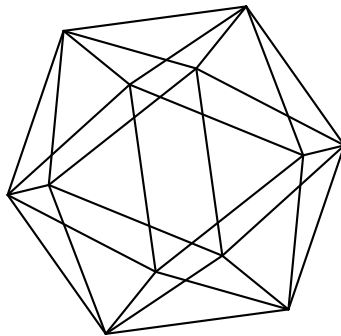# Max-Planck-Institut für Mathematik Bonn

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NORMALITY OF CIRCULAR β-ENSEMBLE

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ABSTRACT. We will prove the Berry-Esseen theorem for the number counting function of the circular β-ensemble (CβE), which will imply the central limit theorem for the number of points in arcs of the unit circle in mesoscopic and macroscopic scales. We will prove the main result by estimating the characteristic functions of the Prüfer phases and the number counting function, which will imply the the uniform upper and lower bounds of their variance. We also show that the similar results hold for the Sine$_\beta$ process. As a direct application of the uniform variance bound, we can prove the normality of the linear statistics when the test function $f(\theta) \in W^{1,p}(S^1)$ for some $p \in (1, +\infty)$.

1. INTRODUCTION

The circular β-ensemble (measure $\mu_{\beta,n}$, $\beta > 0$) is a random process on the unit circle and the joint density of its eigenangles $\theta_j \in [0, 2\pi)$, $1 \leq j \leq n$ with respect to the Lebesgue measure is

$$J(\theta_1, \cdots, \theta_n) = \frac{1}{C_{\beta,n}} \prod_{j<k} |e^{i\theta_j} - e^{i\theta_k}|^\beta,$$

where $\beta > 0$ and $C_{\beta,n} = (2\pi)^n \frac{\Gamma(1+\beta n/2)}{(\Gamma(1+\beta/2))^n}$ is the normalization constant [8].

There are many results regarding the normality of CβE and GβE (we refer to [8] for the definition of GβE). For CβE, Killip [14] proved the central limit theorem for the number of points in the fixed arcs, and the variance is logarithmic in n, where the result can be considered as the macroscopic statistics. For GβE, Costin-Lebowitz proved the normality of eigenvalues in the particular cases $\beta \in \{1, 2, 4\}$ and the variance is also logarithmic with respect to the mean [5]. These results can be extended to more general point processes, we refer to [2, 3, 4, 7, 9, 10, 11, 12, 13, 17, 18, 19, 21, 22, 23, 24] and the references therein.

Recently, in [20], Najnudel-Virág proved the uniform upper bounds on the variance of the number of points in intervals for both CβE and GβE. Their bounds are uniform in n which cover microscopic, mesoscopic and macroscopic scales. And if one rescales the interval or the arc in such a way that the average spacing between the points has order 1, then the upper bounds are logarithmic in the length of the interval or the arc. To be more precise, in the case of CβE, let's write $N_n(a, b)$ for the number of points in a sample from $\mu_{\beta,n}$ that lie in the arc between a and b, Najnudel-Virág proved the following uniform upper bound

$$(1) \qquad \mathbb{E}[|N_n(0, \theta) - n\theta/(2\pi)|^2] \leq C_\beta \ln(2 + n\theta).$$

In this paper, we will study the normality of the number counting function and the linear statistics for CβE. Our first main result is the following Berry-Esseen theorem for the number counting function, which is novel and not proved elsewhere.

Theorem 1. *Let $\theta \in (0, \pi]$ that may depend on n, we have the uniform estimate*

$$\sup_{x \in \mathbb{R}} \left| \mathbb{P}\left[\sqrt{\frac{\pi^2 \beta}{2 \ln(2 + n\theta)}} \left[N_n(0, \theta) - \frac{n\theta}{2\pi} \right] \le x \right] - \int_{-\infty}^{x} \frac{e^{-t^2/2}}{\sqrt{2\pi}} dt \right| \le \frac{C}{(\ln(2 + n\theta))^{\frac{1}{2}}},$$

here $C > 0$ is a constant depending only on β.

As a direct consequence of Theorem 1, we have the following central limit theorem for the number of points in arcs of the unit circle for CβE in both mesoscopic and macroscopic scales.

Corollary 1. *Let $\theta_n \in (0, \pi]$, $n\theta_n \to +\infty$, then*

$$\sqrt{\frac{\pi^2 \beta}{2 \ln(2 + n\theta_n)}} [N_n(0, \theta_n) - \frac{n\theta_n}{2\pi}]$$

converges in law to a Gaussian random variable of mean zero and variance one.

Notice that, Corollary 1 is the main result proved in [14] for the case where $\theta_n = \theta$ is fixed.

To show the key steps to prove Theorem 1, we begin with some preliminary results proved in [15]. Let $\gamma_j \sim \Theta_{\beta(j+1)+1}$ be independent random variables for $j \ge 0$ and let η be a uniform variable on $[0, 2\pi)$ independent of $(\gamma_j)_{j \ge 0}$. We define the so-called Prüfer phases $(\psi_k(\theta))_{\theta \in \mathbb{R}, k \ge 0}$ as follows: $\psi_0(\theta) = \theta$ and for $k \ge 0$,

$$\psi_{k+1}(\theta) = \psi_k(\theta) + \theta + 2\text{Im} \ln \left(\frac{1 - \gamma_k}{1 - \gamma_k e^{i\psi_k(\theta)}} \right).$$

Then the random set

$$\{\theta \in \mathbb{R}, \psi_{n-1}(\theta) \equiv \eta (\text{mod } 2\pi)\}$$

has the same law as the set of all determinations of the arguments of the n points of a CβE. Here, a complex random variable X with values in the unit disk $\mathbb{D}$ is Θ_ν-distributed (for $\nu > 1$) if

$$\mathbb{E}[f(X)] = \frac{\nu - 1}{2\pi} \iint_{\mathbb{D}} f(z)(1 - |z|^2)^{(\nu-3)/2} d^2 z.$$

Simple computations show [14, 15]

$$\mathbb{E}[X] = 0, \ \mathbb{E}[|X|^2] = \frac{2}{\nu + 1}, \ \mathbb{E}[|X|^4] = \frac{8}{(\nu + 1)(\nu + 3)}$$

and

$$(2) \quad \mathbb{E}[(-\ln(1 - |X|^2))^m] = \frac{\nu - 1}{2} \int_0^\infty t^m e^{-(\nu-1)t/2} dt = \Gamma(m + 1) \left(\frac{2}{\nu - 1} \right)^m,$$

where we change the variable $e^{-t} = 1 - |X|^2$.

The above result tells us that

$$N_n(0, \theta) \overset{\text{d}}{=} \lfloor \frac{\psi_{n-1}(\theta) - \eta}{2\pi} \rfloor + 1$$

for $\theta \in (0, 2\pi)$. Here $X \overset{\text{d}}{=} Y$ means that the random variables X, Y have the same distribution, we also used the fact that $\psi_{n-1}(0) = 0$, and that $\psi_{n-1}(\theta)$ is increasing with respect to θ. By rotational invariance we have

$$\mathbb{E}[N_n(0, \theta)] = n\theta/(2\pi)$$

and

$$N_n(0, 2\pi - \theta) \stackrel{\mathrm{d}}{=} N_n(\theta, 2\pi) = n - N_n(0, \theta),$$

i.e., there is a natural symmetry between θ and $2\pi - \theta$, therefore, it is enough to study the case $\theta \in (0, \pi]$.

Throughout the article, we will use $C > 0$ to denote a universal constant depending only on β which may change from line to line.

To prove Theorem 1, the key lemma is the following estimate regarding the characteristic function of the Prüfer phases.

Lemma 1. *Let* $\theta \in (0, \pi]$, $\lambda \in \mathbb{R}$, $\lambda^2 \leq \beta/8$. *There exists a constant* $C > 0$ *depending only on* β *such that*

$$\left| \mathbb{E}[e^{i\lambda(\psi_{n-1}(\theta) - n\theta)}] - e^{-(4\lambda^2/\beta)\ln(2+n\theta)} \right| \leq C\lambda^2 e^{-(4\lambda^2/\beta)\ln(2+n\theta)}.$$

Moreover, as a consequence of Lemma 1, we can prove the following uniform bound first for the variance of $\psi_{n-1}(\theta)$, then for the variance of $N_n(0, \theta)$.

Corollary 2. *There exists a constant* $C > 0$ *depending only on* β *such that for* $\theta \in (0, \pi]$, $n \in \mathbb{Z}$, $n > 0$, $\lambda \in [-2\pi\sqrt{\beta/8}, 2\pi\sqrt{\beta/8}]$, *we have*

$$(3) \qquad \left| \mathbb{E}[(\psi_{n-1}(\theta) - n\theta)^2] - (8/\beta)\ln(2 + n\theta) \right| \leq C,$$

$$(4) \qquad \left| \mathbb{E}[e^{i\lambda(N_n(0,\theta) - n\theta/(2\pi))}] - e^{-\lambda^2/(\beta\pi^2)\cdot\ln(2+n\theta)} \right| \leq C\lambda^2,$$

$$(5) \qquad \left| \mathbb{E}\left[\left| N_n(0, \theta) - \frac{n\theta}{2\pi} \right|^2 \right] - \frac{2\ln(2 + n\theta)}{\pi^2\beta} \right| \leq C.$$

The inequality (5) gives both the upper and lower uniform variance bounds which improves the estimate (1).

The Sine_β point process is the scaling limit of the $C\beta E$, and its central limit theorem has been proved in [16]. In this article, we can further prove the following uniform variance bound and the Berry-Esseen theorem for the Sine_β point process. Let's denote $\mathrm{Card}(A)$ the cardinality of a set A.

