{Q} \cong \mathcal{P} \amalg_{\Sigma \text{Hom}_{\mathcal{P}}(0,m)} \Sigma \text{Hom}_{\mathcal{Q}}(0, m).$$

The ‘‘moreover’’ part follows directly from the facts that, if $F_{0,m}$ is a (trivial) cofibration in $s\text{Set}_{\mathcal{S}}^{\Theta^{\text{op}}}$, then $\Sigma F_{0,m}$ is a (trivial) cofibration in $s\text{Set}_{\mathcal{S}}^{\Theta^{\text{op}}}\text{-Cat}$ by Proposition 1.3.1, and that (trivial) cofibrations are closed under pushout. \square

The following lemmas analyze certain canonical maps of $s\text{Set}^{\Theta^{\text{op}}}$ -enriched categories of the form $\partial \mathfrak{S}_m Y \amalg_{\partial \mathfrak{S}_m X} \mathfrak{S}_m X \rightarrow \mathfrak{S}_m Y$.

Lemma 2.3.5. *Let $f: X \rightarrow Y$ be a (trivial) cofibration in $s\text{Set}_{\mathcal{S}}^{\Theta^{\text{op}}}$. Then the map*

$$I_{0,m}: \text{Hom}_{\partial \mathfrak{S}_m Y \amalg_{\partial \mathfrak{S}_m X} \mathfrak{S}_m X}(0, m) \rightarrow \text{Hom}_{\mathfrak{S}_m Y}(0, m)$$

is a (trivial) cofibration in $s\text{Set}_{\mathcal{S}}^{\Theta^{\text{op}}}$.

Proof. The map $I_{0,m}$ is obtained as the unique map given by the universal property of pushouts as in the following diagram in $s\text{Set}^{\Theta^{\text{op}}}$,

$$\begin{array}{ccc} \partial(\Delta[1]^{\times(m-1)}) \times X^{\times m} & \xrightarrow{\iota \times \text{id}_X^{\times m}} & \Delta[1]^{\times(m-1)} \times X^{\times m} \\ \text{id} \times f^{\times m} \downarrow & & \downarrow \varphi_X(f) \\ \partial(\Delta[1]^{\times(m-1)}) \times Y^{\times m} & \xrightarrow{\varphi_Y} & \text{Hom}_{\partial \mathfrak{S}_m Y \amalg_{\partial \mathfrak{S}_m X} \mathfrak{S}_m X}(0, m) \\ & & \downarrow I_{0,m} \\ & & \Delta[1]^{\times(m-1)} \times Y^{\times m} \end{array} \quad \begin{array}{c} \searrow \text{id} \times f^{\times m} \\ \searrow \end{array}$$

where the pushout is given by Lemma 2.3.3. Note that each non-dashed map of the diagram is a monomorphism, either by definition or because it is a pushout of a monomorphism.

We first show that $I_{0,m}$ is a monomorphism, hence a cofibration in $sSet_S^{\Theta^{op}}$. To this end, suppose we are given $(\sigma, \vec{x}) \in \Delta[1]^{\times(m-1)} \times X^{\times m}$ and $(\tau, \vec{y}) \in \partial(\Delta[1]^{\times(m-1)}) \times Y^{\times m}$ such that

$$(\text{id} \times f^{\times m})(\sigma, \vec{x}) = (\sigma, f\vec{x}) = (\tau, \vec{y}) = (\iota \times \text{id}_Y^{\times m})(\tau, \vec{y}) \in \Delta[1]^{\times(m-1)} \times Y^{\times m}.$$

Then, if we consider $(\tau, \vec{x}) \in \partial(\Delta[1]^{\times(m-1)}) \times X^{\times m}$, we have that

$$(\iota \times \text{id}_X^{\times m})(\tau, \vec{x}) = (\sigma, \vec{x}) \quad \text{and} \quad (\text{id} \times f^{\times m})(\tau, \vec{x}) = (\tau, \vec{y}).$$

Hence (σ, \vec{x}) and (τ, \vec{y}) define equal objects in the pushout $\text{Hom}_{\partial\mathfrak{S}_m Y \amalg_{\partial\mathfrak{S}_m X} \mathfrak{S}_m X}(0, m)$. Since f and ι are monomorphisms, this is sufficient to conclude that $I_{0,m}$ is a monomorphism.

Now, if f is a trivial cofibration in $sSet_S^{\Theta^{op}}$, given that $sSet_S^{\Theta^{op}}$ is cartesian closed, the maps $\text{id} \times f^{\times m}$ are also trivial cofibrations. Hence $\varphi_X(f)$ is also a trivial cofibration in $sSet_S^{\Theta^{op}}$ as a pushout of such. It follows by 2-out-of-3 applied to $\text{id} \times f^{\times m} = I_{0,m} \circ \varphi_X(f)$ that $I_{0,m}$ is also a weak equivalence and hence a trivial cofibration in $sSet_S^{\Theta^{op}}$. \square

Lemma 2.3.6. *If the map $f: X \rightarrow Y$ is a (trivial) cofibration in $sSet_S^{\Theta^{op}}$, then the $sSet_S^{\Theta^{op}}$ -enriched functor*

$$I: \partial\mathfrak{S}_m Y \amalg_{\partial\mathfrak{S}_m X} \mathfrak{S}_m X \rightarrow \mathfrak{S}_m Y$$

is a (trivial) cofibration in $sSet_S^{\Theta^{op}}\text{-Cat}$.

Proof. Let $f: X \rightarrow Y$ be a (trivial) cofibration in $sSet_S^{\Theta^{op}}$. Using Lemma 2.3.3, we have that the map I satisfies the hypotheses of Lemma 2.3.4. Then, by Lemma 2.3.5, the induced map

$$I_{0,m}: \text{Hom}_{\partial\mathfrak{S}_m Y \amalg_{\partial\mathfrak{S}_m X} \mathfrak{S}_m X}(0, m) \rightarrow \text{Hom}_{\mathfrak{S}_m Y}(0, m)$$

is a (trivial) cofibration in $sSet_S^{\Theta^{op}}$. Hence, we conclude from the ‘‘moreover’’ part of Lemma 2.3.4 that $I: \partial\mathfrak{S}_m Y \amalg_{\partial\mathfrak{S}_m X} \mathfrak{S}_m X \rightarrow \mathfrak{S}_m Y$ is a (trivial) cofibration in $sSet_S^{\Theta^{op}}\text{-Cat}$. \square

We can now prove the theorem.

Proof of Theorem 2.3.1. To show that \mathfrak{C} is left Quillen, it is sufficient to show that \mathfrak{C} sends generating (trivial) cofibrations of $(sSet_S^{\Theta^{op}})_{\text{inj}}^{\Delta^{op}}$ to (trivial) cofibrations in $sSet_S^{\Theta^{op}}\text{-Cat}$.

Recall from Remark 1.3.3 that a generating (trivial) cofibration of $(sSet_S^{\Theta^{op}})_{\text{inj}}^{\Delta^{op}}$ is of the form

$$\partial F[m] \times Y \amalg_{\partial F[m] \times X} F[m] \times X \rightarrow F[m] \times Y$$

for $m \geq 0$ and $X \rightarrow Y$ a (trivial) cofibration in $sSet_S^{\Theta^{op}}$. Since \mathfrak{C} preserves colimits, this map is sent to the $sSet_S^{\Theta^{op}}$ -enriched functor

$$\mathfrak{C}(\partial F[m] \times Y) \amalg_{\mathfrak{C}(\partial F[m] \times X)} \mathfrak{C}(F[m] \times X) \rightarrow \mathfrak{C}(F[m] \times Y).$$

We first compute $\mathfrak{C}(F[m] \times X)$:

$$\begin{aligned} \mathfrak{C}(F[m] \times X) &\cong \mathfrak{C}(F[m] \times \text{colim}_{\Theta[\theta] \times \Delta[k] \rightarrow X} (\Theta[\theta] \times \Delta[k])) \\ &\cong \mathfrak{C}(\text{colim}_{\Theta[\theta] \times \Delta[k] \rightarrow X} (F[m] \times \Theta[\theta] \times \Delta[k])) && \times \text{preserves colimits} \\ &\cong \text{colim}_{\Theta[\theta] \times \Delta[k] \rightarrow X} \mathfrak{C}(F[m] \times \Theta[\theta] \times \Delta[k]) && \mathfrak{C} \text{ preserves colimits} \\ &\cong \text{colim}_{\Theta[\theta] \times \Delta[k] \rightarrow X} \mathfrak{S}_m(\Theta[\theta] \times \Delta[k]) && \text{definition of } \mathfrak{C} \\ &\cong \mathfrak{S}_m(\text{colim}_{\Theta[\theta] \times \Delta[k] \rightarrow X} (\Theta[\theta] \times \Delta[k])) && \text{Remark 2.1.3} \\ &\cong \mathfrak{S}_m X. \end{aligned}$$

Similarly, using Notation 2.3.2 we compute $\mathfrak{C}(\partial F[m] \times X)$:

$$\begin{aligned} \mathfrak{C}(\partial F[m] \times X) &= \mathfrak{C}(\text{coeq}(\coprod_{0 \leq i < j \leq m} F[m-2] \rightrightarrows \coprod_{0 \leq i \leq m} F[m-1]) \times X) \\ &\cong \mathfrak{C}(\text{coeq}(\coprod_{0 \leq i < j \leq m} F[m-2] \times X \rightrightarrows \coprod_{0 \leq i \leq m} F[m-1] \times X)) \\ &\cong \text{coeq}(\coprod_{0 \leq i < j \leq m} \mathfrak{C}(F[m-2] \times X) \rightrightarrows \coprod_{0 \leq i \leq m} \mathfrak{C}(F[m-1] \times X)) \\ &\cong \text{coeq}(\coprod_{0 \leq i < j \leq m} \mathfrak{S}_{m-2} X \rightrightarrows \coprod_{0 \leq i \leq m} \mathfrak{S}_{m-1} X) \\ &= \partial \mathfrak{S}_m X. \end{aligned}$$

All-in-all, this says that the image under \mathfrak{C} of the desired map is the map

$$\partial \mathfrak{S}_m Y \amalg_{\partial \mathfrak{S}_m X} \mathfrak{S}_m X \rightarrow \mathfrak{S}_m Y,$$

which is a (trivial) cofibration in $s\text{Set}_S^{\Theta^{\text{op}}}$ -Cat by Lemma 2.3.6. This concludes the proof. \square

Theorem 2.3.7. *The adjunction*

$$s\text{Set}_S^{\Theta^{\text{op}}}\text{-Cat} \xleftarrow[\mathfrak{N}]{\mathfrak{C}} (s\text{Set}_S^{\Theta^{\text{op}}})_{\text{Seg}}^{\Delta^{\text{op}}}$$

is a Quillen pair.

Proof. Recall that $(s\text{Set}_S^{\Theta^{\text{op}}})_{\text{Seg}}^{\Delta^{\text{op}}}$ is the left Bousfield localization of $(s\text{Set}_S^{\Theta^{\text{op}}})_{\text{inj}}^{\Delta^{\text{op}}}$ with respect to the set Seg_{Θ} . Since $\mathfrak{C}: (s\text{Set}_S^{\Theta^{\text{op}}})_{\text{inj}}^{\Delta^{\text{op}}} \rightarrow s\text{Set}_S^{\Theta^{\text{op}}}\text{-Cat}$ is left Quillen by Theorem 2.3.1, by [Hir03, Theorem 3.3.20(1)(a)] it is enough to show that \mathfrak{C} sends maps in Seg_{Θ} to weak equivalences in $s\text{Set}_S^{\Theta^{\text{op}}}\text{-Cat}$. Now recall that a map in Seg_{Θ} is of the form

$$F[1] \times \Theta[\theta] \amalg_{F[0] \times \Theta[\theta]} \dots \amalg_{F[0] \times \Theta[\theta]} F[1] \times \Theta[\theta] \rightarrow F[m] \times \Theta[\theta]$$

for $m \geq 1$ and $\theta \in \Theta$. Since \mathfrak{C} preserves colimits, it is sent to the $s\text{Set}^{\Theta^{\text{op}}}$ -enriched functor

$$\mathfrak{C}(F[1] \times \Theta[\theta]) \amalg_{\mathfrak{C}(F[0] \times \Theta[\theta])} \dots \amalg_{\mathfrak{C}(F[0] \times \Theta[\theta])} \mathfrak{C}(F[1] \times \Theta[\theta]) \rightarrow \mathfrak{C}(F[m] \times \Theta[\theta]).$$

We have that

$$\mathfrak{C}(F[1] \times \Theta[\theta]) = \mathfrak{S}_1 \Theta[\theta] = \Sigma \Theta[\theta] \quad \text{and} \quad \mathfrak{C}(F[0] \times \Theta[\theta]) = [0],$$

and for $m > 1$ we have that

$$\mathfrak{C}(F[m] \times \Theta[\theta]) = \mathfrak{S}_m(\Theta[\theta]).$$

Using the identification from Proposition 1.1.4, we see that the above map is a $s\text{Set}^{\Theta^{\text{op}}}$ -enriched functor of the form

$$\Sigma_m \Theta[\theta] \cong \Sigma \Theta[\theta] \amalg_{[0]} \dots \amalg_{[0]} \Sigma \Theta[\theta] \rightarrow \mathfrak{S}_m \Theta[\theta].$$

In fact, it is precisely the $s\text{Set}^{\Theta^{\text{op}}}$ -enriched functor from Proposition 2.1.5 when taking $X = \Theta[\theta]$, which is a weak equivalence in $s\text{Set}_S^{\Theta^{\text{op}}}\text{-Cat}$, as desired. \square

Remark 2.3.8. Let \mathcal{C} be a fibrant $s\text{Set}_S^{\Theta^{\text{op}}}$ -enriched category. There is a canonical map $N\mathcal{C} \rightarrow \mathfrak{N}\mathcal{C}$ induced by the maps

$$\mathfrak{S}_m(\Theta[\theta] \times \Delta[k]) \rightarrow \Sigma_m(\Theta[\theta] \times \Delta[k])$$

of Remark 2.1.4, for $m \geq 0$, $\theta \in \Theta$, and $k \geq 0$. At $m = 0, 1$ this map induces equalities

$$(N\mathcal{C})_0 = \text{Ob } \mathcal{C} = (\mathfrak{N}\mathcal{C})_0 \quad \text{and} \quad (N\mathcal{C})_1 = \text{Mor } \mathcal{C} = (\mathfrak{N}\mathcal{C})_1.$$

Given $m > 1$, there is a commutative diagram in $s\text{Set}_S^{\Theta^{\text{op}}}$

$$\begin{array}{ccc} (N\mathcal{C})_m & \xrightarrow{\cong} & (N\mathcal{C})_1 \times_{(N\mathcal{C})_0}^{(h)} \dots \times_{(N\mathcal{C})_0}^{(h)} (N\mathcal{C})_1 \\ \downarrow \mathbb{R} & & \downarrow \mathbb{R} \\ (\mathfrak{N}\mathcal{C})_m & \xrightarrow{\simeq} & (\mathfrak{N}\mathcal{C})_1 \times_{(\mathfrak{N}\mathcal{C})_0}^{(h)} \dots \times_{(\mathfrak{N}\mathcal{C})_0}^{(h)} (\mathfrak{N}\mathcal{C})_1 \end{array}$$

where the pullbacks are homotopy pullbacks because they are taken over the discrete object $\text{Ob } \mathcal{C}$ (see [BR13, §4.1]). Note that the horizontal maps are weak equivalences in $s\text{Set}_S^{\Theta^{\text{op}}}$ by Remark 1.2.3 and Theorem 2.3.7 and therefore so is the left-hand map by 2-out-of-3. Since the presheaves $(NC)_m$ and $(\mathfrak{N}\mathcal{C})_m$ are by Proposition 1.3.6 and Theorem 2.3.1 fibrant in $s\text{Set}_S^{\Theta^{\text{op}}}$, the map $(NC)_m \rightarrow (\mathfrak{N}\mathcal{C})_m$ is in fact a weak equivalence in $(s\text{Set}_{(\infty,0)})_{\text{inj}}^{\Theta^{\text{op}}}$. This shows that $NC \rightarrow \mathfrak{N}\mathcal{C}$ is a weak equivalence in $(s\text{Set}_{(\infty,0)})_{\text{inj}}^{\Delta^{\text{op}} \times \Theta^{\text{op}}}$.

