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ASYMPTOTICS AND SIGN PATTERNS FOR COEFFICIENTS IN EXPANSIONS OF HABIRO ELEMENTS

ANKUSH GOSWAMI, ABHASH KUMAR JHA, BYUNGCHAN KIM, AND ROBERT OSBURN

Abstract. We prove asymptotics and study sign patterns for coefficients in expansions of elements in the Habiro ring which satisfy a strange identity. As an application, we prove asymptotics and discuss positivity for the generalized Fishburn numbers which arise from the Kontsevich-Zagier series associated to the colored Jones polynomial for a family of torus knots. This extends Zagier's result on asymptotics for the Fishburn numbers.

1. Introduction

The expression

$$F(q) := \sum_{n\geq 0} (q)_n \tag{1.1}$$

first occurred in a talk entitled "Analytic continuation of Feynman integrals" by Kontsevich as part of the Seminar on Algebra, Geometry and Physics at MPIM Bonn on October 14, 1997. Here and throughout,

$$(a)_n = (a;q)_n := \prod_{k=1}^{n} (1 - aq^{k-1})$$

is the standard q-hypergeometric notation. Note that (1.1) does not converge on any open subset of $\mathbb{C}$, but is well-defined when q is a root of unity (where it is finite) and when q is replaced by $1-q$. Moreover, $F(q)$ is an element of the Habiro ring [10]

$$\mathcal{H} := \varprojlim_{n} \mathbb{Z}[q]/\langle (q)_n \rangle.$$

Motivated by Kontsevich's lecture and Stoimenow's work on regular linearized chord diagrams [13], Zagier determined the asymptotic behavior for the Fishburn numbers $\xi(n)$ which are the coefficients in the formal power series expansion

$$F(1-q) =: \sum_{n\geq 0} \xi(n) q^n = 1 + q + 2q^2 + 5q^3 + 15q^4 + 53q^5 + \cdots .$$

Namely, as $n \to \infty$ [16, Theorem 4]

$$\xi(n) \sim \left(\frac{6}{\pi^2}\right)^n n! \sqrt{n} \left(C_0 + \frac{C_1}{n} + \cdots \right) \tag{1.2}$$

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where $C_0 = \frac{12\sqrt{3}}{\pi^{\frac{5}{2}}}e^{\frac{\pi^2}{12}}$, $C_1 = C_0\left(\frac{3}{8} - \frac{17\pi^2}{144} + \frac{\pi^4}{432}\right)$ and all C_i are effectively computable constants. A key step in proving (1.2) is the "strange identity"

$$F(q) \text{``}=\text{''} -\frac{1}{2}\sum_{n\geq 1} n\left(\frac{12}{n}\right)q^{\frac{n^2-1}{24}} \tag{1.3}$$

where " = " means that the two sides agree to all orders at every root of unity (for further details, see [16, Sections 2 and 5]) and $\left(\frac{12}{*}\right)$ is the quadratic character of conductor 12. The idea is to first express $\xi(n)$ in terms of the Taylor series coefficients of $F(e^{-t})$, then employ (1.3) to, ultimately, obtain estimates for these coefficients. These estimates, in turn, lead to (1.2). "Identities" such as (1.3) are not only important in proving asymptotics, but also play a crucial role in obtaining congruences for $\xi(n)$ modulo prime powers [2] and quantum modularity for $F(q)$ [17]. For developments in these latter two directions, see [1, 2, 3, 5, 6, 7, 8, 9, 14]. The positivity of the Fishburn numbers $\xi(n)$ is a consequence of any of its numerous combinatorial interpretations [15, A022493]. The purpose of this paper is to prove asymptotics and study sign patterns for coefficients in expansions of elements in $\mathcal{H}$ which satisfy a general type of strange identity. Before stating our main result, we introduce some notation.

Let $f : \mathbb{Z} \to \mathbb{C}$ be a function of period $M \geq 2$. For integers $a \geq 0$ and $b > 0$, consider the partial theta series

$$\theta^{(\nu)}_{a,b,f}(q) := \sum_{n\geq 0} n^{\nu} f(n) q^{\frac{n^2-a}{b}}$$

where $\nu \in \{0, 1\}$. Suppose there exists

$$F_f(q) := \sum_{n=0}^{\infty} A_{n,f}(q)(q)_n \in \mathcal{H}$$

where $A_{n,f}(q) \in \mathbb{Z}[q]$ such that

$$F_f(q) \text{``}=\text{''} \theta^{(\nu)}_{a,b,f}(q). \tag{1.4}$$

We write

$$F_f(1-q) =: \sum_{n\geq 0} \xi_f(n) q^n$$

and define

$$G^{(\nu)}_f(k) := \begin{cases} \dfrac{2}{\sqrt{M}} \displaystyle\sum_{m \pmod{M}} f(m)\sin\Big(\frac{2\pi mk}{M}\Big) & \text{if } \nu = 0, \\[2ex] \dfrac{2}{\sqrt{M}} \displaystyle\sum_{m \pmod{M}} f(m)\cos\Big(\frac{2\pi mk}{M}\Big) & \text{if } \nu = 1. \end{cases} \tag{1.5}$$

Assume there exists a smallest positive integer k_ν such that $G^{(\nu)}_f(k_\nu) \neq 0$. Next, we define

$$M_{f,\nu} := 2 \cdot \frac{\#\{0 \leq m \leq M-1 : f(m) \neq 0\}}{|G^{(\nu)}_f(k_\nu)|\sqrt{M}} \max_{0\leq m\leq M-1} |f(m)| \tag{1.6}$$

and let $N_{f,\nu}^{(\max)} \geq 0$ be the smallest integer such that

$$M_{f,\nu}\left(\zeta(2n+\nu+1)-1\right) < 1 \tag{1.7}$$

for $n \geq N_{f,\nu}^{(\max)}$ where $\zeta(s)$ is the Riemann zeta function. Observe that $N_{f,\nu}^{(\max)}$ exists since $\zeta(2n+\nu+1)-1 \to 0$ as $n \to \infty$ and $M_{f,\nu}$ is independent of n. Finally, let $B_k(x)$ denote the kth Bernoulli polynomial for an integer $k \geq 0$. Our main result is now the following.