Corollary 3. *Let* L *be the* Sine_β *point process, there exists a constant* $C > 0$ *depending only on* β *such that for* $x > 0$, *we have*

$$(6) \qquad \left| \mathbb{E}[(\mathrm{Card}(L \cap [0, x]) - x/(2\pi))^2] - 2/(\beta\pi^2) \cdot \ln(2 + x) \right| \leq C,$$

$$(7) \qquad \sup_{y \in \mathbb{R}} \left| \mathbb{P}\left[\sqrt{\frac{\pi^2\beta}{2\ln(2 + x)}} \left[\mathrm{Card}(L \cap [0, x]) - \frac{x}{2\pi} \right] \leq y \right] - \int_{-\infty}^{y} \frac{e^{-t^2/2}}{\sqrt{2\pi}} dt \right|$$
$$\leq C(\ln(2 + x))^{-\frac{1}{2}}.$$

In the end, as a direct application of the uniform variance bound (5), we can prove the normality of the linear statistics for $C\beta E$ when the test function is in $W^{1,p}(S^1)$ for some $p \in (1, +\infty)$, and p will be fixed. Let's denote

$$\xi_n = \sum_{j=1}^{n} \delta_{\theta_j}$$

the empirical measure of a sample from $\mu_{\beta,n}$, and we consider the linear statistics

$$\langle \xi_n, f \rangle = \sum_{j=1}^{n} f(\theta_j).$$

We will prove the following result.

Theorem 2. *Let $f \in W^{1,p}(S^1)$ be real valued and periodic function with $f(0) = f(2\pi)$, and $\int_0^{2\pi} f(x)dx = 0$ where $p \in (1, +\infty)$, then $\langle \xi_n, f \rangle$ converges in law to a Gaussian random variable of mean zero and variance $2\sigma^2$, where*

$$\sigma^2 = \frac{2}{\beta} \sum_{j=1}^{+\infty} j|a_j|^2, \ a_j = \frac{1}{2\pi} \int_0^{2\pi} f(x)e^{-ijx}dx, \ j \in \mathbb{Z}.$$

To prove Theorem 2, we will need the variance estimate of the linear statistics (see Lemma 14 in §6) which is based on the uniform variance bound (5). The rest proof makes use of Lemma 16 (proved in [9]) and the approximation of the $W^{1,p}(S^1)$ space by the Féjer kernel.

There are also some known results on the normality of linear statistics for CβE, we refer to [6, 7, 9, 10, 11, 19, 25] for more details.

The organization of the article is as follows. In §2, we will review some known results on CβE which are proved in [14, 15]. In §3, we will derive Lemma 1 and prove Corollary 2. In §4, we will finish the proof of Theorem 1. In §5, we will prove Corollary 3 for the Sine$_\beta$ process. In §6, as an application of the uniform variance bound (5), we will prove Theorem 2.

2. Preliminary results

In this section, we will collect several properties regarding CβE proved in [14, 15] which will be useful in the proof of Theorem 1.

Now we introduce

$$\Upsilon(\psi, \alpha) = -2\mathrm{Im}\ln[1 - \alpha e^{i\psi}] = \mathrm{Im}\sum_{l=1}^{+\infty} \frac{2}{l}e^{il\psi}\alpha^l,$$

and

$$\Upsilon_1(\psi, \alpha) = \Upsilon(\psi, \alpha) - \Upsilon(0, \alpha).$$

Then we have [15]

$$\psi_{k+1}(\theta) = \psi_k(\theta) + \theta + \Upsilon_1(\psi_k(\theta), \gamma_k).$$

We have the following estimates about Υ (Lemma 2.5 in [14]).

Lemma 2. *Suppose $\phi, \psi \in \mathbb{R}$ and $\alpha \sim \Theta_\nu$. If $\widetilde{\Upsilon}(\psi, \alpha) = 2\mathrm{Im}[\alpha e^{i\psi}]$, then*

$$\mathbb{E}[\Upsilon(\psi, \alpha)] = \mathbb{E}[\widetilde{\Upsilon}(\psi, \alpha)] = 0,$$

$$\mathbb{E}[\widetilde{\Upsilon}(\psi, \alpha)\widetilde{\Upsilon}(\phi, \alpha)] = \frac{4}{\nu + 1}\cos(\psi - \phi),$$

$$\mathbb{E}[\widetilde{\Upsilon}(\psi, \alpha)^4] = \frac{48}{(\nu + 1)(\nu + 3)},$$

$$\mathbb{E}[|\Upsilon(\psi, \alpha) - \widetilde{\Upsilon}(\psi, \alpha)|^2] \leq \frac{16}{(\nu + 1)(\nu + 3)},$$

$$\mathbb{E}[|\Upsilon(\psi, \alpha)|^2] \leq \frac{8}{\nu + 1}.$$

By rotational invariance, we also have $\mathbb{E}[\alpha^3] = \mathbb{E}[|\alpha|^2\alpha] = 0$, which implies

$$(8) \qquad\qquad \mathbb{E}[(\widetilde{\Upsilon}(\psi, \alpha) - \widetilde{\Upsilon}(\phi, \alpha))^3] = 0.$$

We apply Plancharel's theorem to the power series of Υ to get

$$(9) \qquad \mathbb{E}[(\Upsilon(\psi,\alpha) - \widetilde{\Upsilon}(\psi,\alpha))\widetilde{\Upsilon}(\phi,\alpha)] = 0.$$

We also have the following estimate on Υ_1 (see Proposition 2.3 in [15]),

$$(10) \qquad \left|\int_0^{2\pi} \Upsilon_1(\psi, re^{i\theta})\Upsilon_1(\phi, re^{i\theta})\frac{\mathrm{d}\theta}{2\pi}\right| \le 2(|\psi|+|\phi|)\ln\frac{1}{1-r^2}, \ \forall\, r \in [0,1).$$

The following estimates are proved in Corollary 2.4 in [15].

Lemma 3. *For $s \ge 0$, we have*

$$\mathbb{E}[\psi_s(\theta)] = (s+1)\theta, \quad \mathbb{E}[|\psi_s(\theta)|] = (s+1)|\theta|,$$

and for $0 \le k \le m$, we have

$$\mathbb{E}[|\psi_m(\theta) - \psi_k(\theta) - (m-k)\theta|^2] = \sum_{s=k}^{m-1} \mathbb{E}[|\Upsilon_1(\psi_s(\theta), \gamma_s)|^2].$$

By Lemma 3, we can further prove

Lemma 4. *For $0 \le k \le m$, we have*

$$\mathbb{E}[|\psi_m(\theta) - \psi_k(\theta) - (m-k)\theta|^2] \le 8(m-k)|\theta|/\beta.$$

In particular, for $k = 0$ we have

$$\mathbb{E}[|\psi_m(\theta) - (m+1)\theta|^2] \le 8m|\theta|/\beta.$$

Proof. Using Lemma 3, by (2), (10) and the rotational invariance we have

$$\mathbb{E}[|\psi_m(\theta) - \psi_k(\theta) - (m-k)\theta|^2] = \sum_{s=k}^{m-1} \mathbb{E}[|\Upsilon_1(\psi_s(\theta), \gamma_s)|^2]$$

$$\le \sum_{s=k}^{m-1} 4\mathbb{E}\left[|\psi_s(\theta)|\ln\frac{1}{1-|\gamma_s|^2}\right] = \sum_{s=k}^{m-1} 4\mathbb{E}\left[|\psi_s(\theta)|\right]\mathbb{E}\left[\ln\frac{1}{1-|\gamma_s|^2}\right]$$

$$\le \sum_{s=k}^{m-1} 4(s+1)|\theta|\frac{2}{\beta(s+1)} = \sum_{s=k}^{m-1} \frac{8|\theta|}{\beta} = \frac{8(m-k)|\theta|}{\beta}.$$

Here, we take $\nu = \beta(s+1)+1$ for γ_s. This completes the proof. $\qquad\square$

3. The characteristic function and the uniform variance bound

Let's define the characteristic function of the Prüfer phases

$$a_k(\lambda) := a_k(\lambda, \theta, \beta) = \mathbb{E}[e^{i\lambda(\psi_k(\theta)-(k+1)\theta)}].$$

Then $|a_k(\lambda)| \le 1$ for $\lambda \in \mathbb{R}$. In this section, we will derive several estimates regarding the sequence $\{a_k(\lambda)\}_{k=0}^{+\infty}$, then we can prove Lemma 1 and Corollary 2.

We first have

Lemma 5. *Suppose $\phi, \lambda \in \mathbb{R}$ and $\alpha \sim \Theta_\nu$, then*

$$\left|\mathbb{E}[e^{i\lambda\Upsilon_1(\phi,\alpha)}] - 1 + \frac{4\lambda^2(1-\cos\phi)}{\nu+1}\right| \le \frac{64\lambda^2+416\lambda^4}{(\nu+1)(\nu+3)}$$

and

$$\left|\mathbb{E}[e^{i\lambda\Upsilon_1(\phi,\alpha)}] - 1\right| \le \frac{16\lambda^2}{\nu+1}.$$

Proof. Let $X = \Upsilon_1(\phi, \alpha)$, $X_1 = \widetilde{\Upsilon}(\phi, \alpha) - \widetilde{\Upsilon}(0, \alpha)$, $X_3 = \Upsilon(\phi, \alpha) - \widetilde{\Upsilon}(\phi, \alpha)$, $X_4 = \widetilde{\Upsilon}(0, \alpha) - \Upsilon(0, \alpha)$, $X_2 = X_3 + X_4$, here $\widetilde{\Upsilon}(\psi, \alpha) = 2\mathrm{Im}[\alpha e^{i\psi}]$ for every $\psi \in \mathbb{R}$, then we have $X = X_1 + X_3 + X_4 = X_1 + X_2$. By Lemma 2 we have

$$(11) \qquad \mathbb{E}[X_1] = \mathbb{E}[X_3] = \mathbb{E}[X_4] = 0, \quad \mathbb{E}[X_2] = \mathbb{E}[X] = 0,$$

$$(12) \qquad \mathbb{E}[|X_3|^2] \le \frac{16}{(\nu+1)(\nu+3)}, \quad \mathbb{E}[|X_4|^2] \le \frac{16}{(\nu+1)(\nu+3)},$$

$$(13) \qquad \mathbb{E}[|X_2|^2] \le 2\mathbb{E}[|X_3|^2 + |X_4|^2] \le \frac{64}{(\nu+1)(\nu+3)},$$

$$(14) \qquad \mathbb{E}[|X_1|^4] \le 8\mathbb{E}[\widetilde{\Upsilon}(0, \alpha)^4 + \widetilde{\Upsilon}(\phi, \alpha)^4] \le \frac{768}{(\nu+1)(\nu+3)},$$

$$(15) \qquad \mathbb{E}[|X_1|^2] = \frac{8(1 - \cos\phi)}{\nu+1}, \quad \mathbb{E}[|X|^2] \le \frac{32}{\nu+1}.$$

