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A Jacobian mate defines the Jacobian pair.

Leonid Makar-Limanov *

To the memory of Ernest Borisovich Vinberg

Abstract

A polynomial $f \in \mathbb{C}[x, y]$ is a Jacobian mate if the Jacobian $\mathrm{J}(f, g) = 1$ for some $g \in \mathbb{C}[x, y]$. It is not known that then $\mathbb{C}[f, g] = \mathbb{C}[x, y]$ and a conjecture that this is the case is the Jacobian conjecture. In this note we will check that if f is given then the subalgebra $\mathbb{C}[f, g]$ is known and that g can be recovered up to a summand which is a polynomial in f.

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Introduction.

Assume that $f \in \mathbb{C}[x, y]$ (where $\mathbb{C}$ is the field of complex numbers) satisfies $\mathrm{J}(f, g) = \frac{\partial f}{\partial x}\frac{\partial g}{\partial y} - \frac{\partial f}{\partial y}\frac{\partial g}{\partial x} = 1$ for some $g \in \mathbb{C}[x, y]$. This polynomial

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can be the image of x under an automorphism ϕ and it is well known that then $\mathrm{J}(\phi^{-1}(f), \phi^{-1}(g)) = c \in \mathbb{C}^*$, a non-zero complex number. Then from the properties of Jacobian it follows that $\phi^{-1}(g) = cy + p(x)$ and therefore $\mathbb{C}[f, g] = \mathbb{C}[x, y]$.

The JC (Jacobian conjecture) states that $\mathrm{J}(f, g) = 1$ implies $\mathbb{C}[f, g] = \mathbb{C}[x, y]$ (see [K]). This conjecture occasionally becomes a theorem even for many years but today it is a problem. So it is possible that f is not an image of x under an automorphism and then $\mathbb{C}[f, g] \neq \mathbb{C}[x, y]$.

Recall that if $p \in \mathbb{C}[x, y]$ is a polynomial in 2 variables and each monomial of p is represented by a lattice point on the plane with the coordinate vector equal to the degree vector of this monomial then the convex hull $\mathcal{N}(p)$ of the points so obtained is called the Newton polygon.

It is known for many years that for a potential counterexample to JC there exists an automorphism ξ of $\mathbb{C}[x, y]$ such that the Newton polygon $\mathcal{N}(\xi(f))$ of $\xi(f)$ contains a vertex $v = (m, n)$ where $n > m > 0$ and is included in a trapezoid with the vertex v, edges parallel to the y axis and to the bisectrix of the first quadrant adjacent to v, and two edges belonging to the coordinate axes (see [A1], [A2], [AO], [GGV], [H], [J], [L], [MW],[M], [Na1], [Na2], [NN1], [NN2], [Ok]). This was improved with a completely new approach by Pierrette Cassou-Noguès who showed that $\mathcal{N}(f)$ does not have an edge parallel to the bisectrix (see [CN] and [ML]). From now on we will assume that f is "shaped" like this, i.e. that $\mathcal{N}(f)$ is a trapezoid described above.

Our goal is to recover g.

Expansion of g.

Consider the ring $L = \mathbb{C}[x^{-1}, x]$ of Laurent polynomials in x. Define A to be the algebra of asymptotic power series in y with coefficients in L, i.e. the elements of A are $\sum_{-\infty}^{i=k} y_i y^i$ where $y_i \in L,\ y_k \neq 0$. For $a = \sum_{-\infty}^{i=k} y_i y^i$ define the leading form of a as $|a| = y_k y^k$.

The polynomial $f(x, y)$ and a polynomial g which we are trying to find are elements of A and $|f| = cx^m y^n$. If $|g| = g_N y^N$ then since $\mathrm{J}(f, g) = 1$ and $n > 1$ we must have $\mathrm{J}(x^m y^n, g_N y^N) = 0$. (If $\mathrm{J}(x^m y^n, g_N y^N) = (mNx^{m-1}g_N - nx^m g'_N)y^{n+N-1} \neq 0$ then $|\mathrm{J}(f, g)| = cc_1(mNx^{m-1}g_N - nx^m g'_N)y^{n+N-1}$ and $\mathrm{J}(f, g) \neq 1$.) Hence $g_N = c_1 x^M$ and $|g| = c_1 x^M y^N$ where $Mn = mN$, i.e. $x^M y^N = (x^m y^n)^{\lambda_0}$ where $\lambda_0 \in \mathbb{Q}$.

