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Vladimir Dotsenko
Ualbai Umirbaev

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Vladimir Dotsenko
Ualbai Umirbaev

Max-Planck-Institut für Mathematik
Vivatsgasse 7
53111 Bonn
Germany

Institut de Recherche Mathématique Avancée
UMR 7501
Université de Strasbourg et CNRS
7 rue René-Descartes
67000 Strasbourg Cedex
France

Department of Mathematics
Wayne State University
Detroit, MI 48202
USA

Department of Mathematics
Al-Farabi Kazakh National University
Almaty, 050040
Kazakhstan

Institute of Mathematics and
Mathematical Modeling
Almaty, 050010
Kazakhstan

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# AN EFFECTIVE CRITERION FOR NIELSEN–SCHREIER VARIETIES

VLADIMIR DOTSENKO AND UALBAI UMIRBAEV

*To the memory of V. A. Artamonov (1946–2021)*

ABSTRACT. All algebras of a certain type are said to form a Nielsen–Schreier variety if every subalgebra of every free algebra is free. Using methods of the operad theory, we propose an effective combinatorial criterion for that property in the case of algebras over a field of zero characteristic. Using this criterion, we show that the variety of all pre-Lie algebras is Nielsen–Schreier, and that, quite surprisingly, there are already infinitely many Nielsen–Schreier varieties of algebras with one binary operation and identities of degree three.

## 1. INTRODUCTION

Algebras satisfying certain identities are said to form a Nielsen–Schreier variety of algebras if all subalgebras of all free algebras are free; this generalises for algebras over a field the celebrated property of groups established by Nielsen [56] and Schreier [62]. The problem of classifying all varieties having this property was originally recorded in the 1976 edition of Dniester Notebook by V. A. Parfenov (see the easily accessible English translation of a later edition [1, Question 1.179]), and then reiterated in [10, Problem 1.1] and in [54, Problem 11.3.9]. Additionally, the same question is raised in the survey on "niceness theorems" by Hazewinkel [32], who writes (about the existing general freeness result of Fresse [28]):

> I don't think it can be made to take care of the subobject freeness theorems; but there probably is a general theorem, yet to be formulated and proved, that can take care of those.

The Dniester Notebook question of Parfenov also asked whether there existed Nielsen–Schreier varieties of algebras with one binary operation other than all algebras (Kurosh [42]), all Lie algebras (Shirshov and Witt [70, 72]), all commutative or anticommutative algebras (Shirshov [71]), and all algebras with zero product. This latter question was answered by the second author [67] who proved the Nielsen–Schreier property for the variety of all algebras satisfying the identity $xx^2 = 0$.

The Nielsen–Schreier property of a variety has been perceived as very rare. For instance, among varieties of Lie algebras only the variety of all Lie algebras and the variety of Lie algebras with zero Lie bracket are Nielsen–Schreier (Bakhturin [11]). If one considers more general structure operations, the Nielsen–Schreier property was proved for varieties of algebras with structure operations that satisfy *no identities* (Kurosh [43]), and for varieties of algebras with structure operations whose only identities are particular symmetries under permutations of arguments (Polin [59]); more recently, a similar but more complicated result was established by Shestakov and the second author [64] for the so called Akivis algebras. The Nielsen–Schreier property is also true for a number of varieties closely related to that of all Lie algebras: the varieties of all Lie $p$-algebras (Witt [72]), of Lie superalgebras (Mikhalev and Shtern [50, 65]), and of Lie $p$-superalgebras (Mikhalev [51]). Finally, there is an elegant observation of Mikhalev and Shestakov [53] that one can get new Nielsen–Schreier varieties by forming PBW-pairs with known ones; this gives new proofs for many of the above cases, as well as for the variety of Sabinin algebras, first proved to be Nielsen–Schreier in [19].

The second author established [67, 69] that over a field of zero characteristic a variety $\mathfrak{M}$ is Nielsen–Schreier if and only if the following two conditions hold:

(1) for every free $\mathfrak{M}$-algebra $A$, its universal multiplicative enveloping algebra $U_{\mathfrak{M}}(A)$ is a free associative algebra,
(2) for every homogeneous subalgebra $H$ of every free $\mathfrak{M}$-algebra $A$, the universal multiplicative enveloping algebra $U_{\mathfrak{M}}(A)$ is a free $U_{\mathfrak{M}}(H)$-module.

These conditions are not very easy to check, so this criterion has not been used to advance in classification of Schreier varieties. In our present work we prove a criterion of a completely different flavour that is easy to check and at the same time holds for many "new "varieties of algebras. Our main result is the following effective combinatorial criterion expressed in terms of Gröbner bases for operads [15, 24].

**Theorem** (Th. 4.1)**.** *Suppose that the operad $\mathcal{O}$ encoding the given variety of algebras $\mathfrak{M}$ defined over a field $\Bbbk$ of zero characteristic satisfies the following two properties:*

- *for the reverse graded path-lexicographic ordering, each leading term of the reduced Gröbner basis of the corresponding shuffle operad $\mathcal{O}^f$ has the minimal leaf directly connected to the root,*
- *for the graded path-lexicographic ordering, each leading term of the reduced Gröbner basis of the corresponding shuffle operad $\mathcal{O}^f$ is a left comb with the maximal leaf directly connected to the root.*

*Then the variety $\mathfrak{M}$ has the Nielsen–Schreier property.*

Among the new varieties that we show to have the Nielsen–Schreier property using this criterion, the following examples are perhaps most interesting:

- the variety of pre-Lie (also known as right-symmetric) algebras,
- the variety of nonassociative algebras satisfying the identity
$$xx^2+\alpha x^2x=0,$$
for every given $\alpha\neq 1$,
- the variety of nonassociative algebras satisfying the identity
$$x(x(\cdots(xx^2)))=0.$$

The last two examples, both of which generalize the identity $xx^2=0$ of [67], show that, over a field of zero characteristic, the set of Nielsen–Schreier varieties of algebras with just one binary operation is infinite in two different ways, containing both countable families with growing degrees of identities and parametric families with fixed degrees of identities; this suggests that Nielsen–Schreier varieties are not as rare as they were thought to be.

Our work should be viewed as another step in the programme of applying operadic methods to classical questions about varieties of algebras, in the spirit of the first author's work with Tamaroff [25] on a functorial criterion for PBW-pairs of varieties. While our methods may look "foreign" to the reader whose intuition comes from classical ring theory, a big advantage of them lies in applicability of the wealth of methods not available on the level of algebras. That said, using operads forces us to only work with identities equivalent to multilinear ones, and so we focus on varieties of algebras over a field of zero characteristic; developing systematic methods for treating the Nielsen–Schreier property in positive characteristic remains an open problem.

The paper is organized as follows. In Section 2, we give the necessary background, making emphasis on basics of the operad theory for the reader whose intuition comes from the ring theory. In Section 3, we explain a homological approach to Nielsen–Schreier varieties, which in particular provides the reader with useful intuition for the requirements imposed by our combinatorial criterion. In Section 4, we state and prove our combinatorial criterion. Finally, in Sections 5–10, we present numerous applications of our result, exhibiting many new Nielsen–Schreier varieties of algebras.

## 2. Conventions and recollections

All vector spaces in this paper are defined over a field $\Bbbk$ of zero characteristic. We use somewhat freely the language of category theory [47], but go into great detail to explain basics of the operad theory to the reader whose intuition comes from the ring theory. Further details are available in the monographs [15, 46].

2.1. **Nielsen–Schreier varieties of algebras.** Let $\mathfrak{M}$ be a variety of algebras, that is a class of algebras over $\Bbbk$ with certain structure operations satisfying certain identities. We shall assume that the set of basic structure operations of $\mathfrak{M}$ does not include any elements of arity 0 or 1 and has finitely many operations of each arity; thus, we shall not consider unital associative algebras or Rota–Baxter type algebras (which are always equipped with a Rota–Baxter operator $R$ of arity one), or vertex algebras (where one has infinitely many binary operations). Since we work over a field of zero characteristic, every system of identities is equivalent to a system of multilinear ones; sometimes we shall use non-multilinear identities for brevity, and we shall freely move between multilinear and non-multilinear descriptions of the same variety.

