A Jacobian mate defines the Jacobian pair

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To the memory of Ernest Borisovich Vinberg

Abstract

A polynomial $f \in \mathbb{C}[x, y]$ is a Jacobian mate if the Jacobian $J(f, g) = 1$ for some $g \in \mathbb{C}[x, y]$. It is not known that then $\mathbb{C}[f, g] = \mathbb{C}[x, y]$ and a conjecture that this is the case is the Jacobian conjecture. In this note we will check that if $f$ is given then the subalgebra $\mathbb{C}[f, g]$ is known and that $g$ can be recovered up to a summand which is a polynomial in $f$.

Mathematics Subject Classification (2000): Primary 14R15.

Key words: Jacobian mate, Jacobian conjecture.

Introduction.

Assume that $f \in \mathbb{C}[x, y]$ (where $\mathbb{C}$ is the field of complex numbers) satisfies $J(f, g) = \frac{\partial f}{\partial x} \frac{\partial g}{\partial y} - \frac{\partial f}{\partial y} \frac{\partial g}{\partial x} = 1$ for some $g \in \mathbb{C}[x, y]$. This polynomial

*The author is grateful to the Max Planck Institute for Mathematics, where he is presently a visitor.
can be the image of $x$ under an automorphism $\phi$ and it is well known that
then $J(\phi^{-1}(f), \phi^{-1}(g)) = c \in \mathbb{C}^*$, a non-zero complex number. Then from
the properties of Jacobian it follows that $\phi^{-1}(g) = cy + p(x)$ and therefore
$\mathbb{C}[f, g] = \mathbb{C}[x, y]$.

The JC (Jacobian conjecture) states that $J(f, g) = 1$ implies $\mathbb{C}[f, g] = \mathbb{C}[x, y]$ (see [K]). This conjecture occasionally becomes a theorem even for
many years but today it is a problem. So it is possible that $f$ is not an
image of $x$ under an automorphism and then $\mathbb{C}[f, g] \neq \mathbb{C}[x, y]$.

Recall that if $p \in \mathbb{C}[x, y]$ is a polynomial in 2 variables and each monomial
of $p$ is represented by a lattice point on the plane with the coordinate vector
equal to the degree vector of this monomial then the convex hull $\mathcal{N}(p)$ of
the points so obtained is called the Newton polygon.

It is known for many years that for a potential counterexample to JC
there exists an automorphism $\xi$ of $\mathbb{C}[x, y]$ such that the Newton polygon
$\mathcal{N}(\xi(f))$ of $\xi(f)$ contains a vertex $v = (m, n)$ where $n > m > 0$ and is
included in a trapezoid with the vertex $v$, edges parallel to the $y$ axis and
to the bisectrix of the first quadrant adjacent to $v$, and two edges belonging
to the coordinate axes (see [A1], [A2], [AO], [GGV], [H], [J], [L], [MW],[M],
[Na1], [Na2], [NN1], [NN2], [Ok]). This was improved with a completely new
approach by Pierrette Cassou-Noguès who showed that $\mathcal{N}(f)$ does not have
an edge parallel to the bisectrix (see [CN] and [ML]). From now on we will
assume that $f$ is “shaped” like this, i.e. that $\mathcal{N}(f)$ is a trapezoid described
above.

Our goal is to recover $g$. 
Expansion of $g$.

Consider the ring $L = \mathbb{C}[x^{-1}, x]$ of Laurent polynomials in $x$. Define $A$ to be the algebra of asymptotic power series in $y$ with coefficients in $L$, i.e. the elements of $A$ are $\sum_{-\infty}^{i=k} y_i y^i$ where $y_i \in L$, $y_k \neq 0$. For $a = \sum_{-\infty}^{i=k} y_i y^i$ define the leading form of $a$ as $|a| = y_k y^k$.