Lemma on radical. If $r \in \mathbb{Q}$ is a rational number, $|a| = cx^l y^k,\ c \in \mathbb{C}$, and $|a|^r \in A$ then $a^r \in A$.

Proof. From the Newton binomial theorem $a^r = |a|^r \sum_{j=0}^{\infty} \binom{r}{j} (\sum_{-\infty}^{i=k-1} \frac{y_i}{y_k} y^{i-k})^j$ because $a = |a|(1 + \sum_{-\infty}^{i=k-1} \frac{y_i}{y_k} y^{i-k})$. Since all $\frac{y_i}{y_k} \in L$, element $a^r \in A$. $\square$

Since we can divide f by c, we may assume without loss of generality that $|f| = x^m y^n$. Not to be confused with rational powers of complex numbers let us agree that any rational power of f which belongs to A has as the leading form a monomial with coefficient 1.

By lemma on radical $f^{\lambda_0} \in A$ and hence $g_1 = g - c_1 f^{\lambda_0} \in A$. Since $\mathrm{J}(f, g_1) = 1$ either $\mathrm{J}(|f|, |g_1|) = 0$ or $\mathrm{J}(|f|, |g_1|) = 1$. If $\mathrm{J}(|f|, |g_1|) = 0$ then $|g_1| = c_1 |f|^{\lambda_1},\ c_1 \in \mathbb{C},\ \lambda_1 \in \mathbb{Q}$ and we can define $g_2 = g - c_0 f^{\lambda_0} - c_1 f^{\lambda_1}$ which is in A for the same reasons as g_1. We can proceed until we obtain

$g_\kappa = g - \sum_{i=0}^{\kappa-1} c_i f^{\lambda_i} \in A$ for which $\mathrm{J}(|f|, |g_\kappa|) = 1$, i.e. $\mathrm{J}(x^m y^n, |g_\kappa|) = 1$. Therefore $|g_\kappa| = (c_\kappa (x^m y^n)^{\frac{1-n}{n}} - \frac{1}{n-m} x^{1-m} y^{1-n})$ where $c_\kappa \in \mathbb{C}$. If $c_\kappa \neq 0$ then $(x^m y^n)^{\frac{1-n}{n}} \in A$ and $\frac{m}{n} \in \mathbb{Z}$ which is impossible since $0 < m < n$. Thus $|g_\kappa| = \frac{1}{(m-n)} x^{1-m} y^{1-n}$ and

$$g = \sum_{i=0}^{\kappa-1} c_i f^{\lambda_i} + g_\kappa, \ \ c_i \in \mathbb{C} \tag{1}$$

where $\deg_y(|f^{\lambda_i}|) > 1-n$, $\deg_y(|g_\kappa|) = 1-n$, and $|g_\kappa| = \frac{1}{(m-n)} x^{1-m} y^{1-n} = \frac{1}{(m-n)} x^{\frac{n-m}{n}} |f|^{\lambda_\kappa}$ where $\lambda_\kappa = \frac{1-n}{n}$.

We will find g if we find c_i and λ_i for $0 \leq i < \kappa$ since g is a polynomial.

Reductions of f.

Newton introduced the polygon which we call the Newton polygon in order to find a solution y of $p(x, y) = 0$ in terms of x for a polynomial $p(x, y) = \sum_{(i,j) \in \mathcal{N}(p)} p_{ij} x^i y^j$ (see [N]). Here is the process of obtaining such a solution. Consider an edge e of $\mathcal{N}(p)$ which is not parallel to the x axis. Denote by $p(e) = \sum_{(i,j) \in e} p_{ij} x^i y^j$. The form $p(e)$ allows to determine the first summand of the solution as follows. Consider an equation $p(e) = 0$. Since $p(e)$ is a homogeneous form relative to a *weight* given by $w(x) = \alpha (\neq 0)$, $w(y) = \beta$, $w(x^i y^j) = i\alpha + j\beta$, solutions of this equation are $y = c_i x^{\frac{\beta}{\alpha}}$ and $c_i \in \mathbb{C}$. Choose any solution $c_i x^{\frac{\beta}{\alpha}}$ and replace $p(x, y)$ by $p_1(x, y) = p(x, c_i x^{\frac{\beta}{\alpha}} + y)$. Though p_1 is not necessarily a polynomial in x we can define the Newton polygon of p_1 in the same way as it was done for the polynomials; the only difference is that p_1 may contain monomials $x^\mu y^\nu$ where $\mu \in \mathbb{Q}$ rather than in $\mathbb{Z}$. The polygon $\mathcal{N}(p_1)$ contains the *degree* vertex v of e, i.e.