We have then shown the following.

Proposition 2.3.9. *Given a fibrant $s\text{Set}_S^{\Theta^{\text{op}}}$ -enriched category \mathcal{C} , the natural canonical map*

$$NC \rightarrow \mathfrak{N}\mathcal{C}$$

is a weak equivalence in $(s\text{Set}_{(\infty,0)})_{\text{inj}}^{\Delta^{\text{op}} \times \Theta^{\text{op}}}$, and so a weak equivalence in $(s\text{Set}_S^{\Theta^{\text{op}}})_{\text{inj}}^{\Delta^{\text{op}}}$.

We now describe the homotopical limitations of the homotopy coherent nerve \mathfrak{N} .

Notation 2.3.10. For $k \geq 0$, let $I[k]$ denote the contractible groupoid on $k + 1$ objects, i.e., it is the category with $k + 1$ objects and a unique isomorphism between any two objects. We write $E[k] \in \text{Set}^{\Delta^{\text{op}}}$ for the nerve of $I[k]$. Since $\text{Set}^{\Delta^{\text{op}}}$ is naturally included in $\text{Set}^{\Theta^{\text{op}} \times \Delta^{\text{op}}}$ and $s\text{Set}^{\Theta^{\text{op}} \times \Delta^{\text{op}}}$, we regard $E[k]$ as an object of those categories without further specification.

2.3.11. Denote by $(s\text{Set}_S^{\Theta^{\text{op}}})_{\text{CS}}^{\Delta^{\text{op}}}$ the left Bousfield localization of the model structure $(s\text{Set}_S^{\Theta^{\text{op}}})_{\text{Seg}}^{\Delta^{\text{op}}}$ at the set of *constantness maps*

$$\text{Cst}_{\Theta} = \{F[0] \times (\Theta[\theta] \rightarrow \Theta[\theta']) \mid \theta \rightarrow \theta' \in \Theta\}.$$

Denote by $(s\text{Set}_S^{\Theta^{\text{op}}})_{\text{cCS}}^{\Delta^{\text{op}}}$ the left Bousfield localization of the model structure $(s\text{Set}_S^{\Theta^{\text{op}}})_{\text{CS}}^{\Delta^{\text{op}}}$ at the set of *completeness maps*

$$\text{Cpt}_{\Theta} = \{(F[0] \rightarrow E[1]) \times \Theta[\theta] \mid \theta \in \Theta\}.$$

We suggestively call the fibrant objects of the model structure $(s\text{Set}_S^{\Theta^{\text{op}}})_{\text{CS}}^{\Delta^{\text{op}}}$ *pre-complete Segal* objects in S -local Θ -spaces, while the fibrant objects of the model structure $(s\text{Set}_S^{\Theta^{\text{op}}})_{\text{cCS}}^{\Delta^{\text{op}}}$ are *complete Segal* objects in S -local Θ -spaces, as we now make explicit.

Definition 2.3.12. An object $X \in s\text{Set}^{\Theta^{\text{op}} \times \Delta^{\text{op}}}$ is a *pre-complete Segal object in S -local Θ -spaces* if X is a Segal object, as defined in Definition 1.3.5, and the map

$$X_{0,\theta'} \rightarrow X_{0,\theta}$$

is a weak equivalence in $s\text{Set}_{(\infty,0)}$, for all maps $\theta \rightarrow \theta' \in \Theta$. Such an X is called *complete* if, in addition, the map

$$X_0 \rightarrow X_1^{\text{heq}} := \text{Hom}_{s\text{Set}^{\Theta^{\text{op}} \times \Delta^{\text{op}}}}(E[1], X)$$

is a weak equivalence in $s\text{Set}_S^{\Theta^{\text{op}}}$. Here $\text{Hom}_{s\text{Set}^{\Theta^{\text{op}} \times \Delta^{\text{op}}}}(-, -)$ denotes the hom Θ -space functor of $s\text{Set}^{\Theta^{\text{op}} \times \Delta^{\text{op}}}$.

2.3.13. The first instance of this framework in the literature is the case where Θ is the terminal category, and $S = \emptyset$. The resulting model structure $(s\text{Set}_{(\infty,0)})_{\text{cCS}}^{\Delta^{\text{op}}}$ is precisely Rezk's cartesian closed model structure from [Rez01, Theorem 7.2], in which the fibrant objects are the *complete Segal spaces*, which model $(\infty, 1)$ -categories. The idea was later generalized in [Rez10] to the case where Θ is Joyal's cell category Θ_{n-1} from [Joy97] and S is the set $S_{(\infty, n-1)}$, for $n \geq 1$, which are recursively defined as follows.

For $n = 1$, Θ_{n-1} is the terminal category and $S_{(\infty, 0)}$ is the empty set, as mentioned above, and for $n > 1$, Θ_{n-1} is the *wreath product* $\Delta \wr \Theta_{n-2}$ (see e.g. [Ber07a, Definition 3.1]) and the set $S_{(\infty, n-1)}$ consists of the following monomorphisms:

- the *Segal maps*

$$\Theta_{n-1}[1](\theta_1) \sqcup_{[0]} \dots \sqcup_{[0]} \Theta_{n-1}[1](\theta_l) \rightarrow \Theta_{n-1}[l](\theta_1, \dots, \theta_l),$$

for all $l \geq 1$ and $\theta_1, \dots, \theta_l \in \Theta_{n-2}$,

- the *completeness map*

$$F[0] \rightarrow E[1]$$

seen as a map in $\mathcal{S}et^{\Theta_{n-1}^{\text{op}}}$ through the inclusion $\mathcal{S}et^{\Delta^{\text{op}}} \rightarrow \mathcal{S}et^{\Theta_{n-1}^{\text{op}}}$ induced by pre-composition along the projection $\Theta_{n-1} \rightarrow \Delta$ given by $[l](\theta_1, \dots, \theta_l) \mapsto [l]$,

- the *recursive maps*

$$\Theta_{n-1}[1](A) \rightarrow \Theta_{n-1}[1](B),$$

where $A \rightarrow B \in s\mathcal{S}et^{\Theta_{n-2}^{\text{op}}}$ ranges over all monomorphisms in $S_{(\infty, n-2)}$.

Note that by [Rez10, Theorem 8.1] the model structure $s\mathcal{S}et_{(\infty, n-1)}^{\Theta_{n-1}^{\text{op}}}$ obtained by localizing the injective model structure $(s\mathcal{S}et_{(\infty, 0)}^{\text{inj}})^{\Theta_{n-1}^{\text{op}}}$ with respect to the set $S_{(\infty, n-1)}$ is cartesian closed. Moreover, for $n = 2$, we have that $s\mathcal{S}et_{(\infty, 1)}^{\Delta^{\text{op}}}$ is again Rezk's model structure for complete Segal spaces.

The resulting model structure $(s\mathcal{S}et_{(\infty, n-1)}^{\Theta_{n-1}^{\text{op}}})_{\text{cCS}}^{\Delta^{\text{op}}}$ is precisely Bergner–Rezk's model structure from [BR20, Theorem 5.6], in which the fibrant objects are the *complete Segal objects in Θ_{n-1} -spaces*, which model (∞, n) -categories. This notion of completeness is inspired by the one featuring in Barwick's definition of *n-fold complete Segal spaces* [Bar05]. Complete Segal objects in a more general setting have also been studied by Bergner–Rezk in [BR20, §5] and by the second author in [Ras21, §§1.4–1.5].

Proposition 2.3.14. *The adjunction*

$$s\mathcal{S}et_S^{\Theta^{\text{op}}}\text{-Cat} \begin{array}{c} \xleftarrow{\mathfrak{C}} \\ \perp \\ \xrightarrow{\mathfrak{N}} \end{array} (s\mathcal{S}et_S^{\Theta^{\text{op}}})_{\text{cCS}}^{\Delta^{\text{op}}}$$

is a Quillen pair.

Proof. Recall that $(s\mathcal{S}et_S^{\Theta^{\text{op}}})_{\text{cCS}}^{\Delta^{\text{op}}}$ is the left Bousfield localization of $(s\mathcal{S}et_S^{\Theta^{\text{op}}})_{\text{Seg}}^{\Delta^{\text{op}}}$ with respect to the set Cst_{Θ} . Since $\mathfrak{C}: (s\mathcal{S}et_S^{\Theta^{\text{op}}})_{\text{Seg}}^{\Delta^{\text{op}}} \rightarrow s\mathcal{S}et_S^{\Theta^{\text{op}}}\text{-Cat}$ is left Quillen by Theorem 2.3.7, by [Hir03, Theorem 3.3.20(1)(a)] it is enough to show that \mathfrak{C} sends maps in Cst_{Θ} to weak equivalences in $s\mathcal{S}et_S^{\Theta^{\text{op}}}\text{-Cat}$. For this, recall that a map in Cst_{Θ} is of the form $F[0] \times \Theta[\theta] \rightarrow F[0] \times \Theta[\theta']$ for a morphism $\theta \rightarrow \theta' \in \Theta$. Such a map is sent by \mathfrak{C} to the identity at $[0]$ and hence is a weak equivalence in $s\mathcal{S}et_S^{\Theta^{\text{op}}}\text{-Cat}$. \square

Remark 2.3.15. Let \mathcal{C} be a fibrant $s\mathcal{S}et_S^{\Theta^{\text{op}}}$ -enriched category. Combining Proposition 2.3.9 and Theorem 2.3.7, we obtain that the homotopy coherent nerve $\mathfrak{N}\mathcal{C}$ provides a fibrant replacement of $\mathcal{N}\mathcal{C}$ in $(s\mathcal{S}et_S^{\Theta^{\text{op}}})_{\text{Seg}}^{\Delta^{\text{op}}}$ and in $(s\mathcal{S}et_S^{\Theta^{\text{op}}})_{\text{cCS}}^{\Delta^{\text{op}}}$.

Remark 2.3.16. The adjunction

$$s\mathcal{S}et_S^{\Theta^{\text{op}}}\text{-Cat} \begin{array}{c} \xleftarrow{\mathfrak{C}} \\ \perp \\ \xrightarrow{\mathfrak{N}} \end{array} (s\mathcal{S}et_S^{\Theta^{\text{op}}})_{\text{cCS}}^{\Delta^{\text{op}}}$$

is *not* a Quillen pair. Indeed, given a fibrant $s\mathcal{S}et_S^{\Theta^{\text{op}}}$ -enriched category \mathcal{C} , Example 2.3.17 shows that the homotopy coherent nerve $\mathfrak{N}\mathcal{C}$ is generally not fibrant in $(s\mathcal{S}et_S^{\Theta^{\text{op}}})_{\text{cCS}}^{\Delta^{\text{op}}}$.

Example 2.3.17. The $s\mathcal{S}et^{\Theta^{\text{op}}}$ -enriched category $I[1]$ is fibrant in $s\mathcal{S}et_S^{\Theta^{\text{op}}}\text{-Cat}$, however its homotopy coherent nerve $\mathfrak{N}I[1]$ – as well as its strict nerve $NI[1]$ – is not fibrant in $(s\mathcal{S}et_S^{\Theta^{\text{op}}})_{\text{cCS}}^{\Delta^{\text{op}}}$. We show this for $\mathfrak{N}I[1]$, and the same argument works for $NI[1]$, given that they agree in components 0 and 1. To this end, we first observe that the model structure $(s\mathcal{S}et_S^{\Theta^{\text{op}}})_{\text{cCS}}^{\Delta^{\text{op}}}$ is simplicial as a consequence of [Mos19, Theorem 5.4] and [Hir03, Theorem 4.1.1], and we denote by $\text{Map}(-, -)$ its

hom space functor. Now, the map $0: F[0] \rightarrow E[1]$ is a trivial cofibration in $(sSet_S^{\Theta^{op}})_{cCS}^{\Delta^{op}}$, but we argue that the map

$$0^*: \text{Map}(E[1], \mathfrak{N}\mathcal{I}[1]) \rightarrow \text{Map}(F[0], \mathfrak{N}\mathcal{I}[1]) \cong (\mathfrak{N}\mathcal{I}[1])_{0, \top} = \{0, 1\}$$

can not be a trivial fibration in $sSet_{(\infty, 0)}$. If it were, there would be a lift in the following diagram in $sSet$,

$$\begin{array}{ccc} \partial\Delta[1] & \longrightarrow & \text{Map}(E[1], \mathfrak{N}\mathcal{I}[1]) \\ \downarrow & \dashrightarrow & \downarrow 0^* \\ \Delta[1] & \xrightarrow{0} & (\mathfrak{N}\mathcal{I}[1])_{0, \top} = \{0, 1\} \end{array}$$

where the top map picks two maps $E[1] = \mathcal{N}\mathcal{I}[1] \rightarrow \mathfrak{N}\mathcal{I}[1]$ given by the constant map at 0 and the component at $I[1]$ of the canonical fibrant replacement map of Proposition 2.3.9, respectively. This lift could be used to produce a 1-simplex

$$\begin{array}{ccc} \Delta[1] & \dashrightarrow & \text{Map}(E[1], \mathfrak{N}\mathcal{I}[1]) \xrightarrow{1^*} (\mathfrak{N}\mathcal{I}[1])_{0, \top} = \{0, 1\} \\ \uparrow & \nearrow & \\ \partial\Delta[1] & & \end{array} \quad (0, 1)$$

between 0 and 1 in the discrete space $(\mathfrak{N}\mathcal{I}[1])_{0, \top}$, but this is impossible.

3. THE COMPLETE HOMOTOPY COHERENT NERVE AND ITS PROPERTIES

In this section, we specialize the results from the previous sections to the case where Θ is Joyal's cell category Θ_{n-1} from [Joy97] for some $n \geq 1$, and S is the set of maps $S_{(\infty, n-1)}$ from § 2.3.13.

The resulting model structure $sSet_{S_{(\infty, n-1)}}^{\Theta_{n-1}^{op}}$ agrees with Rezk's cartesian closed model structure on $sSet^{\Theta_{n-1}^{op}}$ for *complete Segal Θ_{n-1} -spaces* from [Rez10, Theorem 8.1]. Since this model structure is a model for the homotopy theory of $(\infty, n-1)$ -categories (in the sense of [BSP21]), we use for it the more appropriate notation $sSet_{(\infty, n-1)}^{\Theta_{n-1}^{op}}$. Consequently, we denote by $(sSet_{(\infty, n-1)}^{\Theta_{n-1}^{op}})_{cCS}^{\Delta^{op}}$ the model structure $(sSet_{S_{(\infty, n-1)}}^{\Theta_{n-1}^{op}})_{cCS}^{\Delta^{op}}$, and by $sSet_{(\infty, n-1)}^{\Theta_{n-1}^{op}}\text{-Cat}$ the model structure $sSet_{S_{(\infty, n-1)}}^{\Theta_{n-1}^{op}}\text{-Cat}$. A special feature of this setup is that the model structure $(sSet_{(\infty, n-1)}^{\Theta_{n-1}^{op}})_{cCS}^{\Delta^{op}}$ coincides with the left Bousfield localization of $(sSet_{(\infty, n-1)}^{\Theta_{n-1}^{op}})_{cS}^{\Delta^{op}}$ with respect to just the single map

$$\{F[0] \rightarrow E[1]\}.$$

This can be established with a similar argument to that of [JFS17, Lemma 2.8]; see also [BR20, Proposition 5.10].

In this context, we define in Section 3.1 a preliminary version of a complete homotopy coherent nerve construction $\overline{\mathfrak{N}}: sSet_{(\infty, n-1)}^{\Theta_{n-1}^{op}}\text{-Cat} \rightarrow \mathcal{S}et_{\Theta_{n-1}^{op} \times \Delta^{op}}$, which generalizes Cordier–Porter's nerve from [CP86], and use it to define our preferred complete homotopy coherent nerve construction $\mathfrak{N}^c: sSet_{(\infty, n-1)}^{\Theta_{n-1}^{op}}\text{-Cat} \rightarrow sSet_{\Theta_{n-1}^{op} \times \Delta^{op}}$.