The polynomial $f(x, y)$ and a polynomial $g$ which we are trying to find are elements of $A$ and $|f| = cx^m y^n$. If $|g| = g_N y^N$ then since $J(f, g) = 1$ and $n > 1$ we must have $J(x^m y^n, g_N y^N) = 0$. (If $J(x^m y^n, g_N y^N) = \left(m N x^{m-1} g_N - n x^m g'_N\right)y^{n+N-1} \neq 0$ then $|J(f, g)| = cc_1(m N x^{m-1} g_N - n x^m g'_N)y^{n+N-1}$ and $J(f, g) \neq 1$.) Hence $g_N = c_1 x^M$ and $|g| = c_1 x^M y^N$ where $Mn = m N$, i.e. $x^M y^N = (x^m y^n)^{\lambda_0}$ where $\lambda_0 \in \mathbb{Q}$.

**Lemma on radical.** If $r \in \mathbb{Q}$ is a rational number, $|a| = cx^l y^k$, $c \in \mathbb{C}$, and $|a|^r \in A$ then $a^r \in A$.

**Proof.** From the Newton binomial theorem $a^r = |a|^r \sum_{j=0}^{\infty} \binom{r}{j} \left(\sum_{-\infty}^{i=k-1} \frac{y_k}{y_i} y^{i-k}\right)^j$ because $a = |a|(1 + \sum_{-\infty}^{i=k-1} \frac{y_k}{y_i} y^{i-k})$. Since all $\frac{y_k}{y_i} \in L$, element $a^r \in A$. □

Since we can divide $f$ by $c$, we may assume without loss of generality that $|f| = x^m y^n$. Not to be confused with rational powers of complex numbers let us agree that any rational power of $f$ which belongs to $A$ has as the leading form a monomial with coefficient 1.

By lemma on radical $f^{\lambda_0} \in A$ and hence $g_1 = g - c_1 f^{\lambda_0} \in A$. Since $J(f, g_1) = 1$ either $J(|f|, |g_1|) = 0$ or $J(|f|, |g_1|) = 1$. If $J(|f|, |g_1|) = 0$ then $|g_1| = c_1 |f|^{\lambda_1}$, $c_1 \in \mathbb{C}$, $\lambda_1 \in \mathbb{Q}$ and we can define $g_2 = g - c_0 f^{\lambda_0} - c_1 f^{\lambda_1}$ which is in $A$ for the same reasons as $g_1$. We can proceed until we obtain...
\( g_{\kappa} = g - \sum_{i=0}^{\kappa-1} c_i f^\lambda_i \in A \) for which \( J(|f|, |g_{\kappa}|) = 1 \), i.e. \( J(x^m y^n, |g_{\kappa}|) = 1 \). Therefore \( |g_{\kappa}| = (c_{\kappa}(x^m y^n)^{1-n} - \frac{1}{n-m} x^{1-m} y^{1-n}) \) where \( c_{\kappa} \in \mathbb{C} \). If \( c_{\kappa} \neq 0 \) then \( (x^m y^n)^{1-n} \in A \) and \( \frac{m}{n} \in \mathbb{Z} \) which is impossible since \( 0 < m < n \). Thus \( |g_{\kappa}| = \frac{1}{(m-n)} x^{1-m} y^{1-n} \) and

\[
    g = \sum_{i=0}^{\kappa-1} c_i f^\lambda_i + g_{\kappa}, \quad c_i \in \mathbb{C}
\]

where \( \deg_y(|f^\lambda_i|) > 1-n \), \( \deg_y(|g_{\kappa}|) = 1-n \), and \( |g_{\kappa}| = \frac{1}{(m-n)} x^{1-m} y^{1-n} = \frac{1}{(m-n)} x^{\frac{n-m}{n}} |f|^{\lambda \kappa} \) where \( \lambda_{\kappa} = \frac{1-n}{m} \).

We will find \( g \) if we find \( c_i \) and \( \lambda_i \) for \( 0 \leq i < \kappa \) since \( g \) is a polynomial.

