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AUTOMORPHISM GROUPS OF $\mathbb{P}^1-$BUNDLES OVER A NON-UNIRULED BASE

TATIANA BANDMAN AND YURI G. ZARHIN

ABSTRACT. In this survey we discuss holomorphic $\mathbb{P}^1-$bundles $p : X \to Y$ over a non-uniruled complex compact Kähler manifold Y, paying a special attention to the case when Y is a complex torus. We consider the groups $\mathrm{Aut}(X)$ and $\mathrm{Bim}(X)$ of its biholomorphic and bimeromorphic automorphisms, respectively, and discuss when these groups are bounded, Jordan, strongly Jordan, or very Jordan.

CONTENTS

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1. Introduction

In this survey we consider the groups $\mathrm{Aut}(X)$ and $\mathrm{Bim}(X)$ of all biregular and bimeromorphic self-maps, respectively, for a compact complex connected Kähler manifold X. If X is projective, $\mathrm{Bim}(X) = \mathrm{Bir}(X)$ is the group of all birational transformations of X ([Se56]). The manifolds we are going to deal with are of special type: X has to be a $\mathbb{P}^1-$bundle over a non-uniruled compact complex connected manifold Y.

In general, the groups $\mathrm{Bim}(X)$ may be very huge and non-algebraic (for example Cremona group Cr_n of birational transformation of the $n-$dimensional projective space). Thus one is tempted to study properties of a group via its finite and /or abelian subgroups. Namely, we are interested in the following properties of groups.

Definition 1.1.
1. A group G is called *bounded* if the orders of its finite subgroups are bounded by an universal constant that depends only on G ([Po11, Definition 2.9]).
2. A group G is called *Jordan* if there is a positive integer J such that every finite subgroup B of G contains an abelian subgroup A that is normal in B and such that the index $[B : A] \leq J$. The smallest such J is called the *Jordan constant* of G, denoted by J_G.([Se09, Question 6.1], [Po11, Definition 2.1]).
3. A Jordan group G is called *strongly Jordan* [PS14, BZ17] if there is a positive integer m such that every finite subgroup of G is generated by at most m elements.
4. A group G is *very Jordan* ([BZ20]) if there exist a commutative normal subgroup G_0 of G and a bounded group F that sit in a short exact sequence
$$1 \to G_0 \to G \to F \to 1. \tag{1}$$

In what follows by **Jorfan Properties** we mean one of those described in Definition 1.1. The study of these properties were inspired by the following fundamental results.

Theorem 1.2. *(M.-E.-C. Jordan (1878),* [Jor], [Se16, Theorem 9.9]*) Let $\mathbb{C}$ be the field of complex numbers. Then $\mathrm{GL}_n = \mathrm{GL}_n(\mathbb{C})$ is strongly Jordan.*

Theorem 1.3. *(J.-P. Serre (2009),* [Se09, Theorem 5.3]*)* $\mathrm{Cr}_2 = \mathrm{Bir}(\mathbb{P}^2)$ *is Jordan,* $J_{\mathrm{Cr}_2} \le 2^{10}3^4 5^2 7$.

It was V.L. Popov who asked in [Po11] a question whether for an algebraic variety X the groups $\mathrm{Aut}(X)$ and $\mathrm{Bir}(X)$ are Jordan. The question originated an intensive and fruitful activity. It was proven that there are vast classes of manifolds (varieties) with Jordan groups $\mathrm{Aut}(X), \mathrm{Bim}(X)$, and $\mathrm{Bir}(X)$, see Section 4. In particular, the Cremona group $Cr_n = \mathrm{Bir}(\mathbb{P}^n)$ appeared to be Jordan for all n ([PS14] and [Bi]) (this is the positive answer to a question formulated by J.-P. Serre). In Section 4 we give a glimpse on richness of known facts about Jordan properties of $\mathrm{Aut}(X), \mathrm{Bim}(X)$ or $\mathrm{Bir}(X)$ for various types of varieties X. We do not pretend to give a complete picture. Our aim is to demonstrate that the "worst" manifolds from this point of view are the uniruled but not-rationally connected ones. For example, the group $\mathrm{Bim}(X)$ is not Jordan if X is bimeromorphic to a product of a complex torus of positive algebraic dimension and the projective space $\mathbb{P}^N, N > 0$ ([Zar14], [Zar19]).

In this survey we concentrate on the manifolds of this kind. Namely, our main object of consideration are $\mathbb{P}^1-$bundles over non-uniruled manifolds i.e. triples (X, p, Y) such that

- X, Y are compact complex connected Kähler manifolds;
- $p : X \to Y$ is a holomorphic map from X onto Y;
- Y is not uniruled;
- for every point $y \in Y$ the fiber $p^*(y)$ isomorphic to $\mathbb{P}^1$; in particular, is irreducible and reduced.

We say that such a triple (X, p, Y) has an *almost section* D if an irreducible analytic subset $D \subset X$, $\mathrm{codim}(D) = 1$, meets a general fiber of p at precisely one point (see Definition 6.5). We say that such a triple (X, p, Y) (or X, or morphism p) is *scarce*, if X does *not* admit three distinct almost sections A_1, A_2, A_3 such that $A_1 \cap A_2 = A_1 \cap A_3 = A_2 \cap A_3$ (see Definition 11.5). We say that a connected compact complex manifold Y is *poor* (Definition 13.1) if it contains neither rational curves nor analytic subsets of codim 1.

The facts that we know about Jordan properties of $\mathbb{P}^1-$bundles (X, p, Y) over non-uniruled Kähler manifolds are presented in the following:

Summary

(1) $\mathrm{Aut}(X)$ is always Jordan ([Kim]) and even strongly Jordan (see Remark 4.1);
(2) If morphism p is scarce then $\mathrm{Aut}(X)$ is very Jordan (Theorem 12.1 of this paper).
(3) If Y is a torus and if X is not a projectivization of a decomposable vector bundle of rank 2 on Y, then the group $\mathrm{Aut}(X)$ is strongly Jordan ([Sh19]).

(4) If X, Y are projective, and X is not birational to $Y \times \mathbb{P}^1$, then $\mathrm{Bir}(X)$ is strongly Jordan ([BZ17]);
(5) If Y is a poor manifold (see Definition 13.1) then $\mathrm{Bim}(X) = \mathrm{Aut}(X)$ and is very Jordan([BZ20]).
(6) If Y is a complex torus and there is no almost section of p then $\mathrm{Bim}(X)$ is Jordan ([Sh19]). In particular, if X is not the projectivization of a rank 2 vector bundle on Y, then the group $\mathrm{Bim}(X)$ is strongly Jordan.
(7) If Y is a complex torus of positive algebraic dimension and X is bimeromorphic (birational, if Y is projective) to a direct product $Y \times \mathbb{P}^1$ then the group $\mathrm{Bim}(X)$ (respectively, $\mathrm{Bir}(X)$) is not Jordan ([Zar14, Zar19]).
(8) If Y is a complex torus of positive algebraic dimension, Y_a is its algebraic reduction, $\mathscr{L}$ is the lift to Y of a holomorphic line bundle on Y_a, and X is the projectivization of the rank 2 vector bundle $\mathscr{L} \oplus \mathbf{1}$ then $\mathrm{Bim}(X)$ is not Jordan ([Zar19]).
(9) **Open question.** Assume that Y is a complex torus of positive algebraic dimension and X has no representation as in previous item. Is $\mathrm{Bim}(X)$ Jordan?

Our goal is to give a review of the methods used to prove these facts. The unpublished previously results are provided with full proves.

All manifolds are compact complex, and connected, if not stated otherwise. All algebraic varieties are complex, projective, irreducible, reduced. $\mathbb{P}^n, \mathbb{C}^n$ are complex projective and affine spaces respectively, $\mathbb{P}^n_k, \mathbb{C}^n_k$ are projective and affine spaces respectively over an algebraically closed field k.

The structure of the survey is as follows. In Section 2 we provide facts and examples concerning bounded, Jordan, and very Jordan groups. In Section 3 we enumerate Assumptions and Notation and remind the notions related to manifolds and their maps. In Section 4 we give examples of the known facts about Jordan properties of $\mathrm{Aut}(X), \mathrm{Bim}(X)$ and $\mathrm{Bir}(X)$ for various types of manifolds X. Our aim is to demonstrate a special role of $\mathbb{P}^1-$bundles over a non-uniruled base in this field. In Section 5 we provide some generalities on maps of $\mathbb{P}^1$- bundles. In Section 6 we deal with the group $\mathrm{Bim}(X)$ of a non-trivial rational bundle (in particar, projective conic bundle). In Chapter 3 we deal with certain $\mathbb{P}^1-$bundles over complex tori. We present a unified approach to proving results of [Zar14] and [Zar19]. It is based on sympectic algebra, the highly useful tools for studying line bundles over tori and inspired by the work of D. Mumford [Mum66]. In Chapter 4 we consider $\mathbb{P}^1$-bundles (X, p, Y) with scarce set of sections over a non-uniruled Kähler base. It contains certain generalization and modification of the paper [BZ20]. First, in Section 11, for a $\mathbb{P}^1-$bundle (X, p, Y) we consider the group $\mathrm{Aut}(X)_p$ of those automorphisms of X that leave every fiber of p fixed. In three subsection we describe three different types of such automorphisms. In Section 12, under assumption that Y is Kähler and not uniruled and p is scarce, we prove that the neutral component $\mathrm{Aut}_0(X)$ of the complex Lie group $\mathrm{Aut}(X)$ is commutative, hence $\mathrm{Aut}(X)$ is very Jordan. In Section 13 we prove that if Y is poor then p is scarce and $\mathrm{Aut}(X)$ is very Jordan.

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Chapter 1. Preliminaries

In this chapter we provide some backgrounds: properties of Jordan groups, the Notation and Assumptions and definitions.

2. Jordan properties of groups

In this section we recall the general facts about Jordan properties of groups. The following properties follow easily form the Definition 1.1

1) Every finite group is bounded, Jordan, and very Jordan.

2) Every commutative group is Jordan and very Jordan.

3) Every finitely generated commutative group is bounded. Indeed, such a group is isomorphic to a finite direct sum with every summand isomorphic to $\mathbb{Z}$ or $\mathbb{Z}/n\mathbb{Z}$ where n is positive integer.

4) A subgroup of a Jordan group is Jordan. A subgroup of a very Jordan group is very Jordan.

5) "Bounded" implies "very Jordan", "very Jordan" implies "Jordan".

6) "Bounded" implies "strongly Jordan." On the other hand, "very Jordan" does not imply "strongly Jordan." For example, a direct sum of infinitely many copies of $\mathbb{Z}/2\mathbb{Z}$ is commutative but has finite subgroups with any given minimal number of generators.

Example 2.1. The group $\mathrm{GL}(n,\mathbb{Z})$ is bounded. It follows from the following Theorem of Minkowski [Se16, Section 9.1]):

Theorem 2.2. *(Minkowski, 1887). If an element* $a \in \mathrm{GL}(n,\mathbb{Z})$ *is periodic, and* $a = \mathbf{1} \mod (m)$ *for* $m \geq 3$, *then* $a = 1$.

It follows that every finite subgroup $H \subset \mathrm{GL}(n,\mathbb{Z})$ embeds into $\mathrm{GL}(n,\mathbb{Z}/3\mathbb{Z})$, (there are much more precise bounds, [Se06, Theorem1.1]). Since every finite subgroup of $\mathrm{GL}(n,\mathbb{Q})$ is conjugate to a subgroup of $\mathrm{GL}(n,\mathbb{Z})$ ([Se06, Lecture 1]), the group $\mathrm{GL}(n,\mathbb{Q})$ is bounded as well.

Example 2.3. The multiplicative group $\mathbb{C}^*$ of $\mathbb{C}$ is commutative, very Jordan but not bounded. The same is valid for the group of translations of a complex torus of positive dimension.

Example 2.4. From Theorem 1.2 it follows that the group $\mathrm{GL}(n,k)$ is strongly Jordan for every field k of characteristic zero. Moreover, every linear algebraic group over k is strongly Jordan. On the other hand, $\mathrm{GL}(n,k)$ is obviously not very Jordan if $n \geq 2$.

The following precise values of Jordan constants for groups $\mathrm{GL}(n,\mathbb{C})$ were found by M.J. Collins.

Theorem 2.5. *(*[Col, Theorems **A** and **B**]*). For the Jordan constants of groups* $\mathrm{GL}(n,\mathbb{C})$ *the following relations hold.*

(1) $J_{\mathrm{GL}_n} = (n+1)!$ *if* $n \geq 71$ *or* $n = 63, 65, 67, 69.$

(2) $J_{\mathrm{GL}_n} = 60^r \cdot r!$ *if* $20 \leq n \leq 62$ *or* $n = 64, 66, 68, 70;$

The information on values of Jordan constants for groups $\mathrm{GL}(n,\mathbb{C})$, $n < 20$, is given in extensive tables provided in the same paper.

Example 2.6. We will use below analogues of the Heisenberg groups that were used by D. Mumford [Mum66]. Let

- $\mathbf{K}$ be a finite commutative group of order $N > 1$;
- $\mu_N \subset \mathbb{C}^*$ be the multiplicative group of Nth roots of unity;
- $\hat{\mathbf{K}} = \mathrm{Hom}(\mathbf{K}, \mu_N)$ - the dual of $\mathbf{K}$.

The Mumford's Theta group $\mathfrak{G}_{\mathbf{K}}$ for $\mathbf{K}$ is a group of matrices of the type $\begin{pmatrix} 1 & \alpha & \gamma \\ 0 & 1 & \beta \\ 0 & 0 & 1 \end{pmatrix}$

where $\alpha \in \hat{\mathbf{K}}$, $\gamma \in \mathbb{C}^*$, and $\beta \in \mathbf{K}$. The product of $\alpha \in \hat{\mathbf{K}}$ and $\beta \in \mathbf{K}$ is defined by a certain natural non-degenerate alternating bilinear form $e_{\mathbf{K}}$ on $\mathbf{H}_{\mathbf{K}} = \mathbf{K} \times \hat{\mathbf{K}}$ with values in $\mathbb{C}^*$ [Zar14, p. 302]. This group may be included into a short exact sequence

$$1 \to \mathbb{C}^* \to \mathfrak{G}_{\mathbf{K}} \to \mathbf{H}_{\mathbf{K}} \to 1$$

where the image of $\mathbb{C}^*$ is the center of $\mathfrak{G}_{\mathbf{K}}$.

Properties of $\mathfrak{G}_{\mathbf{K}}$ [Zar14, p. 302] imply that it is a *theta group* attached to the *nondegenerate symplectic pair* $(\mathbf{H}_{\mathbf{K}}, e_{\mathbf{K}})$ in a sense of Chapter 3 below. By Theorem 7.17 below, $\mathfrak{G}_{\mathbf{K}}$ is Jordan and

$$J_{\mathfrak{G}_{\mathbf{K}}} = \sqrt{\#(\mathbf{H}_{\mathbf{K}})} = N = \#(\mathbf{K}).$$

In particular, if $K = \mathbb{Z}/N\mathbb{Z}$ then $J_{\mathfrak{G}_{\mathbb{Z}/N\mathbb{Z}}} = N$.

Example 2.7. The example of a non-Jordan group is given by $\mathrm{SL}(2, \overline{\mathbb{F}}_p)$ where $\overline{\mathbb{F}}_p$ is the algebraic closure of a field $\mathbb{F}_p$ with p elements.

Indeed, if $q = p^n \geq 4$, then $\mathrm{SL}(2, \mathbb{F}_q) \subset \mathrm{SL}(2, \overline{\mathbb{F}_p})$ (Here $\mathbb{F}_q$ is the q-element field.) Group $\mathrm{SL}(2, \mathbb{F}_q)$ is noncommutative, finite, and has order $(q^2-1)q$. Every normal subgroup $C \subsetneq \mathrm{SL}(2, \mathbb{F}_q)$ consists of one or two scalars. thus the indices

$$[\mathrm{SL}(2, \mathbb{F}_q) : C] = (q^2-1)q/2 \text{ or } (q^2-1)q$$

are unbounded when n tends to infinity. Hence $\mathrm{SL}(2, \overline{\mathbb{F}_p})$ is not Jordan.

Remark 2.8. An analogue of the theorem of Jordan holds for matrix groups over fields k of prime characteristic p if one considers only finite subgroups, whose order is prime to p. On the other hand, there are generalizations of the theorem of Jordan (Brauer-Feit [BF], Larsen-Pink [LP]) that deal with arbitrary finite subgroups and take into account the order of their Sylow p-subgroups. Their results led to the following definition [Hu, Definition 1.6] (that will be used in Remark 4.3, part 4 below).

A group G is called *p-Jordan* if there exist positive integers J and e such that every finite subgroup B of G contains an abelian p'-subgroup A that is normal in B and such that the index $[B : A] \leq |B_p|^e J$. Here $|B_p|$ is the order of a Sylow p-subgroup of B.

Remark 2.9. Let G be a group. Assume that it may be included into the following exact sequence of groups
$0 \to H \to G \to F \to 0$.

(1) If F is bounded and H is bounded then G is bounded ;
(2) If H is very Jordan and F is bounded then G is very Jordan;
(3) If F is bounded then G is Jordan if and only if H is Jordan ([Po11, Lemma 2.11]);
(4) If H is bounded and F is strongly Jordan then G is Jordan ([PS14, Lemma 2.8]).
(5) G being Jordan **does not** imply that F is Jordan ;
(6) F and H being Jordan **does not** imply that G is Jordan.

We will need the following modification of [BZ20, Lemma 5.3].

Lemma 2.10. *Consider a short exact sequence of connected complex Lie groups:*
$$0 \to A \xrightarrow{i} B \xrightarrow{j} D \to 0.$$
Here i is a closed holomorphic embedding and j is surjective holomorphic. Assume that D is a complex torus and A is isomorphic as a complex Lie group either to $(\mathbb{C}^+)^n$ or to $\mathbb{C}^*$. Then B is commutative.

Proof. The proof of this lemma coincides verbatim with the proof of [BZ20, Lemma 5.3] where the case $n = 1$ is treated.

Step 1. First, let us prove that A is a central subgroup in B. Take any element $b \in B$. Define a holomorphic map $\phi_b : A \to A$, $\phi_b(a) = bab^{-1} \in A$ for an element $a \in A$. Since it depends holomorphically on b, we have a holomorphic map $\xi : B \to \mathrm{Aut}(A), b \to \phi_b$.

Since A is commutative, for every $c \in A$ we have $\phi_{bc} = \phi_b$. Thus there is a well defined map ψ fitting into the following commutative diagram
$$\begin{tikzcd} & B \arrow[dl, "j"'] \arrow[dr, "\xi"] & \\ D \arrow[rr, "\psi"] & & \mathrm{Aut}(A) \end{tikzcd},$$

The map $\psi = \xi \circ j^{-1}$ is defined at every point of D. It is holomorphic (see, for example, [OV], $ 3).

Since D is a complex torus, and $\mathrm{Aut}(A)$ is either $\mathrm{GL}(n, \mathbb{C})$ (if $A = (\mathbb{C}^+)^n$) or consists of two elements, id and $a \mapsto a^{-1}$ (if $A = \mathbb{C}^*$), we have $\psi(D) = \{id\}$. It follows that A is a central subgroup of B.

Step 2. Let us now show that B is commutative. Consider a holomorphic map $com : B \times B \to A$ defined by $com(x, y) = xyx^{-1}y^{-1}$. Since A is a central subgroup of B, similarly to **Step 1** we get a holomorphic map $D \times D \to A$. It has to be constant since D is a torus and A is either $(\mathbb{C}^+)^n$ or $\mathbb{C}^*$. $\square$

3. Complex manifolds

This section contains preliminaries, Notation, and Assumptions that will be used further on.

By *(projective) variety* we mean an algebraic variety that is Zariski closed subset of a projective space $\mathbb{P}^n$.

Let $U \subset \mathbb{C}^n$ be an open subset. An analytic subset of U is a closed subset $X \subset U$ such that for any $x \in X$ there exists an open neighborhood $x \in V \subset U$ and holomorpphic functions $f_1, \dots, f_k : V \to \mathbb{C}$ such that $X \cap V = \{f_1 = 0, \dots, f_k = 0\}$ ([H, Definition 1.1.23]).

A complex space consist of a Hausdorff topological space X and a sheaf of rings $\mathscr{O}_X$ such that locally $(X, \mathscr{O}_X)$ is isomorphic to an analytic subset $Z \subset U \subset \mathbb{C}^n$ endowed with the sheaf $\mathscr{O}_U/\mathscr{I}$, where $\mathscr{I}$ is a sheaf of holomorphic functions with $Z = Z(\mathscr{I})$([H, Definition 6.2.8]). By the Chow Theorem any closed analytic subset of complex projective space is a projective variety. ([GR, Chapter V, D, 7],[Se56, Proposition13]).

A complex manifold is a complex space which is locally modeled on $Z = U \subset \mathbb{C}^n$ and $\mathscr{I} = \{0\}$ ([H, Example 6.2.9]). In particular, it is smooth.

We will use the following

Notation and Assumptions.

(1) $\mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C}$ stand for the ring of integers and fields of rational, real, and complex numbers, respectively.
(2) In what follows, the ground field is $\mathbb{C}$ if not indicated otherwise.
(3) $\mathrm{Aut}(X)$ stands for the group of all biholomorphic (or biregular, if X is projective) automorphisms of a complex manifold X. The group $\mathrm{Aut}(X)$ of any connected complex compact manifold X carries a natural structure of a complex (not necessarily connected) Lie group such that the action map $\mathrm{Aut}(X) \times X \to X$ is holomorphic (Theorem of Bochner-Montgomery, [BM]) .
(4) $\mathrm{Aut}_0(X)$ stands for the connected identity component of $\mathrm{Aut}(X)$ (as a complex Lie group).
(5) If $p : X \to Y$ is a morphism of complex manifolds, then $\mathrm{Aut}(X)_p$ is the subgroup of all $f \in \mathrm{Aut}(X)$ such that $p \circ f = p$.

(6) For $f \in \mathrm{Aut}(X)$ we denote by $\mathrm{Fix}(f)$ the set of all fixed points of f.
(7) $\cong$ stands for " isomorphic groups" (or isomorphic complex Lie groups if the groups involved are the ones), and also for isomorphic line bundles; $\sim$ for biholomorphically isomorphic complex manifolds; $\approx$ for bimeromorphic or birational complex manifolds.
(8) *id* stands for identity automorphism, **I** stands for identity matrix.
(9) We say that a subset U of a complex manifold X is analytical Zariski open if $U = X \setminus Z$, where Z is an analytic subspace of X.
(10) $\mathbb{P}^n_{(x_0:\ldots:x_n)}$ stands for $\mathbb{P}^n$ with homogeneous coordinates $(x_0 : \ldots : x_n)$.
(11) $\mathbb{C}_z, \overline{\mathbb{C}}_z \sim \mathbb{P}^1$ is the complex line (extended complex line, respectively) with coordinate z.
(12) $\mathbb{C}^+$ and $\mathbb{C}^*$ stand for complex Lie groups $\mathbb{C}$ and $\mathbb{C}^*$ with additive and multiplicative group structure, respectively.
(13) $\dim(X),\ \dim_a(X)$ are the complex and algebraic dimensions of a compact complex manifold X, respectively.
(14) By **pr** we denote the natural projection $Y \times \mathbb{P}^1 \to Y$.
(15) For an element $\psi \in \mathrm{PSL}(2, \mathbb{C})$ we denote $\mathrm{TD}(\psi)$ the number

$$\mathrm{TD}(\psi) := \frac{\mathrm{tr}(F(t))^2}{\det(F)},$$

where $F \in \mathrm{GL}(2, \mathbb{C})$ is any representative of ψ, and tr and det stand for trace and determinant, respectively.
(16) A rational curve in X is the image of a non-constant holomorphic map $\mathbb{P}^1 \to X$.
(17) $\mathbf{1}$ or $\mathbf{1}_Y$ is a trivial holomorphic line bundle $Y \times \mathbb{C}$ over a manifold Y;
(18) For a rank 2 holomorphic vector bundle $\mathscr{E}$ over Y we write $\mathbb{P}(\mathscr{E})$ for the $\mathbb{P}^1$−bundle that is the projectivization of $\mathscr{E}$.
(19) If $\mathscr{L}$ is a holomorphic line bundle over Y and $\mathscr{E} = \mathscr{L} \oplus \mathbf{1}_Y$ then we call $\overline{\mathscr{L}} = \mathbb{P}(\mathscr{E})$ *the closure of (total body) of* $\mathscr{L}$.
(20) $\mathbb{C}(Z)$ stands for the field of rational functions on an irreducible complex projective variety Z.
(21) Let X, Y be two compact connected irreducible reduced analytic complex spaces. A meromorphic map $f : X \to Y$ relates to every point $x \in X$ a subset $f(x) \subset Y$ (the image of x) such that the following conditions are met
   (a) The graph $G_f := \{(x, y)\ |y \in f(x) \subset X \times Y\}$ is a connected irreducible complex analytic subspace of $X \times Y$ with $\dim(G_f) = \dim(X)$;

(b) There exist an open dense subset $X_0 \subset X$ such, that $f(x)$ consists of one point for every $x \in X$.

(22) The *general* point $x \in X$ is a point from an (analytical) Zariski open dense subset of X. The general fiber of a holomorphic map $f : X \to Y$ is the preimage $f^{-1}(y)$ of a general point $y \in Y$.

Definition 3.1. Following [Fu81], we define a covering family of rational curves for a compact complex connected manifold X as a pair of morphisms $p : Z \to T$ and $\psi : Z \to X$ of compact irreducible complex spaces if the following conditions are satisfied:

(1) ψ is surjective;
(2) there is a dense analytical Zariski open subset $U \subset T$ such that for $t \in U$, the fiber $Z_t = g^{-1}(t) \sim \mathbb{P}^1$ and $\dim \psi(Z_t) = 1$.

Manifolds X admitting a covering family with this property are called **uniruled** ([Fu81, Definition 2.1, Lemma 2.2]).

Remark 3.2. The Kodaira dimension $\kappa(X) = -\infty$ if X is unuruled compact complex manifold ([Fu81, Remark, p. 691],[Kol, Corollary IV.1.11]) In low dimensions the converse is true:

Theorem 3.3. *(*[Mi88] *for projective manifolds,* [HP] *for non-projective ones). Let X be a compact Kähler manifold of dimension at most 3. Then X is uniruled if and only if $\kappa(X) = -\infty$.*

Remark 3.4. Fujiki Theorems. It was proven by A. Fujiki for a compact connected complex manifold Y that

(1) If Y contains no rational curves then every meromorphic map $f : X \to Y$ is holomorphic for any complex manifold X ([Fu80]).
(2) $\mathrm{Aut}_0(Y)$ is isomorphic to a complex torus $\mathrm{Tor}(Y)$ (unless it is trivial) if Y is Kähler and either non-uniruled [Fu78, Proposition 5.10]) or has non-negative Kodaira dimension [Fu78, Corollary 5.11]).

The next statement ([BZ20, Proposition 1.4] is similar to Lemma 3.1 of [Kim].

Proposition 3.5. *Let X be a connected complex compact Kähler manifold and $F = \mathrm{Aut}(X)/\mathrm{Aut}_0(X)$. Then the group F is bounded.*

Remark 3.6. Lemma 3.1 of Jin Hong Kim, [Kim], states the following.