For a set $X$, we shall denote by $F_{\mathfrak{M}}\langle X\rangle$ the free $\mathfrak{M}$-algebra generated by $X$. It has a grading with respect to which all elements of $X$ have degree one. For an element $f \in F_{\mathfrak{M}}\langle X\rangle$, we denote by $\hat{f}$ the nonzero homogeneous component of $f$ of maximal degree. A system of elements $f_1,\dots,f_p \in F_{\mathfrak{M}}\langle X\rangle$ is said to be *irreducible* if no element $\hat{f}_i$ belongs to the subalgebra generated by $\hat{f}_j$ with $j \neq i$. Such a system of elements is said to be *algebraically independent* if the obvious map from the free algebra on $p$ generators to the subalgebra these elements generate is an isomorphism.

**Proposition 2.1** ([8, 44])**.** *Over an infinite field, the following properties of a variety $\mathfrak{M}$ are equivalent:*

- *every irreducible system of elements in every free $\mathfrak{M}$-algebra is algebraically independent,*

- *every subalgebra of every free $\mathfrak{M}$-algebra is free.*

We call a variety $\mathfrak{M}$ satisfying either of these equivalent properties a *Nielsen–Schreier variety.* The following necessary and sufficient condition for the Nielsen–Schreier property was proved by the second author.

**Theorem 2.2** ([67, Th. 1])**.** *Over a field of zero characteristic, a variety $\mathfrak{M}$ is Nielsen–Schreier if and only if the following two conditions hold:*

(1) *for every free $\mathfrak{M}$-algebra $A$, its universal multiplicative enveloping algebra $U_{\mathfrak{M}}(A)$ is a free associative algebra,*
(2) *for every homogeneous subalgebra $H$ of every free $\mathfrak{M}$-algebra $A$, the universal multiplicative enveloping algebra $U_{\mathfrak{M}}(A)$ is a free $U_{\mathfrak{M}}(H)$-module.*

One interesting consequence of the Nielsen–Schreier property is the following general theorem of Lewin describing the group of automorphisms of a finitely generated free algebra.

**Theorem 2.3** ([44, Th. 4])**.** *Let $\mathfrak{M}$ be a Nielsen–Schreier variety of algebras, and let $A = F_{\mathfrak{M}}\langle x_1,\ldots,x_n\rangle$ be a finitely generated free algebra in this variety. The group of automorphisms of $A$ is generated by the permutations of $x_1,\ldots,x_n$ together with the automorphisms*

$$x_i \mapsto \begin{cases} \alpha x_i, & i=1,\\ x_i, & i\neq 1,\end{cases} \qquad x_i \mapsto \begin{cases} x_1 + w(x_2,\ldots,x_n), & i=1,\\ x_i, & i\neq 1.\end{cases}$$

*In other words, every automorphism of a finitely generated free $\mathfrak{M}$-algebra is tame.*

Moreover, a result of the second author shows that in this case it is possible to describe the automorphism group of each free algebra by generators and relations [68].

In the case of Lie algebras, one can also use freeness of subalgebras of free Lie algebras to perform certain cohomology computations [31], and to establish analogues of the Schreier formula for groups [58]. It would be interesting to explore similar applications for numerous varieties whose Nielsen–Schreier property is proved in this paper. We hope to address this elsewhere.

2.2. **The language of symmetric operads.** It is well known that over a field of characteristic zero every system of algebraic identities is equivalent to multilinear ones. Let us briefly explain how this leads to the notion of an operad. To a variety of algebras $\mathfrak{M}$ without constants (operations of zero arity), one may associate the datum

$$\mathcal{O} = \mathcal{O}_{\mathfrak{M}} := \{\mathcal{O}(n)\}_{n\geq 1},$$

where $\mathcal{O}(n)$ is the $S_n$-module of multilinear elements (that is, elements of multidegree $(1,1,\ldots,1)$) in the free algebra $F_{\mathfrak{M}}\langle x_1,\ldots,x_n\rangle$. We note that the underlying vector space of each free algebra can be reconstructed from this datum as

$$\mathcal{O}(V) := \bigoplus_{n\geq 1} \mathcal{O}(n)\otimes_{\Bbbk S_n} V^{\otimes n},$$

where $V$ is the vector space spanned by the generators of that algebra.

Each individual free algebra only has its own algebra structure. If we consider all free algebras at the same time, there is something new that emerges: in the language of category theory, we have a *monad.* To see what this means, we regard the assignment to a vector space $V$ the free algebra generated by $V$ as a functor from the category of vector spaces to itself. Nothing prevents us from applying that functor twice, considering $F_{\mathfrak{M}}\langle F_{\mathfrak{M}}\langle V\rangle\rangle$; elements of that vector space are all possible substitution schemes of $\mathfrak{M}$-polynomials into each other. Of course, there is a canonical linear map

$$\tau_V\colon F_{\mathfrak{M}}\langle F_{\mathfrak{M}}\langle V\rangle\rangle \to F_{\mathfrak{M}}\langle V\rangle,$$

which says that for a substitution scheme, we can actually perform a substitution, and write a $\mathfrak{M}$-polynomial of $\mathfrak{M}$-polynomials as a $\mathfrak{M}$-polynomial. This map $\tau$ gives our functor a monad structure, meaning that it is associative: if we apply our functor three times, forming the gigantic algebra

$$F_{\mathfrak{M}}\langle F_{\mathfrak{M}}\langle F_{\mathfrak{M}}\langle V\rangle\rangle\rangle,$$

there are two different maps to $F_{\mathfrak{M}}\langle V\rangle$, depending on the order of substitutions, and those give the same result. (Strictly speaking, to talk about a monad, one should also discuss the unitality, but the compatibility of the maps $\tau_V$ with the obvious embedding maps $\iota_V\colon V \to F_{\mathfrak{M}}\langle V\rangle$ is too trivial to spend time on it.) This structure can be restricted to multilinear elements, and it defines what is called an operad structure on the sequence $\{\mathcal{O}(n)\}_{n\geq 1}$. In the language of operads, our above constraints on varieties of algebras are as follows. Absence of structure operations of arity 0 is described by the word "reduced", absence of structure operations of arity 1 is described by the word "connected" (note that there always exists one "trivial" operation of arity 1, which is identical on every element of every algebra; this operation is not regarded as a structure operation and is permitted). Thus, throughout this paper we work with reduced connected operads.

One key difference between the language of varieties and the language of operads may have already become apparent. Namely, in terms of varieties of algebras, properties like commutativity and associativity are on the same ground: both express certain identities in algebras. In terms of operads, commutativity of an operation is a symmetry type, a consequence of the fact that an operad is in particular a sequence of $S_n$-modules, while associativity of an operation is an identity: a relation between results of substitution of operations into one another. This distinction will be very important for us. A clean interpretation of how an operad structure formalizes the notion of substitutions of multilinear maps uses the language of linear species, which we shall now recall.

The theory of species of structures originated at the concept of a combinatorial species, invented by Joyal [35] and presented in great detail in [13]. The same definitions apply if one changes the target symmetric monoidal category; in particular, if one considers the category of vector spaces, one obtains what is called a linear species. Let us recall some key definitions, referring the reader to [2] for further information.

A *linear species* is a contravariant functor from the groupoid of finite sets (the category whose objects are finite sets and whose morphisms are bijections) to the category of vector spaces. This definition is not easy to digest at a first glance, and a reader with intuition coming from varieties of algebras is invited to think of the value $\mathcal{S}(I)$ of a linear species $\mathcal{S}$ on a finite set $I$ as of the set of multilinear operations of type $\mathcal{S}$ (accepting arguments from some vector space $V_1$ and assuming values in some vector space $V_2$) whose inputs are indexed by $I$. A linear species $\mathcal{S}$ is said to be *reduced* is $\mathcal{S}(\varnothing) = 0$; this means that we do not consider "constant" multilinear operations. (This is perhaps the only situation where several different terminologies clash in our paper: we use the word "reduced" for linear species to indicate that the value on the empty set is zero, and for Gröbner bases to indicate that we consider the unique Gröbner basis of a certain irreducible form.)