**Reductions of \( f \).**

Newton introduced the polygon which we call the Newton polygon in order to find a solution \( y \) of \( p(x, y) = 0 \) in terms of \( x \) for a polynomial \( p(x, y) = \sum_{(i, j) \in \mathcal{N}(p)} p_{ij} x^i y^j \) (see [N]). Here is the process of obtaining such a solution. Consider an edge \( e \) of \( \mathcal{N}(p) \) which is not parallel to the \( x \) axis. Denote by \( p(e) = \sum_{(i, j) \in e} p_{ij} x^i y^j \). The form \( p(e) \) allows to determine the first summand of the solution as follows. Consider an equation \( p(e) = 0 \). Since \( p(e) \) is a homogeneous form relative to a weight given by \( w(x) = \alpha(\neq 0), \quad w(y) = \beta, \quad w(x^i y^j) = i\alpha + j\beta \), solutions of this equation are \( y = c_i x^\beta \) and \( c_i \in \mathbb{C} \). Choose any solution \( c_i x^\beta \) and replace \( p(x, y) \) by \( p_1(x, y) = p(x, c_i x^\beta + y) \). Though \( p_1 \) is not necessarily a polynomial in \( x \) we can define the Newton polygon of \( p_1 \) in the same way as it was done for the polynomials; the only difference is that \( p_1 \) may contain monomials \( x^\mu y^\nu \) where \( \mu \in \mathbb{Q} \) rather than in \( \mathbb{Z} \). The polygon \( \mathcal{N}(p_1) \) contains the degree vertex \( v \) of \( e \), i.e.
the vertex with $y$ coordinate equal to $\deg_y(p(e))$ and an edge $e'$ which is a modification of $e$ ($e'$ may collapse to $v$). Take the order vertex $v_1$ of $e'$, i.e. the vertex with $y$ coordinate equal to the order of $p_1(e')$ as a polynomial in $y$ (if $e' = v$ take $v_1 = v$). Use the edge $e_1$ for which $v_1$ is the degree vertex to determine the next summand and so on. After possibly a countable number of steps we obtain a vertex $v_\mu$ and the edge $e_\mu$ for which $v_\mu$ is not the degree vertex, i.e. either $e_\mu$ is horizontal or the degree vertex of $e_\mu$ has a larger $y$ coordinate than the $y$ coordinate of $v_\mu$. It is possible only if $\mathcal{N}(p_\mu)$ does not have any vertices on the $x$ axis. Therefore $p_\mu(x, 0) = 0$ and a solution is obtained.

When characteristic is zero the process of constructing a solution is more straightforward then it may seem from this description. The denominators of fractional powers of $x$ (if denominators and numerators of these rational numbers are assumed to be relatively prime) do not exceed $\deg_y(p)$. Indeed, for any initial weight there are at most $\deg_y(p)$ solutions while a summand $c x^M \frac{y^N}{}$ can be replaced by $c \varepsilon^M x^\frac{N}{}$ where $\varepsilon^N = 1$ which gives at least $N$ different solutions.

Let $y_k \in \overline{\mathbb{C}[x]}$ be one of the solutions of $f(x, y) = 0$ by the decreasing powers of $x$. Call $f_k(x, y) = f(x, y + y_k) \in \overline{\mathbb{C}(x)}[y]$ a reduction of $f$.

If $g$ exists then $J(f_k, g_k) = 1$ for $g_k(x, y) = g(x, y + y_k)$. Since $f_k(x, 0) = 0$, we have $1 = -\frac{\partial g_k(x, 0)}{\partial x} f_{k,1}$ where $f_{k,1} = \frac{\partial f_k(x, y)}{\partial y}|_{y=0}$. Because of that the lowest vertex of $\mathcal{N}(f_k)$ is $(\mu(k), 1)$ where $\mu(k) \neq 1$ and the lowest vertex of $\mathcal{N}(g_k)$ is either $(1 - \mu(k), 0)$ or $(0, 0)$. Call $(\mu(k), 1)$ the principal vertex and the slanted (non-horizontal) edge of $\mathcal{N}(f_k)$ containing $(\mu(k), 1)$ the principal edge of $f_k$.