After recalling the case $n = 1$ from the literature in Section 3.2, we study in Section 3.3 the homotopical properties of both nerves $\overline{\mathfrak{N}}$ and \mathfrak{N}^c , showing as Theorem 3.3.8 that \mathfrak{N}^c defines a right Quillen functor when considering on its target the model structure for complete Segal objects in $sSet_{(\infty, n-1)}^{\Theta_{n-1}^{op}}$ for $n > 1$.

Afterwards, building on preliminary work and background from Sections 3.4 and 3.5, we show in Section 3.6 that \mathfrak{N}^c gives a right Quillen equivalence between $sSet_{(\infty, n-1)}^{\Theta_{n-1}^{op}}\text{-Cat}$ and $(sSet_{(\infty, n-1)}^{\Theta_{n-1}^{op}})_{cCS}^{\Delta^{op}}$,

and that it can be used to construct a direct right Quillen equivalence $d_* \circ \mathfrak{N}^c$ between $sSet_{(\infty, n-1)}^{\Theta_{n-1}^{\text{op}}}$ -Cat and $sSet_{(\infty, n)}^{\Theta_n^{\text{op}}}$.

3.1. The complete homotopy coherent nerve(s). Fix $n \geq 1$. The definitions of the functors $\overline{\mathfrak{N}}: sSet^{\Theta_{n-1}^{\text{op}}}\text{-Cat} \rightarrow Set^{\Theta_{n-1}^{\text{op}} \times \Delta^{\text{op}}}$ and $\mathfrak{N}^c: sSet^{\Theta_{n-1}^{\text{op}}}\text{-Cat} \rightarrow sSet^{\Theta_{n-1}^{\text{op}} \times \Delta^{\text{op}}}$ rely on a few preliminary adjunctions, which we now describe.

The adjunctions

$$\Theta_{n-1} \times \Delta \xrightleftharpoons[\iota_0]{p} \Theta_{n-1} \quad \text{and} \quad \Delta \times \Theta_{n-1} \times \Delta \xrightleftharpoons[\iota_0]{p} \Delta \times \Theta_{n-1},$$

where $p(x, [k]) := x$ and $\iota_0(x) := (x, [0])$ for x an element of Θ_{n-1} or $\Delta \times \Theta_{n-1}$, respectively, induce by pre-composition adjunctions

$$sSet^{\Theta_{n-1}^{\text{op}}} \xrightleftharpoons[\iota_0^*]{p^*} Set^{\Theta_{n-1}^{\text{op}}} \quad \text{and} \quad sSet^{\Theta_{n-1}^{\text{op}} \times \Delta^{\text{op}}} \xrightleftharpoons[\iota_0^*]{p^*} Set^{\Theta_{n-1}^{\text{op}} \times \Delta^{\text{op}}},$$

where p^* are in fact the canonical inclusions.

Then, recall from Notation 2.3.10 the object $E[k] \in Set^{\Theta_{n-1}^{\text{op}} \times \Delta^{\text{op}}}$, for $k \geq 0$. By taking the left Kan extension of the assignment $t: \Delta \times \Theta_{n-1} \times \Delta \rightarrow Set^{\Theta_{n-1}^{\text{op}} \times \Delta^{\text{op}}}$ given by

$$([m], \theta, [k]) \mapsto F[m] \times \Theta_{n-1}[\theta] \times E[k],$$

we obtain an adjunction

$$Set^{\Theta_{n-1}^{\text{op}} \times \Delta^{\text{op}}} \xrightleftharpoons[\iota^!]{t_!} sSet^{\Theta_{n-1}^{\text{op}} \times \Delta^{\text{op}}}.$$

We define an auxiliary nerve, based on set-valued presheaves. It generalizes Cordier–Porter’s [CP86] homotopy coherent nerve, which we recover as the case $n = 1$.

By taking the left Kan extension of the assignment $\Delta \times \Theta_{n-1} \rightarrow sSet^{\Theta_{n-1}^{\text{op}}}\text{-Cat}$ given by

$$([m], \theta) \mapsto \mathfrak{S}_m(\Theta_{n-1}[\theta]),$$

we obtain an adjunction

$$sSet^{\Theta_{n-1}^{\text{op}}}\text{-Cat} \xrightleftharpoons[\overline{\mathfrak{N}}]{\overline{\mathfrak{C}}} Set^{\Theta_{n-1}^{\text{op}} \times \Delta^{\text{op}}}.$$

The following remark explains the relation between the functors $\overline{\mathfrak{N}}$ and \mathfrak{N} .

Remark 3.1.1. The adjunction $\overline{\mathfrak{C}} \dashv \overline{\mathfrak{N}}$ is the composite of the adjunctions $p^* \dashv \iota_0^*$ and $\mathfrak{C} \dashv \mathfrak{N}$.

$$\overline{\mathfrak{N}}: sSet^{\Theta_{n-1}^{\text{op}}}\text{-Cat} \xrightleftharpoons[\mathfrak{N}]{\mathfrak{C}} sSet^{\Theta_{n-1}^{\text{op}} \times \Delta^{\text{op}}} \xrightleftharpoons[\iota_0^*]{p^*} Set^{\Theta_{n-1}^{\text{op}} \times \Delta^{\text{op}}} : \overline{\mathfrak{C}}$$

By taking the left Kan extension of the assignment $\Delta \times \Theta_{n-1} \times \Delta \rightarrow sSet^{\Theta_{n-1}^{\text{op}}}\text{-Cat}$ given by

$$([m], \theta, [k]) \mapsto \mathfrak{C}(F[m] \times \Theta_{n-1}[\theta] \times E[k]),$$

we obtain an adjunction

$$sSet^{\Theta_{n-1}^{\text{op}}}\text{-Cat} \xrightleftharpoons[\mathfrak{N}^c]{\mathfrak{C}^c} sSet^{\Theta_{n-1}^{\text{op}} \times \Delta^{\text{op}}}.$$

Spelling out the description of the right adjoint, we obtain the following.

Definition 3.1.2. For a $sSet^{\Theta_{n-1}^{\text{op}}}$ -enriched category \mathcal{C} , the *complete homotopy coherent nerve* $\mathfrak{N}^c \mathcal{C}$ is the $(\Delta \times \Theta_{n-1})$ -space given at $m \geq 0$, $\theta \in \Theta_{n-1}$, and $k \geq 0$ by the set

$$(\mathfrak{N}^c \mathcal{C})_{m,\theta,k} := sSet^{\Theta_{n-1}^{\text{op}} \times \Delta^{\text{op}}}(\mathfrak{C}(F[m] \times \Theta_{n-1}[\theta] \times E[k]), \mathcal{C}).$$

The following remark explains the relation between the functor \mathfrak{N}^c and $\overline{\mathfrak{N}}$.

Remark 3.1.3. The adjunction $\mathfrak{C}^c \dashv \mathfrak{N}^c$ is the composite of the adjunctions $t_! \dashv t^!$ and $\overline{\mathfrak{C}} \dashv \overline{\mathfrak{N}}$.

$$\mathfrak{N}^c : sSet^{\Theta_{n-1}^{\text{op}}} \text{-Cat} \begin{array}{c} \xleftarrow{\overline{\mathfrak{C}}} \\ \perp \\ \xrightarrow{\overline{\mathfrak{N}}} \end{array} Set^{\Theta_{n-1}^{\text{op}} \times \Delta^{\text{op}}} \begin{array}{c} \xleftarrow{t_!} \\ \perp \\ \xrightarrow{t^!} \end{array} sSet^{\Theta_{n-1}^{\text{op}} \times \Delta^{\text{op}}} : \mathfrak{C}^c$$

3.2. Homotopical properties of the complete homotopy coherent nerve for $n = 1$. When $n = 1$, we can endow $Set^{\Delta^{\text{op}}}$ with the Joyal model structure $Set_{(\infty,1)}^{\Delta^{\text{op}}}$ for $(\infty, 1)$ -categories from [Joy08], $sSet$ with the Kan–Quillen model structure $sSet_{(\infty,0)}$ for $(\infty, 0)$ -categories from [Qui67], and $sSet\text{-Cat}$ with the resulting model structure being the Bergner model structure $sSet_{(\infty,0)}\text{-Cat}$ from [Ber07b]. With respect to these model structure, it is known (see [Joy07, Theorem 2.10], or [Lur09, Theorem 2.2.5.1], and [DS11, Corollary 8.2]) that the discrete complete homotopy coherent nerve $\overline{\mathfrak{N}}$ defines a Quillen equivalence between models of $(\infty, 1)$ -categories.

Theorem 3.2.1. *The adjunction*

$$sSet_{(\infty,0)}\text{-Cat} \begin{array}{c} \xleftarrow{\overline{\mathfrak{C}}} \\ \perp \\ \xrightarrow{\overline{\mathfrak{N}}} \end{array} Set_{(\infty,1)}^{\Delta^{\text{op}}}$$

is a Quillen equivalence.

If $(sSet_{(\infty,0)})_{\text{cCS}}^{\Delta^{\text{op}}} = sSet_{(\infty,1)}^{\Delta^{\text{op}}}$ denotes Rezk’s model structure on $sSet^{\Delta^{\text{op}}}$ for complete Segal spaces, we then obtain that the complete homotopy coherent nerve \mathfrak{C}^c also gives an equivalence of models of $(\infty, 1)$ -categories.

Theorem 3.2.2 ([JT07, Theorem 4.12]). *The adjunction*

$$Set_{(\infty,1)}^{\Delta^{\text{op}}} \begin{array}{c} \xleftarrow{t_!} \\ \perp \\ \xrightarrow{t^!} \end{array} sSet_{(\infty,1)}^{\Delta^{\text{op}}}$$

is a Quillen equivalence.

Corollary 3.2.3. *The adjunction*

$$sSet_{(\infty,0)}\text{-Cat} \begin{array}{c} \xleftarrow{\mathfrak{C}^c} \\ \perp \\ \xrightarrow{\mathfrak{N}^c} \end{array} sSet_{(\infty,1)}^{\Delta^{\text{op}}}$$

is a Quillen equivalence.

Proof. The adjunction $\mathfrak{C}^c \dashv \mathfrak{N}^c$ is a Quillen pair because by Remark 3.1.3 it is the composite

$$\mathfrak{N}^c : sSet_{(\infty,0)}\text{-Cat} \begin{array}{c} \xleftarrow{\overline{\mathfrak{C}}} \\ \perp \\ \xrightarrow{\overline{\mathfrak{N}}} \end{array} Set_{(\infty,1)}^{\Delta^{\text{op}}} \begin{array}{c} \xleftarrow{t_!} \\ \perp \\ \xrightarrow{t^!} \end{array} sSet_{(\infty,1)}^{\Delta^{\text{op}}} : \mathfrak{C}^c$$

of the Quillen equivalence $\overline{\mathfrak{C}} \dashv \overline{\mathfrak{N}}$ from Theorem 3.2.1 and the Quillen equivalence $t_! \dashv t^!$ from Theorem 3.2.2. \square

3.3. Homotopical properties of the complete homotopy coherent nerve for $n > 1$. We now assume $n > 1$. Recall from § 1.3.4 and § 2.3.11 that we considered the “space-based” model structures $s\text{Set}_{(\infty, n-1)}^{\Theta_{n-1}^{\text{op}}}$, $(s\text{Set}_{(\infty, n-1)}^{\Theta_{n-1}^{\text{op}}})_{\text{inj}}^{\Delta_{\text{op}}}$, $(s\text{Set}_{(\infty, n-1)}^{\Theta_{n-1}^{\text{op}}})_{\text{cCS}}^{\Delta_{\text{op}}}$, and $(s\text{Set}_{(\infty, n-1)}^{\Theta_{n-1}^{\text{op}}})_{\text{cCS}}^{\Delta_{\text{op}}}$. We need to define “set-based” analogs of those: $\text{Set}_{(\infty, n-1)}^{\Theta_{n-1}^{\text{op}}}$, $(\text{Set}_{(\infty, n-1)}^{\Theta_{n-1}^{\text{op}}})_{\text{inj}}^{\Delta_{\text{op}}}$, $(\text{Set}_{(\infty, n-1)}^{\Theta_{n-1}^{\text{op}}})_{\text{cCS}}^{\Delta_{\text{op}}}$, and $(\text{Set}_{(\infty, n-1)}^{\Theta_{n-1}^{\text{op}}})_{\text{cCS}}^{\Delta_{\text{op}}}$.

First, we denote by $\text{Set}_{(\infty, n-1)}^{\Theta_{n-1}^{\text{op}}}$ Ara’s model structure on $\text{Set}_{(\infty, n-1)}^{\Theta_{n-1}^{\text{op}}}$ for $(n-1)$ -quasi-categories from [Ara14, §5.17], which is also a model of $(\infty, n-1)$ -categories.

Theorem 3.3.1 ([Ara14, Theorem 8.4]). *The adjunction*

$$s\text{Set}_{(\infty, n-1)}^{\Theta_{n-1}^{\text{op}}} \begin{array}{c} \xleftarrow{p^*} \\ \xrightarrow[t_0^*]{\perp} \end{array} \text{Set}_{(\infty, n-1)}^{\Theta_{n-1}^{\text{op}}}$$

is a Quillen equivalence.

3.3.2. We denote by $(\text{Set}_{(\infty, n-1)}^{\Theta_{n-1}^{\text{op}}})_{\text{inj}}^{\Delta_{\text{op}}}$ the injective model structure on the category of simplicial objects valued in $\text{Set}_{(\infty, n-1)}^{\Theta_{n-1}^{\text{op}}}$.

We denote by $(\text{Set}_{(\infty, n-1)}^{\Theta_{n-1}^{\text{op}}})_{\text{cCS}}^{\Delta_{\text{op}}}$ the left Bousfield localization of $(\text{Set}_{(\infty, n-1)}^{\Theta_{n-1}^{\text{op}}})_{\text{inj}}^{\Delta_{\text{op}}}$ with respect to the set $\text{Seg}_{\Theta_{n-1}} \cup \text{Cst}_{\Theta_{n-1}}$, where

$$\text{Seg}_{\Theta_{n-1}} = \{(F[1] \amalg_{F[0]} \dots \amalg_{F[0]} F[1] \rightarrow F[m]) \times \Theta_{n-1}[\theta] \mid m \geq 1, \theta \in \Theta_{n-1}\},$$

and

$$\text{Cst}_{\Theta_{n-1}} = \{F[0] \times (\Theta_{n-1}[\theta] \rightarrow \Theta_{n-1}[\theta']) \mid \theta \rightarrow \theta' \in \Theta_{n-1}\}.$$

We further denote by $(\text{Set}_{(\infty, n-1)}^{\Theta_{n-1}^{\text{op}}})_{\text{cCS}}^{\Delta_{\text{op}}}$ the left Bousfield localization of $(\text{Set}_{(\infty, n-1)}^{\Theta_{n-1}^{\text{op}}})_{\text{cCS}}^{\Delta_{\text{op}}}$ with respect to the set

$$\text{Cpt}_{\Theta_{n-1}} = \{(F[0] \rightarrow E[1]) \times \Theta_{n-1}[\theta] \mid \theta \in \Theta_{n-1}\}.$$

Theorem 3.3.1 together with [Lur09, Remark A.2.8.6] and [Hir03, Theorem 3.3.20(1)(b)] implies the following. In particular, since $(s\text{Set}_{(\infty, n-1)}^{\Theta_{n-1}^{\text{op}}})_{\text{cCS}}^{\Delta_{\text{op}}}$ is a model of (∞, n) -categories, so is $(\text{Set}_{(\infty, n-1)}^{\Theta_{n-1}^{\text{op}}})_{\text{cCS}}^{\Delta_{\text{op}}}$.