Let X be a normal compact Kähler variety. Then there exists a positive integer l, depending only on X, such that for any finite subgroup G of $Aut(X)$ acting biholomorphically and meromorphically on X we have $[G : G \cap \mathrm{Aut}_0(X)] \leq l$.

We cannot use straightforwardly this Lemma, since it is not clear why every finite subgroup of $\mathrm{Aut}(X)/\mathrm{Aut}_0(X)$ should be isomorphic to $G/(G \cap \mathrm{Aut}_0(X))$ for some finite subgroup G of $\mathrm{Aut}(X)$.

Corollary 3.7. *Let X be a compact connected complex Kähler manifold either non-uniruled or with Kodaira dimension $\varkappa(X) \geq 0$. Then* $\mathrm{Aut}(X)$ *is very Jordan.*

Proof. In view of Proposition 3.5 it is sufficient to prove that $\mathrm{Aut}_0(X)$ is commutative. But this assertion follows from [Fu78, Proposition 5.10] if X is non-uniruled and [Fu78, Corollary 5.11] if $\varkappa(X) \geq 0$ (see Remark 3.4). □

In general, let Z be a compact complex connected Kähler manifold. The analogue of the Chevalley decomposition for algebraic groups is valid for complex Lie group $\mathrm{Aut}_0(Z)$:

$$1 \to L(Z) \to \mathrm{Aut}_0(Z) \to \mathrm{Tor}(Z) \to 1 \tag{2}$$

where $L(Z)$ is bimeromorphically isomorphic to a linear group, and $\mathrm{Tor}(Z)$ is a complex torus ([Fu78, Theorem 5.5], [Lie, Theorem 3.12], [CP, Theorem 3.28]).

Remark 3.8. If $L(Z)$ in Equation (2) is not trivial, Z contains a rational curve. Moreover, according to [Fu78, Proposition 5.10], Z is bimeromorphic to a fiber space whose general fiber is $\mathbb{P}^1$, i.e X is *uniruled.*

Chapter 2. Rational bundles

In this chapter, in Section 4, we want to persuade the reader that uniruled manifolds (in particular, $\mathbb{P}^1-$bundles) are of special interest from the Jordan properties point of view. To this end we give a very brief and certainly non-complete overview of known facts in this field. In Section 5 we provide general properties of maps of manifolds endowed with fibration over a non-uniruled base with the general fiber $\mathbb{P}^1$. In Section 6 we deal with projective non-trivial conic bundles.

4. Uniruled vs non-uniruled: Jordan properties of groups $\mathrm{Aut}(X), \mathrm{Bim}(X)$, and $\mathrm{Bir}(X)$.

In order to demonstrate the special role of uniruled manifolds from Jordan Properties point of vew, we present samples of results on Jordan Properties of $\mathrm{Aut}(X)$ and $\mathrm{Bim}(X)$ for various types of compact complex manifolds X.

The group $\mathrm{Aut}(X)$ is known to be Jordan if

- X is projective ([MZ]);
- X is a compact complex Kähler manifold ([Kim]):
- X is a compact complex space in Fujiki's Class $\mathscr{C}$ ([MPZ], also [PS19] for Moishezon threefolds);

Remark 4.1. For the group $\mathrm{Aut}(X)$ " Jordan" implies "strongly Jordan" because:

For every compact complex manifold X there is a constant $C = C(X)$ such that every finite subgroup $G \subset \mathrm{Aut}(X)$ may be generated by at most C elements.

The proof of this fact one can find in [MR13, Theorem 1.3]. It is based on the same property for elementary abelian p-groups that was proved for much wider class of topological spaces in [MS], and the group-theoretic arguments (that, according to the author, were explained to him by E. Khukhro and A. Jaikin). Thus the fact is valid in much more general situation.

Moreover, the connected identity component $\mathrm{Aut}_0(X)$ of $\mathrm{Aut}(X)$ is Jordan for every compact complex space X ([Po18, Theorems 5 and 7]). An example of $X = E$, where E is an elliptic curve, shows that $\mathrm{Aut}(X)$ may be Jordan but not bounded. The classification of complex compact surfaces with bounded automorphisms group was done in [PS21].

As follows from Corollary 3.7, the group $\mathrm{Aut}(X)$ is very Jordan for any compact connected complex Kähler non-uniruled manifold X. For uniruled manifolds the situation changes: if $X = E \times \mathbb{P}^1$ then $\mathrm{Aut}(X) \cong \mathrm{PSL}(2, \mathbb{C}) \times \mathrm{Aut}(E)$ is neither bounded nor very Jordan.

The groups $\mathrm{Bir}(X)$ and $\mathrm{Bim}(X)$ of birational and bimeromorphic transformations, respectively, are more complicated. Low-dimensional cases are well understood. Consider the following

LIST

(1) E - an elliptic curve;
(2) A_n - an abelian variety of dimension n;
(3) S_b - a bielliptic surface;
(4) S_{K1} - a surface of Kodaira dimension 1;
(5) S_K - a Kodaira surface (it is not a Kähler surface).

Here are examples of results for low-dimensional cases.

- If X is a complex compact surface with non-negative Kodaira dimension then $\mathrm{Bir}(X)$ is bounded unless it appears in the **LIST** [PS20, Theorem1.1].
- If X is a projective surface then $\mathrm{Bir}(X)$ is Jordan if X is not birational to a product of an elliptic curve and $\mathbb{P}^1$, ([Po11]). (The case of $X = \mathbb{P}^2$ was done earlier by J.-P. Serre, [Se09]).
- If X is birational to a product of an elliptic curve and $\mathbb{P}^1$ then $\mathrm{Bir}(X)$ is not Jordan ([Zar14]).
- If X is a projective threefold then $\mathrm{Bir}(X)$ is not Jordan if and only if X is birational to a direct product $E \times \mathbb{P}^2$ or $S \times \mathbb{P}^1$, where a surface S appears in the **LIST** [PS18].
- The group $\mathrm{Bim}(X)$ is Jordan for any non-uniruled compact complex connected Kähler manifold of dimension 3 ([PS21b],[Gol]).
- If X is a non-algebraic uniruled compact Kähler threefold with non-Jordan group $\mathrm{Bim}(X)$ then X is bimeromorphic to $\mathbb{P}(\mathscr{E})$ for a holomorphic rank 2 vector bundle $\mathscr{E}$ on a two-dimensional

complex torus S with $a(S) = 1$. Moreover, if $a(X) = 2$ then $X \cong S \times \mathbb{P}^1$([PS20b]).

The following Theorem for complex projective varieties was proved by Yu. Prokhorov and C. Shramov (for $\dim X > 3$, assuming a so called *BAB-conjecture* named after A. Borisov, L. Borisov and V. Alexeev), and C. Birkar (who proved this conjecture), ([PS14, Theorem 1.8], [Bi]).

Theorem 4.2. *Let X be a projective variety of dimension n. Then the following hold.*

(i) *The group* $\mathrm{Bir}(X)$ *has bounded finite subgroups provided that X is non-uniruled and has irregularity $q(X) = 0$.*
(ii) *The group* $\mathrm{Bir}(X)$ *is Jordan provided that X is non-uniruled.*
(iii) *The group* $\mathrm{Bir}(X)$ *is Jordan provided that X has irregularity* $q(X) = 0$.

Here $q(X) = \dim_{\mathbb{C}} H^1(X, \mathscr{O}_X)$ is the irregularity of X. In particurar, the Cremona group Cr_n of any rank n is Jordan ([PS16])). The exact value $J_{\mathrm{Cr}_2} = 7200$ (E. Yasinsky, [Ya]). The Jordan constant for $\mathrm{Bir}(X)$ for a rationally connected threefold X may be found in [PS17].

Let us sketch the proof of items (i) and (ii) of Theorem 4.2.

First, using the MMP(Minimal Model Program) the authors reduce the problem to consideration of the group $\mathrm{PAut}(X_m)$, where X_m is a special (relatively minimal) model of X and $\mathrm{PAut}(Z)$ stands for the group of birational selfmaps of a variety Z that are isomorphisms in codimension 1. This means that $f \in \mathrm{PAut}(X_m)$ moves a divisor to a divisor and induces an automorphism $f_* = \psi(f)$ of the finitely generated abelian group $NS^W(X_m) = \mathrm{C}l(X_m)/\mathrm{C}l^0(X_m)$, were $\mathrm{C}l(X_m)$ stands for group of Weil divisors on X_m modulo linear equivalence, and $\mathrm{C}l^0(X_m)$ consists of those ones that are algebraically equivalent to zero.

Thus there is a short exact sequence

$$0 \longrightarrow G_i \xrightarrow{i} G \xrightarrow{\psi} \mathrm{Aut}(NS^W(X)), \tag{3}$$

where $G_i = \ker(\psi)$ acts on each of equivalence classes of $\mathrm{C}l(X_m)$. Since $NS^W(X_m)$ is finitely generated abelian group, $\mathrm{Aut}(NS(X))$ is bounded.

Take a very ample divisor L and denote by $\mathrm{C}l_L(X_m)$ the equivalence class containing L. It is an abelian variety of dimension $q(X_m) = q(X)$.

Let G_L be a the kernel of action of G_i on $\mathrm{C}l_L(X_m)$. Then there is a short exact sequence

$$0 \longrightarrow G_L \longrightarrow G_i \longrightarrow G_{ab} \tag{4}$$

where $G_{ab} \subset \mathrm{Aut}(\mathrm{C}l_L(X_m))$ is a subgroup of automorphisms (as a variety, but not as a group) of abelian variety $\mathrm{C}l_L(X_m)$. The group $\mathrm{Aut}(\mathrm{C}l_L(X_m))$ is strongly Jordan. Let V be a linear space of sections of L and $\mathbb{P}(V)$ its projectivization. Let F_L be the subgroup of those linear transformations of the projective space $\mathbb{P}(V)$ that preserve $X_m \subset \mathbb{P}(V)$.

Since F_L is a linear group and X (and X_m) are **non-uniruled**, F_L has to be finite (see Remark 3.8). Thus $G_L \subset F_L$ is finite.

Therefore

- If $q(X) = 0$, then G_{ab} is trivial and $\mathrm{Bir}(X_m)$ is bounded (see Remark 2.9,(1)).
- If $q(x) > 0$ then G_i is Jordan (see Remark 2.9,(4)) and $\mathrm{Bir}(X)_m$ is Jordan (see Remark 2.9,(3)).

Remark 4.3.
1. One can ask similar questions about the group $\mathrm{Diff}(M)$ of all diffeomorphisms of a smooth manifold M. There was the Conjecture of E. Ghys (1997):

If M is a compact smooth manifold, then $\mathrm{Diff}(M)$ *is Jordan.*

It was answered negatively by B. Csikós, L. Pyber, E.Szabó in [CPS], whose approach was based on an algebraic geometry construction from [Zar14] (see also Chapter 3 below).

In works of J. Winkelmann [W] and V. Popov [Po15] it was proven that there is a connected non-compact Riemann surface M such that $\mathrm{Aut}(M)$ contains an isomorphic copy of every finitely presented (in particular, every finite) group G. In particular, $\mathrm{Diff}(M)$ is not Jordan. B. Zimmerman [Zim] proved that if M is compact and $\dim(M) \leq 3$ then $\mathrm{Diff}(M)$ is Jordan. The Jordan properties of $\mathrm{Diff}(M)$ were deeply studied by I. Mundet i Riera ([MR10], [MR15], [MR16], [MR17], [MR17b], [MR18]). It was proven there, in particular, that $\mathrm{Diff}(M)$ is Jordan if M is one of the following:

(1) open acyclic manifolds,

(2) compact manifolds (possibly with boundary) with nonzero Euler characteristic,

(3) homology spheres.

2. The question on Jordan properties for algebraic groups over various fields was considered in [Po18],[MZ], and [ShVb] (see also [BZ17]).

3. Jordan properties of $\mathrm{Aut}(X)$ and $\mathrm{Bir}(X)$ for open subsets of certain projective $\mathbb{P}^1-$bundles were considered in [BZ15], [BZ18].

4. In the case of algebraic varieties X over algebraically closed fields of prime characteristic p one should not expect the Jordan properties to hold (see Example 2.7). However, there are analogues of several important results over $\mathbb{C}$ that deal instead with p-Jordan properties (see Remark 2.8) of $\mathrm{Aut}(X)$ and $\mathrm{Bir}(X)$ ([Hu], [CS], [Kuz]). On the other hand, it is known that the Cremona group over a *finite* field is Jordan [PS21c].

For compact complex manifolds, roughly speaking, from Jordan properties point of view the uniruled varieties are the worst and may be divided in several categories.

First, manifolds X that are rationally connected (or with $q(X) = 0$). For projective varieties, thanks to Theorem 4.2, $\mathrm{Bir}(X)$ is Jordan.

Second, manifolds that are fibered over a non-uniruled base Y with rationally connected fibers, with $q(X) \neq 0$, that are not bimeromorphic (birational) to a direct product $Y \times \mathbb{P}^N$. In many special cases $\mathrm{Bim}(X)$ (or $\mathrm{Bir}(X)$) is Jordan. Moreover, $\mathrm{Aut}(X)$ appears often to be very Jordan. We discuss some of these special cases in Chapter 4.

Third, X is isomorphic (bimeromorphic) to the direct product $Y \times \mathbb{P}^N$. If Y is a torus, and $a(Y) > 0$ then $\mathrm{Bir}(Y)$ is not Jordan. This case is subject of Chapter 3.

5. Rational bundles

In this section we provide some useful about $\mathbb{P}^1-$ bundles and their morphisms. We start with slightly more general construction.

Definition 5.1. We say that a triple (X, p, Y) is a *rational bundle* over Y if

- X, Y are compact connected complex manifolds endowed with a holomorphic surjective map $p : X \to Y$;
- for a general $y \in Y$ the fiber $p^*(y)$ is reduced and isomorphic to $\mathbb{P}^1$ (where *general* means outside a proper analytic subset of Y, see **Notation and Assumptions** (20)).
- If dim $P_y = 1$, where $P_y := p^{-1}(y)$, for every $y \in Y$ we call (X, p, Y) an *equidimensional rational bundle* over Y.

If for an open subset $U \subset Y$ and for every $y \in U$ the fiber $P_y \sim \mathbb{P}^1$ then, by a theorem of W. Fischer and H. Grauert ([FG]), $p^{-1}(U) \subset X$ is a holomorphically locally trivial fiber bundle over U. If $U = X$ then triple (X, p, Y) is a $\mathbb{P}^1$-bundle over Y.

If (X, p, Y) is a rational bundle over a non-uniruled Kähler manifold Y then $p : X \to Y$ is, by definition, a maximal rational connected (MRC) fibration of X (see [C92, Theorem 2.3, Remark 2.8] and [Kol, IV.5] for the definition and discussion).

Bimeromorphic self-maps preserve the MRC-fibration. This is a well-known fact, but we have not found a suitable reference for the proof of this fact in complex analytic case. We provide it here. In case when the Kodaira dimension $\kappa(Y) \geq 0$, the desired result follows from [Mob, Theorem 1.1.5]. For automorphisms the detailed exposition may be found in [A, Section 2.4].

Lemma 5.2. *Let X, Y, Z be three complex compact connected manifolds, $p : X \to Y$ and $q : X \to Z$ be surjective holomorphic maps. Assume that*

- *Z is non-uniruled ;*
- *there is an analytical Zariski open dense subset $U \subset Y$, such that $P_u = p^{-1}(u) \sim \mathbb{P}^1$ for every $u \in U$.*

Then there is a meromorphic map $\tau : Y \dashrightarrow Z$ such that $\tau \circ p = q$, i.e., the following diagram commutes:

$$\begin{tikzcd} & X \arrow[dl, "p"'] \arrow[dr, "q"] & \\ Y \arrow[rr, dashed, "\tau"] & & Z \end{tikzcd},$$

Proof. Let $\Phi : X \to Y \times Z$ be defined by $\Phi(x) = (p(x), q(x))$. The image $T = \Phi(X)$ is an irreducible compact analytic subspace of $Y \times Z$ (see, e.g. [Nar, Theorem 2, Chapter 7]). We denote by pr_Y and pr_Z the natural projections of T on the first and second factor, respectively. Both projections are evidently surjective. The set

$$T_1 = \{(y, z) \in T \ \mid \dim \Phi^{-1}(y, z) > 0\}$$

is an analytic subset of $T \subset Y \times Z$ ([Re], [Fi, Theorem 3.6, p.137]). Its projections $T_Y = pr_Y(T_1) \subset Y$, and $T_Z = pr_Z(T_1) \subset Z$ to the first and the second factor are analytic subsets of Y and Z, respectively, ([Re], [Nar, Theorem 2, Chapter VII]).

If $T_Y \neq Y$ then $V := (Y \setminus T_Y) \cap U$ is an analytical Zariski open dense subset of Y. For each $y \in V$ we have $p^{-1}(y) \sim \mathbb{P}^1$ and $\dim q(p^{-1}(y)) > 0$. Thus the pair $p : X \to Y,\ q : X \to Z$ would provide a covering family for Z, which is impossible, since Z is not uniruled. Thus $T_Y = Y$.

Take $u \in U$. Since $T_Y = Y$ there is $z \in Z$ such that $(u, z) \in T$ and $\dim \Phi^{-1}(u, z) \geq 1$. Moreover,

$$\Phi^{-1}(u, z) = \{x : p(x) = u, q(x) = z\} \subset P_u \subset X.$$

Since $P_u \sim \mathbb{P}^1$ and $\dim \Phi^{-1}(u, z) \geq 1$, we have $P_u = \Phi^{-1}(u, z)$. Hence, $q\mid_{P_u} = z$ for every $u \in U$ and some $z \in Z$ and there is only one $z \in Z$ such that $(u, z) \in T$. Thus,

(1) T is an irreducible connected subset of $Y \times Z$;
(2) $\dim T = \dim Y$;
(3) for every $u \in U$ there is only one $z \in Z$ such that $(u, z) \in T$.

It follows that T is the graph of a meromorphic map that we denote as τ. $\square$

Remark 5.3. The fact that q contracts every fiber of p over an analytical Zariski open non-empty subset of Y is proven in [GS, Proposition 6.2].

Lemma 5.4. *Let (X, p_X, Y) and (W, p_W, Y) be two rational bundles over a non-uniruled (compact connected) manifold Y. Let $f : X \to W$ be a surjective meromorphic map.*

Then there exists a meromorphic map $\tau(f) : Y \to Y$ that may be included into the following commutative diagram.

$$\begin{array}{ccc} X & \xrightarrow{f} & W \\ {\scriptstyle p_X}\downarrow & & {\scriptstyle p_W}\downarrow \\ Y & \xrightarrow{\tau(f)} & Y \end{array}. \tag{5}$$

In addition, if f is holomorphic, so is $\tau(f)$.

Proof. Let $a : \tilde{X} \to X$ be such a modification of X that the following diagram is commutative

$$\begin{array}{ccc} & \tilde{X} & \\ \swarrow_a & & \searrow^b \\ X & \overset{f}{\dashrightarrow} & W \end{array},$$

where $b : \tilde{X} \to Z$ is a holomorphic map (it always exists, [Pe, Theorem 1.9]).

Consider the holomorphic maps $\tilde{p}_X := p_X \circ a : \tilde{X} \to Y$ and $\tilde{f} := p_W \circ b : \tilde{X} \to Y$. We apply Lemma 5.2 to $\tilde{X}, Y = Z$ and $\tilde{p}_X : \tilde{X} \to Y$, $\tilde{f} := \tilde{X} \to Y$, and obtain the needed map $\tau(f) \in \mathrm{Bim}(Y)$ that may be included into the following commutative diagram

$$\begin{array}{ccc} & \tilde{X} & \\ \swarrow_a & & \searrow^b \\ X & \overset{f}{\to} & W \\ \downarrow_{pr_X} & & \downarrow_{pr_W} \\ Y & \overset{\tau(f)}{\to} & Y \end{array}.$$

If f is holomorphic, one may take $\tilde{X} = X$ and $U = Y$ (in the notation of Lemma 5.2). Thus, $\tau(f)$ will be defined at every point of Y. $\square$

Corollary 5.5. *For a rational bundle (X, p, Y) over a non-uniruled (complex connected compact) manifold Y there are natural group homomorphisms $\tau : \mathrm{Aut}(X) \to \mathrm{Aut}(Y)$ and $\tilde{\tau} : \mathrm{Bim}(X) \to \mathrm{Bim}(Y)$ such that*

$$p \circ f = \tau(f) \circ p, \ p \circ f = \tilde{\tau}(f) \circ p$$

for every $f \in \mathrm{Aut}(X)$ or $f \in \mathrm{Bim}(X)$, respectively.

Remark 5.6. If Y is Kähler non-uniruled, then the restriction group homomorphism

$$\tau \mid_{\mathrm{Aut}_0(X)} : \mathrm{Aut}_0(X) \to \tau(\mathrm{Aut}_0(X))$$

is a holomorphic homomorphism of complex Lie groups and $\tau(\mathrm{Aut}_0(X))$ is a closed complex Lie subgroup of $\mathrm{Aut}(Y)$ (A. Fujiki, [Fu78, Lemma 2.4, 3, Theorem 5.5 and Lemma 4.6]).

Further on we will use heavily the following classical theorems.

Theorem 5.7. *[Remmert-Stein Theorem](see, e.g.,* [Nar, Theorem of Remmert -Stein, ChapterVII]*) Let X be a complex space and Y an analytic subset of X, A an analytic subset of $X \setminus Y$. Suppose that there is an integer $p > 0$ such that* $\dim Y \leq p-1$, *while* $\dim_a A \geq p$ *for any* $a \in A$. *(*$\dim Y \leq -1$ *means that* $Y = \emptyset$.*) Then the closure $\overline{A}$ of A in X is an analytic set in X.*

Theorem 5.8. *[Second Riemann removable singularity theorem] (*[Fi, Chapter 2, Appendix]*) Assume that X is a complex manifold and $A \subset X$ is an analytic subset such that*

$$\operatorname{codim}_x A \geq 2 \text{ for every } x \in X.$$

Then any holomorphic function $f : X \setminus A \to \mathbb{C}$ has a unique holomorphic extension $\tilde{f} : X \to \mathbb{C}$.

Theorem 5.9. *[Levi continuation theorem] (*[Le]*, see also* [Nar, Chapter VII, Theorem 4] *or* [Fi, Section 4.8]*) Let X be a normal complex space and Y an analytic subset of X such that for any $a \in X$ we have* $\dim_a Y \leq \dim_a X - 2$. *Then any meromorphic function on $X \setminus Y$ has an extension to a meromorphic function on X.*

Remark 5.10. It follows from the second Riemann Theorem that a holomorphic map from $f : X \setminus \Sigma \to Z$ where X is a complex manifold, Σ an analytic subset of codimension at least 2, and $Z \subset \mathbb{C}^N$ an affine complex set, may be extended to a holomorphic map $\tilde{f} : X \to Z$.

Indeed, let $z_1, \ldots, z_N$ be coordinates in $\mathbb{C}^N$. The map f consists of N holomorphic functions $z_i(x), i = 1, \ldots, N$ defined on $X \setminus \Sigma$. By Theorem 5.8 the functions z_i may be extended to holomorphic functions $\tilde{z}_i$ defined on X. Since Z is a closed subset of $\mathbb{C}^N$, we have $\tilde{f}(x) = (\tilde{z}_1(y), \ldots, \tilde{z}_N(x)) \in Z$ for every $x \in X$.

This fact is a particular case of the Extension Theorem of A. Andreotti and W. Stoll ([AS69]. Recall the a subset $M \subset X$ of a complex space X is *thin* if in the neighborhood of every point $m \in M$ it is contained in an analytic subset of codimension 1.

Theorem 5.11. *[Andreotti-Stoll Theorem] Let $\tau : A \to Y$ be a holomorphic map of the open subset A of a normal complex space X into a Stein space Y. let $M := X \setminus A$ be a thin set. If M has topological codimension* ≥ 3, *then τ may be extended to a holomorphic map of X into Y.*

We use this fact to prove the next

Lemma 5.12. *Let (X, p, Y) and (Z, q, Y) be two $\mathbb{P}^1-$bundles over a connected complex manifold Y. Let $\Sigma \subset Y$ be an analytic subset of codimension at least 2, $U = Y \setminus \Sigma$, $V_X = p^{-1}(U), V_Z = q^{-1}(U)$. Let $f : X \to Y$ be a meromorphic map such that $q \circ f = p$ and induced map*

$f : V_X \to V_Z$ is an isomorphism. Then $f : X \to Z$ is a biholomorphic isomorphism.

Proof. By construction, for every $u \in U$ the map f induces an isomorphism $f\,|_{P_y}: P_y \to Q_y$, where $P_y = p^{-1}(y), Q_y = q^{-1}(y)$. Consider a point $s \in \Sigma$ and its open neighborhood U_s such that there are isomorphisms $\psi_X : p^{-1}(U_s) \to U_s \times \mathbb{P}^1$, $\psi_Z : q^{-1}(U_s) \to U_s \times \mathbb{P}^1$ compatible with projection maps p and q, respectively. Then for every $y \in U_s \cap U$ we have an element of $\mathrm{PSL}(2,\mathbb{C})$ representing $f\,|_{P_y}: P_y \to Q_y$, which is an automorphism of $\mathbb{P}^1$. Thus we have a holomorphic map $U_s \cap U \to \mathrm{PSL}(2,\mathbb{C})$. Since the last one is an affine set, the map extends to a holomorphic map $U_s \to \mathrm{PSL}(2,\mathbb{C})$. Hence, we have an extension of f to an isomorphism $\tilde{f}_s : p^{-1}(U_s) \to q^{-1}(U_s)$ that coincides with f in $V_X \cap p^{-1}(U_s)$, hence everywhere. □

Lemma 5.13. *Let (X, p, Y) and (Z, q, Y) be two $\mathbb{P}^1-$bundles over a compact connected complex manifold Y with $\dim Y = n$. Let $\Sigma \subset Y$ be an analytic subset of codimension at least 2, $U = Y \setminus \Sigma$, $V_X = p^{-1}(U), V_Z = q^{-1}(U)$. Let $f : V_X \to V_Z$ is a meromorphic map such that $q \circ f = p$. Then there exist a meromorphic map $\tilde{f} : X \to Y$ such that $\tilde{f}\,|_U = f$ and $q \circ \tilde{f} = p$.*

For Kähler manifold Y this Lemma follows from the following general Theorem of Y.-T. Siu ([Siu]).

Theorem 5.14. *[Siu extension Theorem] Let X be a complex manifold, A be a subvariety of codimension ≥ 1 in X, and G be an open subset of X which intersect every branch of A of codimension 1. If M is a compact Kähler manifold, then every meromorphic map f from $(X - A) \cup G$ may be extended to a meromorphic map from X to M.*

At this stage we do not require that Y (and, a fortiori, Z) is Kähler, but we use the fact that X, Z are $\mathbb{P}^1-$bundles.