The polygon $\mathcal{N}(f_k)$ is not compact, it has an infinite edge $[(-\infty, 1), (\mu(k), 1)]$. 

There is also an infinite horizontal edge through the leading vertex, i.e. the vertex \((m, n)\) of \(\mathcal{N}(f_k)\), which contains only this vertex.

Call a slanted edge \(e\) of a Newton polygon semi-positive, zero, semi-negative if its extension intersects the \(x\) axis in a point with a positive, zero, negative abscissa.

Call \(e\) positive if it is semi-positive and the line parallel to \(e\) and containing point \((1, 1)\) intersects the \(x\) axis in a point with a positive abscissa.

Call a positive edge of \(\mathcal{N}(f_k)\) which contains the leading vertex the leading edge. (This is the right edge containing the leading vertex.)

If the principal edge of a reduction is positive call this reduction positive. It is easy to see that for a positive reduction the edges in the chain of slanted edges starting with the leading edge and ending with the principal edge are positive.

In order to say more about reductions we use some extensions of \(\mathbb{C}[x, y]\).

Take a weight on \(\mathbb{C}[x, y]\) given by \(w(x) = 1, \, w(y) = -\alpha\) where \(\alpha \in \mathbb{R}\). Observe that \(\alpha \geq 0\) for the edges of \(\mathcal{N}(f_k)\).

With the help of this weight we can define an extension \(A_w\) of \(\mathbb{C}[x, y]\):

\(A_w\) consists of fractional asymptotic power series \(\sum_{i=k}^{\infty} c_i(z)x^{-\frac{i}{N}}\) where \(z = x^\alpha y, \, k \in \mathbb{Z}, \, c_i(z) \in \mathbb{C}(z), \, N \in \mathbb{Z}^+\) and depends on the element. For any \(a = \sum_{i=k}^{\infty} c_i(z)x^{-\frac{i}{N}} \in A_w\) the leading form \(|a|\) is defined as \(|a| = c_k(z)x^{-\frac{k}{N}}\) (assuming \(c_k \neq 0\)). The Newton binomial formula \((1 + \delta)^\lambda = \sum_{j=0}^{\infty} \binom{\lambda}{j}\delta^j\) insures that if \(|a|^{\lambda} \in A_w\) then \(a^{\lambda} \in A_w\) (see Lemma on radical).

Take an edge \(e\) of \(\mathcal{N}(f_k)\). Denote by \(w_e\) the weight given by \(w_e(x) = 1, \, w_e(y) = -\alpha\) for which all points on \(e\) have the same weigh. Denote by \(f_k(e)\) the leading form of \(f_k\) relative to this weigh and assume that \(\rho = w_e(f_k) \neq 0\). Then \(f_k(e) = x^\rho p(z)\) where \(z = x^\alpha y, \, p(z) \in \mathbb{C}[z], \, \rho \in \mathbb{Q}^*\).
We can present $g_k$ as $g_k = \sum_{i=0}^{s-1} c_i f^{\lambda_i} + g_{k,s}$ where $c_i \in \mathbb{C}$, $\lambda_i \in \mathbb{Q}$, $\lambda_i > \lambda_{i+1}$, and $J(f_k(e), g_{k,s}(e)) = 1$ ($h(e)$ denotes the leading form of $h$ relative to $w_e$).