Theorem 3.3.3. *The adjunction between injective model structures*

$$(s\text{Set}_{(\infty, n-1)}^{\Theta_{n-1}^{\text{op}}})_{\text{inj}}^{\Delta_{\text{op}}} \begin{array}{c} \xleftarrow{p^*} \\ \xrightarrow[t_0^*]{\perp} \end{array} (\text{Set}_{(\infty, n-1)}^{\Theta_{n-1}^{\text{op}}})_{\text{inj}}^{\Delta_{\text{op}}}$$

is a Quillen equivalence, and so the adjunctions between left Bousfield localizations

$$(s\text{Set}_{(\infty, n-1)}^{\Theta_{n-1}^{\text{op}}})_{\text{cCS}}^{\Delta_{\text{op}}} \begin{array}{c} \xleftarrow{p^*} \\ \xrightarrow[t_0^*]{\perp} \end{array} (\text{Set}_{(\infty, n-1)}^{\Theta_{n-1}^{\text{op}}})_{\text{cCS}}^{\Delta_{\text{op}}} \quad \text{and} \quad (s\text{Set}_{(\infty, n-1)}^{\Theta_{n-1}^{\text{op}}})_{\text{cCS}}^{\Delta_{\text{op}}} \begin{array}{c} \xleftarrow{p^*} \\ \xrightarrow[t_0^*]{\perp} \end{array} (\text{Set}_{(\infty, n-1)}^{\Theta_{n-1}^{\text{op}}})_{\text{cCS}}^{\Delta_{\text{op}}}$$

are also Quillen equivalences.

Using the above Quillen equivalences and [Hir03, Theorem 3.3.20(1)(b)], we deduce that the model structure $(\text{Set}_{(\infty, n-1)}^{\Theta_{n-1}^{\text{op}}})_{\text{cCS}}^{\Delta_{\text{op}}}$ coincides with the left Bousfield localization of $(\text{Set}_{(\infty, n-1)}^{\Theta_{n-1}^{\text{op}}})_{\text{cCS}}^{\Delta_{\text{op}}}$ with respect to just the map $F[0] \rightarrow E[1]$ since the analogous statement holds for the space-based model structure $(s\text{Set}_{(\infty, n-1)}^{\Theta_{n-1}^{\text{op}}})_{\text{cCS}}^{\Delta_{\text{op}}}$.

The adjunction $t_! \dashv t^!$ also gives a Quillen equivalence between the last models, and we prove this in Appendix A.

Theorem 3.3.4. *The adjunction*

$$(Set_{(\infty, n-1)}^{\Theta_{n-1}^{op}})_{cCS}^{\Delta^{op}} \xleftarrow[t!]{t_!} (sSet_{(\infty, n-1)}^{\Theta_{n-1}^{op}})_{cCS}^{\Delta^{op}}$$

is a Quillen equivalence.

We are now ready to analyze the homotopical properties of $\overline{\mathfrak{N}}$ when valued in the model structure $(Set_{(\infty, n-1)}^{\Theta_{n-1}^{op}})_{cS}^{\Delta^{op}}$.

Proposition 3.3.5. *The adjunction*

$$sSet_{(\infty, n-1)}^{\Theta_{n-1}^{op}}-Cat \xleftarrow[\overline{\mathfrak{N}}]{\overline{\mathfrak{C}}} (Set_{(\infty, n-1)}^{\Theta_{n-1}^{op}})_{cS}^{\Delta^{op}}$$

is a Quillen pair.

Proof. The adjunction $\overline{\mathfrak{C}} \dashv \overline{\mathfrak{N}}$ is a Quillen pair because by Remark 3.1.1 it is the composite

$$\overline{\mathfrak{N}}: sSet_{(\infty, n-1)}^{\Theta_{n-1}^{op}}-Cat \xleftarrow[\mathfrak{N}]{\mathfrak{C}} (sSet_{(\infty, n-1)}^{\Theta_{n-1}^{op}})_{cS}^{\Delta^{op}} \xleftarrow[t_0^*]{p^*} (Set_{(\infty, n-1)}^{\Theta_{n-1}^{op}})_{cS}^{\Delta^{op}} : \overline{\mathfrak{C}}$$

of the Quillen pair $\mathfrak{C} \dashv \mathfrak{N}$ from Proposition 2.3.14 and the Quillen pair $p^* \dashv t_0^*$ from Theorem 3.3.3. \square

Next, we analyze the homotopical properties of the functor $\overline{\mathfrak{N}}$ when valued in the model structure $(sSet_{(\infty, n-1)}^{\Theta_{n-1}^{op}})_{cCS}^{\Delta^{op}}$. For this, let $\pi_2: \Theta \times \Delta \rightarrow \Delta$ denote the projection given by $\pi_2(\theta, [k]) := [k]$. Then the pre-composition functor $\pi_2^*: sSet \rightarrow sSet^{\Theta_{n-1}^{op}}$ is the canonical inclusion, and the induced base-change functor $(\pi_2^*)_*: sSet-Cat \rightarrow sSet^{\Theta_{n-1}^{op}}-Cat$ is also the canonical inclusion.

Lemma 3.3.6. *The adjunction*

$$sSet_{(\infty, n-1)}^{\Theta_{n-1}^{op}}-Cat \xleftarrow[(\pi_2^*)_*]{(\pi_2^*)_*} sSet_{(\infty, 0)}-Cat$$

obtained by base-change along the adjunction $\pi_2^* \dashv (\pi_2)_*$ is a Quillen pair.

Proof. First note that the adjunction

$$sSet_{(\infty, n-1)}^{\Theta_{n-1}^{op}} \xleftarrow[(\pi_2)_*]{\pi_2^*} sSet_{(\infty, 0)}$$

is a Quillen pair. Indeed, the canonical inclusion π_2^* clearly preserves monomorphisms, and it preserves weak equivalences, since a weak equivalence in $sSet_{(\infty, 0)}$ is in particular a weak equivalence in $(sSet_{(\infty, 0)})_{inj}^{\Theta_{n-1}^{op}}$ and so a weak equivalence in $sSet_{(\infty, n-1)}^{\Theta_{n-1}^{op}}$. Then, by e.g. [MOR22, Proposition 4.3], the induced adjunction by base-change

$$sSet_{(\infty, n-1)}^{\Theta_{n-1}^{op}}-Cat \xleftarrow[(\pi_2^*)_*]{(\pi_2^*)_*} sSet_{(\infty, 0)}-Cat$$

is also a Quillen pair, as desired. \square

Proposition 3.3.7. *The adjunction*

$$s\mathit{Set}_{(\infty, n-1)}^{\Theta_{n-1}^{\text{op}}}\text{-Cat} \begin{array}{c} \xleftarrow{\bar{\mathfrak{C}}} \\ \perp \\ \xrightarrow{\bar{\mathfrak{N}}} \end{array} (\mathit{Set}_{(\infty, n-1)}^{\Theta_{n-1}^{\text{op}}})_{\text{cCS}}^{\Delta_{\text{op}}}$$

is a Quillen pair.

Proof. Recall that the model structure $(\mathit{Set}_{(\infty, n-1)}^{\Theta_{n-1}^{\text{op}}})_{\text{cCS}}^{\Delta_{\text{op}}}$ can be taken to be the left Bousfield localization of $(\mathit{Set}_{(\infty, n-1)}^{\Theta_{n-1}^{\text{op}}})_{\text{cS}}^{\Delta_{\text{op}}}$ with respect to the map $F[0] \rightarrow E[1]$. Hence, by Proposition 3.3.5 and [Hir03, Theorem 3.3.20(1)(a)], it is enough to show that the functor $\bar{\mathfrak{C}}$ sends the map $F[0] \rightarrow E[1]$ to a weak equivalence in $s\mathit{Set}_{(\infty, n-1)}^{\Theta_{n-1}^{\text{op}}}\text{-Cat}$. However, by Theorem 3.2.1, we know that $\bar{\mathfrak{C}}(F[0] \rightarrow E[1])$ is a weak equivalence in $s\mathit{Set}_{(\infty, 0)}\text{-Cat}$ between cofibrant objects. Since the inclusion functor $s\mathit{Set}_{(\infty, 0)}\text{-Cat} \rightarrow s\mathit{Set}_{(\infty, n-1)}^{\Theta_{n-1}^{\text{op}}}\text{-Cat}$ is left Quillen by Lemma 3.3.6, we get that $\bar{\mathfrak{C}}(F[0] \rightarrow E[1])$ is also a weak equivalence in $s\mathit{Set}_{(\infty, n-1)}^{\Theta_{n-1}^{\text{op}}}\text{-Cat}$. \square

We can now deduce the homotopical properties of \mathfrak{N}^c from those of $\bar{\mathfrak{N}}$.

Theorem 3.3.8. *The adjunction*

$$s\mathit{Set}_{(\infty, n-1)}^{\Theta_{n-1}^{\text{op}}}\text{-Cat} \begin{array}{c} \xleftarrow{\mathfrak{C}^c} \\ \perp \\ \xrightarrow{\mathfrak{N}^c} \end{array} (s\mathit{Set}_{(\infty, n-1)}^{\Theta_{n-1}^{\text{op}}})_{\text{cCS}}^{\Delta_{\text{op}}}$$

is a Quillen pair.

Proof. The adjunction $\mathfrak{C}^c \dashv \mathfrak{N}^c$ is a Quillen pair because by Remark 3.1.3 it is the composite

$$\mathfrak{N}^c : s\mathit{Set}_{(\infty, n-1)}^{\Theta_{n-1}^{\text{op}}}\text{-Cat} \begin{array}{c} \xleftarrow{\bar{\mathfrak{C}}} \\ \perp \\ \xrightarrow{\bar{\mathfrak{N}}} \end{array} (\mathit{Set}_{(\infty, n-1)}^{\Theta_{n-1}^{\text{op}}})_{\text{cCS}}^{\Delta_{\text{op}}} \begin{array}{c} \xleftarrow{t_!} \\ \perp \\ \xrightarrow{t^!} \end{array} (s\mathit{Set}_{(\infty, n-1)}^{\Theta_{n-1}^{\text{op}}})_{\text{cCS}}^{\Delta_{\text{op}}} : \mathfrak{C}^c$$

of the Quillen pair $\bar{\mathfrak{C}} \dashv \bar{\mathfrak{N}}$ from Proposition 3.3.7 and the Quillen pair $t_! \dashv t^!$ from Theorem 3.3.4. \square

We conclude by explaining that $\mathfrak{N}^c\mathcal{C}$ should be thought of as a completion of $\mathfrak{N}\mathcal{C}$ (as well as NC).

Proposition 3.3.9. *Given a fibrant $s\mathit{Set}_{(\infty, n-1)}^{\Theta_{n-1}^{\text{op}}}$ -enriched category \mathcal{C} , the natural canonical maps*

$$NC \rightarrow \mathfrak{N}\mathcal{C} \rightarrow \mathfrak{N}^c\mathcal{C}$$

are weak equivalences in $(s\mathit{Set}_{(\infty, 0)})_{\text{inj}}^{\Delta_{\text{op}} \times \Theta_{n-1}^{\text{op}}}$, and so weak equivalences in $(s\mathit{Set}_{(\infty, n-1)}^{\Theta_{n-1}^{\text{op}}})_{\text{inj}}^{\Delta_{\text{op}}}$.

Proof. Let $m \geq 0$, $\theta \in \Theta$, and $k \geq 0$. The map $\Delta[k] \rightarrow \Delta[0]$ is a weak equivalence in $(s\mathit{Set}_{(\infty, 0)})_{\text{inj}}^{\Delta_{\text{op}} \times \Theta_{n-1}^{\text{op}}}$, so $F[m] \times \Theta[\theta] \times \Delta[k] \rightarrow F[m] \times \Theta[\theta] \times \Delta[0] = F[m] \times \Theta[\theta]$ is also one since $(s\mathit{Set}_{(\infty, 0)})_{\text{inj}}^{\Delta_{\text{op}} \times \Theta_{n-1}^{\text{op}}}$ is cartesian closed. By Theorems 2.3.1 and 3.3.8, the functors

$$\mathfrak{C}, \mathfrak{C}^c : (s\mathit{Set}_{(\infty, 0)})_{\text{inj}}^{\Delta_{\text{op}} \times \Theta_{n-1}^{\text{op}}} \rightarrow s\mathit{Set}_{(\infty, n-1)}^{\Theta_{n-1}^{\text{op}}}\text{-Cat}$$

are left Quillen functors, and in particular preserve weak equivalences. So there is a commutative diagram in $s\mathit{Set}_{(\infty, n-1)}^{\Theta_{n-1}^{\text{op}}}\text{-Cat}$,

$$\begin{array}{ccc} \mathfrak{C}(F[m] \times \Theta_{n-1}[\theta] \times \Delta[k]) & \longrightarrow & \mathfrak{C}^c(F[m] \times \Theta_{n-1}[\theta] \times \Delta[k]) \\ \downarrow \mathbb{R} & & \downarrow \mathbb{R} \\ \mathfrak{C}(F[m] \times \Theta_{n-1}[\theta]) & \xlongequal{\quad} & \mathfrak{C}^c(F[m] \times \Theta_{n-1}[\theta]) \end{array}$$

where the vertical maps are weak equivalences. By 2-out-of-3, we obtain that the top map is also a weak equivalence in $s\mathcal{S}et_{(\infty, n-1)}^{\Theta_{n-1}^{\text{op}}}\text{-Cat}$.

Using [Cis19, Corollary 1.3.10] together with Theorems 2.3.1 and 3.3.8, we deduce that, for every (cofibrant) object $X \in (s\mathcal{S}et_{(\infty, 0)})_{\text{inj}}^{\Delta^{\text{op}} \times \Theta_{n-1}^{\text{op}}}$, the canonical map $\mathfrak{C}X \rightarrow \mathfrak{C}^c X$ is a weak equivalence in $(s\mathcal{S}et_{(\infty, 0)})_{\text{inj}}^{\Delta^{\text{op}} \times \Theta_{n-1}^{\text{op}}}$. By [Hov99, Corollary 1.4.4(b)], it follows that, for every $s\mathcal{S}et_{(\infty, n-1)}^{\Theta_{n-1}^{\text{op}}}$ -enriched category \mathcal{C} , there is an adjoint natural canonical map $\mathfrak{N}\mathcal{C} \rightarrow \mathfrak{N}^c\mathcal{C}$, which is a weak equivalence when \mathcal{C} is fibrant in $s\mathcal{S}et_{(\infty, n-1)}^{\Theta_{n-1}^{\text{op}}}\text{-Cat}$.

The statement follows, when combined with Proposition 2.3.9. \square

Remark 3.3.10. Let \mathcal{C} be a fibrant $s\mathcal{S}et_{(\infty, n-1)}^{\Theta_{n-1}^{\text{op}}}$ -enriched category. Combining Proposition 3.3.9 and Theorem 3.3.8, we obtain that the complete homotopy coherent nerve $\mathfrak{N}^c\mathcal{C}$ provides a fibrant replacement of $\mathcal{N}\mathcal{C}$ and of $\mathfrak{N}\mathcal{C}$ in $(s\mathcal{S}et_{(\infty, n-1)}^{\Theta_{n-1}^{\text{op}}})_{\text{cCS}}^{\Delta^{\text{op}}}$.

3.4. The known Quillen equivalences of models of (∞, n) -categories. For $n > 1$, we let $d: \Delta \times \Theta_{n-1} \rightarrow \Theta_n$ be the diagonal functor given by $d([m], \theta) := [m](\theta, \dots, \theta)$. It induces by pre-composition a functor $d^*: s\mathcal{S}et_{(\infty, n-1)}^{\Theta_{n-1}^{\text{op}}} \rightarrow s\mathcal{S}et_{(\infty, n-1) \times \Delta}^{\Theta_{n-1}^{\text{op}}}$ which admits a right adjoint d_* .

Theorem 3.4.1 ([BR20, Corollary 7.1]). *The adjunction*

$$(s\mathcal{S}et_{(\infty, n-1)}^{\Theta_{n-1}^{\text{op}}})_{\text{cCS}}^{\Delta^{\text{op}}} \begin{array}{c} \xleftarrow{d^*} \\ \perp \\ \xrightarrow{d_*} \end{array} s\mathcal{S}et_{(\infty, n)}^{\Theta_n^{\text{op}}}$$

is a Quillen equivalence.

