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VERONESE SUBALGEBRAS AND VERONESE MORPHISMS FOR A CLASS OF YANG-BAXTER ALGEBRAS

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Abstract. We study d-Veronese subalgebras $\mathcal{A}^{(d)}$ of quadratic algebras $\mathcal{A}_X = \mathcal{A}(\mathbf{k}, X, r)$ related to finite nondegenerate involutive set-theoretic solutions (X, r) of the Yang-Baxter equation, where $\mathbf{k}$ is a field and $d \geq 2$ is an integer. We find an explicit presentation of the d-Veronese $\mathcal{A}^{(d)}$ in terms of one-generators and quadratic relations. We introduce the notion of *a d-Veronese solution* (Y, r_Y), canonically associated to (X, r) and use its Yang-Baxter algebra $\mathcal{A}_Y = \mathcal{A}(\mathbf{k}, Y, r_Y)$ to define a Veronese morphism $v_{n,d} : \mathcal{A}_Y \to \mathcal{A}_X$. We prove that the image of $v_{n,d}$ is the d-Veronese subalgebra $\mathcal{A}^{(d)}$, and find explicitly a minimal set of generators for its kernel. The results agree with their classical analogues in the commutative case, which corresponds to the case when (X, r) is the trivial solution. Finally, we show that the Yang-Baxter algebra $\mathcal{A}(\mathbf{k}, X, r)$ is a PBW algebra if and only if (X, r) is a square-free solution. In this case the d-Veronese $A^{(d)}$ is also a PBW algebra.

1. Introduction

It was established in the last three decades that solutions of the linear braid or Yang-Baxter equations (YBE)

$$r^{12}r^{23}r^{12} = r^{23}r^{12}r^{23}$$

on a vector space of the form $V^{\otimes 3}$ lead to remarkable algebraic structures. Here $r : V \otimes V \longrightarrow V \otimes V$, $r^{12} = r \otimes \mathrm{id}$, $r^{23} = id \otimes r$ is a notation and structures include coquasitriangular bialgebras $A(r)$, their quantum group (Hopf algebra) quotients, quantum planes and associated objects, at least in the case of specific standard solutions, see [?, ?]. On the other hand, the variety of all solutions on vector spaces of a given dimension has remained rather elusive in any degree of generality. It was proposed by V.G. Drinfeld [?], to consider the same equations in the category of sets, and in this setting numerous results were found. It is clear that a set-theoretic solution extends to a linear one, but more important than this is that set-theoretic solutions lead to their own remarkable algebraic and combinatoric structures, only somewhat analogous to quantum group constructions. In the present paper we continue our study of set-theoretic solutions and the associated quadratic algebras and monoids that they generate.

In this paper "a solution" means "a nondegenerate involutive set-theoretic solution of YBE", see Definition **??**.

The Yang-Baxter algebras $\mathcal{A}_X = \mathcal{A}(\mathbf{k}, X, r)$ related to solutions (X, r) of finite order n will play a central role in the paper. It was proven in [?] and [?] that these are quadratic algebras with remarkable algebraic, homological and combinatorial properties: they are noncommutative, but preserve the good properties of the commutative polynomial rings $\mathbf{k}[x_1, \cdots, x_n]$. Each such an algebra $\mathcal{A}_X$ has finite global dimension and polynomial growth, it is Koszul and a Noetherian domain. In the special case when (X, r) is square-free, $\mathcal{A}_X$ is a PBW algebra (has a basis of Poincaré-Birkhoff-Witt type) with respect to some enumeration $X = \{x_1, \ldots, x_n\}$, of X. More precisely, $\mathcal{A}_X$ is a binomial skew polynomial ring in the sense of [?] which implies its good combinatorial and computational properties (the use of noncommutative Gröbner bases). Conversely, every binomial skew polynomial ring in the sense of [?] defines via its quadratic relations a square-free solutions (X, r) of YBE. The algebras $\mathcal{A}_X = \mathcal{A}(\mathbf{k}, X, r)$ associated to multipermutation (square-free) solutions of level two were stdudied in [?], we referred to them as

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'quantum spaces'. In this special case a first stage of noncommutative geometry on $\mathcal{A}_X = \mathcal{A}(\mathbf{k}, X, r)$ was proposed, see [?], Section 6. It will be interesting to find more analogues coming from commutative algebra and algebraic geometry.

Given a finitely presented quadratic algebra A it is a classical problem to find presentations of its Veronese subalgebras in terms of generators and relations. This problem was solved in [?] for a class of particular quadratic PBW algebra called "noncommutative projective spaces", and analogues of Veronese morphisms between noncommutative projective spaces were introduced and studied. In the present paper we consider the following problem.

Problem 1.1. (1) Given a finite nondegenerate symmetric set (X, r) of order n and an integer $d \geq 2$, find a presentation of the d-Veronese $\mathcal{A}^{(d)}$ of the Yang-Baxter algebra $\mathcal{A} = \mathcal{A}(\mathbf{k}, X, r)$ in terms of one-generators and quadratic relations.

(2) Introduce analogues of Veronese maps for the class of Yang-Baxter algebras of finite nondegenerate symmetric sets. In particular, study the special case when (X, r) is a square-free solution.

The problem is solved completely. Our approach is entirely algebraic and combinatorial. Our main results are Theorem **??**, Theorem **??** and Theorem **??**. Theorem **??** shows that the Yang-Baxter algebra $\mathcal{A}_X = \mathcal{A}(\mathbf{k}, X, r)$ of a finite solution (X, r) is PBW if and only if (X, r) is a square-free solution. In Theorem **??** we find a presentation of the d-Veronese $\mathcal{A}^{(d)}$ in terms of explicit one-generators and quadratic relations. We introduce an analogue of Veronese morphisms for quantum spaces related to finite nondegenerate symmetric sets. Theorem **??** shows that the image of the Veronese map $v_{n,d}$ is the d-Veronese subalgebra $\mathcal{A}_X^{(d)}$ and describes explicitly a minimal set of generators for its kernel. Moreover, it follows from Theorem **??** and Corollary **??** that analogues of Veronese morphisms between Yang-Baxter algebras related to square-free solutions are not possible.

The paper is organized as follows. In Section 2 we recall basic definitions and facts used throughout the paper. In Section **??** we consider the quadratic algebra $\mathcal{A}(\mathbf{k}, X, r)$ of a finite nondegenerate symmetric set (X, r). We fix the main settings and conventions and collect some of the most important properties of the Yang-Baxter algebras $\mathcal{A}_X = \mathcal{A}(\mathbf{k}, X, r)$ used throughout the paper. We prove one of the main results of the paper Theorem **??**. Proposition **??** gives more information on a special case of PBW quadratic algebras. In Section **??** we study the d-Veronese $\mathcal{A}^{(d)}$ of $\mathcal{A}_X = \mathcal{A}(\mathbf{k}, X, r)$. We use the fact that the algebra $\mathcal{A}$ and its Veronese subalgebras are intimately connected with the braided monoid $S(X, r)$. To solve the main problem we introduce successively three isomorphic solutions associated naturally to (X, r), and involved in the proof of our results. The first and the most natural of the three is *the monomial d-Veronese solution* (S_d, r_d) *associated with* (X, r). It is induced from the graded braided monoid (S, r_S) and depends only on the map r and on d. The monomial d-Veronese solution is intimately connected with the d-Veronese $\mathcal{A}^{(d)}$ and its quadratic relations, but it is not convenient for an explicit description of the relations. We define *the normalized d-Veronese solution* $(\mathcal{N}_d, \rho_d)$ isomorphic to (S_d, r_d), see Definition **??**, we use it to describe the relations of the d-Veronese $\mathcal{A}^{(d)}$ and prove Theorem **??**. In Section **??** we introduce and study analogues of Veronese maps between Yang-Baxter algebras of finite solutions and prove Theorem **??**. In Section **??** we consider two special cases of solutions. We consider Yang-Baxter algebras $\mathcal{A}(\mathbf{k}, X, r)$ of square-free solutions (X, r) and their Veronese subalgebras. In this case $\mathcal{A}$ is a binomial skew polynomial ring and has an explicit $\mathbf{k}$- basis- the set of ordered monomials (terms) in n variables. Then for every $d \geq 2$ the d-Veronese subalgebra $A^{(d)}$ is also a PBW algebra with an explicitly given standard finite presentation in terms of generators and quadratic relations, see Corollary **??**. The important result in this section is Theorem **??** which shows that if (X, r) is a finite square-free solution and $d \geq 2$, then the monomial d-Veronese solution (S_d, r_d) is square-free if and only if (X, r) is a trivial solution. This implies that the notion of Veronese morphisms for the class of Yang-Baxter algebras of finite solutions can not be restricted to the subclass of algebras associated to finite square-free solutions. Finally we consider the particular case when (X, r) is a finite permutation solution. In Section **??** we present two examples which illustrate the results of the paper.

2. Preliminaries

Let X be a non-empty set, and let $\mathbf{k}$ be a field. We denote by $\langle X \rangle$ the free monoid generated by X, where the unit is the empty word denoted by 1, and by $\mathbf{k}\langle X \rangle$-the unital free associative $\mathbf{k}$-algebra generated by X. For a non-empty set $F \subseteq \mathbf{k}\langle X \rangle$, (F) denotes the two sided ideal of $\mathbf{k}\langle X \rangle$ generated by F. When the set X is finite, with $|X| = n$, and ordered, we write $X = \{x_1, \ldots, x_n\}$, and fix the degree-lexicographic order $<$ on $\langle X \rangle$, where we set $x_1 < \cdots < x_n$. As usual, $\mathbb{N}$ denotes the set of all positive integers, and $\mathbb{N}_0$ is the set of all non-negative integers.

We shall consider associative graded $\mathbf{k}$-algebras. Suppose $A = \bigoplus_{m \in \mathbb{N}_0} A_m$ is a graded $\mathbf{k}$-algebra such that $A_0 = \mathbf{k}$, $A_p A_q \subseteq A_{p+q}, p, q \in \mathbb{N}_0$, and such that A is finitely generated by elements of positive degree. Recall that its Hilbert function is $h_A(m) = \dim A_m$ and its Hilbert series is the formal series $H_A(t) = \sum_{m \in \mathbb{N}_0} h_A(m) t^m$. In particular, the algebra $\mathbf{k}[X_n]$ of commutative polynomials satisfies

$$h_{\mathbf{k}[X_n]}(d) = \binom{n+d-1}{d} = \binom{n+d-1}{n-1} \quad \text{and} \quad H_{\mathbf{k}[X_n]} = \frac{1}{(1-t)^n}. \tag{2.1}$$

We shall use the *natural grading by length* on the free associative algebra $\mathbf{k}\langle X \rangle$. For $m \geq 1$, X^m will denote the set of all words of length m in $\langle X \rangle$, where the length of $u = x_{i_1} \cdots x_{i_m} \in X^m$ will be denoted by $|u| = m$. Then

$$\langle X \rangle = \bigsqcup_{m \in \mathbb{N}_0} X^m, \ X^0 = \{1\}, \text{ and } \ X^k X^m \subseteq X^{k+m},$$

so the free monoid $\langle X \rangle$ is naturally *graded by length*.

Similarly, the free associative algebra $\mathbf{k}\langle X \rangle$ is also graded by length:

$$\mathbf{k}\langle X \rangle = \bigoplus_{m \in \mathbb{N}_0} \mathbf{k}\langle X \rangle_m, \qquad \text{where } \ \mathbf{k}\langle X \rangle_m = \mathbf{k}X^m.$$

A polynomial $f \in \mathbf{k}\langle X \rangle$ is *homogeneous of degree* m if $f \in \mathbf{k}X^m$. We denote by

$$\mathcal{T} = \mathcal{T}(X) := \{x_1^{\alpha_1} \cdots x_n^{\alpha_n} \in \langle X \rangle \mid \alpha_i \in \mathbb{N}_0, i \in \{1, \ldots, n\}\}$$

the set of ordered monomials (terms) in $\langle X \rangle$ and by

$$\mathcal{T}_d = \mathcal{T}(X)_d := \left\{ x_1^{\alpha_1} \cdots x_n^{\alpha_n} \in \mathcal{T} \mid \sum_{i=1}^{n} \alpha_i = d \right\}$$

the set of ordered monomials of length d. It is well known that the cardinality $|\mathcal{T}(X)_d|$ is given by the Hilbert function $h_{\mathbf{k}[X]}(d)$ of the (commutative) polynomial ring in the variables $x_1, \cdots, x_n$:

$$|\mathcal{T}(X)_d| = \binom{n+d-1}{n-1} = h_{\mathbf{k}[X]}(d). \tag{2.2}$$

2.1. **Gröbner bases for ideals in the free associative algebra.** In this subsection $X = \{x_1, \ldots, x_n\}$. Suppose $f \in \mathbf{k}\langle X \rangle$ is a nonzero polynomial. Its leading monomial with respect to the degree-lexicographic order $<$ will be denoted by $\mathbf{LM}(f)$. One has $\mathbf{LM}(f) = u$ if $f = cu + \sum_{1 \leq i \leq m} c_i u_i$, where $c, c_i \in \mathbf{k}$, $c \neq 0$ and $u > u_i$ in $\langle X \rangle$, for every $i \in \{1, \ldots, m\}$. Given a set $F \subseteq \mathbf{k}\langle X \rangle$ of non-commutative polynomials, $\mathbf{LM}(F)$ denotes the set

$$\mathbf{LM}(F) = \{\mathbf{LM}(f) \mid f \in F\}.$$

A monomial $u \in \langle X \rangle$ is *normal modulo* F if it does not contain any of the monomials $\mathbf{LM}(f), f \in F$ as a subword. The set of all normal monomials modulo F is denoted by $N(F)$.

Let I be a two sided graded ideal in $K\langle X \rangle$ and let $I_m = I \cap \mathbf{k}X^m$. We shall assume that I *is generated by homogeneous polynomials of degree* ≥ 2 and $I = \bigoplus_{m \geq 2} I_m$. Then the quotient algebra $A = \mathbf{k}\langle X \rangle / I$ is finitely generated and inherits its grading $A = \bigoplus_{m \in \mathbb{N}_0} A_m$ from $\mathbf{k}\langle X \rangle$. We shall work with the so-called *normal* $\mathbf{k}$-*basis of* A. We say that a monomial $u \in \langle X \rangle$ is *normal modulo* I if it is normal modulo $\mathbf{LM}(I)$. We set

$$N(I) := N(\mathbf{LM}(I)).$$

In particular, the free monoid $\langle X \rangle$ splits as a disjoint union

$$\langle X \rangle = N(I) \sqcup \mathbf{LM}(I). \tag{2.3}$$

The free associative algebra $\mathbf{k}\langle X\rangle$ splits as a direct sum of $\mathbf{k}$-vector subspaces

$$\mathbf{k}\langle X\rangle \simeq \mathrm{Span}_{\mathbf{k}} N(I) \oplus I,$$

and there is an isomorphism of vector spaces $A \simeq \mathrm{Span}_{\mathbf{k}} N(I)$.

It follows that every $f \in \mathbf{k}\langle X\rangle$ can be written uniquely as $f = h + f_0$, where $h \in I$ and $f_0 \in \mathbf{k}N(I)$. The element f_0 is called *the normal form of f (modulo I)* and denoted by $\mathrm{Nor}(f)$ We define

$$N(I)_m = \{u \in N(I) \mid u \text{ has length } m\}.$$

Then $A_m \simeq \mathrm{Span}_{\mathbf{k}} N(I)_m$ for every $m \in \mathbb{N}_0$.

A subset $G \subseteq I$ of monic polynomials is a *Gröbner basis* of I (with respect to the ordering $<$) if

(1) G generates I as a two-sided ideal, and
(2) for every $f \in I$ there exists $g \in G$ such that $\mathbf{LM}(g)$ is a subword of $\mathbf{LM}(f)$, that is $\mathbf{LM}(f) = a\mathbf{LM}(g)b$, for some $a, b \in \langle X\rangle$.

A Gröbner basis G of I is *reduced* if (i) the set $G \setminus \{f\}$ is not a Gröbner basis of I, whenever $f \in G$; (ii) each $f \in G$ is a linear combination of normal monomials modulo $G \setminus \{f\}$.

It is well-known that every ideal I of $\mathbf{k}\langle X\rangle$ has a unique reduced Gröbner basis $G_0 = G_0(I)$ with respect to $<$. However, G_0 may be infinite. For more details, we refer the reader to [**?**, **?**, **?**].

The set of leading monomials of the reduced Gröbner basis $G_0 = G_0(I)$

$$W = \{LM(f) \mid f \in G_0(I)\} \tag{2.4}$$

is also called *the set of obstructions* for $A = \mathbf{k}\langle X\rangle/I$, in the sense of Anick, [**?**]. There are equalities of sets $N(I) = N(G_0) = N(W)$.

Bergman's Diamond lemma [**?**, Theorem 1.2] implies the following.

Remark 2.1. Let $G \subset \mathbf{k}\langle X\rangle$ be a set of noncommutative polynomials. Let $I = (G)$ and let $A = \mathbf{k}\langle X\rangle/I$. Then the following conditions are equivalent.

(1) The set G is a Gröbner basis of I.
(2) Every element $f \in \mathbf{k}\langle X\rangle$ has a unique normal form modulo G, denoted by $\mathrm{Nor}_G(f)$.
(3) There is an equality $N(G) = N(I)$, so there is an isomorphism of vector spaces

$$\mathbf{k}\langle X\rangle \simeq I \oplus \mathbf{k}N(G).$$

(4) The image of $N(G)$ in A is a $\mathbf{k}$-basis of A. In this case A can be identified with the $\mathbf{k}$-vector space $\mathbf{k}N(G)$, made a $\mathbf{k}$-algebra by the multiplication $a \bullet b := \mathrm{Nor}(ab)$.

In this paper, we focus on a class of quadratic finitely presented algebras A associated with set-theoretic nondegenerate involutive solutions (X, r) of finite order n. Following Yuri Manin, [**?**], we call them Yang-Baxter algebras.

2.2. **Quadratic algebras.** A quadratic algebra is an associative graded algebra $A = \bigoplus_{i\geq 0} A_i$ over a ground field $\mathbf{k}$ determined by a vector space of generators $V = A_1$ and a subspace of homogeneous quadratic relations $R = R(A) \subset V \otimes V$. We assume that A is finitely generated, so $\dim A_1 < \infty$. Thus $A = T(V)/(R)$ inherits its grading from the tensor algebra $T(V)$.