the vertex with y coordinate equal to $\deg_y(p(e))$ and an edge e' which is a modification of e (e' may collapse to v). Take the *order* vertex v_1 of e', i.e. the vertex with y coordinate equal to the order of $p_1(e')$ as a polynomial in y (if $e' = v$ take $v_1 = v$). Use the edge e_1 for which v_1 is the degree vertex to determine the next summand and so on. After possibly a countable number of steps we obtain a vertex v_μ and the edge e_μ for which v_μ is not the degree vertex, i.e. either e_μ is horizontal or the degree vertex of e_μ has a larger y coordinate than the y coordinate of v_μ. It is possible only if $\mathcal{N}(p_\mu)$ does not have any vertices on the x axis. Therefore $p_\mu(x, 0) = 0$ and a solution is obtained.

When characteristic is zero the process of constructing a solution is more straightforward then it may seem from this description. The denominators of fractional powers of x (if denominators and numerators of these rational numbers are assumed to be relatively prime) do not exceed $\deg_y(p)$. Indeed, for any initial weight there are at most $\deg_y(p)$ solutions while a summand $cx^{\frac{M}{N}}$ can be replaced by $c\varepsilon^M x^{\frac{M}{N}}$ where $\varepsilon^N = 1$ which gives at least N different solutions.

Let $y_k \in \overline{\mathbb{C}(x)}$ be one of the solutions of $f(x, y) = 0$ by the *decreasing* powers of x. Call $f_k(x, y) = f(x, y + y_k) \in \overline{\mathbb{C}(x)}[y]$ a *reduction* of f.

If g exists then $\mathrm{J}(f_k, g_k) = 1$ for $g_k(x, y) = g(x, y+y_k)$. Since $f_k(x, 0) = 0$, we have $1 = -\frac{\partial g_k(x,0)}{\partial x} f_{k,1}$ where $f_{k,1} = \frac{\partial f_k(x,y)}{\partial y}|_{y=0}$. Because of that the lowest vertex of $\mathcal{N}(f_k)$ is $(\mu(k), 1)$ where $\mu(k) \neq 1$ and the lowest vertex of $\mathcal{N}(g_k)$ is either $(1 - \mu(k), 0)$ or $(0, 0)$. Call $(\mu(k), 1)$ the *principal vertex* and the slanted (non-horizontal) edge of $\mathcal{N}(f_k)$ containing $(\mu(k), 1)$ the *principal edge* of f_k.

The polygon $\mathcal{N}(f_k)$ is not compact, it has an infinite edge $[(-\infty, 1), (\mu(k), 1)]$.

There is also an infinite horizontal edge through the *leading* vertex, i.e. the vertex (m,n) of $\mathcal{N}(f_k)$, which contains only this vertex.

Call a slanted edge e of a Newton polygon *semi-positive, zero, semi-negative* if its extension intersects the x axis in a point with a positive, zero, negative abscissa.

Call e positive if it is semi-positive and the line parallel to e and containing point $(1,1)$ intersects the x axis in a point with a positive abscissa.

Call a positive edge of $\mathcal{N}(f_k)$ which contains the leading vertex the *leading edge*. (This is the right edge containing the leading vertex.)

If the principal edge of a reduction is positive call this reduction positive. It is easy to see that for a positive reduction the edges in the chain of slanted edges starting with the leading edge and ending with the principal edge are positive.

In order to say more about reductions we use some extensions of $\mathbb{C}[x,y]$. Take a weight on $\mathbb{C}[x,y]$ given by $w(x)=1,\ w(y)=-\alpha$ where $\alpha\in\mathbb{R}$. Observe that $\alpha\geq 0$ for the edges of $\mathcal{N}(f_k)$.

With the help of this weight we can define an extension A_w of $\mathbb{C}[x,y]$: A_w consists of fractional asymptotic power series $\sum_{i=k}^{\infty}c_i(z)x^{\frac{-i}{N}}$ where $z=x^{\alpha}y,\ k\in\mathbb{Z},\ c_i(z)\in\mathbb{C}(z),\ N\in\mathbb{Z}^+$ and depends on the element. For any $a=\sum_{i=k}^{\infty}c_i(z)x^{\frac{-i}{N}}\in A_w$ the leading form $|a|$ is defined as $|a|=c_k(z)x^{\frac{-k}{N}}$ (assuming $c_k\neq 0$). The Newton binomial formula $(1+\delta)^{\lambda}=\sum_{j=0}^{\infty}\binom{\lambda}{j}\delta^j$ insures that if $|a|^{\lambda}\in A_w$ then $a^{\lambda}\in A_w$ (see Lemma on radical).