By (8), (9) we have $\mathbb{E}[X_1^3] = \mathbb{E}[X_1 X_2] = 0$. Notice that

$$e^{i\lambda X} - e^{i\lambda X_1} = e^{i\lambda X_1}(e^{i\lambda X_2} - 1)$$
$$= e^{i\lambda X_1}(e^{i\lambda X_2} - i\lambda X_2 - 1) + (e^{i\lambda X_1} - i\lambda X_1 - 1)(i\lambda X_2) + i\lambda X_2(i\lambda X_1 + 1),$$

and that $|e^{ix} - ix - 1| \le |x|^2/2$ for $x \in \mathbb{R}$ by Taylor expansion, we have

$$|e^{i\lambda X} - e^{i\lambda X_1} - i\lambda X_2(i\lambda X_1 + 1)| \le |\lambda X_2|^2/2 + |\lambda X_1|^2|\lambda X_2|/2,$$

which together with $\mathbb{E}[X_2] = \mathbb{E}[X_1 X_2] = 0$ and (13), (14) gives

$$(16) \qquad |\mathbb{E}[e^{i\lambda X} - e^{i\lambda X_1}]| = |\mathbb{E}[e^{i\lambda X} - e^{i\lambda X_1} - i\lambda X_2(i\lambda X_1 + 1)]|$$

$$\le \mathbb{E}[|\lambda X_2|^2/2 + |\lambda X_1|^2|\lambda X_2|/2] \le \mathbb{E}[|\lambda X_2|^2 + |\lambda X_1|^4/2]$$

$$\le \frac{64\lambda^2}{(\nu+1)(\nu+3)} + \frac{384\lambda^4}{(\nu+1)(\nu+3)}.$$

Since $|e^{ix} + ix^3/6 + x^2/2 - ix - 1| \le |x|^4/24$ for $x \in \mathbb{R}$ by Taylor expansion and $\mathbb{E}[X_1^3] = \mathbb{E}[X_1] = 0$, by (14) we have

$$|\mathbb{E}[e^{i\lambda X_1}] - 1 + \lambda^2\mathbb{E}[|X_1|^2]/2| = |\mathbb{E}[e^{i\lambda X_1} + i\lambda^3 X_1^3/6 + \lambda^2 X_1^2/2 - i\lambda X_1 - 1]|$$

$$\le \mathbb{E}[|\lambda X_1|^4]/24 \le \frac{32\lambda^4}{(\nu+1)(\nu+3)},$$

which together with (15), (16) gives

$$\left| \mathbb{E}[e^{i\lambda X}] - 1 + \frac{4\lambda^2(1 - \cos\phi)}{\nu+1} \right|$$

$$\le |\mathbb{E}[e^{i\lambda X} - e^{i\lambda X_1}]| + |\mathbb{E}[e^{i\lambda X_1}] - 1 + \lambda^2\mathbb{E}[|X_1|^2]/2|$$

$$\le \frac{64\lambda^2 + 384\lambda^4}{(\nu+1)(\nu+3)} + \frac{32\lambda^4}{(\nu+1)(\nu+3)} = \frac{64\lambda^2 + 416\lambda^4}{(\nu+1)(\nu+3)}.$$

This completes the proof of the first inequality. Since $|e^{ix} - ix - 1| \le |x|^2/2$ for $x \in \mathbb{R}$ by Taylor expansion and $\mathbb{E}[X] = 0$, by (15) we have

$$|\mathbb{E}[e^{i\lambda X}] - 1| = |\mathbb{E}[e^{i\lambda X} - i\lambda X - 1]| \le \mathbb{E}[|\lambda X|^2/2] \le \frac{16\lambda^2}{\nu+1}.$$

This completes the proof of the second inequality. $\qquad\square$

We need the following estimate of the sequence $\{a_k(\lambda)\}_{k=0}^{+\infty}$.

Lemma 6. *Let* $\theta \in (0, \pi]$, $\lambda \in \mathbb{R}$, *then we have*

$$\left| a_{k+1}(\lambda) - a_k(\lambda) + \frac{\lambda^2(4a_k(\lambda) - 2e^{i(k+1)\theta}a_k(\lambda+1) - 2e^{-i(k+1)\theta}a_k(\lambda-1))}{\beta(k+1)+2} \right|$$

$$\leq \frac{64\lambda^2 + 416\lambda^4}{(\beta(k+1)+2)(\beta(k+1)+4)}$$

and

$$|a_{k+1}(\lambda) - a_k(\lambda)| \leq \frac{16\lambda^2}{\beta(k+1)+2}.$$

Proof. Let $X_k := \psi_k(\theta)$, $Y_k := \Upsilon_1(\psi_k(\theta), \gamma_k)$, then

$$X_{k+1} = X_k + \theta + Y_k.$$

For the sigma algebras

$$\mathcal{M}_{k-1} := \sigma(\gamma_0, \cdots, \gamma_{k-1}),$$

γ_k is independent of $\mathcal{M}_{k-1}$ and X_k is measurable in $\mathcal{M}_{k-1}$. By Lemma 5 we have (taking $\nu = \beta(k+1)+1$ for γ_k)

$$(17) \quad \left| \mathbb{E}[e^{i\lambda Y_k}|\mathcal{M}_{k-1}] - 1 + \frac{4\lambda^2(1-\cos X_k)}{\beta(k+1)+2} \right| \leq \frac{64\lambda^2 + 416\lambda^4}{(\beta(k+1)+2)(\beta(k+1)+4)}$$

and

$$\left| \mathbb{E}[e^{i\lambda Y_k}|\mathcal{M}_{k-1}] - 1 \right| \leq \frac{16\lambda^2}{\beta(k+1)+2}.$$

Let's denote

$$Z_k(\lambda) := e^{i\lambda(X_k - (k+1)\theta)},$$

then we have

$$Z_k(\lambda) = e^{i\lambda(\psi_k(\theta) - (k+1)\theta)}, \quad Z_{k+1}(\lambda) = e^{i\lambda Y_k}Z_k(\lambda), \quad a_k(\lambda) = \mathbb{E}[Z_k(\lambda)]$$

and

$$2Z_k(\lambda)\cos X_k = Z_k(\lambda)(e^{iX_k} + e^{-iX_k}) = e^{i(\lambda+1)X_k - i\lambda(k+1)\theta} + e^{i(\lambda-1)X_k - i\lambda(k+1)\theta}$$

$$= Z_k(\lambda+1)e^{i(k+1)\theta} + Z_k(\lambda-1)e^{-i(k+1)\theta}.$$

Let's denote

$$V_k(\lambda) := Z_{k+1}(\lambda) - Z_k(\lambda) + Z_k(\lambda)\frac{4\lambda^2(1-\cos X_k)}{\beta(k+1)+2}$$

$$= e^{i\lambda Y_k}Z_k(\lambda) - Z_k(\lambda) + Z_k(\lambda)\frac{4\lambda^2(1-\cos X_k)}{\beta(k+1)+2}$$

$$= Z_{k+1}(\lambda) - Z_k(\lambda) + \frac{\lambda^2(4Z_k(\lambda) - 2Z_k(\lambda+1)e^{i(k+1)\theta} - 2Z_k(\lambda-1)e^{-i(k+1)\theta})}{\beta(k+1)+2}.$$

Then by (17) and the fact that $|Z_k(\lambda)| = 1$, $a_k(\lambda) = \mathbb{E}[Z_k(\lambda)]$ we have

$$|\mathbb{E}[V_k(\lambda)|\mathcal{M}_{k-1}]| = \left| \mathbb{E}[e^{i\lambda Y_k}|\mathcal{M}_{k-1}]Z_k(\lambda) - Z_k(\lambda) + Z_k(\lambda)\frac{4\lambda^2(1-\cos X_k)}{\beta(k+1)+2} \right|$$

$$= |Z_k(\lambda)| \left| \mathbb{E}[e^{i\lambda Y_k}|\mathcal{M}_{k-1}] - 1 + \frac{4\lambda^2(1-\cos X_k)}{\beta(k+1)+2} \right| \leq \frac{64\lambda^2 + 416\lambda^4}{(\beta(k+1)+2)(\beta(k+1)+4)}$$

and

$$\left| a_{k+1}(\lambda) - a_k(\lambda) + \frac{\lambda^2(4a_k(\lambda) - 2e^{i(k+1)\theta}a_k(\lambda+1) - 2e^{-i(k+1)\theta}a_k(\lambda-1))}{\beta(k+1)+2} \right|$$

$$= |\mathbb{E}[V_k(\lambda)]| \le \mathbb{E}\left|\mathbb{E}\left[V_k(\lambda)|\mathcal{M}_{k-1}\right]\right| \le \frac{64\lambda^2 + 416\lambda^4}{(\beta(k+1)+2)(\beta(k+1)+4)},$$

which is the first inequality. Similarly, we have

$$|a_{k+1}(\lambda) - a_k(\lambda)| = |\mathbb{E}[Z_{k+1}(\lambda) - Z_k(\lambda)]| \le \mathbb{E}\left|\mathbb{E}\left[Z_{k+1}(\lambda) - Z_k(\lambda)|\mathcal{M}_{k-1}\right]\right|$$

$$= \mathbb{E}\left|\mathbb{E}\left[e^{i\lambda Y_k}|\mathcal{M}_{k-1}\right]Z_k(\lambda) - Z_k(\lambda)\right| = \mathbb{E}\left|\mathbb{E}[e^{i\lambda Y_k}|\mathcal{M}_{k-1}] - 1\right| \le \frac{16\lambda^2}{\beta(k+1)+2},$$

which is the second inequality. This completes the proof. $\qquad\square$

Lemma 7. *Let* $\theta \in (0, \pi]$, $\delta \in [-\pi, \pi] \setminus \{0\}$, $\lambda \in \mathbb{R}$, *then we have*

$$\left|\sum_{j=k}^{n-1} \frac{e^{ij\delta}a_j(\lambda)}{\beta(j+1)+2}\right| \le \frac{2 + 16\lambda^2/\beta}{|1 - e^{i\delta}|(\beta(k+1)+2)}.$$