We now consider the full subcategory $\mathcal{P}Cat(s\mathcal{S}et_{(\infty, n-1) \times \Delta}^{\Theta_{n-1}^{\text{op}}})$ of $s\mathcal{S}et_{(\infty, n-1) \times \Delta}^{\Theta_{n-1}^{\text{op}}}$ spanned by the objects $W \in s\mathcal{S}et_{(\infty, n-1) \times \Delta}^{\Theta_{n-1}^{\text{op}}}$ such that W_0 is discrete, meaning that W_0 lies in the image of $\mathcal{S}et \hookrightarrow s\mathcal{S}et_{(\infty, n-1)}^{\Theta_{n-1}^{\text{op}}}$. There is a ‘‘projective-like’’ model structure on $\mathcal{P}Cat(s\mathcal{S}et_{(\infty, n-1)}^{\Theta_{n-1}^{\text{op}}})$, denoted by $\mathcal{P}Cat(s\mathcal{S}et_{(\infty, n-1)}^{\Theta_{n-1}^{\text{op}}})_{\text{Seg}}$ and constructed in [BR13, Theorem 6.12]. The inclusion $I: \mathcal{P}Cat(s\mathcal{S}et_{(\infty, n-1)}^{\Theta_{n-1}^{\text{op}}}) \rightarrow s\mathcal{S}et_{(\infty, n-1) \times \Delta}^{\Theta_{n-1}^{\text{op}}}$ has a right adjoint R .

The following is a combination of [BR20, Theorem 9.6] with [BR13, Proposition 7.1].

Theorem 3.4.2. *The adjunction*

$$(s\mathcal{S}et_{(\infty, n-1)}^{\Theta_{n-1}^{\text{op}}})_{\text{cCS}}^{\Delta^{\text{op}}} \begin{array}{c} \xleftarrow{I} \\ \perp \\ \xrightarrow{R} \end{array} \mathcal{P}Cat(s\mathcal{S}et_{(\infty, n-1)}^{\Theta_{n-1}^{\text{op}}})_{\text{Seg}}$$

is a Quillen equivalence.

Finally, the strict nerve construction $N: s\mathcal{S}et_{(\infty, n-1)}^{\Theta_{n-1}^{\text{op}}}\text{-Cat} \rightarrow s\mathcal{S}et_{(\infty, n-1) \times \Delta}^{\Theta_{n-1}^{\text{op}}}$ restricts to a functor $N: s\mathcal{S}et_{(\infty, n-1)}^{\Theta_{n-1}^{\text{op}}}\text{-Cat} \rightarrow \mathcal{P}Cat(s\mathcal{S}et_{(\infty, n-1)}^{\Theta_{n-1}^{\text{op}}})$ since $(\mathcal{N}\mathcal{C})_0$ is discrete, for every $s\mathcal{S}et_{(\infty, n-1)}^{\Theta_{n-1}^{\text{op}}}$ -enriched category \mathcal{C} .

Theorem 3.4.3 ([BR13, Theorem 7.6]). *The adjunction*

$$s\mathcal{S}et_{(\infty, n-1)}^{\Theta_{n-1}^{\text{op}}}\text{-Cat} \begin{array}{c} \xleftarrow{c} \\ \perp \\ \xrightarrow{N} \end{array} \mathcal{P}Cat(s\mathcal{S}et_{(\infty, n-1)}^{\Theta_{n-1}^{\text{op}}})_{\text{Seg}}$$

is a Quillen equivalence.

3.5. Recognizing cells in models of (∞, n) -categories. For $0 \leq j \leq n$, we denote by C_j the j -cell, which can be seen as an n -category and an object of Θ_n .

Barwick–Schommer–Pries identify what it means for an object of a model of (∞, n) -categories (in the sense of Barwick–Schommer–Pries [BSP21]) to be a representative for the j -cell.

For instance, the j -cells in $s\mathcal{S}et_{(\infty, n)}^{\Theta_n^{\text{op}}}$ have been identified by Barwick and Schommer–Pries.

Proposition 3.5.1 ([BSP21, §13]). *For $0 \leq j \leq n$, the Θ_n -space $\Theta_n[C_j]$ is a representative of the j -cell in $sSet_{(\infty, n)}^{\Theta_n^{\text{op}}}$.*

This notion is useful to apply the following criterion to establish that a given Quillen pair is a Quillen equivalence between models of (∞, n) -categories.

Proposition 3.5.2 ([BSP21, Proposition 15.10]). *Let \mathcal{M} and \mathcal{N} be model categories that are models for (∞, n) -categories, and $L: \mathcal{M} \rightleftarrows \mathcal{N}: R$ a Quillen pair between them. Then the Quillen pair (L, R) is a Quillen equivalence if and only if the derived functor of L sends j -cells to j -cells for all $0 \leq j \leq n$.*

The following helps one identify cells through a known left Quillen equivalence. The proof is analogous to [BOR21, Lemma 3.2], where the case $n = 2$ is treated.

Remark 3.5.3. Let $F: \mathcal{M} \rightarrow \mathcal{N}$ be a left Quillen equivalence between models of (∞, n) -categories, and let X be a cofibrant object in \mathcal{M} . Then X is a j -cell in \mathcal{M} for some $0 \leq j \leq n$ if and only if $F(X)$ is a j -cell in \mathcal{N} .

We recalled in the previous subsection that there is a zig zag of Quillen equivalences relating the model structures $(sSet_{(\infty, n-1)}^{\Theta_{n-1}^{\text{op}}})_{\text{cCS}}^{\Delta^{\text{op}}}$ and $sSet_{(\infty, n-1)}^{\Theta_{n-1}^{\text{op}}}\text{-Cat}$, which we now use to identify the cell representatives.

To identify the j -cells in $(sSet_{(\infty, n-1)}^{\Theta_{n-1}^{\text{op}}})_{\text{cCS}}^{\Delta^{\text{op}}}$, we use the Quillen equivalence of Theorem 3.4.1.

Proposition 3.5.4. *For $0 \leq j \leq n$, the $(\Delta \times \Theta_{n-1})$ -space $d^*(\Theta_n[C_j])$ is a representative of the j -cell in $(sSet_{(\infty, n-1)}^{\Theta_{n-1}^{\text{op}}})_{\text{cCS}}^{\Delta^{\text{op}}}$.*

Proof. Let $0 \leq j \leq n$. We know that $\Theta_n[C_j]$ is cofibrant in $sSet_{(\infty, n)}^{\Theta_n^{\text{op}}}$, as all objects are cofibrant, and by Proposition 3.5.1 it is a representative of the j -cell in $sSet_{(\infty, n)}^{\Theta_n^{\text{op}}}$. So, by Proposition 3.5.2, its derived image $d^*(\Theta_n[C_j])$ under the left Quillen equivalence d^* is a representative of the j -cell in $(sSet_{(\infty, n-1)}^{\Theta_{n-1}^{\text{op}}})_{\text{cCS}}^{\Delta^{\text{op}}}$. \square

Lemma 3.5.5. *For $j = 0$ we have that $d^*(\Theta_n[0]) = F[0]$, and for $1 \leq j \leq n$ we have that the $(\Delta \times \Theta_{n-1})$ -space $d^*(\Theta_n[C_j])$ can be computed as the following pushout in $sSet_{n-1}^{\Theta_{n-1}^{\text{op}}} \times \Delta^{\text{op}}$.*

$$\begin{array}{ccc} \Theta_{n-1}[C_{j-1}] \amalg \Theta_{n-1}[C_{j-1}] & \longrightarrow & F[1] \times \Theta_{n-1}[C_{j-1}] \\ \downarrow & & \downarrow \\ F[0] \amalg F[0] & \longrightarrow & d^*(\Theta_n[C_j]) \end{array}$$

Proof. We compute the (m, θ) -component of $d^*(\Theta_n[C_j])$ for $m \geq 0$ and $\theta \in \Theta_{n-1}$:

$$\begin{aligned} (d^*(\Theta_n[C_j]))_{m, \theta} &\cong (d^*(\Theta_n[1](C_{j-1})))_{m, \theta} \\ &\cong \Theta_n([m](\theta, \dots, \theta), [1](C_{j-1})) \\ &\cong \Delta([m], [0]) \amalg \Delta([m], [0]) \amalg (\Delta^{\text{nc}}([m], [1]) \times \Theta_{n-1}(\theta, C_{j-1})) \\ &\cong (F[0] \amalg F[0] \amalg (\Delta^{\text{nc}}([m], [1]) \times \Theta_{n-1}[C_{j-1}]))_{m, \theta}, \end{aligned}$$

where $\Delta^{\text{nc}}([m], [1])$ denotes the set of non-constant maps from $[m]$ to $[1]$ in Δ . Observe that this is precisely the (m, θ) -component of the given pushout. \square

To identify the j -cells in $\mathcal{P}Cat(sSet_{(\infty, n-1)}^{\Theta_{n-1}^{\text{op}}})_{\text{Seg}}$, we use the Quillen equivalence of Theorem 3.4.2. Note that $d^*(\Theta_n[C_j])$ lies in the image of $I: \mathcal{P}Cat(sSet_{n-1}^{\Theta_{n-1}^{\text{op}}}) \hookrightarrow sSet_{n-1}^{\Theta_{n-1}^{\text{op}}} \times \Delta^{\text{op}}$, and hence we can see it as an object in $\mathcal{P}Cat(sSet_{n-1}^{\Theta_{n-1}^{\text{op}}})$.

Lemma 3.5.6. *For $0 \leq j \leq n$, the object $d^*(\Theta_n[C_j])$ is cofibrant in $\mathcal{P}Cat(sSet_{(\infty, n-1)}^{\Theta_{n-1}^{\text{op}}})_{\text{Seg}}$.*

Proof. For $j = 0$, the fact that the unique map $\emptyset \rightarrow F[0] = d^*(\Theta_{n-1}[C_0])$ is a cofibration in $\mathcal{P}Cat(sSet_{(\infty, n-1)}^{\Theta_{n-1}^{\text{op}}})_{\text{Seg}}$, namely that $d^*(\Theta_{n-1}[C_0])$ is cofibrant, is mentioned in the proof of [BR13, Lemma 6.5].

Now let $1 \leq j \leq n$. For every $m \geq 0$ and every cofibration $A \rightarrow B$ in $sSet_{(\infty, n-1)}^{\Theta_{n-1}^{\text{op}}}$, by [BR13, §6.2] the map $A_{[m]} \rightarrow B_{[m]}$ is a cofibration in $\mathcal{P}Cat(sSet_{(\infty, n-1)}^{\Theta_{n-1}^{\text{op}}})_{\text{Seg}}$. For the reader's convenience, we recall that $A_{[m]}$ is defined as the following pushout in $sSet_{(\infty, n-1)}^{\Theta_{n-1}^{\text{op}} \times \Delta^{\text{op}}}$.

$$\begin{array}{ccc} \coprod_{m+1} A & \longrightarrow & F[m] \times A \\ \downarrow & & \downarrow \\ \coprod_{m+1} F[0] & \longrightarrow & A_{[m]} \end{array}$$

In particular, by Lemma 3.5.5 we have an isomorphism

$$\Theta_n[C_j]_{[1]} \cong d^*\Theta_{n-1}[C_j].$$

Since $\Theta_{n-1}[C_j]$ is cofibrant in $sSet_{(\infty, n-1)}^{\Theta_{n-1}^{\text{op}}}$, then the map

$$F[0] \amalg F[0] \cong \emptyset_{[1]} \rightarrow \Theta_n[C_j]_{[1]} \cong d^*\Theta_{n-1}[C_j]$$

is a cofibration in $\mathcal{P}Cat(sSet_{(\infty, n-1)}^{\Theta_{n-1}^{\text{op}}})_{\text{Seg}}$. Since $F[0]$ is cofibrant in $\mathcal{P}Cat(sSet_{(\infty, n-1)}^{\Theta_{n-1}^{\text{op}}})_{\text{Seg}}$, then so is $F[0] \amalg F[0]$ and we conclude that $d^*\Theta_{n-1}[C_j]$ is cofibrant in $\mathcal{P}Cat(sSet_{(\infty, n-1)}^{\Theta_{n-1}^{\text{op}}})_{\text{Seg}}$. \square

Proposition 3.5.7. *For $0 \leq j \leq n$, the object $d^*(\Theta_n[C_j])$ is a representative of the j -cell in $\mathcal{P}Cat(sSet_{(\infty, n-1)}^{\Theta_{n-1}^{\text{op}}})_{\text{Seg}}$.*

Proof. Let $0 \leq j \leq n$. By Lemma 3.5.6 we have that $d^*(\Theta_n[C_j])$ is cofibrant in $\mathcal{P}Cat(sSet_{(\infty, n-1)}^{\Theta_{n-1}^{\text{op}}})_{\text{Seg}}$. So its derived image under the left Quillen equivalence I is $Id^*(\Theta_n[C_j]) \cong d^*(\Theta_n[C_j])$, which is a representative of the j -cell in $(sSet_{(\infty, n-1)}^{\Theta_{n-1}^{\text{op}}})_{\text{cCS}}^{\Delta^{\text{op}}}$ by Proposition 3.5.4. Hence we conclude by Remark 3.5.3 that $d^*(\Theta_n[C_j])$ is a representative of the j -cell in $\mathcal{P}Cat(sSet_{(\infty, n-1)}^{\Theta_{n-1}^{\text{op}}})_{\text{Seg}}$. \square

To identify the j -cells in $sSet_{(\infty, n-1)}^{\Theta_{n-1}^{\text{op}}}\text{-Cat}$, we use the Quillen equivalence of Theorem 3.4.3.

Proposition 3.5.8. *The terminal category $[0]$ is a representative of the 0-cell in $sSet_{(\infty, n-1)}^{\Theta_{n-1}^{\text{op}}}\text{-Cat}$, and, for $1 \leq j \leq n$, the $sSet_{(\infty, n-1)}^{\Theta_{n-1}^{\text{op}}}$ -enriched category $\Sigma(\Theta_{n-1}[C_{j-1}])$ is a representative of the j -cell in $sSet_{(\infty, n-1)}^{\Theta_{n-1}^{\text{op}}}\text{-Cat}$.*

Proof. Let $0 \leq j \leq n$. By Lemma 3.5.6 we have that $d^*(\Theta_n[C_j])$ is cofibrant in $\mathcal{P}Cat(sSet_{(\infty, n-1)}^{\Theta_{n-1}^{\text{op}}})_{\text{Seg}}$, and by Proposition 3.5.7 it is a representative of the j -cell in $\mathcal{P}Cat(sSet_{(\infty, n-1)}^{\Theta_{n-1}^{\text{op}}})_{\text{Seg}}$. So, by Proposition 3.5.2, its derived image $c(d^*(\Theta_n[C_j]))$ under the left Quillen equivalence c from Theorem 3.4.3 is a representative of the j -cell in $sSet_{(\infty, n-1)}^{\Theta_{n-1}^{\text{op}}}\text{-Cat}$.

When $j = 0$, it is straightforward to see that $c(d^*(\Theta_n[0])) = c(F[0]) \cong [0]$, and we now prove that for $1 \leq j \leq n$ there is an isomorphism

$$c(d^*(\Theta_n[C_j])) \cong \Sigma(\Theta_{n-1}[C_{j-1}]).$$

Let $1 \leq j \leq n$. Since c preserves pushouts, then by Lemma 3.5.5 $c(d^*(\Theta_n[C_j]))$ can be computed as the following pushout in $sSet_{(\infty, n-1)}^{\Theta_{n-1}^{\text{op}}}\text{-Cat}$.

$$\begin{array}{ccc}
c(\Theta_{n-1}[C_{j-1}] \amalg \Theta_{n-1}[C_{j-1}]) & \longrightarrow & c(F[1] \times \Theta_{n-1}[C_{j-1}]) \\
\downarrow & & \downarrow \\
c(F[0] \amalg F[0]) & \longrightarrow & c(d^*(\Theta_n[C_j]))
\end{array}$$

Since the left-hand map is the isomorphism

$$c(\Theta_{n-1}[C_{j-1}] \amalg \Theta_{n-1}[C_{j-1}]) \cong [0] \amalg [0] \cong c(F[0] \amalg F[0]),$$

the right-hand map gives an isomorphism

$$c(d^*(\Theta_n[C_j])) \cong c(F[1] \times \Theta_{n-1}[C_{j-1}]) \cong \Sigma(\Theta_{n-1}[C_{j-1}]),$$

where the last isomorphism holds by definition of c . This concludes the proof. \square

3.6. New Quillen equivalence of models of (∞, n) -categories. The goal of this subsection is to show that the functor \mathfrak{N}^c defines a Quillen equivalence for $n > 1$.