Following the classical tradition (and a recent trend), we take a combinatorial approach to study A. The properties of A will be read off a presentation $A = \mathbf{k}\langle X\rangle/(\Re)$, where by convention X is a fixed finite set of generators of degree 1, $|X| = n$, and $(\Re)$ is the two-sided ideal of relations, generated by a *finite* set $\Re$ of homogeneous polynomials of degree two.

Definition 2.2. A quadratic algebra A is *a Poincarè–Birkhoff–Witt type algebra* or shortly *a PBW algebra* if there exists an enumeration $X = \{x_1, \cdots, x_n\}$ of X, such that the quadratic relations $\Re$ form a (noncommutative) Gröbner basis with respect to the degree-lexicographic ordering $<$ on $\langle X\rangle$. In this case the set of normal monomials (mod $\Re$) forms a $\mathbf{k}$-basis of A called a *PBW basis* and $x_1, \cdots, x_n$ (taken exactly with this enumeration) are called *PBW-generators of A*.

The notion of a *PBW* algebra was introduced by Priddy, [?]. His *PBW basis* is a generalization of the classical Poincaré-Birkhoff-Witt basis for the universal enveloping of a finite dimensional Lie algebra. PBW algebras form an important class of Koszul algebras. The interested reader can find information on quadratic algebras and, in particular, on Koszul algebras and PBW algebras in [?]. A special class of PBW algebras important for this paper, are the *binomial skew polynomial rings.*

The binomial skew polynomial rings were introduced by the author in [?], initially they were called "skew polynomial rings with binomial relations". They form a class of quadratic PBW algebras with remarkable properties: they are noncommutative, but preserve the good algebraic and homological properties of the commutative polynomial rings $\mathbf{k}[x_1, \cdots, x_n]$, each such an algebra A is a Noetherian Artin-Schelter regular domain, it is Koszul. Moreover, each skew polynomial ring defines via its relation a solution of the Yang-Baxter equation, see [?], [?], and Fact **??**. We recall the definition.

Definition 2.3. [?, ?] A *binomial skew polynomial ring* is a quadratic algebra $A = \mathbf{k}\langle x_1, \cdots, x_n\rangle/(\Re_0)$ with precisely $\binom{n}{2}$ defining relations

$$\Re_0 = \{f_{ji} = x_j x_i - c_{ij} x_{i'} x_{j'} \mid 1 \leq i < j \leq n\} \tag{2.5}$$

such that (a) $c_{ij} \in \mathbf{k}^{\times}$; (b) For every pair $i, j,\ 1 \leq i < j \leq n$, the relation $x_j x_i - c_{ij} x_{i'} x_{j'} \in \Re_0$, satisfies $j > i',\ i' < j'$; (c) Every ordered monomial $x_i x_j$, with $1 \leq i < j \leq n$ occurs (as a second term) in some relation in $\Re_0$; (d) $\Re_0$ is the *reduced Gröbner basis* of the two-sided ideal $(\Re_0)$, with respect to the degree-lexicographic order $<$ on $\langle X\rangle$, or equivalently the overlaps $x_k x_j x_i$, with $k > j > i$ do not give rise to new relations in A.

Note that the leading monomial of each relation in (**??**) satisfy

$$\mathbf{LM}(f_{ji}) = x_j x_i,\ 1 \leq i < j \leq n,$$

so a monomial u is normal modulo the relations $\Re_0$ if and only if $u \in \mathcal{T}$.

Example 2.4. Let $A = \mathbf{k}\langle x_1, x_2, x_3, x_4\rangle/(\Re_0)$, where

$$\Re_0 = \{x_4x_2 - x_1x_3,\ x_4x_1 - x_2x_3,\ x_3x_2 - x_1x_4,\ x_3x_1 - x_2x_4,\ x_4x_3 - x_3x_4,\ x_2x_1 - x_1x_2\}.$$

The algebra A is a binomial skew-polynomial ring. It is a PBW algebra with PBW generators $X = \{x_1, x_2, x_3, x_4\}$. The relations of A define in a natural way a solution of YBE.

2.3. **Quadratic sets and their algebraic objects.** The notion of *a quadratic set* was introduced in [?], see also [?], as a set-theoretic analogue of quadratic algebras.

Definition 2.5. [?] Let X be a nonempty set (possibly infinite) and let $r : X \times X \longrightarrow X \times X$ be a bijective map. In this case we use notation (X, r) and refer to it as *a quadratic set.* The image of (x, y) under r is presented as

$$r(x, y) = ({}^x y, x^y).$$

This formula defines a "left action" $\mathcal{L} : X \times X \longrightarrow X$, and a "right action" $\mathcal{R} : X \times X \longrightarrow X$, on X as: $\mathcal{L}_x(y) = {}^x y$, $\mathcal{R}_y(x) = x^y$, for all $x, y \in X$. (i) (X, r) is *non-degenerate*, if the maps $\mathcal{L}_x$ and $\mathcal{R}_x$ are bijective for each $x \in X$. (ii) (X, r) is *involutive* if $r^2 = id_{X\times X}$. (iii) (X, r) is *square-free* if $r(x, x) = (x, x)$ for all $x \in X$. (iv) (X, r) is *a set-theoretic solution of the Yang–Baxter equation* (YBE) if the braid relation

$$r^{12}r^{23}r^{12} = r^{23}r^{12}r^{23}$$

holds in $X \times X \times X$, where $r^{12} = r \times \mathrm{id}_X$, and $r^{23} = \mathrm{id}_X \times r$. In this case we refer to (X, r) also as *a braided set.* (v) A braided set (X, r) with r involutive is called *a symmetric set.* (vi) A nondegenerate symmetric set will be called simply *a solution.*

(X, r) is *the trivial solution* on X if $r(x, y) = (y, x)$ for all $x, y \in X$.

Remark 2.6. [?] Let (X, r) be quadratic set. Then r obeys the YBE, that is (X, r) is a braided set *iff* the following three conditions hold for all $x, y, z \in X$:

$$\mathbf{l1}:\quad {}^x({}^y z) = {}^{{}^x y}({}^{x^y} z), \qquad \mathbf{r1}:\quad (x^y)^z = (x^{{}^y z})^{y^z}, \qquad \mathbf{lr3}:\quad ({}^x y)^{({}^{x^y} z)} = {}^{(x^{{}^y z})}(y^z).$$

The map r is involutive *iff*

$$\mathbf{inv}:\quad {}^{{}^x y}(x^y) = x, \text{ and } ({}^x y)^{x^y} = y.$$

Convention 2.7. In this paper we shall always assume that (X, r) is nondegenerate. "*A solution*" means "*a non-degenerate symmetric set*" (X, r), where X is a set of arbitrary cardinality.

As a notational tool, we shall identify the sets $X^{\times m}$ of ordered m-tuples, $m \geq 2$, and X^m, the set of all monomials of length m in the free monoid $\langle X \rangle$. Sometimes for simplicity we shall write $r(xy)$ instead of $r(x, y)$.

Definition 2.8. [?, ?] To each quadratic set (X, r) we associate canonically algebraic objects generated by X and with quadratic relations $\Re = \Re(r)$ naturally determined as

$$xy = y'x' \in \Re(r) \text{ iff } r(x, y) = (y', x') \text{ and } (x, y) \neq (y', x') \text{ hold in } X \times X.$$

The monoid $S = S(X, r) = \langle X;\ \Re(r) \rangle$ with a set of generators X and a set of defining relations $\Re(r)$ is called *the monoid associated with* (X, r). The *group* $G = G(X, r) = G_X$ *associated with* (X, r) is defined analogously. For an arbitrary fixed field $\mathbf{k}$, *the* $\mathbf{k}$*-algebra associated with* (X, r) is defined as

$$A = A(\mathbf{k}, X, r) = \mathbf{k}\langle X \rangle / (\Re_0) \simeq \mathbf{k}\langle X;\ \Re(r) \rangle, \text{ where } \Re_0 = \{xy - y'x' \mid xy = y'x' \in \Re(r)\}.$$

Clearly, A is a quadratic algebra generated by X and with defining relations $\Re_0(r)$, which is isomorphic to the monoid algebra $\mathbf{k}S(X, r)$. When (X, r) is a solution of YBE, following Yuri Manin, [?], the algebra $\mathcal{A} = \mathcal{A}(\mathbf{k}, X, r)$ is also called *an Yang-Baxter algebra*, or shortly *YB algebra*.

Suppose (X, r) is a finite quadratic set. Then $A = A(\mathbf{k}, X, r)$ is *a connected graded* $\mathbf{k}$-algebra (naturally graded by length), $A = \bigoplus_{i \geq 0} A_i$, where $A_0 = \mathbf{k}$, and each graded component A_i is finite dimensional. Moreover, the associated monoid $S = S(X, r)$ *is naturally graded by length*:

$$S = \bigsqcup_{i \geq 0} S_i, \text{ where } S_0 = 1,\ S_1 = X,\ S_i = \{u \in S \mid \ |u| = i\},\ S_i.S_j \subseteq S_{i+j}. \tag{2.6}$$

In the sequel, by "*a graded monoid* S", we shall mean that S is generated by $S_1 = X$ and graded by length. The grading of S induces a canonical grading of its monoid algebra $\mathbf{k}S(X, r)$. The isomorphism $A \cong \mathbf{k}S(X, r)$ agrees with the canonical gradings, so there is an isomorphism of vector spaces $A_m \cong Span_{\mathbf{k}} S_m$.

By [?, Proposition 2.3.] If (X, r) is a nondegenerate involutive quadratic set of finite order $|X| = n$ then the set $\Re(r)$ consists of precisely $\binom{n}{2}$ quadratic relations. In this case the associated algebra $A = A(\mathbf{k}, X, r)$ satisfies $\dim A_2 = \binom{n+1}{2}$.

Remark 2.9. [?] Let (X, r) be an involutive quadratic set, and let $S = S(X, r)$ be the associated monoid.

(i) By definition, two monomials $w, w' \in \langle X \rangle$ are equal in S *iff* w can be transformed to w' by a finite sequence of replacements each of the form

$$axyb \longrightarrow ar(xy)b, \qquad \text{where } x, y \in X, a, b \in \langle X \rangle.$$

Clearly, every such replacement preserves monomial length, which therefore descends to $S(X, r)$. Furthermore, replacements coming from the defining relations are possible only on monomials of length ≥ 2, hence $X \subset S(X, r)$ is an inclusion. For monomials of length 2, $xy = zt$ holds in $S(X, r)$ *iff* $zt = r(xy)$ is an equality of words in X^2.

(ii) It is convenient for each $m \geq 2$ to refer to the subgroup $D_m = D_m(r)$ of the symmetric group $\mathrm{Sym}(X^m)$ generated concretely by the maps

$$r^{ii+1} : X^m \longrightarrow X^m,\ r^{ii+1} = \mathrm{id}_{X^{i-1}} \times r \times \mathrm{id}_{X^{m-i-1}},\ i = 1, \cdots, m-1. \tag{2.7}$$

One can also consider the free groups

$$\mathcal{D}_m(r) = {}_{\mathrm{gr}}\langle r^{ii+1} \mid i = 1, \cdots, m-1 \rangle,$$

where the r^{ii+1} are treated as abstract symbols, as well as various quotients depending on the further type of r of interest. These free groups and their quotients act on X^m via the actual maps r^{ii+1}, so that the image of $\mathcal{D}_m(r)$ in $Sym(X^m)$ is $D_m(r)$. In particular, $D_2(r) = \langle r \rangle \subset Sym(X^2)$ is the cyclic group generated by r. It follows straightforwardly from part (i) that $w, w' \in \langle X \rangle$ are equal as words in $S(X, r)$

iff they have the same length, say m, and belong to the same orbit $\mathcal{O}_{\mathcal{D}_m}$ of $\mathcal{D}_m(r)$ in X^m. In this case the equality $w = w'$ holds in $S(X,r)$ and in the algebra $A(\mathbf{k}, X, r)$.

An effective part of our combinatorial approach is the exploration of the action of the group $\mathcal{D}_2(r) = \langle r \rangle$ on X^2, and the properties of the corresponding orbits. In the literature a $\mathcal{D}_2(r)$-orbit $\mathcal{O}$ in X^2 is often called "*an r-orbit*" and we shall use this terminology.

In notation and assumption as above, let (X, r) be a finite quadratic set with $S = S(X, r)$ graded by length. Then the order of of each component S_m, in (**??**) equals the number of $\mathcal{D}_m(r)$-orbits in X^m.

Notation 2.10. [**?**] Suppose (X, r) is a quadratic set. The element $xy \in X^2$ is *an r-fixed point* if $r(x, y) = (x, y)$. *The set of r-fixed points* in X^2 will be denoted by $\mathcal{F}(X, r)$:

$$\mathcal{F}(X,r) = \{xy \in X^2 \mid r(x,y) = (x,y)\}. \tag{2.8}$$

The following corollary is a consequence of [**?**, Lemma 3.7]

Corollary 2.11. *Let (X, r) be a nondegenerate symmetric set of finite order $|X| = n$.*

(1) For every $x \in X$ there exists a unique $y \in X$ such that $r(x, y) = (x, y)$, so $\mathcal{F} = \mathcal{F}(X, r) = \{x_1y_1, \cdots, x_ny_n\}$. In particular, $|\mathcal{F}(X, r)| = |X| = n$. In the special case, when (X, r) is a square-free solution, one has $\mathcal{F}(X, r) = \Delta_2$, the diagonal of X^2.
(2) The number of non-trivial r-orbits is exactly $\binom{n}{2}$.
(3) The set $X \times X$ splits into $\binom{n+1}{2}$ r-orbits.

3. The quadratic algebra $\mathcal{A}(\mathbf{k}, X, r)$ of a finite nondegenerate symmetric set (X, r)

It was proven through the years that the Yang-Baxter algebras $\mathcal{A}(\mathbf{k}, X, r)$ coresponding to finite nondegenerate symmetric sets have remarkable algebraic and homological properties. They are noncommutative, but have many of the "good" properties of the commutative polynomial ring $\mathbf{k}[x_1, \cdots, x_n]$, see Facts **??** and **??**. This motivates us to look for more analogues coming from commutative algebra and algebraic geometry.

3.1. **Basic facts about the YB algebras $\mathcal{A}(\mathbf{k}, X, r)$ of finite solutions** (X, r)**.** Suppose (X, r) is a finite solution of order n, and let $\mathcal{A}$ be its Yang-Baxter algebra. In the case, when (X, r) is square-free there exists an enumeration $X = \{x_1, \cdots, x_n\}$, so that $\mathcal{A}$ is a binomial skew-polynomial ring, see Fact **??**, and by convention we shall always fix such an enumeration of X. If (X, r) is not square-free then the algebra $\mathcal{A}$ is not PBW with respect to any enumeration of X, see Theorem **??**, so by convention we fix an arbitrary enumeration $X = \{x_1, \cdots, x_n\}$. We extend the fixed enumeration on X to the degree-lexicographic ordering $<$ on $\langle X \rangle$.

By definition the Yang-Baxter algebra $\mathcal{A} = \mathcal{A}_X = \mathcal{A}(\mathbf{k}, X_n, r)$ is presented as

$$\begin{array}{c} \mathcal{A} = \mathcal{A}(\mathbf{k}, X, r) = \mathbf{k}\langle X \rangle / (\Re_0) \simeq \mathbf{k}\langle X;\ \Re(r) \rangle, \text{ where} \\ \Re_0 = \Re_0(r) = \{xy - y'x' \mid r(x,y) = (y', x'), \text{and } xy > y'x'.\}. \end{array} \tag{3.1}$$

Consider the two-sided ideal $I = (\Re_0)$ of $\mathbf{k}\langle X \rangle$, let $G = G(I)$ be the unique reduced Gröbner basis of I with respect to $<$. It follows from the shape of the relations $\Re_0$ that $G(I)$ is finite, or countably infinite, and consists of homogeneous binomials $f_j = u_j - v_j$, with $\mathbf{LM}(f_j) = u_j > v_j$, $|u_j| = |v_j|$.

The set of all normal monomials modulo I is denoted by $\mathcal{N}$. As we mentioned in Section 2, $\mathcal{N} = \mathcal{N}(I) = \mathcal{N}(G)$. An element $f \in \mathbf{k}\langle X \rangle$ is in normal form (modulo I), if $f \in \mathrm{Span}_{\mathbf{k}}\mathcal{N}$. The free monoid $\langle X \rangle$ splits as a disjoint union $\langle X \rangle = \mathcal{N} \sqcup \mathbf{LM}(I)$. The free associative algebra $\mathbf{k}\langle X \rangle$ splits as a direct sum of $\mathbf{k}$-vector subspaces $\mathbf{k}\langle X \rangle \simeq \mathrm{Span}_{\mathbf{k}}\mathcal{N} \oplus I$, and there is an isomorphism of vector spaces $\mathcal{A} \simeq \mathrm{Span}_{\mathbf{k}}\mathcal{N}$. We define

$$\mathcal{N}_m = \{u \in \mathcal{N}(I) \mid u \text{ has length } m\}. \tag{3.2}$$

Then $\mathcal{A}_m \simeq \mathrm{Span}_{\mathbf{k}}\mathcal{N}_m$ for every $m \in \mathbb{N}_0$. In particular $\dim \mathcal{A}_m = |\mathcal{N}_m|$, $\forall m \geq 0$.

Note that since $\Re_0$ consists of a finite set of homogeneous polynomials, the elements of the reduced Gröbner basis $G = G(I)$ of degree $\leq m$ can be found effectively, (using the standard strategy for constructing a Gröbner basis) and therefore the set of normal monomials $\mathcal{N}_m$ can be found inductively for $m = 1, 2, 3, \cdots$. Here we do not need an explicit description of the reduced Gröbner basis $G(I)$ of I.

We can also determine the set $\mathcal{N}_m$ of normal monomials of degree m in a natural (and direct) way using the discussion in Remark **??** and avoiding the standard Gröbner basis techniques. Recall that in our settings the normal form of a monomial $v \in \langle X \rangle$ is a monomial of the same length, there is an equality in $\mathcal{A}$ (and in S) $v = \mathrm{Nor}(v)$ and $v \geq \mathrm{Nor}(v)$, as words in X^m. Consider the set of all distinct $\mathcal{D}_m(r)$-orbits in X^m, say $\mathcal{O}^i = \mathcal{O}^i_{\mathcal{D}_m}, 1 \leq i \leq p$, where $p = |S_m| = \dim \mathcal{A}_m$. Each orbit is finite and has unique minimal element $u_i \in \mathcal{O}$ (w.r.t. $<$). Then $u_i \in \mathcal{N}_m$, and every $v \in \mathcal{O}^i$ satisfies $v \geq u_i$ (as words), $\mathrm{Nor}(v) = u_i$, and the equality $v = u_i$ holds in A (and in S). In particular,

$$\mathcal{N}_m = \{u_i \mid u_i \text{ is a minimal element of } \mathcal{O}^i, 1 \leq i \leq p\}.$$

The following conventions will be kept in the sequel.

