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REPRESENTATIONS AND COHOMOLOGIES OF KLEINIAN 4-RINGS

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Abstract. We introduce a new class of algebras over discrete valuation rings, called *Kleinian 4-rings*, which generalize the group algebra of the Kleinian 4-group. For these algebras we describe the lattices and their cohomologies. In the case of *regular lattices* we obtain an explicit form of cocycles defining the cohomology classes.

Introduction

Integral representations of the Kleinian 4-group G (or G-lattices) were described by Nazarova [9]. Another description was proposed by Plakosh [10]. In the papers [6] and [7] cohomologies of these lattices were calculated. In this paper we consider a class of rings that generalizes group rings of the Kleinian 4-group. We call them *Kleinian 4-rings*. We give a description of lattices over such rings and calculate cohomologies of these lattices. In a special case of *regular lattices* we obtain an explicit form of cocycles defining cohomology classes.

1. Lattices over Kleinian 4-rings

In what follows R denotes a complete discrete valuation ring with a prime element p, the field of fractions Q and the field of residues $\mathbb{k} = R/pR$. We write $\otimes$ instead of $\otimes_R$. If A is an R-algebra, we call an A-module M an *A-lattice* if it is finitely generated and free as R-module. Then we identify M with its image $1 \otimes M$ in the vector space $Q \otimes M$ and an element $v \in M$ with $1 \otimes v \in Q \otimes M$. We denote by A-lat the category of A-lattices.

Definition 1.1. The *Kleinian 4-ring* over R is the R-algebra $K = R[x, y]/(x(x - p), y(y - p))$.

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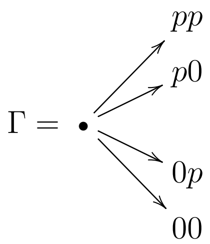
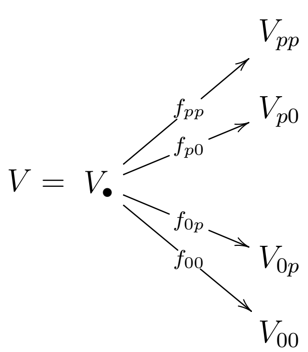
Note that if $p = 2$ this is just the group algebra over R of the *Kleinian 4-group* $G = \langle a, b \mid a^2 = b^2 = 1,\ ab = ba \rangle$. One has to set $x = a + 1$, $y = b + 1$.

One easily sees that $Q \otimes K$ is isomorphic to Q^4: just map x to $\bar{x} = (p, p, 0, 0)$ and y to $\bar{y} = (p, 0, p, 0)$. We consider K as embedded into Q^4 identifying x with $\bar{x}$ and y with $\bar{y}$. We also set $z = (p, 0, 0, 0) \in Q^4$ (note that $z \notin K$ and $z^2 = xy$). The maximal ideal $\mathfrak{r}$ of K is (p, x, y) and $K/\mathfrak{r} \simeq \Bbbk$. Let $A = \{a \in Q^4 \mid a\mathfrak{r} \subset K\}$. One easily verifies that $A = K + Rz$ and $A/K \simeq \Bbbk$. Hence K is a *Gorenstein ring* [3, Proposition 6], i.e. $\mathrm{inj.dim}_K K = 1$. Therefore, A is its unique minimal over-ring and every K-lattice is isomorphic to a direct sum of a free K-module and an A-lattice (see [5, Lemma 2.9] or [4, Lemma 3.2]). Note that the ring A is also local with the maximal ideal $\mathfrak{m} = (p, x, y, z)$ and $A/\mathfrak{m} \simeq \Bbbk$. Moreover, as the submodule of Q^4, $\mathfrak{m} = pA^\sharp = \mathrm{rad}\, A^\sharp$, where $A^\sharp = R^4$ is hereditary. Thus A is a *Backström order* in the sense of [11]. Therefore, A-lattices can be described by the representations of the quiver



over the field $\Bbbk$. Namely, denote by $R_{\alpha\beta}$, where $\alpha, \beta \in \{0, p\}$ the A-lattice such that $R_{\alpha\beta} = R$ as R-module, $xv = \alpha v$ and $yv = \beta v$ for all $v \in R_{\alpha\beta}$. For any A-lattice M and $\alpha, \beta \in \{0, p\}$ set $M_{\alpha\beta} = \{v \in M \mid xv = \alpha v,\ yv = \beta v\}$. If M is an A-lattice, $M^\sharp = A^\sharp M$ is an $A^\sharp$-module, hence $M^\sharp = \bigoplus_{\alpha,\beta} M^\sharp_{\alpha\beta}$. Let $V_\bullet = M/\mathfrak{m}M$ and $V_{\alpha\beta} = M^\sharp_{\alpha\beta}/pM^\sharp_{\alpha\beta}$. Note that $M^\sharp \supset M \supset \mathfrak{m}M = pM^\sharp$. So the natural maps $f_{\alpha\beta} : V_\bullet \to V_{\alpha\beta}$ are defined and we obtain a representation V of the quiver Γ:

(1)


We denote this representation by $\Phi(M)$. It gives a functor $\Phi : A\text{-lat} \to \operatorname{rep}\Gamma$. The next result follows from [11].