Take an edge e of $\mathcal{N}(f_k)$. Denote by w_e the weight given by $w_e(x)=1,\ w_e(y)=-\alpha$ for which all points on e have the same weigh. Denote by $f_k(e)$ the leading form of f_k relative to this weigh and assume that $\rho=w_e(f_k)\neq 0$. Then $f_k(e)=x^{\rho}p(z)$ where $z=x^{\alpha}y,\ p(z)\in\mathbb{C}[z],\ \rho\in\mathbb{Q}^*$.

We can present g_k as $g_k = \sum_{i=0}^{s-1} c_i f^{\lambda_i} + g_{k,s}$ where $c_i \in \mathbb{C},\ \lambda_i \in \mathbb{Q},\ \lambda_i > \lambda_{i+1}$, and $\mathrm{J}(f_k(e), g_{k,s}(e)) = 1$ ($h(e)$ denotes the leading form of h relative to w_e).

Indeed, if $\mathrm{J}(f_k(e), g_k(e)) = 0$ then $g_k(e) = c_0 f_k(e)^{\lambda_0}$ for some $c_0 \in \mathbb{C}^*,\ \lambda_0 \in \mathbb{Q}$. Since $g_k(e)$ is a polynomial in z, $\lambda_0 = \frac{a_0}{N},\ a_0 \in \mathbb{Z}$ where $f^{\frac{1}{N}}$ is a polynomial in z and $N \in \mathbb{Z}$ is maximal possible under this condition. In this case $g_k = c_0 f_k^{\lambda_0} + c_1 g_{k,1}$ where $g_{k,1} \in A_w$. If $\mathrm{J}(f_k(e), g_{k,1}(e)) = 0$ then $g_{k,1}(e) = c_1 f_k(e)^{\lambda_1}$ and $\lambda_1 = \frac{a_1}{N},\ a_1 \in \mathbb{Z}$ because $g_{k,1}(e)$ is a rational function in y, and so on. After a finite number of steps we will get $g_{k,s}$ for which $\mathrm{J}(f_k(e), g_{k,s}(e)) = 1$.

The form $g_{k,s}(e) = x^\sigma q(z)$ where $q(z) \in \mathbb{C}(z)$ and $\mathrm{J}(f_k(e), g_{k,s}(e)) = 1$ corresponds to $\rho p q' - \sigma p' q = 1$.

It is more convenient to look at $p(z)$ and $r(z) = p(z)q(z)$ which satisfy

$$\rho p r' - \tau p' r = p \tag{2}$$

where $\tau = \rho + \sigma$.

(2) can be rewritten as $\ln(r^\rho p^{-\tau})' = \frac{1}{r}$. This allows us to obtain a Dixmier relation

$$p^\tau = r^\rho \exp\Big(\int \frac{-d\,z}{r}\Big) \tag{3}$$

between p and r (see [D]). Thus all roots of a rational function r must be simple.

If e is a positive edge then $\rho > 0,\ \tau > 0,\ r$ cannot have poles, and r is a polynomial with simple roots. Assume that e is positive.

If $\deg(r) = 1$ then $p = \mu r^\kappa = (c_1 z + c_2)^d,\ d \in \mathbb{Z}^+$ and $c_2 = 0$ since $f_k(e)(x, 0) = 0$, i.e. e collapses to a vertex.

If e is not a vertex then $\deg(r) > 1$ and $\frac{1}{r} = \sum_i \frac{c_i}{z-\mu_i}$ where $\sum_i c_i = 0$. (Of course, $\frac{\rho - c_i}{\tau} \in \mathbb{Z}^+$ since $p(z)$ is a polynomial.) So $\deg(p) = \frac{\rho}{\tau}\deg(r)$ while the multiplicity of any root of p is $\frac{\rho-c_i}{\tau} \neq \frac{\rho}{\tau}$ since $c_i \neq 0$, i. e. for any edge e' obtained from e in the resolution process, the order vertex ofe' does not belong to the bisectrix. We can also see that there is a root with the multiplicity larger than $\frac{\rho}{\tau}$: since $\sum_i c_i = 0$ there is a negative c_j

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Lemma on positive reductions. There exists a positive reduction.