Indeed, if $J(f_k(e), g_k(e)) = 0$ then $g_k(e) = c_0 f_k(e)^{\lambda_0}$ for some $c_0 \in \mathbb{C}^*$, $\lambda_0 \in \mathbb{Q}$. Since $g_k(e)$ is a polynomial in $z$, $\lambda_0 = \frac{a_0}{N}$, $a_0 \in \mathbb{Z}$ where $f_k^\frac{1}{N}$ is a polynomial in $z$ and $N \in \mathbb{Z}$ is maximal possible under this condition. In this case $g_k = c_0 f_k^{\lambda_0} + c_1 g_{k,1}$ where $g_{k,1} \in A_w$. If $J(f_k(e), g_{k,1}(e)) = 0$ then $g_{k,1}(e) = c_1 f_k(e)^{\lambda_1}$ and $\lambda_1 = \frac{a_1}{N}$, $a_1 \in \mathbb{Z}$ because $g_{k,1}(e)$ is a rational function in $y$, and so on. After a finite number of steps we will get $g_{k,s}$ for which $J(f_k(e), g_{k,s}(e)) = 1$.

The form $g_{k,s}(e) = x^\sigma q(z)$ where $q(z) \in \mathbb{C}(z)$ and $J(f_k(e), g_{k,s}(e)) = 1$ corresponds to $\rho pq' - \sigma p'q = 1$.

It is more convenient to look at $p(z)$ and $r(z) = p(z)q(z)$ which satisfy

$$\rho p' - \tau p' r = p$$

(2)

where $\tau = \rho + \sigma$.

(2) can be rewritten as $\ln(r^p p^{-\tau})' = \frac{1}{r}$. This allows us to obtain a Dixmier relation

$$p^r = r^\rho \exp\left(\int \frac{-d z}{r}\right)$$

(3)

between $p$ and $r$ (see [D]). Thus all roots of a rational function $r$ must be simple.

If $e$ is a positive edge then $\rho > 0$, $\tau > 0$, $r$ cannot have poles, and $r$ is a polynomial with simple roots. Assume that $e$ is positive.

If $\deg(r) = 1$ then $p = \mu r^\kappa = (c_1 z + c_2)^d$, $d \in \mathbb{Z}^+$ and $c_2 = 0$ since $f_k(e)(x,0) = 0$, i.e. $e$ collapses to a vertex.
If $e$ is not a vertex then $\deg(r) > 1$ and $\frac{1}{r} = \sum_i \frac{c_i}{z - \mu_i}$ where $\sum_i c_i = 0$. (Of course, $\frac{c_i}{r} \in \mathbb{Z}^+$ since $p(z)$ is a polynomial.) So $\deg(p) = \frac{\rho}{\tau} \deg(r)$ while the multiplicity of any root of $p$ is $\frac{\rho - c_i}{\tau} \neq \frac{\rho}{\tau}$ since $c_i \neq 0$, i.e. for any edge $e'$ obtained from $e$ in the resolution process, the order vertex of $e' \neq e$ does not belong to the bisectrix. We can also see that there is a root with the multiplicity larger than $\frac{\rho}{\tau}$: since $\sum_i c_i = 0$ there is a negative $c_j$.

**Lemma on positive reductions.** There exists a positive reduction.

Proof. The leading edge $e$ is positive. Therefore there are roots of $f(e)$ with the multiplicity larger than $\frac{\text{we}(e)}{\text{we}(xy)}$. Use any of these roots in the Newton resolution process. Denote by $e_1$ the edge attached to the modification of $e$. Its degree vertex is above the bisectrix. If $e_1$ is positive we can find a root of the form supported on $e_1$ with a sufficiently large multiplicity and proceed with a resolution process. Suppose after several steps we obtain a non-positive edge $e_i$. Its degree vertex is above the bisectrix.

After that proceed with the resolution process to obtain $f_k$. Consider also $g_k$ which, we assumed, exists and recall expansion (1). The leading vertices of $\mathcal{N}(f_k)$ and $\mathcal{N}(g_k)$ are homothetic with the coefficient $\lambda_0$ and their leading edges are also homothetic with the same coefficient because $J(f_k(e), g_k(e)) = 0$. Since $J(f_k(e_j), g_k(e_j)) = 0$ for $1 \leq j < i$ we will get the chain of edges of $\mathcal{N}(f_k)$ staring with the leading edge up to, but not including $e_i$, such that its homothetic image with the coefficient $\lambda_0$ is a chain of edges of $\mathcal{N}(g_k)$.