Theorem 1.2. *Let* $\operatorname{rep}_+\Gamma$ *be the full subcategory of* $\operatorname{rep}\Gamma$ *consisting of such representations* V *that all maps* $f_{\alpha\beta}$ *are surjective and the map* $f_+ : V_\bullet \to V_+$ *is injective. The image of the functor* Φ *is in* $\operatorname{rep}_+\Gamma$ *and, considered as the functor* $A\text{-lat} \to \operatorname{rep}_+\Gamma$*, the functor* Φ *is an* epivalence.

Recall that the term *epivalence* means that Φ is full, maps non-isomorphic objects to non-isomorphic and every representation $V \in \operatorname{rep}_+\Gamma$ is isomorphic to some $\Phi(M)$ (then Φ maps indecomposable objects to indecomposable). Actually, this M can be reconstructed as follows. Set $d_{\alpha\beta} = \dim V_{\alpha\beta}$, $V_+ = \bigoplus_{\alpha\beta} V_{\alpha\beta}$ and $\bar{V}$ be the image of the map $V_\bullet \to V_+$ with the components $f_{\alpha\beta}$. Then $V_+ \simeq M^\sharp/pM^\sharp$, where $M^\sharp = \bigoplus_{\alpha\beta} R_{\alpha\beta}^{d_{\alpha\beta}}$. Let $\Psi(V)$ be the preimage of V_+ in $M^\sharp$. It is an A-lattice and $\Phi(M) \simeq V$. Moreover, $M^\sharp = A^\sharp M$. Note also that the kernel of the map $\operatorname{Hom}_A(M,N) \to \operatorname{Hom}_\Gamma(\Phi(M),\Phi(N))$ coincides with $\operatorname{Hom}_A(M,\mathfrak{m}N)$.

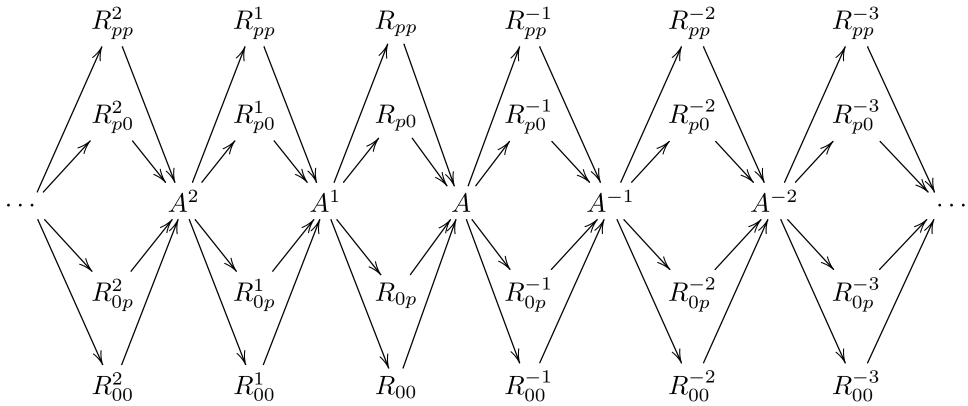
The quintuple $(d_\bullet \mid d_{pp}, d_{p0}, d_{0p}, d_{00})$, where $d_\bullet = \dim V_\bullet$, is called the *vector dimension* of the representation V. We also call it the *vector rank* of the lattice $M = \Psi(V)$ and denote it by $\mathsf{Rk}\, M$. For instance, $\mathsf{Rk}\, R_{pp} = (1 \mid 1,0,0,0)$ and $\mathsf{Rk}\, A = (1 \mid 1,1,1,1)$. Note that the rank of M as of R-module equals $\sum_{\alpha\beta} d_{\alpha\beta}$, while $d_\bullet = \dim_{\Bbbk} M/\mathfrak{m}M$.