Proof. (Proof of Lemma 5.13). Consider a fiber product

$$W = X \times_Y Z = \{(x, z) \in X \times Z \mid p_X(x) = p_Z(z)\} \subset X \times Z$$

and its subsets:

$$\Gamma_f = \{(x, z) \in V_X \times V_Z \mid p_X(x) = p_Z(z),\ z \in f(x)\} \subset W,$$

$$\tilde{\Sigma} = \{(x, z) \in X \times Z \mid p_X(x) = p_Z(z) \in \Sigma\} \subset W.$$

By construction $\dim(\tilde{\Sigma}) \leq n$, $\dim \Gamma_f = \dim X = n+1$. Thus, according to the Remmert-Stein Theorem (Theorem 5.7) the closure of $\overline{\Gamma}_f$ of Γ_f in W is an analytic subset in W. Let $U_1 \subset U$ be an open subset such that f is defined at every point of $V_1 := p_X^{-1}(U_1)$. We have

- $\overline{\Gamma}_f$ is an irreducible (since Γ_f, being the graph of a meromorphic map is irreducible) analytic subset of $X \times Z$;
- $\dim(\overline{\Gamma}_f) = \dim X$;

- for every $v \in V_1$ there is unique $z \in Z$ such that $(v, z) \in \overline{\Gamma}_f$.
- the natural projection $\tau : \overline{\Gamma}_f \to X$ is proper, since both sets are compact.

It follows that $\overline{\Gamma}_f$ is a graph of a meromomorphic map $\tilde{f} : X \to Z$ (see [AS71, page 75]). ☐

We will use also the following

Lemma 5.15. *Assume that Y is a compact connected complex manifold, $\Sigma \subset Y$ is an analytic subset of codimension at least 2, $U = Y \setminus \Sigma$. Let $(\mathscr{L}, \pi, Y)$ be a holomorphic line bundle over Y such that $\mathscr{L}|_U$ is trivial. Then $\mathscr{L}$ is trivial.*

Proof. Indeed, $V := \pi^{-1}(U) \sim U \times \mathbb{C}_z$, thus $z = F(v)$ is a holomorphic function on V. The set $\tilde{\Sigma} := \pi^{-1}(\Sigma)$ has codimension at least two in $\mathscr{L}$. By the Second Riemann removable singularity theorem (Theorem 5.8), F may be extended to a holomorphic function $\overline{F}$ on $\mathscr{L}$. Thus we have a holomorphic map $\Phi : \mathscr{L} \to Y \times \mathbb{C}_z, \quad x \in \mathscr{L} \to (p(x), \overline{F}(x))$, that is an isomorphism outside $\tilde{\Sigma}$. Let S is the set of all points in $\mathscr{L}$ where the differential $d\Phi$ of Φ does not have the maximal rank. The sets S and $\tilde{S} = p(S)$ are analytic subsets of $\mathscr{L}$ and Y, respectively (see, for instance, [Nar, Theorem 2, Chapter VII], [PR, Therem 1.22], [Re]). Moreover, $\operatorname{codim}(\tilde{S}) = 1$ ([Ra]). But $\tilde{S} \subset \Sigma$, hence $\tilde{S} = \emptyset$. It follows that Φ is an isomorphism. ☐

6. Non-trivial rational bundles

In this section we consider non-trivial $\mathbb{P}^1-$ bundles over a non-uniruled base. It appears that the fact that $X \not\sim Y \times \mathbb{P}^1$ imposes the significant restrictions on the structure of the groups $\operatorname{Aut}(X)$ and $\operatorname{Bim}(X)$. We will start with projective case.

Definition 6.1. A regular surjective map $f : X \to Y$ of smooth irreducible projective complex varieties is a *conic bundle* over Y if there is a Zariski-open dense subset $U \subset Y$ such that the fiber $f^{-1}(y) \sim \mathbb{P}^1$ for all $y \in U$.

The *generic fiber* of f is an irreducible smooth projective curve $\mathscr{X}_f$ over the field $K := \mathbb{C}(Y)$ such that its field of rational functions $K(\mathscr{X}_f)$ coincides with $\mathbb{C}(X)$. (The genus of $\mathscr{X}_f$ is 0.)

Theorem 6.2. *(*[BZ17]*) Let X be a conic bundle over a non-uniruled smooth irreducible projective variety Y with* $\dim(Y) \geq 2$. *If X is not birational to $Y \times \mathbb{P}^1$ then* $\operatorname{Bir}(X)$ *is strongly Jordan.*

Let us sketch the proof of Theorem 6.2.

Let $f : X \to Y$ be a conic bundle and assume that Y is non-uniruled. According to Corollary 5.5. every $\phi \in \operatorname{Bir}(X)$ is fiberwise: there is a homomorphism $\tilde{\tau} : \operatorname{Bir}(X) \to \operatorname{Bir}(Y)$ such that $\tilde{\tau}(\phi) \circ f = f \circ \phi :$

$$\begin{array}{ccc} X & \xrightarrow{\phi} & X \\ f\downarrow & & \downarrow f \\ Y & \xrightarrow{\tilde{\tau}(\phi)} & Y \end{array}.$$

It follows that there is an exact sequence of groups:

$$0 \to \mathrm{Bir}_{\mathbb{C}(Y)}(\mathscr{X}_f) \to \mathrm{Bir}(X) \to \mathrm{Bir}(Y); \tag{6}$$

Since Y is non-uniruled the group $\mathrm{Bir}(Y)$ is strongly Jordan thanks to Theorem 4.2 (see also [BZ17, Cor. 3.8 and its proof]).

Let us compute $\mathrm{Bir}_K(\mathscr{X}_f)$. We have

1. $\mathrm{Bir}(\mathscr{X}_f) = \mathrm{Aut}(\mathscr{X}_f)$ since $\dim(\mathscr{X}_f) = 1$.
2. Since $X \not\sim Y \times \mathbb{P}^1$ the genus 0 curve $\mathscr{X}_f$ has **no** K-points and therefore there exists a ternary quadratic form
$$q(T) = a_1T_1^2 + a_2T_2^2 + a_3T_3^2$$
over K such that
— all a_i are nonzero elements of K;
— $q(T) = 0$ if and only if $T = (0, 0, 0))$ (this means that q is *anisotropic*);
— $\mathscr{X}_f$ is biregular over K to the plane projective quadric
$$\mathbf{X}_q := \{(T_1 : T_2 : T_3) \mid q(T) = 0\} \subset \mathbb{P}^2_K.$$
3. K is a field of characteristic zero that contains all roots of unity.

Now we consider a quadric, i.e., a hypersurface in a projective space defined by one irreducible quadratic equation over K. It is anisotropic if it has no point defined over K. In [BZ17] proven was the following

Theorem 6.3. *(*[BZ17]*) Suppose that K is a field of characteristic zero that contains all roots of unity, $d \geq 3$ an odd integer, V a d-dimensional K-vector space and let $q : V \to K$ be a quadratic form such that $q(v) \neq 0$ for all nonzero $v \in V$. Let us consider the projective quadric $X_q \subset \mathbb{P}(V)$ defined by the equation $q = 0$, which is a smooth projective irreducible $(d-2)$-dimensional variety over K. Let $\mathrm{Aut}(X_q)$ be the group of biregular automorphisms of X_q. Let G be a finite subgroup in $\mathrm{Aut}(X_q)$. Then G is commutative, all its non-identity elements have order 2 and the order of G divides 2^{d-1}.*

Thus if G is a nontrivial finite subgroup of $\mathrm{Aut}(\mathscr{X}_f)$ then either $G \cong \mathbb{Z}/2\mathbb{Z}$ or $G \cong (\mathbb{Z}/2\mathbb{Z})^2$.

Now applying Remark 2.9(4) we get from Equation (6) that $\mathrm{Bir}(X)$ is Jordan.

Remark 6.4. Actually in Theorem 6.2 the variety X is considered as a pointless ($X(K) = \emptyset$) rational curve defined over a field K, where field K contains all roots of unity. The "pointless surfaces" were studied by C. Shramov and V. Vologodsky in [ShV], [ShVb].

For complex compact manifolds the absence of a point in generic fiber has to be reformulated in terms of sections.

Let (X, p, Y) be a rational bundle over a compact complex connected non-uniruled manifold Y (see Definition 5), i.e.,

- X, Y are compact connected manifolds;
- Y is non-uniruled;
- $p : X \to Y$ is a surjective holomorphic map;
- $p^{-1}(U)$ is a holomorphic locally trivial fiber bundle over a dense analytical Zariski open subset $U \subset Y$ with fiber $\mathbb{P}^1$ and with the corresponding projection map $p : p^{-1}(U) \to U$.

According to Lemma 5.4, every map $f \in \mathrm{Bim}(X)$ maps the general fiber of p to a fiber of p. Let

$\mathrm{Aut}(X)_p = \{f \in \mathrm{Aut}(X) \mid \tau(f) = id\}$, $\mathrm{Bim}(X)_p = \{f \in \mathrm{Bim}(X) \mid \tilde{\tau}(f) = id\}$,

be the kernels of τ and $\tilde{\tau}$, respectively.

Then we have the following short exact sequences

$$0 \to \mathrm{Aut}(X)_p \to \mathrm{Aut}(X) \xrightarrow{\tau} \mathrm{Aut}(Y), \tag{7}$$

$$0 \to \mathrm{Bim}(X)_p \to \mathrm{Bim}(X) \xrightarrow{\tilde{\tau}} \mathrm{Bim}(Y). \tag{8}$$

Definition 6.5. Let (X, p, Y) be an equidimensional rational bundle over a compact complex connected non-uniruled manifold Y. We will call an irreducible analytic subspace D of X *almost section* if the intersection number (D, F) of D with a fiber $F = p^{-1}(y)$, $y \in Y$ is 1.

Remark 6.6. For $f \in \mathrm{Bim}(X)_p$ let $\tilde{S}_f$ be the indeterminancy locus of f that is an analytic subspace of X of codimension at least 2 ([Re, page 369]). Let $S_f = p(\tilde{S}_f)$, which is an analytic subset of Y ([Re], [Nar, Theorem 2, Chapter VII]). Since the dimension of a fiber of p is one, $Y \setminus S_f$ is an analytical Zariski open dence subset U of Y. Hence the restriction $f\mid_{P_y}$ of f onto the fiber $P_y = p^{-1}(y)$ of p over a general point $y \in Y$ belongs to $\mathrm{Aut}(P_y)$. Thus f induces an automorphism of $V = p^{-1}(U)$ onto itself.

Let D be an almost section of X.

(1) Let $a : \tilde{X} \to X$ be such a modification of X that the following diagram is commutative

$$\begin{tikzcd} & \tilde{X} \arrow[dl, "a"'] \arrow[dr, "b"] & \\ X \arrow[rr, dashed, "f"] & & X \end{tikzcd},$$

where $b : \tilde{X} \to X$ is a holomorphic map (it always exists, [Pe, Theorem 1.9]). Then $f(D) = ba^{-1}(D)$ is an analytic subset ([Re],[Fi, Theorem 3.6]) that is a union of finite number of irreducible components $D_1, \dots, D_n$.

(2) We may assume (may be after shrinking U) that D meets every fiber $P_y, y \in U$ at precisely one point. Thus $f(D)$ meets $P_y, y \in U$ at precisely one point as well.
(3) It follows from (2) that precisely one of irreducible components of $f(D)$, say, D_1, meets a fiber $P_y, y \in U$. The intersection $D_1 \cap P_y, y \in U$ consists of precisely one point.

Thus D_1 is an almost section. It follows that the image of an almost section under $f \in \mathrm{Bim}(X)_p$ contains precisely one almost section. In particular, f cannot contract an almost section.

Similarly, if $\Phi : X \to Z$ is a bimeromorphic map of a $\mathbb{P}^1-$bundle (X, p, Y) to a $\mathbb{P}^1-$bundle (Z, q, Y) such that $q \circ \Phi = p$, then the image of an almost section contains an almost section.

The following results were proved by Yu. Prokhorov and C. Shramov in more general setting, we formulate below its application for the case of $\mathbb{P}^1-$ bundles.

Theorem 6.7. *Let (X, p, Y) be a $\mathbb{P}^1$-bundle over a compact complex connected non-uniruled manifold Y. Let $P_y = p^{-1}(y)$ be a fiber of p over a general point $y \in Y$. Then*

1. *Every countable union of finite subgroups of* $\mathrm{Bim}(X)_p$ *may be embedded into* $\mathrm{Bim}(P_y)$. *(*[PS20b, Lemma 4.1]*).*
2. *If X is Kähler, then* $\mathrm{Bim}(X)_p$ *is Jordan ((*[PS20b, Corollary 4.3]
3. *If there exists an almost section D on X then $X \sim \mathbb{P}(\mathscr{E})$ for some rank two holomorphic vector bundle $\mathscr{E}$ on Y.* [Sh19, Lemma 3.5].
4. *Assume that no almost section exists on X. Assume that* $\mathrm{Bim}(Y)$ *is strongly Jordan. Then* $\mathrm{Bim}(X)$ *is Jordan* [Sh19, Corollary 5.8].
5. *If there exist $f \in \mathrm{Bim}(X)_p$ of finite order $d > 2$ then there exist at least two distinct almost sections on X. If f is biholomorphic, the almost sections may be chosen to be disjoint.* [Sh19, Lemma 4.1]

Let us add to this the following

Lemma 6.8. *In the Notation of Theorem 6.7, assume that there exists precisely one almost section on X. Then if* $\mathrm{Bim}(Y)$ *is Jordan, so is* $\mathrm{Bim}(X)$.

Proof. Assume that D is the only almost section. Let $f \in \mathrm{Bim}(X)_p$, $f \neq id$. The set $f(D)$ contains an irreducible component D_1 that is an almost section (see Remark 6.6). Therefore $D = D_1$ and D is contained in the set $\mathrm{Fix}(f)$ of fixed points of f. Let $V \subset Y$ be an analytical Zariski open dense subset such that the restriction f_v of f onto the fiber P_v is a non-identical automorphism of P_v for all $v \in V$. Since f_v has at most two fixed points, we have:

—either $\mathrm{Fix}(f)\cap P_v = D\cap P_v$ contains one point, and f_v has infinite order;

—or $(\mathrm{Fix}(f)\cap P_v)\setminus(D\cap P_v)$ contains a point for the general $v\in V$ and $\mathrm{Fix}(f)$ contains an almost section distinct from D, which is impossible.

Thus every element $f\in \mathrm{Bim}(X)_p$ different from id has infinite order. Therefore $G\cap \mathrm{Bim}(X)_p=\{id\}$ for every finite group $G\subset \mathrm{Bim}(X)$ and $\tilde{\tau}: G\to \mathrm{Bim}(Y)$ is a group embedding. Hence, the Jordan index $J_{\mathrm{Bim}(X)}\le J_{\mathrm{Bim}(Y)}$. □

The opposite case, when the $\mathbb{P}^1-$bundle has many almost sections, is when $X\cong Y\times\mathbb{P}^1$. It will be considered in the next chapter.

Chapter 3. $\mathbb{P}^1-$bundles over complex tori

In this section we deal with $\mathbb{P}^1-$bundles of a special type, namely $(\overline{\mathscr{L}},p,T)$, where $\mathscr{L}$ is a holomorphic line bundle over a complex torus T and $\overline{\mathscr{L}}=\mathbb{P}(\mathscr{L}\oplus\mathbf{1}_T)$. Most examples of compact complex connected manifolds with a non-Jordan group $\mathrm{Bim}(X)$ (at least for dimensions greater than 3) are $\mathbb{P}^1-$ bundles of this type. Manifolds of this type were studied by one of the authors in papers [Zar14](projective case) and [Zar19] (non-algebraic case). The goal of this chapter is to present a unified approach for both situations. It is based on a construction motivated by symplective geometry and inspired by an algebraic approach to theta functions developed by [Mum66]. The chapter starts with symplectic constructions, then the theta groups follow, then we arrive to description of certain subgroups of $\mathrm{Bim}(\overline{\mathscr{L}})$.

7. Symplectic Group Theory

This section contains elementary but useful facts about Jordan properties of central extensions of commutative groups by $\mathbb{C}^*$.

Traditionally, some groups are written in the multiplicative form, and some in the additive one. We hope that no confusion will arise.

Definition 7.1. A *symplectic pair* is a pair (A,e) that consists of a commutative group A and an alternating bilinear pairing

$$e: A\times A\to\mathbb{C}^*.$$

Here *alternating* means that

$$e(a,a)=1\ \forall a\in A.$$

The bilinearity means that

$$e(a_1{+}a_2,b)=e(a_1,b)e(a_2,b),\ e(a,b_1{+}b_2)=e(a,b_1)e(a,b_2)\ \forall a,a_1,a_2,b,b_1,b_2\in A.$$

These property implies that for all $a,b\in A$

$$1=e(a+b,a+b)=e(a,a)e(a,b)e(b,a)e(b,b)=e(a,b)e(b,a),$$

i.e.,

$$e(a,b)=e(b,a)^{-1}\ \forall a,b\in A.$$

As usual, e gives rise to the group homomorphism

$$\Psi_e : A \to \mathrm{Hom}(A, \mathbb{C}^*), \ b \mapsto \{\Psi_e(b) : A \to \mathbb{C}^*, \ a \mapsto e(a,b)\}. \tag{9}$$

A subgroup B of A is called *isotropic* with respect to e if

$$e(B,B) = \{1\}.$$

We define the *kernel* of e as

$$\ker(e) := \{a \in A \mid e(a,A) = \{1\}\} = \ker(\Psi_e),$$

which is a subgroup of A that is isotropic with respect to e.

We say that e is *nondegenerate* if $\ker(e) = \{0\}$, i.e.,

$$\Psi_e : A \to \mathrm{Hom}(A, \mathbb{C}^*)$$

is an *injective* homomorphism. If e is nondegenerate then we call (A,e) a *nondegenerate symplectic pair*.

Example 7.2. Let d be a positive integer, $\mathbf{S}_d = \left(\frac{1}{d}\mathbb{Z}/\mathbb{Z}\right)^2 \cong (\mathbb{Z}/d\mathbb{Z})^2$,

$\mathbf{e}_d : \mathbf{S}_d \times \mathbf{S}_d \to \mathbb{C}^*, \ (a_1+\mathbb{Z}, b_1+\mathbb{Z}), (a_2+\mathbb{Z}, b_2+\mathbb{Z}) \mapsto \exp(2\pi \mathbf{i} d(a_1b_2 - a_2b_1)).$

Then $(\mathbf{S}_d, \mathbf{e}_d)$ is a *nondegenerate symplectic pair*.

Remark 7.3. Let (A_1e_1) and (A_2,e_2) be nondegenerate symplectic pairs. Let us consider the bilinear alternating form

$$e_1e_2 : (A_1 \oplus A_2) \times (A_1 \oplus A_2) \to \mathbb{C}^*,$$

$$(a_1,a_2),(b_1,b_2) \mapsto e_1(a_1,b_1) \cdot e_2(a_2,b_2).$$

Then $(A_1 \oplus A_2, e_1e_2)$ is a *nondegenerate symplectic pair*.

Remark 7.4. If (A,e) is a symplectic pair and B is a subgroup of A then $(B, e\mid_B)$ is also a symplectic pair. Here $e\mid_B$ is the restriction of e to $B \times B$.

Remark 7.5.
(i) Each symplectic pair (A,e) gives rise to a nondegenerate symplectic pair $(\bar{A},\bar{e})$ where

$$\bar{A} = A/\ker(e), \ \bar{e}(a \ \ker(e), b \ \ker(e)) = e(a,b) \ \forall a,b \in A. \tag{10}$$

(ii) Clearly, a subgroup B of A is isotropic with respect to e if and only if its image $\bar{B}$ in $\bar{A}$ is isotropic with respect to $\bar{e}$. In particular, B is isotropic if and only if $B + \ker(e)$ is isotropic.

(iii) Let B be a subgroup of A. One may restate a property of B to be isotropic with respect to e as follows. The composition of $\Psi_e : A \to \mathrm{Hom}(A,\mathbb{C}^*)$ with the *restriction map* $\mathrm{Hom}(A,\mathbb{C}^*) \to \mathrm{Hom}(B,\mathbb{C}^*)$ is the group homomorphism

$$A \xrightarrow{\Psi_e} \mathrm{Hom}(A,\mathbb{C}^*) \to \mathrm{Hom}(B,\mathbb{C}^*) \tag{11}$$

that kills B. Clearly, the kernel $B^{\perp}$ of this homomorphism (which is the *orthogonal complement* of B in A with respect to e) contains B if and only if B is isotropic.

(iv) Suppose that B coincides with $B^{\perp}$. This means that if $a \in A \backslash B$ then $e(B, a) \neq \{1\}$. In other words, B is a *maximal* isotropic subgroup of A with respect to e.

Conversely, suppose that B is a *maximal* isotropic subgroup of A with respect to e. Since B is isotropic,

$$B \subset B^{\perp} \subset A, \ e(B^{\perp}, B) = \{1\}.$$

If $B^{\perp} \neq B$ then there is $a \in B^{\perp} \in B$ such that $e(a, B) = \{1\}$. This implies that the subgroup B_1 of A generated by B and a is isotropic, which contradicts the maximality of B.

It follows that $B = B^{\perp}$ if and only if B is a *maximal isotropic* subgroup of A.

Remark 7.6. Suppose that A is finite. Then the finite groups A and $\mathrm{Hom}(A, \mathbb{C}^*)$ are isomorphic (non-canonically); in particular, they have the same order. It follows that in the case of finite A the pairing e is nondegenerate if and only if Ψ_e is a group *isomorphism*.

Lemma 7.7 (Useful Lemma)**.** *Let (A, e) be a symplectic pair such that $A/\ker(e)$ is a finite group. If B is a maximal isotropic subgroup of A then the index $[A : B]$ equals $\sqrt{\#(A/\ker(e))}$. In particular, if e is nondegenerate then*

$$[A : B] = \sqrt{\#(A)} = \#(B).$$

Proof of Usefull Lemma. In light of Remark 7.5, B contains $\ker(e)$ and therefore it suffices to prove the desired result for nondegenerate $(\bar{A}, \bar{e})$ (instead of (A, e)). In other words, without loss of generality, we may assume that $\ker(e) = \{0\}$, i.e., $A = \bar{A}$ is *finite* and $e = \bar{e}$ is *nondegenerate.*

Since $\mathbb{C}^*$ is a divisible group, every group homomorphism $B \to \mathbb{C}^*$ extends to a group homomorphism $A \to \mathbb{C}^*$. This means that the restriction map $\mathrm{Hom}(A, \mathbb{C}^*) \to \mathrm{Hom}(B, \mathbb{C}^*)$ is surjective. Since A is finite, the nongeneracy of e means (in light of Remark 7.6) that $\mathrm{Hom}(A, \mathbb{C}^*) = \Psi_e(A)$. On the other hand, the maximality of B means that the kernel of the *surjective* composition

$$A \overset{\Psi_e}{\cong} \mathrm{Hom}(A, \mathbb{C}^*) \twoheadrightarrow \mathrm{Hom}(B, \mathbb{C}^*)$$

coincides with B (see Remark 7.5) and therefore there is an *injective* group homomorphism

$$A/B \hookrightarrow \mathrm{Hom}(B, \mathbb{C}^*),$$

which is also surjective and therefore is an isomorphism. This implies that

$$\#(A/B) = \#\big(\mathrm{Hom}(B, \mathbb{C}^*)\big) = \#(B),$$

which ends the proof if we take into account that $\#(A/B) = \#(A)/\#(B)$. $\square$

Remark 7.8. Suppose that $\ker(e)$ is either *finite* or *divisible*. Then every finite subgroup $\bar{B}$ of $\bar{A}$ is the image of a finite subgroup $B \subset A$ under $A \twoheadrightarrow \bar{A}$. Indeed, if $\ker(e)$ is finite then one may take as B the preimage of $\bar{B}$ in A. If $\ker(e)$ is divisible then it is a direct summand of A, i.e., A splits into a direct sum $A = \ker(e) \oplus A'$ and the map $A \to \bar{A}$ induces an isomorphism $A' \cong \bar{A}$. Now one may take as B the (isomorphic) preimage of $\bar{B}$ in A'.

Definition 7.9. A symplectic pair (A, e) is called *almost isotropic* if there exists a positive integer D that enjoys the following property.

Each finite subgroup $\mathscr{B}$ of A contains an *isotropic* (with respect to e) subgroup $\mathscr{A}$ such that the index $[\mathscr{B} : \mathscr{A}] \le D$. Such a smallest D is called the *isotropy defect* of (A, e) and denoted by $D_{A,e}$.

Example 7.10. If $e \equiv 1$ then every subgroup is isotropic and therefore $D_{A,e} = 1$.

Remark 7.11. Suppose that $\ker(e)$ is either finite or divisible.

(i) It follows from Remarks 7.8 and 7.5 that (A, e) is almost isotropic if and only if $(\bar{A}, \bar{e})$ is almost isotropic. In addition, if this is the case then
$$D_{A,e} = D_{\bar{A},\bar{e}}. \tag{12}$$
Indeed, let $\mathscr{A}$ be a finite subgroup of A and B an isotropic subgroup of largest possible order in $\mathscr{A}$. In particular, B is a maximal isotropic subgroup of $\mathscr{A}$. Since $B_1 = B + (\mathscr{A} \cap \ker(e))$ is an isotropic subgroup of $\mathscr{A}$ that contains B, the maximality of B implies that $B_1 = B$, i.e., $B \supset \mathscr{A} \cap \ker(e)$. This implies that the index $(\mathscr{A} : B)$ equals the index $[\bar{\mathscr{A}} : \bar{B}]$ where the subgroups $\bar{\mathscr{A}}$ and $\bar{B}$ are the images in $\bar{A}$ of $\mathscr{A}$ and B respectively. Taking into account that $\bar{B}$ is an isotropic (with respect to $\bar{e}$) subgroup of finite group $\bar{\mathscr{A}} \subset \bar{A}$, we conclude that
$$D_{A,e} \ge D_{\bar{A},\bar{e}}.$$
Conversely, suppose that $\bar{B}$ is an isotropic (with respect to $\bar{e}$) subgroup of maximal order in a finite group $\bar{\mathscr{A}} \subset \bar{A}$. As above, this implies that $\bar{B}$ is a maximal isotropic subgroup of $\bar{\mathscr{A}}$. By Remark 7.8, A contains a finite subgroup $\mathscr{A}$, whose image in $\bar{A}$ coincides with $\bar{\mathscr{A}}$. Let B the preimage of $\bar{B}$ in $\mathscr{A}$. Then B is isotropic with respect to e and the index $[\mathscr{A} : B]$ coincides with the index $[\bar{\mathscr{A}} : \bar{B}]$. This implies that
$$D_{A,e} \le D_{\bar{A},\bar{e}},$$
which ends the proof.

(ii) Assume additionally that $\bar{A}$ is *finite*. Applying Lemma 7.7 to subgroups of $\bar{A}$ and using (12), we conclude that
$$D_{A,e} = D_{\bar{A},\bar{e}} = \sqrt{\#(\bar{A})}. \tag{13}$$

Definition 7.12. A *theta group* attached to a symplectic pair (A, e) is a group G that sits in a short exact sequence

$$1 \to \mathbb{C}^* \xrightarrow{i} G \xrightarrow{j} A \to 0 \tag{14}$$

that enjoys the following properties.

The image of $\mathbb{C}^*$ is a *central* subgroup of G, and the alternating *commutator* pairing

$$A \times A \to \mathbb{C}^*, \ j(g_1), j(g_2) \mapsto i^{-1}\left(g_1 g_2 g_1^{-1} g_2^{-1}\right) \in \mathbb{C}^* \ \forall g_1, g_2 \in G$$

attached to exact sequence (14) coincides with e.