Theorem 1.1. *Assume (1.4) is true. Then as $n \to \infty$, we have*

$$\xi_f(n) \sim (-1)^{\nu} \left(\frac{M}{2\pi k_\nu}\right)^{2n+\nu+1} \frac{G_f^{(\nu)}(k_\nu) 2^{2n+\nu} n! n^{\nu-\frac{1}{2}}}{b^n \sqrt{\pi M}} e^{\frac{b k_\nu^2 \pi^2}{2M^2}}. \tag{1.8}$$

Moreover, there exists an integer $0 \leq N_{f,\nu} \leq N_{f,\nu}^{(max)}$ such that if

$$(-1)^{n+1} \sum_{m=1}^{M} f(m) B_{2n+\nu+1}\left(\frac{m}{M}\right) \tag{1.9}$$

has the same sign as $(-1)^{\nu} G_f^{(\nu)}(k_\nu)$ for all $0 \leq n < N_{f,\nu}$, then for all non-negative integers ℓ, $\xi_f(\ell)$ has the same sign as $(-1)^{\nu} G_f^{(\nu)}(k_\nu)$.

The paper is organized as follows. In Section 2, we prove Theorem 1.1. In Section 3, we give some applications, including asymptotics and positivity statements for the generalized Fishburn numbers $\xi_t(n)$ which arise from the Kontsevich-Zagier series $\mathcal{F}_t(q)$ associated to the colored Jones polynomial for the family of torus knots $T(3,2^t)$, $t \geq 2$ [5]. This extends (1.2) and gives an alternative proof of the positivity of the Fishburn numbers $\xi(n)$ (see Corollary 3.1 and Remark 3.2). In Section 4, we comment on asymptotics for other expansions of $F_f(q)$ and then conclude with conjectures concerning the positivity of coefficients for these expansions in three situations: $\mathcal{F}_t(q)$, the Kontsevich-Zagier series associated to the colored Jones polynomial for the family of torus knots $T(2,2m+1)$, $m \geq 1$ and a "Habiro-type" q-series.

2. Proof of Theorem 1.1

Proof of Theorem 1.1. We follow the strategy of [16]. To find the asymptotics of $\xi_f(n)$, we first consider the expansions

$$F_f(e^{-t}) = \sum_{n=0}^{\infty} \frac{B_{n,f}}{n!} t^n \tag{2.1}$$

and

$$e^{\frac{-ta}{b}} F_f(e^{-t}) = \sum_{n=0}^{\infty} \frac{C_{n,f}}{n!} \left(\frac{t}{b}\right)^n. \tag{2.2}$$

Let

$$\mathcal{P}_{a,b,f}^{(\nu)}(q) := q^{\frac{a}{b}} \theta_{a,b,f}^{(\nu)}(q)$$

and define the L-function

$$L(s,f)=\sum_{n=1}^{\infty}\frac{f(n)}{n^s}.$$

The Mellin transform of $\mathcal{P}^{(\nu)}_{a,b,f}(e^{-t})$ is

$$\begin{aligned}\int_0^{\infty} t^{s-1}\mathcal{P}^{(\nu)}_{a,b,f}(e^{-t})dt &= \sum_{n\geq 0} n^{\nu} f(n)\int_0^{\infty} t^{s-1}e^{\frac{-tn^2}{b}}dt\\ &= b^s\Gamma(s)\sum_{n\geq 0}\frac{f(n)}{n^{2s-\nu}}\\ &= b^s\Gamma(s)L(2s-\nu,f) \qquad (2.3)\end{aligned}$$

where $\Gamma(s)$ is the usual Gamma function. Applying Mellin inversion to (2.3), we obtain

$$\mathcal{P}^{(\nu)}_{a,b,f}(e^{-t})=\frac{1}{2\pi i}\int_{\mathrm{Re}(s)=c} b^s\Gamma(s)L(2s-\nu,f)\frac{ds}{t^s}$$

where $\mathrm{Re}(s)=c$ is the abscissa of absolute convergence of $\frac{b^s\Gamma(s)L(2s-\nu,f)}{t^s}$. It is well-known that $L(s,f)$ can be analytically continued to the whole complex plane except for a possible simple pole at $s=1$ with residue

$$R_{f,M}:=\frac{1}{M}\sum_{m \,(\mathrm{mod}\; M)} f(m).$$

By a standard complex analytic computation, we have

$$\mathcal{P}^{(\nu)}_{a,b,f}(e^{-t})\sim\sum_{n=0}^{\infty}\frac{(-1)^nL(-2n-\nu,f)}{b^n n!}t^n+\frac{b^{\frac{\nu+1}{2}}\Gamma\left(\frac{\nu+1}{2}\right)R_{f,M}}{t^{\frac{\nu+1}{2}}}$$

as $t\to 0^+$. By (1.4) and (2.2), we compare coefficients to obtain

$$C_{n,f}=(-1)^nL(-2n-\nu,f). \qquad (2.4)$$

Also, it is clear from (1.4) and (2.2) that $R_{f,M}=0$. Via [12, Eqn. (24)] or [4, Chapter 12], we have

$$L(-2n-\nu,f)=(-1)^{n+\nu}\left(\frac{M}{2\pi}\right)^{2n+\nu+1}\frac{(2n+\nu)!}{\sqrt{M}}\sum_{k=1}^{\infty}\frac{G^{(\nu)}_f(k)}{k^{2n+\nu+1}}. \qquad (2.5)$$

Let $k_\nu\geq 1$ be the smallest integer such that $G^{(\nu)}_f(k_\nu)\neq 0$. Then

$$L(-2n-\nu,f)=(-1)^{n+\nu}\left(\frac{M}{2\pi}\right)^{2n+\nu+1}\frac{G^{(\nu)}_f(k_\nu)(2n+\nu)!}{k_\nu^{2n+\nu+1}\sqrt{M}}\left(1+O\left(\left(\frac{k_\nu}{k_\nu+1}\right)^{2n-\varepsilon}\right)\right) \qquad (2.6)$$

for any $\varepsilon>0$[1]. Using (2.4) and (2.6), it follows

$$C_{n,f}=(-1)^{\nu}\left(\frac{M}{2\pi}\right)^{2n+\nu+1}\frac{G^{(\nu)}_f(k_\nu)(2n+\nu)!}{k_\nu^{2n+\nu+1}\sqrt{M}}\left(1+O\left(\left(\frac{k_\nu}{k_\nu+1}\right)^{2n-\varepsilon}\right)\right). \qquad (2.7)$$

[1]We can take $\varepsilon=0$ when $\nu=1$.