Remark 1.3. Note that the only indecomposable representations of Γ that do not belong to $\operatorname{rep}_+\Gamma$ are "*trivial representations*" V^j, where $j \in \{\bullet,\, \alpha\beta \mid \alpha,\beta \in \{0,p\}\}$ such that $V^j_j = \Bbbk$ and $V^j_{j'} = 0$ if $i \neq j$. Therefore, the A-lattices are indeed classified by the representations of the quiver Γ.

Let τ_K (τ_A) denote the *Auslander-Reitentranslate* in the category K-lat (respectively, A-lat). Recall that $\tau_K M$ for a non-projective indecomposable K-lattice M is an indecomposable K-lattice N such that there is an *almost split sequence* $0 \to N \to E \to M \to 0$ [1]. The next result follows from [4].

Proposition 1.4.
1. $\tau_K M \simeq \tau_A M$ *for any indecomposable A-lattice* $M \not\simeq A$.
2. $\tau_K A \simeq \mathfrak{r}$ *and it is a unique indecomposable A-lattice N such that* $\operatorname{inj.dim}_N = 1$.
3. $\tau_K M \simeq \Omega M$ *for any A-lattice M, where ΩM denote the* syzygy *of M as of K-module.*

Following [12], we can also restore the Auslander-Reiten quiver $\mathcal{Q}(A)$ of the category A-lat from the Auslander-Reiten quiver $\mathcal{Q}(\Gamma)$r of the category $\operatorname{rep}\Gamma$. Recall that the quiver $\mathcal{Q}(\Gamma)$ consists of the *preprojective, preinjective* and *regular* components. The quiver $\mathcal{Q}((A)$ is obtained from $\mathcal{Q}(\Gamma)$ by glueing the preprojective and preinjective components omitting trivial representations. The resulting *preprojective-preinjective* component is the following:



Here M^k denotes $\tau_K^k M$. Note that $A^1 \simeq \mathfrak{r} \simeq A^\vee$, where $M^\vee = \operatorname{Hom}_K(M, K)$. The representations belonging to this component are uniquely determined by their vector-ranks. One can verify that

$$\mathsf{Rk}\, A^k = \begin{cases} (2k-1 \mid k,k,k,k) & \text{if } k>0,\\ (1-2k \mid 1-k,1-k,1-k,1-k) & \text{if } k<0; \end{cases}$$

$$\mathsf{Rk}\, R^k_{pp} = \begin{cases} (k+1 \mid \left[\frac{k}{2}\right]-(-1)^k, \left[\frac{k}{2}\right], \left[\frac{k}{2}\right], \left[\frac{k}{2}\right]) & \text{if } k>0,\\ (-k \mid \left[\frac{1-k}{2}\right]+(-1)^k, \left[\frac{1-k}{2}\right], \left[\frac{1-k}{2}\right], \left[\frac{1-k}{2}\right]) & \text{if } k<0. \end{cases}$$

$\mathsf{Rk}\, R^k_{\alpha\beta}$ is obtained from $\mathsf{Rk}\, R^k_{pp}$ by permutation of d_{pp} with $d_{\alpha\beta}$.

The remaining (regular) components are *tubes*, where τ_K acts periodically. They are parametrized by the set

$$\mathbb{P} = \{\text{irreducible unital polynomials } f(t) \in \Bbbk[t]\} \cup \{\infty\}.$$

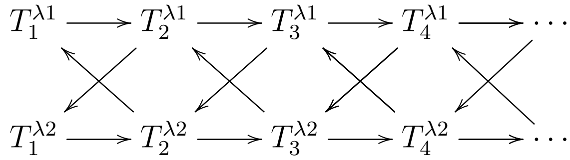
Actually, it is the set of closed points of the projective line over the field $\Bbbk$, that is of the projective scheme $\operatorname{Proj}\Bbbk[x,y]$. If $f(t) = t-\lambda$ ($\lambda \in \Bbbk$), we write $\mathcal{T}^\lambda$ instead of $\mathcal{T}^f$.

If $f \in \mathbb{P} \setminus \{t, t-1, \infty\}$, the corresponding tube $\mathcal{T}^f$ is *homogeneous*, which means that $\tau_K M \simeq M$ for all $M \in \mathcal{T}^f$. It has the form

$$T_1^f \rightleftarrows T_2^f \rightleftarrows T_3^f \rightleftarrows \cdots$$

and $\mathsf{Rk}\, T_n^f = (2dn \mid dn, dn, dn, dn)$, where $d = \deg f(t)$. In this diagram all maps $T_n^f \to T_{n+1}^f$ are monomorphisms with the cokernels T_1^f, while all maps $T_{n+1}^f \to T_n^f$ are epimorphisms with the kernels T_1^f.