Proof. Let $\epsilon_j = 1/(\beta(j+1)+2)$, $a_j = a_j(\lambda)$, using summation by parts

$$\sum_{j=k}^{n-1} e^{ij\delta}\epsilon_j a_j(\lambda) = \sum_{j=k}^{n-1} \frac{e^{ij\delta}\epsilon_j a_j(\lambda) - e^{i(j+1)\delta}\epsilon_j a_j(\lambda)}{1 - e^{i\delta}}$$

$$= \frac{e^{ik\delta}\epsilon_k a_k - e^{in\delta}\epsilon_n a_n}{1 - e^{i\delta}} + \sum_{j=k}^{n-1} \frac{(\epsilon_{j+1} - \epsilon_j)e^{i(j+1)\delta}a_j}{1 - e^{i\delta}} + \sum_{j=k+1}^{n} \epsilon_j \frac{e^{ij\delta}(a_j - a_{j-1})}{1 - e^{i\delta}},$$

and using $|a_j(\lambda)| \le 1$ we have

$$\left|\sum_{j=k}^{n-1} e^{ij\delta}\epsilon_j a_j\right| \le \frac{|\epsilon_k| + |\epsilon_n| + \sum_{j=k+1}^{n}|\epsilon_j - \epsilon_{j-1}| + \sum_{j=k}^{n-1}|\epsilon_j(a_j(\lambda) - a_{j-1}(\lambda))|}{|1 - e^{i\delta}|}.$$

Since $\epsilon_{j-1} > \epsilon_j > 0$, we have

$$\sum_{j=k+1}^{n} |\epsilon_j - \epsilon_{j-1}| = \epsilon_k - \epsilon_n$$

and

$$|\epsilon_k| + |\epsilon_n| + \sum_{j=k+1}^{n} |\epsilon_j - \epsilon_{j-1}| = 2\epsilon_k.$$

By Lemma 6 we have

$$|a_j(\lambda) - a_{j-1}(\lambda)| \le 16\lambda^2 \epsilon_{j-1},$$

this together with $\epsilon_{j-1} - \epsilon_j = \beta\epsilon_{j-1}\epsilon_j > 0$ implies that

$$\sum_{j=k}^{n-1} |\epsilon_j(a_j(\lambda) - a_{j-1}(\lambda))| \le \sum_{j=k}^{n-1} 16\lambda^2|\epsilon_j\epsilon_{j-1}| = (16\lambda^2/\beta)(\epsilon_k - \epsilon_n).$$

Summing up we conclude that

$$\left|\sum_{j=k}^{n-1} e^{ij\delta}\epsilon_j a_j(\lambda)\right| \le \frac{2\epsilon_k + (16\lambda^2/\beta)\epsilon_k}{|1 - e^{i\delta}|} = \frac{2 + 16\lambda^2/\beta}{|1 - e^{i\delta}|(\beta(k+1)+2)}.$$

This completes the proof. $\qquad\square$

Lemma 8. *Given complex valued sequences ϵ_j, a_j, b_j, c_j and $n \in \mathbb{Z}$, $n > 0$, $\lambda \in \mathbb{R}$ such that $|a_j| \le 1$, $\epsilon_j > 0$, $a_{j+1} - a_j + \lambda^2(\epsilon_j a_j - b_j) = c_j$, let $s_k = \sum_{j=0}^{k-1} \epsilon_j$, $t_k = \sum_{j=k}^{n-1} b_j$, then we have (for $k \in [0, n-1] \cap \mathbb{Z}$)*

$$\left| e^{\lambda^2 s_k} a_k - e^{\lambda^2 s_n} a_n \right| \le e^{\lambda^2 s_k} \lambda^2 |t_k| + \sum_{j=k}^{n-1} e^{\lambda^2 s_{j+1}} (|c_j| + \lambda^4 \epsilon_j^2/2 + \lambda^4 |\epsilon_j t_j|).$$

Proof. By the definition of t_k we have $b_j = t_j - t_{j+1}$, inserting this into the equation of c_j we have $a_{j+1} + \lambda^2 t_{j+1} - a_j + \lambda^2(\epsilon_j a_j - t_j) = c_j$. Let $\widetilde{a}_j = a_j + \lambda^2 t_j$ then

$$\widetilde{a}_{j+1} - e^{-\lambda^2 \epsilon_j} \widetilde{a}_j = c_j + (1 - \lambda^2 \epsilon_j - e^{-\lambda^2 \epsilon_j}) a_j + \lambda^2 (1 - e^{-\lambda^2 \epsilon_j}) t_j.$$

Since $|1 - x - e^{-x}| \le |x|^2/2$, $|1 - e^{-x}| \le |x|$ for $x \ge 0$ by Taylor expansion and $|a_j| \le 1$, we have

$$|\widetilde{a}_{j+1} - e^{-\lambda^2 \epsilon_j} \widetilde{a}_j| \le |c_j| + \lambda^4 \epsilon_j^2/2 + \lambda^4 |\epsilon_j t_j|.$$

By the definition of s_k we have $s_{j+1} = s_j + \epsilon_j$, thus

$$|e^{\lambda^2 s_k} \widetilde{a}_k - e^{\lambda^2 s_n} \widetilde{a}_n| \le \sum_{j=k}^{n-1} |e^{\lambda^2 s_{j+1}} \widetilde{a}_{j+1} - e^{\lambda^2 s_j} \widetilde{a}_j| = \sum_{j=k}^{n-1} e^{\lambda^2 s_{j+1}} |\widetilde{a}_{j+1} - e^{-\lambda^2 \epsilon_j} \widetilde{a}_j|$$

$$(18) \qquad\qquad \le \sum_{j=k}^{n-1} e^{\lambda^2 s_{j+1}} (|c_j| + \lambda^4 \epsilon_j^2/2 + \lambda^4 |\epsilon_j t_j|).$$

Notice that $t_n = 0$, $e^{\lambda^2 s_k} \widetilde{a}_k - e^{\lambda^2 s_n} \widetilde{a}_n = e^{\lambda^2 s_k} a_k - e^{\lambda^2 s_n} a_n + e^{\lambda^2 s_k} \lambda^2 t_k$, and

$$|e^{\lambda^2 s_k} a_k - e^{\lambda^2 s_n} a_n| \le e^{\lambda^2 s_k} \lambda^2 |t_k| + |e^{\lambda^2 s_k} \widetilde{a}_k - e^{\lambda^2 s_n} \widetilde{a}_n|,$$

which together with (18) concludes the proof. $\qquad \square$

Lemma 9. *Let $\theta \in (0, \pi]$, $\lambda \in \mathbb{R}$, $\epsilon_k = 4/(\beta(k+1)+2)$, $s_k = \sum_{j=0}^{k-1} \epsilon_j$, then (for $n, k \in \mathbb{Z}$, $n > k \ge 0$)*

$$\left| e^{\lambda^2 s_k} a_k(\lambda) - e^{\lambda^2 s_n} a_n(\lambda) \right| \le \frac{C}{\theta}(\lambda^2 + \lambda^4) e^{\lambda^2 s_k} \epsilon_k + \frac{C}{\theta}(\lambda^2 + \lambda^6) \sum_{j=k}^{n-1} e^{\lambda^2 s_{j+1}} \epsilon_j^2,$$

here $C > 0$ is a constant depending only on β.

Proof. Let $a_k = a_k(\lambda)$, $b_k = \dfrac{2e^{i(k+1)\theta} a_k(\lambda+1) + 2e^{-i(k+1)\theta} a_k(\lambda-1)}{\beta(k+1)+2}$, and $c_k = a_{k+1} - a_k + \lambda^2(\epsilon_k a_k - b_k)$, by Lemma 6 we have $|c_k| \le (4\lambda^2 + 26\lambda^4)\epsilon_k^2$. We can write $t_k = \sum_{j=k}^{n-1} b_j = 2(t_{k,1} + t_{k,2})$ such that

$$t_{k,1} = \sum_{j=k}^{n-1} \frac{e^{i(j+1)\theta} a_j(\lambda+1)}{\beta(j+1)+2}, \quad t_{k,2} = \sum_{j=k}^{n-1} \frac{e^{-i(j+1)\theta} a_j(\lambda-1)}{\beta(j+1)+2}.$$

By Lemma 7 we have

$$|t_{k,1}| \le \frac{2 + 16(\lambda+1)^2/\beta}{|1 - e^{i\theta}|(\beta(k+1)+2)}, \quad |t_{k,2}| \le \frac{2 + 16(\lambda-1)^2/\beta}{|1 - e^{-i\theta}|(\beta(k+1)+2)},$$

thus

$$|t_k| \le 2(|t_{k,1}| + |t_{k,2}|) \le 4 \cdot \frac{2 + 16(\lambda^2+1)/\beta}{|1 - e^{i\theta}|(\beta(k+1)+2)} \le \frac{C(\lambda^2+1)\epsilon_k}{|1 - e^{i\theta}|}.$$

Summing up we have

$$(|c_j| + \lambda^4 \epsilon_j^2/2 + \lambda^4 |\epsilon_j t_j|) \le (4\lambda^2 + 26\lambda^4)\epsilon_j^2 + \lambda^4 \epsilon_j^2/2 + \frac{C\lambda^4 \epsilon_j^2(\lambda^2+1)}{|1 - e^{i\theta}|}$$

$$\le C(\lambda^2 + \lambda^4)\epsilon_j^2 + C\lambda^4 \epsilon_j^2(\lambda^2+1)/\theta \le C(\lambda^2 + \lambda^6)\epsilon_j^2/\theta.$$

By Lemma 8 we have

$$\left| e^{\lambda^2 s_k} a_k(\lambda) - e^{\lambda^2 s_n} a_n(\lambda) \right| \le e^{\lambda^2 s_k} \lambda^2 |t_k| + \sum_{j=k}^{n-1} e^{\lambda^2 s_{j+1}} (|c_j| + \lambda^4 \epsilon_j^2/2 + \lambda^4 |\epsilon_j t_j|)$$

$$\le e^{\lambda^2 s_k} \frac{C\lambda^2(\lambda^2+1)\epsilon_k}{|1 - e^{i\theta}|} + C \sum_{j=k}^{n-1} e^{\lambda^2 s_{j+1}} (\lambda^2 + \lambda^6)\epsilon_j^2/\theta.$$

This completes the proof. $\square$

Lemma 10. *Let* $\epsilon_k = 4/(\beta(k+1)+2)$, $s_k = \sum_{j=0}^{k-1} \epsilon_j$, *then* $|s_k - (4/\beta) \ln(k+1)| \le C$, *here* $C > 0$ *is a constant depending only on* β.