From (2.1), we have

$$B_{n,f}=\frac{1}{b^n}\sum_{k=0}^{n}\binom{n}{k}a^{n-k}C_{k,f}=\frac{1}{b^n}\left(C_{n,f}+naC_{n-1,f}+\frac{n(n-1)a^2}{2}C_{n-2,f}+\cdots\right). \tag{2.8}$$

From (2.7) and (2.8), we deduce

$$B_{n,f}=\frac{C_{n,f}}{b^n}\left(1+\frac{na\left(\frac{2\pi k_\nu}{M}\right)^2}{(2n+\nu-1)(2n+\nu)}\right.$$
$$\left.+\frac{n(n-1)a^2\left(\frac{2\pi k_\nu}{M}\right)^4}{2(2n+\nu-3)(2n+\nu-2)(2n+\nu-1)(2n+\nu)}+\cdots\right). \tag{2.9}$$

An application of Stirling's formula

$$n!=\sqrt{2\pi n}\left(\frac{n}{e}\right)^n\left(1+\frac{1}{12n}+\frac{1}{288n^2}+\cdots\right) \tag{2.10}$$

implies

$$\frac{(2n+\nu)!}{n!^2}=\frac{2^{2n+\nu}n^{\nu-\frac{1}{2}}}{\sqrt{\pi}}\left(1+\frac{(-1)^{\nu+1}(2\nu+1)}{8n}+\frac{(-1)^{\nu}(6\nu+1)}{128n^2}+\cdots\right). \tag{2.11}$$

Combining (2.7), (2.9) and (2.11) yields

$$B_{n,f}=(-1)^\nu\left(\frac{M}{2\pi k_\nu}\right)^{2n+\nu+1}\frac{G_f^{(\nu)}(k_\nu)2^{2n+\nu}n!^2n^{\nu-\frac{1}{2}}}{b^n\sqrt{\pi M}}\left(1+O\left(\left(\frac{k_\nu}{k_\nu+1}\right)^{2n-\varepsilon}\right)\right)$$
$$\times\left(1+\frac{(-1)^{\nu+1}(2\nu+1)}{8n}+\frac{(-1)^{\nu}(6\nu+1)}{128n^2}+\cdots\right)\left(1+\frac{na\left(\frac{2\pi k_\nu}{M}\right)^2}{(2n+\nu-1)(2n+\nu)}\right.$$
$$\left.+\frac{n(n-1)a^2\left(\frac{2\pi k_\nu}{M}\right)^4}{2(2n+\nu-3)(2n+\nu-2)(2n+\nu-1)(2n+\nu)}+\cdots\right)$$

and so

$$B_{n,f}\sim(-1)^\nu\left(\frac{M}{2\pi k_\nu}\right)^{2n+\nu+1}\frac{G_f^{(\nu)}(k_\nu)2^{2n+\nu}n!^2n^{\nu-\frac{1}{2}}}{b^n\sqrt{\pi M}}\left(1+\frac{\alpha_{1,f,\nu}}{n}+\frac{\alpha_{2,f,\nu}}{n^2}+\cdots\right) \tag{2.12}$$

where $\alpha_{1,f,\nu}:=\frac{a\left(\frac{2\pi k_\nu}{M}\right)^2}{4}+\frac{(-1)^{\nu+1}(2\nu+1)}{8}$ and all the remaining constants $\alpha_{i,f,\nu}$ are effectively computable. Now, we recall

$$\frac{t^m}{m!}=\sum_{n=m}^{\infty}S_{n,m}\frac{(1-e^{-t})^n}{n!} \tag{2.13}$$

where $S_{n,m}$ denotes the Stirling numbers of the first kind. From (2.1) and (2.13), we interchange sums

$$F_f(e^{-t})=\sum_{m=0}^{\infty}B_{m,f}\sum_{n=m}^{\infty}S_{n,m}\frac{(1-e^{-t})^n}{n!}=B_{0,f}+\sum_{n=1}^{\infty}\frac{(1-e^{-t})^n}{n!}\sum_{m=1}^{n}S_{n,m}B_{m,f}$$

and thus

$$\xi_f(n) = \frac{1}{n!} \sum_{m=0}^{n-1} S_{n,n-m} B_{n-m,f}. \tag{2.14}$$

Next, from $S_{n,n} = 1$ and the recursion $S_{n+1,m} = S_{n,m-1} + nS_{n,m}$ we have

$$S_{n,n-m} = \frac{n^{2m}}{2^m m!} \left(1 - \frac{\beta_1(m)}{n} + \frac{\beta_2(m)}{n^2} + \cdots\right) \tag{2.15}$$

with computable coefficients $\beta_1(m) = \frac{2m^2+m}{3}$, $\beta_2(m), \cdots$. From (2.12), it follows

$$\frac{B_{n-m,f}}{B_{n,f}} = \left(\frac{bk_\nu^2 \pi^2}{M^2}\right)^m \left(\frac{(n-m)!}{n!}\right)^2 \left(1 + O\left(\frac{1}{n}\right)\right) \tag{2.16}$$

and by (2.10)

$$\left(\frac{(n-m)!}{n!}\right)^2 = \frac{1}{n^{2m}} \left(1 + \frac{m(m-1)}{n} + \frac{3m^4 - 4m^3 + m}{6n^2} + \cdots\right). \tag{2.17}$$

Finally, we combine (2.14)–(2.17) to obtain

$$\begin{aligned}
\xi_f(n) &= \frac{B_{n,f}}{n!} \sum_{m=0}^{n-1} \frac{1}{m!} \left(\frac{bk_\nu^2 \pi^2}{2M^2}\right)^m \left(1 + O\left(\frac{1}{n}\right)\right) \\
&\qquad \times \left(1 - \frac{\beta_1(m)}{n} + \frac{\beta_2(m)}{n^2} + \cdots\right) \left(1 + \frac{m(m-1)}{n} + \frac{3m^4 - 4m^3 + m}{6n^2} + \cdots\right) \\
&= \frac{B_{n,f}}{n!} \left(\sum_{m=0}^{\infty} \frac{1}{m!} \left(\frac{bk_\nu^2 \pi^2}{2M^2}\right)^m + O\left(\frac{1}{n}\right)\right) \\
&= \frac{B_{n,f}}{n!} e^{\frac{bk_\nu^2 \pi^2}{2M^2}} \left(1 + O\left(\frac{1}{n}\right)\right).
\end{aligned}$$