The *exceptional tubes* $\mathcal{T}^\lambda$ $(\lambda \in \{0, 1, \infty\})$ are of he form

(2) 

Here $\tau_K T_n^{\lambda 1} = T_n^{\lambda 2}$ and $\tau_K T_n^{\lambda 2} = T_n^{\lambda 1}$. In this diagram all maps $T_n^{\lambda i} \to T_{n+1}^{\lambda i}$ are monomorphisms with the cokernels $T_1^{\lambda j}$, where $j = i$ if n is even and $j \neq i$ if n is odd. All maps $T_{n+1}^{\lambda i} \to T^{\lambda j}$ $(j \neq i)$ are epimorphisms with the kernels $T_1^{\lambda i}$.

For $\lambda = 1$ we have

$$
\begin{aligned}
&\mathsf{Rk}\, T_{2m}^{1j} = (2m \mid m, m, m, m) \ \text{ for both } j = 1 \text{ and } j = 2,\\
&\mathsf{Rk}\, T_{2m-1}^{11} = (2m - 1 \mid m, m, m - 1, m - 1),\\
&\mathsf{Rk}\, T_{2m-1}^{12} = (2m - 1 \mid m - 1, m - 1, mm).
\end{aligned} \tag{3}
$$

The vector-ranks for the tubes $\mathcal{T}^0$ and $\mathcal{T}^\infty$ are obtained from those for $\mathcal{T}^1$ by permutation of d_{p0}, respectively, with d_{00} and with d_{0p}.

2. Cohomologies

A Kleinian 4-ring is a *supplemented R-algebra* in the sense of [2, Ch. X] with respect to the surjection $\pi : K \to K/(x - p, y - p) \simeq R$. Therefore, for any K-module M the homologies $H_n(K, M) = \mathrm{Tor}_N^K(R, M)$ and cohomologies $H^n(K, M) = \mathrm{Ext}_K^n(R, M)$ are defined. Moreover, if we consider M as K-bimodule setting $mx = my = pm$ for all $m \in M$, they coincide with the Hochschild homologies and cohomologies:

$$H_n(K, M) \simeq HH_n(K, M) \ \text{ and } \ H^n(K, M) \simeq HH^n(K, M).$$

(see [2, Theorem X.2.1]).

Remark 2.1. We have chosen the augmentation $K \to R$ such that if $p = 2$, hence $K \simeq RG$ for the Kleinian 4-group G, it coincides with the usual augmentation $RG \to R$ mapping all elements of the group to 1. Thus in this case $H_n(K, M) = H_n(G, M)$.

Proposition 2.2. *For every K-module M and $n \neq 0$*

$$xyH_n(K, M) = xyH^n(K, M) = p^2 H^n(K, M) = p^2 H_n(K, M) = 0$$

Proof. The map $\mu : r \mapsto rxy$ is a homomorphism of K-modules $R \to K$ such that $\pi\mu : R \to R$ is the multiplication by xy or, the same, by p^2. Therefore, the multiplication by xy or by p^2 in $\mathrm{Ext}^n_K(R, M)$ or in $\mathrm{Tor}^K_n(R, M)$ factors, respectively, through $\mathrm{Ext}^n_K(K, M) = 0$ or through $\mathrm{Tor}^K_n(K, M) = 0$. □

Note that $K \simeq \bar{K} \otimes_R \bar{K}$, where $\bar{K} = R[x]/(x(x-p))$ A projective resolution $\bar{\mathbf{P}}$ for R as of $\bar{K}$-module, where $xr = pr$ for all $r \in R$, is obtained if we set $\bar{P}_n = \bar{K}u^n$ and

$$du^n = C_n(x)u^{n-1}, \text{ where } C_i(x) = \begin{cases} x & \text{if } n \text{ is even,} \\ x-p & \text{if } n \text{ is odd.} \end{cases}$$

Then $\mathbf{P} = \bar{\mathbf{P}} \otimes_R \bar{\mathbf{P}}$ is a projective resolution of R as of K-module. Here P_n is the module of homogeneous polynomials of degree n from $K[u, v]$ and

$$d(x^i y^j) = C_i(x)u^{i-1}v^j + (-1)^i C_j(y)u^i v^{j-1}.$$

Denote $H_n(\bar{K}, M) = \mathrm{Tor}^{\bar{K}}_n(R, M)$. Then

$$H_n(\bar{K}, M) = \begin{cases} M/(x-p)M & \text{if } n = 0, \\ \mathrm{Ker}(x-p)_M/xM & \text{if } n \text{ is odd,} \\ \mathrm{Ker}\, x_M/(x-p)M & \text{if } n \text{ is even,} \end{cases}$$