Proof. By definition we have $s_0 = 0$ and $s_k - s_{k-1} = \epsilon_{k-1}$ for $k \ge 1$. Let $\widetilde{s}_k = s_k - (4/\beta) \ln(k+1)$, then we have $\widetilde{s}_0 = 0$ and $\widetilde{s}_k - \widetilde{s}_{k-1} = \epsilon_{k-1} - (4/\beta) \ln(1 + 1/k)$ for $k \ge 1$. Thus

$$|\widetilde{s}_k - \widetilde{s}_{k-1}| \le |\epsilon_{k-1} - 4/(\beta k)| + |4/(\beta k) - (4/\beta) \ln(1 + 1/k)|$$

$$= |4/(\beta k + 2) - 4/(\beta k)| + (4/\beta)|\ln(1 + 1/k) - 1/k|$$

$$\le 8/(\beta k)^2 + (4/\beta)(1/k)^2/2 = (8/\beta^2 + 2/\beta)/k^2,$$

and

$$|s_k - (4/\beta) \ln(k+1)| = |\widetilde{s}_k| \le \sum_{j=1}^{k} |\widetilde{s}_j - \widetilde{s}_{j-1}| \le \sum_{j=1}^{k} (8/\beta^2 + 2/\beta)/j^2$$

$$\le (8/\beta^2 + 2/\beta)(\pi^2/6).$$

This completes the proof. $\square$

Lemma 11. *Let* $\theta \in (0, \pi]$, $\lambda \in \mathbb{R}$, $\lambda^2 \le \beta/8$, $\epsilon_k = 4/(\beta(k+1)+2)$, $s_k = \sum_{j=0}^{k-1} \epsilon_j$, *then (for* $n, k \in \mathbb{Z}$, $n \ge k \ge 0$*)*

$$|e^{\lambda^2 s_k} a_k(\lambda) - e^{\lambda^2 s_n} a_n(\lambda)| \le C\lambda^2 e^{\lambda^2 s_k}/(\theta(k+1)),$$

here $C > 0$ *is a constant depending only on* β.

Proof. If $n = k$ the result is clearly true, now we assume $n > k \ge 0$. By Lemma 10 we have

$$\sum_{j=k}^{n-1} e^{\lambda^2 s_{j+1}} \epsilon_j^2 \le C \sum_{j=k}^{n-1} (j+2)^{(4/\beta)\lambda^2} (j+1)^{-2} \le C \sum_{j=k}^{n-1} (j+2)^{(4/\beta)\lambda^2 - 2}$$

$$\le C(k+1)^{(4/\beta)\lambda^2 - 1} \le C e^{\lambda^2 s_k} (k+1)^{-1}.$$

Here we used the fact that $\lambda^2 \le \beta/8$, $(4/\beta)\lambda^2 \le 1/2 < 1$, which also implies that $\lambda^2 + \lambda^4 \le C\lambda^2$, $\lambda^2 + \lambda^6 \le C\lambda^2$. By Lemma 9 we have

$$\left| e^{\lambda^2 s_k} a_k(\lambda) - e^{\lambda^2 s_n} a_n(\lambda) \right| \le \frac{C}{\theta}(\lambda^2 + \lambda^4) e^{\lambda^2 s_k} \epsilon_k + \frac{C}{\theta}(\lambda^2 + \lambda^6) \sum_{j=k}^{n-1} e^{\lambda^2 s_{j+1}} \epsilon_j^2$$

$$\leq \frac{C}{\theta} \lambda^2 e^{\lambda^2 s_k} (k+1)^{-1} + \frac{C}{\theta} \lambda^2 e^{\lambda^2 s_k} (k+1)^{-1} \leq \frac{C\lambda^2 e^{\lambda^2 s_k}}{\theta(k+1)}.$$

This completes the proof. □

Lemma 12. *Let* $\theta \in (0, \pi]$, $\lambda \in \mathbb{R}$, *then*

$$|a_k(\lambda) - 1| \leq 4\lambda^2 k|\theta|/\beta.$$

Proof. Let $X_k = \psi_k(\theta) - (k+1)\theta$. By Lemma 3 and Lemma 4 we have $\mathbb{E}[X_k] = 0$, $\mathbb{E}[X_k^2] \leq 8k|\theta|/\beta$, which together with Taylor expansion $|e^{ix} - ix - 1| \leq |x|^2/2$ for $x \in \mathbb{R}$ gives

$$|a_k(\lambda) - 1| = |\mathbb{E}[e^{i\lambda X_k}] - 1| = |\mathbb{E}[e^{i\lambda X_k} - i\lambda X_k - 1]| \leq \mathbb{E}[|\lambda X_k|^2/2] \leq 4\lambda^2 k|\theta|/\beta.$$

This completes the proof. □

3.1. **Proof of Lemma 1.** Now we are ready to prove Lemma 1. The proof relies on Lemma 11 and Lemma 12 with n replaced by $n-1$.

Proof. Let's denote

$$b_k(\lambda) = e^{-(4\lambda^2/\beta)\ln(2+k\theta)}, \quad \epsilon_k = 4/(\beta(k+1)+2), \quad s_k = \sum_{j=0}^{k-1} \epsilon_j$$

for every $k \in \mathbb{Z}$, $k \geq 0$.

If $n\theta \leq 2$, by Lemma 12 we have

$$|a_{n-1}(\lambda) - 1| \leq 4\lambda^2(n-1)|\theta|/\beta \leq 8\lambda^2/\beta.$$

By Taylor expansion we have

$$|b_n(\lambda) - 1| \leq (4\lambda^2/\beta)\ln(2+n\theta) \leq (4\lambda^2/\beta)\ln 4 \leq (1/2)\ln 4$$

and

$$e^{(4\lambda^2/\beta)\ln(2+n\theta)} \leq e^{(1/2)\ln 4} = 2.$$

Thus we have

$$|\mathbb{E}[e^{i\lambda(\psi_{n-1}(\theta)-n\theta)}] - e^{-(4\lambda^2/\beta)\ln(2+n\theta)}| = |a_{n-1}(\lambda) - b_n(\lambda)|$$

$$\leq |a_{n-1}(\lambda) - 1| + |b_n(\lambda) - 1| \leq 8\lambda^2/\beta + (4\lambda^2/\beta)\ln 4$$

$$\leq (8 + 4\ln 4)(\lambda^2/\beta)(2e^{-(4\lambda^2/\beta)\ln(2+n\theta)}) \leq C\lambda^2 e^{-(4\lambda^2/\beta)\ln(2+n\theta)}.$$

If $n\theta \geq 2$, we take $k = \lfloor 1/\theta \rfloor$, then we have $0 \leq k \leq 1/\theta \leq n/2 < n$, thus $k \leq n-1$. By Lemma 12 we have

$$|a_k(\lambda) - 1| \leq 4\lambda^2 k|\theta|/\beta \leq 4\lambda^2/\beta.$$

By Lemma 10 we have

$$|(s_{n-1} - s_k) - (4/\beta)\ln(n/(k+1))| \leq C.$$

We also have $k\theta \leq 1 \leq (k+1)\theta \leq 1 + \theta \leq 1 + \pi$, $0 \leq \ln((k+1)\theta) \leq C$, $0 \leq \ln(2/(n\theta) + 1) \leq \ln 2$, and $|\ln(2+n\theta) - \ln(n/(k+1))| = |\ln(2/(n\theta) + 1) + \ln((k+1)\theta)| \leq C$, thus

$$|(s_{n-1} - s_k) - (4/\beta)\ln(2+n\theta)| \leq C,$$

therefore, we have

$$|e^{\lambda^2(s_k - s_{n-1})} - e^{-(4\lambda^2/\beta)\ln(2+n\theta)}| \leq C\lambda^2 e^{-(4\lambda^2/\beta)\ln(2+n\theta)}.$$

By Lemma 11, we have

$$|e^{\lambda^2 s_k}a_k(\lambda) - e^{\lambda^2 s_{n-1}}a_{n-1}(\lambda)| \leq C\lambda^2 e^{\lambda^2 s_k}/(\theta(k+1)),$$

and thus we have

$$|e^{\lambda^2(s_k-s_{n-1})}a_k(\lambda) - a_{n-1}(\lambda)| \leq C\lambda^2 e^{\lambda^2(s_k-s_{n-1})}/(\theta(k+1))$$

$$\leq C\lambda^2 e^{\lambda^2(s_k-s_{n-1})} \leq C\lambda^2 e^{-(4\lambda^2/\beta)\ln(2+n\theta)}.$$

Now we have (recall $b_n(\lambda) = e^{-(4\lambda^2/\beta)\ln(2+n\theta)}$ and $|a_k(\lambda) - 1| \leq 4\lambda^2/\beta$)

$$|e^{\lambda^2(s_k-s_{n-1})}a_k(\lambda) - b_n(\lambda)| \leq |e^{\lambda^2(s_k-s_{n-1})} - b_n(\lambda)| + |e^{\lambda^2(s_k-s_{n-1})}(a_k(\lambda) - 1)|$$

$$\leq C\lambda^2 b_n(\lambda) + Cb_n(\lambda)|a_k(\lambda) - 1| \leq C\lambda^2 b_n(\lambda)$$

and

$$|e^{\lambda^2(s_k-s_{n-1})}a_k(\lambda) - a_{n-1}(\lambda)| \leq C\lambda^2 b_n(\lambda).$$

Therefore, we have

$$|a_{n-1}(\lambda) - b_n(\lambda)| \leq C\lambda^2 b_n(\lambda).$$

Now the result follows by the definitions of $a_{n-1}(\lambda)$ and $b_n(\lambda)$. □

3.2. **Proof of Corollary 2.** As a consequence of Lemma 1, we now give the proof of Corollary 2.