The result (1.8) now follows from (2.12). Now for fixed f, we have from (2.4) and (2.5)

$$C_{n,f} = (-1)^\nu G_f^{(\nu)}(k_\nu) \left(\frac{M}{2\pi}\right)^{2n+\nu+1} \frac{(2n+\nu)!}{k_\nu^{2n+\nu+1}\sqrt{M}} \left(1 + \frac{1}{G_f^{(\nu)}(k_\nu)} \sum_{k=k_\nu+1}^{\infty} \frac{G_f^{(\nu)}(k)}{k^{2n+\nu+1}}\right). \tag{2.18}$$

Next, (1.5) implies

$$\left|\frac{G_f^{(\nu)}(k)}{G_f^{(\nu)}(k_\nu)}\right| \leq M_{f,\nu}$$

and this yields

$$\left|\frac{1}{G_f^{(\nu)}(k_\nu)} \sum_{k=k_\nu+1}^{\infty} \frac{G_f^{(\nu)}(k)}{k^{2n+\nu+1}}\right| \leq M_{f,\nu} \left(\zeta(2n+\nu+1) - 1\right) < 1 \tag{2.19}$$

for $n \geq N_{f,\nu}^{(\max)}$ where $N_{f,\nu}^{(\max)}$ is as in (1.7). Clearly, we can choose $0 \leq N_{f,\nu} \leq N_{f,\nu}^{(\max)}$ satisfying (2.19) for $n \geq N_{f,\nu}$. For such an $N_{f,\nu}$, it now follows from (2.18) that $C_{n,f}$ and $(-1)^\nu G_f^{(\nu)}(k_\nu)$ have the same sign for all $n \geq N_{f,\nu}$. Next, we note using (2.4) and [2, Lemma 3.2] that

$$C_{n,f} = (-1)^{n+1} \frac{M^{2n+\nu}}{2n+\nu+1} \sum_{m=1}^{M} f(m) B_{2n+\nu+1}\left(\frac{m}{M}\right) \tag{2.20}$$

where for $k \geq 0$, $B_k(x)$ denotes the kth Bernoulli polynomial. Hence (2.20) implies that if $C_{n,f}$, or equivalently (1.9), has the same sign as $(-1)^\nu G_f^{(\nu)}(k_\nu)$ for $0 \leq n < N_{f,\nu}$, then (2.8) and (2.14) imply that $\xi_f(\ell)$ has the same sign as $(-1)^\nu G_f^{(\nu)}(k_\nu)$ for all $\ell \geq 0$. □

3. Examples

In this section, we illustrate Theorem 1.1 with three examples.

3.1. Kontsevich-Zagier series for torus knots $T(3, 2^t)$.

For $t \geq 2$, consider the *Kontsevich-Zagier series associated to the family of torus knots* $T(3,2^t)$[2]

$$\mathcal{F}_t(q) = (-1)^{h''(t)} q^{-h'(t)} \sum_{n \geq 0} (q)_n \sum_{3\sum_{\ell=1}^{m(t)-1} j_\ell \ell \equiv 1 \ (\mathrm{mod}\ m(t))} (-1)^{\sum_{\ell=1}^{m(t)-1} j_\ell} q^{\frac{-a(t)+\sum_{\ell=1}^{m(t)-1} j_\ell \ell}{m(t)} + \sum_{\ell=1}^{m(t)-1} \binom{j_\ell}{2}}$$
$$\times \sum_{k=0}^{m(t)-1} \prod_{\ell=1}^{m(t)-1} \begin{bmatrix} n + I(\ell \leq k) \\ j_\ell \end{bmatrix} \tag{3.1}$$

where

$$h''(t) = \begin{cases} \frac{2^t-1}{3}, & \text{if } t \text{ is even,} \\ \frac{2^t-2}{3}, & \text{if } t \text{ is odd,} \end{cases} \qquad h'(t) = \begin{cases} \frac{2^t-4}{3}, & \text{if } t \text{ is even,} \\ \frac{2^t-5}{3}, & \text{if } t \text{ is odd,} \end{cases} \qquad a(t) = \begin{cases} \frac{2^{t-1}+1}{3}, & \text{if } t \text{ is even,} \\ \frac{2^t+1}{3}, & \text{if } t \text{ is odd,} \end{cases}$$

$m(t) = 2^{t-1}$, $I(*)$ is the characteristic function and

$$\begin{bmatrix} n \\ k \end{bmatrix} = \frac{(q)_n}{(q)_{n-k}(q)_k}$$

is the q-binomial coefficient. The expression $\mathcal{F}_t(q)$ matches the Nth colored Jones polynomial for $T(3,2^t)$ at a root of unity $q = e^{\frac{2\pi i}{N}}$, converges in a similar manner as $F(q)$ and is an element of $\mathcal{H}$ (see [5] for further details). The *generalized Fishburn numbers* $\xi_t(n)$ are defined by

$$\mathcal{F}_t(1-q) = \sum_{n=0}^{\infty} \xi_t(n) q^n.$$

An application of Theorem 1.1 is the following. Note that (1.2) follows after taking $t = 1$ and simplifying (for brevity, we only state the leading term).

[2]For $t = 1$, one may define the sum over the j_ℓ to be 1 in (3.1) to recover (1.1).

Corollary 3.1. *Let $t \geq 1$. As $n \to \infty$, we have*

$$\xi_t(n) \sim \frac{\sin(\frac{\pi}{2^t})}{2^t\sqrt{3\pi}} \left(\frac{3 \cdot 2^t}{\pi}\right)^{2n+2} \frac{2^{2n+1} n! \sqrt{n}}{(3 \cdot 2^{t+2})^n} e^{\frac{\pi^2}{3 \cdot 2^{t+1}}}. \tag{3.2}$$

Moreover, let $N_t \geq 0$ be the smallest integer such that

$$\zeta(2n+2) < \sin\left(\frac{\pi}{2^t}\right) + 1 \tag{3.3}$$

for $n \geq N_t$. If

$$(-1)^n \left[B_{2n+2}\left(\frac{2^{t+1}-3}{3 \cdot 2^{t+1}}\right) - B_{2n+2}\left(\frac{2^{t+1}+3}{3 \cdot 2^{t+1}}\right)\right] \geq 0 \tag{3.4}$$

for all $0 \leq n < N_t$, then $\xi_t(\ell) > 0$ for all $\ell \geq 0$ and $t \geq 1$.