where a_M denotes the multiplication by a in the module M. Let $R_0 = \bar{K}/(x)$, $R_p = \bar{K}/(x-p)$. Then $R_{\alpha\beta} \simeq R_\alpha \otimes_R R_\beta$. As the ring R is hereditary, the Künneth formula [2, Theorem VI.3.2] implies that

$$\begin{aligned} H_n(K, R_{\alpha\beta}) \simeq & \big(\textstyle\bigoplus_{i+j=n} H_i(\bar{K}, R_\alpha) \otimes_R H_j(\bar{K}, R_\beta)\big) \oplus \\ & \oplus \big(\textstyle\bigoplus_{i+j=n-1} \mathrm{Tor}^R_1(H_i(\bar{K}, R_\alpha), H_j(\bar{K}, R_\beta))\big). \end{aligned}$$

Since

$$H_n(\bar{K}, R_0) = \begin{cases} 0 & \text{if } n \text{ is odd,} \\ \mathbb{k} & \text{if } n \text{ is even;} \end{cases}$$

$$H_n(\bar{K}, R_p) = \begin{cases} R & \text{if } n = 0, \\ \mathbb{k} & \text{if } n \text{ is odd,} \\ 0 & \text{if } n \text{ is even,} \end{cases}$$

we obtain

$$H_n(K, R_{pp}) = \begin{cases} R & \text{if } n = 0, \\ (R/p)^{[(n+3)/2]} & \text{if } n \text{ is odd,} \\ (R/p)^{n/2} & \text{if } n \text{ is even;} \end{cases} \tag{4}$$

and if $(\alpha,\beta)\neq(p,p)$

$$H_n(K,R_{\alpha\beta})=(R/p)^{[(n+2)/2]}. \tag{5}$$

On the other hand, the exact sequence $0\to K\to A\to \Bbbk\to 0$ implies that for $n>0$

$$H_n(K,A)\simeq H_n(K,\Bbbk)\simeq P_n\otimes_K\Bbbk\simeq \Bbbk^{n+1}, \tag{6}$$

since $H_n(K,K)=0$ and the differential in $\mathbf{P}\otimes_K\Bbbk$ is zero.

As K is Gorenstein, the functor $M\mapsto M^\vee=\operatorname{Hom}_K(M,K)$ is an exact duality in the category K-lat, i.e. the natural map $M\mapsto M^{\vee\vee}$ is an isomorphism. If P is projective, then $P\otimes_K M\simeq \operatorname{Hom}_K(P^\vee,M)$. Therefore, homologies of a module M can be obtained as $H_n(\operatorname{Hom}_K(\mathbf{P}^\vee,M)$. Note that the embedding $R\to P_0^\vee\simeq K$ maps 1 to xy. Hence, just as for finite groups, we can consider a *full resolution* $\hat{\mathbf{P}}$ setting

$$\hat P_n=\begin{cases}P_n & \text{if } n\geq 0,\\ P^\vee_{-n-1} & \text{if } n<0\end{cases}$$

and defining $d_0:K=\hat P_0\to\hat P_{-1}\simeq K$ as multiplication by xy. Thus the *Tate cohomologies* $\hat H^n(K,M)$ are defined as $H^n(\operatorname{Hom}_K(\hat{\mathbf{P}},M))$ with the usual properties

$$\hat H^n(K,M)=\begin{cases}H^n(K,M) & \text{if } n>0,\\ H_{-1-n}(K,M) & \text{if } n<-1,\\ M_{pp}/xyM & \text{if } n=0,\\ \{m\mid xym=0\}/((x-p)M+(y-p)M) & \text{if } n=-1,\end{cases}$$

where $M_{pp}=\{m\mid xm=ym=pm\}$. In particular, $xy\hat H^n(K,M)=p^2\hat H^n(K,M)=0$ for all M. Note also that, if M is an A-lattice, $M_{pp}=\{m\mid zm=pm\}$ and $xyM=z^2M$.

A basis of $\hat P_{-n}$ $(n>0)$ can be chosen as $\{\hat u^i\hat v^j\mid i+j=n-1\}$, where $(\hat u^i\hat v^j)(u^kv^l)=\delta_{ik}\delta_{jl}$. Then

$$d(\hat u^i\hat v^j)=C_{i+1}\hat u^{i+1}\hat v^j+(-1)^iC_{j+1}\hat u^i\hat v^{j+1}.$$

Proposition 2.3. *If M is an A-lattice that has no direct summands isomorphic to R_{pp}, then*

$$\hat H^0(K,M)=M_{pp}/pM_{pp}\simeq\Bbbk^{d_{pp}},$$

where $(d_\bullet\mid d_{pp},d_{p0},d_{0p},d_{00})=\mathsf{Rk}\,M$.