Proof. Let $X = \psi_{n-1}(\theta) - n\theta$, $Z = \lfloor\frac{\psi_{n-1}(\theta)-\eta}{2\pi}\rfloor + 1$, then $N_n(0,\theta) \overset{\mathrm{d}}{=} Z$. Taking the real part in Lemma 1 we have

$$|\mathbb{E}[\cos(\lambda X)] - e^{-(4\lambda^2/\beta)\ln(2+n\theta)}| \leq C\lambda^2 e^{-(4\lambda^2/\beta)\ln(2+n\theta)} \leq C\lambda^2,$$

$$|\mathbb{E}[(1 - \cos(\lambda X))/\lambda^2] - (1 - e^{-(4\lambda^2/\beta)\ln(2+n\theta)})/\lambda^2| \leq C,$$

for $\lambda \in [-\sqrt{\beta/8}, \sqrt{\beta/8}] \setminus \{0\}$. Letting $\lambda \to 0$ we conclude that

$$|\mathbb{E}[X^2/2] - (4/\beta)\ln(2 + n\theta)| \leq C,$$

which implies (3). Since η is a uniform variable on $[0, 2\pi)$, we have

$$\mathbb{E}[\lfloor x - \eta/(2\pi)\rfloor + 1] = \frac{1}{2\pi}\int_0^{2\pi}(\lfloor x - \eta/(2\pi)\rfloor + 1)d\eta = \int_{x-1}^x(\lfloor y\rfloor + 1)dy$$

$$= \int_{x-1}^{\lfloor x\rfloor}\lfloor x\rfloor dy + \int_{\lfloor x\rfloor}^x(\lfloor x\rfloor + 1)dy = \lfloor x\rfloor(\lfloor x\rfloor - x + 1) + (\lfloor x\rfloor + 1)(x - \lfloor x\rfloor) = x,$$

for $x \in \mathbb{R}$. Since η is independent of $\psi_{n-1}(\theta)$ and $N_n(0,\theta) \overset{\mathrm{d}}{=} Z = \lfloor\frac{\psi_{n-1}(\theta)-\eta}{2\pi}\rfloor + 1$, we have $\mathbb{E}[Z|\psi_{n-1}(\theta)] = \psi_{n-1}(\theta)/(2\pi)$. Let $Z_1 := Z - \psi_{n-1}(\theta)/(2\pi)$ then $|Z_1| \leq 1$, $\mathbb{E}[Z_1|\psi_{n-1}(\theta)] = 0$. For $\lambda \in \mathbb{R}$ we first have

$$|\mathbb{E}[e^{i\lambda(Z-n\theta/(2\pi))}] - \mathbb{E}[e^{i\lambda(\psi_{n-1}(\theta)-n\theta)/(2\pi)}]| = |\mathbb{E}[e^{i\lambda Z} - e^{i\lambda\psi_{n-1}(\theta)/(2\pi)}]|$$

$$\leq \mathbb{E}|\mathbb{E}[e^{i\lambda Z} - e^{i\lambda\psi_{n-1}(\theta)/(2\pi)}|\psi_{n-1}(\theta)]| = \mathbb{E}|(\mathbb{E}[e^{i\lambda Z_1}|\psi_{n-1}(\theta)] - 1)e^{i\lambda\psi_{n-1}(\theta)/(2\pi)}|$$

$$= |\mathbb{E}[e^{i\lambda Z_1}|\psi_{n-1}(\theta)] - 1| = |\mathbb{E}[e^{i\lambda Z_1} - 1 - i\lambda Z_1|\psi_{n-1}(\theta)]|$$

$$\leq \mathbb{E}[(\lambda Z_1)^2/2|\psi_{n-1}(\theta)] \leq \lambda^2/2.$$

On the other hand, for $\lambda \in [-2\pi\sqrt{\beta/8}, 2\pi\sqrt{\beta/8}]$, let $\lambda_1 = \lambda/(2\pi)$ then $\lambda_1^2 \leq \beta/8$, by Lemma 1 we have

$$|\mathbb{E}[e^{i\lambda(\psi_{n-1}(\theta)-n\theta)/(2\pi)}] - e^{-\lambda^2/(\beta\pi^2)\cdot\ln(2+n\theta)}|$$

$$=|\mathbb{E}[e^{i\lambda_1(\psi_{n-1}(\theta)-n\theta)}]-e^{-(4\lambda_1^2/\beta)\cdot\ln(2+n\theta)}|\le C\lambda_1^2 e^{-(4\lambda_1^2/\beta)\cdot\ln(2+n\theta)}\le C\lambda_1^2\le C\lambda^2.$$

Therefore, we have

$$|\mathbb{E}[e^{i\lambda(Z-n\theta/(2\pi))}]-e^{-\lambda^2/(\beta\pi^2)\cdot\ln(2+n\theta)}|\le C\lambda^2,$$

which implies (4). We also have

$$\begin{equation}\tag{19}\mathbb{E}[|N_n(0,\theta)-n\theta/(2\pi)|^2]=\mathbb{E}[|Z-n\theta/(2\pi)|^2]\end{equation}$$
$$=\mathbb{E}[(\psi_{n-1}(\theta)-n\theta)^2/(2\pi)^2]+\mathbb{E}[|Z-\psi_{n-1}(\theta)/(2\pi)|^2]$$

and

$$\begin{equation}\tag{20}0\le\mathbb{E}[|Z-\psi_{n-1}(\theta)/(2\pi)|^2]=\mathbb{E}(|Z_1|^2)\le 1.\end{equation}$$

Using (3), (19) and (20), we conclude (5). □

4. Proof of Theorem 1

In this section, we will finish the proof of Theorem 1.

Let $F(x)$ be the distribution function of a random variable X and let

$$\begin{equation}\tag{21}G(x):=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^x e^{-t^2/2}dt\end{equation}$$

be the Gaussian distribution function. Let's denote

$$M=\sup_{x\in\mathbb{R}}|F(x)-G(x)|,\ \ \delta=M(\pi/2)^{1/2}$$

and let

$$\phi(t):=\mathbb{E}[e^{itX}]=\int_{\mathbb{R}}e^{itx}dF(x),\ \ \psi(t):=\int_{\mathbb{R}}e^{itx}dG(x)=e^{-t^2/2}$$

be the characteristic functions.

For every $T>0$ we have the following bound (see (30) in [1])

$$\begin{equation}\tag{22}A(T\delta)\le\int_0^T(T-t)\frac{|\phi(t)-\psi(t)|}{t}dt\le T\int_0^T\frac{|\phi(t)-\psi(t)|}{t}dt,\end{equation}$$

where

$$A(u)=(2\pi)^{1/2}\cdot u\cdot\left(3\int_0^u\frac{1-\cos x}{x^2}dx-\pi\right).$$

Now we take

$$T=\sqrt{\ln(2+n\theta)},\ \ X=\sqrt{\beta/8}(\psi_{n-1}(\theta)-n\theta)/T,$$

for $\theta\in(0,\pi]$. Let $a_k(\lambda)=\mathbb{E}[e^{i\lambda(\psi_k(\theta)-(k+1)\theta)}]$ as in §3, then we have

$$\phi(t)=\mathbb{E}[e^{itX}]=a_{n-1}(\sqrt{\beta/8}t/T).$$

By Lemma 1 we have (for $\lambda\in\mathbb{R}$, $\lambda^2\le\beta/8$)

$$|a_{n-1}(\lambda)-e^{-(4\lambda^2/\beta)\ln(2+n\theta)}|\le C\lambda^2 e^{-(4\lambda^2/\beta)\ln(2+n\theta)}.$$

Notice that if $t\in[0,T]$, $\lambda=\sqrt{\beta/8}t/T$, then $\lambda^2\le\beta/8$, $(4\lambda^2/\beta)\ln(2+n\theta)=t^2/2$. Thus we have

$$|\phi(t)-\psi(t)|=|a_{n-1}(\sqrt{\beta/8}t/T)-e^{-t^2/2}|\le C\beta t^2/(8T^2)\cdot e^{-t^2/2}$$

and

$$T \int_0^T \frac{|\phi(t) - \psi(t)|}{t} dt \le CT \int_0^T \beta t^2/(8T^2) \cdot e^{-t^2/2} dt \le C/T \le C.$$

By (22) we have $A(T\delta) \le C$. As $\lim_{u \to +\infty} A(u) = +\infty$, we have $T\delta \le C$. Recall that $\delta = M(\pi/2)^{1/2}$, we have $\delta \le C/T$, $M \le C/T$. Recall that $T = \sqrt{\ln(2 + n\theta)}$, $M = \sup_{x \in \mathbb{R}} |F(x) - G(x)|$, $F(x) = \mathbb{P}[X \le x]$, $X = \sqrt{\beta/8}(\psi_{n-1}(\theta) - n\theta)/T$, now we have proven the following result.

Lemma 13. *Let $\theta \in (0, \pi]$, $n > 0$, $n \in \mathbb{Z}$, then*

$$\sup_{x \in \mathbb{R}} |\mathbb{P}[\sqrt{\beta/(8 \ln(2 + n\theta))}(\psi_{n-1}(\theta) - n\theta) \le x] - G(x)| \le C(\ln(2 + n\theta))^{-1/2}.$$

Here, $C > 0$ is a constant depending only on β.

4.1. **Proof of Theorem 1.** Now we give the proof of Theorem 1.