Proof. The Kontsevich-Zagier series $\mathcal{F}_t(q)$ satisfies the strange identity [5, Proposition 2.4][3]

$$\mathcal{F}_t(q) \text{``}=\text{''} \theta^{(1)}_{(2^{t+1}-3)^2, 3 \cdot 2^{t+2}, \chi_t}(q) \tag{3.5}$$

where

$$\chi_t(n) := \begin{cases} -\frac{1}{2} & \text{if } n \equiv 2^{t+1}-3,\ 3+2^{t+2} \pmod{3 \cdot 2^{t+1}}, \\ \frac{1}{2} & \text{if } n \equiv 2^{t+1}+3,\ 2^{t+2}-3 \pmod{3 \cdot 2^{t+1}}, \\ 0 & \text{otherwise.} \end{cases} \tag{3.6}$$

Note that χ_t is an even function with period $M = 3 \cdot 2^{t+1}$. Next, we claim that $k_1 = 1$. To see this, observe that (1.5) and (3.6) yield

$$\begin{aligned} G^{(1)}_{\chi_t}(1) &= -\frac{1}{\sqrt{3 \cdot 2^{t+1}}} \left\{\cos\left(\frac{2\pi(2^{t+1}-3)}{3 \cdot 2^{t+1}}\right) + \cos\left(\frac{2\pi(3+2^{t+2})}{3 \cdot 2^{t+1}}\right) \right. \\ &\qquad \left. - \cos\left(\frac{2\pi(2^{t+1}+3)}{3 \cdot 2^{t+1}}\right) - \cos\left(\frac{2\pi(2^{t+2}-3)}{3 \cdot 2^{t+1}}\right)\right\} \\ &= -\frac{1}{\sqrt{2^{t-1}}} \sin\left(\frac{\pi}{2^t}\right) \end{aligned} \tag{3.7}$$

which is non-zero for any $t \geq 1$. By Theorem 1.1, (3.5) and (3.7), (3.2) follows. To deduce the positivity statement for $\xi_t(n)$, we first note using (1.6) and (3.7) that

$$M_{\chi_t,1} = \frac{8}{\sqrt{3 \cdot 2^{t+1}} \left|G^{(1)}_{\chi_t}(k_\nu)\right|} = \frac{4}{\sqrt{3}\sin\left(\frac{\pi}{2^t}\right)}.$$

Thus, (1.7) implies that $N_t^{(\max)} := N^{(\max)}_{\chi_t,1} \geq 0$ is the smallest integer satisfying

$$\zeta(2n+2) < \frac{\sqrt{3}}{4}\sin\left(\frac{\pi}{2^t}\right) + 1$$

for $n \geq N_t^{(\max)}$. In fact,

$$\left|G^{(1)}_{\chi_t}(k)\right| \leq \frac{1}{\sqrt{2^{t-1}}}$$

[3]Taking $t = 1$ in (3.5) and (3.6) recovers (1.3).

and thus we obtain using (3.7)

$$\left|\frac{1}{G^{(1)}_{\chi_t}(1)}\sum_{k=2}^{\infty}\frac{G^{(1)}_{\chi_t}(k)}{k^{2n+2}}\right| \leq \frac{\zeta(2n+2)-1}{\sin\left(\frac{\pi}{2^t}\right)} < 1$$

for $n \geq N_t$ with $0 \leq N_t \leq N_t^{(\max)}$. Using Theorem 1.1 and the fact that

$$B_k(x) = (-1)^k B_k(1-x) \tag{3.8}$$

for $k \geq 0$, it now follows that for all non-negative integers ℓ, $\xi_t(\ell) > 0$ if (3.4) is non-negative for $0 \leq n < N_t$. $\square$

Remark 3.2. Using Corollary 3.1, we verify (3.4) for $0 \leq n < N_t$ to confirm that $\xi_t(\ell) > 0$ for all $\ell \geq 0$ and $1 \leq t \leq 500$. In particular, this shows that $\xi(\ell) > 0$ for all $\ell \geq 0$. In Table 1, we list values of N_t for $1 \leq t \leq 10$.

t	1	2	3	4	5	6	7	8	9	10
N_t	0	0	1	1	1	2	2	3	3	4

TABLE 1. List of values of N_t for $1 \leq t \leq 10$

3.2. **Kontsevich-Zagier series for torus knots $T(2, 2m+1)$.** Let $m \in \mathbb{N}$. For $0 \leq \ell \leq m-1$, define the Kontsevich-Zagier series for the torus knot $T(2, 2m+1)$ as follows:

$$X_m^{(\ell)}(q) := \sum_{k_1,k_2,\ldots,k_m=0}^{\infty} (q)_{k_m} q^{k_1^2+\cdots+k_{m-1}^2+k_{\ell+1}+\cdots+k_{m-1}} \prod_{i=1}^{m-1} \begin{bmatrix} k_{i+1}+\delta_{i,\ell} \\ k_i \end{bmatrix}$$

where $\delta_{i,\ell}$ is the characteristic function. The expression $X_m^{(\ell)}(q)$ matches the Nth colored Jones polynomial for $T(2, 2m+1)$ when $\ell = 0$ and $q = e^{\frac{2\pi i}{N}}$ and is an element of $\mathcal{H}$. Write

$$X_m^{(\ell)}(1-q) =: \sum_{n\geq 0} \xi_{\ell,m}(n) q^n.$$

Another application of Theorem 1.1 is the following. Observe that (1.2) also follows by choosing $m = 1$ and $\ell = 0$ and simplifying as $X_1^{(0)}(q) = F(q)$.

Corollary 3.3. *Let $m \in \mathbb{N}$ and $0 \leq \ell \leq m-1$. As $n \to \infty$, we have*

$$\xi_{\ell,m}(n) \sim \sin\left(\frac{\pi(\ell+1)}{2m+1}\right)\left(\frac{2m+1}{\pi^2}\right)^{n+1}\frac{2^{n+3}n!\sqrt{n}}{\sqrt{\pi}}e^{\frac{\pi^2}{8m+4}}. \tag{3.9}$$

Moreover, let $N_{m,\ell} \geq 0$ be the smallest integer such that

$$\zeta(2n+2) < \sin\left(\frac{\pi(\ell+1)}{2m+1}\right) + 1 \tag{3.10}$$

for $n \geq N_{m,\ell}$. If

$$(-1)^n\left[B_{2n+2}\left(\frac{2m-2\ell-1}{8m+4}\right) - B_{2n+2}\left(\frac{2m+2\ell+3}{8m+4}\right)\right] \geq 0 \tag{3.11}$$

for all $0 \leq n < N_{m,\ell}$, then $\xi_{\ell,m}(k) > 0$ for all $k \geq 0$ and $0 \leq \ell \leq m-1$.