Proof. Set $M^\sharp=A^\sharp M=\bigoplus_{\alpha\beta}M^\sharp_{\alpha\beta}$. Note that $xyA=Rxy=xyA^\sharp$, hence $xyM=xyM^\sharp=p^2M^\sharp_{pp}$. On the other hand, $M^\sharp_{pp}\simeq R_{pp}^{d_{pp}}$ and $pM^\sharp_{pp}\subset M_{pp}\subset M^\sharp_{pp}$. If $M_{pp}\neq pM^\sharp_{pp}$, M_{pp} contains a direct summand

$L \simeq R_{pp}$ of $M^{\sharp}_{pp}$. Then $M^{\sharp} = L \oplus L'$ and $M = L \oplus (L' \cap M)$, which is impossible. Therefore, $M_{pp} = pM^{\sharp}_{pp}$, $xyM = pM_{pp}$ and $\hat{H}^0(K, M) = M_{pp}/pM_{pp} \simeq \Bbbk^{d_{pp}}$. □

Denote $T = Q/R$, $DM = \mathrm{Hom}_R(M, T)$. It is the Matlis duality between noetherian and artinian R-modules, as well as K-modules [8]. We have the following dualities for cohomologies.

Proposition 2.4. *Let M be a K-module. Then*

$$\hat{H}^n(K, DM) \simeq D\hat{H}^{-n-1}(K, M), \tag{7}$$

and if M is a lattice

$$\hat{H}^n(K, DM) \simeq \hat{H}^{n+1}(K, M^\vee), \tag{8}$$

$$\hat{H}^n(K, M^\vee) \simeq D\hat{H}^{-n}(K, M). \tag{9}$$

Proof. Note first that, since K is local and Gorenstein, $\mathrm{Hom}_R(K, R) \simeq K$, whence $M^\vee \simeq \mathrm{Hom}_R(M, R)$ and we identify these modules. As T is an injective R-module,

$$\mathrm{Ext}^n_K(R, \mathrm{Hom}_R(M, T)) \simeq \mathrm{Hom}_R(\mathrm{Tor}^K_n(R, M), T),$$

(see [2, Proposition VI.5.1]), which is just (7).

The exact sequence $0 \to R \to Q \to T \to 0$ gives, for any lattice M, the exact sequence

$$0 \to M^\vee \to \mathrm{Hom}_R(M, Q) \to DM \to 0.$$

As multiplication by p^2 is an automorphism of $\mathrm{Hom}_R(M, Q)$, Proposition 2.2 implies that $\hat{H}^n(\mathrm{Hom}_R(M, Q)) = 0$. Then the long exact sequence for cohomologies implies (8).

(9) is a combination of (7) and (8). □

Note also that $\hat{H}^n(K, F) = 0$ for any projective (hence free) K-module F. Therefore, Proposition 1.4 implies that, for any indecomposable A-lattice M,

$$\hat{H}^n(K, M) \simeq \hat{H}^{n+1}(K, \tau_K M) \simeq \hat{H}^{n-1}(K, \tau_K^{-1} M). \tag{10}$$

Hence from the formulae (4)-(6) and the duality (9) we obtain a complete description of cohomologies of K-lattices belonging to the preprojective-preinjective component.

Theorem 2.5.

$$\hat{H}^n(K, A^k) \simeq \begin{cases} \mathbb{k}^{n-k+1} & \text{if } n \geq k, \\ \mathbb{k}^{k-n} & \text{if } n < k; \end{cases}$$

$$\hat{H}^n(K, R^k_{pp}) \simeq \begin{cases} \mathbb{k}^{(|n-k|/2+1)} & \text{if } n-k \neq 0 \text{ is even}, \\ \mathbb{k}^{|n-k|/2]} & \text{if } n-k \text{ is odd}, \\ R/xyR & \text{if } n = k; \end{cases}$$

$$\hat{H}^n(K, R^k_{\alpha\beta}) \simeq \mathbb{k}^{[(|n-k|+1)/2]} \quad \text{if } (\alpha, \beta) \neq (p, p).$$

The description of cohomologies of A-lattices belonging to tubes are obtained from Proposition 2.3, since $\hat{H}^n(K, M) \simeq \hat{H}^0(K, \tau_K^{-n} M)$ and we know the action of τ_K in tubes.