Convention 3.1. Let (X, r) be a finite nondegenerate symmetric set of order n, and Let $A = A(\mathbf{k}, X, r)$ be the associated Yang-Baxter algebra, with presentation (**??**). (a) If (X, r) is not square-free we fix an arbitrary enumeration $X = \{x_1, \cdots, x_n\}$ on X and extend it to degree-lexicographic ordering $<$ on $\langle X \rangle$; (b) If (X, r) is square-free we fix an enumeration such that $X = \{x_1, \cdots, x_n\}$ is a set of PBW generators of A.

Let $\mathcal{N}$ be the set of normal monomials modulo the ideal $I = (\Re_0)$. It follows from Bergman's Diamond lemma, [**?**, Theorem 1.2], that if we consider the spase $\mathbf{k}\mathcal{N}$ endowed with multiplication defined by

$$f \bullet g := \mathrm{Nor}(fg), \quad \text{for every } f, g \in \mathbf{k}\mathcal{N}$$

then $(\mathbf{k}\mathcal{N}, \bullet)$ has a well-defined structure of a graded algebra, and there is an isomorphism of graded algebras

$$A = A(\mathbf{k}, X_n, r) \cong (\mathbf{k}\mathcal{N}, \bullet). \tag{3.3}$$

By convention we shall identify the algebra $\mathcal{A}$ with $(\mathbf{k}\mathcal{N}, \bullet)$. Similarly, we consider an operation $\bullet$ on the set $\mathcal{N}$, with $a \bullet b := \mathrm{Nor}(ab)$, for every $a, b \in \mathcal{N}$ and identify the monoid $S = S(X, r)$ with $(\mathcal{N}, \bullet)$, see [**?**], Section 6.

In the case when (X, r) is square-free, the set of normal monomials is exactly $\mathcal{T}$, so $\mathcal{A}$ is identified with $(\mathbf{k}\mathcal{T}, \bullet)$ and $S(X, r)$ is identified with $(\mathcal{T}, \bullet)$.

The identification (**??**) gives

$$\mathcal{A} = \bigoplus_{m \in \mathbb{N}_0} A_m \cong \bigoplus_{m \in \mathbb{N}_0} \mathbf{k}\mathcal{N}_m.$$

We shall recall some important properties of the Yang-Baxter algebras which will be used in the sequel, but first we need a lemma.

Lemma 3.2. *Suppose (X, r) is a nondegenerate involutive quadratic set (not necessarily finite). Then the following condition hold*

***O**: Given $a, b \in X$ there exist unique $c, d \in X$, such that $r(c, a) = (d, b)$. Furthermore, if $a = b$ then $c = d$.*

In particular, r is 2-cancellative.

Proof. Let (X, r) be a nondegenerate involutive quadratic set (not necessarily finite). Let $a, b \in X$. We have to find unique pair c, d, such that $r(c, a) = (d, b)$. By the nondegeneracy there is unique $c \in X$, such that $c^a = b$. Let $d = {}^c a$, then $r(c, a) = ({}^c a, c^a) = (d, b)$, as desired. It also follows from the nondegeneracy that the pair c, d with this propery is unique. Assume now that $a = b$. The equality $r(c, a) = (d, a)$ implies $({}^c a, c^a) = (d, a)$, so $c^a = a$. But r is involutive, thus $(c, a) = r(d, a) = ({}^d a, d^a)$, and therefore $d^a = a$. It follows that $c^a = d^a$, and, by the nondegeneracy, $c = d$. □

The following results are extracted from [**?**].

Facts 3.3. Suppose (X, r) is a nondegenerate symmetric set of order n, $X = \{x_1, \cdots, x_n\}$, let $S = S(X, r)$ be the associated monoid and $A = A(\mathbf{k}, X, r)$ the associated Yang-Baxter algebra (A is isomorphic to the monoid algebra $\mathbf{k}S$). Then the following conditions hold.

(1) S is a semigroup of I-type, that is there is a bijective map $v : \mathcal{U} \mapsto S$, where $\mathcal{U}$ is the free n-generated abelian monoid $\mathcal{U} = [u_1, \cdots, u_n]$ such that $v(1) = 1$, and such that

$$\{v(u_1 a), \cdots, v(u_n a)\} = \{x_1 v(a), \cdots, x_n v(a)\}, \text{ for all } a \in \mathcal{U}.$$

(2) The Hilbert series of A is $H_A(t) = 1/(1-t)^n$.
(3) [?, Theorem 1.4] (a) A has finite global dimension and polynomial growth; (b) A is Koszul; (c) A is left and right Noetherian; (d) A satisfies the Auslander condition and is is Cohen-Macaulay, (e) A is finite over its center.
(4) [?, Corollary 1.5] A is a domain, and in particular the monoid S is cancellative.

Note that (1) is a consequence of Theorem 1.3 in [?] the second part of which states: if (X, r) is an involutive solution of YBE, which satisfies condition **O**, see Lemma **??**, then S is a semigroup of I-type. Lemma **??** shows that even weaker assumptions that (X, r) is a nondegenerate and involutive quadratic set imply the needed condition **O**, so S is a semigroup of I-type. Part (2) is straightforward from (1).

Corollary 3.4. *In notation and convensions as above. Let (X, r) be a nondegenerate symmetric set of order n. Then for every integer $d \geq 1$ there are equalities*

$$\dim \mathcal{A}_d = \binom{n+d-1}{d} = |\mathcal{N}_d|. \tag{3.4}$$

We recall some important properties of the square-free solutions, especially interesting is the implication (**??**)$\Longrightarrow$ (**??**).

Fact 3.5. [?, Theorem 1.2] Suppose (X, r) is a nondegenerate, square-free, and involutive quadratic set of order $|X| = n$, and let $A = A(\mathbf{k}, X, r)$ be its quadratic algebra. The following conditions are equivalent:

(1) (X, r) is a solution of the Yang-Baxter equation.
(2) A is a binomial skew polynomial ring, with respect to an enumeration of X.
(3) A is an Artin–Schelter regular PBW algebra, that is
  (a) A has polynomial growth of degree n (equivalently, GKdim $A = n$);
  (b) A has finite global dimension $\operatorname{gl\,dim} \mathcal{A} = n$;
  (c) A is Gorenstein, meaning that $Ext^q_A(\mathbf{k}, A) = 0$ if $q \neq n$ and $Ext^n_A(\mathbf{k}, A) \cong \mathbf{k}$.
(4) The Hilbert series of A is $H_A(t) = 1/(1-t)^n$.

Each of these conditions implies that A is Koszul and a Noetherian domain.

Question 3.6. Suppose (X, r) is a finite non-degenerate symmetric set, and assume that the Yang-Baxter algebra $A = A(\mathbf{k}, X, r)$ is PBW, where $X = \{x_1, x_2, \cdots, x_n\}$ is a set of PBW generators. Is it true that (X, r) is square-free?

3.2. **Every finite solution (X, r) whose Yang-Baxter algebra $\mathcal{A}(\mathbf{k}, X, r)$ is PBW is square-free.** In this subsection we give a positive answer to Question **??**.

Suppose (X, r) is a finite non-degenerate symmetric set, and assume that the Yang-Baxter algebra $A = A(\mathbf{k}, X, r)$ is PBW, where $X = \{x_1, x_2, \cdots, x_n\}$ is a set of PBW generators. Then $\mathcal{A} = \mathbf{k}\langle X\rangle/(\Re_0)$, where the set of (quadratic) defining relations $\Re_0$ of $\mathcal{A}$ coinsides with the reduced Gröbner basis of the ideal $(\Re_0)$ modulo the degree-lexicographic ordering on $\langle X\rangle$. Recall that the set of leading monomials

$$W = \{LM(f) \mid f \in \Re_0\} \tag{3.5}$$

is called *the set of obstructions* for $\mathcal{A}$, in the sense of Anick, [?], see (**??**).

Lemma 3.7. *Suppose (X, r) is a nondegenerate symmetric set of order n, and assume the Yang-Baxter algebra $A = A(\boldsymbol{k}, X, r)$ is PBW, where $X = \{x_1, x_2, \cdots, x_n\}$ is a set of PBW generators. Then there exists a permutation*

$$y_1 = x_{s_1},\ y_2 = x_{s_2},\ \cdots,\ y_n = x_{s_n} \text{ of } x_1, x_2, \cdots, x_n,$$

such that the following conditions hold.

(1) The set of obstructions $W = \{LM(f) \mid f \in \Re_0\}$ consists of $\binom{n}{2}$ monomials given below

$$W = \{y_j y_i \mid 1 \leq i < j \leq n\}. \tag{3.6}$$

(2) The normal $\boldsymbol{k}$-basis of $\mathcal{A}$ modulo $I = (\Re_0)$ is the set

$$\mathcal{N} = \{y_1^{\alpha_1} y_2^{\alpha_2} \cdots y_n^{\alpha_n} \mid \alpha_i \geq 0, \text{ for } 1 \leq i \leq n\}. \tag{3.7}$$

Proof. Let W be the set of obstructions defined via (**??**) and let A_W be *the associated monomial algebra* defined as

$$A_W := \mathbf{k}\langle X\rangle/(W) \tag{3.8}$$

It is well known that a word $u \in \langle X\rangle$ is normal modulo $I = (\Re_0)$ *iff* u is normal modulo the set of obstructions W. Therefore the two algebras $\mathcal{A}$ and A_W share the same normal $\mathbf{k}$-basis $\mathcal{N} = \mathcal{N}(I) = \mathcal{N}(W)$ and their Hilbert series are equal. By Facts **??** part (2), The Hilbert series of $\mathcal{A}$ is $H_{\mathcal{A}}(t) = 1/(1-t)^n$, therefore

$$H_{A_W}(t) = H_{\mathcal{A}}(t) = 1/(1-t)^n. \tag{3.9}$$

Thus the Hilbert series of A_W satisfies condition (5) of [**?**, Theorem 3.7] (see page 2163), and it follows from the theorem that there exists a permutation $y_1 = x_{s_1},\ y_2 = x_{s_2},\ \cdots,\ y_n = x_{s_n}$ of the generators $x_1, x_2, \cdots, x_n$, such that the set of obstructions W satisfies (**??**). The Diamond Lemma, [**?**] and the explicit description (**??**) of the obstruction set W imply that the set of normal words $\mathcal{N} = \mathcal{N}(I) = \mathcal{N}(W)$ is described in (**??**). □

It is clear that if the permutation given in the lemma is not trivial there is an inversion, that is a pair i, j with $i < j$ and $y_j < y_i$.

Theorem 3.8. *Suppose (X, r) is a nondegenerate symmetric set of order n, and $\mathcal{A} = \mathcal{A}(\boldsymbol{k}, X, r)$ is its Yang-Baxter algebra. Then $\mathcal{A}$ is a PBW algebra with a set of PBW generators $X = \{x_1, x_2, \cdots, x_n\}$ (enumerated properly) if and only if (X, r) is a square-free solution.*

Proof. It well known that if (X, r) is square-free then there exists an enumeration $X = \{x_1, \cdots, x_n\}$, so that $\mathcal{A}$ is a binomial skew-polynomial ring in the sense of [**?**], and therefore $\mathcal{A}$ is PBW, see Fact **??**

Assume now that (X, r) is a finite solution of order n whose YB-algebra $A = A(\mathbf{k}, X, r)$ is PBW, where $X = \{x_1, x_2, \cdots, x_n\}$ is a set of PBW generators. We have to show that (X, r) is square-free that is $r(x, x) = (x, x)$, for all $x \in X$.

It follows from our assumptions that in the presentation $\mathcal{A} = \mathbf{k}\langle X\rangle/(\Re_0)$ the set of (quadratic) defining relations $\Re_0$ of $\mathcal{A}$ is the reduced Gröbner basis of the ideal $(\Re_0)$ modulo the degree-lexicographic ordering on $\langle X\rangle$. By Lemma **??** there exists exists a permutation $y_1 = x_{s_1}, y_2 = x_{s_2}, \cdots, y_n = x_{s_n}$ of $x_1, x_2, \cdots, x_n$ such that the obstruction set $W = \{\mathbf{LM}(f) \mid f \in \Re_0\}$ satisfies (**??**) and the set of normal monomials $\mathcal{N}$ described in (**??**) is a PBW basis of $\mathcal{A}$.

We use some properties of (X, r) and the relations of $\mathcal{A}$ listed below.

(i) (X, r) is 2-cancellative. This follows from Lemma **??**; (ii) There are exactly n fixed points $xy \in X^2$ with $r(x, y) = (x, y)$. This follows from [**?**, Lemma 3.7], part (3), since (X, r) is nondegenerate and 2-cancellative. (iii) Every monomial of the shape $y_j y_i,\ 1 \le i < j \le n$ is the leading monomial of some polynomial $f_{ji} \in \Re_0$. (It is possible that $y_j < y_i$ for some $j > i$.) It follows from [**?**, Proposition 2.3.] that for a nondegenerate involutive quadratic set (X, r) of order n the set $\Re_0$ consists of exactly $\binom{n}{2}$ relations. (iv) Therefore the algebra $\mathcal{A}$ has a presentation

$$\mathcal{A} = \mathbf{k}\langle x_1, \cdots, x_n\rangle/(\Re_0)$$

with precisely $\binom{n}{2}$ defining relations

$$\Re_0 = \{f_{ji} = y_j y_i - u_{ij} \mid 1 \le i < j \le n\} \tag{3.10}$$

such that

1. For every pair $i, j,\ 1 \le i < j \le n$, the monomial u_{ij} satisfies $u_{ij} = y_{i'} y_{j'}$, where $i' \le j'$, and $y_j > y_{i'}$ (since $\mathbf{LM}(f_{ji}) = y_j y_i > y_{i'} y_{j'}$, and since (X, r) is 2-cancellative);
2. Each monomial $y_i y_j$ with $1 \le i \le j \le n$ occurs at most once in $\Re_0$ (since r is a bijective map).
3. $\Re_0$ is the *reduced Gröbner basis* of the two-sided ideal $(\Re_0)$, with respect to the degree-lexicographic order $<$ on $\langle X\rangle$.

In terms of the relations $\Re_0$ our claim that $r(x, x) = (x, x)$, for all $x \in X$, is equivalent to

$$u_{ij} \ne xx, \quad \text{where } x \in X, \text{ and } 1 \le i < j \le n. \tag{3.11}$$

So far we know that (X, r) has exactly n fixed points, and each monomial $y_j y_i, 1 \leq i < j \leq n$ is not a fixed point. Therefore it will be enough to show that a monomial $y_i y_j$, with $1 \leq i < j \leq n$, can not be a fixed point.

Assume on the contrary, that $r(y_i, y_j) = (y_i, y_j)$, for some $1 \leq i < j \leq n$. We claim that in this case $\Re_0$ contains two relations of the shape

$$(a) \quad y_p y_q - y_j y_j, \text{ where } p > q, y_p > y_j, \text{ and } (b) \quad y_s y_t - y_i y_i, \text{ where } s > t, y_s > y_i. \tag{3.12}$$

Consider the increasing chain of left ideals of $\mathcal{A}$

$$I_1 \subseteq I_2 \subseteq \cdots \subseteq I_k \subseteq \cdots,$$

where for $k \geq 1$, I_k is the left ideal

$$I_k = {}_{\mathcal{A}}(y_i y_j,\ y_i y_j^2,\ \cdots\ y_i y_j^k).$$

By [**?**, Theorem 1.4], see also Facts **??** (3) the algebra $\mathcal{A}$ is left Noetherian hence there exists $k > 1$, such that $I_{k-1} = I_k = I_{k+1} = \cdots$, and therefore $y_i y_j^k \in I_{k-1}$. This implies

$$w \bullet (y_i y_j^r) = y_i y_j^k \in \mathcal{N}, \text{ for some } r,\ 1 \leq r \leq k-1, \text{ and some } w \in \mathcal{N}, |w| = k - r. \tag{3.13}$$

It follows from (**??**) that the monomial $v_0 = y_i y_j^k$ can be obtained from the monomial $w(y_i y_j^r)$ by applying a finite sequence of replacements (reductions) in $\langle X \rangle$. More precisely, there exists a sequence of monomials

$$v_0 = y_i y_j^k,\ v_1, \cdots,\ v_{t-1},\ v_t = w(y_i y_j^r) \in \langle X \rangle$$

and replacements

$$v_t \to v_{t-1} \to \cdots \to v_1 \to v_0 = y_i y_j^k \in \mathcal{N}, \tag{3.14}$$

where each replacement comes from some quadratic relation $f_{pq} = y_p y_q - u_{qp}$ in (**??**) and has the shape

$$a[y_p y_q]b \to a(u_{qp})b, \text{ where } n \geq p > q \geq 1,\ a, b \in \langle X \rangle.$$

We have assumed that $y_i y_j$ is a fixed point, so it can not occur in a relation in (**??**). Thus the rightmost replacement in (**??**) is of the form

$$u_1 = y_i y_j \cdots y_j [y_p y_q] \cdots y_j \to y_i y_j \cdots y_j (u_{qp}) \cdots y_j = y_i y_j \cdots y_j (y_j y_j) \cdots y_j = v_0$$

where p, q is a pair with, $1 \leq q < p \leq n$, $u_{qp} = y_j y_j$ and $y_p > y_j$. In other words the set $\Re_0$ contains a relation of type (a) $y_p y_q - y_j y_j$, where $p > q, y_p > y_j$.

Analogous argument proves the existence of a relation of the type (b) in (**??**). This time we consider an increasing chain of right ideals $I_1 \subseteq I_2 \subseteq \cdots \subseteq I_k \subseteq \cdots$, where I_k is the right ideal $I_k = (y_i y_j, y_i^2 y_j, \cdots, y_i^k y_j)_{\mathcal{A}}$ and apply the right Noetherian property of $\mathcal{A}$.