Sometimes, a "skeletal definition" is preferable: the category of linear species is equivalent to the category of symmetric sequences $\{\mathcal{S}(n)\}_{n\geq 0}$, where each $\mathcal{S}(n)$ is a right $S_n$-module, a morphisms between the sequences $\mathcal{S}_1$ and $\mathcal{S}_2$ in this category is a sequence of $S_n$-equivariant maps $f_n\colon \mathcal{S}_1(n)\to\mathcal{S}_2(n)$. While this definition may seem more appealing, the functorial definition simplifies the definitions of operations on linear species: it is harder to comprehend the two following definitions skeletally.

The *Cauchy product* of two linear species $\mathcal{S}_1$ and $\mathcal{S}_2$ is defined by the formula

$$(\mathcal{S}_1\cdot\mathcal{S}_2)(I):=\bigoplus_{I=I_1\sqcup I_2}\mathcal{S}_1(I_1)\otimes\mathcal{S}_2(I_2).$$

One may consider monoids with respect to the Cauchy product which are called *twisted associative algebras* [15], and are useful when working with universal multiplicative enveloping algebras of algebras in different varieties of algebras; we shall use them meaningfully below. Additionally, the crucial *composition product* of linear species is compactly expressed via the Cauchy product as

$$\mathcal{S}_1\circ\mathcal{S}_2:=\bigoplus_{n\geq 0}\mathcal{S}_1(\{1,\ldots,n\})\otimes_{\Bbbk S_n}\mathcal{S}_2^{\cdot n},$$

that is, if one unwraps the definitions,

$$(\mathcal{S}_1\circ\mathcal{S}_2)(I)=\bigoplus_{n\geq 0}\mathcal{S}_1(\{1,\ldots,n\})\otimes_{\Bbbk S_n}\left(\bigoplus_{I=I_1\sqcup\cdots\sqcup I_n}\mathcal{S}_2(I_1)\otimes\cdots\otimes\mathcal{S}_2(I_n)\right).$$

The linear species $\mathbb{1}$ which vanishes on a finite set $I$ unless $|I|=1$, and whose value on $I=\{a\}$ is given by $\Bbbk a$ is the unit for the composition product: we have $\mathbb{1}\circ\mathcal{S}=\mathcal{S}\circ\mathbb{1}=\mathcal{S}$.

Formally, a *symmetric operad* is a monoid with respect to the composition product. It is just the multilinear version of substitution schemes of free algebras discussed above, but re-packaged in a certain way. The advantage is that existing intuition of monoids and modules over them, available in any monoidal category [47], can be used for studying varieties of algebras. In particular, one can talk about left or right modules over operads, a notion which does not emerge too frequently in the context of varieties of algebras (though right ideals of an operad have been extensively studied in the theory of PI-algebras under the name "T-spaces").

The free symmetric operad generated by a linear species $\mathcal{X}$ is defined as follows. Its underlying linear species is the species $\mathcal{T}(\mathcal{X})$ for which $\mathcal{T}(\mathcal{X})(I)$ is spanned by decorated rooted trees (including the rooted tree without internal vertices and with just one leaf, which corresponds to the unit of the operad): the leaves of a tree must be in bijection with $I$, and each internal vertex $v$ of a tree must be decorated by an element of $\mathcal{X}(I_v)$, where $I_v$ is the set of incoming edges of $v$. Such decorated trees should be thought of as tensors: they are linear in each vertex decoration. The operad structure is given by grafting of trees onto each other. We remark that if one prefers the skeletal definition, one can talk about the free operad generated by a collection of $S_n$-modules, but the formulas will become heavier.

2.3. **Shuffle operads and Gröbner bases.** We shall now recall how to develop a workable theory of normal forms in operads using the theory of Gröbner bases developed by the first author and Khoroshkin [24]. It is important to emphasize that it is in general extremely hard to find convenient normal forms in free algebras for a given variety $\mathfrak{M}$. However, focusing on multilinear elements simplifies the situation quite drastically: for instance, for a basis in multilinear elements for the operad controlling Lie algebras one may take all left-normed commutators of the form

$$[[[a_1, a_{i_2}], \cdots], a_{i_n}],$$

where $i_2,\ldots,\ i_n$ is a permutation of $2,\ldots,n$; by contrast, all known bases in free Lie algebras [60] are noticeably harder to describe.

To define Gröbner bases for operads, one builds, step by step, an analogue of the theory of Gröbner bases for noncommutative associative algebras. To do this, one has to abandon the universe that has symmetries, for otherwise there is not even a good notion of a monomial that leads to a workable theory. The kind of monoids that have a good theory of Gröbner bases are *shuffle operads*. A rigorous definition of a shuffle operad uses ordered species [13], which we shall now discuss in the linear context.

An *ordered linear species* is a contravariant functor from the groupoid of finite ordered sets (the category whose objects are finite totally ordered sets and whose morphisms are order preserving bijections) to the category of vector spaces. In terms of the intuition with multilinear maps, this more or less corresponds to choosing a basis of multilinear operations whose inputs are indexed by an ordered set $I$. An ordered linear species $\mathcal{S}$ is said to be *reduced* is $\mathcal{S}(\varnothing) = 0$.

The *shuffle Cauchy product* of two ordered linear species $\mathcal{S}_1$ and $\mathcal{S}_2$ is defined by the same formula as in the symmetric case:

$$(\mathcal{S}_1 \cdot_{\text{III}} \mathcal{S}_2)(I) := \bigoplus_{I=I_1\sqcup I_2} \mathcal{S}_1(I_1)\otimes\mathcal{S}_2(I_2).$$

One may consider monoids with respect to the Cauchy product; they are called *shuffle algebras* [15, 61] or *permutads* [45]. Moreover, even in the absence of symmetric group actions, the extra datum of an order in our category allows one to define *divided powers* of reduced ordered linear species by

$$\mathcal{S}^{(n)}(I) := \bigoplus_{\substack{I=I_1\sqcup\cdots\sqcup I_n,\\ I_1,\ldots,I_n\neq\varnothing,\\ \min(I_1)<\cdots<\min(I_n)}} \mathcal{S}(I_1)\otimes\cdots\otimes\mathcal{S}(I_n).$$

Using those, the *shuffle composition product* of two reduced ordered linear species $\mathcal{S}_1$ and $\mathcal{S}_2$ is defined by the formula

$$\mathcal{S}_1 \circ_{\text{III}} \mathcal{S}_2 := \bigoplus_{n\geq 1} \mathcal{S}_1(\{1,\ldots,n\})\otimes\mathcal{S}_2^{(n)},$$

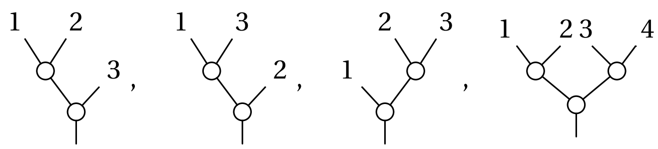
that is, if one unwraps the definitions,

$$(\mathcal{S}_1 \circ_{\text{Ш}} \mathcal{S}_2)(I) = \bigoplus_{n\geq 1} \mathcal{S}_1(\{1,\ldots,n\}) \otimes \left( \bigoplus_{\substack{I=I_1\sqcup\cdots\sqcup I_n,\\ I_1,\ldots,I_n\neq\varnothing,\\ \min(I_1)<\cdots<\min(I_n)}} \mathcal{S}_2(I_1)\otimes\cdots\otimes\mathcal{S}_2(I_n) \right).$$

The linear species $\mathbb{1}$ discussed above may be regarded as an ordered linear species; as such, it is the unit of the shuffle composition product.

Formally, a *shuffle operad* is a monoid with respect to the shuffle composition product. As we shall see below, each symmetric operad gives rise to a shuffle operad, and that is the main reason to care about shuffle operads. However, we start with explaining how to develop a theory of Gröbner bases of ideals in free shuffle operads.