Remark 7.13. Every *central* extension G of a commutative group A by $\mathbb{C}^*$ gives rise to the symplectic pair (A, e) where $e(a_1, a_2) \in \mathbb{C}^*$ is the commutator of preimages of a_1, a_2 in G (for all $a_1, a_2 \in A$). This makes G a theta group attached to (A, e).

Remark 7.14.
(i) Clearly, an element g of the theta group G lies in the center of G if and only if

$$e(j(g), j(h)) = 1 \ \forall h \in G.$$

Since $j(G) = A$, the element g is central if and only if $j(g) \in \ker(e)$. This implies that the center of G coincides with $j^{-1}(\ker(e))$.

(ii) Clearly, a subgroup H of G is commutative if and only if its image $j(H) \subset A$ is an isotropic subgroup of A with respect to e.

Remark 7.15. Let G be a theta group that sits in the short exact sequence (14). If B is a subgroup of A then obviously the preimage $j^{-1}(B)$ is a theta group attached to the symplectic pair $(B, e\,|_B)$.

Lemma 7.16. *Let B be a finite subgroup of A. Then there exists a finite subgroup $\tilde{B}$ of the theta group G such that $j(\tilde{B}) = B$.*

Proof. In what follows, we identify $\mathbb{C}^*$ with its image in G and view it as a certain central subgroup of G. Let d be the *exponent* of B.

Let us consider the finite multiplicative subgroups μ_d and μ_{d^2} of all dth roots of unity and d^2th roots of unity, respectively, in $\mathbb{C}^*$. We have

$$\mu_d \subset \mu_{d^2} \subset \mathbb{C}^* \subset G;$$

in addition,

$$e(B, B) \subset e(B, A) \subset \mu_d. \tag{15}$$

For every $b \in B$ choose its *lifting* $\tilde{b} \in G$ such that

$$\tilde{b}^d = 1, \ \tilde{b}^{-1} = \widetilde{b^{-1}} \ \forall b \in B; \tag{16}$$

this is possible, since $\mathbb{C}^*$ is a central divisible subgroup of $\mathbb{C}^*$. Indeed, let $\tilde{b}_1 \in G$ be any lifting of b to G, i.e., $j(\tilde{b}_1) = b$. Then

$$z_1 := \tilde{b}_1^d \in \ker(j) = \mathbb{C}^*.$$

Let us choose any
$$z = \sqrt[d]{z_1} \in \mathbb{C}^*$$
and put $\tilde{b} = z^{-1}\tilde{b}_1 \in G$. We have
$$j(\tilde{b}) = j(z^{-1}) + j(\tilde{b}_1) = 0 + b = b; \ \tilde{b}^d = (z^{-1})^d \tilde{b}_1^d = z_1^{-1} z_1 = 1.$$

Let us put
$$\tilde{B} := \{\gamma\tilde{b} \mid \gamma \in \mu_{d^2}, b \in B\} \subset G.$$
Clearly, $\tilde{B}$ is finite, $j(\tilde{B}) = B$, and
$$1 \in \mu_{d^2} \subset \tilde{B} = \tilde{B}^{-1} := \{u^{-1} \mid u \in \tilde{B}\}$$
(the latter equality follows from the invariance of the central subgroup μ_{d^2} and the subset $\{\tilde{b} \mid b \in B\}$ under the map $u \mapsto u^{-1}$).

So, in order to prove that $\tilde{B}$ is a subgroup of G, it suffices to check that $\tilde{B}$ is closed under multiplication in G. Let $b_1, b_2 \in B$ and $b_3 = b_1 + b_2 \in B$. We need to compare $\tilde{b}_1\tilde{b}_2$ and $\tilde{b}_3$ in G. Clearly, there is $\gamma \in \mathbb{C}^*$ such that
$$\tilde{b}_3 = \gamma\tilde{b}_1\tilde{b}_2.$$

Notice that
$$\tilde{b}_1^d = \tilde{b}_2^d = \tilde{b}_3^d = 1 \in \mathbb{C}^* \subset G.$$
On the other hand, in light of (15),
$$\gamma_0 := \tilde{b}_1\tilde{b}_2\tilde{b}_1^{-1}\tilde{b}_2^{-1} = e(b_1, b_2) \in \mu_d \subset \mathbb{C}^* \subset G.$$

It follows that the images of $\tilde{b}_1$ and $\tilde{b}_2$ in the quotient G/μ_d do commute and therefore the image of $\tilde{b}_1\tilde{b}_2$ in G/μ_d has order that divides d. This means that
$$\left(\tilde{b}_1\tilde{b}_2\right)^d \in \mu_d$$
and therefore
$$\left(\tilde{b}_1\tilde{b}_2\right)^{d^2} = 1.$$
It follows that
$$1 = \tilde{b}_3^{d^2} = \left(\gamma \cdot \tilde{b}_1\tilde{b}_2\right)^{d^2} = \gamma^{d^2}\left(\tilde{b}_1\tilde{b}_2\right)^{d^2} = \gamma^{d^2} \cdot 1 = \gamma^{d^2}.$$

This implies that $\gamma^{d^2} = 1$, i.e., $\gamma \in \mu_{d^2}$ and therefore
$$\tilde{b}_1\tilde{b}_2 = \gamma^{-1}\tilde{b}_3 \in \tilde{B}.$$
This ends the proof. $\square$

Theorem 7.17. *Let (A, e) be a symplectic pair. Suppose that $\bar{A} = A/\ker(e)$ is finite. Assume also that either $\ker(e)$ is divisible or A is finite. Let G be a theta group attached to (A, e).*

Then G is a Jordan group and its Jordan index equals $\sqrt{\#(\bar{A})}$.

Proof. Assume that G sits in a short exact sequence (14). We may view $\mathbb{C}^*$ as a central subgroup of G. Let $\tilde{\mathscr{A}}$ be a finite subgroup of G and $\tilde{B}$ a commutative subgroup of *maximal order* in $\tilde{\mathscr{A}}$. Then $\tilde{B}$ contains the intersection $\tilde{\mathscr{A}} \cap \mathbb{C}^*$ and therefore the index $[\tilde{\mathscr{A}} : \tilde{B}]$ coincides with the index $[j(\tilde{\mathscr{A}}) : j(\tilde{B})]$. The commutativeness of $\tilde{B}$ means that $j(\tilde{B})$ is an *isotropic* subgroup in $j(\tilde{\mathscr{A}})$. This implies that

$$J_G \geq D_{A,e}.$$

Conversely, let $\mathscr{A}$ be a finite subgroup of A and B is an isotropic subgroup of maximal order in $\mathscr{A}$. By Lemma 7.16, there is a finite subgroup $\tilde{\mathscr{A}}$ of G such that

$$j(\tilde{\mathscr{A}}) = \tilde{\mathscr{A}}.$$

Let $\tilde{B}$ be the preimage of B in $\tilde{\mathscr{A}}$. Then

$$j(\tilde{B}) = B,\ [\mathscr{A} : B] = [\tilde{\mathscr{A}} : \tilde{B}].$$

By Remark 7.14(ii), $\tilde{B}$ is commutative, because its image B is isotropic. The equality of indices implies that

$$J_G \leq D_{A,e},$$

which, combined with the previous opposite inequality, implies that $J_G = D_{A,e}$. Now the explicit formula for J_G follows from Remark 7.11. □

8. Symplectic linear algebra

In this section we construct theta groups that arise from (non necessarily nondegenerate) alternating bilinear form on integral lattices.

Definition 8.1. (i) An *admissible triple* is a triple (V, E, Π) that consists of a nonzero real vector space V of finite positive even dimension $2g$, an alternating $\mathbb{R}$-bilinear form

$$E : V \times V \to \mathbb{R}$$

on V, and a discrete lattice Π of rank $2g$ in V such that $E(\Pi, \Pi) \subset \mathbb{Z}$. Let us put

$$\Pi_E^{\perp} := \{v \in V \mid E(v, l) \in \mathbb{Z}\ \forall l \in \Pi.$$

By definition, $\Pi_E^{\perp}$ is a closed real Lie subgroup of V that contains Π as a discrete subgroup.

(ii) A symplectic pair attached to the admissible triple (V, E, Π) is a pair $(K_{E,\Pi}, e_E)$ where $K_{E,\Pi} := \Pi_E^{\perp}/\Pi$ and the bilinear pairing e_E is defined as follows.

$$e_E : \Pi_E^{\perp}/\Pi \times \Pi_E^{\perp}/\Pi \to \mathbb{C}^*,\ (v_1 + \Pi, v_2 + \Pi) \mapsto \exp(2\pi \mathbf{i} E(v_1, v_2)).$$

Definition 8.2. Recall that a subgroup C of a commutative group D is called *saturated* if it enjoys the following equivalent properties.

- There are no elements of finite order in the quotient D/C except 0.
- If x is an element of D such that there is a positive integer m with $mx \in C$ then $x \in C$.

Our goal is to find the isotropy index of $(K_{E,\Pi}, e_E)$. In order to do that, let us consider the *kernel* of E, i.e., the subset

$$\ker(E) = \{v \in V \mid E(v, V) = \{0\} \subset V.$$

Clearly, $\ker(E)$ is a real even-dimensional (recall that E is alternating) vector subspace of V containing $\Pi_E^{\perp}$. Let us put

$$\Pi_0 := \Pi \bigcap \ker(E) \subset \ker(E).$$

Clearly, Π_0 is a *saturated* subgroup of Π. The integrality property of E implies that the natural homomorphism of real vector spaces

$$\Pi_0 \otimes \mathbb{R} \to \ker(E),\ l_0 \otimes \lambda \mapsto \lambda \cdot l_0\ \forall l_0 \in \Pi_0, \lambda \in \mathbb{R}$$

is an isomorphism. In particular, the following conditions are equivalent.

(a) E is nondegenerate, i.e., $\ker(E) = \{0\}$.
(b) $\Pi_0 = \{0\}$.

Let us consider several cases.

Case I If $E \equiv 0$ then

$$\Pi_E^{\perp} = V, K_{E,\Pi} = \Pi_E^{\perp}/\Pi = V/\Pi, e_E \equiv 1,$$

$\ker(e_E) = K_{E,\Pi}$ is divisible and $K_{E,\Pi}/\ker(e) = \{0\}$ is finite. By Remark 7.11, the isotropy defect $D_{K_{E,\Pi},e_E} = 1$.

Case II Suppose that E is a *nondegenerate* form. Let $\{s_1, \dots, s_{2g}\}$ be any basis of the $\mathbb{Z}$-module Π. Clearly, it is also a basis of the $\mathbb{R}$-vector space space V. Let

$$\tilde{E} = \Big(E(s_j, s_k)\Big)_{j,k=1} \in \mathrm{Mat}_g(\mathbb{Z})$$

be the $2g \times 2g$ *skew-symmetric* matrix of E with respect to this basis with *integer* entries. Let $\det(\tilde{E})$ and $\mathrm{Pf}(\tilde{E})$ be the *determinant* of $\tilde{E}$ and the *pfaffian* of $\tilde{E}$ respectively. Then

$$\det(\tilde{E}) \in \mathbb{Z}, \mathrm{Pf}(\tilde{E}) \in \mathbb{Z};\ 0 \ne \det(\tilde{E}) = \mathrm{Pf}(\tilde{E})^2.$$

In particular, $\det(\tilde{E})$ is a *positive integer*. Clearly, $\det(\tilde{E})$ does not depend on the choice of a basis of Π and therefore $|\mathrm{Pf}(\tilde{E})|$ does not depend on this choice as well. That is why we denote $\det(\tilde{E})$ by $\det(E, \Pi)$ and $|\mathrm{Pf}(\tilde{E})|$ by $|\mathrm{Pf}(E, \Pi)|$.

We claim that $\Pi_E^{\perp}/\Pi$ *is finite, the form*

$$e_E : \Pi_E^{\perp}/\Pi \times \Pi_E^{\perp}/\Pi \to \mathbb{C}^*$$

is nondegenerate and its isotropy defect is $|\mathrm{Pf}(E, \Pi)|$.

Indeed, there is a basis $\{f_1, h_1, \dots, f_g, h_g\}$ of Π such that

$$E(f_j, h_k) = -E(h_k, f_j) = 0 \ \forall j \neq k \ (1 \leq j, k \leq g)$$

([Lang, Ch. XV, Ex. 17 on p. 598]). Let us put

$$d_j = E(f_j, h_j) \in \mathbb{Z} \ \forall j = 1, \dots, g.$$

The nondegeneracy of E means that all $d_j \neq 0$. Replacing if necessary, h_j by $-h_j$, we may and will assume that all $d_j > 0$. If $\tilde{E}$ is the matrix of E with respect to this basis then the *pfaffian* $\mathrm{Pf}(\tilde{E})$ of $\tilde{E}$ is $\pm \prod_{j=1}^{g} d_j$ and therefore

$$|\mathrm{Pf}(E, \Pi)| = \prod_{j=1}^{g} d_j.$$

We claim that

$$\Pi_E^{\perp} = \oplus_{j=1}^{g} \frac{1}{d_j} \left(\mathbb{Z} \cdot f_j \oplus \mathbb{Z} \cdot h_j\right). \tag{17}$$

Indeed, a vector

$$v = \left(\sum_{j=1}^{g} \lambda_j f_j\right) + \left(\sum_{j=1}^{g} \mu_j g_j\right) \text{ with all } \ \lambda_j, \mu_j \in \mathbb{R}$$

lies in $\Pi_E^{\perp}$ if and only if

$$\mathbb{Z} \ni E(f_j, v) = d_j \mu_j, \ \mathbb{Z} \ni (h_j, v) = -d_j \mu_j \ \forall j,$$

i.e., if and only if $dv \in \oplus_{j=1}^{g} (\mathbb{Z} \cdot f_j \oplus \mathbb{Z} \cdot h_j)$, which is obviously equivalent to (17).

It follows from (17) that

$$\Pi_E^{\perp}/\Pi = \oplus_{j=1}^{g} \frac{1}{d_j} \left(\mathbb{Z} \cdot f_j \oplus \mathbb{Z} \cdot h_j\right) / \left(\mathbb{Z} \cdot f_j \oplus \mathbb{Z} \cdot h_j\right). \tag{18}$$

Clearly, different summands of $\Pi_E^{\perp}/L$ are mutually orthogonal with respect to e_E while the restriction of e_E to each

$$\frac{1}{d_j} \left(\mathbb{Z} \cdot f_j \oplus \mathbb{Z} \cdot h_j\right) / \left(\mathbb{Z} \cdot f_j \oplus \mathbb{Z} \cdot h_j\right)$$

is isomorphic to $(\mathbf{S}_{d_j}, \mathbf{e}_{d_j})$. In particular, this restriction is a nondegenerate symplectic pair. This implies that the direct sum $(\Pi_E^{\perp}/\Pi, e_E)$ is also a nondegenerate symplectic pair. On the other hand, clearly,

$$\Pi_E^{\perp}/\Pi \cong \oplus_{j=1}^{g} \left(\frac{1}{d_j} \mathbb{Z}/\mathbb{Z}\right)^2.$$

This implies that

$$\#(\Pi_E^{\perp}/\Pi) \cong \prod_{j=1}^{g} d_j^2, \ \sqrt{\#(\Pi_E^{\perp}/\Pi)} = \prod_{j=1}^{g} d_j = |\mathrm{Pf}(E, \Pi)|.$$

This implies that $(K_{E,\Pi}, e_E)$ is almost isotropic and its *isotropy defect* is $|\mathrm{Pf}(E, \Pi)|$.

Case IIbis We keep the notation and assumptions of **Case II**. Let us consider the form nE where n is a positive integer. Then

$$\Pi_{nE}^{\perp} = \frac{1}{n}\Pi_{E}^{\perp} = \oplus_{j=1}^{g}\frac{1}{nd_j}\left(\mathbb{Z}\cdot f_j \oplus \mathbb{Z}\cdot h_j\right),$$

$$\Pi_{nE}^{\perp}/\Pi \cong \oplus_{j=1}^{g}\left(\frac{1}{nd_j}\mathbb{Z}/\mathbb{Z}\right)^2,$$

$$\#(\Pi_{nE}^{\perp}/\Pi) = \prod_{j=1}^{g}(nd_j)^2,\ \sqrt{\#(\Pi_{E}^{\perp}/\Pi} = n^g\prod_{j=1}^{g}d_j = n^g\cdot \mathrm{Pf}(E, \Pi).$$

Hence, the corresponding isotropy index

$$D_{K_{nE,\Pi},e_{nE}} = n^g \cdot |\mathrm{Pf}(E, \Pi)|$$

for all positive integers n.

Case III Now let us consider the case of degenerate nonzero E, i.e., the case when

$$\{0\} \ne \Pi_0 \ne \Pi.$$

Clearly, Π_0 is a free abelian group of a certain positive even rank $2g_0 < 2g$. Since Π_0 is a saturated subgroup of Π, it is a *direct summand* of Π, i.e., there is a (nonzero saturated) subgroup Π_1 in Π that is a free abelian group of rank $2g - 2g_0$ and such that

$$\Pi = \Pi_0 \oplus \Pi_1.$$

In other words, there is a basis $\{u_1, \dots, u_{2g_0}; v_1, \dots, v_{2g-2g_0}\}$ of the $\mathbb{Z}$-module Π such that $\{u_1, \dots, u_{2g_0}\}$ is a basis of Π_0 and $\{v_1, \dots, v_{2g-2g_0}\}$ is a basis of Π_1. Let us consider the real vector subspaces

$$V_0 := \sum_{j=1}^{2g_0}\mathbb{R}u_j \subset V,\ V_1 := \sum_{k=1}^{2g_1}\mathbb{R}v_k \subset V.$$

Clearly,

$$V = V_0 \oplus V_1;\ \Pi_0 = V_0 \cap \Pi,\ \Pi_1 = V_1 \cap \Pi;$$

in addition, $V_0 = \ker(E)$, the subspaces V_0 and V_1 are mutually orthogonal with respect to E and the restriction of E to V_1

$$E_1 : V_1 \times V_1 \to \mathbb{R},\ u, v \mapsto E(u, v)$$

is a *nondegenerate* alternating bilinear form. It is also clear that

$$E_1(\Pi_1, \Pi_1) = E(\Pi_1, \Pi_1) \subset E(\Pi, \Pi) \subset \mathbb{Z}.$$

On the other hand, the restriction of E to V_0, which we denote by E_0, is identically 0. This implies that (as the symplectic pair)

$$(K_{E,\Pi}, e_E) = (K_{E_0,\Pi_0}, e_{E_0}) \oplus (K_{E_1,\Pi_1}, e_{E_1}).$$

By **Case I** applied to (V_0, E_0, Π_0), the group $K_{E_0,\Pi_0} = V_0/\Pi_0$ is *divisible* as a quotient of a complex vector space, and $e_{E_0} \equiv 1$. By **Case II** applied to (V_1, E_1, Π_1), the group K_{E_1,Π_1} is *finite* of order $|\mathrm{Pf}(E,\Pi)|^2$ and the pairing

$$e_{E_1} : K_{E_1,\Pi_1} \times K_{E_1,\Pi_1} \to \mathbb{C}^*$$

is *nondegenerate*. This implies that $\ker(e_E) = K_{E_0,\Pi_0}$ and therefore $\ker(e_E)$ is divisible and

$$K_{E,\Pi}/\ker(e_E) = K_{E_1,\Pi_1}$$

is a finite group. This implies that $(K_{E,\Pi}, e_E)$ is almost isotropic and its isotropy defect, by Theorem 7.17,

$$d_{K_{E,\Pi},e_E} = \sqrt{\#(K_{E,\Pi}/\ker(e_E))} = \sqrt{\#(K_{E_1,\Pi_1})} = |\mathrm{Pf}(E_1,\Pi_1)|. \quad (19)$$

Case IIIbis We keep the notation and assumptions of **Case III**. Let

$$M : V \times V \to \mathbb{R}$$

be an alternating bilineaer form that enjoys the following properties.

(1) $M(\Pi,\Pi) \subset \mathbb{Z}$.
(2) $\ker(E) \subset \ker(M)$.

If n is an integer then we write $\mathbf{M}(n)$ for the alternating bilnear form $nE + M$ on V. Clearly,

$$\mathbf{M}(n)(\Pi,\Pi) \subset nE(\Pi,\Pi) + M(\Pi,\Pi) \subset n\mathbb{Z} + \mathbb{Z} = \mathbb{Z}.$$

Lemma 8.3. *There exists a degree $(g-g_0)$ polynomial $\mathscr{P}(t) \in \mathbb{Z}[t]$ of degree $g-g_0$ with integer coefficients and leading coefficient $|\mathrm{Pf}(E_1,\Pi_1)|$ that enjoys the following property.*

For all but finitely many positive integers n the symplectic pair $(K_{\mathbf{M}(n),\Pi}, e_{\mathbf{M}(n)})$ is almost isotropic and its isotropy defect

$$D_{K_{\mathbf{M}(n),\Pi},e_{\mathbf{M}(n)}} = \mathscr{P}(n). \quad (20)$$

Proof. Indeed, let $M_1 : V_1 \times M_1 \to \mathbb{R}$ be the restriction of M to V_1. Let $\tilde{E}_1$ and $\tilde{M}_1$ be the matrices of E_1 andM_1 with respect to the basis $\{f_1, \dots, f_{2g-2g_0}\}$ of Π_1. The nondegeneracy of E_1 implies that $\det(\tilde{E}_1) \neq 0$ and therefore

$$\det(n\tilde{E}_1 + \tilde{M}_1) = \det(\tilde{E}_1)\det(n\mathrm{I}_{2g-2g_0} + \tilde{E}_1^{-1}\tilde{M}_1)$$

does *not* vanish for all but finitely many integers n. (Hereafter I_{2g-2g_0} is the identity square matrix of size $2g-2g_0$.) Taking into account that $n\tilde{E}_1 + \tilde{M}_1$ is the matrix of the restriction of $nE + M = \mathbf{M}(n)$, we obtain that for all but finitely many integers n

$$\ker(\mathbf{M}(n)) = \ker(nE + M) = \ker(E) = V_0. \tag{21}$$

In what follows, we assume that n is any integer that enjoys the property (21) (this assumption excludes only finitely many integers n). Now we may apply results of **Case III** to $\mathbf{M}(n) = nE + M$ (instead of E) and get that $(K_{\mathbf{M}(n),\Pi}, e_{\mathbf{M}(n)})$ is almost isotropic and its isotropy defect is

$$|\mathrm{Pf}(nE_1{+}M_1, \Pi_1)| = \sqrt{\det(nE_1 + M_1, \Pi_1)} = \sqrt{\det(\tilde{E}_1)\det(n\mathrm{I}_{2g-2g_0} + \tilde{E}_1^{-1}\tilde{M}_1)} =$$

$$|\mathrm{Pf}(E_1, \Pi_1)|\sqrt{\det(n\mathrm{I}_{2g-2g_0} + \tilde{E}_1^{-1}\tilde{M}_1)}.$$

Clearly, there is a polynomial $\mathscr{Q}(t) \in \mathbb{Z}[t]$ with integer coefficients such that for *all* our n

$$\mathscr{Q}(n) = \mathrm{Pf}(n\tilde{E}_1 + \tilde{M}_1).$$

This implies that

$$\mathscr{Q}(n)^2 = \det(n\tilde{E}_1 + \tilde{M}_1) = \det(\tilde{E}_1)\det(n\mathrm{I}_{2g-2g_0} + \tilde{E}_1^{-1}\tilde{M}_1).$$

It is also clear that there exists a *monic* degree $(2g - 2g_0)$ polynomial $\mathscr{R}(t) \in \mathbb{Q}[t]$ with rational coefficients such that for all our n

$$\mathscr{R}(n) = \det(n\mathrm{I}_{2g-2g_0} + \tilde{E}_1^{-1}\tilde{M}_1).$$

This implies that

$$\mathscr{Q}(n)^2 = \det(\tilde{E}_1)\mathscr{R}(n) = |\mathrm{Pf}(E_1, \Pi_1)|^2\mathscr{R}(n).$$

Since $\mathscr{R}(t)$ is monic of degree $(2g - 2g_0)$, we have

$$\deg(\mathscr{Q}) = (g - g_0)$$

and the leading coefficient of $\mathscr{Q}(t)$ is $\pm|\mathrm{Pf}(E_1, \Pi_1)|$.

Let $\mathscr{P}(t)$ be the polynomial with posittive leading coefficient that coincides either with $\mathscr{Q}(t)$ or with $-\mathscr{Q}(t)$. Then $\mathscr{P}(t)$ is a degree $(g - g_0)$ polynomial with integer coefficients and leading coefficient $|\mathrm{Pf}(E_1, \Pi_1)|$ such that

$$\mathscr{P}(n) = \pm\mathrm{Pf}(n\tilde{E}_1 + \tilde{M}_1).$$

Since the leading coefficient of $\mathscr{P}(t)$ is positive, $\mathscr{P}(n)$ is positive for all but finitely many positive integers n. This implies that

$$\mathscr{P}(n) = |\mathrm{Pf}(n\tilde{E}_1 + \tilde{M}_1)| = |\mathrm{Pf}(nE_1 + M_1, \Pi_1)|$$

for all such n. This ends the proof. $\square$

Theorem 8.4. *Let g be a positive integer, V a $2g$-dimensional real vector space, (V, E, Π) and (V, M, Π) are admissible triples such that*

$$E \not\equiv 0, \ \ker(E) \subset \ker(M).$$

If n is an integer then we write $\mathbf{M}(n)$ for the alternating bilinear form $nE + M$ on V.

Let $\mathcal{G}$ be a group that enjoys the following properties.

There are infinitely many positive integers n such that $\mathcal{G}$ contains a subgroup G_n that is a theta group attached to $(K_{\mathbf{M}(n),\Pi}, e_{\mathbf{M}(n)})$.

Then $\mathcal{G}$ is not Jordan.

Proof. It suffices to check that the Jordan index of G_n tends to infinity while n tends to infinity. But this assertion follows from results of **Cases II, III, IIIbis** of this section combined with Theorem 7.17. □

9. Line bundles over tori and theta groups

In this section we use results from previous two sections in order to compute the Jordan index of certain automorphism groups of holomorphic line bundles on complex tori.

Let V be a complex vector space of finite positive dimension g, Π a discrete lattice of rank $2g$ in V,

$$H : V \times V \to \mathbb{C}$$

an Hermitian form on V such that its imaginary part

$$E : V \times V \to \mathbb{R}, \ (v_1, v_2) \mapsto \operatorname{Im}(H(v_1, v_2))$$

satisfies

$$E(\Pi, \Pi) \subset \mathbb{Z}.$$

One may view V as the $2g$-dimensional real vector space. Then E becomes an alternating $\mathbb{R}$-bilinear form on V such that

$$E(\mathbf{i}v_1, \mathbf{i}v_2) = E(v_1, v_2) \ \forall v_1, v_2 \in V.$$

In addition,

$$H(v_1, v_2) = E(\mathbf{i}v_1, v_2) + \mathbf{i}E(v_1, v_2) \ \forall v_1, v_2 \in V$$

(see [CAV, Lemma 2.1.7]). This implies that H and E have the same kernels, i.e.,

$$\ker(H) := \{w \in V \mid H(w, V) = 0\} = \{w \in V \mid E(w, V) = 0\} =: \ker(E).$$

Definition 9.1 (see [BL], [Ke]). A pair (H, α) is called an *Appel-Humbert data* (A.-H. data) on (V, Π) if H, E, Π are as above and α is a map ("semicharacter")

$$\alpha : \Pi \to \mathbf{U}(1) = \{z \in \mathbb{C}, |z| = 1\} \subset \mathbb{C}^*$$

such that

$$\alpha(l_1 + l_2) = (-1)^{E(l_1, l_2)} \alpha(l_1)\alpha(l_2) \ \forall l_1, l_2 \in \Pi. \tag{22}$$

In particular, if $l_1 = l_2 = 0$ then $\alpha(0) = \alpha(0)^2$, i.e.,

$$\alpha(0) = 1.$$

Notice that a classical theorem of Appel-Humbert ([Ke, Theorem 1.5], [BL, Theorem 21.1]) classifies holomorphic line bundles on the complex torus V/Π in terms of A.-H. data.