Consider now the subset of fixed points

$$\mathcal{F}_0(X, r) = \{y_i y_j \in X^2 \text{ such that } \ i < j \text{ and } \ r(y_i, y_j) = (y_i, y_j)\},$$

which by our assumtion is not empty. Then $\mathcal{F}_0(X, r)$ has cardinality $m \geq 1$ and $\Re_0$ contains at least $m + 1$ (distinct) relations of the type

$$y_p y_q - xx, \text{ where } x \in X, p > q \text{ and } y_p > x. \tag{3.15}$$

The set $\mathcal{N}_2$ of normal monomials of length 2 contains $\binom{n}{2}$ elements of the shape $y_s y_t, 1 \leq s < t \leq n$, and we have assummed that m of them are fixed. Then there are $\binom{n}{2} - m$ distinct monomials $y_i y_j \in \mathcal{N}_2, 1 \leq i < j \leq n$, which are not fixed. Each of these monomials occurs in exactly one relation

$$y_s y_t - y_i y_j, \text{ where } r(y_s, y_t) = (y_i, y_j),\ s > t,\ y_s > y_i.$$

Thus $\Re_0$ contains $\binom{n}{2} - m$ distinct square-free relations and at least $m+1$ relations which contain squares as in (**??**) . Therefore the set of relations has cardinality

$$|\Re_0| \geq \binom{n}{2} - m + m + 1 > \binom{n}{2},$$

which is a contradiction.

We have shown that a monomial y_iy_j with $1 \leq i < j \leq n$ can not be a fixed point, and therefore occurs in a relation in $\Re_0$. But (X,r) has exactly n fixed points, so these are the elements of the diagonal of X^2 , $x_ix_i, 1 \leq i \leq n$. It follows that (X,r) is square-free. □

Proposition 3.9. *Let (X,r) be a finite non-degenerate involutive quadratic set, and let $\mathcal{A} = \mathcal{A}(\boldsymbol{k}, X, r) = \boldsymbol{k}\langle X\rangle/(\Re_0)$ be its quadratic algebra. Assume that there is an enumeration $X = \{x_1, x_2, \cdots, x_n\}$ of X such that the set*

$$\mathcal{N} = \{x_1^{\alpha_1}x_2^{\alpha_2}\cdots x_n^{\alpha_n} \mid \alpha_i \geq 0 \text{ for } 1 \leq i \leq n\}$$

is a normal $\boldsymbol{k}$-basis of $\mathcal{A}$ modulo the ideal $I = (\Re_0)$. Then $\mathcal{A} = \mathcal{A}(\boldsymbol{k}, X, r)$ is a PBW algebra, where $X = \{x_1, x_2, \cdots, x_n\}$ is a set of PBW generators of $\mathcal{A}$ and the set of relations $\mathcal{R}_0$ is a quadratic Gröbner basis of the two-sided ideal $(\Re_0)$. The following conditions are equivalent.

1. *The algebra $\mathcal{A}$ is left and right Noetherian.*
2. *The quadratic set (X,r) is square-free.*
3. *(X,r) is a solution of YBE.*
4. *$\mathcal{A}$ is a binomial skew polymomial ring in the sense of* [**?**].

Proof. The quadratic set (X,r) and the relations of $\mathcal{A}$ satisfy conditions similar to those listed in the proof of Theorem **??**. More precisely: (i) (X,r) is 2-cancellative. This follows from Lemma **??**; (ii) There are exactly n fixed points $xy \in X^2$ with $r(x,y) = (x,y)$. This follows from [**?**, Lemma 3.7], part (3), since (X,r) is nondegenerate and 2-cancellative. (iii) It follows from the hypothesis that every monomial of the shape $x_jx_i, 1 \leq i < j \leq n$, is not in the normal $\mathbf{k}$ -basis $\mathcal{N}$, and therefore it is the highest monomial of some polynomial $f_{ji} \in \Re_0$. [**?**, Proposition 2.3.] implies that if (X,r) is a nondegenerate involutive quadratic set of order n then the set $\Re_0$ consists of exactly $\binom{n}{2}$ relations. Therefore the algebra $\mathcal{A}$ has a presentation

$$\mathcal{A} = \mathbf{k}\langle x_1, \cdots, x_n\rangle/(\Re_0)$$

with precisely $\binom{n}{2}$ defining relations

$$\Re_0 = \{f_{ji} = x_jx_i - x_{i'}x_{j'} \mid 1 \leq i < j \leq n\} \tag{3.16}$$

such that

(a) For every pair $i,j,\ 1 \leq i < j \leq n$, one has $i' \leq j'$, and $j > i'$ (since $\mathbf{LM}(f_{ji}) = x_jx_i > x_{i'}x_{j'}$, and since (X,r) is 2-cancellative);
(b) Each ordered monomial (term) of length 2 occurs at most once in $\Re_0$ (since r is a bijective map).
(c) $\Re_0$ is the *reduced Gröbner basis* of the two-sided ideal $(\Re_0)$, with respect to the degree-lexicographic order $<$ on $\langle X\rangle$, or equivalently the overlaps $x_kx_jx_i$, with $k > j > i$ do not give rise to new relations in $\mathcal{A}$.

We give a sketch of the proof of the equivalence of conditions (1) through (4).

(1) $\Rightarrow$ (2) . The proof is analogous to the proof of Theorem **??**. It is enough to show that a monomial x_ix_j with $1 \leq i < j \leq n$, can not be a fixed point. Assuming the contrary, and applying an argument similar to the proof of Theorem **??**, in which we involve the left and right Noetherian properties of $\mathcal{A}$, we get a contradiction. Thus every monomial x_ix_j with $1 \leq i < j \leq n$ occurs in a relation in $\Re_0$. At the same time the monomials x_jx_i with $1 \leq i < j \leq n$ are also involved in the relations $\Re_0$, hence they are not fixed points. But (X,r) has exactly n fixed points, so these are the elements of the diagonal of X^2 , $x_ix_i, 1 \leq i \leq n$. It follows that (X,r) is square-free.

(2) $\Rightarrow$ (4). If (X,r) is square-free then the relations $\Re_0$ given in (**??**) are exactly the defining relations of a binomial skew polynomial ring, which form a reduced Gröbner basis, therefore $\mathcal{A}$ is a skew polynomial ring with binomial relations in the sence of [**?**].

The implication (4) $\Rightarrow$ (3) follows from [**?**, Theorem 1.1].

The implication (3) $\Rightarrow$ (1) follows from [**?**, Theorem 1.4], see also Facts **??** (3). □

4. The d-Veronese subalgebra $\mathcal{A}^{(d)}$ of $\mathcal{A}(\mathbf{k}, X, r)$, its generators and relations

In this section (X,r) is a finite solution (a nondegenerate symmetric set) , $d \geq 2$ is an integer. We shall study the d-Veronese subalgebras $\mathcal{A}^{(d)}$ of the Yang-Baxter algebra $\mathcal{A} = \mathcal{A}(\mathbf{k}, X, r)$. This is an algebraic construction which mirrors the Veronese embedding. Some of the first results on Veronese subalgebras of noncommutative graded algebras appeared in [**?**] and [**?**]. Our main reference here is [**?**, Section 3.2]. The main result of this section is Theorem **??** which presents the d-Veronese subalgebra $\mathcal{A}^{(d)}$ in terms of generators and quadratic relations.

4.1. **Veronese subalgebras of graded algebras.** We recall first some basic definitions and facts about Veronese subalgebras of general graded algebras.

Definition 4.1. Let $A = \bigoplus_{k\in\mathbb{N}_0} A_k$ be a graded algebra. For any integer $d \geq 1$, the *d-Veronese subalgebra* of A is the graded algebra

$$A^{(d)} = \bigoplus_{k\in\mathbb{N}_0} A_{kd}.$$

By definition the algebra $A^{(d)}$ is a subalgebra of A. However, the embedding is not a graded algebra morphism. The Hilbert function of $A^{(d)}$ satisfies

$$h_{A^{(d)}}(t) = \dim(A^{(d)})_t = \dim(A_{td}) = h_A(td).$$

It follows from [**?**, Proposition 2.2, Chapter 3] that if A is a one-generated quadratic Koszul algebra, then its Veronese subalgebras are also one-generated quadratic and Koszul.

Corollary 4.2. *Let (X, r) be a solution of order n, and let $\mathcal{A} = \mathcal{A}(\boldsymbol{k}, X, r)$ be its Yang-Baxter algebra, let $d \geq 2$ be an integer. (1) The d-Veronese algebra $\mathcal{A}^{(d)}$ is one-generated, quadratic and Koszul. (2) $\mathcal{A}^{(d)}$ is a Noetherian domain.*

Proof. (1) If (X, r) is a solution of order n then, by definition the Yang-Baxter algebra $\mathcal{A} = \mathcal{A}(\mathbf{k}, X, r)$ is one-generated and quadratic. Moreover, $\mathcal{A}$ is Koszul, see Facts **??**. It follows straightforwardly from [**?**, Proposition 2.2, Ch 3] that $\mathcal{A}^{(d)}$ is one-generated, quadratic and Koszul. (2) The d-Veronese $\mathcal{A}^{(d)}$ is a subalgebra of $\mathcal{A}$ which is a domain, see Facts **??**. Theorem **??** implies that $A^{(d)}$ a homomorphic image of the Yang-Baxter algebra $\mathcal{A}_Y = \mathcal{A}(\mathbf{k}, Y, r_Y)$, where (Y, r_Y) is the d-Veronese solution associated with (X, r). The algebra $\mathcal{A}_Y$ is Noetherian, since the solution (Y, r_Y) is finite, see Facts **??**. □

In the assumptions of Corollary **??**, it is clear, that the d-Veronese subalgebra $\mathcal{A}^{(d)}$ satisfies

$$\mathcal{A}^{(d)} = \bigoplus_{m\in\mathbb{N}_0} A_{md} \cong \bigoplus_{m\in\mathbb{N}_0} \mathbf{k}\mathcal{N}_{md}. \tag{4.1}$$

Moreover, the normal monomials $w \in \mathcal{N}_d$ of length d are degree one generators of $A^{(d)}$, and by Corollary **??** there are equalities

$$|\mathcal{N}_d| = \dim \mathcal{A}_d = n + d - 1d.$$

We set

$$N = \binom{n+d-1}{d}$$

and order the elements of $\mathcal{N}_d$ lexicographically:

$$\mathcal{N}_d := \{w_1 < w_2 < \cdots < w_N\}. \tag{4.2}$$

The d-Veronese $\mathcal{A}^{(d)}$ is a quadratic algebra (one)-generated by $w_1, w_2, \ldots, w_N$. We shall find a minimal set of quadratic relations for $\mathcal{A}^{(d)}$, each of which is a linear combination of products $w_i w_j$ for some $i, j \in \{1, \ldots, N\}$. The relations are intimately connected with the properties of the braided monoid $S(X, r)$. As a first step we shall introduce a nondegenerate symmetric set (S_d, r_d) of order N, induced in a natural way by the braided monoid $S(X, r)$.

4.2. **The braided monoid $S = S(X, r)$ of a braided set.** Matched pairs of monoids, M3-monoids and braided monoids in a most general setting were studied in [**?**], where the interested reader can find the necessary definitions and the original results. Here we extract only some facts which will be used in the paper.

Fact 4.3. ([**?**], Theor. 3.6, Theor. 3.14.) Let (X, r) be a braided set and let $S = S(X, r)$ be the associated monoid. Then

(1) The left and the right actions ${}^{(\ )}\circ : X \times X \longrightarrow X$, and $\circ^{(\ )} : X \times X \longrightarrow X$ defined via r can be extended in a unique way to a left and a right action

$$ {}^{(\ )}\circ : S \times S \longrightarrow S, \ \ (a, b) \mapsto {}^{a}b, \text{ and } \circ^{(\ )} : S \times S \longrightarrow S, \ \ (a, b) \mapsto a^{b} $$

which make S *a strong graded* **M3**-*monoid*. In particular, the following equalities hold in S for all $a, b, u, v \in S$.

$$ \begin{array}{llll} ML0: & {}^{a}1 = 1, \quad {}^{1}u = u; & MR0: & 1^{u} = 1, \quad a^{1} = a \\ ML1: & {}^{(ab)}u = {}^{a}({}^{b}u), & MR1: & a^{(uv)} = (a^{u})^{v} \\ ML2: & {}^{a}(u.v) = ({}^{a}u)({}^{a^{u}}v), & MR2: & (a.b)^{u} = (a^{{}^{b}u})(b^{u}) \\ M3: & {}^{u}v u^{v} = uv. & & \end{array} \tag{4.3} $$

These actions define an associated bijective map

$$ r_S : S \times S \longrightarrow S \times S, \quad r_S(u, v) = ({}^{u}v, u^{v}) $$

which obeys the Yang-Baxter equation, so (S, r_S) is *a braided monoid*. In particular, (S, r_S) is a set-theoretic solution of YBE, and the associated bijective map r_S restricts to r.

(2) The following conditions hold.
   (a) (S, r_S) is *a graded braided monoid*, that is the actions agree with the grading of S:

$$ |{}^{a}u| = |u| = |u^{a}|, \forall\, a, u \in S. \tag{4.4} $$

   (b) (S, r_S) is non-degenerate *iff* (X, r) is non-degenerate.
   (c) (S, r_S) is involutive *iff* (X, r) is involutive.
   (d) (S, r_S) is square-free *iff* (X, r) is a trivial solution.

Let (X, r) be a non-degenerate symmetric set, let (S, r_S) be the associated graded braided monoid, where we consider the natural grading by length given in (**??**):

$$ S = \bigsqcup_{d \in \mathbb{N}_0} S_d, \ \ S_0 = \{1\}, S_1 = X, \ \text{and} \ S_k S_m \subseteq S_{k+m}. $$

Each of the graded components S_d, $d \geq 1$, is r_S-invariant, that is $r_S(S_d \times S_d) \subseteq S_d \times S_d$.

Consider the restriction $r_d = (r_S)_{|S_d \times S_d}$, where r_d is the map $r_d : S_d \times S_d \longrightarrow S_d \times S_d$.

Corollary 4.4. *In notation as above the following conditions hold.*

(1) For every positive integer $d \geq 1$, (S_d, r_d) is a nondegenerate symmetric set. Moreover, if (X, r) is of finite order n, then (S_d, r_d) is a finite nondegenerate symmetric set of order

$$ |S_d| = \binom{n + d - 1}{d} = N. \tag{4.5} $$

(2) The number of fixed points is $|\mathcal{F}(S_d, r_d)| = N$.

Definition 4.5. We call (S_d, r_d) *the monomial d-Veronese solution associated with* (X, r).

The monomial d-Veronese solution (S_d, r_d) depends only on the map r and on d, it is invariant with respect to the enumeration of X. Although it is intimately connected with the d-Veronese $\mathcal{A}^{(d)}$ and its quadratic relations, this solution is not convenient for an explicit description of the relations. Its rich structure inherited from the braiding in (S, r_S) is used in the proof of Theorem **??**.

The solution (S_d, r_d) induces in a natural way an isomorphic solution $(\mathcal{N}_d, \rho_d)$ and the fact that $\mathcal{N}_d$ is ordered lexicographically makes this solution convenient for our description of the relations of $\mathcal{A}^{(d)}$. Note that the set $\mathcal{N}_d$, as a subset of the set of normal monomials $\mathcal{N}$, depends on the initial enumeration of X. We shall construct $(\mathcal{N}_d, \rho_d)$ below.

Remark 4.6. Note that given the monomials $a = a_1 a_2 \cdots a_p \in X^p$, and $b = b_1 b_2 \cdots b_q \in X^q$ we can find effectively the monomials ${}^{a}b \in X^q$ and $a^{b} \in X^p$. Indeed, as in [**?**], we use the conditions (**??**) to extend the left and the right actions inductively:

$$
\begin{aligned}
&{}^{c}(b_1b_2\cdots b_q) = ({}^{c}b_1)({}^{c^{b_1}}b_2)\cdots({}^{(c^{(b_1\cdots b_{q-1})})}b_q)), \quad \text{for all } c\in X\\
&{}^{(a_1a_2\cdots a_p)}b = {}^{a_1}({}^{(a_2\cdots a_p)}b).
\end{aligned}
\tag{4.6}
$$

We proceed similarly with the right action.

Lemma 4.7. *Notation as in Remark ??. Suppose $a, a_1 \in X^p, a_1 \in \mathcal{O}_{\mathcal{D}_p}(a)$, and $b, b_1 \in X^q, b_1 \in \mathcal{O}_{\mathcal{D}_q}(b)$,*

(1) The following are equalities of words in the free monoid $\langle X\rangle$:

$$\mathrm{Nor}({}^{a_1}b_1) = \mathrm{Nor}({}^{a}b), \quad \mathrm{Nor}(a_1{}^{b_1}) = \mathrm{Nor}(a^{b}). \tag{4.7}$$

In partricular, if $a, a_1 \in X^p$ and $b, b_1 \in X^q$ the equalities $a = a_1$ in S and $b = b_1$ in S imply that ${}^{a_1}b_1 = {}^{a}b$ and $a_1^{b_1} = a^b$ hold in S.

(2) The following are equalities in the monoid S:

$$ab = {}^{a}ba^{b} = \mathrm{Nor}({}^{a}b)\mathrm{Nor}(a^{b}). \tag{4.8}$$

Proof. By Remark **??** there is an equality $a = a_1$ in S *iff* $a_1 \in \mathcal{O}_{\mathcal{D}_p}(a)$, in this case $\mathcal{O}_{\mathcal{D}_p}(a) = \mathcal{O}_{\mathcal{D}_p}(a_1)$. At the same time $a = a_1$ in S *iff* $\mathrm{Nor}(a_1) = \mathrm{Nor}(a)$ as words in X^p, in particular, $\mathrm{Nor}(a) \in \mathcal{O}_{\mathcal{D}_p}(a)$. Similarly, $b_1 = b$ in S *iff* $b_1 \in \mathcal{O}_{\mathcal{D}_q}(b)$, and in this case $\mathrm{Nor}(b) = \mathrm{Nor}(b_1) \in \mathcal{O}_{\mathcal{D}_q}(b)$. Part (1) follows from the properties of the actions in (S, r_S) studied in [**?**], Proposition 3.11.