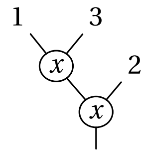
To describe free shuffle operads, we first define shuffle trees. Combinatorially, a *shuffle tree* is a planar rooted tree whose leaves are indexed by a finite ordered set $I$ in such a way that the following “local increasing condition” is satisfied: for every vertex of the tree, the minimal leaves of trees grafted at that vertex increase from the left to the right. The free shuffle operad generated by an ordered linear species $\mathcal{X}$ can be defined as follows. It is an ordered linear species $\mathcal{T}_{\text{Ш}}(\mathcal{X})$ for which $\mathcal{T}_{\text{Ш}}(\mathcal{X})(I)$ is spanned by decorated shuffle trees: each internal vertex $v$ of a tree must be decorated by an element of $\mathcal{X}(I_v)$, where $I_v$ is the set of incoming edges of $v$, ordered from the left to the right according to the planar structure. Such decorated trees should be thought of as tensors: they are linear in each vertex decoration. The operad structure is given by grafting of trees onto each other. One particular class of shuffle trees we shall consider are the so called *left combs*: trees for which all vertices appear on the unique path from the root to the minimal leaf. For instance, among the shuffle trees



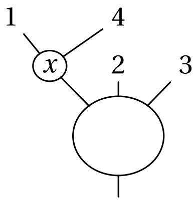

the first two are left combs, and the last two are not.

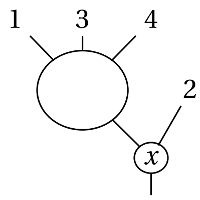
Given a basis of the vector space of an ordered linear species $\mathcal{X}$, one may consider all shuffle trees whose vertices are decorated by those basis elements. Such shuffle trees with leaves in a bijection with the given ordered set $I$ form a basis of $\mathcal{T}_{\text{Ш}}(\mathcal{X})(I)$, and we shall think of them as monomials in the free shuffle operad.

The next step in developing a theory of Gröbner bases is to define divisibility of monomials. Suppose that we have a shuffle tree $S$. We can insert another shuffle tree $S'$ into an internal vertex of $S$, and connect its leaves to the children of that vertex so that the order of leaves agrees with the left-to-right order of the children. We say that the thus obtained shuffle tree is divisible by $S'$, and use this notion of divisibility to define divisibility of decorated shuffle trees, that is of monomials in the free operad. For example, if we work in the free operad generated by the ordered linear species $\mathcal{X}$ such that $\mathcal{X}(I)$ is nonzero only for

$|I| = 2$, and is spanned by one element $x$, we may insert the tree $U =$ 


into the ternary vertices of the trees



and


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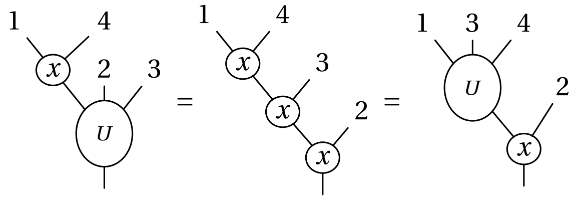
We obtain



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Once divisibility is understood, the usual Gröbner–Shirshov method of computing S-polynomials (in the language of Shirshov, one would say "compositions", which has the huge disadvantage in the case of operads where the same word is used to talk about the monoid structure), normal forms, etc. works in the usual way, if one has an *admissible ordering of monomials*, that is a total ordering of shuffle trees with the given set of leaf labels which is compatible with the shuffle operad structure. Such orderings exist, and we invite the reader to consult [15, 22] for definitions and examples. For us the so called graded path-lexicographic ordering and reverse graded path-lexicographic ordering will be of particular importance. With respect to the former, the trees are first compared by the depth of their leaves, while with respect to the latter, one reverses the comparison with respect to the depth of the leaves (in both cases, leaves are considered one by one in their given order). Throughout the paper, we say "the (reverse) graded path-lexicographic ordering" (with the definite article), though such an ordering depends on some ordering of generators, which one may choose freely.

2.4. **From symmetric operads to shuffle operads.** Note that there is a forgetful functor $\mathcal{S} \mapsto \mathcal{S}^f$ from all linear species to ordered linear species; it is defined by the formula $\mathcal{S}^f(I) := \mathcal{S}(I^f)$, where $I$ is a finite totally ordered set and $I^f$ is the same set but with the total order ignored. The reason to consider ordered linear species, shuffle algebras and shuffle operads is explained by the following proposition.

**Proposition 2.4** ([15, 24])**.** *For any two linear species $\mathcal{S}_1$ and $\mathcal{S}_2$, we have ordered linear species isomorphisms*

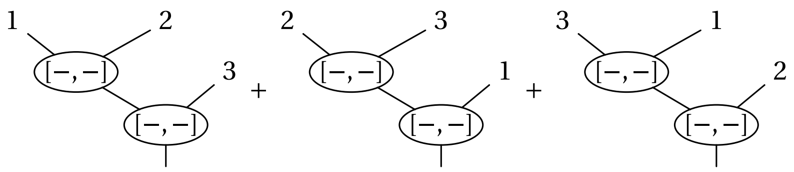
$$(\mathcal{S}_1 \cdot \mathcal{S}_2)^f \cong \mathcal{S}_1^f \cdot_{\mathrm{Ш}} \mathcal{S}_2^f,$$
$$(\mathcal{S}_1 \circ \mathcal{S}_2)^f \cong \mathcal{S}_1^f \circ_{\mathrm{Ш}} \mathcal{S}_2^f.$$

*In particular, applying the forgetful functor to a twisted associative algebra produces a shuffle algebra, and applying a forgetful functor to a reduced symmetric operad gives a shuffle operad. The forgetful functor sends modules over symmetric operads to modules over shuffle operads, ideals to ideals, free symmetric operads to free shuffle operads, etc.*

As an example, let us suppose that we consider multilinear operations that one may define starting from one binary operation $a_1, a_2 \mapsto [a_1, a_2]$ which is skew-symmetric. Then we work with binary trees (each vertex is either a leaf or has two children), and each binary vertex is now decorated by our only structure operation. Applying the forgetful functor means rewriting each such tree in terms of monomials in the free shuffle operad, using the skew-symmetry of the operation. For instance, the "Jacobiator" (the element encoding the Jacobi identity)

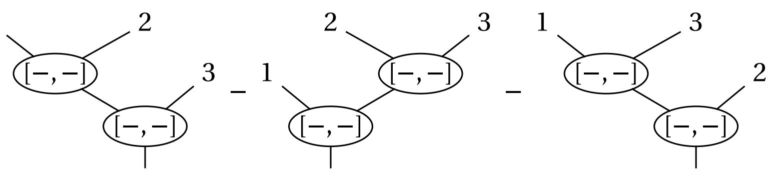
$$[[a_1, a_2], a_3] + [[a_2, a_3], a_1] + [[a_3, a_1], a_2]$$

corresponds to the element

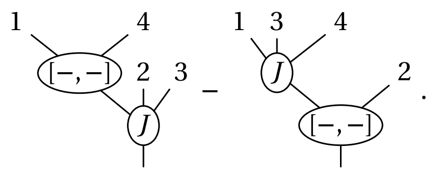


in the free symmetric operad, and then to the element

$J :=$



in the free shuffle operad. (For an operation $f(a_1, a_2)$ without symmetries, one has to introduce another operation $f^{\circ}(a_1, a_2) := f(a_2, a_1)$ to perform such rewriting: in the world of shuffle operads, we work with $\Bbbk$-modules, not $\Bbbk S_n$-modules.) There exists an admissible ordering for which the last of the three monomials in $J$ is the largest one, and in order to compute the Gröbner basis, we should form the S-polynomial corresponding to the self-overlap $T$ of that leading monomial with itself, which is



(If we denote $[-,-]$ by $x$, we note that we saw these monomials when discussing divisibility.) The leading terms in an S-polynomial always cancel, and one needs to check if it has a non-zero reduced form with respect to the existing elements: if it does, that reduced form needs to be adjoined to the existing elements in the course of computing the Gröbner basis. In our particular case, the S-polynomial gets reduced to zero, and so the Jacobiator forms a Gröbner basis.