Proof. The leading edge e is positive. Therefore there are roots of $f(e)$ with the multiplicity larger than $\frac{w_e(f)}{w_e(xy)}$. Use any of these roots in the Newton resolution process. Denote by e_1 the edge attached to the modification of e. Its degree vertex is above the bisectrix. If e_1 is positive we can find a root of the form supported on e_1 with a sufficiently large multiplicity and proceed with a resolution process. Suppose after several steps we obtain a non-positive edge e_i. Its degree vertex is above the bisectrix.

After that proceed with the resolution process to obtain f_k. Consider also g_k which, we assumed, exists and recall expansion (1). The leading vertices of $\mathcal{N}(f_k)$ and $\mathcal{N}(g_k)$ are homothetic with the coefficient λ_0 and their leading edges are also homothetic with the same coefficient because $\mathrm{J}(f_k(e), g_k(e)) = 0$. Since $\mathrm{J}(f_k(e_j), g_k(e_j)) = 0$ for $1 \leq j < i$ we will get the chain of edges of $\mathcal{N}(f_k)$ staring with the leading edge up to, but not including e_i, such that its homothetic image with the coefficient λ_0 is a chain of edges of $\mathcal{N}(g_k)$.

If e_i is not a principal edge then $\mathrm{J}(f_k(e_i), g_k(e_i)) = 0$. In this case both $\mathcal{N}(f_k)$ and $\mathcal{N}(g_k)$ belong to the sector bounded by the ray connecting the origin and $\mathrm{dv}(e_i) = (\mu, \nu)$ (the degree vertex of e_i) and the negative ray of the x axis.

We can rewrite f_k and g_k in the variables $u = \frac{\nu}{\nu-\mu} x^{\frac{\nu-\mu}{\nu}}, \ z = x^{\frac{\mu}{\nu}} y$. Since $\mathrm{dv}(e_i)$ is above the bisectrix $\mu < \nu$ and in coordinates $u, \ z$ the Newton polygons of $f_k, \ g_k$ belong to the second quadrant. But then $\mathrm{J}(f_k, g_k) = 1$ is impossible. Hence e_i is the principal edge of f_k, its order vertex is the principal vertex $(\mu(k), 1)$ where $\mu(k) < 1$, and $\mathcal{N}(g_k)$ has a vertex $(1 - \mu(k), 0)$.

Consider the non-horizontal edge e_i' adjacent to $(1-\mu(k), 0)$. If the slope of e_i' is larger than the slope of e_i then $g_k(e_i) = x^{1-\mu(k)}$ and $\mathrm{J}(f_k(e_i), g_k(e_i))$ is neither zero nor one. So the slope of e_i' does not exceed the slope of e. In this case the degree vertex of e_i' cannot be proportional to the degree or to the order vertex of e_i since these edges are separated by the ray with the vertex in the origin and parallel to e_i. Again $\mathrm{J}(f_k(e_i), g_k(e_i))$ is neither zero nor one.

Thus e_i is positive and Lemma is proved. $\square$.

As we know, $\mathcal{N}(f_k)$ has a finite number of slanted edges, connecting the leading and principal vertices. With a resolution process described above, all vertices of these edges are above the bisectrix of the first quadrant.

Now we can find all c_i and λ_i from (1). Indeed, if e_p is the principal edge of f_k then $\mathrm{J}(f_k(e_p), g_k(e_p)) = 1$. Hence $w(f_k g_k) = w(xy)$ for the weight $w = w_{e_p}$. The degree vertices of $f_k(e_p)$ and $g_k(e_p)$ are proportional with the coefficient λ_0 so $(1 + \lambda_0) w(f_k) = w(xy)$. Therefore the point $(\frac{1}{1+\lambda_0}, \frac{1}{1+\lambda_0})$ belongs to the line which contain e and we know λ_0.

To find c_0 we should find $g_k(e_p)$ using (3) and compare coefficients with the leading monomials of $f_k(e_p)$ and $g_k(e_p)$. As we saw above, finding $g_k(e_p)$ boils dawn to finding a polynomial $q(z)$ for which $\rho p q' - \sigma p' q = 1$ where ρ and $p(z) \in \mathbb{C}[z]$ are known ($\rho = w(f_k)$, $f_k(e_p) = x^\rho p(z)$) and $\sigma = 1 - \mu(k)$. We can replace ρ and σ by relatively prime integers: $apq' - bp'q = k$. Finding q is possible only if neither a nor b is one. Indeed, if $\deg(p) = d_1 > 1$, $\deg(q) = d_2$ then $ad_2 = bd_1$. If, say, $a = 1$ then $p(q - cp^b)' - bp'(q - cp^b) = k$ for any $c \in \mathbb{C}$ and we can find c such that $\deg(q - cp^b) < d_2$. This means that λ_0 is neither integer nor the reciprocal of an integer.