(2) (S, r_S) is an M3- braided monoid, see Fact **??**, so condition M3 implies the first equality in (**??**). Now (**??**) implies the second equality in (**??**). □

Definition-Notation 4.8. In notation and conventions as above. Let $d \geq 1$ be an integer. Suppose (X, r) is a solution of order n, $\mathcal{A} = \mathcal{A}(\mathbf{k}, X, r)$, is the associated Yang-Baxter algebra, and (S, r_S) is the associated braided monoid. By convention we identify $\mathcal{A}$ with $(\mathbf{k}\mathcal{N}, \bullet)$ and S with $(\mathcal{N}, \bullet)$. Define a left "action" and a right "action" on $\mathcal{N}_d$ as follows.

$$
\begin{aligned}
&\triangleright : \mathcal{N}_d \times \mathcal{N}_d \longrightarrow \mathcal{N}_d, \quad a\triangleright b := Nor({}^{a}b) \in \mathcal{N}_d, \quad \forall a, b \in \mathcal{N}_d\\
&\triangleleft : \mathcal{N}_d \times \mathcal{N}_d \longrightarrow \mathcal{N}_d, \quad a\triangleleft b := Nor(a^{b}) \in \mathcal{N}_d, \quad \forall a, b \in \mathcal{N}_d.
\end{aligned}
\tag{4.9}
$$

It follows from Lemma **??** (1) that the two actions are well-defined.

Define the map

$$\rho_d : \mathcal{N}_d \times \mathcal{N}_d \longrightarrow \mathcal{N}_d \times \mathcal{N}_d, \quad \rho_d(a, b) := (a\triangleright b, a\triangleleft b). \tag{4.10}$$

For simplicity of notation (when there is no ambiguity) we shall often write $(\mathcal{N}_d, \rho)$, where $\rho = \rho_d$.

Definition 4.9. We call $(\mathcal{N}_d, \rho_d)$ *the normalized d-Veronese solution associated with* (X, r).

Proposition 4.10. *In assumption and notation as above.*

(1) Let $\rho_d : \mathcal{N}_d \times \mathcal{N}_d \longrightarrow \mathcal{N}_d \times \mathcal{N}_d$ be the map defined as $\rho_d(a, b) = (a\triangleright b, a\triangleleft b)$. Then $(\mathcal{N}_d, \rho_d)$ is a nondegenerate symmetric set of order $N = \binom{n+d-1}{d}$.

(2) The symmetric sets $(\mathcal{N}_d, \rho_d)$ and (S_d, r_d) are isomorphic.

Proof. (1) By Corollary **??** (S_d, r_d) is a nondegenerate symmetric set. Thus by Remark **??** the left and the right actions associated with (S_d, r_d) satisfy conditions **l1**, **r1**, **lr3**, and **inv**. Consider the actions $\triangleright$ and $\triangleleft$ on $\mathcal{N}_d$, given in Definition-Notation **??**. It follows from (**??**) and Lemma **??** that these actions also satisfy **l1**, **r1**, **lr3** and **inv**. Therefore, by Remark **??** again, ρ_d obeys YBE, and is involutive, so $(\mathcal{N}_d, \rho_d)$ is a symmetric set. Moreover, the nondegeneracy of (S_d, r_d) implies that $(\mathcal{N}_d, \rho_d)$ is nondegenerate. By Corollary **??** there are equalities $|\mathcal{N}_d| = |S_d| = \binom{n+d-1}{d} = N$.

(2) We shall prove that the map $\mathrm{Nor} : S_d \longrightarrow \mathcal{N}_d, \quad u \mapsto \mathrm{Nor}(u)$ is an isomorphism of solutions. It is clear that the map is bijective. We have to show that Nor is a homomorphism of solutions, that is

$$(\mathrm{Nor} \times \mathrm{Nor}) \circ r_d = \rho_d \circ (\mathrm{Nor} \times \mathrm{Nor}). \tag{4.11}$$

Let $(u, v) \in S_d \times S_d$, then the equalities $u = \mathrm{Nor}(u)$ and $v = \mathrm{Nor}(v)$ hold in S_d, so

$$\mathrm{Nor}({}^{u}v) = \mathrm{Nor}({}^{\mathrm{Nor}(u)}\mathrm{Nor}(v)), \quad \mathrm{Nor}(u^{v}) = \mathrm{Nor}(\mathrm{Nor}(u)^{\mathrm{Nor}(v)})$$

and by by (**??**)

$$\begin{aligned}(\mathrm{Nor}\times\mathrm{Nor})\circ r_d(u,v) &= \mathrm{Nor}\times\mathrm{Nor}({}^uv,u^v) = (\mathrm{Nor}({}^uv),\mathrm{Nor}(u^v))\\ &= (\mathrm{Nor}(u)\triangleright\mathrm{Nor}(v),\mathrm{Nor}(u)\triangleleft\mathrm{Nor}(v)) = \rho_d(\mathrm{Nor}(u),\mathrm{Nor}(v)).\end{aligned}$$

This implies (**??**). □

Recall that the monomials in $\mathcal{N}_d$ are ordered lexicographically, see **??**, and $w_i < w_j$ *iff* $i < j$, $1 \le i, j \le N$.

Notation 4.11. Denote by $\mathcal{H}(n,d)$ the set

$$\mathcal{H}(n,d) = \{(j,i), 1\le i,j\le n \mid \rho_d(w_j,w_i) = (w_{i'},w_{j'}), \text{ where } \ w_j > w_{i'} \text{ holds in } \langle X\rangle\}. \tag{4.12}$$

Equivalently, $\mathcal{H}(n,d)$ is the set of all pairs (j,i) such that $(w_j,w_i)\in(\mathcal{N}_d\times\mathcal{N}_d)\setminus\mathcal{F}(\mathcal{N}_d,\rho_d)$, and $w_j\triangleright w_i < w_j$. Here $\mathcal{F}(\mathcal{N}_d,\rho_d)$ is the set of fixed points defined in (**??**).

Clearly, $w_j > w_{i'}$ implies that $w_jw_i > w_{i'}w_{j'}$ in $\langle X\rangle$.

Proposition 4.12. *In assumption and notation as above. Let $(\mathcal{N}_d,\rho_d)$ be the normalized d-Veronese solution, see Definition* **??**. *Then the Yang-Baxter algebra $B = \mathcal{A}(\boldsymbol{k},\mathcal{N}_d,\rho_d)$ is generated by the set $\mathcal{N}_d$ and has $\binom{N}{2}$ quadratic defining relations given below:*

$$\Re = \{g_{ji} = w_jw_i - w_{i'}w_{j'} \quad\mid\ (j,i)\in\mathcal{H}(n,d),\ 1\le i,j\le n\}. \tag{4.13}$$

Moreover,

(i) for every pair $(a,b)\in(\mathcal{N}_d\times\mathcal{N}_d)\setminus\mathcal{F}(\mathcal{N}_d,\rho_d)$ the monomial ab occurs exactly once in $\Re$;
(ii) for every pair $(j,i)\in\mathcal{H}(n,d)$ the equality $\mathbf{LM}(g_{ji}) = w_jw_i$ holds in $\boldsymbol{k}\langle X\rangle$.

Proof. There is a one-to-one correspondence between the set of relations of the algebra B and the set of nontrivial orbits of ρ_d. Each nontrivial relation of B corresponds to a nontrivial orbit of ρ_d, say

$$\mathcal{O} = \{(w_j,w_i),\rho_d(w_j,w_i) = (w_{i'},w_{j'})\} = \{(w_{i'},w_{j'}),\rho_d(w_{i'},w_{j'}) = (w_j,w_i)\},$$

so without loss of generality we may assume that the relation is

$$w_jw_i - w_{i'}w_{j'}, \text{ where } w_jw_i > w_{i'}w_{j'}.$$

By Lemma **??** (2) the equality $w_jw_i = w_{i'}w_{j'}$ holds in S. The monoid $S = S(X,r)$ is cancellative, see Facts **??**. hence an assumption that $w_j = w_{i'}$ would imply $w_i = w_{j'}$, a contradiction. Therefore $w_j > w_{i'}$, and so $(j,i)\in\mathcal{H}(n,d)$.

Conversely, for each $(j,i)\in\mathcal{H}(n,d)$, one has $\cdot\rho(w_j,w_i) = w_{i'}w_{j'} \ne w_jw_i$ hence g_{ji} is a (nontrivial) relation of the algebra $B = \mathcal{A}(\mathbf{k},\mathcal{N}_d,\rho_d)$.

Clearly, $w_j > w_{i'}$ implies $w_jw_i > w_{i'}w_{j'}$ in $\langle X\rangle$, so $\mathbf{LM}(g_{ji}) = w_jw_i$, and the number of relations g_{ji} is exactly $|\mathcal{H}(n,d)| = \binom{N}{2}$, see (**??**). □

4.3. **The d-Veronese $\mathcal{A}^{(d)}$ presented in terms of generators and relations.** We shall need more notation. For convenience we add in the list some of the notation that are already in use.

Notation 4.13.

$$\begin{array}{ll}\cdot(a,b) & := ab,\ \forall a,b\in\langle X\rangle,\\ N & := \binom{n+d-1}{d}\\ \mathcal{N} := \mathcal{N}(X) & \text{is the set of normal monomials in } \langle X\rangle.\\ \mathcal{N}_d & = \{w_1 < w_2 < \cdots < w_N\}, \text{ the set of normal monomials of length } d.\\ (\mathcal{N}_d,\rho) & = (\mathcal{N}_d,\rho_d) \text{ is the normalized } d\text{-Veronese solution see Definition ??}.\\ \mathcal{H}(n,d) & = \{(j,i),\ 1\le i,j\le n \mid \rho(w_j,w_i) = (w_{i'},w_{j'}), \text{ where } w_j > w_{i'} \text{ in } \langle X\rangle\}\\ \mathrm{P}(n,d) & = \{(i,j) \mid \cdot(\rho(w_i,w_j)) \ge w_iw_j, w_i,w_j\in\mathcal{N}_d\}\\ \mathrm{C}(n,d) & = \{(i,j)\in\mathrm{P}(n,d) \mid w_iw_j\in\mathcal{N}_{2d}\}\\ \mathrm{MV}(n,d) & = \{(i,j)\in\mathrm{P}(n,d) \mid w_iw_j\notin\mathcal{N}_{2d}\}.\end{array} \tag{4.14}$$

Clearly, $\cdot\rho(w_j,w_i) = w_{i'}w_{j'}$, whenever $\rho(w_j,w_i) = (w_{i'},w_{j'})$.

The following lemma is a generalization of [**?**, Lemma 4.4].

Lemma 4.14. *In notation* **??** *the following conditions hold.*

(1) *The map*

$$\Phi : \mathrm{C}(n,d) \to \mathcal{N}_{2d} \quad (i,j) \mapsto w_i w_j$$

is bijective. Therefore

$$|\mathrm{C}(n,d)| = |\mathcal{N}_{2d}| = \binom{n+2d-1}{n-1} = \binom{n+2d-1}{2d}. \tag{4.15}$$

(2) *The set of all pairs* $\{(i,j) \mid 1 \le i,j \le n\}$ *splits as a union of disjoint sets:*

$$\{(i,j) \mid 1 \le i,j \le n\} = \mathcal{H}(n,d) \sqcup \mathrm{P}(n,d).$$

Every nontrivial ρ*-orbit has exactly one element* (w_j, w_i) *with* $(j,i) \in \mathcal{H}(n,d)$ *and a second element* $(w_{i'}, w_{j'}) = \rho(w_j, w_i)$*, with* $(i', j') \in \mathrm{P}(n,d)$*. For each one-element* ρ*-orbit* $\{(w_i, w_j) = \rho(w_i, w_j)\}$ *one has* $(i,j) \in \mathrm{P}(n,d)$*.*

(3) *The set* $\mathrm{P}(n,d)$ *is a disjoint union*

$$\mathrm{P}(n,d) = \mathrm{C}(n,d) \sqcup \mathrm{MV}(n,d).$$

(4) *The following equalities hold:*

$$|\mathcal{H}(n,d)| = \tbinom{N}{2}, \quad |\mathrm{P}(n,d)| = \tbinom{N+1}{2}, \quad \textit{and} \quad |\mathrm{MV}(n,d)| = \tbinom{N+1}{2} - \tbinom{n+2d-1}{n-1}. \tag{4.16}$$

Proof. By Proposition **??** $(\mathcal{N}_d, \rho)$ is a nondegenerate symmetric set of order $N = \binom{n+d-1}{d}$.

(1) Given $w_i, w_j \in \mathcal{N}_d$, the word $w = w_i w_j$ belongs to $\mathcal{N}_{2d}$ if and only if $(i,j) \in \mathrm{C}(n,d)$, hence Φ is well-defined. Observe that every $w \in \mathcal{N}_{2d}$ can be written uniquely as

$$w = x_{i_1} \dots x_{i_d} x_{j_1} \dots x_{j_d}, \text{ where all } x_{i_k}, x_{j_k} \in X. \tag{4.17}$$

It follows that w has a unique presentation as a product $w = w_i w_j$, where

$$w_i = x_{i_1} \dots x_{i_d} \in \mathcal{N}_d,\ w_j = x_{j_1} \dots x_{j_d} \in \mathcal{N}_d, \text{ and } (i,j) \in \mathrm{C}(n,d).$$

This implies that Φ is a bijection, and (**??**) holds.

(2) and (3) are clear.

(4) Note that the set $\mathrm{P}(n,d)$ contains exactly one element of each ρ-orbit. Indeed, the map ρ is involutive, so every non-trivial ρ - orbit in $\mathcal{N}_d \times \mathcal{N}_d$ consists of two elements: (w_i, w_j) and $\rho(w_i, w_j)$, where $(w_i, w_j) \ne \rho(w_i, w_j)$. Without loss of generality we may assume that $w_i w_j < \cdot \rho(w_i, w_j)$ in $\langle X \rangle$, in this case $(i,j) \in \mathrm{P}(n,d)$. By definition a pair $(w_i, w_j) \in \mathcal{F}$ *iff* it belongs to a one-element ρ-orbit, and in this case $(i,j) \in \mathrm{P}(n,d)$. Therefore each ρ - orbit determines unique element $(i,j) \in \mathrm{P}(n,d)$, and, conversely, each $(i,j) \in \mathrm{P}(n,d)$ determines unique ρ-orbit in $\mathcal{N}_d \times \mathcal{N}_d$. Hence the order $|\mathrm{P}(n,d)|$ equals the total number of ρ-orbit in $\mathcal{N}_d \times \mathcal{N}_d$. By Corollary **??** (3) the set $\mathcal{N}_d \times \mathcal{N}_d$ has exactly $\binom{N+1}{2}$ ρ-orbits, thus

$$|\mathrm{P}(n,d)| = \binom{N+1}{2}. \tag{4.18}$$

The order $|\mathcal{H}(n,d)|$ equals the number of nontrivial ρ-orbit, and since by Corollary **??** there are exactly N one-element orbits, one has $|\mathcal{H}(n,d)| = \binom{N+1}{2} - N = \binom{N}{2}$.

By part (3) $\mathrm{P}(n,d) = \mathrm{C}(n,d) \sqcup \mathrm{MV}(n,d)$ is a union of disjoint sets, which together with (**??**) and (**??**) imply

$$|\mathrm{MV}(n,d)| = |\mathrm{P}(n,d)| - |\mathrm{C}(n,d)| = \binom{N+1}{2} - \binom{n+2d-1}{n-1}. \qquad \square$$

Suppose $(a,b) \in (\mathcal{N}_d \times \mathcal{N}_d) \setminus \mathcal{F}$ then $(a,b) \ne ({}^a b, a^b)$, and the equality $ab = ({}^a b)(a^b)$ holds in $\mathcal{A}^{(d)}$.

In Convention **??** and Notation **??**, the following result describes the d-Veronese subalgebra $\mathcal{A}^{(d)}$ of the Yang-Baxter algebra $\mathcal{A}$ in terms of one-generators and quadratic relations.

Theorem 4.15. *Let* $d \ge 2$ *be an integer. Let* (X,r) *be a finite solution of order* n*,* $X = X_n = \{x_1, \cdots, x_n\}$*, let* $\mathcal{A} = \mathcal{A}(\boldsymbol{k}, X_n, r)$ *be the associated quadratic algebra, and let* $(\mathcal{N}_d, \rho)$ *be the normalized* d*-Veronese solution from Definition* **??***.*

The d-Veronese subalgebra $\mathcal{A}^{(d)} \subseteq \mathcal{A}$ is a quadratic algebra with $N = \binom{n+d-1}{d}$ one-generators, namely the set $\mathcal{N}_d$ of normal monomials of length d, subject to $N^2 - \binom{n+2d-1}{n-1}$ linearly independent quadratic relations $\mathcal{R}$ described below.

(1) The relations $\mathcal{R}$ split into two disjoint subsets $\mathcal{R} = \mathcal{R}_a \bigcup \mathcal{R}_b$, as follows.

(a) The set $\mathcal{R}_a$ contains $\binom{N}{2}$ relations corresponding to the non-trivial ρ-orbits:

$$\mathcal{R}_a = \{g_{ji} = w_j w_i - w_{i'} w_{j'} \mid \text{where } (j,i) \in \mathcal{H}(n,d), (w_{i'}, w_{j'}) = \rho(w_j, w_i), w_j w_i > w_{i'} w_{j'}\}. \quad (4.19)$$

Each monomial $w_i w_j$, such that (w_i, w_j) is in a nontrivial ρ-orbit occurs exatly once in $\mathcal{R}_a$. In particular, for each $(j,i) \in \mathcal{H}(n,d)$, $\mathbf{LM}(g_{ji}) = w_j w_i > w_{i'} w_{j'}$.

(b) The set $\mathcal{R}_b$ contains $\binom{N+1}{2} - \binom{n+2d-1}{n-1}$ relations

$$\mathcal{R}_b = \{g_{ij} = w_i w_j - w_{i_0} w_{j_0} \mid (i,j) \in \mathrm{MV}(n,d),\ (i_0, j_0) \in \mathrm{C}(n,d)\}, \quad (4.20)$$

where for each $(i,j) \in \mathrm{MV}(n,d)$, $w_{i_0} w_{j_0} = \mathrm{Nor}(w_i w_j) \in \mathcal{N}_{2d}$ is the normal form of $w_i w_j$. In particular, $\mathbf{LM}(g_{ij}) = w_i w_j > w_{i_0} w_{j_0}$.

(2) The d-Veronese subalgebra $\mathcal{A}^{(d)}$ has a second set of linearly independent quadratic relations, $\mathcal{R}_1$, which splits into two disjoint subsets $\mathcal{R}_1 = \mathcal{R}_{1a} \bigcup \mathcal{R}_b$ as follows.