Proof. Since $N_n(0, \theta) \overset{\mathrm{d}}{=} Z$ for $Z = \lfloor \frac{\psi_{n-1}(\theta) - \eta}{2\pi} \rfloor + 1$ and $\eta \in [0, 2\pi)$, we have $|Z - \frac{\psi_{n-1}(\theta)}{2\pi}| \le 1$. Let $T_1 = \sqrt{\frac{\pi^2 \beta}{2 \ln(2 + n\theta)}}$, then we have $T_1/(2\pi) = \sqrt{\beta/(8 \ln(2 + n\theta))}$. Thus for $x \in \mathbb{R}$, by Lemma 13 we have

$$\mathbb{P}[T_1(N_n(0, \theta) - n\theta/(2\pi)) \le x] = \mathbb{P}[T_1(Z - n\theta/(2\pi)) \le x]$$
$$\le \mathbb{P}[T_1(\psi_{n-1}(\theta)/(2\pi) - 1 - n\theta/(2\pi)) \le x]$$
$$= \mathbb{P}[T_1/(2\pi) \cdot (\psi_{n-1}(\theta) - n\theta) \le x + T_1] \le G(x + T_1) + C(\ln(2 + n\theta))^{-1/2}$$
$$\le G(x) + T_1/\sqrt{2\pi} + C(\ln(2 + n\theta))^{-1/2} \le G(x) + C(\ln(2 + n\theta))^{-1/2},$$

here we used the fact that $0 \le G'(x) = e^{-x^2/2}/\sqrt{2\pi}$ for $x \in \mathbb{R}$ which implies that $|G(x) - G(y)| \le |x - y|/\sqrt{2\pi}$ for $x, y \in \mathbb{R}$. Similarly, we have

$$\mathbb{P}[T_1(N_n(0, \theta) - n\theta/(2\pi)) \le x] \ge \mathbb{P}[T_1/(2\pi) \cdot (\psi_{n-1}(\theta) - n\theta) \le x - T_1]$$
$$\ge G(x - T_1) - C(\ln(2 + n\theta))^{-1/2} \ge G(x) - C(\ln(2 + n\theta))^{-1/2}.$$

Combining the upper and lower bounds we conclude that

$$\sup_{x \in \mathbb{R}} |\mathbb{P}[T_1(N_n(0, \theta) - n\theta/(2\pi)) \le x] - G(x)| \le C(\ln(2 + n\theta))^{-1/2}.$$

This completes the proof of Theorem 1 by the definitions of T_1 and $G(x)$.   □

5. Results for Sine$_\beta$ process

Now we give the proof of Corollary 3.

Proof. Since the Sine$_\beta$ point process is the scaling limit of the CβE, by Skorokhod's representation theorem, one can construct point processes L_n and L such that the point measure corresponding to L_n converges locally weakly to the measure corresponding to L almost surely [20], and

$$\mathrm{Card}(L_n \cap [0, x]) \overset{\mathrm{d}}{=} N_n(0, x/n), \ 0 < x < 2\pi n.$$

Let $x > 0$, $\lambda \in [-2\pi\sqrt{\beta/8}, 2\pi\sqrt{\beta/8}]$. Since L almost surely does not contain the points 0 and x, we have almost surely

$$\mathrm{Card}(L_n \cap [0, x]) \to \mathrm{Card}(L \cap [0, x]),$$

and
$$e^{i\lambda(\mathrm{Card}(L_n\cap[0,x])-x/(2\pi))} \to e^{i\lambda(\mathrm{Card}(L\cap[0,x])-x/(2\pi))}.$$

By dominated convergence theorem we have
$$\mathbb{E}[e^{i\lambda(\mathrm{Card}(L_n\cap[0,x])-x/(2\pi))}] \to \mathbb{E}[e^{i\lambda(\mathrm{Card}(L\cap[0,x])-x/(2\pi))}].$$

For $n > x/\pi$ we have $\pi n > x$, and by (4) in Corollary 2 we have
$$|\mathbb{E}[e^{i\lambda(\mathrm{Card}(L_n\cap[0,x])-x/(2\pi))}] - e^{-\lambda^2/(\beta\pi^2)\cdot\ln(2+x)}|$$
$$=|\mathbb{E}[e^{i\lambda(N_n(0,x/n)-x/(2\pi))}] - e^{-\lambda^2/(\beta\pi^2)\cdot\ln(2+x)}| \le C\lambda^2,$$

which implies
$$|\mathbb{E}[e^{i\lambda(\mathrm{Card}(L\cap[0,x])-x/(2\pi))}] - e^{-\lambda^2/(\beta\pi^2)\cdot\ln(2+x)}| \le C\lambda^2.$$

Taking the real part we have
$$|\mathbb{E}[(1-\cos(\lambda(\mathrm{Card}(L\cap[0,x])-x/(2\pi))))/\lambda^2] - (1 - e^{-\lambda^2/(\beta\pi^2)\cdot\ln(2+x)})/\lambda^2| \le C,$$

for $\lambda \in [-2\pi\sqrt{\beta/8}, 2\pi\sqrt{\beta/8}] \setminus \{0\}$. Letting $\lambda \to 0$ we conclude that
$$|\mathbb{E}[(\mathrm{Card}(L\cap[0,x])-x/(2\pi))^2/2] - 1/(\beta\pi^2)\cdot\ln(2+x)| \le C,$$

which implies (6).

Now let $x > 0$, $y \in \mathbb{R}$, $X_n = \mathrm{Card}(L_n \cap [0,x])$, $X = \mathrm{Card}(L \cap [0,x])$, $T_1 = \sqrt{\frac{\pi^2\beta}{2\ln(2+x)}}$, then we have $X_n \to X$ almost surely. For $n > x/\pi$ we have $\pi n > x$, and $X_n \overset{\mathrm{d}}{=} N_n(0, x/n)$, by Theorem 1, we have
$$|\mathbb{P}[T_1(X_n - x/(2\pi)) \le y] - G(y)| \le C(\ln(2+x))^{-1/2},$$

where the function G is defined in (21). For every $a > 0$ we have
$$\mathbb{P}[T_1(X - x/(2\pi)) \le y] \le \liminf_{n\to+\infty} \mathbb{P}[T_1(X_n - x/(2\pi)) \le y + a]$$
$$\le G(y+a) + C(\ln(2+x))^{-1/2}.$$

Since G is continuous we have
$$\mathbb{P}[T_1(X - x/(2\pi)) \le y] \le G(y) + C(\ln(2+x))^{-1/2}.$$

Similarly, we have
$$\mathbb{P}[T_1(X - x/(2\pi)) \le y] \ge G(y) - C(\ln(2+x))^{-1/2}.$$

Combining the upper and lower bounds we conclude that
$$\sup_{y\in\mathbb{R}} |\mathbb{P}[T_1(X - x/(2\pi)) \le y] - G(y)| \le C(\ln(2+x))^{-1/2},$$

which gives (7) by the definitions of T_1, X and $G(y)$. This completes the proof. $\square$

6. Application: normality of linear statistics

In this section, we will prove Theorem 2.

6.1. **Variance bound.** We first need the following estimate on the variance of the linear statistics. We write $\|g\|_{L^p} = \|g\|_{L^p(0,2\pi)}$.

Lemma 14. *Let $f \in W^{1,p}(S^1)$ be real valued and $\int_0^{2\pi} f(x)dx = 0$, then*

$$\mathbb{E}[\langle \xi_n, f \rangle] = 0, \ \mathbb{E}|\langle \xi_n, f \rangle|^2 \le C\|f'\|_{L^p}^2,$$

here $p \in (1, +\infty)$, and $C > 0$ is a constant depending only on β, p.

To prove Lemma 14, we first need the following lemma which is the consequence of the uniform variance bound (5) in Corollary 2.

Let $\widetilde{N}_n(a,b) = N_n(a,b) - n(b-a)/(2\pi)$, $\widetilde{N}_n(b,a) = -\widetilde{N}_n(a,b)$ for $0 \le a \le b < 2\pi$. As $N_n(a,b) = N_n(0,b) - N_n(0,a)$, for $0 \le a \le b < 2\pi$, we have $\widetilde{N}_n(a,b) = \widetilde{N}_n(0,b) - \widetilde{N}_n(0,a)$ for $a, b \in [0, 2\pi)$.

Lemma 15. *For $a, b \in [0, 2\pi)$, $a \ne b$ we have*

$$|\mathbb{E}[\widetilde{N}_n(a,b)^2] - 2\ln n/(\pi^2 \beta)| \le C(1 - \ln \sin(|a-b|/2)),$$

here $C > 0$ is a constant depending only on β.

Proof. By symmetry we only need to consider the case $0 \le a < b < 2\pi$. For $x \in (0, \pi]$, by (5) in Corollary 2 we have

$$|\mathbb{E}[\widetilde{N}_n(0,x)^2] - 2\ln(2+nx)/(\pi^2 \beta)| \le C.$$

Thus we have

$$\mathbb{E}[\widetilde{N}_n(0,x)^2] \le 2\ln(2+nx)/(\pi^2 \beta) + C \le 2\ln(2+n\pi)/(\pi^2 \beta) + C$$
$$\le 2\ln(6n)/(\pi^2 \beta) + C \le 2\ln n/(\pi^2 \beta) + C$$

and

$$\mathbb{E}[\widetilde{N}_n(0,x)^2] \ge 2\ln(2+nx)/(\pi^2 \beta) - C \ge 2\ln(nx)/(\pi^2 \beta) - C$$
$$\ge 2\ln n/(\pi^2 \beta) + 2\ln(2\sin(x/2))/(\pi^2 \beta) - C$$
$$\ge 2\ln n/(\pi^2 \beta) - C(1 - \ln \sin(x/2)),$$

here we used the fact that $\sin(x/2) \le 1$, $2\sin(x/2) \le x$, $\ln \sin(x/2) \le 0$. Combining the upper and lower bounds we conclude that

$$(23) \qquad\qquad |\mathbb{E}[\widetilde{N}_n(0,x)^2] - 2\ln n/(\pi^2 \beta)| \le C(1 - \ln \sin(x/2))$$

for $x \in (0, \pi]$. If $x \in [\pi, 2\pi)$, by rotational invariance we have $\widetilde{N}_n(0,x) = -\widetilde{N}_n(x, 2\pi) \overset{d}{=} -\widetilde{N}_n(0, 2\pi - x)$, $2\pi - x \in (0, \pi]$ and

$$|\mathbb{E}[\widetilde{N}_n(0,x)^2] - 2\ln n/(\pi^2 \beta)| = |\mathbb{E}[\widetilde{N}_n(0, 2\pi - x)^2] - 2\ln n/(\pi^2 \beta)|$$
$$\le C(1 - \ln \sin((2\pi - x)/2)) = C(1 - \ln \sin(x/2)).$$

Thus (23) is true for $x \in (0, 2\pi)$. Now for $0 \le a < b < 2\pi$, by rotational invariance we have $\widetilde{N}_n(a,b) \overset{d}{=} \widetilde{N}_n(0, b-a)$, and by (23) we have

$$|\mathbb{E}[\widetilde{N}_n(a,b)^2] - 2\ln n/(\pi^2 \beta)| = |\mathbb{E}[\widetilde{N}_n(0, b-a)^2] - 2\ln n/(\pi^2 \beta)|$$
$$\le C(1 - \ln \sin((b-a)/2)).$$

This completes the proof. $\square$

Now we give the proof of Lemma 14.