Using other edges of $\mathcal{N}(f_k)$ we can find other λ_i and c_i including $c_{\kappa-1}$, $\lambda_{\kappa-1}$ defined by the leading edge, but not all of them. It is possible to have some intermediate values of λ_i corresponding to the vertices of $\mathcal{N}(f_k)$.

If $d = (m, n)$ (greatest common divisor) and $\lambda_0 = \frac{m_0}{d}$ then $g = [\sum_{i=0}^{m_0} c_i f^{\frac{m_0 - i}{d}}]$, the polynomial part of the expression, and finding c_i is a linear algebra problem. Since $\sum_{i=0}^{m_0} c_i [f, [f^{\frac{m_0-i}{d}}]] = 1$ we should express 1 as a linear combination of known polynomials.

If a g is recovered and $\mathrm{J}(f, h) = 1$ then $h - g \in \mathbb{C}[f]$ (see [No]). Here is a somewhat different proof that $\mathrm{J}(p, f) = 0$ only when $p \in \mathbb{C}[f]$. If $\mathrm{J}(p, f) = 0$ then p and f are algebraically dependent, i.e. $Q(p, f) = 0$ for some polynomial Q. Therefore $\mathrm{J}(Q(p, f), g) = Q_p \mathrm{J}(p, g) + Q_f = 0$ and $\mathrm{J}(p, g) \in \mathbb{C}(p, f)$. Therefore $\mathrm{J}(\mathrm{J}(p, g), f) = 0$, i.e Jacobian with g acts on $C(f)$, a subalgebra of elements algebraically dependent with f.

Lemma on degree. $\deg_y(\mathrm{J}(p, g)) - \deg_y(p) \le -\deg_y(f)$.

Proof. It is clear that $\deg_y(\mathrm{J}(p, g)) < \deg_y(p) + \deg_y(g)$. Hence we can find $q \in C(f)$ for which $D(q) = \deg_y(\mathrm{J}(q, g)) - \deg_y(q)$ is maximal possible. Assume that $D(q) > -\deg_y(f)$. Since $\mathrm{J}(f,q) = 0$ the leading form of q is proportional to a fractional power of $|f|$. Hence we can find $a,\ b \in \mathbb{Z}$ and $c \in \mathbb{C}$ for which $\deg_y(q^a - cf^b) < \deg_y(q^a)$. Now, $D(q^a) = \deg_y(\mathrm{J}(q^a, g)) - \deg_y(q^a) = \deg_y(aq^{a-1}\mathrm{J}(q,g)) - \deg_y(q^a)) = D(q)$ and $D(q^a - cf^b) = \deg_y(\mathrm{J}(q^a - cf^b, g)) - \deg_y((q^a - cf^b)) > D(q)$. Indeed, $\mathrm{J}(q^a - cf^b, g) = aq^{a-1}\mathrm{J}(q,g) - cbf^{b-1}$ and $\deg_y(\mathrm{J}(q^a - cf^b, g)) = \deg_y(aq^{a-1}\mathrm{J}(q,g) - cbf^{b-1}) = \deg_y(aq^{a-1}\mathrm{J}(q,g))$ since $\deg_y(q^{a-1}\mathrm{J}(q,g)) = \deg_y(q^a) + D(q) = \deg_y(f^b) + D(q) > \deg_y(f^{b-1})$ by our assumption while $\deg_y(q^a - cf^b) < \deg_y(q^a)$. We have a contradiction which proves the lemma. □

Assume now that there are elements in $C(f)$ which are not polynomials in f. Say, $p \in C(f)$ is such an element with minimal degree possible. Then $\mathrm{J}(p,g) = \sum_{i=0}^{d} p_i f^i$ since $\deg_y(\mathrm{J}(p,g)) < \deg_y(p)$. Therefore $\mathrm{J}(p - \sum_{i=0}^{d} \frac{p_i}{i+1} f^{i+1}, g) = 0$, i.e. $p' = p - \sum_{i=0}^{d} \frac{p_i}{i+1} f^{i+1}$ is algebraically dependent with g. Since p' is also algebraically dependent with f and f and g are algebraically independent, $p' \in \mathbb{C}$ and $p \in \mathbb{C}[f]$.

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