(a) The set $\mathcal{R}_{1a}$ is a reduced version *of $\mathcal{R}_a$ and contains exactly $\binom{N}{2}$ relations*

$$\mathcal{R}_{1a} = \{f_{ji} = w_j w_i - w_{i''} w_{j''} \mid (j,i) \in \mathcal{H}(n,d), (i'', j'') \in \mathrm{C}(n,d)\}, \quad (4.21)$$

where $w_{i''} w_{j''} = \mathrm{Nor}(w_j w_i)$, for each $(j,i) \in \mathcal{H}(n,d)$, $LM(f_{ji}) = w_j w_i > w_{i''} w_{j''} \in \mathcal{N}_{2d}$.

*(b) The set $\mathcal{R}_b$ is given in (**??**).*

(3) The two sets of relations $\mathcal{R}$ and $\mathcal{R}_1$ are equivalent: $\mathcal{R} \Longleftrightarrow \mathcal{R}_1$.

Proof. By Convention **??** we identify the algebra $\mathcal{A}$ with $(\mathbf{k}\mathcal{N}, \bullet)$. We know that the d-Veronese subalgebra $\mathcal{A}^{(d)}$ is one-generated and quadratic, see Corollary **??**. Moreover, by (**??**)

$$\mathcal{A}^{(d)} = \bigoplus_{m \in \mathbb{N}_0} \mathcal{A}_{md} \cong \bigoplus_{m \in \mathbb{N}_0} \mathbf{k}\mathcal{N}_{md}.$$

The ordered monomials $w \in \mathcal{N}_d$ of length d are degree one generators of $\mathcal{A}^{(d)}$, there are equalities

$$\dim \mathcal{A}_d = |\mathcal{N}_d| = \binom{n+d-1}{d} = N.$$

Moreover,

$$\dim(\mathcal{A}^{(d)})_2 = \dim(\mathcal{A}_{2d}) = \dim(\mathbf{k}\mathcal{N}_{2d}) = |\mathcal{N}_{2d}| = \binom{n+2d-1}{n-1}.$$

We compare dimensions to find the number of quadratic linearly independent relations for the d-Veronese $\mathcal{A}^{(d)}$. Suppose R is a set of linearly independent quadratic relations defining $\mathcal{A}^{(d)}$. Then we must have $|R| + \dim \mathcal{A}_2^{(d)} = N^2$, so

$$|R| = N^2 - \binom{n+2d-1}{n-1}. \quad (4.22)$$

We shall prove that the set of quadratic polynomials $\mathcal{R} = \mathcal{R}_a \bigcup \mathcal{R}_b$ given above consists of relations of $\mathcal{A}^{(d)}$, it has order $|\mathcal{R}| = N^2 - \binom{n+2d-1}{n-1}$, and is linearly independent.

(a) Consider an element $g_{ji} \in \mathcal{R}_a$, where $(j,i) \in \mathcal{H}(n,d)$. We have to show that $w_j w_i - w_{i'} w_{j'} = 0$, or equivalently, $w_j w_i = w_{i'} w_{j'}$ holds in $\mathcal{A}^{(d)}$. Since$\mathcal{A}^{(d)}$ is a subalgebra of $\mathcal{A} = \mathbf{k}S$, it will be enough to prove that

$$w_j w_i = w_{i'} w_{j'} \quad \text{is an equality in } S. \quad (4.23)$$

Note that $\mathcal{N}$ is a subset of $\langle X \rangle$ and $a = b$ in $\mathcal{N}$ is equivalent to $a, b \in \mathcal{N}$ and $a = b$ as words in $\langle X \rangle$. Clearly, each equality of words in $\langle X \rangle$ holds also in S.

By assumption

$$\rho(w_j, w_i) = (w_{i'}, w_{j'}) \quad \text{holds in } \mathcal{N}_d \times \mathcal{N}_d. \quad (4.24)$$

By Definition-Notation **??**, see (**??**) and (**??**) one has

$$\rho(w_j, w_i) = (\mathrm{Nor}({}^{w_j} w_i), \mathrm{Nor}(w_j^{w_i})), \text{ in } \mathcal{N}_d \times \mathcal{N}_d \quad (4.25)$$

and comparing (**??**) with (**??**) we obtain that

$$\mathrm{Nor}({}^{w_j}w_i) = w_{i'}, \text{ and } \mathrm{Nor}(w_j^{w_i}) = w_{j'} \text{ are equalities of words in } \mathcal{N}_d \subset X^d. \tag{4.26}$$

The equality $u = \mathrm{Nor}(u)$ holds in S and in $\mathcal{A}$, for every $u \in \langle X \rangle$, therefore the following are equalities in S:

$$\begin{array}{lcl} \mathrm{Nor}({}^{w_j}w_i) = {}^{w_j}w_i, & & \mathrm{Nor}(w_j^{w_i}) = w_j^{w_i} \\ (\mathrm{Nor}({}^{w_j}w_i))(\mathrm{Nor}(w_j^{w_i})) & = & ({}^{w_j}w_i)(w_j^{w_i}). \end{array} \tag{4.27}$$

Now (**??**) and (**??**) imply that

$$w_{i'}w_{j'} = ({}^{w_j}w_i)(w_j^{w_i}) \quad \text{holds in } S. \tag{4.28}$$

But S is an M3- braided monoid, so by condition (**??**) M3, the following is an equality in S :

$$w_jw_i = ({}^{w_j}w_i)(w_j^{w_i}). \tag{4.29}$$

This together with (**??**) imply the desired equality $w_jw_i = w_{i'}w_{j'}$ in S. It follows that $g_{ji} = w_jw_i - w_{i'}w_{j'}$ is identically 0 in $\mathcal{A}$ and therefore in $\mathcal{A}^{(d)}$.

Observe that for every $(j,i) \in \mathcal{H}(n,d)$ the leading monomial $\mathbf{LM}(g_{ji})$ is w_jw_i, so the polynomials g_{ji} are pairwise distinct relations. This together with (**??**) implies

$$|\mathcal{R}_a| = |\mathcal{H}(n,d)| = \binom{N}{2}. \tag{4.30}$$

(b) Next we consider the elements $g_{ij} \in \mathcal{R}_b$, where $(i,j) \in \mathrm{MV}(n,d)$. Each $g_{ij} = w_iw_j - w_{i_0}w_{j_0}$ is a homogeneous polynomial of degree $2d$ which is identically 0 in $\mathcal{A}$. Indeed, by the description of $\mathrm{MV}(n,d)$ see (**??**), the monomial w_iw_j is not in normal form. Clearly, $w_iw_j = \mathrm{Nor}(w_iw_j)$ is an identity in $\mathcal{A}$, (and in $(\mathbf{k}\mathcal{N}, \bullet)$). The normal form $\mathrm{Nor}(w_iw_j)$ is a monomial of length $2d$, so it can be written as a product $\mathrm{Nor}(w_iw_j) = w_{i_0}w_{j_0}$, where $w_{i_0}, w_{j_0} \in \mathcal{N}_d$, moreover, since $w_{i_0}w_{j_0} \in \mathcal{N}_{2d}$ is a normal monomial one has $(i_0, j_0) \in \mathrm{C}(n,d)$. It follows that

$$g_{ij} = w_iw_j - w_{i_0}w_{j_0} = 0$$

holds in $\mathcal{A}^{(d)}$, for every $(i,j) \in \mathrm{MV}(n,d)$.

Note that all polynomials in $\mathcal{R}_b$ are pairwise distinct, since they have distinct leading monomials $\mathbf{LM}(g_{ij}) = w_iw_j$, for every $(i,j) \in \mathrm{MV}(n,d)$. Thus, using (**??**) again we obtain

$$|\mathcal{R}_b| = |\mathrm{MV}(n,d)| = \binom{N+1}{2} - \binom{n+2d-1}{n-1}. \tag{4.31}$$

The sets $\mathcal{R}_a$ and $\mathcal{R}_b$ are disjoint, since $\{\mathbf{LM}(g) \mid g \in \mathcal{R}_a\} \cap \{\mathbf{LM}(g) \mid g \in \mathcal{R}_b\} = \emptyset$. Therefore there are equalities:

$$|\mathcal{R}| = |\mathcal{R}_a| + |\mathcal{R}_b| = \binom{N}{2} + (\binom{N+1}{2} - \binom{n+2d-1}{n-1}) = N^2 - \binom{n+2d-1}{n-1}, \tag{4.32}$$

hence the set $\mathcal{R}$ has exactly the desired number of relations given in (**??**). It remains to show that $\mathcal{R}$ consits of linearly independent elements of $\mathbf{k}\langle X \rangle$.

Lemma 4.16. *Under the hypothesis of Theorem* **??**, *the set of polynomials $\mathcal{R} \subset \mathbf{k}\langle X \rangle$ is linearly independent.*

Proof. It is well known that the set of all words in $\langle X \rangle$ forms a basis of $\mathbf{k}\langle X \rangle$ (considered as a vector space), in particular every finite set of distinct words in $\langle X \rangle$ is linearly independent. All words occurring in $\mathcal{R}$ are elements of X^{2d}, but some of them occur in more than one relation, e.g. every w_iw_j, with $(i,j) \in \mathrm{MV}(n,d)$ which is not a fixed point occurs as a second term of a polynomial $g_{pq} = w_pw_q - w_iw_j \in \mathcal{R}_a$, where $\cdot\rho(w_i, w_j) = w_pw_q > w_iw_j$, and also as a leading term of $g_{ij} \in \mathcal{R}_b$. We shall prove the lemma in three steps.

(1) The set of polynomials $\mathcal{R}_a \subset \mathbf{k}\langle X \rangle$ is linearly independent.

Notice that the polynomials in $\mathcal{R}_a$ are in 1-to-1 correspondence with the nontrivial ρ-oprbits in $\mathcal{N}_d \times \mathcal{N}_d$: Each polynomial $g_{ji} = w_jw_i - w_{i'}w_{j'}$ is formed out of the two monomials in the nontrivial ρ-oprbit $\{(w_j, w_i), (w_{i'}, w_{j'}) = \rho(w_j, w_i)\}$. But the ρ-orbits are disjoint, hence each monomial $\cdot(a,b)$, with $(a,b) \neq \rho(a,b)$ occurs exactly once in $\mathcal{R}_a$. A linear relation

$$\sum_{(j,i)\in\mathcal{H}(n,d)} \alpha_{(j,i)} g_{ji} = \sum_{(j,i)\in\mathcal{H}(n,d)} \alpha_{(j,i)}(w_j w_i - w_{i'} w_{j'}) = 0, \text{where all } \alpha_{(j,i)} \in \mathbf{k}$$

involves only pairwise distinct monomials in X^{2d} and therefore it must be trivial: $\alpha_{(j,i)} = 0, \forall (j,i) \in \mathcal{H}(n,d)$. It follows that $\mathcal{R}_a$ is linearly independent.

(2) The set $\mathcal{R}_b \subset \mathbf{k}\langle X \rangle$ is linearly independent.

Assume the contrary. Then there exists a nontrivial linear relation for the elements of $\mathcal{R}_b$:

$$\sum_{(i,j)\in\mathrm{MV}(n,d)} \beta_{(i,j)} g_{ij} = \sum_{(i,j)\in\mathrm{MV}(n,d)} \beta_{(i,j)}(w_i w_j - w_{i_0} w_{j_0}) = 0, \text{ with } \beta_{(i,j)} \in \mathbf{k}. \tag{4.33}$$

Recall that the leading monomials $\mathbf{LM}(g_{ij}) = w_i w_j, (i,j) \in \mathrm{MV}(n,d)$ are pairwise distionct. Let g_{pq} be the polynomial with $\beta_{(p,q)} \neq 0$ whose leading monomial is the highest among all leading monomials of polynomials g_{ij}, with $\beta_{(i,j)} \neq 0$, so we have

$$\mathbf{LM}(g_{pq}) = w_p w_q > \mathbf{LM}(g_{ij}), \text{ for all } (i,j) \in \mathrm{MV}(n,d), \text{ where } \beta_{(i,j)} \neq 0. \tag{4.34}$$

We use (**??**) to find the following equality in $\mathbf{k}\langle X \rangle$:

$$w_p w_q = w_{p_0} w_{q_0} - \sum_{(i,j)\in\mathrm{MV}(n,d), \mathbf{LM}(g_{ij}) < w_p w_q} \frac{\beta_{(i,j)}}{\beta_{(p,q)}} g_{ij}.$$

It follows from (**??**) that the right-hand side of this equality is a linear combination of monomials strictly less than $w_p w_q$, which is impossible. It follows that the set $\mathcal{R}_b \subset \mathbf{k}\langle X \rangle$ is linearly independent.

(3) The set $\mathcal{R} \subset \mathbf{k}\langle X \rangle$ is linearly independent. Assume the polynomials in $\mathcal{R}$ satisfy a linear relation

$$\sum_{(j,i)\in\mathcal{H}(n,d)} \alpha_{(j,i)} g_{ji} + \sum_{(i,j)\in\mathrm{MV}(n,d)} \beta_{(i,j)} g_{ij} = 0, \text{ where } \alpha_{(j,i)}, \beta_{(i,j)} \in \mathbf{k}. \tag{4.35}$$

This gives the following equality in the free associative algebra $\mathbf{k}\langle X \rangle$:

$$S_1 = \sum_{(j,i)\in\mathcal{H}(n,d)} \alpha_{(j,i)} w_j w_i = \sum_{(j,i)\in\mathcal{H}(n,d)} \alpha_{(j,i)} w_{i'} w_{j'} - \sum_{(i,j)\in\mathrm{MV}(n,d)} \beta_{(i,j)} g_{ij} = S_2. \tag{4.36}$$

The element $S_1 = \sum_{(j,i)\in\mathcal{H}(n,d)} \alpha_{(j,i)} w_j w_i$ on the left-hand side of (**??**) is in the space $V = \mathrm{Span} B_1$, where $B_1 = \{w_j w_i \mid (j,i) \in \mathcal{H}(n,d)\}$ is linearly independent. The element

$$S_2 = \sum_{(j,i)\in\mathcal{H}(n,d)} \alpha_{(j,i)} w_{i'} w_{j'} - \sum_{(i,j)\in\mathrm{MV}(n,d)} \beta_{(i,j)} g_{(i,j)}$$

on the right-hand side of the equality is in the space $W = \mathrm{Span} B$, where

$$B = \{w_{i'} w_{j'} \mid (j,i) \in \mathcal{H}(n,d)\} \cup \{w_i w_j, w_{i_0} w_{j_0} \mid (i,j) \in \mathrm{MV}(n,d)\}.$$

Take a subset $B_2 \subset B$ which forms a basis of W. Note that $B_1 \cap B = \emptyset$, hence $B_1 \cap B_2 = \emptyset$. Moreover each of the sets B_1, and B_2 consists of pairwise distinct monomials and it is easy to show that $V \cap W = 0$. Thus the equality $S_1 = S_2 \in V \cap W = 0$ implies a linear relation

$$S_1 = \sum_{(j,i)\in\mathcal{H}(n,d)} \alpha_{(j,i)} w_j w_i = 0,$$

for the set of leading monomials of $\mathcal{R}_a$ which are pairwise distinct, and therefore independent. It follows that $\alpha_{(j,i)} = 0$, for all $(j,i) \in \mathcal{H}(n,d)$. This together with (**??**) implies the linear relation

$$\sum_{(i,j)\in\mathrm{MV}(n,d)} \beta_{(i,j)} g_{ij} = 0,$$

and since by (2) $\mathcal{R}_b$ is linearly independent we get again $\beta_{(i,j)} = 0, \forall\, (i,j) \in \mathrm{MV}(n,d)$. It follows that the linear relation (**??**) must be trivial, and therefore $\mathcal{R}$ is a linearly independent set of polynomials.

□

We have proven part (1) of the theorem.

Analogous argument proves part (2). Note that the polynomials of $\mathcal{R}_{1a}$ are reduced from $\mathcal{R}_a$ using $\mathcal{R}_b$. It is not difficult to prove the equivalence $\mathcal{R} \Longleftrightarrow \mathcal{R}_1$. □

5. Veronese maps

In this section we shall introduce an analogue of Veronese maps between quantum spaces (Yang-Baxter algebras) associated to finite solutions of YBE. We keep the notation and all conventions from the previous sections. As usual, (X,r) is a finite solution of order n, $\mathcal{A} = \mathcal{A}(\mathbf{k}, X, r)$ is the associated algebra, where we fix an enumeration, $X = X_n = \{x_1, \cdots, x_n\}$ as in Convention **??**, $d \geq 2$ is an integer, $N = \binom{n+d-1}{d}$, and $\mathcal{N}_d = \{w_1 < w_2 < \cdots < w_N\}$ is the set of all normal monomials of length d in X^d ordered lexicographically, as in (**??**).

5.1. The d-Veronese solution of YBE associated to a finite solution (X,r).

We have shown that the braided monoid (S, r_S) associated to (X,r) induces the normalized d-Veronese solution $(\mathcal{N}_d, \rho_d)$ of order $N = \binom{n+d-1}{d}$, see Definition **??**. We shall use this construction to introduce the notion of *a d-Veronese solution of YBE associated to* (X,r).

Definition-Notation 5.1. In notation as above. Let (X,r) be a finite solution, $X = \{x_1, \cdots, x_n\}$, let $\mathcal{N}_d = \{w_1 < w_2 < \cdots w_N\}$ be the set of normal monomials of length d, and let $(\mathcal{N}_d, \rho) = (\mathcal{N}_d, \rho_d)$ be the normalized d-Veronese solution.

Let $Y = \{y_1, y_2, \cdots, y_N\}$ be an abstract set and consider the quadratic set (Y, r_Y) , where the map $r_Y : Y \times Y \longrightarrow Y \times Y$ is defined as

$$r_Y(y_j, y_i) := (y_{i'}, y_{j'}) \quad \textit{iff} \quad \rho(w_j, w_i) = (w_{i'}, w_{j'}),\ 1 \leq i, j, i', j' \leq N. \tag{5.1}$$

It is straightforward that (Y, r_Y) is a nondegenerate symmetric set of order N isomorphic to $(\mathcal{N}_d, \rho_d)$. We shall refer to it as *the d-Veronese solution of YBE associated to* (X,r).

By Corollary **??** the set $Y \times Y$ splits into $\binom{N}{2}$ two-element r_Y- orbits and N one-element r_Y-orbits.