The construction of Section 8 gives us the symplectic pair $(K_{E,\Pi}, e_E)$. The aim of this section is to constuct a certain theta group $\mathfrak{G}(H,\alpha)$ attached to this pair that corresponds to any A.-H. data (H,α). We define $\tilde{\mathfrak{G}}(H,V)$ as a certain group of biholomorphic automorphisms of $\mathscr{L}(H,\alpha)$. Here $\mathscr{L}(H,\alpha)$ is the *total body* of the holomorphic line bundle $\mathscr{L}(H,\alpha)$ over V/Π that corresponds to A.-H. data (H,α).

First, we start with a certain theta group $\tilde{\mathfrak{G}}(H,V)$ attached to the symplectic pair $(V, \tilde{e}_E)$ where

$$\tilde{e}_E : V \times V \to \mathbb{C}^*, \ (v_1, v_2) \mapsto \exp\left(2\pi\mathbf{i}E(v_1,v_2)\right).$$

We define $\tilde{\mathfrak{G}}(H,V)$ as a certain group of holomorphic automorphisms of

$$V_{\mathbb{L}} := V \times \mathbb{L}$$

where $\mathbb{L}$ is a one-dimensional $\mathbb{C}$-vector space. Namely, $\tilde{\mathfrak{G}}(H,V)$ consists of automorphisms $\mathscr{B}_{H,u,\lambda}$ indexed by $u \in V, \lambda \in \mathbb{C}^*$ that are defined as follows.

$$\mathscr{B}_{H,u,\lambda} : (v,c) \mapsto \big(v+u, \lambda\exp(\pi H(v,u)c\big) \ \forall v \in V, c \in \mathbb{L}.$$

One may easily check (see [Zar19, Sect. 2.1]) that the composition

$$\mathscr{B}_{H,u_1,\lambda_1} \circ \mathscr{B}_{H,u_2,\lambda_2} = \mathscr{B}_{H,u_1+u_2,\lambda_1\lambda_2\mu} \text{ where } \mu = \exp(\pi H(u_1,u_2)) \quad (23)$$

and the inverse

$$\mathscr{B}^{-1}_{H,u,\lambda} = \mathscr{B}_{H,-u,\nu/\lambda} \text{ where } \nu = \exp(-\pi H(u,u)). \quad (24)$$

This implies that $\tilde{\mathfrak{G}}(H,V)$ is indeed a subgroup of the group of biholomorphic automorphisms of $V_{\mathbb{L}}$. (Our $\mathfrak{G}(H,\alpha)$ will be defined as a subquotient of $\tilde{\mathfrak{G}}(H,V)$.) Notice that for all $\lambda \in \mathbb{C}^*$ the automorphism $\mathscr{B}_{H,0,\lambda}$ sends every (u,c) to $(u,\lambda c)$. This implies that the map

$$\mathrm{mult} : \mathbb{C}^* \to \tilde{\mathfrak{G}}(H,V), \ \lambda \mapsto \mathscr{B}_{H,0,\lambda}$$

is an injective group homomorphism, whose image lies in the center of $\tilde{\mathfrak{G}}(H,V)$. This allows us to include $\tilde{\mathfrak{G}}(H,V)$ in a short exact sequence of groups

$$1 \to \mathbb{C}^* \overset{\mathrm{mult}}{\to} \tilde{\mathfrak{G}}(H,V) \overset{\tilde{j}}{\to} V \to 0$$

where $\tilde{j}$ sends $\mathscr{B}_{H,u,\lambda}$ to u. It follows from (23) and (24) (see also [Zar19, Sect. 2.1]) that

$$\mathscr{B}_{H,u_1,\lambda_2} \circ \mathscr{B}_{H,u_2,\lambda_2} \circ \mathscr{B}^{-1}_{H,u_1,\lambda_1} \circ \mathscr{B}^{-1}_{H,u_2,\lambda_2} = \mathrm{mult}(\exp(2\pi\mathbf{i}E(u_1,u_2)) = \mathrm{mult}(\tilde{e}_E(u_1,u_2)). \quad (25)$$

This implies that $\tilde{\mathfrak{G}}(H,V)$ is a theta group attached to the symplectic pair $(V,\tilde{e}_E)$.

Let us consider the following subgroups of $\tilde{\mathfrak{G}}(H,V)$.

$$\tilde{\mathfrak{G}}(H,\Pi)=\tilde{j}^{-1}(\Pi)=\{\mathscr{B}_{H,u,\lambda}\mid \lambda\in\mathbb{C}^*, u\in\Pi\}; \tag{26}$$

$$\tilde{\mathfrak{G}}(H,\Pi_E^{\perp})=\tilde{j}^{-1}(\Pi_E^{\perp})=\{\mathscr{B}_{H,u,\lambda}\mid \lambda\in\mathbb{C}^*, u\in\Pi_E^{\perp}\}. \tag{27}$$

By Remark 7.15, $\tilde{\mathfrak{G}}(H,\Pi)$ and $\tilde{\mathfrak{G}}(H,\Pi_E^{\perp})$ are theta groups attached to the symplectic pairs $(\Pi,\tilde{e}\mid_{\Pi})$ and $(\Pi_E^{\perp},\tilde{e}\mid_{\Pi_E^{\perp}})$ respectively. Since $\Pi\subset\Pi_E^{\perp}$, the group $\tilde{\mathfrak{G}}(H,\Pi)$ is a subgroup of $\tilde{\mathfrak{G}}(H,\Pi_E^{\perp})$. It follows from (25) that $\tilde{\mathfrak{G}}(H,\Pi)$ is actually a *central* subgroup of $\tilde{\mathfrak{G}}(H,\Pi_E^{\perp})$, because

$$E(\Pi,\Pi_E^{\perp})=\{0\}.$$

We will define $\mathfrak{G}(H,\alpha)$ as a quotient of $\tilde{\mathfrak{G}}(H,\Pi_E^{\perp})$ by a certain central subgroup that depends on the "semicharacter" α. In order to define this subgroup, let us consider the *discrete free* action of the group Π on $V_{\mathbb{L}}$ by holomorphic automorphisms defined as follows. An element l of Π acts as

$$\mathscr{A}_{H,\alpha,l}:V_{\mathbb{L}}\to V_{\mathbb{L}},\ (v,c)\mapsto\left(v+l, c\alpha(l)\exp\left(\pi H(v,l)+\pi H(l,l)/2)\right)\right)\ \forall v\in V, c\in\mathbb{L}, \tag{28}$$

i.e.,

$$\mathscr{A}_{H,\alpha,l}=\mathrm{mult}(\alpha(l))\mathscr{B}_{H,l,1}\in\tilde{\mathfrak{G}}(H,\Pi). \tag{29}$$

Direct calculations that are based on (22) show that

$$\mathscr{A}_{H,\alpha,l_1}\mathscr{A}_{H,\alpha,l_2}=\mathscr{A}_{H,\alpha,l_1+l_2}\ \forall l_1,l_2\in\Pi,$$

i.e.,

$$\mathbf{A}^{\Pi}:\Pi\to\tilde{\mathfrak{G}}(H,\Pi),\ l\mapsto\mathscr{A}_{H,\alpha,l}$$

is an *injective* group homomorphism, whose image we denote by

$$\tilde{\Pi}=\tilde{\Pi}(H,\alpha):=\mathbf{A}^{\Pi}(\Pi)\subset\tilde{\mathfrak{G}}(H,\Pi)\subset\tilde{\mathfrak{G}}(H,\Pi_E^{\perp}).$$

Notice that $\tilde{\Pi}$ meets $\mathrm{mult}(\mathbb{C}^*)$ precisely at the identity element of $\tilde{\mathfrak{G}}(H,\Pi_E^{\perp})$. Notice that the quotient $V_{\mathbb{L}}/\tilde{\Pi}(H,\alpha)$ is precisely the total body $\mathscr{L}(H,\alpha)$ of the holomorphic vector bundle $\mathscr{L}(H,\alpha)$ over V/Π attached to the A.-H. data (H,α) where the structure map

$$p:\mathscr{L}(H,\alpha)=V_{\mathbb{L}}/\tilde{\Pi}(H,\alpha)\to V/\Pi$$

is induced by the projection map

$$V_{\mathbb{L}}=V\times\mathbb{L}\to V$$

[CAV, Ch. 2, Sect. 2.2, p. 30]. Let us put

$$\mathfrak{G}(H,\alpha):=\tilde{\mathfrak{G}}(H,\Pi_E^{\perp})/\tilde{\Pi}(H,\alpha). \tag{30}$$

The faithful action of $\tilde{\mathfrak{G}}(H,\Pi_E^{\perp})$ on $V_{\mathbb{L}}$ induces the *faithful* action of $\mathfrak{G}(H,\alpha)$ on $\mathscr{L}(H,\alpha)$. Under this action, each coset

$$\mathscr{B}_{H,u,\lambda}\tilde{\Pi}\in\tilde{\mathfrak{G}}(H,\Pi_E^{\perp})/\tilde{\Pi}(H,\alpha)=\mathfrak{G}(H,\alpha)$$

maps $\mathbb{C}$-*linearly* and isomorphically the fiber of p over $v+\Pi \in V/\Pi$ to the fiber over $(v+u)\Pi \in V/\Pi$ for any pair

$$u+\Pi \in \Pi_E^{\perp}/\Pi \subset V/\Pi, \quad \text{and} \quad v+\Pi \in V/\Pi, \quad \text{and} \quad \lambda \in \mathbb{C}^*.$$

In particular, $\mathrm{mult}(\lambda)\tilde{\Pi}$ acts as the automorphism $[\lambda]$ that leaves invariant each fiber of $p : \mathscr{L}(H,\alpha) \to V/\Pi$ and acts on this fiber (which is a one-dimensional $\mathbb{C}$-vector space) as multiplication by λ (for all $\lambda \in \mathbb{C}^*$). Clearly, each $[\lambda]$ lies in the *center* of $\mathfrak{G}(H,\alpha)$.

Lemma 9.2. *The group $\mathfrak{G}(H,\alpha)$ is a theta group attached to the symplectic pair $(K_{E,\Pi}, e_E)$.*

Proof. Clearly,

$$[\mathrm{mult}] : \mathbb{C}^* \to \mathfrak{G}(H,\alpha), \ \lambda \mapsto [\lambda]$$

is an *injective* group homomorphism, whose image $[\mathrm{mult}](\mathbb{C}^*)$ is a *central* subgroup of $\mathfrak{G}(H,\alpha)$. On the other hand, $\tilde{j}$ induces the surjective group homomorphism

$$j : \mathfrak{G}(H,\alpha) = \tilde{\mathfrak{G}}(H,\Pi_E^{\perp})/\tilde{\Pi} \twoheadrightarrow \Pi_E^{\perp}/\Pi = K_{E,\Pi},$$

$$\mathscr{B}_{H,u,\lambda}\tilde{\Pi} \mapsto u+\Pi \in \Pi_E^{\perp}/\Pi.$$

Clearly, the kernel of j consists of all $\mathscr{B}_{H,0,\lambda}\tilde{\Pi} = [\mathrm{mult}](\lambda)$, i.e., coincides with $[\mathrm{mult}](\mathbb{C}^*)$. Hence, $\mathfrak{G}(H,\alpha)$ sits in the short exact sequence

$$1 \to \mathbb{C}^* \stackrel{[\mathrm{mult}]}{\to} \mathfrak{G}(H,\alpha) \stackrel{j}{\to} \Pi_E^{\perp}/\Pi \to 0.$$

It follows from (25) that $\mathfrak{G}(H,\alpha)$ is a theta group attached to the symplectic pair $(K_{E,\Pi}, e_E)$. □

Remark 9.3. It is well known [CAV, Lemma 2.2.1] that if (H_1,α_1) and (H_2,α_2) are A.H. data on (V,Π) then $(H_1+H_2,\alpha_1\alpha_2)$ is also an A.H. data on (V,Π) and holomorphic vector bundles $\mathscr{L}(H_1+H_2,\alpha_1\alpha_2)$ and $\mathscr{L}(H_1,\alpha_1)\otimes\mathscr{L}(H_2,\alpha_2)$ are canonically isomorphic.

10. $\mathbb{P}^1$-bundles bimeromorphic to the direct product

In this section we prove the non-Jordanness of the groups of bimeromorphic selfmaps of certain $\mathbb{P}^1$-bundles over complex tori of positive algebraic dimension.

Let V be a complex vector space of finite positive dimension g, Π a discrete lattice of rank $2g$ in V and $T = V/\Pi$ the corresponding complex torus. Recall that $\mathbf{1}_T$ stands for the trivial holomorphic line bundle $T \times \mathbb{C}$ over T. If x is point of T then we write$\mathscr{L}_x$ for the fiber of a holomorphic vector bundle $\mathscr{L}$ over T, which is a one-dimensional complex vector space. We write $\bar{\mathscr{L}}$ for the projectivization $\mathbb{P}(\mathscr{E})$ of the two-dimensional holomorphic vector bundle $\mathscr{E} = \mathscr{L} \oplus \mathbf{1}_T$. The fiber $\mathscr{E}_x$ of $\mathscr{E}$ over x is the set of pairs (s_x, c) where $s_x \in \mathscr{L}_x, c \in \mathbb{C}$ and the fiber $\bar{\mathscr{L}}_x$ of $\bar{\mathscr{L}}$ over x is the set of equivalence classes of $(s_x : c)$ where either $s_x \neq 0$ or $c \neq 0$ and the equivalence class of $(s_x : c)$ is the set of all

$$(\mu s_x : \mu c), \ \mu \in \mathbb{C}^*.$$

Lemma 10.1. *Suppose that $\mathscr{L} = \mathscr{L}(H, \alpha)$ where (H, α) is an A.-H. data. Then there is a natural group embedding*

$$\mathfrak{G}(H, \alpha) \hookrightarrow \mathrm{Aut}(\overline{\mathscr{L}(H, \alpha)}).$$

Proof. First, let us define the group embedding

$$\mathfrak{G}(H, \alpha) \hookrightarrow \mathrm{Aut}(\mathscr{L}(H, \alpha) \oplus \mathbf{1}_T) \tag{31}$$

by the formula

$$g : \big(s_x, (x, c)\big) \mapsto \big(g(s_x), (x+j(g), c)\big) \ \forall g \in \mathfrak{G}(H, \alpha), x \in V/\Pi = T, c \in \mathbb{C}, \ s_x \in \mathscr{L}_x \subset \mathscr{L}. \tag{32}$$

In particular, g induces an isomorphism of two-dimensional complex vector spaces between the fibers of $\mathscr{L}(H, \alpha) \oplus \mathbf{1}_T$ over x and over $x + j(g)$. Since $\mathfrak{G}(H, \alpha) \to \mathrm{Aut}(\mathscr{L}(H, \alpha))$ is a group *embedding*, we conclude that if $j(g) = 0$ then g_x is multiplication by a scalar if and only if g is the identity element of $\mathfrak{G}(H, \alpha)$. This implies that (31) and (32) induce a group embedding

$$\mathfrak{G}(H, \alpha) \hookrightarrow \mathrm{Aut}(\mathbb{P}(\mathscr{L}(H, \alpha) \oplus \mathbf{1}_T)) = \mathrm{Aut}(\overline{\mathscr{L}(H, \alpha)}) \tag{33}$$

such that each $g \in \mathfrak{G}(H, \alpha)$ sends every $(s_x : c) \in \mathscr{L}(H, \alpha)_x$ to $(g(s_x) : c) \in \mathscr{L}(H, \alpha)_{x+j(g)}$. This ends the proof. $\square$

Let $\mathscr{L}$ be a holomorphic line bundle over the complex torus $T = V/\Pi$. Then $\mathscr{L} \cong \mathscr{L}(H, \alpha)$ for a certain (actually, precisely one) A.-H. data $H, \alpha)$ on (V, Π) ([Ke, Theorem 1.5]). Let us denote by $\mathfrak{G}(\mathscr{L})$ the group $\mathfrak{G}(H, \alpha)$. By Lemma 10.1, there exists a *group embedding*

$$\mathfrak{G}(\mathscr{L}) \hookrightarrow \mathrm{Aut}(\bar{\mathscr{L}}). \tag{34}$$

Lemma 10.2. *Let $\mathscr{L}$ and $\mathscr{N}$ be holomorphic line bundles over $T = V/\Pi$. Assume that $\mathscr{L}$ admits a nonzero holomorphic section. Then the compact complex manifolds $\bar{\mathscr{N}}$ and $\overline{\mathscr{L}^n \otimes \mathscr{N}}$ are bimeromorphic for all positive integers n. In particular, for all such n there is a group embedding*

$$\mathfrak{G}(\mathscr{L}^n \otimes \mathscr{N}) \hookrightarrow \mathrm{Bim}(\bar{\mathscr{N}}). \tag{35}$$

Proof. Let t be a nonzero section of $\mathscr{L}$. Then t^n is a nonzero section of $\mathscr{L}^n$. So, it suffices to prove the Lemma for $n = 1$, i.e., to prove that $\bar{\mathscr{L}}$ and $\overline{\mathscr{L} \otimes \mathscr{N}}$ are *bimeromorphic*.

The holomorphic $\mathbb{C}$-linear map of rank 2 vector bundles

$$\mathscr{N} \oplus \mathbf{1}_T \to (\mathscr{L} \otimes \mathscr{N}) \oplus \mathbf{1}_T, \ \big(s_x; (x, c)\big) \mapsto \big(s_x \otimes t(x; (x, c)\big) \ \forall x \in T, s_x \in \mathscr{N}_x, c \in \mathbb{C}$$

induces a bimeromorphic isomorphism of their projectivizations $\bar{\mathscr{N}}$ and $\overline{\mathscr{L} \otimes \mathscr{N}}$. Hence, the groups $\mathrm{Bim}(\bar{\mathscr{N}})$ and $\mathrm{Bim}(\overline{\mathscr{L} \otimes \mathscr{N}})$ are isomorphic. Now the second assertion of our Lemma follows from Lemma 10.1.

□

Corollary 10.3. *We keep the notation and assumptions of Lemma 10.2. In particular, $\mathscr{L}$ is isomorphic to $\mathscr{L}(H,\alpha)$ and admits a nonzero holomorphic section.*

Suppose that $\mathscr{N}$ is isomorphic to $\mathscr{L}(H_0,\beta)$ where the kernel $\ker(H_0)$ *of the Hermitian form H_0 contains the kernel* $\ker(H)$ *of the Hermitian form H.*

Then the group $\mathrm{Bim}(\bar{\mathscr{N}})$ *is not Jordan.*

Proof. Let us consider the alternating $\mathbb{R}$-bilinear forms $E := \mathrm{Im}(H)$ and $M := \mathrm{Im}(H_0)$ on V. We have

$$\ker(E) = \ker(H) \subset \ker(H_0) = \ker(M)$$

and therefore $\ker(E) \subset \ker(M)$. Notice also that the alternating form $\mathbf{M}(n) = nE+M$ is the imaginary part of the Hermitian form $nH+H_0$ for all positive integers n; in addition, obviously, the holomorphic line bundle

$$\mathscr{L}^n \otimes \mathscr{N} \cong \mathscr{L}(H,\alpha)^n \otimes \mathscr{L}(H_0,\beta) = \mathscr{L}(nH+H_0,\alpha\beta^n) = \mathscr{L}(\mathbf{M}(n),\alpha\beta^n).$$

In light of Lemma 10.2, there is a group embedding

$$\mathfrak{G}(nH+H_0,\alpha\beta^n) \hookrightarrow \mathrm{Bim}(\bar{\mathscr{N}}).$$

On the other hand, applying Lemma 9.2 to $(nH+H_0,\alpha\beta^n)$ (instead of (H,α)), we conclude that $\mathfrak{G}(nH+H_0,\alpha\beta^n)$ is a *theta group* attached to the *symplectic pair* $\big(K_{\mathbf{M}(n),\Pi}, e_{\mathbf{M}(n)}\big)$. Now the desired result follows from Theorem 8.4.

□

Definition 10.4. Let $T = V/\Gamma$ be a complex torus. We write T_a for its algebraic model, which is also a complex torus (even an abelian variety) provided with a surjective holomorphic homomorphism of complex tori

$$\pi_a : T \twoheadrightarrow T_a$$

with connected kernel (actually, all the fibers of π_a are connected) [BL, Ch. 2, Sect. 6]. We write $\dim_a(T)$ for $\dim(T_a)$ and call it the *algebraic dimension* of T.

Clearly,

$$\dim(T_a) \leq \dim(T);$$

the equality holds if and only if $T = T_a$, i.e., T is an *abelian variety*.

Theorem 10.5 (Theorem 1.7 of [Zar19])**.** *Suppose that a complex torus $T = V/\Pi$ has positive algebraic dimension. Then* $\mathrm{Bim}(T \times \mathbb{P}^1)$ *is not Jordan.*

Proof. Take $\mathscr{N} = \mathbf{1}_T$. Then $\bar{\mathscr{N}} = T \times \mathbb{P}^1$. On the other hand, $\mathscr{N} = \mathbf{1}_T \cong \mathscr{L}(\mathbf{0},\mathbf{1})$ where $\mathbf{0}$ is the *zero* Hermitian form on V and

$$\mathbf{1}_\Pi : \Pi \to \{1\} \subset \mathbf{U}(1) \subset \mathbb{C}^*$$

is the constant semicharacter (actually, a character) of Π that identically equals 1. Clearly,

$$\ker(\mathbf{0}) = V.$$

Since $\dim_a(T) > 0$, the algebraic model T_a is a *positive-dimensional* abelian variety. Then T_a admits an *ample* holomorphic line bundle $\mathscr{L}_a$ with a nonzero section. Since $\psi : T \to T_a$ is surjective, the inverse image $\mathscr{L} = \psi^* \mathscr{L}_a$ is a holomorphic line bundle on T that also admits a nonzero section. We have $\mathscr{L} \cong \mathscr{L}(H, \alpha)$ for some A.-H. data (H, α). Obviously,

$$\ker(H) \subset V = \ker(\mathbf{0}).$$

Therefore we may apply Corollary 10.3 and obtain that the group $\mathrm{Bim}(\bar{\mathscr{N}})$ is *not* Jordan. It remains to recall that $\bar{\mathscr{N}} = T \times \mathbb{P}^1$.

□

The following assertion is a generalization of Theorem 10.5.

Theorem 10.6 (A special case of Theorem 1.8 in [Zar19])**.** *Let $\psi : T \to A$ be a surjective holomorphic group homomorphism from a complex torus $T = V/\Pi$ to a positive-dimensional complex abelian variety A. Let $\mathscr{M}$ be a holomorphic line bundle over A and $\mathscr{F}$ be a holomorphic line bundle over T that is isomorphic to the inverse image $\psi^*\mathscr{M}$.*

Then the group $\mathrm{Bim}(\bar{\mathscr{F}})$ is not Jordan.

Proof. A positive-dimensional complex abelian variety A is a complex torus $A = W/\Gamma$ (where W is a complex vector space of finite positive dimension m and Γ a discrete lattice of rank $2m$ in W) that admits a *polarization*, i.e., a *positive* (and therefore *nondegenerate*) Hermitian form

$$\mathbf{H}_A : W \times W \to \mathbb{C},$$

whose imaginary part

$$\mathbf{E}_A : W \times W \to \mathbb{R},\ (w_1, w_2) \mapsto \mathrm{Im}(\mathbf{H}_A(w_1, w_2))$$

satisfies the condition

$$\mathbf{E}_A(\Gamma, \Gamma) \subset \mathbb{Z}.$$

Replacing if necessary, $\mathbf{H}_A$ by $2\mathbf{H}_A$, we may and will assume that

$$E_A(\Gamma, \Gamma) \subset 2 \cdot \mathbb{Z}.$$

Then obviously $(\mathbf{H}_A, \mathbf{1}_\Gamma)$ is an A.H. data on (W, Γ). The positiveness of $\mathbf{H}_A$ implies that the corresponding holomorphic line bundle $\mathscr{L}(\mathbf{H}_A, \mathbf{1})$ over A has a *nonzero* holomorphic section (the corresponding theta function) (see [Ke, Theorem 2.1]).

It follows from [BL, Lemma 2.3.4 on p. 33] that every surjective holomorphic homomorphism $\psi : T \to A$ is induced by a certain surjective $\mathbb{C}$-linear map $\bar{\psi} : V \to W$ in the sense that

$$\bar{\psi}(\Pi) \subset \Gamma; \quad \psi(v + \Pi) = \bar{\psi}(v) + \Gamma \in W/\Gamma = A\ \forall v + \Pi \in V/\Pi = T.$$

The surjectiveness of ψ implies that the induced holomorphic line bundle $\mathscr{L} = \psi^*\mathscr{L}(\mathbf{H}_A, \mathbf{1}_\Gamma)$ over T also has a *nonzero* holomorphic section.

Let (H_A, β) be an A.-H. data on (W, Γ) and $\mathscr{L}(H_A, \beta)$ the corresponding holomorphic line bundle over $A = W/\Gamma$. Then the inverse image $\psi^*\mathscr{L}(H_A, \beta)$ is isomorphic to $\mathscr{L}(H_A \circ \bar{\psi}, \beta \circ \bar{\psi})$ where the A.-H. data $(H_A \circ \bar{\psi}, \beta \circ \bar{\psi})$ for (V, Γ) is as follows (see [Ke, Lemma 2.3.4]).

$$H_A \circ \bar{\psi} : V \times V \to \mathbb{C},\ (v_1, v_2) \mapsto H_A(\bar{\psi}v_1, \bar{\psi}v_2);\ \ \beta \circ \bar{\psi} : \Pi \to \mathbf{U}(1),\ l \mapsto \beta(\bar{\psi}(l)). \tag{36}$$

In light of the nondegeneracy of $\mathbf{H}_A$, this implies that

$$\ker(\mathbf{H}_A \circ \psi) = \ker(\bar{\psi}) \subset \ker(H_A \circ \bar{\psi}) \subset V. \tag{37}$$

Now let (H_A, β) be the A.-H. data on (W, Γ) such that $\mathscr{M}$ is isomorphic to $\mathscr{L}(H_A, \beta)$. In light of (36), $\mathscr{F}$ is isomorphic to $\mathscr{L}(H_A \circ \bar{\psi}, \beta \circ \bar{\psi})$. In particular, $\mathscr{L} = \psi^*\mathscr{L}(\mathbf{H}_A, \mathbf{1}_\Gamma)$ is isomorphic to $\mathscr{L}(\mathbf{H}_A \circ \bar{\psi}, \mathbf{1}_\Pi)$. (Here

$$\mathbf{1}_\Pi = \mathbf{1}_\Gamma \circ \bar{\psi} : \Pi \to \{1\} \subset \mathbf{U}(1)$$

is the trivial character of Π.) Since $\mathscr{L}$ admits a nonzero holomorphic section, the inclusion (37) allows us to apply Corollary 10.3 to $\mathscr{N} = \mathscr{F}$ and $H_0 = H_A \circ \bar{\psi}$, and conclude that $\mathrm{Bim}(\bar{\mathscr{F}})$ is *not* Jordan.