Theorem 2.6. (1) *If $f \notin \{t, t-1\}$, then $\hat{H}^n(K, T^f_m) \simeq \mathbb{k}^{dm}$, where $d = \deg f$.*

(2) *If $\lambda \in \{0, 1, \infty\}$, then*

$$\hat{H}^n(K, T^{\lambda i}_m) \simeq \begin{cases} \mathbb{k}^{m/2} & \text{if } m \text{ is even}, \\ \mathbb{k}^{(m-(-1)^{n+i})/2} & \text{if } m \text{ is odd}. \end{cases}$$

3. Regular lattices

An A-lattice M is called *regular* if all its indecomposable direct summands belong to tubes. As neither regular lattice is projective, $\tau_K M = \tau_A M = \Omega M$. Note that if M is regular, then

$$2d_\bullet(M) = \sum\nolimits_{\alpha\beta} d_{\alpha\beta}(M). \tag{11}$$

Therefore,

$$d_\bullet(\Omega M) = d_\bullet(M) \quad \text{and} \quad d_{\alpha\beta}(\Omega M) = d_\bullet(M) - d_{\alpha\beta}(M). \tag{12}$$

These formulae imply the following fact.

Lemma 3.1. *Every exact sequence of regular A-lattices $0 \to M \to N \to L \to 0$ induces exact sequences*

$$0 \to \Omega M \to \Omega N \to \Omega L \to 0, \tag{13}$$

$$0 \to \Omega^{-1} M \to \Omega^{-1} N \to \Omega^{-1} L \to 0. \tag{14}$$

Proof. Obviously, there is an exact sequence $0 \to \Omega M \to \Omega N \oplus P \to \Omega L \to 0$ for some projective module P. On the other hand, as $d_{\alpha\beta}(N) = d_{\alpha\beta}(M) + d_{\alpha\beta}(L)$, the formulae (11) and (12) imply that $\mathsf{Rk}\,\Omega N = \mathsf{Rk}\,\Omega M + \mathsf{Rk}\,L$. Hence $P = 0$ and we obtain (13). By duality, we also have (14). □

Corollary 3.2. *Every exact sequence of regular A-lattices* $0 \to M \to N \to L \to 0$ *induces exact sequences of cohomologies*

$$0 \to \hat{H}^n(K,M) \to \hat{H}^n(K,N) \to \hat{H}^n(K,L) \to 0.$$

Proof. If $n = 0$, it follows from Proposition 2.3. For other n it is obtained by an easy induction using Lemma 3.1. □

For regular lattices we can give an explicit form of cocycles defining cohomology classes. Namely, for an indecomposable regular lattice M and an integer n we set

$$M(n) = \begin{cases} M_{pp} & \text{if } n \text{ is even,} \\ M_{0p} & \text{if } n \text{ is odd and } M \notin \mathcal{T}^\infty, \\ M_{p0} & \text{if } n \text{ is odd and } M \in \mathcal{T}^\infty. \end{cases}$$

For $n > 0$ we define a homomorphism $M(n) \to \hat{H}^n(K,M)$ mapping an element $a \in M(n)$ to the class of the cocycle $\xi_a : P_n \to M$ defined as follows:

- If $M \notin \mathcal{T}^\infty$, then

$$\xi_a(u^k v^{n-k}) = \begin{cases} a & \text{if } k = n, \\ 0 & \text{otherwise.} \end{cases}$$

- If $M \in \mathcal{T}^\infty$, then

$$\xi_a(u^k v^{n-k}) = \begin{cases} a & \text{if } k = 0, \\ 0 & \text{otherwise.} \end{cases}$$

Theorem 3.3. *The map* $a \mapsto \xi_a$ *induces an isomorphism*

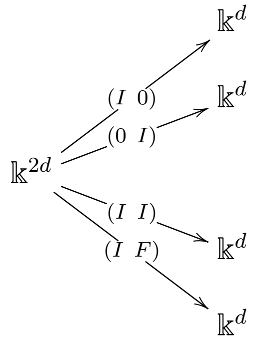
$$\xi : M(n)/pM(n) \simeq \hat{H}^n(K,M)$$

for every $n > 0$ *and every regular indecomposable A-lattice* M*.*

Proof. One easily sees that ξ_a is a cocycle. Theorem 2.6 and formulae (3) show that $\hat{H}^n(K,M) \simeq M(n)/pM(n)$. Hence we only have to prove that ξ_a is not a coboundary if $a \notin pM(n)$. First we check it for the lattices T_1^f and $T_1^{\lambda i}$.

We consider the case when n is even and $M = T_1^f$, where $\deg f = d$ and $f \notin \{t, t-1\}$. The other cases are quite similar or even easier.