As usual, we consider the degree-lexicographic ordering on the free monoid $\langle Y \rangle$ extending $y_1 < y_2 < \cdots < y_N$. The Yang-Baxter algebra $\mathcal{A}_Y = \mathcal{A}(\mathbf{k}, Y, r_Y) \simeq \mathbf{k}\langle Y;\ \mathcal{R}_Y\rangle$ has exactly $\binom{N}{2}$ quadratic relations which can be written explicitly as

$$\mathcal{R}_Y = \{\gamma_{ji} = y_j y_i - y_{i'} y_{j'} \mid (j,i) \in \mathcal{H}(n,d),\ (y_{i'}, y_{j'}) = r_Y(y_j, y_i)\}, \tag{5.2}$$

where $\mathcal{H} = \mathcal{H}(n,d)$ is the set defined in (**??**), and each relation corresponds to a non-trivial r_Y-orbit. The leading monomials satisfy $\mathbf{LM}(\gamma_{ji}) = y_j y_i > y_{i'} y_{j'}$.

5.2. The Veronese map $v_{n,d}$ and its kernel.

Lemma 5.2. *In notation as above. Let (X,r) be a solution of order n, $\mathcal{A}_X = \mathcal{A}(\boldsymbol{k}, X, r)$, let $d \geq 2$, be an integer, and let $N = \binom{n+d-1}{d}$. Suppose (Y, r_Y) is the associated d-Veronese solution, $Y = \{y_1, \cdots, y_N\}$, and $\mathcal{A}_Y = \mathcal{A}(\boldsymbol{k}, Y, r_Y)$, is the corresponding Yang-Baxter algebra.*

The assignment

$$y_1 \mapsto w_1,\ \ y_2 \mapsto w_2,\ \ \ldots,\ \ y_N \mapsto w_N$$

extends to an algebra homomorphism $v_{n,d} : \mathcal{A}_Y \to \mathcal{A}_X$.

Proof. Naturally we set $v_{n,d}(y_{i_1} \cdots y_{i_p}) := w_{i_1} \cdots w_{i_p}$, for all words $y_{i_1} \cdots y_{i_p} \in \langle Y \rangle$ and then extend this map linearly. Note that for each polynomial $\gamma_{ji} \in \mathcal{R}_Y$ one has

$$v_{n,d}(\gamma_{ji}) = g_{ji} \in \mathcal{R}_a,$$

where the set $\mathcal{R}_a$ is a part of the relations of $\mathcal{A}_X^{(d)}$ given in (**??**). Indeed, let $\gamma_{ji} \in \mathcal{R}_Y$, so $(j,i) \in \mathcal{H}(n,d)$ and $\gamma_{ji} = y_j y_i - y_{i'} y_{j'}$, where $(y_{i'}, y_{j'}) = r_Y(y_j, y_i)$, see also (**??**). Then

$$\begin{aligned} v_{n,d}(\gamma_{ji}) &= v_{n,d}(y_j y_i - y_{i'} y_{j'}) \\ &= w_j w_i - w_{i'} w_{j'},\ \text{where } (w_{i'}, w_{j'}) = \rho(w_j, w_i), \\ &= g_{ji} \in \mathcal{R}_a. \end{aligned}$$

We have shown that g_{ji} equals identically 0 in $\mathcal{A}_X$, so the map $v_{n,d}$ agrees with the relations of the algebra $\mathcal{A}_X$. It follows that $v_{n,d} : \mathcal{A}_Y \to \mathcal{A}_X$ is a well-defined homomorphism of algebras.

The image of $v_{n,d}$ is the subalgebra of $\mathcal{A}_X$ generated by the normal monomials $\mathcal{N}_d$, which by Theorem **??** is exactly the d-Veronese $\mathcal{A}_X^{(d)}$. □

Definition 5.3. We call the map $v_{n,d}$ from Lemma **??** *the* (n,d)*-Veronese map.*

Theorem 5.4. *In assumption and notation as above. Let* (X, r) *be a solution of order* n*, with* $X = X_n = \{x_1, \cdots, x_n\}$*, let* $\mathcal{A}_X = \mathcal{A}(\boldsymbol{k}, X_n, r)$ *be its Yang-Baxter algebra. Let* $d \geq 2$ *be an integer,* $N = \binom{n+d-1}{d}$*, and suppose that* (Y, r_Y) *is the associated* d*-Veronese solution of YBE with corresponding Yang-Baxter algebra* $\mathcal{A}_Y = \mathcal{A}(\boldsymbol{k}, Y, r_Y)$*,*

Let $v_{n,d} : \mathcal{A}_Y \to \mathcal{A}_X$ *be the Veronese map (homomorphism of algebras) extending the assignment*

$$y_1 \mapsto w_1,\ y_2 \mapsto w_2,\ \ldots,\ y_N \mapsto w_N.$$

Then the following conditions hold.

1. *The image of* $v_{n,d}$ *is the* d*-Veronese subalgebra* $\mathcal{A}_X^{(d)}$ *of* $\mathcal{A}_X$*.*
2. *The kernel* $\mathfrak{K} := \ker(v_{n,d})$ *of the Veronese map is generated by the set of* $\binom{N+1}{2} - \binom{n+2d-1}{n-1}$ *linearly independent quadratic binomials:*

$$\mathcal{R}_Y^{\mathrm{v}} := \{\gamma_{ij} = y_i y_j - y_{i_0} y_{j_0} \mid (i,j) \in \mathrm{MV}(n,d),\ (i_0, j_0) \in \mathrm{C}(n,d)\}, \tag{5.3}$$

where for each pair $(i,j) \in \mathrm{MV}(n,d)$*,* $w_{i_0} w_{j_0} = \mathrm{Nor}(w_i w_j) \in \mathcal{N}_{2d}$*. In particular,*

$$\mathbf{LM}(\gamma_{ij}) = y_i y_j > y_{i_0} y_{j_0}.$$

Proof. (1) The image of $v_{n,d}$ is the subalgebra of $\mathcal{A}_X$ generated by the normal monomials $\mathcal{N}_d$, which by Theorem **??** is exactly the d-Veronese subalgebra $\mathcal{A}_X^{(d)}$.

Part (2). We have to verify that the set $\mathcal{R}_Y^{\mathrm{v}}$ generates $\mathfrak{K}$. Note first that $\mathcal{R}_Y^{\mathrm{v}} \subset \mathfrak{K}$. Indeed, by direct computation, one shows that $v_{n,d}(\mathcal{R}_Y^{\mathrm{v}}) = \mathcal{R}_b$, the set of relations of the d-Veronese $(\mathcal{A}_X)^{(d)}$ given in (**??**), so $\mathcal{R}_Y^{\mathrm{v}} \subset \mathfrak{K}$. Moreover, for each pair $(i,j) \in \mathrm{MV}(n,d)$, the monomial $y_i y_j$ occurs exactly once in the set $\mathcal{R}_Y^{\mathrm{v}}$, namely in $\gamma_{ij} = y_i y_j - y_{i_0} y_{j_0}$. Here $(i_0, j_0) \in \mathrm{C}(n,d)$, and $w_{i_0} w_{j_0} = \mathrm{Nor}(w_i w_j) \in \mathcal{N}_{2d}$, see Theorem **??**.

The polynomials γ_{ij} are pairwise distinct, since they have pairwise distinct highest monomials. Therefore the cardinality of $\mathcal{R}_Y^{\mathrm{v}}$ satisfies $|\mathcal{R}_Y^{\mathrm{v}}| = |\mathrm{MV}(n,d)|$, and (**??**) implies

$$|\mathcal{R}_Y^{\mathrm{v}}| = \binom{N+1}{2} - \binom{n+2d-1}{n-1}. \tag{5.4}$$

The Yang-Baxter algebra $\mathcal{A}_Y$ is a quadratic algebra with N generators and $\binom{N}{2}$ defining quadratic relations which are linearly independent, so

$$\dim(\mathcal{A}_Y)_2 = N^2 - \binom{N}{2} = \binom{N+1}{2}.$$

By the First Isomorphism Theorem $(\mathcal{A}_Y/\mathfrak{K})_2 \cong (\mathcal{A}_X^{(d)})_2 = (\mathcal{A}_X)_{2d}$, hence

$$\dim(\mathcal{A}_Y)_2 = \dim(\mathfrak{K})_2 + \dim(\mathcal{A}_X)_{2d}.$$

We know that $\dim(\mathcal{A}_X)_{2d} = |\mathcal{N}_{2d}| = \binom{n+2d-1}{n-1}$, hence

$$\binom{N+1}{2} = \dim(\mathfrak{K})_2 + \binom{n+2d-1}{n-1}.$$

This together with (**??**) implies that

$$\dim(\mathfrak{K})_2 = \binom{N+1}{2} - \binom{n+2d-1}{n-1} = |\mathcal{R}_Y^{\mathrm{v}}|.$$

The set $\mathcal{R}_Y^{\mathrm{v}}$ is linearly independent, since $v_{n,d}(\mathcal{R}_Y^{\mathrm{v}}) = \mathcal{R}_b$, and by Lemma **??** the set $\mathcal{R}_b$ is linearly independent. Thus the set $\mathcal{R}_Y^{\mathrm{v}}$ is a basis of the graded component $\mathfrak{K}_2$, and $\mathfrak{K}_2 = \mathbf{k}\mathcal{R}_Y^{\mathrm{v}}$. But the ideal $\mathfrak{K}$ is generated by homogeneous polynomials of degree 2, therefore

$$\mathfrak{K} = (\mathfrak{K}_2) = (\mathcal{R}_Y^{\mathrm{v}}). \tag{5.5}$$

We have proven that $\mathcal{R}^{\mathrm{v}}_Y$ is a minimal set of generators for the kernel $\mathfrak{K}$. □

6. Special cases

6.1. **Veronese subalgebras of the Yang-Baxter algebra of a square-free solution.** In this subsection (X,r) is a finite square-free solution of YBE of order n, $d \geq 2$ is an integer. We keep the conventions and notation from the previous sections. We apply Fact **??** and fix an appropriare enumeration $X = X_n = \{x_1, \cdots, x_n\}$, such that the algebra the Yang-Baxter algebra $\mathcal{A} = \mathcal{A}(\mathbf{k}, X, r)$ is a binomial skew polynomial ring. More precisely, $\mathcal{A}$ is a PBW algebra $\mathcal{A} = \mathbf{k}\langle x_1, \cdots, x_n\rangle/(\Re_0)$, where

$$\Re_0 = \Re_0(r) = \{f_{ji} = x_jx_i - x_{i'}x_{j'} \mid 1 \leq i < j \leq n\}, \tag{6.1}$$

is such that for every pair i, j, $1 \leq i < j \leq n$, the relation $f_{ji} = x_jx_i - x_{i'}x_{j'} \in \Re_0$, satisfies $j > i'$, $i' < j'$ and every term $x_ix_j, 1 \leq i < j \leq n$, occurs in some relation in $\Re_0$. In particular

$$\mathbf{LM}(f_{ji}) = x_jx_i,\ 1 \leq i < j \leq n. \tag{6.2}$$

The set $\Re_0$ is a quadratic Gröbner basis of the ideal $I = (\Re_0)$ w.r.t the degree-lexicographic ordering $<$ on $\langle X\rangle$. It follows from the shape of the elements of the Gröbner basis $\Re_0$, and (**??**) that the set $\mathcal{N} = \mathcal{N}(I)$ of normal monomials modulo $I = (\Re_0)$ coincides with the set $\mathcal{T}$ of ordered monomials (terms) in X,

$$\mathcal{N} = \mathcal{T} = \{x_1^{\alpha_1}\cdots x_n^{\alpha_n} \mid \alpha_i \in \mathbb{N}_0, i \in \{0,\dots,n\}\}. \tag{6.3}$$

All definitions, notation, and results from Sections **??** and **??** are valid but they can be rephrased in more explicit terms replacing the abstract sets $\mathcal{N} = \mathcal{N}(I)$, $\mathcal{N}_d$, and $\mathcal{N}_{2d}$ with the explicit set of ordered monomials $\mathcal{T} = \mathcal{T}(X)$, $\mathcal{T}_d$, and $\mathcal{T}_{2d}$.

As usual, we consider the space $\mathbf{k}\mathcal{T}$ endowed with multiplication defined by

$$f \bullet g := \mathrm{Nor}_{\Re_0}(fg), \quad \text{for every } f, g \in \mathbf{k}\mathcal{T}.$$

Then there is an isomorphism of graded algebras

$$\mathcal{A} = \mathcal{A}(\mathbf{k}, X, r) \cong (\mathbf{k}\mathcal{T}, \bullet), \tag{6.4}$$

and we identify the PBW algebra $\mathcal{A}$ with $(\mathbf{k}\mathcal{T}, \bullet)$. Similarly, the monoid $S(X,r)$ is identified with $(\mathcal{T}, \bullet)$.

We order the elements of $\mathcal{T}_d$ lexicographically, so

$$\mathcal{T}_d = \{w_1 = (x_1)^d < w_2 = (x_1)^{d-1}x_2 < \cdots < w_N = (x_n)^d\},\ \text{where } N = \binom{n+d-1}{d} \tag{6.5}$$

The d-Veronese $\mathcal{A}^{(d)}$ is a quadratic algebra (one)-generated by $w_1, w_2, \dots, w_N$.

It follows from [**?**], Proposition 4.3, Ch 4, that if $x_1, \cdots, x_n$ is a set of PBW generators of a quadratic algebra A, then the elements of the PBW-basis of degree d , taken in lexicographical order are PBW-generators of the Veronese subalgebra $A^{(d)}$.

Corollary 6.1. *Let (X,r) be a finite square-free solution of order n, let $X = \{x_1, \cdots, x_n\}$, be enumerated so that the algebra $\mathcal{A} = \mathcal{A}(\boldsymbol{k}, X, r)$ is a binomial skew polynomial ring. Let $d \geq 2$ be an integer, and $N = \binom{n+d-1}{d}$. The d-Veronese subalgebra $\mathcal{A}^{(d)} \subseteq \mathcal{A}$ is a quadratic PBW algebra with PBW generators the set of ordered monomials (terms) in X of length d, $\mathcal{T}_d = \{w_1 = x_1^d, \cdots, w_N = x_n^d\}$, ordered lexicographically and $N^2 - \binom{n+2d-1}{n-1}$ linearly independent quadratic relations $\mathcal{R} = \mathcal{R}_a \cup \mathcal{R}_b$ given in Theorem **??**. Thus $\mathcal{A}^{(d)}$ has a standard finite presentation*

$$\mathcal{A}^{(d)} \simeq \boldsymbol{k}\langle w_1, \cdots, w_N\rangle/(\mathcal{R}),$$

where the set of defining relations $\mathcal{R}$ forms a Gröbner basis of the ideal $(\mathcal{R})$ of $\boldsymbol{k}\langle w_1, \cdots, w_N\rangle$ (with respect to the degree-lexicographic order on $\langle w_1, \cdots, w_N\rangle$).

For a square-free solution (X,r) as above, the normalized d-Veronese solution is denoted by $(\mathcal{T}_d, \rho_d)$. The d-Veronese solution (Y, r_Y), associated to (X,r), is defined in Definition-Notation **??**. One has $Y = \{y_1, y_2, \cdots, y_N\}$, and the map $r_Y : Y \times Y \longrightarrow Y \times Y$ is determined by

$$r_Y(y_j, y_i) := (y_{i'}, y_{j'}) \quad \textit{iff} \quad \rho(w_j, w_i) = (w_{i'}, w_{j'}),\ 1 \leq i, j, i', j' \leq n. \tag{6.6}$$

By definition (Y, r_Y) is isomorphic to the solution $(\mathcal{T}_d, \rho_d)$. Its Yang-Baxter algebra $\mathcal{A}_Y = \mathcal{A}(\mathbf{k}, Y, r_Y)$ is needed to define the Veronese homomorphism

$$v_{n,d} : \mathcal{A}_Y \to \mathcal{A}_X$$

extending the assignment

$$y_1 \mapsto w_1,\ y_2 \mapsto w_2,\ \dots,\ y_N \mapsto w_N.$$

Theorem **??** shows that the image of $v_{n,d}$ is the d-Veronese subalgebra $\mathcal{A}^{(d)}$ and determines a minimal set of generators of its kernel.

Remark 6.2. The finite square-free solutions (X, r) form an important sublcass of the class of all finite solutions, see for example [**?**]. It is natural to ask "can we define analogue of Veronese morphisms between Yang-Baxter algebras of square-free solutions?" In fact, it is not possible to restrict the defininition of Veronese maps introduced for Yang-Baxter algebras of finite solutions to the subclass of Yang-Baxter algebras of finte square-free solutions. Indeed, if we assume that (X, r) is square-free then the algebra A_Y involved in the definition of the map $v_{n,d}$ is associated with the d-Veronese solution (Y, r_Y), which, in general is not square-free, see Corollary **??**.

To prove the following result we work with the monomial d-Veronese solution (S_d, r_d) keeping in mind that it has special "hidden" properties induced by the braided monoid (S, r_S).

Theorem 6.3. *Let $d \geq 2$ be an integer. Suppose (X, r) is a finite square-free solution of order $n \geq 2$, (S, r_S) is the associated braided monoid, and (S_d, r_d) is the monomial d-Veronese solution induced by (S, r_S), see Def.* **??**. *Then (S_d, r_d) is a square-free solution if and only if (X, r) is a trivial solution.*

Proof. Assume (S_d, r_d) is a square-free solution. We shall prove that (X, r) is a trivial solution.

Observe that if (Z, r_Z) is a solution, then (i) (Z, r_Z) is square-free if and only if

$${}^{z}z = z, \text{ for all } z \in Z;$$

and (ii) (Z, r_Z) is the trivial solution if and only if

$${}^{y}x = x, \text{ for all } x, y \in Z.$$

Let $x, y \in X, x \neq y$ and consider the monomial $a = x^{d-1}y \in S_d$. Our assumption that (S_d, r_d) is square-free implies that ${}^{a}a = a$ holds in S_d, and therefore in S. It follows from Remark **??** that the words a and ${}^{a}a$ (considered as elements of X^d) belong to the orbit $\mathcal{O} = \mathcal{O}_{\mathcal{D}_d}(a)$ of $a = x^{d-1}y$ in X^d. We analyze the orbit $\mathcal{O} = \mathcal{O}(x^{d-1}y)$ to find that it contains two type of elements:

$$u = ({}^{x^{d-1}}y)b, \text{ where } b \in X^{d-1}; \tag{6.7}$$

and

$$v = x^i c, \text{ where } 1 \leq i \leq d-1 \text{ and } c \in X^{d-i}. \tag{6.8}$$

A reader who is familiar with the techniques and properties of square-free solutions such as "cyclic conditions" and condition "lri" may compute that $b = (x^y)^{d-1}$ and $c = ({}^{x^{d-i-1}}y)(x^y)^{d-i-1}$, but these details are not used in our proof. We use condition ML2, see (**??**) to yield the following equality in S:

$${}^{a}a = {}^{(x^{d-1}y)}(x^{d-1}y) = ({}^{x^{d-1}y}x)({}^{(x^{d-1}y)^x}x) \cdots ({}^{(x^{d-1}y)^{x^{d-1}}}y) = \omega \tag{6.9}$$

The word ω, considered as an element of X^d is in the orbit $\mathcal{O}$, and therefore two cases are possible.