Proof. By definition and $\int_0^{2\pi} f(x)dx = 0$ and integration by parts we have

$$\langle \xi_n, f \rangle = \int_0^{2\pi} f(x)dN_n(0,x) = \int_0^{2\pi} f(x)d(N_n(0,x) - nx/(2\pi))$$

$$= -\int_0^{2\pi} f'(x)(N_n(0,x) - nx/(2\pi))dx.$$

By rotational invariance we have $\mathbb{E}[N_n(0,x)] = nx/(2\pi)$, which implies $\mathbb{E}[\langle \xi_n, f \rangle] = 0$. By the definition of $\widetilde{N}_n(a,b)$ we have

$$|\langle \xi_n, f \rangle|^2 = \int_0^{2\pi} \int_0^{2\pi} f'(x)f'(y)\widetilde{N}_n(0,x)\widetilde{N}_n(0,y)dxdy$$

$$= -\frac{1}{2} \int_0^{2\pi} \int_0^{2\pi} f'(x)f'(y)(\widetilde{N}_n(0,x) - \widetilde{N}_n(0,y))^2 dxdy$$

$$+ \int_0^{2\pi} \int_0^{2\pi} f'(x)f'(y)\widetilde{N}_n(0,x)^2 dxdy$$

$$= -\frac{1}{2} \int_0^{2\pi} \int_0^{2\pi} f'(x)f'(y)\widetilde{N}_n(y,x)^2 dxdy,$$

here we used the fact that $\int_0^{2\pi} f'(y)dy = f(2\pi) - f(0) = 0$, which also implies that

$$\mathbb{E}|\langle \xi_n, f \rangle|^2 = -\frac{1}{2} \int_0^{2\pi} \int_0^{2\pi} f'(x)f'(y)\mathbb{E}[\widetilde{N}_n(y,x)^2]dxdy$$

$$= -\frac{1}{2} \int_0^{2\pi} \int_0^{2\pi} f'(x)f'(y)(\mathbb{E}[\widetilde{N}_n(y,x)^2] - 2\ln n/(\pi^2 \beta))dxdy.$$

By Lemma 15 we have

$$\mathbb{E}|\langle \xi_n, f \rangle|^2 \leq \frac{1}{2} \int_0^{2\pi} \int_0^{2\pi} |f'(x)f'(y)||\mathbb{E}[\widetilde{N}_n(y,x)^2] - 2\ln n/(\pi^2 \beta)|dxdy$$

$$\leq C \int_0^{2\pi} \int_0^{2\pi} |f'(x)f'(y)|(1 - \ln\sin(|x-y|/2))dxdy.$$

Notice that

$$\int_0^{2\pi} (1 - \ln\sin(|x-y|/2))^{p'} dy = \int_0^{2\pi} (1 - \ln\sin(y/2))^{p'} dy = C_p < +\infty,$$

for $x \in [0, 2\pi]$, where $p' = p/(p-1)$ and C_p is a constant depending only on p. By Hölder's inequality we have

$$\int_0^{2\pi} |f'(y)|(1 - \ln\sin(|x-y|/2))dy \leq \|f'\|_{L^p} C_p^{1-1/p}$$

for $x \in [0, 2\pi]$, and

$$\mathbb{E}|\langle \xi_n, f \rangle|^2 \leq C\|f'\|_{L^p} C_p^{1-1/p} \int_0^{2\pi} |f'(x)|dx \leq C\|f'\|_{L^p} C_p^{1-1/p}\|f'\|_{L^p}.$$

This completes the proof. $\qquad\square$

6.2. **Proof of Theorem 2.** Now we are ready to prove Theorem 2. The proof is based on the following result of Jiang-Matsumoto for the case $f(x)$ a finite sum of $\{e^{ikx}\}_{k \in \mathbb{Z}}$ (see Corollary 3 in [9]).

Lemma 16. *Let* $(\theta_1, \cdots, \theta_n)$ *be a sample from* $\mu_{\beta,n}$. *Let* $g(z) = \sum_{k=0}^{m} c_k z^k$ *with fixed* m *and* $c_k \in \mathbb{C}$ *for all* k. *Set* $X_n = \sum_{j=1}^{n} g(e^{i\theta_j})$. *then* $X_n - \mu_n$ *converges in law to a complex Gaussian random variable* $\sim \mathbb{C}N(0, \sigma^2)$, *where*

$$\mu_n = nc_0, \ \sigma^2 = \frac{2}{\beta} \sum_{j=1}^{+\infty} j|c_j|^2.$$

Lemma 16 tells us that if $c_0 = 0$, $f(x) = g(e^{ix}) + \overline{g(e^{ix})}$ then $X_n + \overline{X_n} = \langle \xi_n, f \rangle$ converges in law to a real Gaussian random variable $\sim N(0, 2\sigma^2)$.

Now we give the proof of Theorem 2.

Proof. It is enough to prove of the convergence of the characteristic functions

(24) $$\lim_{n \to +\infty} \mathbb{E}[e^{i\lambda\langle \xi_n, f \rangle}] = e^{-\lambda^2 \sigma^2}, \ \forall \ \lambda \in \mathbb{R}.$$

Given a function $f \in W^{1,p}(S^1)$, we will prove that $f_N := f * K_N$ approximates f in $W^{1,p}(S^1)$, where $f_1 * f_2(x) := \int_0^{2\pi} f_1(y) f_2(x-y) dy$, and $K_N(x)$ is the Féjer kernel

$$K_N(x) = \frac{1}{2\pi} \sum_{j=-N}^{N} \left(1 - \frac{|j|}{N}\right) e^{ijx} = \frac{N}{2\pi} \left(\frac{\sin(Nx/2)}{N\sin(x/2)}\right)^2, \ N > 0, \ N \in \mathbb{Z}.$$

In fact $K_N(x) \geq 0$, $\|K_N\|_{L^1} = 1$, $f_N' = f' * K_N$, and

$$\lim_{N \to +\infty} \|K_N\|_{L^1(\delta, 2\pi - \delta)} = 0, \ \forall \ \delta \in (0, \pi).$$

The following results are classical

$$\|g * K_N\|_{L^p} \leq \|g\|_{L^p}, \ \lim_{N \to +\infty} \|g * K_N - g\|_{L^p} = 0, \ \forall \ g \in L^p(0, 2\pi).$$

Thus we have

(25) $$\|f_N'\|_{L^p} \leq \|f'\|_{L^p}, \ \lim_{N \to +\infty} \|f_N' - f'\|_{L^p} = 0.$$

We also have

$$f_N(x) = \sum_{j=-N}^{N} \left(1 - \frac{|j|}{N}\right) a_j e^{ijx},$$

where a_j is defined in Theorem 2. Since f is real valued and $\int_0^{2\pi} f(x) dx = 0$, we have $a_0 = 0$, $a_{-j} = \overline{a_j}$, $f_N(x) = g_N(e^{ix}) + \overline{g_N(e^{ix})}$ with $g_N(z) = \sum_{j=1}^{N} (1 - j/N) a_j z^j$. By Lemma 16, $\langle \xi_n, f_N \rangle$ converges in law to $J_N \sim N(0, 2\sigma_N^2)$ as $n \to +\infty$ for every fixed N, where

$$\sigma_N^2 = \frac{2}{\beta} \sum_{j=1}^{N} j (1 - j/N)^2 |a_j|^2$$

with $\sigma_{N+1} \geq \sigma_N$. Thus

(26) $$\lim_{n \to +\infty} \mathbb{E}[e^{i\lambda\langle \xi_n, f_N \rangle}] = e^{-\lambda^2 \sigma_N^2}, \ \forall \ \lambda \in \mathbb{R}.$$

As $a_0 = 0$, $\int_0^{2\pi} f_N(x)dx = 0$, by Lemma 14 and Fatou's Lemma we have

$$2\sigma_N^2 = \mathbb{E}[J_N^2] \leq \liminf_{n \to +\infty} \mathbb{E}[|\langle \xi_n, f_N \rangle|^2] \leq C\|f_N'\|_{L^p}^2,$$

which implies

$$\sigma_N^2 \leq C\|f_N'\|_{L^p}^2 \leq C\|f'\|_{L^p}^2.$$

Thus by monotone convergence theorem we have

$$(27) \qquad\qquad \sigma^2 = \lim_{N \to +\infty} \sigma_N^2 \leq C\|f'\|_{L^p}^2 < +\infty,$$

where σ is defined in Theorem 2. By Lemma 14 again we have

$$\mathbb{E}[|\langle \xi_n, f_N \rangle - \langle \xi_n, f \rangle|^2] \leq C\|f_N' - f'\|_{L^p}^2,$$

and thus

$$|\mathbb{E}[e^{i\lambda\langle \xi_n, f_N \rangle} - e^{i\lambda\langle \xi_n, f \rangle}]| \leq \mathbb{E}[|\lambda||\langle \xi_n, f_N \rangle - \langle \xi_n, f \rangle|] \leq C|\lambda|\|f_N' - f'\|_{L^p}, \ \forall \ \lambda \in \mathbb{R},$$

which together with (26) gives

$$(28) \qquad \limsup_{n \to +\infty} |\mathbb{E}[e^{i\lambda\langle \xi_n, f \rangle}] - e^{-\lambda^2\sigma^2}| \leq C|\lambda|\|f_N' - f'\|_{L^p} + |e^{-\lambda^2\sigma_N^2} - e^{-\lambda^2\sigma^2}|$$

for every $\lambda \in \mathbb{R}$, $N > 0$, $N \in \mathbb{Z}$. By (25), (27), (28) we have

$$\limsup_{n \to +\infty} |\mathbb{E}[e^{i\lambda\langle \xi_n, f \rangle}] - e^{-\lambda^2\sigma^2}|$$

$$\leq \limsup_{N \to +\infty} (C|\lambda|\|f_N' - f'\|_{L^p} + |e^{-\lambda^2\sigma_N^2} - e^{-\lambda^2\sigma^2}|) = 0, \ \forall \ \lambda \in \mathbb{R},$$

which implies (24). This completes the proof. $\qquad\qquad\qquad\qquad\qquad$ $\square$

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