To conclude this section, the forgetful functor from symmetric operads to shuffle operads allows one to go from the universe of "interesting" objects (actual varieties of algebras) to the universe of "manageable" objects (shuffle operads). Besides the symmetric group actions, it does not really lose any information, and, in particular, if certain properties can be expressed by saying that certain vector spaces are equal to zero, one can prove that in the context of shuffle operads (zero is zero with or without the symmetric group actions). For instance, all results on freeness (of operads or modules over operads) are like that, as we shall see in the next section.

2.5. **Homological criterion of freeness.** We shall now recall an important technical tool, a homological criterion of freeness of operadic modules. We refer the reader to [29] for details on operadic modules and their homotopy theory.

Recall that for an operad $\mathcal{O}$, its left module $\mathcal{L}$, and its right module $\mathcal{R}$, there is a two-sided bar construction $\mathsf{B}_\bullet(\mathcal{R},\mathcal{O},\mathcal{L})$. In somewhat concrete terms, it is spanned by rooted trees where for each tree the root vertex is decorated by an element of $\mathcal{R}$, the internal vertices whose all children are leaves are decorated by elements of $\mathcal{L}$, and other internal vertices are decorated by elements of $\mathcal{O}$; the differential contracts edges of the tree and uses the operadic composition and the module action maps. For an operad with unit, $\mathsf{B}_\bullet(\mathcal{O},\mathcal{O},\mathcal{O})$ is acyclic; moreover, for an augmented operad $\mathcal{O}$ with the augmentation ideal $\mathcal{O}_+$, the two-sided bar construction $\mathsf{B}_\bullet(\mathcal{O},\mathcal{O}_+,\mathcal{O})$ is acylic. This leads to a free resolution of any left $\mathcal{O}$-module $\mathcal{L}$ as

$$\mathsf{B}_\bullet(\mathcal{O},\mathcal{O}_+,\mathcal{O})\circ_{\mathcal{O}}\mathcal{L}\cong\mathsf{B}_\bullet(\mathcal{O},\mathcal{O}_+,\mathcal{L}).$$

This resolution can be used to prove the following result. (A similar result for right modules is slightly simpler, it was proved and used in [25].)

**Proposition 2.5.** *Let $\mathcal{O}$ be a (reduced connected) operad, and let $\mathcal{L}$ be a left $\mathcal{O}$-module. Then $\mathcal{L}$ is free as a left $\mathcal{O}$-module if and only if the positive degree homology of the bar construction $\mathsf{B}_\bullet(\mathbb{1},\mathcal{O}_+,\mathcal{L})$ vanishes; in the latter case, $\mathcal{R}$ is generated by the degree zero homology of $\mathsf{B}_\bullet(\mathbb{1},\mathcal{O}_+,\mathcal{L})$.*

*Proof.* This immediately follows from the existence and uniqueness up to isomorphism of the minimal free $\mathcal{O}$-module resolution of $\mathcal{L}$, which, for connected operads over a field of characteristic zero, is done similarly to the case of modules over rings in [27], even though the category of left $\mathcal{O}$-modules is not abelian. □

In particular, if $\mathcal{L}^f$ is free as a left $\mathcal{O}^f$-module, this means that the vanishing condition of this criterion holds, and we may use that very condition in the universe of symmetric operads to conclude that $\mathcal{L}$ is free as a left $\mathcal{O}$-module. This idea, first indicated in [21], is one of the key features of our approach.

2.6. **Universal multiplicative enveloping algebras.** For any variety of algebras $\mathfrak{M}$ and any $\mathfrak{M}$-algebra $A$, one can define the universal multiplicative enveloping algebra $U_{\mathfrak{M}}(A)$ to consist of all actions of elements of $A$: formally, one keeps one dedicated slot of an operation open, and inserts elements of $A$ in all other slots. This object has a natural associative algebra structure; moreover, the category of left modules over that algebra is equivalent to the category of $\mathfrak{M}$-bimodules over the algebra $A$. A classical exposition is given in [34, Sec. II.7] and in [73, §3.3], and a presentation in the language of operads is available in [30, Sec. 1.6]. In fact, it is possible to view the universal multiplicative enveloping algebra as the arity one part of the *enveloping operad*, see [12] and [29, Chapter 4].

We shall use the viewpoint on universal multiplicative enveloping algebras and on enveloping operads that encodes them via particular right operadic modules, mainly following [29, Chapter 10], but slightly re-casting it in the language of linear species.

Let us recall that the *derivative* $\partial(\mathcal{S})$ of a species $\mathcal{S}$ is defined by the formula

$$\partial(\mathcal{S})(I) := \mathcal{S}(I \sqcup \{\star\}),$$

so that in the case of linear species of multilinear operations of some type, it forms multilinear operations with one extra dedicated input that does not mix with the others. In the view of the above discussion of multiplicative universal envelopes, this is very appropriate. If $\mathcal{O}$ is an operad, then $\partial(\mathcal{O})$ has two structures: it is a right $\mathcal{O}$-module (via substitutions into the non-dedicated input) and a *twisted associative algebra*, that is a left module over the associative operad (via concatenating operations, substituting them into the dedicated inputs of one another). These two structures commute: $\partial(\mathcal{O})$ is a twisted associative algebra in the symmetric monoidal category of right $\mathcal{O}$-modules. The universal enveloping algebra is obtained from this via a relative composite product construction [29, 36]:

$$U_{\mathcal{O}}(A) \cong \partial(\mathcal{O}) \circ_{\mathcal{O}} A.$$

This has been used in a crucial way in [38] to establish the following result: there is a PBW type theorem for universal multiplicative enveloping algebras over the given operad $\mathcal{O}$ if and only if $\partial(\mathcal{O})$ is free as a right $\mathcal{O}$-module. Moreover, in this case, for the linear species $\mathcal{Y}$ that freely generates that right module we have an isomorphism

$$U_{\mathcal{O}}(A) \cong \mathcal{Y}(A)$$

that is functorial with respect to $\mathcal{O}$-algebra morphisms.

Similarly to the passage from symmetric operads to shuffle operads, it is possible to pass from left modules over the associative operad to a certain shuffle version. We shall not discuss this passage in detail, but rather briefly explain what this means for $\partial(\mathcal{O})$. In the symmetric context, the operation $\partial$ makes one of the inputs of the operation "special". Once we apply the forgetful functor to ordered linear species, the inputs are linearly ordered. A good "canonical" way to make one of them special is take the first one. We shall recall the corresponding construction, referring the reader to [38] for a slightly different viewpoint. For a shuffle operad $\mathcal{P}$, we let $\partial^{\text{Ш}}(\mathcal{P})(I) = \mathcal{P}(\{-\infty_I\} \sqcup I)$, where $-\infty_I$ denotes a new element that is smaller than all elements of $I$. For each ordered set $K$ partitioned as $K = I \sqcup J$, the product

$$\mu_{I,J} \colon \partial^{\text{Ш}}(\mathcal{P})(I) \otimes \partial^{\text{Ш}}(\mathcal{P})(J) \to \partial^{\text{Ш}}(\mathcal{P})(K),$$

is defined as follows. Suppose that

$$\alpha \otimes \beta \in \partial^{\text{Ш}}(\mathcal{P})(I) \otimes \partial^{\text{Ш}}(\mathcal{P})(J) = \mathcal{P}(\{-\infty_I\} \sqcup I) \otimes \mathcal{P}(\{-\infty_J\} \sqcup J\}).$$

We set

$$\mu_{I,J}(\alpha \otimes \beta) = \alpha \circ_{-\infty_I} \beta.$$

(Strictly speaking, this way one obtains an element in $\mathcal{P}(-\infty_J\} \sqcup K\})$, and one has to use the unique bijection of ordered sets to land in $\mathcal{P}(-\infty_K\} \sqcup K)$.) We also obtain a designated element in $\partial^{\text{Ш}}(\mathcal{P})(\varnothing) = \mathcal{P}(\{-\infty_\varnothing\})$ corresponding to the operadic unit.