Lemma 3.6.1. *For $1 \leq j \leq n$, there is an isomorphism in $sSet^{\Theta_{n-1}^{\text{op}}}\text{-Cat}$*

$$\mathfrak{C}^c(d^*(\Theta_n[C_j])) \cong \Sigma(\Theta_{n-1}[C_{j-1}]).$$

Proof. For $1 \leq j \leq n$, since \mathfrak{C}^c preserves pushouts, then by Lemma 3.5.5 $\mathfrak{C}^c(d^*(\Theta_n[C_j]))$ can be computed as the following pushout in $sSet^{\Theta_{n-1}^{\text{op}}}\text{-Cat}$.

$$\begin{array}{ccc}
\mathfrak{C}^c(\Theta_{n-1}[C_{j-1}] \amalg \Theta_{n-1}[C_{j-1}]) & \longrightarrow & \mathfrak{C}^c(F[1] \times \Theta_{n-1}[C_{j-1}]) \\
\downarrow & & \downarrow \\
\mathfrak{C}^c(F[0] \amalg F[0]) & \longrightarrow & \mathfrak{C}^c(d^*(\Theta_n[C_j]))
\end{array}$$

The left-hand map is the isomorphism

$$\mathfrak{C}^c(\Theta_{n-1}[C_{j-1}] \amalg \Theta_{n-1}[C_{j-1}]) \cong [0] \amalg [0] \cong \mathfrak{C}^c(F[0] \amalg F[0])$$

so the right-hand map yields an isomorphism

$$\mathfrak{C}^c(d^*(\Theta_n[C_j])) \cong \mathfrak{C}^c(F[1] \times \Theta_{n-1}[C_{j-1}]),$$

where the right-hand side can be computed as

$$\mathfrak{C}^c(F[1] \times \Theta_{n-1}[C_{j-1}]) = \mathfrak{C}(F[1] \times \Theta_{n-1}[C_{j-1}]) = \mathfrak{S}_1(\Theta_{n-1}[C_{j-1}]) = \Sigma(\Theta_{n-1}[C_{j-1}])$$

by definition of \mathfrak{C}^c . This concludes the proof. \square

Theorem 3.6.2. *For $n > 1$, the adjunction*

$$sSet_{(\infty, n-1)}^{\Theta_{n-1}^{\text{op}}}\text{-Cat} \xleftarrow[\mathfrak{N}^c]{\mathfrak{C}^c} (sSet_{(\infty, n-1)}^{\Theta_{n-1}^{\text{op}}})_{\text{cCS}}^{\Delta^{\text{op}}}$$

is a Quillen equivalence.

Proof. We show that the derived functor of \mathfrak{C}^c sends j -cells to j -cells, for all $0 \leq j \leq n$. Since \mathfrak{C}^c is left Quillen by Theorem 3.3.8, it then follows from Proposition 3.5.2 that the functor \mathfrak{C}^c is a left Quillen equivalence, as desired.

By Lemma 3.5.6 and Proposition 3.5.7, we know that $d^*(\Theta_n[C_j])$ is cofibrant in $(sSet_{(\infty, n-1)}^{\Theta_{n-1}^{\text{op}}})_{\text{cCS}}^{\Delta^{\text{op}}}$ and that it is a j -cell. For $j = 0$, we have that \mathfrak{C}^c sends the 0-cell $d^*(\Theta_n[0]) \cong F[0]$ to the terminal category $[0]$, which is by Proposition 3.5.8 a 0-cell in $sSet_{(\infty, n-1)}^{\Theta_{n-1}^{\text{op}}}\text{-Cat}$. For $1 \leq j \leq n$, Lemma 3.6.1 shows that \mathfrak{C}^c sends the j -cell $d^*(\Theta_n[C_j])$ to the $sSet_{(\infty, n-1)}^{\Theta_{n-1}^{\text{op}}}$ -enriched category $\Sigma(\Theta_{n-1}[C_{j-1}])$, which is by Proposition 3.5.8 a j -cell in $sSet_{(\infty, n-1)}^{\Theta_{n-1}^{\text{op}}}\text{-Cat}$. \square

Combining Remark 3.1.3, Proposition 3.3.5, and Theorem 3.6.2, we also obtain that $\overline{\mathfrak{M}}$ is a Quillen equivalence.

Corollary 3.6.3. *For $n > 1$, the adjunction*

$$s\mathcal{S}et_{(\infty, n-1)}^{\Theta_{n-1}^{\text{op}}}\text{-Cat} \begin{array}{c} \xleftarrow{\overline{\mathfrak{c}}} \\ \xrightarrow[\overline{\mathfrak{M}}]{\perp} \end{array} (\mathcal{S}et_{(\infty, n-1)}^{\Theta_{n-1}^{\text{op}}})_{\text{cS}}^{\Delta^{\text{op}}}$$

is a Quillen equivalence.

Combining Theorems 3.4.1 and 3.6.2, we also obtain a direct Quillen equivalence between the model categories $s\mathcal{S}et_{(\infty, n-1)}^{\Theta_{n-1}^{\text{op}}}\text{-Cat}$ and $s\mathcal{S}et_{(\infty, n)}^{\Theta_n^{\text{op}}}$.

Corollary 3.6.4. *The adjunction*

$$s\mathcal{S}et_{(\infty, n-1)}^{\Theta_{n-1}^{\text{op}}}\text{-Cat} \begin{array}{c} \xleftarrow{\mathfrak{C}^c \circ d^*} \\ \xrightarrow[d_* \circ \mathfrak{N}^c]{\perp} \end{array} s\mathcal{S}et_{(\infty, n)}^{\Theta_n^{\text{op}}}$$

is a Quillen equivalence.

APPENDIX A. SET- VS SPACE-BASED MODEL STRUCTURES FOR (∞, n) -CATEGORIES

Recall from Section 3.1 the functor $t: \Delta \times \Theta_{n-1} \times \Delta \rightarrow \mathcal{S}et^{\Theta_{n-1}^{\text{op}} \times \Delta^{\text{op}}}$ given by

$$([m], \theta, [k]) \mapsto F[m] \times \Theta_{n-1}[\theta] \times E[k],$$

and the induced adjunction

$$\mathcal{S}et^{\Theta_{n-1}^{\text{op}} \times \Delta^{\text{op}}} \begin{array}{c} \xleftarrow{t_!} \\ \xrightarrow[t^!]{\perp} \end{array} s\mathcal{S}et^{\Theta_{n-1}^{\text{op}} \times \Delta^{\text{op}}}$$

obtained by left Kan extending along t . The goal of this appendix is to show – in Appendix A.3 – that this gives a Quillen equivalence

$$(\mathcal{S}et_{(\infty, n-1)}^{\Theta_{n-1}^{\text{op}}})_{\text{cCS}}^{\Delta^{\text{op}}} \begin{array}{c} \xleftarrow{t_!} \\ \xrightarrow[t^!]{\perp} \end{array} (s\mathcal{S}et_{(\infty, n-1)}^{\Theta_{n-1}^{\text{op}}})_{\text{cCS}}^{\Delta^{\text{op}}}$$

between the model structures $(\mathcal{S}et_{(\infty, n-1)}^{\Theta_{n-1}^{\text{op}}})_{\text{cCS}}^{\Delta^{\text{op}}}$ and $(s\mathcal{S}et_{(\infty, n-1)}^{\Theta_{n-1}^{\text{op}}})_{\text{cCS}}^{\Delta^{\text{op}}}$ introduced in § 3.3.2 and § 2.3.11, respectively. To this end, we first study point-set properties of the adjunction $t_! \dashv t^!$ in Appendix A.1 and the enrichment of the model structure $(\mathcal{S}et_{(\infty, n-1)}^{\Theta_{n-1}^{\text{op}}})_{\text{cS}}^{\Delta^{\text{op}}}$ over $\mathcal{S}et_{(\infty, 1)}^{\Delta^{\text{op}}}$ in Appendix A.2.

A.1. Auxiliary adjunctions. To better understand the adjunction $t_! \dashv t^!$, we make use of the auxiliary adjunction $\kappa_! \dashv \kappa^!$ from [JT07, §2], and study how they compare.

By left Kan extending along the assignment $\kappa: \Delta \rightarrow \mathcal{S}et^{\Delta^{\text{op}}}$ given by

$$[k] \mapsto E[k] = NI[k]$$

from Notation 2.3.10, we obtain an adjunction

$$\mathcal{S}et^{\Delta^{\text{op}}} \begin{array}{c} \xleftarrow{\kappa_!} \\ \xrightarrow[\kappa^!]{\perp} \end{array} s\mathcal{S}et.$$

The following is [Joy08, Theorem 6.22] (also re-stated as [JT07, Theorem 1.19]).

Theorem A.1.1. *The adjunction*

$$\mathcal{S}et_{(\infty, 1)}^{\Delta^{\text{op}}} \begin{array}{c} \xleftarrow{\kappa_!} \\ \xrightarrow[\kappa^!]{\perp} \end{array} s\mathcal{S}et_{(\infty, 0)}$$

is a Quillen pair.

The following notations are inspired by the notations used in [JT07, §2].

Notation A.1.2. We denote by

- $(-)\square(-): \mathcal{S}et^{\Theta_{n-1}^{\text{op}} \times \Delta^{\text{op}}} \times s\mathcal{S}et \rightarrow s\mathcal{S}et^{\Theta_{n-1}^{\text{op}} \times \Delta^{\text{op}}}$ the bi-functor sending a pair (A, K) to the product $A \times K \in s\mathcal{S}et^{\Theta_{n-1}^{\text{op}} \times \Delta^{\text{op}}}$,
- $A \setminus (-): s\mathcal{S}et^{\Theta_{n-1}^{\text{op}} \times \Delta^{\text{op}}} \rightarrow s\mathcal{S}et$ the right adjoint of the functor $A \square (-): s\mathcal{S}et \rightarrow s\mathcal{S}et^{\Theta_{n-1}^{\text{op}} \times \Delta^{\text{op}}}$, for $A \in \mathcal{S}et^{\Theta_{n-1}^{\text{op}} \times \Delta^{\text{op}}}$,
- $A \pitchfork (-): \mathcal{S}et^{\Theta_{n-1}^{\text{op}} \times \Delta^{\text{op}}} \rightarrow \mathcal{S}et^{\Delta^{\text{op}}}$ the right adjoint of $A \times (-): \mathcal{S}et^{\Delta^{\text{op}}} \rightarrow \mathcal{S}et^{\Theta_{n-1}^{\text{op}} \times \Delta^{\text{op}}}$, for $A \in \mathcal{S}et^{\Theta_{n-1}^{\text{op}} \times \Delta^{\text{op}}}$.

The following lemmas are analog statements to the ones of [JT07, Lemma 2.11].

Lemma A.1.3. *Let $A \in \mathcal{S}et^{\Theta_{n-1}^{\text{op}} \times \Delta^{\text{op}}}$ and $K \in s\mathcal{S}et$. There is a natural isomorphism in $\mathcal{S}et^{\Theta_{n-1}^{\text{op}} \times \Delta^{\text{op}}}$*

$$t_!(A \square K) \cong A \times \kappa_!(K).$$

Proof. Write

$$A \cong \text{colim}_{(\Delta \times \Theta_{n-1}) \downarrow A} F[m] \times \Theta_{n-1}[\theta] \quad \text{and} \quad K \cong \text{colim}_{\Delta \downarrow K} \Delta[k].$$

We have the following natural isomorphisms:

$$\begin{aligned} t_!(A \square K) &\cong t_!(\text{colim}_{(\Delta \times \Theta_{n-1}) \downarrow A} F[m] \times \Theta_{n-1}[\theta] \times (\text{colim}_{\Delta \downarrow K} \Delta[k])) \\ &\cong t_!(\text{colim}_{(\Delta \times \Theta_{n-1}) \downarrow A} \text{colim}_{\Delta \downarrow K} (F[m] \times \Theta_{n-1}[\theta] \times \Delta[k])) && \times \text{preserves colimits} \\ &\cong \text{colim}_{(\Delta \times \Theta_{n-1}) \downarrow A} \text{colim}_{\Delta \downarrow K} t_!(F[m] \times \Theta_{n-1}[\theta] \times \Delta[k]) && t_! \text{ preserves colimits} \\ &\cong \text{colim}_{(\Delta \times \Theta_{n-1}) \downarrow A} \text{colim}_{\Delta \downarrow K} (F[m] \times \Theta_{n-1}[\theta] \times E[k]) && t_! \text{ on representables} \\ &\cong (\text{colim}_{(\Delta \times \Theta_{n-1}) \downarrow A} (F[m] \times \Theta_{n-1}[\theta])) \times (\text{colim}_{\Delta \downarrow K} E[k]) && \times \text{preserves colimits} \\ &\cong (\text{colim}_{(\Delta \times \Theta_{n-1}) \downarrow A} (F[m] \times \Theta_{n-1}[\theta])) \times (\text{colim}_{\Delta \downarrow K} \kappa_! \Delta[k]) && \kappa_! \Delta[k] \cong E[k] \\ &\cong (\text{colim}_{(\Delta \times \Theta_{n-1}) \downarrow A} (F[m] \times \Theta_{n-1}[\theta])) \times \kappa_!(\text{colim}_{\Delta \downarrow K} \Delta[k]) && \kappa_! \text{ preserves colimits} \\ &\cong A \times \kappa_!(K). \end{aligned}$$

as desired. \square

Lemma A.1.4. *Let $A, X \in \mathcal{S}et^{\Theta_{n-1}^{\text{op}} \times \Delta^{\text{op}}}$. Then there is a natural isomorphism in $s\mathcal{S}et$*

$$A \setminus t^!(X) \cong \kappa^!(A \pitchfork X).$$

Proof. We have a square of adjunctions

$$\begin{array}{ccc} \mathcal{S}et^{\Theta_{n-1}^{\text{op}} \times \Delta^{\text{op}}} & \begin{array}{c} \xleftarrow{A \times (-)} \\ \perp \\ \xrightarrow{A \pitchfork (-)} \end{array} & \mathcal{S}et^{\Delta^{\text{op}}} \\ \begin{array}{c} \uparrow t_! \\ \downarrow t^! \end{array} & & \begin{array}{c} \uparrow \kappa_! \\ \downarrow \kappa^! \end{array} \\ s\mathcal{S}et^{\Theta_{n-1}^{\text{op}} \times \Delta^{\text{op}}} & \begin{array}{c} \xleftarrow{A \square (-)} \\ \perp \\ \xrightarrow{A \setminus (-)} \end{array} & s\mathcal{S}et \end{array}$$

where the diagram of left adjoint functors commutes up to isomorphism by Lemma A.1.3. Hence, the diagram of right adjoints also commutes up to isomorphism, yielding the desired result. \square

A.2. Auxiliary homotopical facts. Let $(s\text{Set}_{(\infty, n-1)}^{\Theta_{n-1}^{\text{op}}})_{\text{CS}}^{\Delta^{\text{op}}}$ and $(\text{Set}_{(\infty, n-1)}^{\Theta_{n-1}^{\text{op}}})_{\text{CS}}^{\Delta^{\text{op}}}$ denote the left Bousfield localizations of $(s\text{Set}_{(\infty, n-1)}^{\Theta_{n-1}^{\text{op}}})_{\text{inj}}^{\Delta^{\text{op}}}$ and $(\text{Set}_{(\infty, n-1)}^{\Theta_{n-1}^{\text{op}}})_{\text{inj}}^{\Delta^{\text{op}}}$, respectively, with respect to the set $\text{Seg}_{\Theta_{n-1}} \cup \text{Cpt}_{\Theta_{n-1}}$.

The following is a consequence of Theorem 3.3.3 and [Hir03, Theorem 3.3.20(1)(b)].