Case 1. The following is an equality of words in X^d:

$$\omega = ({}^{x^{d-1}y}x)({}^{(x^{d-1}y)^x}x) \cdots ({}^{(x^{d-1}y)^{x^{d-1}}}y) = ({}^{x^{d-1}}y)b, \quad b \in X^{d-1}.$$

Then there is an equality of elements of X:

$${}^{(x^{d-1}y)}x = {}^{x^{d-1}}y. \tag{6.10}$$

Now we use condition ML1, see (**??**) to obtain

$${}^{(x^{d-1}y)}x = {}^{(x^{d-1})}({}^{y}x)$$

which together with (**??**) gives

$${}^{(x^{d-1})}({}^{y}x) = {}^{(x^{d-1})}y. \tag{6.11}$$

The nondegeneracy implies that ${}^{y}x = y$. At the same time ${}^{y}y = y$, since (X, r) is square-free, and using the nondegeneracy again one gets $x = y$, a contradiction. It follows that Case 1 is impossible, whenever $x \neq y$.

Case 2. The following is an equality of words in X^d :

$$\omega = ({}^{x^{d-1}y}x)({}^{(x^{d-1}y)^x}x)\cdots({}^{(x^{d-1}y)^{x^{k-1}}}y) = x^i c, \quad \text{where } 1 \leq i \leq d-1, c \in X^{d-i}.$$

Then

$$ {}^{(x^{d-1}y)}x = x. \tag{6.12}$$

At the same time the equality ${}^{x}x = x$ and condition ML1 imply ${}^{x^{d-1}}x = x$, which together with (**??**) and ML1 (again) gives

$${}^{x^{d-1}}x = {}^{(x^{d-1}y)}x = {}^{x^{d-1}}({}^{y}x).$$

Thus, by the nondegeneracy again ${}^{y}x = x$. We have shown that ${}^{y}x = x$, for all $x, y \in X, y \neq x$. But (X, r) is square-free, so ${}^{y}y = y$ for all $y \in X$. It follows that ${}^{y}x = x$ holds for all $x, y \in X$ and therefore (X, r) is the trivial solution. □

By construction the (abstract) d-Veronese solution (Y, r_Y) associated to (X, r) is isomorphic to the normalized d-Veronese solution $(\mathcal{N}_d, \rho_d)$ and therefore it is isomorphic to the monomial d-Veronese solution (S_d, r_d). Theorem **??** implies straightforwardly the following corollary.

Corollary 6.4. *Let $d \geq 2$ be an integer, suppose (X, r) is a square-free solution of finite order. Then the d-Veronese solution (Y, r_Y) is square-free if and only if (X, r) is a trivial solution.*

It follows that the notion of Veronese morphisms introduced for the class of Yang-Baxter algebras of finite solutions can not be restricted to the subclass of algebras associated to finite square-free solutions.

6.2. **Involutive permutation solutions.** Recall that a symmetric set (X, r) is *an involutive permutation solution* of Lyubashenko (or shortly *a permutation solution*) if there exists a permutation $f \in \mathrm{Sym}(X)$, such that $r(x, y) = (f(y), f^{-1}(x))$. In this case we shall write (X, f, r), see [?], and [?], p. 691.

Proposition 6.5. *Suppose (X, f, r) is an involutive permutation solution of finite order n defined as $r(x, y) = (f(y), f^{-1}(x))$, where f is a permutation of X and let $\mathcal{A}$ be the associated Yang-Baxter algebra.*

(1) For every integer $d \geq 2$ the monomial d-Veronese solution (S_d, r_d) is an involutive permutation solution.

(2) If the permutation f has order m then for every integer d divisible by m the monomial d-Veronese solution (S_d, r_d) is the trivial solution and the d-Veronese subalgebra $\mathcal{A}^{(d)}$ of $\mathcal{A}$ is a quotient of the commutative polynomial ring $\boldsymbol{k}[y_1, y_2, \cdots, y_N]$, where $N = \binom{n+d-1}{d}$.

Proof. (1) The condition ML1 in (**??**) implies that

$${}^{a}t = f^q(t), \text{ and } t^a = f^{-q}(t), \quad \text{for all monomials } a \in S_q, \text{ and all } t \in X. \tag{6.13}$$

Moreover, since S is a graded braided monoid the monomials a, ${}^{b}a$ and a^b have the same length, therefore

$${}^{a}t = {}^{a^b}t = f^q(t), \quad t^a = t^{{}^{b}a} = f^{-q}(t) \text{ for all } a \in S_q, \ b \in S, \text{ and all } t \in X. \tag{6.14}$$

It follows then from (**??**) ML2 that S acts on itself (on the left and on the right) as automorphisms. In particular, for $a, t_1t_2\cdots t_d \in S_d$ one has

$$\begin{aligned} {}^{a}(t_1t_2\cdots t_d) &= ({}^{a}t_1)({}^{a}t_2)\cdots({}^{a}t_d) = f^d(t_1)f^d(t_2)\cdots f^d(t_d). \\ (t_1t_2\cdots t_d)^a &= (t_1{}^a)(t_2{}^a)\cdots(t_d{}^a) = f^{-d}(t_1)f^{-d}(t_2)\cdots f^{-d}(t_d). \end{aligned} \tag{6.15}$$

Therefore (S_d, r_d) is a permutation solution, (S_d, f_d, r_d), where the permutation $f_d \in \mathrm{Sym}(S_d)$ is defined as $f_d(t_1t_2\cdots t_d) := f^d(t_1)f^d(t_2)\cdots f^d(t_d)$. One has $f_d^{-1}(t_2t_2\cdots t_d) := f^{-d}(t_1)f^{-d}(t_2)\cdots f^{-d}(t_d)$.

(2) Assume now that $d = km$ for some integer $k \geq 1$, then $f^d = id_X$. It will be enough to prove that the monomial d-Veronese solution (S_d, r_d) is the trivial solution. It follows from (**??**) that if $a \in S_d$ then

$${}^{a}(t_1t_2\cdots t_d) = t_1t_2\cdots t_d, \text{ where } t_i \in X, 1 \leq i \leq n. \tag{6.16}$$

This implies ${}^a b = b$ for all $a, b \in S_d$. Similarly, $a^b = a$ for all $a, b \in S_d$. It follows that (S_d, r_d) is the trivial solution. But the associated d-Veronese solution (Y, r_Y) is isomorphic to (S_d, r_d), hence (Y, r_Y) is also a trivial solution, and therefore its Yang-Baxter algebra $\mathcal{A}(\mathbf{k}, \mathbf{Y}, \mathbf{r_Y})$ is the commutative polynomial ring $\mathbf{k}[y_1, y_2, \cdots, y_N]$. It follows from Theorem **??** that the d-Veronese subalgebra $\mathcal{A}^{(d)}$ is isomorphic to the quotient $\mathbf{k}[y_1, y_2, \cdots, y_N]/(\mathfrak{K})$ where $\mathfrak{K}$ is the kernel of the Veronese map $v_{n,d}$. □

7. Examples

We shall present two examples which illustrates the results of the paper. We use the notation of the previous sections.

Example 7.1. Let $n = 3$, consider the solution (X, r), where

$$\begin{array}{ll} X = \{x_1, x_2, x_3\}, & \\ r(x_3, x_1) = (x_2, x_3) & r(x_2, x_3) = (x_3, x_1) \\ r(x_3, x_2) = (x_1, x_3) & r(x_1, x_3) = (x_3, x_2) \\ r(x_2, x_1) = (x_1, x_2) & r(x_1, x_2) = (x_2, x_1) \\ r(x_i, x_i) = (x_i, x_i), & 1 \le i \le 3. \end{array}$$

Then

$$\begin{array}{l} A(\mathbf{k}, X, r) = \mathbf{k}\langle X \rangle/(\Re_0) \;\text{ where} \\ \Re_0 = \Re_0(r) = \{x_3x_2 - x_1x_3,\; x_3x_1 - x_2x_3,\; x_2x_1 - x_1x_2.\}. \end{array}$$

The algebra $A = A(\mathbf{k}, X, r)$ is a PBW algebra with PBW generators $X = \{x_1, x_2, x_3\}$, in fact it is a binomial skew-polynomial algebra.

We first give an explicit presentation of the 2-Veronese $\mathcal{A}^{(2)}$ in terms of generators and quadratic relations. In this case $N = \binom{3+1}{2} = 6$ and the 2-Veronese subalgebra $\mathcal{A}^{(2)}$ is generated by $\mathcal{T}_2$, the terms of length 2 in $\mathbf{k}\langle x_1, x_2, x_3 \rangle$. These are all normal (modulo $\Re_0$) monomials of length 2 ordered lexicographically:

$$\mathcal{T}_2 = \{w_1 = x_1x_1,\; w_2 = x_1x_2,\; w_3 = x_1x_3,\; w_4 = x_2x_2,\; w_5 = x_2x_3,\; w_6 = x_3x_3\}. \tag{7.1}$$

Determine the normaized 2-Veronese solution $(\mathcal{T}_2, \rho_2) = (\mathcal{T}_2, \rho)$, where $\rho(a, b) = (\mathrm{Nor}({}^a b), \mathrm{Nor}(a^b))$. An explicit description of ρ is given below:

$$\begin{array}{ll} (x_3x_3, w_i) \longleftrightarrow (w_i, x_3x_3), & 1 \le i \le 5 \\ (x_2x_3, x_2x_3) \longleftrightarrow (x_1x_3, x_1x_3), & (x_2x_3, x_2x_2) \longleftrightarrow (x_1x_1, x_2x_3), \\ (x_2x_3, x_1x_2) \longleftrightarrow (x_1x_2, x_2x_3), & (x_2x_3, x_1x_1) \longleftrightarrow (x_2x_2, x_2x_3), \\ (x_2x_2, x_1x_3) \longleftrightarrow (x_1x_3, x_1x_1), & (x_2x_2, x_1x_2) \longleftrightarrow (x_1x_2, x_2x_2), \\ (x_2x_2, x_1x_1) \longleftrightarrow (x_1x_1, x_2x_2), & (x_1x_3, x_2x_2) \longleftrightarrow (x_1x_1, x_1x_3), \\ (x_1x_3, x_1x_2) \longleftrightarrow (x_1x_2, x_1x_3), & (x_1x_2, x_1x_1) \longleftrightarrow (x_1x_1, x_1x_2). \end{array} \tag{7.2}$$

The fixed points $\mathcal{F} = \mathcal{F}(\mathcal{T}_2, \rho_2)$ are the monomials ab determined by the one-element orbits of ρ, one has $(a, b) = ({}^a b, a^b)$. There are exactly 6 fixed points:

$$\begin{array}{rl} \mathcal{F} = & \{w_1w_1 = (x_1x_1)(x_1x_1) \in \mathcal{T}_4,\; w_4w_4 = (x_2x_2)(x_2x_2) \in \mathcal{T}_4,\; w_6w_6 = (x_3x_3)(x_3x_3) \in \mathcal{T}_4, \\ & w_2w_2 = (x_1x_2)(x_1x_2) \notin \mathcal{T}_4,\; w_3w_5 = (x_1x_3)(x_2x_3) \notin \mathcal{T}_4,\; w_5w_3 = (x_2x_3)(x_1x_3) \notin \mathcal{T}_4.\}. \end{array} \tag{7.3}$$

There are exactly $15 = \binom{N}{2}$ nontrivial ρ-orbits in $\mathcal{T}_2 \times \mathcal{T}_2$ determined by (**??**). These orbits imply the following equalities in $\mathcal{A}^{(2)}$:

$$\begin{array}{ll} (x_3x_3)w_i = w_i(x_3x_3) \in \mathcal{T}_4, 1 \le i \le 5, & \\ (x_2x_3)(x_2x_3) = (x_1x_3)(x_1x_3) \notin \mathcal{T}_4, & (x_2x_3)(x_2x_2) = (x_1x_1)(x_2x_3) \in \mathcal{T}_4, \\ (x_2x_3)(x_1x_2) = (x_1x_2)(x_2x_3) \in \mathcal{T}_4, & (x_2x_3)(x_1x_1) = (x_2x_2, x_2x_3) \in \mathcal{T}_4, \\ (x_2x_2)(x_1x_3) = (x_1x_3)(x_1x_1) \notin \mathcal{T}_4, & (x_2x_2)(x_1x_2) = (x_1x_2)(x_2x_2) \in \mathcal{T}_4, \\ (x_2x_2)(x_1x_1) = (x_1x_1)(x_2x_2) \in \mathcal{T}_4, & (x_1x_3)(x_2x_2) = (x_1x_1)(x_1x_3) \in \mathcal{T}_4, \\ (x_1x_3)(x_1x_2) = (x_1x_2)(x_1x_3) \notin \mathcal{T}_4, & (x_1x_2)(x_1x_1) = (x_1x_1)(x_1x_2) \in \mathcal{T}_4. \end{array} \tag{7.4}$$

Note that for every pair $(w_i, w_j) \in \mathcal{T}_2 \times \mathcal{T}_2 \setminus \mathcal{F}$ the monomial w_iw_j occurs exactly once in (**??**) .

Six additional quadratic relations of $\mathcal{A}^{(2)}$ arise from (**??**), (**??**), and the obvious equality $a = \mathrm{Nor}(a) \in \mathcal{T}$, which hold in $\mathcal{A}^{(2)}$ for every $a \in X^2$. In this case we simply pick up all monomials which occur in

(**??**), or (**??**) but are not in $\mathcal{T}_4$ and equalize each of them with its normal form. This way we get the six relations which determine $\mathcal{R}_b$:

$$\begin{array}{lll}(x_1x_2)(x_1x_2) = (x_1x_1)(x_2x_2), & (x_1x_3)(x_2x_3) = (x_1x_1)(x_3x_3), & (x_2x_3)(x_1x_3) = (x_2x_2)(x_3x_3)\\ (x_1x_3)(x_1x_3) = (x_1x_2)(x_3x_3), & (x_2x_2)(x_1x_3) = (x_1x_2)(x_2x_3), & (x_1x_2)(x_1x_3) = (x_1x_1)(x_2x_3).\end{array} \tag{7.5}$$

The 2-Veronese algebra $\mathcal{A}^{(2)}$ has 6 generators $w_1, \cdots, w_6$ written explicitly in (**??**) and a set of 21 relations presented as a disjoint union $\mathcal{R} = \mathcal{R}_a \bigcup \mathcal{R}_b$ described below.

(1) The relations $\mathcal{R}_a$ are:

$$\begin{array}{ll} w_6w_i - w_iw_6,\ w_iw_6 \in \mathcal{T}_4,\ 1 \leq i \leq 5, & \\ w_5w_5 - w_3w_3,\ w_3w_3 \notin \mathcal{T}_4, & w_5w_4 - w_1w_5,\ w_1w_5 \in \mathcal{T}_4,\\ w_5w_2 - w_2w_5,\ w_2w_5 \in \mathcal{T}_4, & w_5w_1 - w_4w_5,\ w_4w_5 \in \mathcal{T}_4,\\ w_4w_3 - w_3w_1,\ w_3w_1 \notin \mathcal{T}_4, & w_4w_2 - w_2w_4,\ w_2w_4 \in \mathcal{T}_4,\\ w_4w_1 - w_1w_4,\ w_1w_4 \in \mathcal{T}_4, & w_3w_4 - w_1w_3,\ w_1w_3 \in \mathcal{T}_4,\\ w_3w_2 - w_2w_3,\ w_2w_3 \notin \mathcal{T}_4, & w_2w_1 - w_1w_2,\ w_1w_2 \in \mathcal{T}_4.\end{array} \tag{7.6}$$

(2) The relations $\mathcal{R}_b$ are:

$$\begin{array}{lll} w_2w_2 - w_1w_4, & w_3w_5 - w_1w_6, & w_5w_3 - w_4w_6,\\ w_3w_3 - w_2w_6, & w_3w_1 - w_2w_5, & w_2w_3 - w_1w_5.\end{array} \tag{7.7}$$

The elements of $\mathcal{R}_b$ correspond to the generators of the kernel of the Veronese map.

Thus the 2-Veronese $\mathcal{A}^{(2)}$ of the algebra $\mathcal{A}$ is a PBW algebra with a standard finite presentation

$$\mathcal{A}^{(2)} \simeq \mathbf{k}\langle w_1, \cdots, w_6\rangle/(\mathcal{R}),$$

where, $w_1, \cdots, w_6$ is a set of PBW generators and $\mathcal{R} = \mathcal{R}_a \cup \mathcal{R}_b$ is a set of defining relations which forms a quadratic Gröbner basis of the ideal $(\mathcal{R})$ in $\mathbf{k}\langle w_1, \cdots, w_6\rangle$.

Another standard finite presentation is

$$\mathcal{A}^{(2)} \simeq \mathbf{k}\langle w_1, \cdots, w_6\rangle/(\mathcal{R}_1),$$

where $\mathcal{R}_1 = \mathcal{R}_{a1} \cup \mathcal{R}_b$, and the relations $\mathcal{R}_{a1}$ are:

$$\begin{array}{ll} w_6w_i - w_iw_6,\ w_iw_6 \in \mathcal{T}_4,\ 1 \leq i \leq 5, & \\ w_5w_5 - w_2w_6,\ w_2w_6 \in \mathcal{T}_4, & w_5w_4 - w_1w_5,\ w_1w_5 \in \mathcal{T}_4,\\ w_5w_2 - w_2w_5,\ w_2w_5 \in \mathcal{T}_4, & w_5w_1 - w_4w_5,\ w_4w_5 \in \mathcal{T}_4,\\ w_4w_3 - w_2w_5,\ w_2w_5 \in \mathcal{T}_4, & w_4w_2 - w_2w_4,\ \in \mathcal{T}_4,\\ w_4w_1 - w_1w_4,\ w_1w_4 \in \mathcal{T}_4, & w_3w_4 - w_1w_3,\ w_1w_3 \in \mathcal{T}_4,\\ w_3w_2 - w_1w_5,\ w_1w_5 \in \mathcal{T}_4, & w_2w_1 = w_1w_2,\ w_1w_2 \in \mathcal{T}_4.\end{array} \tag{7.8}$$

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