Remark 10.7. Let V, Π, T and $\mathscr{F}$ be as in Theorem 10.6. Suppose that $\mathscr{F} \cong \mathscr{L}(H, \alpha)$. Let $\alpha' : \Pi \to \mathbf{U}(1)$ be a map such that (H, α') is also an A.H. data on (V, Π). Let $\mathscr{F}'$ be a holomorphic line bundle on T that is isomorphic to $\mathscr{L}(H, \alpha')$. Then the same arguments as in the proof of Theorem 10.6 prove that $\mathrm{Bim}(\overline{\mathscr{F}'})$ is also non-Jordan (see Theorem 1.8 of [Zar19]).

□

Chapter 4. Non-trivial $\mathbb{P}^1$−bundles over non-uniruled base

In this chapter we consider the group $\mathrm{Aut}(X)$ for a non-trivial $\mathbb{P}^1$−bundle over a non-uniruled compact complex connected Kähler manifold Y. Recall that there is homomorphism $\tau : \mathrm{Aut}(X) \to \mathrm{Aut}(Y)$ and its kernel is denoted by $\mathrm{Aut}(X)_p$. First we classify automorphisms $f \in \mathrm{Aut}(X)_p$, i.e. those automorphisms that do not move fibers of p. We get that if $\mathrm{Aut}(X)_p \neq \{id\}$ then either X or its double cover is a projectivization $\mathbb{P}(\mathscr{E})$ of rank two vector bundle over Y or its double cover, respectively. Thus, if Y is Kähler, so is X ([Vo, Proposition 3.5]). Thus the group $\mathrm{Aut}(X)$ is Jordan by a Theorem of Jin Hong Kim ([Kim]). It appears that if X is scarce, (i.e. it does not have many sections, see Definition 11.5 below), then $\mathrm{Aut}_0(X)$ is commutative and $\mathrm{Aut}(X)$ is very Jordan. This is, for example, the case when Y is torus of algebraic dimension zero.

11. Automorphisms of $\mathbb{P}^1-$bundles that preserve fibers

This section contains the classification of those automorphisms of a $\mathbb{P}^1$-bundle X that preserve the fibers of $p : X \to Y$. There are three different types, each one is described in a separate subsection.

Let (X, p, Y) be a $\mathbb{P}^1$-bundle over a compact complex connected manifold Y, i.e.,

- X, Y are compact connected complex manifolds of positive dimension;
- $p : X \to Y$ is a surjective holomorphic map;
- X is a holomorphically locally trivial fiber bundle over Y with fiber $\mathbb{P}^1$ and with the corresponding projection map $p : X \to Y$.

Let P_y stand for the fiber $p^{-1}(y)$. Let $U \subset Y$ be an open non-empty subset of Y. We call a covering $U = \cup U_i, i \in I$, by open subsets of U_i of Y to be *fine* if for every $i \in I$ there exists an isomorphism $\phi_i : V_i = p^{-1}(U_i) \to U_i \times \mathbb{P}^1_{(x_i:y_i)}$ such that:

– $(u, z_i),\ u \in U_i, z_i = \frac{x_i}{y_i} \in \overline{\mathbb{C}}$, are local coordinates in $V_i := p^{-1}(U_i) \subset X$;

– $\mathbf{pr} \circ \phi_i = p$, where $\mathbf{pr} : U_i \times \mathbb{P}^1 \to \mathbb{P}^1$ is the natural projection (see **Notation and Assumptions**(14)).

Definition 11.1. An $k-$section S of p is a codimension 1 analytic subset $D \subset X$ such that the intersection $X \cap P_y$ is finite for every $y \in Y$ and consists of k distinct points for a general $y \in Y$. We call *bisection* a $2-$section that meets **every** fiber at two distinct points. Obviously, *usual* holomorphic section S of p is a 1-section. A section S is defined by the set $\mathbf{a} = \{a_i(y)\}$ of functions $a_i : U_i \to \mathbb{P}^1$ such that $p(y, a_i(y)) = id,\ y \in U_i$. We will denote this by $S = \mathbf{a}$.

Lemma 11.2. *Let A_1, A_2, A_3 be 3 distinct almost sections of p (see Definition 6.5). Assume that there is an analytic subspace $\Sigma \subset Y$ of codimension at least 2 such that $A_k, k = 1, 2, 3,$ are pairwise disjoint in $V = p^{-1}(U)$, where $U = Y \setminus \Sigma$.*

Then there exists an isomorphism $\Phi : X \to Y \times \mathbb{P}^1$ such that $\mathbf{pr} \circ \Phi = p$ where $\mathbf{pr} : Y \times \mathbb{P}^1 \to \mathbb{P}^1$ is the natural projection (see **Notation and Assumptions***(14)).*

Proof. Indeed, let $\{U_i\}$ be a fine covering of Y and let

$$a_{ki}(u)x_i - b_{ki}(u)y_i = 0, u \in U_i$$

be the eqaution of $A_k \cap U, k = 1, 2, 3,$ over U_i. We define a meromorphic function $F(x)$ in every V_i by

$$F(x) = \frac{(a_{1i}(u)x_i - b_{1i}(u)y_i)(a_{2i}(u)b_{3i}(u) - a_{3i}(u)b_{2i}(u))}{(a_{2i}(u)x_i - b_{2i}(u)y_i)(a_{1i}(u)b_{3i}(u) - a_{3i}(u)b_{3i}(u))},\ u = p(x). \tag{38}$$

Then $F(x)$ is globally everywhere defined and meromorphic in V. Its restrictions to $A_1 \cap V, A_2 \cap V, A_3 \cap V$ are equal to $0, \infty, 1$, respectively.

The fiber of p has dimension 1, thus $X \setminus V = p^{-1}(U)$ has codimension 2 in X. Thus the function F may be extended to a meromorphic function on the whole X by the Levi's continuation theorem (Theorem 5.9). Thus, we have the bimeromorphic map $\Phi : X \to Y \times \mathbb{P}^1, \Phi(x) = (p(x), F(x))$ that induces an isomorphism of V onto $U \times \mathbb{P}^1$ that is compatible with p. According to Lemma 5.12, Φ is an isomorphism. $\square$

Remark 11.3. In particular, if there are three disjoint sections in X then $X \sim Y \times \mathbb{P}^1$.

Remark 11.4. Note that a section is an almost section. If A is an almost section but not a section then the set

$$\Sigma(A) = \{y \in Y \mid p^{-1}(y) \subset A\} \subset Y$$

has codimension at least two because

–$\tilde{\Sigma} := p^{-1}(\Sigma(A))$ is a proper analytic subset of A with $\dim(A) = \dim(Y) = n$; thus $\dim(\tilde{\Sigma}) \leq n - 1$;

– every fiber of restriction of p to $\tilde{\Sigma}$ has dimension 1.

Definition 11.5. We say that three sections S_1, S_2, S_3 in X are good configuration if $S_1 \cap S_2 = S_1 \cap S_3 = \emptyset$ and $S_2 \cap S_3 \neq \emptyset$. We say that three almost sections A_1, A_2, A_3 in X are a special configuration if $A_1 \cap A_2 = A_1 \cap A_3 = A_2 \cap A_3$. We say that X is *scarce* if X admits no special configurations.

Lemma 11.6. *Let S_1, S_2, S_3, S_4 be 4 distinct sections of p such that $S_1 \cap S_2 = \emptyset$, $S_3 \cap S_4 = \emptyset$. Then $X \sim Y \times \mathbb{P}^1$.*

Proof. If $S_3 \cap (S_1 \cup S_2) = \emptyset$, then $X \sim Y \times \mathbb{P}^1$ (Remark 11.3). Assume that $X \not\sim Y \times \mathbb{P}^1$. Let $\emptyset \neq S_3 \cap S_2 = D \subset S_2$. Let $\{U_i\}, i \in I$ be a fine covering of Y. In every $V_i = p^{-1}(U_i)$ we choose coordinates (y, z_i) in such a way that $S_2 \cap V_i = \{z_i = 0\}, S_1 \cap V_i = \{z_i = \infty\}$. Then $z_j = \lambda_{ij} z_i$ in $V_i \cap V_j$, where λ_{ij} are non-vanishing in $U_i \cap U_j$ holomorphic functions.

Let $S_3 \cap V_i = \{(y, z_i = p_i(y)), y \in U_i\}$, where $p_j = \lambda_{ij} p_i$, and $S_4 \cap V_i = \{(y, z_i = q_i(y)), y \in U_i\}$, where $q_j = \lambda_{ij} q_i$. Then $r(y) := \frac{p_i(y)}{q_i(y)}$ is a globally defined meromorphic function on Y that omits value 1 (since $S_3 \cap S_4 = \emptyset$). Thus, $r := r(y) = constant$. But then q_i vanishes at D and $S_3 \cap S_4 \supset D$. Contradiction. $\square$

Remark 11.7. We proved also the following fact: If X contains two disjoint sections S_1 and S_2, then

- there is a holomorphic line bundle $\mathscr{L} := \mathscr{L}(S_1, S_2)$ such that $X \sim \mathbb{P}(\mathscr{L} \oplus \mathbf{1}_Y)$;
- there is a fine covering $\cup U_i, i \in I$ of Y and coordinates $(u, z_i), u \in U_i, z_i \in \overline{\mathbb{C}}$ in V_i, such that

$$S_1 \cap V_i = \{z_i = \infty\}, S_2 \cap V_i = \{z_i = 0\}.$$

- $z_j = a_{ij} z_i$, and cocycle $\mathbf{a} = \{a_{ij}\}$ defines $\mathscr{L}$.

Lemma 11.8. *If there exist 3 distinct almost sections A_1, A_2, A_3 of p then there exist a bimeromorphic map $\Phi : X \to Y \times \mathbb{P}^1$ such that $\mathbf{pr} \circ \Phi = p$.*

Proof. We maintain the notation of the proof of Lemma 11.2

Let

$$\Sigma(A_i) = \{y \in Y \ |p^{-1}(y) \subset A_i\},\ i = 1, 2, 3, \quad \text{and } \Sigma = \bigcup_1^3 \Sigma(A_i).$$

Let $\tilde{\Sigma} = p^{-1}(\Sigma)$.

The function $F(x)$ defined by Equation (38) is defined and meromporhic at every point outside the set

$$D = (A_1 \cap A_3) \cup (A_2 \cap A_3) \cup (A_1 \cap A_2) \cup \tilde{\Sigma}.$$

Since codimension of D is at least 2, the function F may be extended to a meromorphic function on X by the Levi Theorem. Consider a map $\Phi : X \to Y \times \mathbb{P}^1$. $x \mapsto (p(x), F(x))$. It is meromorphic and induces an isomorphism on every fiber $P_u, u \notin p(D)$ to $\mathbb{P}^1$. Thus Φ is bimeromorphic. $\square$

Lemma 11.9. *If X admits a good configuration S_1, S_2, S_3, then X admits a special configuration.*

Proof. By assumption $S_1 \cap S_2 = S_1 \cap S_3 = \emptyset, S_3 \cap S_2 \neq \emptyset$. Recall that S_2 is a zero section of the line bundle $\mathscr{L}(S_1, S_2)$ (see Remark 11.7). Let $\{U_i\}, i \in I$ be a fine covering of Y and $(u, z_i), u \in U_i, z_i \in \overline{\mathbb{C}}$ be coordinates in V_i, such that $S_1 \cap V_i = \{z_i = \infty\}, S_2 \cap V_i = \{z_i = 0\}$. Let the non-zero section of $\mathscr{L}$, namely, S_3 have the equation $z_i = h_i(u)$ in V_i. For any $c \in \mathbb{C}^*$ the equations $z_i = ch_i$ will also define a section $S_4 \neq S_3$ of $\mathscr{L}$. By construction, $S_2 \cap S_3 = S_2 \cap S_4 = S_3 \cap S_4 = \bigcup_{i \in I} \{h_i = 0\}$. Thus, S_2, S_3, S_4 is a special configuration. $\square$

We now consider the subgroup $\operatorname{Aut}(X)_p$ of those automorphisms f of X that do not move fibers of p, i.e., such that $p \circ f = f$. Similarly to Lemma 11.2, every $f \in \operatorname{Aut}(X)_p$ defines locally a holomorphic map $\psi_f : Y \to \mathrm{PSL}(2, \mathbb{C})$ and the function

$$\mathrm{TD}(y), y \to \mathrm{TD}(\psi_f(y)) = \frac{\mathrm{tr}^2(\psi_f(y))}{\det(\psi_f(y))}$$

(see **Notation**) is everywhere defined and holomorphic, hence constant on Y ([BZ20, Remark 4.9]). We denote this constant by $\mathrm{TD}(f)$.

Assume that $X \not\sim Y \times \mathbb{P}^1$. Let $f \in \operatorname{Aut}(X)_p, f \neq id$. Recall that $\operatorname{Fix}(f)$ is the set of all fixed points of f. Let $\{U_i\}, i \in I$ be a fine covering of Y. We summarize in Lemma 11.10 and Lemma 11.11 below the properties of *non-identity* automorphisms $f \in \operatorname{Aut}(X)_p$ with $\mathrm{TD}(f) \neq 4$ ([BZ20]).

Lemma 11.10. *Assume that (X, p, Y) is a $\mathbb{P}^1-$bundle and $X \not\sim Y \times \mathbb{P}^1$. Let $f \in \mathrm{Aut}(X)_p, f \neq id$, and $\mathrm{TD}(f) \neq 4$. Then one of two following cases holds.*

Case A. *The set $\mathrm{Fix}(f)$ consists of exactly two disjoint sections S_1, S_2 of p. We say that f is of type **A** with data (S_1, S_2), an ordered pair. In notation of Remark 11.7, let $\{U_i\}, i \in I$, $\mathscr{L}(S_1, S_2)$, and $\mathbf{a} = \{a_{ij}\}$ be the corresponding fine covering, holomorphic line bundle and cocycle, respectively. Then*

- *Defined is the number $\lambda_f \in \mathbb{C}^*$ such that in every V_i*

$$f(u, z_i) = (u, \lambda_f z_i); \tag{39}$$

- *If $G_0 \subset Aut(X)_p$ be the subgroup of all $f \in \mathrm{Aut}(X)_p$ such that $f(S_1) = S_1$, $f(S_2) = S_2$, then $G_0 \cong \mathbb{C}^*$;*
- *The restriction $f \to f\mid_{P_y}$ defines a group embedding of G_0 into $\mathrm{Aut}(P_y)$.*

Case C. *The set $\mathrm{Fix}(f)$ is a smooth unramified double cover S of Y. We will call such f an automorphism of type **C** with data S. Here S is a bisection of p.*

Proof. $\mathrm{TD}(f) \neq 4$ implies that f has exactly two distinct fixed points at **every** fiber $P_y = p^{-1}(y), y \in Y$. Thus $\mathrm{Fix}(f)$ is either a union of two disjoint sections or is a 2-section of p. Equation (39) follows from the fact that

$$f(u, z_i) = \lambda_i z_i,\ f(u, z_j) = \lambda_j z_j = \lambda_j a_{ij} z_i = a_{ij} \lambda_i z_i.$$

The constant $\lambda_f = \lambda_i \neq 0$ does not depend on the choice of the fiber, hence f is determined uniquely by its restriction to every given fiber. On the other hand for every $\lambda \in \mathbb{C}^*$ there exists an automorphism $f_\lambda \in \mathrm{Aut}(X)_p$ defined in every V_i by

$$(u, z_i) \to (u, \lambda z_i).$$

Therefore $G_0 \cong \mathbb{C}^*$. □

Lemma 11.11. *(see* [BZ20]*) Let S be a bisection of the $\mathbb{P}^1-$bundle (X, p, Y).*

Consider

$$\tilde{X} := \tilde{X}_S := S \times_Y X = \{(s, x) \in S \times X \subset X \times X \quad | \ p(s) = p(x)\}.$$

We denote the restriction of p to S by the same letter p, while p_X and $\tilde{p}$ stand for the restrictions to $\tilde{X}$ of the natural projections $S \times X \to X$ and $S \times X \to S$ respectively. We write $\mathrm{inv} : S \to S$ for the involution (the only non-trivial deck transformation for $p\,\big|_S$). Then $(\tilde{X}, \tilde{p}, S)$ is a $\mathbb{P}^1-$bundle with the following properties:

a) *The following diagram commutes*

$$\begin{array}{ccc} \tilde{X} & \xrightarrow{p_X} & X \\ {\scriptstyle \tilde{p}}\Big\downarrow & & {\scriptstyle p}\Big\downarrow . \\ S & \xrightarrow{p|_S} & Y \end{array} \tag{40}$$

b) $p_X : \tilde{X} \to X$ *is an unramified double cover of* X;

c) *Every fiber* $\tilde{p}^{-1}(s), s \in S$ *is isomorphic to*

$$P_{p(s)} = p^{-1}(p(s)) \sim \mathbb{P}^1;$$

d) *The* $\mathbb{P}^1$*-bundle* $\tilde{X}$ *over* S *has two disjoint sections, namely:*

$$S_+ := S_+(f) := \{(s, s) \in \tilde{X},\ s \in S \subset X\}$$

and

$$S_- := S_-(f) := \{(s, \mathrm{inv}(s)) \in \tilde{X},\ s \in S \subset X\}.$$

They are mapped onto S *isomorphically by* p_X.

e) *Every* $h \in \mathrm{Aut}(X)_p$ *induces an automorphism* $\tilde{h} \in \mathrm{Aut}(\tilde{X})_{\tilde{p}}$ *defined by*

$$\tilde{h}(s, x) = (s, h(x)).$$

f) *The involution* $s \to \mathrm{inv}(s)$ *may be extended from* S *to a holomorphic involution of* $\tilde{X}$ *by*

$$\mathrm{inv}(s, x) = (\mathrm{inv}(s), x);$$

g) *Every section* $N = \{y, \sigma(y)\}$ *of* p *in* X *induces the section* $\tilde{N} := \{(s, \sigma(p(s))\}$ *of* $\tilde{p}$ *in* $\tilde{X}$. *We have* $p_X(\tilde{N}) = N$ *is a section of* p, *thus* $\tilde{N}$ *cannot coincide* S_+ *or* S_-.

11.1. **Automorphisms with** $TD = 4$**.** If $f \in \mathrm{Aut}(X)_p, f \neq id$ and $\mathrm{TD}(f) = 4$, then there is precisely one fixed point of f in the fiber $P_y = p^{-1}(y)$ over the general point $y \in Y$. That means that $\mathrm{Fix}(f)$ contains precisely one almost section D of p. In this case we say that f is of type **B** with data D.

Lemma 11.12. *Let* (X, p, Y) *be a* $\mathbb{P}^1-$*bundle, where* X, Y *are compact connected complex manifolds,* $\dim(Y) = n$, $f \in \mathrm{Aut}(X)_p, f \neq id$, *and* $\mathrm{TD}(f) = 4$. *Let* D *be the only almost section contained in* $\mathrm{Fix}(f)$. *Let* $\Sigma = \{y \in Y \mid P_y \subset D\}$ *and* $U = Y \setminus \Sigma$, $V = p^{-1}(U) \subset X$. *Let* $\tilde{S}$ *be the union of all irreducible distinct from* D *components of* $\mathrm{Fix}(f)$ *and* $S = p(\tilde{S})$.

Then

(1) *there is a fine covering* $U_i, i \in J$ *of* U *and coordinates* (u, z_i) *in* $V_i = p^{-1}(U_i)$ *such that* $D \cap V_i = \{z_i = \infty\}$.

(2) $f(u, z_i) = (u, z_i + \tau_i(u))$, *where* τ_i *are holomorphic functions on* U_i;

(3) *if $i, j \in J$ then $z_j = \mu_{ij} z_i + \nu_{ij}$ where μ_{ij} and ν_{ij} are holomorphic functions in $U_i \cap U_j$ and μ_{ij} does not vanish. Moreover, μ_{ij} depend on D and the choice of coordinates in V_i but not on f.*
(4) *if $i, j \in J$ then $\tau_j = \mu_{ij} \tau_i$ in $U_i \cap U_j$.*
(5) *S has pure codimension 1 in Y.*

Proof. Recall that the set Σ has codimension at least two in Y. (Remark 11.4).

(1) follows from the fact that D is a section of p over U.

(2) follows from the fact that $D \subset \mathrm{Fix}(f)$, thus the restriction of f onto a fiber $P_y, y \in U_i$ is an automorphism of $\mathbb{P}^1$ which is either identity or has the only fixed point $z_i = \infty$.

(3) follows from the fact that z_j is obtained from z_i by an automorphism of $\mathbb{P}^1$ with $z = \infty$ fixed.

Since X admits an almost section, $X \sim \mathbb{P}(\mathscr{E})$ for some rank two holomorphic vector bundle $\mathscr{E}$ on Y with projection $\pi : \mathscr{E} \to Y$ ([Sh19, Lemma 3.5],Theorem 6.7). That means that we have a fine covering U_i and a cocycle $A_{ij} \in \mathrm{GL}(2, \mathscr{O}(U_i \cap U_j))$ of two by two transition matrices of $\mathscr{E}$ such that

- $\pi^{-1}(U_i) \sim U_i \times \mathbb{C}^2_{x_i, y_i}$;
- if $U_i \cap U_j \neq \emptyset$ then
$$A_{ij} \begin{bmatrix} x_i \\ y_i \end{bmatrix} = \begin{bmatrix} x_j \\ y_j \end{bmatrix}.$$

Since $D \cap V$ is a section of p over U we may choose a basis in $\mathbb{C}^2_{x_i, y_i}$ in such a way that the preimage of $D \cap U_i$ in $U_i \times \mathbb{C}^2_{x_i, y_i}$ is $U_i \times \{(x_i, 0)\},\ x_i \in \mathbb{C}$. For these coordinates

-
$$A_{ij} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \lambda_{i,j} \\ 0 \end{bmatrix}$$
-
$$A_{ij} = \begin{bmatrix} \lambda_{i,j} & b_{ij} \\ 0 & \tilde{\lambda}_{i,j} \end{bmatrix}, \tag{41}$$

  where $b_{ij}, \lambda_{i,j}, \tilde{\lambda}_{i,j}$ and
$$d_{ij} = \lambda_{i,j} \tilde{\lambda}_{i,j} = \det(A_{ij}) \tag{42}$$
  are holomorphic functions in $U_i \cap U_j$.

Let now $z_j = \frac{x_j}{y_j}, z_i = \frac{x_i}{y_i}$. Then
$$z_j = \frac{\lambda_{i,j} x_i + b_{ij} y_i}{y_i \tilde{\lambda}_{i,j}} = \mu_{ij} z_i + \nu_{ij}. \tag{43}$$

Thus $\mu_{ij} = \frac{\lambda^2_{i,j}}{d_{ij}} = \frac{\lambda_{i,j}}{\tilde{\lambda}_{i,j}}$ depends on the choice of D, and is defined by the eigenvalue of the basis vector in the invariant subspace representing D. It does not depend on the choice of f with the given data D.

Note that $\lambda_{i,j}$ and $\tilde{\lambda}_{i,j}$ form cocycles for the covering of U.

(4) follows from the fact that f is globally defined, and D is fixed, thus

$$f(u, z_j) = (u, z_j + \tau_j(u)) = (u, \mu_{ij} z_i + \nu_{ij} + \tau_j(u)) = (u, \mu_{ij}(z_i + \tau_i(u)) + \nu_{ij}).$$

(5) follows from the fact that τ_i are holomorphic and $S \cap U_i = \{\tau_i = 0\}$. Indeed, let $\tilde{S}_1 \subset \tilde{S}$ be an irreducible component of $\tilde{S}$. It cannot be an almost section, thus $S_1 = p(\tilde{S}_1)$ is a proper analytic subset of Y. Moreover, since $\tilde{\Sigma} \subset D$, we have: $\tilde{S}_1 \not\subset \tilde{\Sigma}$, $S_1 \not\subset \Sigma$. Thus, $S_1 \cap U$ is a dense open subset of S_1. Since $S \cap U_i = \{\tau_i = 0\}$ has pure codimension 1 (if $S \cap U_i \neq \emptyset$), the same is valid for every its component that intersect U_i. Thus, $\dim(S_1) = n - 1$.

□

Proposition 11.13. *We maintain the notation of Lemma 11.12. Let $S_1, \dots, S_k$ be all irreducible components of S. Then*

- *For every $l, 1 \leq l \leq k$, defined is a non-negative number n_l, that is the order of zero of τ_i along the component S_l if $S_l \cap U_i \neq \emptyset$. It depends on l but not on i. The holomorphic line bundle $\mathscr{L}(f)$ corresponding to the effective divisor $\Delta_f := \sum_1^k n_l S_l$ restricts to U to the holomorphic line bundle defined by the cocycle μ_{ij}.*
- *Let G_D be the subgroup of $\mathrm{Aut}(X)_p$ of all those $g \in \mathrm{Aut}(X)_p$, that have $\mathrm{TD}(g) = 4$ and $D \subset \mathrm{Fix}(g)$. Then G_D is isomorphic to the additive group of $H^0(Y, \mathscr{L}(f))$. Thus $G_D \cong (\mathbb{C}^+)^n$, $n > 0$.*

Proof. Let S_l be an irreducible component of S. For every U_i such that $S_l \cap U_i \neq \emptyset$ defined is the order n_{li} of zero of τ_i along S_l. In $U_i \cap U_j$ we have $\tau_j = \tau_i \mu_{ij}$. Since μ_{ij} does not vanish, τ_j has the same order of zero along $S_l \cap U_j$. Since S_l is irreducible and $U \cap S_l$ is open and dense in S_l, the order n_l is well defined (see, for example [H, Remarks 2.3.6]). By construction, the divisor of τ_i in U_i is $\Delta_f \cap U_i$, thus the transition, functions for $\mathscr{L}(f)$ in $U_i \cap U_j$ are $\tau_j / \tau_i = \mu_{ij}$.

Let $h \in \mathrm{Aut}(X)_p$, and $\mathrm{TD}(h) = 4$, and $D \subset \mathrm{Fix}(g)$. Applying item (3) of Lemma 11.12, we get $h(u, z_i) = (u, z_i + h_i(u))$ where $h_j = \mu_{ij} h_i$. Thus the function defined in every U_i by $G_h(u) = \frac{h_i}{\tau_i}$ is meromorphic in U. By the Levi Theorem, $G_h(u)$ is meromorphic on Y. By construction, its divisor $(G_h) \geq -\Delta_f$, thus $G \in H^0(Y, \mathscr{L}(f))$.