Proposition A.2.1. *The adjunction*

$$(s\text{Set}_{(\infty, n-1)}^{\Theta_{n-1}^{\text{op}}})_{\text{CS}}^{\Delta^{\text{op}}} \begin{array}{c} \xleftarrow{p^*} \\ \xrightarrow[\iota_0^*]{\perp} \end{array} (\text{Set}_{(\infty, n-1)}^{\Theta_{n-1}^{\text{op}}})_{\text{CS}}^{\Delta^{\text{op}}}$$

is a Quillen equivalence.

The reason that we consider these model structures – rather than their localizations at $\text{Cst}_{\Theta_{n-1}}$ $(s\text{Set}_{(\infty, n-1)}^{\Theta_{n-1}^{\text{op}}})_{\text{cCS}}^{\Delta^{\text{op}}}$ and $(\text{Set}_{(\infty, n-1)}^{\Theta_{n-1}^{\text{op}}})_{\text{cCS}}^{\Delta^{\text{op}}}$ – is that unlike the latter, the former are cartesian closed, as we now record.

Proposition A.2.2. *The model structures $(s\text{Set}_{(\infty, n-1)}^{\Theta_{n-1}^{\text{op}}})_{\text{CS}}^{\Delta^{\text{op}}}$ and $(\text{Set}_{(\infty, n-1)}^{\Theta_{n-1}^{\text{op}}})_{\text{CS}}^{\Delta^{\text{op}}}$ are cartesian closed.*

Proof. It is shown as [BR20, Proposition 5.9] that the model structure $(s\text{Set}_{(\infty, n-1)}^{\Theta_{n-1}^{\text{op}}})_{\text{CS}}^{\Delta^{\text{op}}}$ is cartesian closed. For $(\text{Set}_{(\infty, n-1)}^{\Theta_{n-1}^{\text{op}}})_{\text{CS}}^{\Delta^{\text{op}}}$, this follows from [Ara14, Proposition 2.8] applied to the left Quillen equivalence $p^*: (\text{Set}_{(\infty, n-1)}^{\Theta_{n-1}^{\text{op}}})_{\text{CS}}^{\Delta^{\text{op}}} \rightarrow (s\text{Set}_{(\infty, n-1)}^{\Theta_{n-1}^{\text{op}}})_{\text{CS}}^{\Delta^{\text{op}}}$ from Proposition A.2.1, which preserves binary products and creates weak equivalences. \square

Let $\pi_1: \Delta \times \Theta_{n-1} \rightarrow \Delta$ denote the projection given by $\pi_1([m], \theta) := [m]$. Then the precomposition functor $\pi_1^*: \text{Set}^{\Delta^{\text{op}}} \rightarrow \text{Set}^{\Theta_{n-1}^{\text{op}} \times \Delta^{\text{op}}}$ is the canonical inclusion.

Lemma A.2.3. *The adjunction*

$$(\text{Set}_{(\infty, n-1)}^{\Theta_{n-1}^{\text{op}}})_{\text{CS}}^{\Delta^{\text{op}}} \begin{array}{c} \xleftarrow{\pi_1^*} \\ \xrightarrow[(\pi_1)_*]{\perp} \end{array} \text{Set}_{(\infty, 1)}^{\Delta^{\text{op}}}$$

is a Quillen pair.

Proof. The canonical inclusion π_1^* , being also a right adjoint, preserves monomorphisms, namely cofibrations. To show that it is left Quillen, by [Joy08, Proposition E.2.14 and Theorem 5.22] it is enough to show that the inner horn inclusions

$$L^t[m] \rightarrow F[m]$$

for $m > 1$ and $0 < t < m$, and either inclusion

$$F[0] \rightarrow E[1]$$

are trivial cofibrations in $(\text{Set}_{(\infty, n-1)}^{\Theta_{n-1}^{\text{op}}})_{\text{CS}}^{\Delta^{\text{op}}}$.

However, by [JT07, Lemma 3.5] the map $L^t[m] \rightarrow F[m]$ is in the saturated class of monomorphisms generated by the inclusions $F[1] \amalg_{F[0]} \dots \amalg_{F[0]} F[1] \rightarrow F[m] \in \text{Seg}_{\Theta_{n-1}}$ for $m \geq 1$, so it is also a trivial cofibration in $(\text{Set}_{(\infty, n-1)}^{\Theta_{n-1}^{\text{op}}})_{\text{CS}}^{\Delta^{\text{op}}}$. Clearly, the map $F[0] \rightarrow E[1] \in \text{Cpt}_{\Theta_{n-1}}$ is a trivial cofibration in $(\text{Set}_{(\infty, n-1)}^{\Theta_{n-1}^{\text{op}}})_{\text{CS}}^{\Delta^{\text{op}}}$. This concludes the proof. \square

Proposition A.2.4. *The model structure $(\text{Set}_{(\infty, n-1)}^{\Theta_{n-1}^{\text{op}}})_{\text{CS}}^{\Delta^{\text{op}}}$ is enriched over $\text{Set}_{(\infty, 1)}^{\Delta^{\text{op}}}$ with $\text{Set}^{\Delta^{\text{op}}}$ -enriched hom functor given by $(-) \pitchfork (-)$.*

Proof. This follows directly from Proposition A.2.2 and Lemma A.2.3, using [GMMO19, Proposition 3.8]. \square

A.3. The Quillen equivalence. We are now ready to study the homotopical properties of the adjunction $t_! \dashv t^!$.

Proposition A.3.1. *The functor $t_! : s\text{Set}^{\Theta_{n-1}^{\text{op}} \times \Delta^{\text{op}}} \rightarrow \text{Set}^{\Theta_{n-1}^{\text{op}} \times \Delta^{\text{op}}}$ preserves monomorphisms.*

Proof. It is enough to show that $t_!$ sends pushout-product maps

$$(A \rightarrow B) \widehat{\times} (\partial\Delta[k] \rightarrow \Delta[k])$$

with $A \rightarrow B$ a monomorphism in $\text{Set}^{\Theta_{n-1}^{\text{op}} \times \Delta^{\text{op}}}$ and $k \geq 0$ to monomorphisms in $\text{Set}^{\Theta_{n-1}^{\text{op}} \times \Delta^{\text{op}}}$. By Lemma A.1.3, the image under $t_!$ of the above map is given by

$$t_!((A \rightarrow B) \widehat{\times} (\partial\Delta[k] \rightarrow \Delta[k])) = (A \rightarrow B) \widehat{\times} \kappa_! (\partial\Delta[k] \rightarrow \Delta[k]).$$

By Theorem A.1.1, the functor $\kappa_!$ preserves monomorphisms. Hence the above map is a pushout-product of two monomorphisms, and so it is a monomorphism as well. \square

Remark A.3.2. Let $A \in \text{Set}^{\Theta_{n-1}^{\text{op}} \times \Delta^{\text{op}}}$ and $Y \in s\text{Set}^{\Theta_{n-1}^{\text{op}} \times \Delta^{\text{op}}}$. There is a natural isomorphism in $s\text{Set}$

$$A \setminus Y \cong \text{Map}(A, Y),$$

where $\text{Map}(-, -)$ denotes the hom space functor of $s\text{Set}^{\Theta_{n-1}^{\text{op}} \times \Delta^{\text{op}}}$. Furthermore, if Y is a fibrant object in $(s\text{Set}_{(\infty, 0)})_{\text{inj}}^{\Delta^{\text{op}} \times \Theta_{n-1}^{\text{op}}}$, then the same hom space computes the derived mapping space

$$A \setminus Y \cong \text{Map}^h(A, Y).$$

Lemma A.3.3. *If X is a fibrant object in $(\text{Set}_{(\infty, n-1)}^{\Theta_{n-1}^{\text{op}}})_{\text{CS}}^{\Delta^{\text{op}}}$, and $A \rightarrow B$ is a (trivial) cofibration in $(\text{Set}_{(\infty, n-1)}^{\Theta_{n-1}^{\text{op}}})_{\text{CS}}^{\Delta^{\text{op}}}$, then the induced map between mapping spaces*

$$\text{Map}(B, t^!(X)) \rightarrow \text{Map}(A, t^!(X))$$

is a (trivial) fibration in $s\text{Set}_{(\infty, 0)}$.

Proof. Let $X \in (\text{Set}_{(\infty, n-1)}^{\Theta_{n-1}^{\text{op}}})_{\text{CS}}^{\Delta^{\text{op}}}$ be fibrant. First note that, since $(\text{Set}_{(\infty, n-1)}^{\Theta_{n-1}^{\text{op}}})_{\text{CS}}^{\Delta^{\text{op}}}$ is enriched over $\text{Set}_{(\infty, 1)}^{\Delta^{\text{op}}}$ by Proposition A.2.4, then for every (trivial) cofibration $A \rightarrow B$ in $(\text{Set}_{(\infty, n-1)}^{\Theta_{n-1}^{\text{op}}})_{\text{CS}}^{\Delta^{\text{op}}}$, the induced map $B \pitchfork X \rightarrow A \pitchfork X$ is a (trivial) fibration in $\text{Set}_{(\infty, 1)}^{\Delta^{\text{op}}}$. Then, since the functor $\kappa^! : \text{Set}_{(\infty, 1)}^{\Delta^{\text{op}}} \rightarrow s\text{Set}_{(\infty, 0)}$ is right Quillen by Theorem A.1.1, the induced map

$$\kappa^!(B \pitchfork X) \rightarrow \kappa^!(A \pitchfork X)$$

is a (trivial) fibration in $s\text{Set}_{(\infty, 0)}$. By Lemma A.1.4, the above map is isomorphic to the map

$$B \setminus t^!(X) \rightarrow A \setminus t^!(X)$$

and by Remark A.3.2 to the map

$$\text{Map}(B, t^!(X)) \rightarrow \text{Map}(A, t^!(X)),$$

which are therefore also (trivial) fibrations in $s\text{Set}_{(\infty, 0)}$. \square

Lemma A.3.4. *A map $X \rightarrow Y$ is a fibration in $(s\text{Set}_{(\infty, 0)})_{\text{inj}}^{\Delta^{\text{op}} \times \Theta_{n-1}^{\text{op}}}$ if and only if, for every monomorphism $A \rightarrow B$ in $\text{Set}^{\Theta_{n-1}^{\text{op}} \times \Delta^{\text{op}}}$, the induced map*

$$\text{Map}(B, X) \rightarrow \text{Map}(A, X) \times_{\text{Map}(A, Y)} \text{Map}(B, Y)$$

is a fibration in $s\text{Set}_{(\infty, 0)}$. In particular, an object X is fibrant in $(s\text{Set}_{(\infty, 0)})_{\text{inj}}^{\Delta^{\text{op}} \times \Theta_{n-1}^{\text{op}}}$ if and only if, for every monomorphism $A \rightarrow B$ in $\text{Set}^{\Theta_{n-1}^{\text{op}} \times \Delta^{\text{op}}}$, the induced map

$$\text{Map}(B, X) \rightarrow \text{Map}(A, X)$$

is a fibration in $s\text{Set}_{(\infty, 0)}$.

Proof. As a consequence of [Hir03, Corollary 15.7.2], we have that the injective model structure $(sSet_{(\infty,0)})_{\text{inj}}^{\Delta^{\text{op}} \times \Theta_{n-1}^{\text{op}}}$ coincides with the Reedy model structure $(sSet_{(\infty,0)})_{\text{Reedy}}^{\Delta^{\text{op}} \times \Theta_{n-1}^{\text{op}}}$. In particular by [Hir03, Theorem 15.3.4(1)], a map $X \rightarrow Y$ is a fibration in $(sSet_{(\infty,0)})_{\text{inj}}^{\Delta^{\text{op}} \times \Theta_{n-1}^{\text{op}}}$ if and only if, for all $m \geq 0$ and $\theta \in \Theta_{n-1}$, the induced map

$$\begin{array}{c} \text{Map}(F[m] \times \Theta_{n-1}[\theta], X) \\ \downarrow \\ \text{Map}(\partial(F[m] \times \Theta_{n-1}[\theta]), X) \times_{\text{Map}(\partial(F[m] \times \Theta_{n-1}[\theta]), Y)} \text{Map}(F[m] \times \Theta_{n-1}[\theta], Y) \end{array}$$

is a fibration in $sSet_{(\infty,0)}$, where

$$\partial(F[m] \times \Theta_{n-1}[\theta]) := \partial F[m] \times \Theta_{n-1}[\theta] \amalg_{\partial F[m] \times \partial \Theta_{n-1}[\theta], X} F[m] \times \partial \Theta_{n-1}[\theta].$$

This holds if and only if, for every monomorphism $A \rightarrow B$ in $\text{Set}^{\Theta_{n-1}^{\text{op}} \times \Delta^{\text{op}}}$, the induced map

$$\text{Map}(B, X) \rightarrow \text{Map}(A, X) \times_{\text{Map}(A, Y)} \text{Map}(B, Y)$$

is a fibration in $sSet_{(\infty,0)}$. Indeed, the direct implication follows from the fact that for all $m \geq 0$ and $\theta \in \Theta_{n-1}$ the pushout-product

$$(\partial F[m] \rightarrow F[m]) \widehat{\times} (\partial \Theta_{n-1}[\theta] \rightarrow \Theta_{n-1}[\theta])$$

is a monomorphism in $\text{Set}^{\Theta_{n-1}^{\text{op}} \times \Delta^{\text{op}}}$. The converse implication follows from the fact that the model structure $(sSet_{(\infty,0)})_{\text{inj}}^{\Delta^{\text{op}} \times \Theta_{n-1}^{\text{op}}}$ is simplicial (see e.g. [Mos19, Theorem 5.4]). \square

Proposition A.3.5. *The functor $t^! : (\text{Set}_{(\infty, n-1)}^{\Theta_{n-1}^{\text{op}}})_{\text{CS}}^{\Delta^{\text{op}}} \rightarrow (s\text{Set}_{(\infty, n-1)}^{\Theta_{n-1}^{\text{op}}})_{\text{CS}}^{\Delta^{\text{op}}}$ preserves fibrant objects.*

Proof. By applying Lemma A.3.3 to specific monomorphisms $A \rightarrow B$, we get that $t^!(X)$ is fibrant in $(s\text{Set}_{(\infty, n-1)}^{\Theta_{n-1}^{\text{op}}})_{\text{CS}}^{\Delta^{\text{op}}}$. Indeed, by taking $A \rightarrow B$ to be

- a generic monomorphism in $\text{Set}^{\Theta_{n-1}^{\text{op}} \times \Delta^{\text{op}}}$, we get that

$$\text{Map}(B, t^!(X)) \rightarrow \text{Map}(A, t^!(X))$$

is a fibration in $sSet_{(\infty,0)}$, which combined with Lemma A.3.4 shows that $t^!(X)$ is fibrant in $(s\text{Set}_{(\infty,0)})_{\text{inj}}^{\Delta^{\text{op}} \times \Theta_{n-1}^{\text{op}}}$,

- the trivial cofibration $F[m] \times C \rightarrow F[m] \times D$ in $(\text{Set}_{(\infty, n-1)}^{\Theta_{n-1}^{\text{op}}})_{\text{inj}}^{\Delta^{\text{op}}}$ with $m \geq 0$ and $C \rightarrow D$ a monomorphism in $S_{(\infty, n-1)}$, we get that

$$\text{Map}^{(h)}(F[m] \times D, t^!(X)) \rightarrow \text{Map}^{(h)}(F[m] \times C, t^!(X))$$

is a trivial fibration in $sSet_{(\infty,0)}$, which shows that $t^!(X)_m$ is fibrant in $s\text{Set}_{(\infty, n-1)}^{\Theta_{n-1}^{\text{op}}}$ for all $m \geq 0$, and so $t^!(X)$ is fibrant in $(s\text{Set}_{(\infty, n-1)}^{\Theta_{n-1}^{\text{op}}})_{\text{inj}}^{\Delta^{\text{op}}}$,

- the trivial cofibration $(F[1] \amalg_{F[0]} \dots \amalg_{F[0]} F[1]) \times \Theta_{n-1}[\theta] \rightarrow F[m] \times \Theta_{n-1}[\theta]$ in $(\text{Set}_{(\infty, n-1)}^{\Theta_{n-1}^{\text{op}}})_{\text{CS}}^{\Delta^{\text{op}}}$ with $m \geq 1$ and $\theta \in \Theta_{n-1}$, we get that

$$\text{Map}^{(h)}(F[m] \times \Theta_{n-1}[\theta], t^!(X)) \rightarrow \text{Map}^{(h)}((F[1] \amalg_{F[0]} \dots \amalg_{F[0]} F[1]) \times \Theta_{n-1}[\theta], t^!(X))$$

is a trivial fibration in $sSet_{(\infty,0)}$, which shows that $t^!(X)$ is fibrant in $(s\text{Set}_{(\infty, n-1)}^{\Theta_{n-1}^{\text{op}}})_{\text{Seg}}^{\Delta^{\text{op}}}$,

- the trivial cofibration $F[0] \times \Theta_{n-1}[\theta] \rightarrow E[1] \times \Theta_{n-1}[\theta]$ in $(\text{Set}_{(\infty, n-1)}^{\Theta_{n-1}^{\text{op}}})_{\text{CS}}^{\Delta^{\text{op}}}$ with $\theta \in \Theta_{n-1}$, we get that

$$\text{Map}^{(h)}(E[1] \times \Theta_{n-1}[\theta], t^!(X)) \rightarrow \text{Map}^{(h)}(F[0] \times \Theta_{n-1}[\theta], t^!(X))$$

is a trivial fibration in $sSet_{(\infty,0)}$, which shows that $t^!(X)$ is fibrant in $(s\text{Set}_{(\infty, n-1)}^{\Theta_{n-1}^{\text{op}}})_{\text{CS}}^{\Delta^{\text{op}}}$.