The corresponding representation of the quiver Γ is



where I is $d \times d$ unit matrix and F is the Frobenius matrix with the characteristic polynomial $f(t)$. Therefore, M is the submodule of $\bigoplus_{\alpha\beta} M^\sharp_{\alpha\beta}$, where $M^\sharp_{\alpha\beta} = R^d_{\alpha\beta}$ and M consists of the quadruples $a = (a_{pp}, a_{p0}, a_{0p}, a_{00}) \equiv (r, r', r + r', r + \tilde{F}r') \pmod p$, where $r, r' \in R^d$ and $\tilde{F}$ is a $d\times d$ matrix over R such that $F = \tilde{F} \pmod p$. Hence, $M_{\alpha\beta} = pM^\sharp_{\alpha\beta}$. In particular, elements $a \in M(n)$ are of the form $(pr, 0, 0, 0)$. Let $\xi_a = d\gamma$, where $\gamma(x^{k-1}y^{n-k}) = \gamma_k \equiv (r_k, r'_k, r_k + r'_k, r_k + \tilde{F}r'_k) \pmod p$ for $1 \le k \le n$. Then

$$d\gamma(v^n) = 0 = y\gamma_1 \equiv (pr_1, 0, p(r_1 + r'_1), 0) \pmod{p^2},$$

hence $\gamma_1 \equiv 0 \pmod p$. Suppose that $\gamma_{k-1} \equiv 0 \pmod p$ for $1 < k \le n$. If k is odd, then

$$d\gamma(u^{k-1}v^{n-k+1}) = 0 = x\gamma_{k-1} + y\gamma_k \equiv \\ \equiv (pr_k, 0, p(r_k + r'_k), 0) \pmod{p^2},$$

If k is even, then

$$d\gamma(u^{k-1}v^{n-k+1}) = 0 = (x - p)\gamma_{k-1} - (y - p)\gamma_k \equiv \\ \equiv (0, pr'_k, 0, p(r_k + \tilde{F}r'_k)) \pmod{p^2}.$$

In both cases $\gamma_k \equiv 0 \pmod p$. Therefore, $\gamma_k \equiv 0 \pmod p$ for all $1 \le k \le n$. Then

$$d\gamma(u^n) = (a, 0, 0, 0) = x\gamma_n \equiv 0 \pmod{p^2},$$

so $a \in pM(n)$.

Suppose now that the theorem is valid for all T^f_{k-1} and for all $T^{\lambda i}_{k-1}$. If $M = T^f_k$ or $M = T^{\lambda i}_k$, there is an exact sequence $0 \to M' \to M \to M'' \to 0$, where, respectively, $M' = T^f_1$, $M'' = T^f_{k-1}$ or $M' = T^{\lambda i}$, $M'' =$

$T_{k-1}^{\lambda j}$ $(j \neq i)$. It gives a commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M'(n) & \longrightarrow & M(n) & \longrightarrow & M''(n) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \xi} & & \downarrow{\scriptstyle \xi} & & \downarrow{\scriptstyle \xi} & & \\ 0 & \longrightarrow & \hat{H}^n(K,M') & \longrightarrow & \hat{H}^n(K,M) & \longrightarrow & \hat{H}^n(K,M'') & \longrightarrow & 0 \end{array}$$

Using induction, we may suppose that the first and the third homomorphisms ξ satisfy the theorem. Therefore, so does the second, which accomplishes the proof. □

Dualizing this construction, we obtain an explicit description of Tate cohomologies with negative indices. Namely, for $n < 0$ we define a homomorphism $M(n) \to \hat{H}^n(K,M)$ mapping an element $a \in M(n)$ to the class of the cocycle $\hat{\xi}_a : P_n \to M$ defined as follows:

- If $M \notin \mathcal{T}^\infty$, then
$$\hat{\xi}_a(\hat{u}^k\hat{v}^{|n|-1-k}) = \begin{cases} a & \text{if } k = |n|-1, \\ 0 & \text{otherwise.} \end{cases}$$
- If $M \in \mathcal{T}^\infty$, then
$$\hat{\xi}_a(\hat{u}^k\hat{v}^{|n|-1-k}) = \begin{cases} a & \text{if } k = 0, \\ 0 & \text{otherwise.} \end{cases}$$

Theorem 3.4. *The map $a \mapsto \hat{\xi}_a$ induces an isomorphism*
$$\hat{\xi} : M(n)/pM(n) \simeq \hat{H}^n(K,M)$$
for every $n < 0$ and every regular indecomposable A-lattice M.

The proof just repeats that of Theorem 3.3, so we omit it.

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