Proof. Hikami [11, Eqn. (15)] established the strange identity

$$X_m^{(\ell)}(q) \text{``}=\text{''} \theta^{(1)}_{(2m-2\ell-1)^2, 8(2m+1), \chi_m^{(\ell)}}(q) \tag{3.12}$$

where

$$\chi_m^{(\ell)}(n) := \begin{cases} -\frac{1}{2} & \text{if } n \equiv 2m-2\ell-1,\ 6m+2\ell+5 \pmod{8m+4}, \\ \frac{1}{2} & \text{if } n \equiv 2m+2\ell+3,\ 6m-2\ell+1 \pmod{8m+4}, \\ 0 & \text{otherwise.} \end{cases} \tag{3.13}$$

Note that $\chi_m^{(\ell)}(n)$ is an even function with period $M = 8m+4$. Next, we claim that $k_1 = 1$. To see this, observe that (1.5) and (3.13) yield

$$\begin{aligned} G^{(1)}_{\chi_m^{(\ell)}}(1) &= -\frac{1}{\sqrt{8m+4}} \left\{ \cos\left(\frac{2\pi(2m-2\ell-1)}{8m+4}\right) + \cos\left(\frac{2\pi(6m+2\ell+5)}{8m+4}\right) \right. \\ &\qquad \left. - \cos\left(\frac{2\pi(2m+2\ell+3)}{8m+4}\right) - \cos\left(\frac{2\pi(6m-2\ell+1)}{8m+4}\right) \right\} \\ &= -\frac{2}{\sqrt{2m+1}} \sin\left(\frac{\pi(\ell+1)}{2m+1}\right) \end{aligned} \tag{3.14}$$

which is non-zero for any $m \in \mathbb{N}$ and $0 \leq \ell \leq m-1$. By Theorem 1.1, (3.12) and (3.14), (3.9) follows. To deduce the positivity statement for $\xi_{\ell,m}(n)$, we first note using (1.6) and (3.14) that

$$M_{\chi_m^{(\ell)},1} = \frac{8}{\sqrt{8m+4}\left|G^{(1)}_{\chi_m^{(\ell)}}(k_\nu)\right|} = \frac{2}{\sin\left(\frac{\pi(\ell+1)}{2m+1}\right)}.$$

Thus, (1.7) implies that $N_{m,\ell}^{(\max)} := N^{(\max)}_{\chi_{8m+4}^{(\ell)},1} \geq 0$ is the smallest integer satisfying

$$\zeta(2n+2) < \frac{1}{2}\sin\left(\frac{\pi(\ell+1)}{2m+1}\right) + 1 \tag{3.15}$$

for $n \geq N_{m,\ell}^{(\max)}$. In fact,

$$\left|G^{(1)}_{\chi_m^{(\ell)}}(k)\right| \leq \frac{2}{\sqrt{2m+1}}$$

and thus we obtain using (3.14)

$$\left| \frac{1}{G^{(1)}_{\chi_m^{(\ell)}}(1)} \sum_{k=2}^{\infty} \frac{G^{(1)}_{\chi_m^{(\ell)}}(k)}{k^{2n+2}} \right| \leq \frac{\zeta(2n+2)-1}{\sin\left(\frac{\pi(\ell+1)}{2m+1}\right)} < 1$$

for $n \geq N_{m,\ell}$ with $0 \leq N_{m,\ell} \leq N_{m,\ell}^{(\max)}$. Using Theorem 1.1 and (3.8), it now follows that for all non-negative integers k, $\xi_{\ell,m}(k) > 0$ if (3.11) is non-negative for $0 \leq n < N_{m,\ell}$. □

Remark 3.4. Using Corollary 3.3, we verify (3.11) for $0 \leq n < N_{m,\ell}$ to confirm that $\xi_{\ell,m}(k) > 0$ for all $k \geq 0$, $1 \leq m \leq 500$ and $0 \leq \ell \leq m-1$. In Table 2, we list values of $N_{m,\ell}$ for $1 \leq m \leq 5$ and $0 \leq \ell \leq m-1$.

m	1	2		3			4				5				
ℓ	0	0	1	0	1	2	0	1	2	3	0	1	2	3	4
$N_{m,\ell}$	0	1	0	1	0	0	1	1	0	0	1	1	0	0	0

TABLE 2. List of values of $N_{m,\ell}$ for $1 \leq m \leq 5$ and $0 \leq \ell \leq m-1$

Remark 3.5. In fact, we can determine an infinite number of m and $0 \leq \ell \leq m-1$ such that $\xi_{\ell,m}(k)$ is positive for every $k \geq 0$. Let us put $\ell = cm + d$. Then

(1) It turns out that in order for $\ell \in \mathbb{Z}$, c and d must be rational numbers in their reduced forms such that $c = \frac{p_1}{q_1}$ and $d = \frac{p_2}{q_1}$ so that
$$p_1 m \equiv -p_2 \pmod{q_1}. \tag{3.16}$$

(2) Let m_0 be the smallest non-negative integer satisfying the congruence in (3.16). Then with the choices of c and d as in (1) and using (3.10) with $N_{m,\ell} = 0$, we have
$$\max\left(0, \frac{2m_0 - 3}{4}\right) \leq cm_0 + d \leq m_0 - 1, \quad \text{and} \quad \frac{1}{2} \leq c \leq 1. \tag{3.17}$$

Equations (3.16) and (3.17) can now be used to determine an infinite family of m and $0 \leq \ell \leq m-1$ such that $\xi_{\ell,m}(k) > 0$ for all $k \geq 0$ and $m \geq 1$. For example, let us choose $c = 1$ ($p_1 = 1,\ q_1 = 1$). Then (3.16) and (3.17) force $m_0 = 1$ and $d = -1$. Thus, $\xi_{m-1,m}(k) > 0$ for all k. Similarly, if we choose $c = \frac{1}{2}$ ($p_1 = 1,\ q_1 = 2$), then (3.16) implies that $m \equiv 1 \pmod 2$. This combined with (3.17) force $m_0 = 1$ and $d = -\frac{1}{2}$ so that we have $\xi_{\frac{m-1}{2},m}(k) > 0$ for all $k \geq 0$ and integers $m \equiv 1 \pmod 2$.