This concludes the proof. \square

Proposition A.3.6. *The functor $t^! : (\mathcal{S}et_{(\infty, n-1)}^{\Theta_{n-1}^{\text{op}}})_{\text{CS}}^{\Delta^{\text{op}}} \rightarrow (s\mathcal{S}et_{(\infty, n-1)}^{\Theta_{n-1}^{\text{op}}})_{\text{CS}}^{\Delta^{\text{op}}}$ preserves fibrations between fibrant objects.*

Proof. Let $f : X \rightarrow Y$ be a fibration in $(\mathcal{S}et_{(\infty, n-1)}^{\Theta_{n-1}^{\text{op}}})_{\text{CS}}^{\Delta^{\text{op}}}$ between fibrant objects. By Proposition A.3.5, the objects $t^!(X)$ and $t^!(Y)$ are fibrant in $(s\mathcal{S}et_{(\infty, n-1)}^{\Theta_{n-1}^{\text{op}}})_{\text{CS}}^{\Delta^{\text{op}}}$. By construction of the left Bousfield localization, it follows that $t^!(f) : t^!(X) \rightarrow t^!(Y)$ is a fibration in $(s\mathcal{S}et_{(\infty, n-1)}^{\Theta_{n-1}^{\text{op}}})_{\text{CS}}^{\Delta^{\text{op}}}$ if and only if it is a fibration in $(s\mathcal{S}et_{(\infty, 0)}^{\Theta_{n-1}^{\text{op}}})_{\text{inj}}^{\Delta^{\text{op}} \times \Theta_{n-1}^{\text{op}}}$. We show that $t^!(f)$ is a fibration in $(s\mathcal{S}et_{(\infty, 0)}^{\Theta_{n-1}^{\text{op}}})_{\text{inj}}^{\Delta^{\text{op}} \times \Theta_{n-1}^{\text{op}}}$.

Since $(\mathcal{S}et_{(\infty, n-1)}^{\Theta_{n-1}^{\text{op}}})_{\text{CS}}^{\Delta^{\text{op}}}$ is enriched over $\mathcal{S}et_{(\infty, 1)}^{\Delta^{\text{op}}}$ by Proposition A.2.4, for every monomorphism $A \rightarrow B$ in $\mathcal{S}et_{(\infty, 1)}^{\Theta_{n-1}^{\text{op}} \times \Delta^{\text{op}}}$, the map

$$B \pitchfork X \rightarrow A \pitchfork X \times_{A \pitchfork Y} B \pitchfork Y$$

is a fibration in $\mathcal{S}et_{(\infty, 1)}^{\Delta^{\text{op}}}$. As $\kappa^! : \mathcal{S}et_{(\infty, 1)}^{\Delta^{\text{op}}} \rightarrow s\mathcal{S}et_{(\infty, 0)}$ is right Quillen by Theorem A.1.1, we get that the map

$$\kappa^!(B \pitchfork X \rightarrow A \pitchfork X \times_{A \pitchfork Y} B \pitchfork Y)$$

is a fibration in $s\mathcal{S}et_{(\infty, 0)}$. By Lemma A.1.4, the above map is isomorphic to the map

$$B \setminus t^!(X) \rightarrow A \setminus t^!(X) \times_{A \setminus t^!(Y)} B \setminus t^!(Y)$$

and by Remark A.3.2 to the map

$$\text{Map}(B, t^!(X)) \rightarrow \text{Map}(A, t^!(X)) \times_{\text{Map}(A, t^!(Y))} \text{Map}(B, t^!(Y)),$$

which is therefore also a fibration in $s\mathcal{S}et_{(\infty, 0)}$. By Lemma A.3.4 this shows that $t^!(f) : t^!(X) \rightarrow t^!(Y)$ is a fibration in $(s\mathcal{S}et_{(\infty, 0)}^{\Theta_{n-1}^{\text{op}}})_{\text{inj}}^{\Delta^{\text{op}} \times \Theta_{n-1}^{\text{op}}}$, as desired. \square

Theorem A.3.7. *The adjunction*

$$(\mathcal{S}et_{(\infty, n-1)}^{\Theta_{n-1}^{\text{op}}})_{\text{CS}}^{\Delta^{\text{op}}} \begin{array}{c} \xleftarrow{t_!} \\ \xrightarrow[t^!]{\perp} \end{array} (s\mathcal{S}et_{(\infty, n-1)}^{\Theta_{n-1}^{\text{op}}})_{\text{CS}}^{\Delta^{\text{op}}}$$

is a Quillen equivalence.

Proof. Using [Joy08, Proposition E.2.14], it follows from Propositions A.3.1 and A.3.6 that $t_! \dashv t^!$ is a Quillen pair. By Lemma A.1.3, the composite of left adjoints

$$\text{id} : (\mathcal{S}et_{(\infty, n-1)}^{\Theta_{n-1}^{\text{op}}})_{\text{CS}}^{\Delta^{\text{op}}} \begin{array}{c} \xleftarrow{t_!} \\ \xrightarrow[t^!]{\perp} \end{array} (s\mathcal{S}et_{(\infty, n-1)}^{\Theta_{n-1}^{\text{op}}})_{\text{CS}}^{\Delta^{\text{op}}} \begin{array}{c} \xleftarrow{p^*} \\ \xrightarrow[t_0^*]{\perp} \end{array} (\mathcal{S}et_{(\infty, n-1)}^{\Theta_{n-1}^{\text{op}}})_{\text{CS}}^{\Delta^{\text{op}}} : \text{id}$$

is the identity. Hence, by Proposition A.2.1 and 2-out-of-3 for Quillen equivalences, we conclude that $t_! \dashv t^!$ is also a Quillen equivalence. \square

We finally obtain the desired result as a direct consequence of the above theorem and [Hir03, Theorem 3.3.20(1)(b)].

Theorem 3.3.4. *The adjunction*

$$(\mathcal{S}et_{(\infty, n-1)}^{\Theta_{n-1}^{\text{op}}})_{\text{cCS}}^{\Delta^{\text{op}}} \begin{array}{c} \xleftarrow{t_!} \\ \xrightarrow[t^!]{\perp} \end{array} (s\mathcal{S}et_{(\infty, n-1)}^{\Theta_{n-1}^{\text{op}}})_{\text{cCS}}^{\Delta^{\text{op}}}$$

is a Quillen equivalence.

REFERENCES

- [Ara14] Dimitri Ara, *Higher quasi-categories vs higher Rezk spaces*, J. K-Theory **14** (2014), no. 3, 701–749. MR 3350089
- [Bar05] Clark Barwick, *(∞, n) -Cat as a closed model category*, ProQuest LLC, Ann Arbor, MI, 2005, Thesis (Ph.D.)—University of Pennsylvania. MR 2706984
- [Ber07a] Clemens Berger, *Iterated wreath product of the simplex category and iterated loop spaces*, Adv. Math. **213** (2007), no. 1, 230–270. MR 2331244
- [Ber07b] Julia E. Bergner, *A model category structure on the category of simplicial categories*, Trans. Amer. Math. Soc. **359** (2007), no. 5, 2043–2058. MR 2276611
- [BOR21] Julia E. Bergner, Viktoriya Ozornova, and Martina Rovelli, *An explicit comparison between 2-complicial sets and Θ_2 -spaces*, arXiv:2104.13292, 2021.
- [BR13] Julia E. Bergner and Charles Rezk, *Comparison of models for (∞, n) -categories, I*, Geom. Topol. **17** (2013), no. 4, 2163–2202. MR 3109865
- [BR20] ———, *Comparison of models for (∞, n) -categories, II*, J. Topol. **13** (2020), no. 4, 1554–1581. MR 4186138
- [BSP21] Clark Barwick and Christopher Schommer-Pries, *On the unicity of the theory of higher categories*, J. Amer. Math. Soc. **34** (2021), no. 4, 1011–1058. MR 4301559
- [Cis19] Denis-Charles Cisinski, *Higher categories and homotopical algebra*, Cambridge Studies in Advanced Mathematics, Cambridge University Press, 2019.
- [cM20] tsllil clingman and Lyne Moser, *2-terminal objects and 2-limits are too different*, arXiv:2004.01313, 2020.
- [cM22] ———, *Bi-initial objects and bi-representations are not so different*, Cahiers Topologie Géom. Différentielle Catég. **63** (2022), no. 3, 259–330.
- [CP86] Jean-Marc Cordier and Timothy Porter, *Vogt’s theorem on categories of homotopy coherent diagrams*, Math. Proc. Cambridge Philos. Soc. **100** (1986), no. 1, 65–90. MR 838654
- [DS11] Daniel Dugger and David I. Spivak, *Mapping spaces in quasi-categories*, Algebr. Geom. Topol. **11** (2011), no. 1, 263–325. MR 2764043
- [Dug01] Daniel Dugger, *Combinatorial model categories have presentations*, Adv. Math. **164** (2001), no. 1, 177–201. MR 1870516
- [GH15] David Gepner and Rune Haugseng, *Enriched ∞ -categories via non-symmetric ∞ -operads*, Adv. Math. **279** (2015), 575–716. MR 3345192
- [Gin19] Harry Gindi, *Θ -sets and higher-quasicategories*, MathOverflow answer available at <https://mathoverflow.net/questions/252037/theta-sets-and-higher-quasicategories>, retrieved July 2022, 2019.
- [Gin21] ———, *Coherent nerves for higher quasicategories*, Theory Appl. Categ. **37** (2021), Paper No. 23, 709–817. MR 4276763
- [GMMO19] Bertrand Guillou, J. Peter May, Mona Merling, and Angélica M. Osorno, *A symmetric monoidal and equivariant Segal infinite loop space machine*, J. Pure Appl. Algebra **223** (2019), no. 6, 2425–2454. MR 3906557
- [GP99] Marco Grandis and Robert Pare, *Limits in double categories*, Cahiers Topologie Géom. Différentielle Catég. **40** (1999), no. 3, 162–220. MR 1716779
- [Hau15] Rune Haugseng, *Rectification of enriched ∞ -categories*, Algebr. Geom. Topol. **15** (2015), no. 4, 1931–1982. MR 3402334
- [Hir03] Philip S. Hirschhorn, *Model categories and their localizations*, Mathematical Surveys and Monographs, vol. 99, American Mathematical Society, Providence, RI, 2003. MR 1944041
- [Hov99] Mark Hovey, *Model categories*, Mathematical Surveys and Monographs, vol. 63, American Mathematical Society, Providence, RI, 1999. MR 1650134 (99h:55031)
- [JFS17] Theo Johnson-Freyd and Claudia Scheimbauer, *(Op)lax natural transformations, twisted quantum field theories, and “even higher” Morita categories*, Adv. Math. **307** (2017), 147–223. MR 3590516
- [Joy97] André Joyal, *Disks, Duality and Θ -categories*, preprint available at <https://ncatlab.org/nlab/files/JoyalThetaCategories.pdf>, 1997.
- [Joy02] ———, *Quasi-categories and Kan complexes*, J. Pure Appl. Algebra **175** (2002), no. 1–3, 207–222, Special volume celebrating the 70th birthday of Professor Max Kelly. MR 1935979
- [Joy07] ———, *Quasi-categories vs simplicial categories*, preprint available at [http://www.math.uchicago.edu/~may/IMA/Incoming/Joyal/QvsDJJan9\(2007\).pdf](http://www.math.uchicago.edu/~may/IMA/Incoming/Joyal/QvsDJJan9(2007).pdf), 2007.
- [Joy08] ———, *The theory of quasi-categories and its applications*, preprint available at <http://mat.uab.cat/~kock/crm/hocat/advanced-course/Quadern45-2.pdf>, 2008.
- [JT07] André Joyal and Myles Tierney, *Quasi-categories vs Segal spaces*, Categories in algebra, geometry and mathematical physics, Contemp. Math., vol. 431, Amer. Math. Soc., Providence, RI, 2007, pp. 277–326. MR 2342834

- [Lur09] Jacob Lurie, *Higher topos theory*, Annals of Mathematics Studies, vol. 170, Princeton University Press, Princeton, NJ, 2009. MR 2522659
- [Mae20] Yuki Maehara, *Inner horns for 2-quasi-categories*, Adv. Math. **363** (2020), 107003, 56. MR 4056974
- [MOR22] Lyne Moser, Viktoriya Ozornova, and Martina Rovelli, *Model independence of $(\infty, 2)$ -categorical nerves*, arXiv:2206.00660, 2022.
- [Mos19] Lyne Moser, *Injective and projective model structures on enriched diagram categories*, Homology Homotopy Appl. **21** (2019), no. 2, 279–300. MR 3923784
- [NS17] Thomas Nikolaus and Steffen Sagave, *Presentably symmetric monoidal ∞ -categories are represented by symmetric monoidal model categories*, Algebr. Geom. Topol. **17** (2017), no. 5, 3189–3212. MR 3704256
- [OR20] Viktoriya Ozornova and Martina Rovelli, *Model structures for (∞, n) -categories on (pre)stratified simplicial sets and prestratified simplicial spaces*, Algebr. Geom. Topol. **20** (2020), no. 3, 1543–1600. MR 4105558
- [Our10] David Oury, *Duality for joyal’s category Θ and homotopy concepts for Θ_2 -sets*, PhD thesis, 2010.
- [Qui67] Daniel G Quillen, *Homotopical algebra, volume 43 of lecture notes in mathematics*, 1967.
- [Ras17] Nima Rasekh, *Yoneda lemma for simplicial spaces*, arXiv:1711.03160v3, 2017.
- [Ras21] ———, *Yoneda lemma for \mathcal{D} -simplicial spaces*, arXiv:2108.06168, 2021.
- [Rez01] Charles Rezk, *A model for the homotopy theory of homotopy theory*, Trans. Amer. Math. Soc. **353** (2001), no. 3, 973–1007 (electronic). MR 1804411 (2002a:55020)
- [Rez10] ———, *A Cartesian presentation of weak n -categories*, Geom. Topol. **14** (2010), no. 1, 521–571. MR 2578310
- [Rov21] Martina Rovelli, *Weighted limits in an $(\infty, 1)$ -category*, Appl. Categ. Structures **29** (2021), no. 6, 1019–1062. MR 4324373
- [RV20] Emily Riehl and Dominic Verity, *Recognizing quasi-categorical limits and colimits in homotopy coherent nerves*, Applied Categorical Structures (2020), 1–48.
- [Shu06] Michael Shulman, *Homotopy limits and colimits and enriched homotopy theory*, arXiv:0610194, 2006.

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