If $e_i$ is not a principal edge then $J(f_k(e_i), g_k(e_i)) = 0$. In this case both $\mathcal{N}(f_k)$ and $\mathcal{N}(g_k)$ belong to the sector bounded by the ray connecting the origin and $\text{dv}(e_i) = (\mu, \nu)$ (the degree vertex of $e_i$) and the negative ray of
the $x$ axis.

We can rewrite $f_k$ and $g_k$ in the variables $u = \frac{\nu}{\nu - \mu} x^{\nu - \mu}$, $z = x^\mu y$. Since $dv(e_i)$ is above the bisectrix $\mu < \nu$ and in coordinates $u, z$ the Newton polygons of $f_k, g_k$ belong to the second quadrant. But then $J(f_k, g_k) = 1$ is impossible. Hence $e_i$ is the principal edge of $f_k$, its order vertex is the principal vertex $(\mu(k), 1)$ where $\mu(k) < 1$, and $\mathcal{N}(g_k)$ has a vertex $(1 - \mu(k), 0)$.

Consider the non-horizontal edge $e'_i$ adjacent to $(1 - \mu(k), 0)$. If the slope of $e'_i$ is larger than the slope of $e_i$ then $g_k(e_i) = x^{1-\mu(k)}$ and $J(f_k(e_i), g_k(e_i))$ is neither zero nor one. So the slope of $e'_i$ does not exceed the slope of $e$. In this case the degree vertex of $e'_i$ cannot be proportional to the degree or to the order vertex of $e_i$ since these edges are separated by the ray with the vertex in the origin and parallel to $e_i$. Again $J(f_k(e_i), g_k(e_i))$ is neither zero nor one.

Thus $e_i$ is positive and Lemma is proved. $\Box$.

As we know, $\mathcal{N}(f_k)$ has a finite number of slanted edges, connecting the leading and principal vertices. With a resolution process described above, all vertices of these edges are above the bisectrix of the first quadrant.

Now we can find all $c_i$ and $\lambda_i$ from (1). Indeed, if $e_p$ is the principal edge of $f_k$ then $J(f_k(e_p), g_k(e_p)) = 1$. Hence $w(f_kg_k) = w(xy)$ for the weight $w = w_{e_p}$. The degree vertices of $f_k(e_p)$ and $g_k(e_p)$ are proportional with the coefficient $\lambda_0$ so $(1 + \lambda_0)w(f_k) = w(xy)$. Therefore the point $(\frac{1}{1+\lambda_0}, \frac{1}{1+\lambda_0})$ belongs to the line which contain $e$ and we know $\lambda_0$.

To find $c_0$ we should find $g_k(e_p)$ using (3) and compare coefficients with
the leading monomials of $f_k(e_p)$ and $g_k(e_p)$. As we saw above, finding $g_k(e_p)$ boils down to finding a polynomial $q(z)$ for which $\rho q' - \sigma p'q = 1$ where $\rho$ and $p(z) \in \mathbb{C}[z]$ are known ($\rho = w(f_k)$, $f_k(e_p) = x^\rho p(z)$) and $\sigma = 1 - \mu(k)$. We can replace $\rho$ and $\sigma$ by relatively prime integers: $apq' - bp'q = k$. Finding $q$ is possible only if neither $a$ nor $b$ is one. Indeed, if $\deg(p) = d_1 > 1$, $\deg(q) = d_2$ then $ad_2 = bd_1$. If, say, $a = 1$ then $p(q - cp^b)' - bp'(q - cp^b) = k$ for any $c \in \mathbb{C}$ and we can find $c$ such that $\deg(q - cp^b) < d_2$. This means that $\lambda_0$ is neither integer nor the reciprocal of an integer.