On the other hand, let G be a meromorphic function on Y with divisor $(G) \geq -\Delta_f$ (i.e., $G \in H^0(Y, \mathscr{L}(f))$). For every i the function $h_i = G\tau_i$ is holomorphic in U_i, hence we can define a holomorphic automorphism of every $V_i = p^{-1}(U_i)$ by

$$h(u, z_i) = (u, z_i + h_i(u)). \tag{44}$$

Since $h_j := \mu_{ij}h_i$, the map h is an automorphism of V. Moreover, all the points of $D \cap V = \cup\{z_i = \infty\}$ are fixed by h. By Lemma 5.13 it may be extended to a bimeromorphic map of X.

By Lemma 5.12, $h \in \mathrm{Aut}(X)_p$. Moreover, $\mathrm{Fix}(\tilde{h})$ contains the closure of $D \cap V$, that is D. In the general fiber P_y of p it has precisely one fixed point $D \cap P_y$, thus $\mathrm{TD}(h) = 4$.

Thus, we get a one-to-one map

$$\phi : G_D \to H^0(Y, \mathscr{L}(f)), \quad h \in G_D \mapsto G_h \in H^0(Y, \mathscr{L}(f)).$$

From item (3) of Lemma 11.12 we get that the composition of $g, h \in \mathrm{Aut}(X)_p$ is defined by the cocycle $g_i + h_i$ of corresponding cocycles, which implies that

$$\phi(h \circ g) = \phi(h) + \phi(g).$$

□

The next Lemma answers the question when an almost section $D \subset \mathrm{Fix}(f)$ is the section. We used this fact in [BZ20] while dealing with automorphisms of type **B**.

Lemma 11.14. *We maintain the notation of Lemma 11.12 and Proposition 11.13. If $\Delta_f = 0$ then D is a section.*

Proof. First, let us note that $\Delta_f = 0$ implies that corresponding line bundle $\mathscr{L}_f$ is trivial and that $f \neq id$ in a fiber $F_y = p^{-1}(y)$ if $y \notin \Sigma$.

Since X admits an almost section, $X \sim \mathbb{P}(\mathscr{E})$ for some rank two holomorphic vector bundle $\mathscr{E}$ on Y ([Sh19, Lemma 3.5], Theorem 6.7). That means that we have a fine covering $\{U_i\}_{i\in I}$ of Y and a cocycle A_{ij} of two by two matrices (with holomorphic in $U_i \cap U_j$ entrees) such that

(1) $p^{-1}(U_i) = V_i \sim U_i \times \mathbb{P}^1_{x_i:y_i}, z_i = \frac{x_i}{y_i}$ and if $U_i \cap U_j \neq \emptyset$ then

$$A_{ij}\begin{bmatrix} x_i \\ y_i \end{bmatrix} = \begin{bmatrix} x_j \\ y_j \end{bmatrix}$$

(2) In every U_i defined is a 2×2 matrix F_i (representing f) with holomorphic functions (in $u \in U_i$) as entries and with $\mathrm{TD}(F_i) = 4, \det(F_i) = d_i \neq 0$, and such that $f(u, (x_i : y_i)) = (u, (x_i' : y_i'))$, where

$$\begin{bmatrix} x_i' \\ y_i' \end{bmatrix} = F_i \begin{bmatrix} x_i \\ y_i \end{bmatrix}$$

(3)

$$F_j(u)A_{ij}(u) = A_{ij}(u)F_i(u)\frac{d_j}{d_i}.$$

Since $4d_i = \mathrm{tr}(F_i)^2$ is a square we may divide F_i by $\mathrm{tr}(F_i)/2 = \sqrt{d_i}$ and assume that $d_i = 1$ (we use that $(x_i : y_i)$ are homogeneous coordinates in $\mathbb{P}^1_{x_i:y_i}$).

Assume that D is not a section, i.e., $\Sigma = \{y \in Y \mid p^{-1}(y) \subset D\} \neq \emptyset$.

Let a fine covering of Y consist of open sets $U_0, \dots, U_N$ and $U_0, \dots, U_k$ intersect Σ while $U = Y \setminus \Sigma = \bigcup_{k+1}^{N} U_i$.

Then for each $i > k$ we may assume that

- $$F_i = \begin{bmatrix} 1 & \tau_i \\ 0 & 1 \end{bmatrix} = I + \tau_i V$$
  with I being the identity matrix, τ_i holomorphic functions in U_i, and $V = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ (by Lemma 11.12(2)).
- Recall that $\mathscr{L}_f \mid_U$ is defined on U by cocycle $\{\mu_{ij}\}$, where $\mu_{ij} = \tau_j/\tau_i$ is holomorphic non-vanishing function on $U_i \cap U_j$, if $U_i \subset U$ and $U_j \subset U$ (by Lemma 11.12). Since $\mathscr{L}_f$ is trivial, we may assume that cocycle $\{\mu_{ij}\}$ is trivial , i.e., $\mu_{ij} = 1$ and $\tau_i = 1$ do not depend on i for $U_i \subset U = Y \setminus \Sigma$. Moreover from Equations (42) and (43) we get that A_{ij} are triangular matrices, and for the eigenvalues $\lambda_{ij}, \tilde{\lambda}_{ij}$ of matrices A_{ij} we have $\lambda_{ij} = \tilde{\lambda}_{ij}$. hence, $\det(A_{ij}) = \lambda_{ij}^2$.

  Thus if both $i, j > k$, we may assume that
  $$A_{ij} = \begin{bmatrix} \lambda_{ij} & \nu_{ij} \\ 0 & \lambda_{ij} \end{bmatrix}$$
  where λ_{ij}, ν_{ij} are holomorphic functions in $U_i \cap U_j$.

Take a point $\mathbf{s} \in \Sigma$ and let U_0 be a neighborhood of $\mathbf{s}$. Let $\tilde{r}(\mathbf{s})$ be the number of those neighborhoods U_i with $i > k$ in our fine cover that have $U_i \cap U_0 \neq \emptyset$. Let $r = \tilde{r}(\mathbf{s})$. Let

$$U_t, \dots, U_{t+r}, t > k$$

those neighborhoods for which $U_i \cap U_0 \neq \emptyset, t \le i \le t + r$. For $t \le i, j \le t + r$ we have :

- $$F_0 = A_{i0}(u) F_i A_{i0}(u)^{-1} = I + W_i = I + A_{j0}(u) V A_{j0}(u)^{-1} = I + W_j,$$
  where $W_i = A_{i0}(u) V A_{i0}(u)^{-1}, t \le i \le t + r$. It follows that the matrix function W_i defined apriori in $U_0 \cap U_i$ may be extended as a matrix function with holomorphic entries to all U_0 and
  $$W_i = W_j. \tag{45}$$
- $$A_{i0}(u) A_{j0}^{-1}(u) = A_{ij}(u) \tag{46}$$
  whenever $U_i \cap U_j \cap U_0 \neq \emptyset$.
- Let
  $$A_{i0}(u) = \begin{bmatrix} \alpha_1(u) & \beta_1(u) \\ \gamma_1(u) & \delta_1(u) \end{bmatrix}, A_{j0}(u) = \begin{bmatrix} \alpha_2(u) & \beta_2(u) \\ \gamma_2(u) & \delta_2(u) \end{bmatrix}$$

Then

$$W_i(u) = \begin{bmatrix} -\alpha_1(u)\gamma_1(u) & \alpha_1^2(u) \\ -\gamma_1^2(u) & \alpha_1(u)\gamma_1(u) \end{bmatrix} = W_j(u) = \begin{bmatrix} -\alpha_2(u)\gamma_2(u) & \alpha_2^2(u) \\ -\gamma_2^2(u) & \alpha_2(u)\gamma_2(u) \end{bmatrix}, \tag{47}$$

$$A_{i0}(u)A_{j0}^{-1}(u) = \frac{1}{d_{j0}} \begin{bmatrix} \alpha_1\delta_2 - \beta_1\gamma_2 & -\alpha_1\beta_2 + \beta_1\alpha_2 \\ \gamma_1\delta_2 - \delta_1\gamma_2 & -\gamma_1\beta_2 + \delta_1\alpha_2 \end{bmatrix} = \begin{bmatrix} \lambda_{ij} & \nu_{ij} \\ 0 & \lambda_{ij} \end{bmatrix}. \tag{48}$$

Let $\tilde{U}_{ij} = U_i \cap U_j \cap U_0 \neq \emptyset$. From Equation (47) we get that in $\tilde{U}_{ij}$ we have $\alpha_1^2 = \alpha_2^2$ and $\alpha_1(u)\gamma_1(u) = \alpha_2(u)\gamma_2(u)$. Note that these equations are valid in all U_0, since W_i, W_j are defined there.

In $\tilde{U}_{ij}$ the following three cases are possible: $\alpha_1 = \alpha_2, \gamma_1 = \gamma_2$, or $\alpha_1 = -\alpha_2, \gamma_1 = -\gamma_2$, or $\alpha_1 = \alpha_2 = 0$.

Case 1. $\alpha_1 = \alpha_2, \gamma_1 = \gamma_2$ in $\tilde{U}_{ij}$. Plugging in this into Equation (48) we get the following:

$$\frac{1}{d_{j0}} \begin{bmatrix} \alpha_1\delta_2 - \beta_1\gamma_2 & -\alpha_1\beta_2 + \beta_1\alpha_2 \\ \gamma_1\delta_2 - \delta_1\gamma_2 & -\gamma_1\beta_2 + \delta_1\alpha_2 \end{bmatrix} = \frac{1}{d_{j0}} \begin{bmatrix} \alpha_1\delta_2 - \beta_1\gamma_1 & -\alpha_1\beta_2 + \beta_1\alpha_1 \\ \gamma_1\delta_2 - \delta_1\gamma_1 & -\gamma_2\beta_2 + \delta_1\alpha_2 \end{bmatrix} =$$

$$\frac{1}{d_{j0}} \begin{bmatrix} d_{i0} + \alpha_1(\delta_2 - \delta_1) & \alpha_1(\beta_1 - \beta_2) \\ \gamma_1(\delta_2 - \delta_1) & d_{j0} - \alpha_2(\delta_2 - \delta_1) \end{bmatrix} = \begin{bmatrix} \lambda_{ij} & \nu_{ij} \\ 0 & \lambda_{ij} \end{bmatrix}.$$

Thus there are once more two cases.

Case 1.1 $\gamma_1 \equiv 0$ in $\tilde{U}_{ij}$, hence $\gamma_1^2 = 0$ in U_0. Then in all U_0

$$F_0 = \begin{bmatrix} 1 & \alpha_1^2(u) \\ 0 & 1 \end{bmatrix}$$

and $\alpha_1^2(u)$ does not vanish in U_0 since $\operatorname{codim}(\Sigma) \leq 2$ and $\Delta_f = 0$, i.e $F_0(u) \neq \mathbf{I}$ if $u \notin \Sigma$. Thus $D \cap V_0 = \{y_0 = 0\}$ and $\Sigma \cap U_0 = \emptyset$. This contradicts to $\mathbf{s} \in \Sigma$.

Case 1.2 $\gamma_1 \not\equiv 0$, $\delta_2 \equiv \delta_1$ in $\tilde{U}_{ij}$. Then $1 = \lambda_{ij} = \frac{d_{i0}}{d_{j0}}$. Moreover $\beta_1 = \frac{\alpha_1\delta_1 - d_{i0}}{\gamma_1} = \beta_2 = \frac{\alpha_2\delta_2 - d_{j0}}{\gamma_2}$ and $\nu_{ij} = 0$ in $\tilde{U}_{ij} \cap \{\gamma_1 \neq 0\}$. Since this set is open in $U_i \cap U_j$ we have $\nu_{ij} \equiv 0$ and

$$A_{ij} \equiv \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

It follows that that there is a compatible with p isomorphism $V_i \cup V_j \sim (U_i \cup U_j) \times \mathbb{P}^1_z$, where $z = \frac{x_i}{y_i} = \frac{x_j}{y_j}$. Thus we can replace U_i, U_j by $U_i \cup U_j$ and obtain a new fine covering of Y consisting of $N-1$ open subsets and such that $\tilde{r}(\mathbf{s}) = r - 1$. Since U_0 is connected we can repeat this process (recall that $\gamma_1 = \gamma_2 \not\equiv 0$ in $U_i \cup U_j$ so we will stay in **Case 1.2**) till we get a covering with $\tilde{r}(\mathbf{s}) = 1$.

Thus, since $U_0 \setminus \Sigma$ was contained in $U_t \cup \ldots U_{t+r}$ we get $p^{-1}(U_0 \setminus \Sigma) \sim (U_0 \setminus \Sigma) \times \mathbb{P}^1_z$. By Lemma 5.12 and Lemma 5.13 it extends to an isomorphism and D is the preimage of $\{z = \infty\}$.

Case 2. $\alpha_1 = -\alpha_2, \gamma_1 = -\gamma_2$.

Plugging in this into Equation (48) we get the following:

$$\frac{1}{d_{j0}}\begin{bmatrix}\alpha_1\delta_2-\beta_1\gamma_2 & -\alpha_1\beta_2+\beta_1\alpha_2\\ \gamma_1\delta_2-\delta_1\gamma_2 & -\gamma_1\beta_2+\delta_1\alpha_2\end{bmatrix}=\frac{1}{d_{j0}}\begin{bmatrix}\alpha_1\delta_2+\beta_1\gamma_1 & -\alpha_1\beta_2-\beta_1\alpha_1\\ \gamma_1\delta_2+\delta_1\gamma_1 & \gamma_2\beta_2+\delta_1\alpha_2\end{bmatrix}=$$

$$\frac{1}{d_{j0}}\begin{bmatrix}-d_{i0}+\alpha_1(\delta_2+\delta_1) & -\alpha_1(\beta_1+\beta_2)\\ \gamma_1(\delta_2+\delta_1) & -d_{j0}-\alpha_1(\delta_2+\delta_1)\end{bmatrix}=\begin{bmatrix}\lambda_{ij} & \nu_{ij}\\ 0 & \lambda_{ij}\end{bmatrix}.$$

Similarly to **Case 1** we have

Case 2.1 $\gamma_1\equiv 0$. Then

$$F_0=\begin{bmatrix}1 & \alpha_1^2(u)\\ 0 & 1\end{bmatrix}$$

and D is a section of p over U_0.

Case 2.2 $\gamma_1\not\equiv 0, \quad \delta_2\equiv-\delta_1$ in $\tilde{U}_{ij}$. Then $-1=\lambda_{ij}=\frac{-d_{i0}}{d_{j0}}$.

Then $\beta_1=\frac{\alpha_1\delta_1-d_{i0}}{\gamma_1}=-\beta_2=-\frac{\alpha_2\delta_2-d_{j0}}{\gamma_2}$ and $\nu_{ij}=0$. Similarly to **Case 1.2** we get that $p^{-1}(U_0\setminus\Sigma)\sim(U_0\setminus\Sigma)\times\mathbb{P}^1_z$ and D is a section of p over U_0.

Case 3. $\alpha_1=\alpha_2=0$. According to Equation (47)

$$F_0=I+W_i=\begin{bmatrix}1 & 0\\ -\gamma_1^2(u) & 1\end{bmatrix}$$

and $\gamma_1^2(u)$ does not vanish in U_0 since $\Delta_f=0$. Thus $D\cap V_0=\{z=0\}$ that contradicts to $\mathbf{s}\in\Sigma$.

□

Remark 11.15. We may assume that a fine covering of Y contains a finite covering of U since $U_0\setminus\Sigma$ may be covered by two neighborhoods $U_0\cap\{\alpha_i\neq 0\}$ and $U_0\cap\{\gamma_i\neq 0\}$ (see Equation (47)).

Lemma 11.16. *Let $f\in\mathrm{Aut}(X)_p, f\neq id$ be an automorphism of type* **B** *with data D. Assume that there exists an almost section A of p distinct from D. Then X contains a special configuration.*

Proof. Since $A\neq D$, and $A\not\subset\mathrm{Fix}(f)$, we have $A_1:=f(A)\neq D$ and $A_1\neq A$. Similarly, $A_2:=f(A_1)\neq D$ and $A_2\neq A_1$. Let us show that $A_2\neq A$.

If $A_2=A$, then in the fiber $P_y=p^{-1}(y)$ over the general point $y\in Y$ there is point $a=A\cap P_y$ such that $f(a)\neq a$ but $f(f(a))=a$. But along the general fiber P_y the map f act as translation $z\to z+\tau$ where $\tau\neq 0$. This map has no periodic points except $z\neq\infty$. This contradiction shows that $A_2\neq A$.

Let us show that A, A_1, A_2 is a special configuration. For a fiber P_y we have the following options.

- $f\mid_{P_y}=id$. Then $P_y\cap A=P_y\cap A_1=P_y\cap A_2$;
- $f\mid_{P_y}$ is translation $z\to z+\tau$ and $P_y\cap A\neq P_y\cap D$. Then $P_y\cap A,\ P_y\cap A_1,\ P_y\cap A_2$ are pairwise disjoint sets.

- $f\,|_{P_y}$ is translation $z \to z+\tau$ and $a := P_y \cap A = P_y \cap D$. Then $P_y \cap A_1 = a, P_y \cap A_2 = a$.

It follows that $A\cap A_1 = A\cap A_2 = A_1 \cap A_2$ and A, A_1, A_2 is a special configuration. □

Corollary 11.17. *In the notation of Lemma 11.16, if X is scarce and $\mathrm{Aut}(X)_p$ contains an automorphism f of type* **B** *with data D then it contains no automorphisms of type* **B** *with another data and no automorphisms of type* **A**.

Proof. Indeed, the existence of such automorphisms would imply the existence of an almost section (in particular, section in case of type **A**) distinct from one contained in $\mathrm{Fix}(f)$. □

11.2. **Automorphisms of type A.**

Lemma 11.18. *Assume that $X \not\sim Y \times \mathbb{P}^1$. Let S_1, S_2 be two sections of p such that $S_1 \cap S_2 = \emptyset$. Let $f \in \mathrm{Aut}(X)_p$. Then one of the following holds.*

(1) $f(S_1) \subset S_1 \cup S_2$;
(2) $f(S_2) \subset S_1 \cup S_2$;
(3) $f(S_1 \cup S_2) = S_1 \cup S_2$.

Proof. Note that a fiberwise automorphism moves a section to a section. Let $S_3 = f(S_1), S_4 = f(S_2)$. Since $S_1 \cap S_2 = \emptyset$, we have $S_3 \cap S_4 = \emptyset$. According to Lemma 11.6 it may happen only if the pairs (S_3, S_4) and (S_1, S_2) share a section. This may happen only if one of the sections of the pair (S_3, S_4) coincides with either S_1 or S_2. □

Recall that the group G_0 of all those $f \in \mathrm{Aut}(X)_p$ that have data (S_1, S_2) is isomorphic to $\mathbb{C}^*$ (see Lemma 11.10).

Assume that the holomorphic line bundle $\mathscr{L}(S_1, S_2)$ is defined by cocycle $\{\lambda_{ij}\}$ and $\mathscr{L}(S_1, S_2)^{\otimes 2}$ has a section $T \subset X$ defined by $\mathbf{a} := \{a_i(y)\},\ a_j(y) = \lambda_{ij}^2 a_i(y)\}$.

Define
$$\phi_T : X \to X, \phi_T(y, z_i) = (y, \frac{a_i(y)}{z_i}).$$
The fixed point set $\mathrm{Fix}(\phi_T) = \{\phi_T(y, z_i) = (y, z_i)\}$ is defined by $T \cap V_i = \{z_i^2 = a_i\}$. If $\phi_T \in \mathrm{Aut}(X)_p$, then a_i do not vanish. In this case $\mathbf{a} := \{a_i\}$ provide a section of $\mathscr{L}_p^{\otimes 2}$ that does not meet the zero section, thus $\mathscr{L}_p^{\otimes 2}$ is a trivial bundle and we may define z_i in such a way that $a_i = a = const \neq 0$. We will then write $T = T_a$ and $\phi_a := \phi_T$.

Proposition 11.19. *Let (X, p, Y) be a $\mathbb{P}^1-$bundle, where X, Y are compact connected complex manifolds, and $X \not\sim Y \times \mathbb{P}^1$. Let S_1, S_2 be two sections of p such that $S_1 \cap S_2 = \emptyset$. Let $\mathscr{L} := \mathscr{L}(S_1, S_2)$ be the corresponding holomorphic line bundle over Y. Let*

- *$G_1 \subset \mathrm{Aut}(X)_p$ be the subgroup of all $f \in \mathrm{Aut}(X)_p$ such that $f(S_1) = S_1$;*

- $G_2 \subset \mathrm{Aut}(X)_p$ *be the subgroup of all* $f \in \mathrm{Aut}(X)_p$ *such that* $f(S_2) = S_2$;
- $G \subset \mathrm{Aut}(X)_p$ *be the subgroup of all* $f \in \mathrm{Aut}(X)_p$ *such that* $f(S_1 \cup S_2) = S_1 \cup S_2$;
- F_1 *be the additive group of* $H^0(Y, \mathscr{O}(\mathscr{L}))$.
- F_2 *be the additive group of* $H^0(Y, \mathscr{O}(\mathscr{L}^{-1}))$.

Then

(1) X *does not admit a good configuration (see Definition 13.3) if and only if* $F_1 = F_2 = \{0\}$;
(2) $G_1 \cong \mathbb{C}^* \rtimes F_1$;
(3) $G_2 \cong \mathbb{C}^* \rtimes F_2$;
(4) *either* $G = G_0 = G_1 \cap G_2 \cong \mathbb{C}^*$ *or* $\mathscr{L}_p^{\otimes 2}$ *is a trivial bundle and* $G = G_0 \sqcup \phi_a \cdot G_0$ *for some* $a \in \mathbb{C}^*$.

Proof. Let $\lambda = \{\lambda_{ij}\}$ be the cocycle corresponding to $\mathscr{L}$. Take $f \in G_1$. Since $S_1 = \{z_i = \infty\}$ is f-invariant, we have

$$f(y, z) = (y, a_i z_i + b_i) \tag{49}$$

in V_i, where both a_i and b_i are holomorphic functions in U_i. Since f is globally defined,we have

$$\lambda_{ij}(a_i z_i + b_i) = a_j \lambda_{ij} z_i + b_j.$$

It follows that $a_i = a_j := a$ is constant (as globally defined holomorphic function) and $b_j = \lambda_{ij} b_i$, hence $\mathbf{b} := \{b_i\}$ is a section of $\mathscr{L}$. On the other hand, every section $\mathbf{b} := \{b_i\}$ of $\mathscr{L}$ defines $f \in G_1$ by formula (49). Thus, G_1 is isomorphic to the group of matrices

$$\begin{bmatrix} a & \mathbf{b} \\ 0 & 1 \end{bmatrix},$$

where $a \in \mathbb{C}^*$ and $\mathbf{b} \in F_1$. We also showed that if $f \in G_1$ is defined by $\mathbf{b} := \{b_i\} \neq 0$ then $f(S_2) \neq S_2$, and $f(S_2) \cap S_1 = \emptyset$. If $f(S_2) \cap S_2 = \emptyset$, then $S_1, f(S_2), S_2$ would be three pairwise disjoint section, which contradicts to $X \not\sim Y \times \mathbb{P}^1$.

Thus $S_1, f(S_2), S_2$ is a good configuration.

In opposite direction: consider a good configuration S_1, S_2, S_3 such that $S_3 \cap S_1 = \emptyset, S_3 \cap S_2 \neq \emptyset$. Since S_3 is a section of p and does not meet S_1 it is defined by a section $\mathbf{b} := \{b_i\}$ as $z_i = b_i(y), y \in U_i$. Thus, $F_1 \neq \{0\}$.

The case of G_2 and sections that meet S_1 but do not meet S_2 may be treated in the same way, interchanging S_2 with S_1 and F_1 with F_2. This proves (1-3).

Let us prove (4). If for each $f \in G$ all the points in $(S_1 \cup S_2)$ are fixed then, by Lemma 11.10, $G = G_0 \cong \mathbb{C}^*$. If it is not the case, take $\phi \in G \setminus G_0$. Then $\phi(S_1) = S_2$ and $\phi(S_2) = S_1$. Thus, $\phi(y, z_i) = \frac{a_i(y)}{z_i}$ in every V_i and

$$\lambda_{ij}\frac{a_i(y)}{z_i} = \frac{a_j(y)}{\lambda_{ij}z_i} \tag{50}$$

where $a_i(y)$ are non-vanishing holomorphic functions in U_i. Thus $\{a_i(y)\}$ define a section of $\mathscr{L}^{\otimes 2}$. Since $a_i(y)$ never vanish, we get that $\mathscr{L}^{\otimes 2}$ is trivial. Therefore, we may choose z_i in such a way that $a_i = a \in \mathbb{C}^*$. Then $\phi = \phi_a$.

For any other $f \in G \setminus G_0$ the composition $f \circ \phi \in G_0$, hence $G = G_0 \sqcup \phi_a \cdot G_0$. □

Corollary 11.20. *Let (X, p, Y) be a $\mathbb{P}^1-$bundle, where X, Y are compact connected manifolds and $X \not\sim Y \times \mathbb{P}^1$. Assume that p admits no good configurations but admits two disjoint sections S_1, S_2. Then one of the following holds.*

(1) $\mathrm{Aut}(X)_p \sim \mathbb{C}^*$;
(2) *the holomorphic line bundle* $\mathscr{L}(S_1, S_2)^{\otimes 2}$ *is trivial and* $\mathrm{Aut}(X)_p = G_0 \sqcup \phi_a \cdot G_0$, *for some $a \in \mathbb{C}^*$. Here $G_0 \sim \mathbb{C}^*$ and $a \in \mathbb{C}^*$*

The restriction map $\mathrm{Aut}(X)_p \to \mathrm{Aut}(P_y)$, $f \to f\,|_{P_y}$ *is a group embedding.*

Proof. It follows from Proposition 11.19 that $F_1 = F_2 = \{0\}$, thus $\mathrm{Aut}(X)_p = G$. □

11.3. **Automorphisms of type C.** Let (X, p, Y) be a $\mathbb{P}^1-$bundle where X, Y are complex compact connected manifolds. Assume that $X \not\sim Y \times \mathbb{P}^1$ and $f \in \mathrm{Aut}(X)_p, f \neq id$ has type **C**. The analytic subset $F \subset X$ of all fixed points of f contains no sections, but contains a bisection S that is a smooth unramified double cover of Y (see Lemma 11.10). Further on we use the notation of Lemma 11.10 and Lemma 11.11.

Lemma 11.21. *Assume that $\tilde{X} := \tilde{X}_S \not\sim S \times \mathbb{P}^1$. Let $N \subset \tilde{X}$ be a section of $\tilde{p}$ distinct from S_+ and S_-. Then $N_X := p_X(N)$ is a section of p and (S_+, S_-, N) is not a good configuration.*

Proof. Let us show that $p_X : N \to N_X$ is an unramified double cover. Indeed, assume that it is not the case. Since $\tilde{X}$ is the unramified double cover of X, the preimage $p_X^{-1}(x)$ contains precisely two points for every $x \in N_X$. Thus if $p_X^{-1}(N_X) \neq N$, the preimage $p_X^{-1}(N_X)$ consists of two irreducible components, N and N_1 . Moreover, since p_X is unramified, $N \cap N_1 = \emptyset$. It follows that there are two distinct pairs of non-intersecting sections of $\tilde{p}$, namely, S_+, S_- and N, N_1. According the Lemma 11.6, $\tilde{X} \sim S \times \mathbb{P}^1$, which gives us a contradiction. It follows that N is a double cover of N_X. Let $s \in S, y = p(s) = p(\mathrm{inv}(s))$. Then

$$p_X^{-1}(N_X \cap P_y) = N \cap p_X^{-1}(P_y) = N \cap (\tilde{p}^{-1}(s) \cup \tilde{p}^{-1}(\mathrm{inv}(s))$$

contains two points (since N meets every fiber of $\tilde{p}$ at a single point.)