3.3. **An example with** $\nu = 0$**.** For $k \geq 1$, let $\mathcal{G}_k(q)$ denote the q-series
$$\mathcal{G}_k(q) := \sum_{n_k \geq n_{k-1} \geq \cdots \geq n_1 \geq 0} q^{n_k + 2n_{k-1}^2 + n_{k-1} + \cdots + 2n_1^2 + 2n_1} (q;q^2)_{n_k} \begin{bmatrix} n_k \\ n_{k-1} \end{bmatrix}_{q^2} \cdots \begin{bmatrix} n_2 \\ n_1 \end{bmatrix}_{q^2} \tag{3.18}$$
and write
$$\mathcal{G}_k(1-q) =: \sum_{n \geq 0} \xi_{\mathcal{G}_k}(n) q^n.$$

Corollary 3.6. *As $n \to \infty$, we have*
$$\xi_{\mathcal{G}_k}(n) \sim \frac{\cos\left(\frac{\pi}{2(2k+1)}\right) 2^{2n+2} n!}{\pi^{\frac{3}{2}} \sqrt{n}} \left(\frac{2k+1}{\pi^2}\right)^n e^{\frac{\pi^2}{8(2k+1)}}. \tag{3.19}$$
Moreover, $\xi_{\mathcal{G}_k}(n) > 0$ for all $n \geq 0$ and $k \geq 1$.

Proof. It was shown in [2, Example 5.2] that
$$\mathcal{G}_k(q) = \sum_{n \geq 0} \chi_k(n) q^{\frac{n^2 - k^2}{2k+1}} \tag{3.20}$$
where
$$\chi_k(n) := \begin{cases} 1 & n \equiv k,\ k+1 \pmod{4k+2}, \\ -1 & n \equiv -k,\ -k-1 \pmod{4k+2}, \\ 0 & \text{otherwise}. \end{cases} \tag{3.21}$$

Observe that (3.20) is not a strange identity but an actual identity valid for $|q| < 1$ and every odd order root of unity q (see [2, Example 3.2]). Although $\mathcal{G}_k(q) \notin \mathcal{H}$, we note that when $q = e^{-t}$ with $t \to 0^+$, the expression $(q;q^2)_{n_k}$ in the right-hand side of (3.18) will have an asymptotic expansion starting with t^{n_k}. Hence, the expansions (2.1) and (2.2) for this q-series as $t \to 0^+$ are still valid and we can apply Theorem 1.1. First, we have that $\chi_k(n)$ is an odd function with period $M = 4k+2$. Next, we claim that $k_0 = 1$. To see this, observe that for any $\ell \geq 1$, (1.5) and (3.21) yield

$$\begin{aligned} G_{\chi_k}^{(0)}(\ell) &= \frac{4}{\sqrt{4k+2}}\left[\sin\left(\frac{\pi k \ell}{2k+1}\right) + \sin\left(\frac{\pi(k+1)\ell}{2k+1}\right)\right] \\ &= \frac{8}{\sqrt{4k+2}} \sin\left(\frac{\pi \ell}{2}\right) \cos\left(\frac{\pi\ell}{2(2k+1)}\right) \end{aligned} \tag{3.22}$$

and thus

$$G_{\chi_k}^{(0)}(1) = \frac{8}{\sqrt{4k+2}} \cos\left(\frac{\pi}{2(2k+1)}\right)$$

which is non-zero for all $k \geq 1$. By Theorem 1.1, (3.20) and (3.22), (3.19) follows. Next, using (1.6) we get

$$M_{\chi_k,0} = \frac{1}{\cos\left(\frac{\pi}{2(2k+1)}\right)}.$$

As $\cos\left(\frac{\pi}{2(2k+1)}\right)$ is an increasing function for $k \geq 1$ and

$$M_{\chi_1,0} \cdot (\zeta(3) - 1) = 0.233\cdots < 1,$$

we can choose $N_{\chi_k,0} = 1$ for all $k \geq 1$. To deduce the positivity statement for $\xi_{\mathcal{G}_k}(n)$, we need only show that

$$\sum_{m=1}^{4k+2} \chi_k(m) B_1\left(\frac{m}{4k+2}\right) \leq 0 \tag{3.23}$$

for all $k \geq 1$. To prove (3.23), we first note that $B_1(x) = x - \frac{1}{2}$. Hence, (3.21) implies

$$\sum_{m=1}^{4k+2} \chi_k(m) B_1\left(\frac{m}{4k+2}\right) = -1.$$

$\square$

4. Other expansions and conjectures

Other expansions for $F(q)$ frequently appear throughout the combinatorics literature. For example, we have [15, A138265]

$$F\left(\frac{1}{1+q}\right) = 1 + q + q^2 + 2q^3 + 5q^4 + 16q^5 + 61q^6 + 271q^7 + 1372q^8 + \cdots$$

and [15, A289312]

$$F\left(\frac{1-q}{1+q}\right) = 1 + 2q + 6q^2 + 26q^3 + 142q^4 + 946q^5 + 7446q^6 + 67658q^7 + 697118q^8 + \cdots.$$

Using Theorem 1.1, we may deduce asymptotics for the coefficients of $F_f\left(\frac{1}{1+q}\right)$ and $F_f\left(\frac{1-q}{1+q}\right)$. Namely, if we write

$$F_f\left(\frac{1}{1+q}\right) =: \sum_{n\geq 0} g_f(n)q^n,$$

then

$$g_f(n) \sim (-1)^\nu \left(\frac{M}{2\pi k_\nu}\right)^{2n+\nu+1} \frac{G_f^{(\nu)}(k_\nu)2^{2n+\nu}n!n^{\nu-\frac{1}{2}}}{b^n\sqrt{\pi M}} e^{-\frac{bk_\nu^2\pi^2}{2M^2}}. \tag{4.1}$$

This follows upon first noting

$$F_f\left(\frac{1}{1+q}\right) = F_f\left(1 - \frac{q}{1+q}\right) = \sum_{j\geq 0} \xi_f(j)q^j \sum_{m\geq 0}(-1)^m\binom{j+m-1}{m}q^m$$

and so

$$g_f(n) = \sum_{\ell=0}^{n-1}(-1)^\ell\binom{n-1}{\ell}\xi_f(n-\ell),$$

then applying Theorem 1.1. Similarly, if

$$F_f\left(\frac{1-q}{1+q}\right) =: \sum_{n\geq 0} h_f(n)q^n,$$

then one can check

$$h_f(n) \sim (-1)^\nu \left(\frac{M}{2\pi k_v}\right)^{2n+\nu+1} \frac{G_f^{(\nu)}(k_\nu)2^{3n+\nu}n!n^{\nu-\frac{1}{2}}}{b^n\sqrt{\pi M}}. \tag{4.2}$$

Asymptotics for the coefficients of $\mathcal{F}_t(q)$, $X_m^{(\ell)}(q)$ and $\mathcal{G}_k(q)$ with q replaced by $\frac{1}{1+q}$ or $\frac{1-q}{1+q}$ now follow readily from (4.1), (4.2) and Corollaries 3.1, 3.3 and 3.6. Thus, all but finitely many coefficients are positive for $\mathcal{F}_t(q)$, $X_m^{(\ell)}(q)$ and $\mathcal{G}_k(q)$ where q is replaced by $\frac{1}{1+q}$ or $\frac{1-q}{1+q}$. Interestingly, it appears numerically that more is true. Some supporting data is given below in Tables 3–8.