**Proposition 2.6.** *For each shuffle operad $\mathcal{P}$, the thus defined map*

$$\partial^{\text{Ш}}(\mathcal{P}) \cdot_{\text{Ш}} \partial^{\text{Ш}}(\mathcal{P}) \to \partial^{\text{Ш}}(\mathcal{P})$$

*satisfies the associativity axiom and the unitality axiom with respect to the designated elements, so $\partial^{\text{Ш}}(\mathcal{P})$ becomes a shuffle algebra. Moreover, if the shuffle operad $\mathcal{P}$ is of the form $\mathcal{P} = \mathcal{O}^f$, where $\mathcal{O}$ is a symmetric operad, then the shuffle algebra $\partial^{\text{Ш}}(\mathcal{P})$ is isomorphic to $\partial(\mathcal{O})^f$.*

*Proof.* Both the associativity and the unitality immediately follow from the corresponding axioms for shuffle operads. (Note that since we compose only at minima, only the sequential axiom of the operad, corresponding to the genuine associativity, will be used.) The statement about the forgetful functor is essentially tautological and holds by direct inspection. □

## 3. A HOMOLOGICAL APPROACH TO THE NIELSEN–SCHREIER PROPERTY

Before we move on to proving our main result, let us discuss a homological interpretation of the criterion of Theorem 2.2, which will allow us to see that this criterion is much stronger than one would *a priori* expect. Recall that according to that theorem, the Nielsen–Schreier property of the variety $\mathfrak{M}$ is equivalent to the following two conditions:

(1) for every free $\mathfrak{M}$-algebra $A$, the universal multiplicative enveloping algebra $U_{\mathfrak{M}}(A)$ is a free associative algebra,
(2) for every homogeneous subalgebra $H$ of every free $\mathfrak{M}$-algebra $A$, the universal multiplicative enveloping algebra $U_{\mathfrak{M}}(A)$ is a free $U_{\mathfrak{M}}(H)$-module.

Let us make two additional assumptions on the variety of algebras that we are considering:

(i) there is a homology theory for $\mathfrak{M}$-algebras leading to a homological criterion of freeness of positively graded $\mathfrak{M}$-algebras, that is, a positively graded $\mathfrak{M}$-algebra $A$ is free if its homology with trivial coefficients vanishes in degree two and higher,
(ii) the homology can be computed via the Tor functors over the universal multiplicative enveloping algebra $U_{\mathfrak{M}}(A)$.

For the interested reader, we would like to clarify that we insist on the homological criterion of freeness for *positively graded* algebras, since this is the best thing to hope within the reach of current knowledge: even for Lie algebras freeness of algebras of homological dimension one is an open question in characteristic zero and is false in positive characteristic, as shown by Mikhalev, Zolotykh and the second author [52].

Let us show how to re-prove Theorem 2.2 for a variety $\mathfrak{M}$ satisfying these assumptions. The crucial observation here is a generalization of the trick of Shirshov used in the case of Lie algebras [70]. Namely, let us consider the free $\mathfrak{M}$-algebra $F_n$ with generators $y, x_1, \dots, x_n$. This algebra admits an obvious homomorphism to the one-dimensional vector space spanned by $y$, viewed as an $\mathfrak{M}$-algebra with zero structure operations. Let us denote by $H_n$ the kernel of that homomorphism, viewed as an $\mathfrak{M}$-subalgebra of the free algebra $F_n$.

**Proposition 3.1.** *If the $\mathfrak{M}$-algebra $H_n$ is free for each $n \geq 0$, the variety $\mathfrak{M}$ has the Nielsen–Schreier property.*

*Proof.* Let $H$ be a subalgebra of the free $\mathfrak{M}$-algebra $F$ with $k$ generators. Without loss of generality, we may assume $H$ to be finitely generated by an irreducible system of elements $h_1, \dots, h_q$. We shall prove our claim by Noetherian induction on $(\deg(h_1), \dots, \deg(h_q)) \in \mathbb{N}^q$, where we use the partial order of $\mathbb{N}^q$ for which $\mathbf{u} < \mathbf{v}$ if and only if $\mathbf{u} \neq \mathbf{v}$ and all coordinates of $\mathbf{v} - \mathbf{u}$ are nonnegative. The basis of induction is the case of all degrees equal to one, in which case we consider a subalgebra generated by several generators, and the claim is clear. Otherwise, the system of elements $h_1, \dots, h_q$ cannot contain all generators of $F$; we may assume that it does not contain the generator $x_k$, but that $x_k$ nontrivially appears in some of the $\mathfrak{M}$-monomials used to define the elements $h_1, \dots, h_q$, for otherwise we could find a counterexample in a smaller free algebra. If we denote $x_k$ by $y$, thus identifying our ambient free algebra $F$ with the free algebra generated by $y$, $x_1, \dots, x_{k-1}$, we see that all elements $h_1, \dots, h_q$ belong to $H_{k-1}$. By our assumption, this algebra is free; moreover, each element $h_i$ for which $y$ nontrivially appears in some of the $\mathfrak{M}$-monomials has smaller degree when expressed in terms of generators of $H_{k-1}$, so the induction hypothesis applies. □

We are now ready to re-prove Theorem 2.2. Each algebra $H_n$ is positively graded, so to establish its freeness it is enough, according to our assumption (i), to show that its homology with trivial coefficients vanishes in degree two and higher. To establish that, we shall use our assumption (ii) and compute homology via the Tor functors over universal multiplicative enveloping algebras. Since $U_{\mathfrak{M}}(F_n)$ is a free associative algebra, its trivial coefficients (co)representation has a free $U_{\mathfrak{M}}(F_n)$-resolution supported in homological degrees 0 and 1. Since $U_{\mathfrak{M}}(A_n)$ is a free $U_{\mathfrak{M}}(H_n)$-module, this resolution is a free $U_{\mathfrak{M}}(H_n)$-resolution. Thus, the homology of $H_n$ with trivial coefficients vanishes in degree two and higher, and so $H_n$ is free.

Let us look closely at the assumptions we made. First of all, a good homology theory leading to criteria of freeness is not readily available for an arbitrary variety of algebras, and some work is required here. More importantly, even when such a homology theory is available, it is not necessarily the homology theory computed via the Tor functors over the universal multiplicative enveloping algebra. One possible condition of when this assumption is true is given in [29, Th. 17.3.4], which, when applied to our situation (reduced connected operads operads without differential concentrated in homological degree zero), requires that the right $\mathcal{O}$-module $\partial(\mathcal{O})$ is free, meaning a PBW type theorem for universal multiplicative enveloping algebras. This suggests that, if one wishes to apply operadic methods, it is probably reasonable to weaken the necessary and sufficient conditions of Theorem 2.2. We take this observation on board, and do just that in the next section. In the language of varieties of algebras, the our combinatorial criterion gives the following three sufficient conditions for the Nielsen–Schreier property:

(1) for every free $\mathfrak{M}$-algebra $A$, its universal multiplicative enveloping algebra $U_{\mathfrak{M}}(A)$ is a free associative algebra,
(2) the variety $\mathfrak{M}$ has the PBW property for universal multiplicative enveloping algebras,

(3) for every $\mathfrak{M}$-algebra $A$ with zero structure operations and every subspace $H \subset A$, the universal multiplicative enveloping algebra $U_{\mathfrak{M}}(A)$ is a free $U_{\mathfrak{M}}(H)$-module.

## 4. A COMBINATORIAL CRITERION FOR THE NIELSEN–SCHREIER PROPERTY

In this section, we prove the main theoretical result of this paper: a combinatorial criterion for the Nielsen–Schreier property in terms of Gröbner bases for operads. The combinatorics of our criterion somewhat resembles that from a work of Burgin [17] who introduced a certain "property (S)" of a variety of algebras that he claimed to be equivalent to the Nielsen–Schreier property. We begin our section by showing that this claim is not true at face value even under the most lenient interpretation. (There are some doubts about this work already in the `MathReviews` review [7] by Artamonov, who however questioned the proof, not the result itself.) Our new combinatorial criterion for Nielsen–Schreier varieties that is stated and proved below suggests that, once Gröbner bases of operads are available as a technical tool, one can prove mathematically sound statements that are superficially similar to the assertions of Burgin.