Since N is double cover of N_X it follows that $(N_X \cap P_y)$ contains precisely one point. Therefore, N_X is a section of p.

Assume that N meets S_+ at a point $a = (s,s) \in \tilde{X}, s \in S$. Then it meets S_- at the point $\mathrm{inv}(a) = (\mathrm{inv}(s), s)$ since $p_X(a) = p_X(\mathrm{inv}(a))$. Thus, N meets both S_+ and S_- and the configuration is not good. $\square$

Corollary 11.22. *Assume that (X,p,Y) is a $\mathbb{P}^1-$bundle that admits a non-identity automorphism $f \in \mathrm{Aut}(X)_p$ of type* **C** *with data S. Assume that the corresponding double cover $\tilde{X}_S \not\sim S \times \mathbb{P}^1$.Then*

(1) *one of the following holds:*
- $\mathrm{Aut}(\tilde{X})_{\tilde{p}} \sim \mathbb{C}^*$;
- $\mathrm{Aut}(\tilde{X})_{\tilde{p}} = \tilde{G}_0 \sqcup \phi_a \cdot \tilde{G}_0$, *where* $\tilde{G}_0 \sim \mathbb{C}^*$ *and* $\phi \in \mathrm{Aut}(\tilde{X})_{\tilde{p}}$ *interchanges S_+ with S_-.*

(2) *The restriction map* $\mathrm{Aut}(X)_p \to \mathrm{Aut}(P_y)$, $f \to f\mid_{P_y}$ *is a group embedding for every $y \in Y$.*

(3) *the map $h \mapsto \tilde{h}$ is a group embedding of* $\mathrm{Aut}(X)_p$ *to* $\mathrm{Aut}(\tilde{X})_{\tilde{p}}$.

Proof. Since, by Lemma 11.21, there are no good configurations in $\tilde{X}_f$, item (1) follows from Corollary 11.20 applied to $\tilde{X}$.

Take $u \in S,\ t \in Y, t = p(u)$. If $f\mid_{P_t} = id$, then, by construction,
- $\tilde{f}\mid_{P_s} = id$, , hence
- $\tilde{f} = id$, (by Corollary 11.20 applied to $\tilde{X}$), hence
- $\tilde{f}\mid_{P_s} = id$ for every $s \in S$, hence
- $f\mid_{P_y} = id$ for $y = p(s) \in Y$.

Hence f is uniquely determined by its restriction to the fiber $P_t = p^{-1}(t)$. This proves (2).

On the other hand, in (2) was shown that $\tilde{h} = id$ implies $f\mid_{P_y} = id$ for every $y \in Y$, i.e. $h = id$. Therefore $h \mapsto \tilde{h}$ is an embedding. This proves (3). $\square$

Lemma 11.23. *Assume that $f \in \mathrm{Aut}(X)_p$, $f \neq id$, and f is of type* **C** *with Data (bisection) S.*

(1) If the corresponding double cover (see case **C***) $\tilde{X} := \tilde{X}_S$ is not isomorphic to $S \times \mathbb{P}^1$ then the group $\mathrm{Aut}(X)_p$ has exponent 2 and consists of 2 or 4 elements.*

(2) If $\tilde{X}$ is isomorphic to $S \times \mathbb{P}^1$ then there are two disjoint sections $S_1, S_2 \subset X$ of p. Moreover, if $X \not\sim Y \times \mathbb{P}^1$ then $\mathrm{Aut}(X)_p$ is a disjoint union of its abelian complex Lie subgroup $\Gamma \cong \mathbb{C}^*$ of index 2 and its coset Γ'. The subgroup Γ consists of those $f \in \mathrm{Aut}(X)_p$ that fix S_1 and S_2. The coset Γ' consists of those $f \in \mathrm{Aut}(X)_p$ that interchange S_1 and S_2. Moreover, the restriction homomorphism $\mathrm{Aut}(X)_p \to \mathrm{Aut}(P_y)$, $f \to f\mid_{P_y}$ is a group embedding for every $y \in Y$.

Proof. We modify the proof of [BZ20, Lemma 4.7].

Choose a point $a \in S$. Let $b = p(a) \in Y$. It means that a sits in the two elements set $S \cap P_b$. The lift $\tilde{f}$ of f onto $\tilde{X}$ has type **A** with Data $(S_+, S_-) \subset \tilde{X}$, since points of S are fixed by f. It is determined uniquely by its restriction to P_a (see Proposition 11.19). For the corresponding holomorphic line bundle $\tilde{\mathscr{L}} := \tilde{\mathscr{L}}(S_-, S_+)$ the section S_+ is the zero section. Let

- $\{\tilde{U}_j\}$ be a fine covering of S;
- (u, z_j) be local coordinates in $\tilde{V}_j = \tilde{p}^{-1}(\tilde{U}_j)$, such that $z_j \mid_{S_+} = 0, z_j \mid_{S_-} = \infty$;
- $a \in \tilde{U}_i,\ \mathrm{inv}(a) \in \tilde{U}_k$ and $\tilde{U}_k \cap \tilde{U}_i = \emptyset$;
- $b = p(a) = p(\mathrm{inv}(a)) \in Y$.

It was shown in [BZ20, Lemma 4.7] that

A. If we define the isomorphism $\alpha : \overline{\mathbb{C}}_{z_i} \to \overline{\mathbb{C}}_{z_k}$ in such a way that the following diagram is commutative

$$\begin{array}{ccccc} P_b & \xrightarrow{(a,id)} & a \times P_b & \xrightarrow{z_i} & \overline{\mathbb{C}}_{z_i} \\ {\scriptstyle id}\big\downarrow & & \big\downarrow & & {\scriptstyle \alpha}\big\downarrow \\ P_b & \xrightarrow{(\mathrm{inv}(a),id)} & \mathrm{inv}(a) \times P_b & \xrightarrow{z_k} & \overline{\mathbb{C}}_{z_k} \end{array}, \tag{51}$$

then

$$z_k = \alpha(z_i) = \frac{\nu}{z_i}$$

for some $\nu = \nu(a) \neq 0$.

B. Consider an automorphism $h \in \mathrm{Aut}(X)_p$. Let $\tilde{h}$ be its pullback to $\mathrm{Aut}(\tilde{X})_{\tilde{p}}$ defined by $\tilde{h}(s, x) = (s, h(x))$. Let $n_1(z_i) = \tilde{h} \mid_{\tilde{P}_a}$, which means that $h(a, z_i) = (a, n_1(z_i))$. Let $n_2(z_k) = \tilde{h} \mid_{\tilde{P}_{\mathrm{inv}(a)}}$, which means that $h(\mathrm{inv}(a), z_k) = (a, n_2(z_k))$. Then

$$\frac{\nu}{n_1(z_i)} = \alpha(n_1(z_i)) = n_2(\alpha(z_i)) = n_2(\frac{\nu}{z_i}). \tag{52}$$

Proof of *(1)*. Assume that $\tilde{X} \not\sim S \times \mathbb{P}^1$.

According to Proposition 11.22, if $\tilde{h} \in \mathrm{Aut}(\tilde{X})_{\tilde{p}}$ then either $\tilde{h}(s, z_j) = \lambda z_j$, or $h(s, z_j) = \frac{\lambda}{z_j}$ in every $\tilde{U}_j$ of our fine covering, where $\lambda \in \mathbb{C}^*$ does not depend on s or j.

Fix $a \in S$. According to item B one of following two conditions holds.

(a) $n_1(z_i) = \lambda z_i,\ n_2(z_k) = \lambda z_k,\ \ z_k = \frac{\nu(a)}{z_i}$ and from (52)

$$\frac{\nu(a)}{\lambda z_i} = \lambda \frac{\nu(a)}{z_i}.$$

(b) $n_1(z_i) = \frac{\lambda}{z_i},\ n_2(z_k) = \frac{\lambda}{z_k},\ z_k = \frac{\nu}{z_i}$ and from (52)

$$\frac{\nu z_i}{\lambda} = \frac{\lambda z_i}{\nu}.$$

In the former case $\lambda = \pm 1$, in the latter case $\lambda = \pm\nu$. Hence, at most 4 maps are possible. Clearly, the squares of all these maps are the identity map.

Note, that all the calculations are done for the fiber of $\tilde{p}$ over the point a. We use the fact that the map $\tilde{h}$ is defined by its restriction to a fiber. Apriori, ν could depend on a fiber. But since λ does not, we got as a byproduct that the same is valid for ν.

Proof of *(2)*. Assume that $\tilde{X} \sim S \times \mathbb{P}^1$. Let $\zeta : S \times \mathbb{P}^1 \to \mathbb{P}^1$ be the projection on the second factor, let $\zeta_1 = \zeta \mid_{S_+}, \zeta_2 = \zeta \mid_{S_-}$. Since $S_+ \cap S_- = \emptyset$, the function $z = \frac{\zeta - \zeta_1}{\zeta - \zeta_2}$ is well defined on $\tilde{X}$.

Since $z = 0$ on $S_+ = \{(s, s)\}$ and $z = \infty$ on $S_- = \{(s, \mathrm{inv}(s))\}$ we may assume that $z_j = z$ for all j. Recall that for every s

$$\mathrm{inv}(s, z) = (\mathrm{inv}(s), \alpha(z)) = (\mathrm{inv}(s), \frac{\nu(s)}{z}). \tag{53}$$

This implies that $\nu(s)$ is a holomorphic function on S, hence $\nu = const$. From (53) we get that two disjoint sections $N_1 = \{(s, z = \sqrt{\nu})\}$ and $N_2 = \{(s, z = -\sqrt{\nu})\}$ (for some choice of $\sqrt{\nu}$) are invariant under the involution, which means that their images are two disjoint sections S_1, S_2, respectively, in X.

Thus, X has two disjoint sections. Let us show that there is no good configuration in X. Assume that S_3 is a third section (of p) in X. On $\tilde{S}_3 = p_X^{-1}(S_3) \subset \tilde{X}$ the function z is either constant or get all values in $\overline{\mathbb{C}}$. If it is constant, then X has three disjoint sections(S_1, S_2, S_3), thus $X = Y \times \mathbb{P}^1$. If z takes on all the values on $\tilde{S}_3$, then S_3 meets both S_1 and S_2, thus S_1, S_2, S_3 is not a good configuration.

Now *(2)* follows from Corollary 11.20.

□

We have proved (see Lemma 11.12) that if $X \not\sim Y \times \mathbb{P}^1$ and there is $f \in \mathrm{Aut}(X)_p, f \neq id$, of type **B** then $\mathrm{Aut}(X)_p$ contains a subgroup isomorphic to $(\mathbb{C}^+)^n$ for some positive integer n.

Corollary 11.24. *Assume that $X \not\sim Y \times \mathbb{P}^1$ and $\mathrm{Aut}(X)_p$, contains an automorphism $f \neq id$ of type* **B**. *Then $\mathrm{Aut}(X)_p$ contains no automorphisms of type* **C**.

Proof. Assume that $\mathrm{Aut}(X)_p$, contains an automorphism of type **C**. Then by Lemma 11.23 $\mathrm{Aut}(X)_p$ is either finite or consists of two cosets isomorphic to $\mathbb{C}^*$; in both cases $\mathrm{Aut}(X)_p$ does not contain a Lie subgroup $\Gamma \cong (\mathbb{C}^+)^n$ with $n > 0$. □

Proposition 11.25. *Let (X, p, Y) be a $\mathbb{P}^1-$bundle, where X, Y are complex compact connected manifolds, and Y is Kähler and not uniruled. Then $\mathrm{Aut}(X)$ is Jordan.*

Proof. Indeed, we proved that three cases are possible.

(1) $\mathrm{Aut}(X)_p = \{id\}$. Then $\mathrm{Aut}(X)$ embeds into $\mathrm{Aut}(Y)$ that is Jordan according to [Kim].
(2) $\mathrm{Aut}(X)_p$ contains an automorphisms of type **A** or **B**. Then $X = \mathbb{P}(\mathscr{E})$ for some rank 2 vector bundle $\mathscr{E}$ on Y. Thus, X is Kähler ([Vo, Proposition 3.5]).
(3) $\mathrm{Aut}(X)_p$ contains an automorphisms of type **C**. Then the double cover $\tilde{X}$ of X fits into Case 2. Thus, X is Kähler.

In Cases 2 and 3 $\mathrm{Aut}(X)$ is Jordan, once more, according to [Kim]. □

12. Structure of $\mathrm{Aut}_0(X)$ and $\mathrm{Aut}(X)$

In this section we prove the main result of this chapter. Namely, that the group $\mathrm{Aut}(X)$ is very Jordan provided that the $\mathbb{P}^1-$bundle (X, p, Y) is scarce.

Theorem 12.1. *Let (X, p, Y) be a $\mathbb{P}^1-$bundle, where X, Y are complex compact connected manifolds, X is not biholomorphic to the direct product $Y \times \mathbb{P}^1$ and Y is Kähler and not uniruled. Assume that (X, p, Y) is scarce. Then:*

a) *The connected identity component $\mathrm{Aut}_0(X)$ of the complex Lie group $\mathrm{Aut}(X)$ is* **commutative***;*
b) *The group $\mathrm{Aut}(X)$ is very Jordan. More precisely, there is a short exact sequence*
$$1 \to \mathrm{Aut}_0(X) \to \mathrm{Aut}(X) \to F \to 1, \tag{54}$$
where F is a bounded group.
c) *The commutative group $\mathrm{Aut}_0(X)$ sits in a short exact sequence of complex Lie groups*
$$1 \to \Gamma \to \mathrm{Aut}_0(X) \to H \to 1, \tag{55}$$
where H is a complex torus and one of the following conditions holds:
 – *$\Gamma = \{id\}$, the trivial group;*
 – *$\Gamma \cong (\mathbb{C}^+)^n$.*
 – *$\Gamma \cong \mathbb{C}^*$.*

Proof. We know that the set of almost sections is either infinite or contains at most 2 of them (by Lemma 11.8 and Remark 6.6).

Consider cases.

Case 1. There are no almost sections of p. Then, by Lemma 11.23, $\mathrm{Aut}(X)_p$ is finite.

Case 2. p has only two almost sections, A_1, A_2, that meet.

Assume that $f \in \mathrm{Aut}(X)_p$, $f \neq id$. Since f moves almost sections to almost sections, $A_1 \cup A_2$ is invariant under f. According to Proposition 11.19, the following cases are possible:

- Points of A_1 are fixed points of f. Then the same is true for A_2. Since A_1 and A_2 meet, f is neither of type **A** or of type **C**. Since they are distinct, f cannot be of type **B** (see Lemma 11.16). Thus $f = id$ and $\mathrm{Aut}(X)_p = \{id\}$.
- Not all points of A_1 are fixed points of f. That means $f(A_1) = A_2, f(A_2) = A_1$. Assume that $g \neq f \in \mathrm{Aut}(X)_p, g \neq id$. Since $g \neq id$, it too does not fix points of A_1(due to the previous case). Then for $h := g \circ f$ we have $h(A_1) = A_1, h(A_2) = A_2$. Hence, as in previous item, $h = id$. It follows that $f^2 = id$, $g = f = f^{-1}$.

$\mathrm{Aut}(X)_p$ is finite.

Case 3. p has precisely one almost section. Then there are no automorphisms of type **A**, since there are no two disjoint sections. If $\mathrm{Aut}(X)_p$ contains no automorphisms of type **B** then, by Lemma 11.23, $\mathrm{Aut}(X)_p$ is finite. If $\mathrm{Aut}(X)_p$ contains an automorphism of type **B**, then, thanks to Corollary 11.24, $\mathrm{Aut}(X)_p$ contains no automorphisms of type **C**. Since all automorphisms of type **B** have to share this section in their sets of fixed points, $\mathrm{Aut}(X)_p \cong (\mathbb{C}^+)^n$ by Proposition 11.13 (unless $\mathrm{Aut}(X)_p = \{id\}$).

Case 4. p admits precisely two almost sections S_1, S_2 and they do not meet. Than they are sections. But X admits no good configuration. Thus, by Proposition 11.19 group $\mathrm{Aut}(X)_p$ contains a subgroup isomorphic to $\mathbb{C}^*$ of index at most 2.

Case 5. X is scarce and all almost sections pairwise meet (in particular, all sections pairwise meet). Then $\mathrm{Aut}(X)_p$ contains no automorphism of type **A**. If $\mathrm{Aut}(X)_p$ contains an automorphism of type **B** then, by Lemma 11.16 the set of sections cannot be scarce (assuming that there more than 1 of them), contradicition. Hence, by Lemma 11.23, $\mathrm{Aut}(X)_p$ is finite.

Case 6. X is scarce and admits two disjoint sections S_1, S_2. By Lemma 11.9, X admits no good configurations, and by Lemma 11.16 no automorphisms of type **B** . By Corollary 11.20 $\mathrm{Aut}(X)_p$ contains a subgroup isomorphic to $\mathbb{C}^*$ of index at most 2 .

The proof now repeats the proof of [BZ20, Theorem 5.4] with only one modification: $\mathbb{C}^+$ should be changed to $(\mathbb{C}^+)^n$ and,accordingly Lemma 2.10 should be applied. The group $\mathrm{Aut}(X)_p$ may be included into the short exact sequence

$$1 \to (\mathrm{Aut}(X)_p \cap \mathrm{Aut}_0(X)) \to \mathrm{Aut}_0(X) \xrightarrow{\tau} H \to 1, \tag{56}$$

where $H = \tau(\mathrm{Aut}_0(X)) \subset \mathrm{Tor}(Y)$ is a torus (see Remark 5.6). According to Cases **1-6**, one of the following holds:

- $\mathrm{Aut}(X)_p \cap \mathrm{Aut}_0(X)$ is finite;
- $\mathrm{Aut}(X)_p \cap \mathrm{Aut}_0(X) \cong (\mathbb{C}^+)^n$;
- $\mathrm{Aut}(X)_p \cap \mathrm{Aut}_0(X) \cong (\mathbb{C}^*)$;

Thus, due to Lemma 2.10, the group $\operatorname{Aut}_0(X)$ is commutative. Now the theorem follows from the fact that $\operatorname{Aut}(X)/\operatorname{Aut}_0(X)$ is bounded (see Proposition 3.5). □

13. Rational bundles over poor manifolds

In this section we consider rational bundles over poor manifolds. We prove that if Y is poor then p is scarce and the results of the previous section may be applied.

Definition 13.1. We say that a compact connected complex manifold Y of positive dimension is *poor* if it enjoys the following properties.

- Y does not contain analytic subspaces of codimension 1 (*a fortiori*, the algebraic dimension $a(Y)$ of Y is 0).
- Y does not contain rational curves, i.e., it is meromorphically hyperbolic in the sense of Fujiki [Fu80].

A complex torus T with $\dim(T) \geq 2$ and $a(T) = 0$ is a poor Kähler manifold. Indeed, a complex torus T is a Kähler manifold that does not contain rational curves. If $a(T) = 0$, it contains no analytic subsets of codimension 1 [BL, Corollary 6.4, Chapter 2]. An explicit example of such a torus of dimension 2 is given in [BL, Example 7.4]. Explicit examples of poor tori of any dimension are presented [BZ22]. Another example of a poor manifold is provided by a non-algebraic $K3$ surface S with $\mathrm{NS}(S) = 0$ (see [BHPV, Proposition 3.6, Chapter VIII]).

Further on Y is assumed to be a compact connected complex manifold.

Proposition 13.2. *(*[BZ20, Proposition 3.6]*). Let (X, p, Y) be an equidimensional rational bundle. Assume that Y contains no analytic subsets of codimension 1. Then (X, p, Y) is a $\mathbb{P}^1-$bundle.*

Proof. Let $\dim Y = n$, and

$$S = \{x \in X \mid \operatorname{rk}(dp)(x) < n\}$$

be the set of all points in X where the differential dp of p does not have the maximal rank. Then S and $\tilde{S} = p(S)$ are analytic subsets of X and Y, respectively (see, for instance, [Nar, Theorem 2, Chapter VII], [PR, Theorem 1.22], [Re]). Moreover, $\operatorname{codim}(\tilde{S}) = 1$ ([Ra]). Since Y contains no analytic subsets of codimension 1, we obtain: $\tilde{S} = \emptyset$. Thus the holomorphic map p has no singular fibers. □

Lemma 13.3. *Let (X, p, Y) be a $\mathbb{P}^1-$bundle, and* $\dim(Y) = n$*. For an almost section A we denote $\Sigma(A) = \{y \in Y \mid p^{-1}(y) \subset A\}$. If Y contains no analytic subsets of codimension 1, then*

1. *a $n-$section has no ramification points (i.e the intersection $X \cap P_y$ consists in n distinct point for every $y \in Y$);*
2. *if A_1, A_2 are two almost sections then $p(A_1 \cap A_2) \subset \Sigma(A_1) \cap \Sigma(A_2)$.*

(3) *any two distinct sections of* p *in* X *are disjoint;*
(4) *if there is an alomost section* $A \subset X$ *that is not a section then* X *contain neither sections nor* $n-$*sections;*

Proof. (1) Let R be an $n-$section of p, let A be the set of all points $x \in R$ where the restriction $p\mid_R: R \to Y$ of p onto R is not locally biholomorphic. Then the image $p(A)$ is either empty or has pure codimension 1 in Y ([DG, Section 1, 9], [Pe, Theorem1.6], [Re]). Since Y carries no analytic subsets of codimension 1, $p(A) = \emptyset$. Hence, $A = \emptyset$.

(2) Let B be an irreducible component of $A_1 \cap A_2$. Since $\dim(B) = n-1$, and $\dim p(B) \le n-2$, we have $p^{-1}(p(b)) \subset B$ for every point $b \in B$. Thus, $p(b) \in \Sigma(A_1) \cap \Sigma(A_2)$.

(3) In particular, if A_1, A_2 are distinct sections, then $\Sigma(A_1) = \Sigma(A_2) = \emptyset$ and $A := A_1 \cap A_2 = \emptyset$.

(4) Since A is not a section, there is a point $y \in Y$ such that $P_y = p^{-1}(y) \in A$. Thus for any $n-$section S we have $S \cap A \neq \emptyset$. This contradicts item (2), since $\Sigma(S) = \emptyset$. Hence, such an S does not exist. □

Corollary 13.4. *Let* (X, p, Y) *be a* $\mathbb{P}^1-$*bundle,* $\dim(Y) = n$. *If* Y *contains no analytic subsets of codimension 1, then one of the following holds.*

(1) $X \sim Y \times \mathbb{P}^1$;
(2) *X admits two disjoint sections,* $\mathrm{Aut}(X)_p$ *contains a subgroup* $G \cong \mathbb{C}^*$ *of index at most 2;*
(3) *X admits two meeting almost sections,* $\mathrm{Aut}(X)_p$ *is finite.*
(4) *X admits precisely one almost section D, then* $\mathrm{Aut}(X)_p \cong \mathbb{C}^+$ *(and D, by Lemma 11.14, is a section) or* $\mathrm{Aut}(X)_p = \{id\}$;
(5) *X admits no almost sections,* $\mathrm{Aut}(X)_p$ *is finite;*

Proof. First, note that since Y does not admit meromorphic functions, for a line bundle $\mathscr{L}$ on Y either $H^0(\mathscr{L}) = \{0\}$ or $\mathscr{L}$ is trivial and $H^0(\mathscr{L}) \cong \mathbb{C}$.

Item (1): Assume that X admits $n \ge 3$ almost sections. By Lemma 13.3 they are disjoint over an open set $U \subset Y$ that has complement of codimension 2. Thus $X \sim Y \times \mathbb{P}^1$ by Lemma 11.2.

Item (2) follows from Corollary 11.20.

Item (3) is proven in **Case 3** of the proof of Theorem 12.1.

Item (4) follows from Lemma 11.13: if $\mathrm{Aut}(X)_p \neq \{id\}$ then $\mathrm{Aut}(X)_p$ is isomorphic to the additive group of $\mathbb{C}^n$. That means that for corresponding line bundle $0 < n = H^0(\mathscr{L})$. Hence, $n = 1$.

Item (5) follows from Lemma 11.23. □

Lemma 13.5. *Let* (X, p, Y) *be a* $\mathbb{P}^1-$*bundle,* $\dim(Y) = n$. *If* Y *is poor then* $\mathrm{Bim}(X) = \mathrm{Aut}(X)$.

Proof. Since Y contains no rational curves, it is not uniruled. According to Corollary 5.5, every map $f \in \mathrm{Bim}(X)$ is $p-$fiberwise, i.e. there exists a group homomorphism $\tilde{\tau} : \mathrm{Bim}(X) \to \mathrm{Bim}(Y)$ (see Lemma 5.4) such that for all $f \in \mathrm{Bim}(X)$

$$p \circ f = \tilde{\tau}(f) \circ p.$$

Since Y contains no rational curves, every meromophic map into Y is holomorphic ([Fu80], see Remark 3.4). Thus $\tilde{\tau}(f) \in \mathrm{Aut}(Y)$.

For $f \in \mathrm{Bim}(X)$ let $\tilde{S}_f$ be the indeterminancy locus of f that is an analytic subspace of X of codimension at least 2 ([Re, page 369]). Let $S_f = p(\tilde{S}_f)$, which is an analytic subset of Y ([Re], [Nar, Theorem 2, Chapter VII],). Since Y contains no analytic subsets of codimension 1, $\operatorname{codim} S_f \geq 2$. Moreover, f is defined at all points of $X \setminus p^{-1}(S_f)$. By Lemma 5.12 both $f \in \mathrm{Bim}(X)$ and $f^{-1} \in \mathrm{Bim}(X)$ may be holomorphically extended to X, hence we get $\mathrm{Bim}(X) = \mathrm{Aut}(X)$. $\square$

We summarize the result in the following

Theorem 13.6. *Let (X, p, Y) be an equidimensional rational bundle over a poor Kähler manifold Y. Then:*

- (X, p, Y) *is a $\mathbb{P}^1-$bundle (see Proposition 13.2);*
- $\mathrm{Bim}(X) = \mathrm{Aut}(X)$ *(see Corollary 13.5);*

Assume additionally that Y is Kähler and X is not isomorphic to the direct product $Y \times \mathbb{P}^1$. Then:

- *X admits at most two almost sections (Corollary 13.4).*
- *The connected identity component* $\mathrm{Aut}_0(X)$ *of complex Lie group* $\mathrm{Aut}(X)$ *is* **commutative** *(Theorem 12.1);*
- *Group* $\mathrm{Aut}(X)$ *is very Jordan (Theorem 12.1); .*
- *The commutative group* $\mathrm{Aut}_0(X)$ *sits in a short exact sequence of complex Lie groups*

$$1 \to \Gamma \to \mathrm{Aut}_0(X) \to H \to 1, \tag{57}$$

  where H is a complex torus and one of the following conditions holds (Lemma 13.4)):
  - $\Gamma = \{id\}$*, the trivial group;*
  - $\Gamma \cong \mathbb{C}^+$*, the additive group of complex numbers;*
  - $\Gamma \cong \mathbb{C}^*$*, the multiplicative group of complex numbers.*

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