$t=1$	$1+q+q^2+2q^3+5q^4+16q^5+61q^6+271q^7+1372q^8+7795q^9+49093q^{10}+\dots$
$t=2$	$1+3q+8q^2+31q^3+160q^4+1029q^5+7910q^6+70658q^7+718687q^8+\dots$
$t=3$	$1+7q+42q^2+329q^3+3395q^4+43638q^5+670663q^6+11980513q^7+\dots$
$t=4$	$1+15q+190q^2+3005q^3+61885q^4+1587420q^5+48722721q^6+1739070735q^7+\dots$
$t=5$	$1+31q+806q^2+25637q^3+1054465q^4+54008696q^5+3311724885q^6+\dots$

TABLE 3. Coefficients for $\mathcal{F}_t(\frac{1}{1+q})$ for $1 \leq t \leq 5$

$t=1$	$1+2q+6q^2+26q^3+142q^4+946q^5+7446q^6+67658q^7+697118q^8+8031586q^9+\dots$
$t=2$	$1+6q+38q^2+318q^3+3406q^4+44790q^5+699126q^6+12630702q^7+\dots$
$t=3$	$1+14q+182q^2+2982q^3+62734q^4+1630174q^5+50474886q^6+1813113398q^7+\dots$
$t=4$	$1+30q+790q^2+25590q^3+1064590q^4+54905390q^5+3382387174q^6+\dots$
$t=5$	$1+62q+3286q^2+211606q^3+17496462q^4+1797007566q^5+220762565542q^6+\dots$

Table 4. Coefficients for $\mathcal{F}_t(\frac{1-q}{1+q})$ for $1 \leq t \leq 5$

$\ell=0$	$1+5q+25q^2+180q^3+1725q^4+20538q^5+291571q^6+4801844q^7+\dots$
$\ell=1$	$2+9q+45q^2+330q^3+3195q^4+38286q^5+545949q^6+9020385q^7+\dots$
$\ell=2$	$3+12q+60q^2+446q^3+4350q^4+52374q^5+749294q^6+12410001q^7+\dots$
$\ell=3$	$4+14q+70q^2+525q^3+5145q^4+62139q^5+890925q^6+14779290q^7+\dots$
$\ell=4$	$5+15q+75q^2+565q^3+5550q^4+67134q^5+963578q^6+15997212q^7+\dots$

Table 5. Coefficients for $X_5^{(\ell)}(\frac{1}{1+q})$ for $0 \leq \ell \leq 4$

$\ell=0$	$1+10q+110q^2+1650q^3+32230q^4+776666q^5+22237534q^6+737031746q^7+\dots$
$\ell=1$	$2+18q+198q^2+3018q^3+59598q^4+1446210q^5+41605014q^6+1383694074q^7+\dots$
$\ell=2$	$3+24q+264q^2+4072q^3+81048q^4+1976760q^5+57067560q^6+1902795528q^7+\dots$
$\ell=3$	$4+28q+308q^2+4788q^3+95788q^4+2344076q^5+67828068q^6+2265402148q^7+\dots$
$\ell=4$	$5+30q+330q^2+5150q^3+103290q^4+2531838q^5+73345162q^6+2451727038q^7+\dots$

Table 6. Coefficients for $X_5^{(\ell)}(\frac{1-q}{1+q})$ for $0 \leq \ell \leq 4$

$k=1$	$1+q+3q^2+11q^3+50q^4+280q^5+1892q^6+15052q^7+137957q^8+\dots$
$k=2$	$1+2q+8q^2+42q^3+293q^4+2630q^5+29054q^6+380894q^7+5773064q^8+\dots$
$k=3$	$1+3q+15q^2+103q^3+977q^4+12137q^5+186601q^6+3411009q^7+72158001q^8+\dots$
$k=4$	$1+4q+24q^2+204q^3+2454q^4+39000q^5+768720q^6+18028512q^7+\dots$
$k=5$	$1+5q+35q^2+355q^3+5180q^4+100346q^5+2413318q^6+69085190q^7+\dots$

Table 7. Coefficients for $\mathcal{G}_k(\frac{1}{1+q})$ for $1 \leq k \leq 5$

$k=1$	$1+2q+6q^2+34q^3+278q^4+2978q^5+39302q^6+615554q^7+11151446q^8+\dots$
$k=2$	$1+4q+20q^2+180q^3+2420q^4+42916q^5+940244q^6+24478804q^7+\dots$
$k=3$	$1+6q+42q^2+518q^3+9674q^4+239302q^5+7323946q^6+266553414q^7+\dots$
$k=4$	$1+8q+72q^2+1128q^3+26952q^4+855240q^5+33608136q^6+1571210280q^7+\dots$
$k=5$	$1+10q+110q^2+2090q^3+60830q^4+2355562q^5+113032942q^6+6454755274q^7+\dots$

TABLE 8. Coefficients for $\mathcal{G}_k\left(\frac{1-q}{1+q}\right)$ for $1 \leq k \leq 5$

Based on the evidence in Remarks 3.2 and 3.4, the computations in Remark 3.5 and Tables 3–8, we make the following

Conjecture 4.1. *We have*

(1) *the coefficients of* $\mathcal{F}_t(1-q)$, $\mathcal{F}_t(\frac{1}{1+q})$ *and* $\mathcal{F}_t(\frac{1-q}{1+q})$ *are positive for all* $t \geq 1$.
(2) *the coefficients of* $X_m^{(\ell)}(1-q)$, $X_m^{(\ell)}(\frac{1}{1+q})$ *and* $X_m^{(\ell)}(\frac{1-q}{1+q})$ *are positive for all* $m \in \mathbb{N}$ *and* $0 \leq \ell \leq m-1$.
(3) *the coefficients of* $\mathcal{G}_k(\frac{1}{1+q})$ *and* $\mathcal{G}_k(\frac{1-q}{1+q})$ *are positive for all* $k \geq 1$.

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