Using other edges of $N(f_k)$ we can find other $\lambda_i$ and $c_i$ including $c_{\kappa - 1}$, $\lambda_{\kappa - 1}$ defined by the leading edge, but not all of them. It is possible to have some intermediate values of $\lambda_i$ corresponding to the vertices of $N(f_k)$.

If $d = (m, n)$ (greatest common divisor) and $\lambda_0 = \frac{mn}{d}$ then $g = \sum_{i=0}^{m_0} c_i f^{\frac{m_0 - i}{d}}$, the polynomial part of the expression, and finding $c_i$ is a linear algebra problem. Since $\sum_{i=0}^{m_0} c_i [f, \left[f^{\frac{m_0 - i}{d}}\right]] = 1$ we should express 1 as a linear combination of known polynomials.

If a $g$ is recovered and $J(f, h) = 1$ then $h - g \in \mathbb{C}[f]$ (see [No]). Here is a somewhat different proof that $J(p, f) = 0$ only when $p \in \mathbb{C}[f]$. If $J(p, f) = 0$ then $p$ and $f$ are algebraically dependent, i.e. $Q(p, f) = 0$ for some polynomial $Q$. Therefore $J(Q(p, f), g) = Q_p J(p, g) + Q_f = 0$ and $J(p, g) \in \mathbb{C}(p, f)$. Therefore $J(J(p, g), f) = 0$, i.e Jacobian with $g$ acts on $C(f)$, a subalgebra of elements algebraically dependent with $f$.

**Lemma on degree.** $\deg_y(J(p, g)) - \deg_y(p) \leq - \deg_y(f)$.

Proof. It is clear that $\deg_y(J(p, g)) < \deg_y(p) + \deg_y(g)$. Hence we can find $q \in C(f)$ for which $D(q) = \deg_y(J(q, g)) - \deg_y(q)$ is maximal pos-
sible. Assume that $D(q) > -\deg_y(f)$. Since $J(f, q) = 0$ the leading form of $q$ is proportional to a fractional power of $|f|$. Hence we can find $a, b \in \mathbb{Z}$ and $c \in \mathbb{C}$ for which $\deg_y(q^a - cf^b) < \deg_y(q^a)$. Now, $D(q^a) = \deg_y(J(q^a, g)) - \deg_y(q^a) = \deg_y(aq^{a-1}J(q, g)) - \deg_y(q^a)) = D(q)$ and $D(q^a - cf^b) = \deg_y(J(q^a - cf^b, g)) - \deg_y((q^a - cf^b)) > D(q)$. Indeed, $J(q^a - cf^b, g) = aq^{a-1}J(q, g) - cbf^{b-1}$ and $\deg_y(J(q^a - cf^b, g)) = \deg_y(aq^{a-1}J(q, g) - cbf^{b-1}) = \deg_y(aq^{a-1}J(q, g))$ since $\deg_y(q^{a-1}J(q, g)) = \deg_y(q^a) + D(q) = \deg_y(f^b) + D(q) > \deg_y(f^{b-1})$ by our assumption while $\deg_y(q^a - cf^b) < \deg_y(q^a)$. We have a contradiction which proves the lemma. $\square$

Assume now that there are elements in $C(f)$ which are not polynomials in $f$. Say, $p \in C(f)$ is such an element with minimal degree possible. Then $J(p, g) = \sum_{i=0}^d p_i f^i$ since $\deg_y(J(p, g)) < \deg_y(p)$. Therefore $J(p - \sum_{i=0}^d \frac{p_i}{i+1} f^{i+1}, g) = 0$, i.e. $p' = p - \sum_{i=0}^d \frac{p_i}{i+1} f^{i+1}$ is algebraically dependent with $g$. Since $p'$ is also algebraically dependent with $f$ and $f$ and $g$ are algebraically independent, $p' \in \mathbb{C}$ and $p \in \mathbb{C}[f]$.

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