4.1. **A non-criterion of Nielsen–Schreier varieties.** Following [17], we say that a variety of algebras has the property (S) if in every minimal identity (that is, an identity that does not follow from identities of smaller degrees) each variable appears as an argument of the top level operation in at least one of the monomials in the identity. The main result of [17] asserts that a homogeneous variety has the property (S) if and only if it is a Nielsen–Schreier variety.

We start with highlighting one intrinsic issue of [17]. In the introduction to that paper, it is claimed that the variety of (left) Leibniz algebras, that is algebras satisfying the identity

$$a_1(a_2a_3) = (a_1a_2)a_3 + a_2(a_1a_3),$$

satisfies the property (S). (At that point, the name "Leibniz algebras" was not yet invented, but the corresponding variety of algebras was studied by Bloh in [14] under the name "left D-algebras".) This claim already makes the reader a little bit concerned, since the identity above does satisfy the combinatorial condition of the property (S), but its consequence of the same degree

$$(a_1a_2)a_3 + (a_2a_1)a_3 = 0$$

obviously fails the corresponding combinatorial condition. Indeed, the variety of Leibniz algebras does not have the Nielsen–Schreier property, see [55].

The example of Leibniz algebras indicates an unfortunate ambiguity in how one may interpet the property (S). There exists however an example of a variety that does not have the Nielsen–Schreier property but satisfies the property (S), whatever interpretation of that property one may choose. It is the variety of mock Lie algebras [74], known also under the names Jordan–Lie algebras [57] and Jacobi–Jordan algebras [16], defined as the variety of *commutative* algebras satisfying the identity

$$(a_1a_2)a_3 + (a_2a_3)a_1 + (a_3a_1)a_2 = 0,$$

resembling the Jacobi identity, but, due to the commutativity of the operation, equivalent to the nil identity $x^2x = 0$. In this case, the defining multilinear identity transforms under the action of $S_3$ as the trivial representation, so the space of identities of degree 3 is one-dimensional, and the combinatorial condition of the property (S) is satisfied; in fact, combinatorially there is no difference between the multilinear mock-Lie identity and the Jacobi identity. It is however very easy to see that the variety of mock-Lie algebras does not have the Nielsen–Schreier property. Indeed, let us consider the free algebra on one generator $x$. Note that substituting $a_1 = a_2 = x$, $a_3 = x^2$ into the mock Lie identity gives us

$$x^2x^2 + (x(x^2))x + (x^2x)x = 0,$$

and since our operation is commutative and $x^2x = 0$, we conclude that $x^2x^2 = 0$. Thus, our free algebra is the two-dimensional vector space spanned by $x$ and $x^2$, and the subalgebra generated by $x^2$ is not free.

Let us remark that from the operad point of view, it is possible to unravel the mystery behind this phenomenon: the mock-Lie identity does not form a Gröbner basis of the corresponding operad! In fact, for one of the possible orderings, the reduced Gröbner basis for that operad contains the element

$$((a_1a_2)a_3)a_4 + ((a_1a_2)a_4)a_3 + ((a_1a_3)a_2)a_4 \\ + ((a_1a_3)a_4)a_2 + ((a_1a_4)a_2)a_3 + ((a_1a_4)a_3)a_2$$

which fails the corresponding combinatorial condition. This suggests that perhaps one can repair Burgin's criterion by re-defining the word "follow" in "does not follow from identities of smaller degrees" using Gröbner bases for operads, and then replacing the property (S) by a combinatorial condition of similar flavour. One possible implementation of this plan leads to the combinatorial criterion proved below.

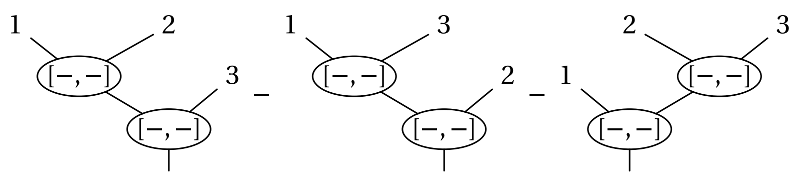
4.2. **A criterion of Nielsen–Schreier varieties.** We are now prepared and motivated to state and prove the main result of this paper.

**Theorem 4.1.** *Suppose that the operad $\mathcal{O}$ encoding the given variety of algebras $\mathfrak{M}$ satisfies the following two properties:*

- *for the reverse graded path-lexicographic ordering, each leading term of the reduced Gröbner basis of the corresponding shuffle operad $\mathcal{O}^f$ has the minimal leaf directly connected to the root,*
- *for the graded path-lexicographic ordering, each leading term of the reduced Gröbner basis of the corresponding shuffle operad $\mathcal{O}^f$ is a left comb with the maximal leaf directly connected to the root.*

*Then the variety $\mathfrak{M}$ has the Nielsen–Schreier property.*

Before proving this theorem, let us give an example of how it can be applied. Consider the operad encoding Lie algebras. It is immediate to check that the shuffle Jacobi identity




forms the reduced Gröbner basis for both orderings; its last monomial is the leading term for the reverse graded path-lexicographic ordering, and its first monomial is the leading term for the graded path-lexicographic ordering, and the combinatorial conditions of our theorem clearly hold. Thus, our result in particular gives a new one-line proof of the Shirshov–Witt theorem on subalgebras of free Lie algebras.

*Proof.* Let us begin with outlining the general plan of the proof. We shall use the first combinatorial condition to prove that for every free $\mathfrak{M}$-algebra $A$, its universal multiplicative enveloping algebra $U_{\mathfrak{M}}(A)$ is a free associative algebra, one half of the second combinatorial condition (the left comb condition) to prove that the variety $\mathfrak{M}$ has the PBW property for universal multiplicative enveloping algebras, and the other half of the second combinatorial condition (the maximal leaf condition) to prove that for every $\mathfrak{M}$-algebra $A$ with zero structure operations and every subspace $H \subset A$, the universal multiplicative enveloping algebra $U_{\mathfrak{M}}(A)$ is a free $U_{\mathfrak{M}}(H)$-module. Finally, we shall combine the two latter assertions to recover the second condition of Theorem 2.2.

**Lemma 4.2.** *For every free $\mathcal{O}$-algebra $A$, the universal multiplicative enveloping algebra $U_{\mathcal{O}}(A)$ is a free associative algebra.*

*Proof.* For the free algebra $A = \mathcal{O}(V)$, we have

$$U_{\mathcal{O}}(A) \cong \partial(\mathcal{O}) \circ_{\mathcal{O}} A = \partial(\mathcal{O}) \circ_{\mathcal{O}} \mathcal{O}(V) \cong \partial(\mathcal{O})(V),$$

with the product of $U_{\mathcal{O}}(A)$ induced from that of $\partial(\mathcal{O})$ on the twisted associative algebra level, so it is enough to show that $\partial(\mathcal{O})$ is free as a twisted associative algebra. To establish that, it is sufficient to consider the associated shuffle operad $\mathcal{O}^f$ and prove the corresponding result for the shuffle algebra $\partial^{\text{Ш}}(\mathcal{O}^f)$. Indeed, we are working with connected operads over a field of characteristic zero, so the homological criterion of freeness implies that $\partial(\mathcal{O})$ is a free twisted associative algebra if and only if $\partial(\mathcal{O})^f$ is a free shuffle algebra, and we know that $\partial(\mathcal{O})^f \cong \partial^{\text{Ш}}(\mathcal{O}^f)$.

Let us denote by $G_r$ the reduced Gröbner basis of the shuffle operad $\mathcal{O}^f$ for the reverse graded path-lexicographic ordering, and by $\mathcal{N}_r$ the ordered species of monomials that are normal with respect to $G_r$. From the definition of the shuffle algebra structure on $\partial^{\text{Ш}}(\mathcal{O}^f)$, it immediately follows that it is generated by “min-indecomposable” elements of $\mathcal{N}_r$, that is shuffle trees which have their minimal leaf directly connected to the root. Moreover, if $\alpha \in \partial^{\text{Ш}}(\mathcal{O}^f)(I)$ and $\beta \in \partial^{\text{Ш}}(\mathcal{O}^f)(J)$ are two normal forms, then $\mu_{I,J}(\alpha \otimes \beta)$ is a normal form. Indeed, composing two normal forms at the minimal leaf of a normal form cannot create an element divisible by a leading term of the Gröbner basis, by our assumption on the leading terms. □

The following result follows from [38, Th. 5.16], but we give a proof to make the exposition self-contained.

**Lemma 4.3.** *The right $\mathcal{O}$-module $\partial(\mathcal{O})$ is free. Accordingly, there is a PBW-type theorem for multiplicative universal envelopes of $\mathcal{O}$-algebras: there exists a linear species $\mathcal{Y}$ such that for every $\mathcal{O}$-algebra $V$, the underlying vector space of $U_{\mathcal{O}}(V)$ is isomorphic to $\mathcal{Y}(V)$ functorially with respect to $\mathcal{O}$-algebra morphisms.*

*Proof.* Let us denote by $G_l$ the reduced Gröbner basis of the shuffle operad $\mathcal{O}^f$ for the graded path-lexicographic ordering, by $\mathcal{N}_l$ the ordered species of monomials that are normal with respect to $G_l$, and by $\mathcal{N}_l^{(0)} \subset \mathcal{N}_l$ the ordered subspecies of normal monomials that are left combs. It is clear that the right $\mathcal{O}^f$-module $\partial^{\text{Ш}}(\mathcal{O}^f)$ is freely generated by $\partial^{\text{Ш}}(\mathcal{N}_l^{(0)})$. Indeed, this follows from the fact that composing two normal forms at a non-minimal leaf cannot create an element divisible by a leading term of the Gröbner basis, by our assumption on the leading terms. Since we are working with connected operads over a field of characteristic zero, the homological criterion of freeness implies that $\partial(\mathcal{O})$ is free as a right $\mathcal{O}$-module, which in turn implies that the operad $\mathcal{O}$ has the PBW property for universal multiplicative enveloping algebras: if $\partial(\mathcal{O}) \cong \mathcal{Y} \circ \mathcal{O}$, then

$$U_{\mathcal{O}}(V) \cong \partial(\mathcal{O}) \circ_{\mathcal{O}} V \cong (\mathcal{Y} \circ \mathcal{O}) \circ_{\mathcal{O}} V \cong \mathcal{Y}(V).$$

□

The next result is perhaps the least obvious of the three key steps of the proof.

**Lemma 4.4.** *For every $\mathcal{O}$-algebra $A$ with zero operations and any subspace $H \subset A$, viewed as a subalgebra with zero operations, the universal enveloping algebra $U_{\mathcal{O}}(A)$ is free as a $U_{\mathcal{O}}(H)$-module.*

*Proof.* We already know that the right $\mathcal{O}$-module $\partial(\mathcal{O})$ is free, so that

$$\partial(\mathcal{O}) \cong \mathcal{Y} \circ \mathcal{O}$$

for some linear species $\mathcal{Y}$. To comprehend universal multiplicative envelopes for algebras with zero operations functorially, it is convenient to think of such an algebra $A$ as $\mathbb{1}(A)$, where $\mathbb{1}$ is given a left $\mathcal{O}$-module structure via the augmentation map. This way,

$$\partial(\mathcal{O}) \circ_{\mathcal{O}} A = \partial(\mathcal{O}) \circ_{\mathcal{O}} (\mathbb{1} \circ A) \cong (\partial(\mathcal{O}) \circ_{\mathcal{O}} \mathbb{1}) \circ A,$$

and we see that the suitable formula for the species $\mathcal{Y}$ is

$$\mathcal{Y} := \partial(\mathcal{O}) \circ_{\mathcal{O}} \mathbb{1};$$

this way, $\mathcal{Y}(A)$ literally corresponds to the universal envelope of the algebra with zero operations. The advantage of this viewpoint is that $\mathcal{Y}$ is, by construction, a twisted associative algebra, and the associative algebra structure of $U_{\mathcal{O}}(A) \cong \mathcal{Y}(A)$ is induced from that twisted associative algebra structure.

From the formula $\mathcal{Y} = \partial(\mathcal{O}) \circ_{\mathcal{O}} \mathbb{1}$ it follows that any right action of a nontrivial structure operation vanishes, so as a twisted associative algebra, $\mathcal{Y}$ is generated by $\partial(\mathcal{X})$, where $\mathcal{X}$ is the species of generators of $\mathcal{O}$, and therefore as a shuffle algebra, $\mathcal{Y}^f$ is generated by $\partial(\mathcal{X})^f$. Let us describe a Gröbner basis of relations for this shuffle algebra. To the Gröbner basis $G_l$, we may associate the subset $\bar{G}_l$ of the free shuffle algebra generated by $\partial(\mathcal{X})^f$ consisting of elements obtained from elements of $G_l$ by deleting all monomials that are not left combs. It is clear that $\bar{G}_l$ consists of relations of $\mathcal{Y}^f$, since the deleted monomials vanish in $\partial(\mathcal{O}) \circ_{\mathcal{O}} \mathbb{1}$. Moreover, according to our assumption about the leading terms of $G_l$, the elements of $\bar{G}_l$ have the same leading terms for the appropriate graded lexicographic order of monomials in the free shuffle algebra. This immediately implies that $\bar{G}_l$ forms a Gröbner basis: the cosets of elements from $\partial^{\text{Ш}}(\mathcal{N}_l^{(0)})$ form a basis of $\mathcal{Y}^f$, and these are precisely the normal forms with respect to $\bar{G}_l$.

The last step of the proof requires to slightly extend the combinatorics with which we work. We shall need the language of two-sorted linear species. Informally, a two-sorted linear species is a canonical rule to associate a vector space to each pair of finite sets, see [13] for details. In our case, our goal is to prove that for an algebra $A$ with zero operations and its subspace $H$, viewed as a subalgebra with zero operations, $U_{\mathcal{O}}(A)$ is free as a $U_{\mathcal{O}}(H)$-module. We write $A = H \oplus H'$ for some subspace $H'$, and we wish to make this splitting propagate in a certain way to universal enveloping algebras. The universal enveloping algebra $U_{\mathcal{O}}(A)$ can be calculated as $\mathcal{Y}(A)$, and now we shall use two-sorted species to distinguish between elements coming from $H$ and from $H'$.

To be precise, we consider the two-sorted species $\mathcal{Y}^{(2)}$ and $\mathcal{X}^{(2)}$ defined as

$$\mathcal{Y}^{(2)}(I,J) = \mathcal{X}(\{\star\} \sqcup I \sqcup J), \quad \mathcal{X}^{(2)}(I,J) = \mathcal{X}(\{\star\} \sqcup I \sqcup J)$$

The meaning of these species is as follows. The species $\mathcal{Y}^{(2)}$ is a functorial version of the universal multiplicative envelope $U_{\mathcal{O}}(A)$ when written as $U_{\mathcal{O}}(H\oplus H')$. The species $\mathcal{X}^{(2)}$ is the functorial species of generators of that algebra: we consider structure operations of our algebras for which we have a special input that will be used to define the twisted associative algebra structure, some inputs of the first type (where we shall later substitute elements of $H$), and some inputs of the second type (where we shall later substitute elements of $H'$). We have $\mathcal{X}^{(2)} = \mathcal{X}_0^{(2)} \oplus \mathcal{X}_1^{(2)}$, where

$$\mathcal{X}_0^{(2)}(I,J) = \begin{cases} \mathcal{X}^{(2)}(I,J), & J = \varnothing, \\ 0, & J \neq \varnothing, \end{cases} \qquad \mathcal{X}_1^{(2)}(I,J) = \begin{cases} 0, & J = \varnothing, \\ \mathcal{X}^{(2)}(I,J), & J \neq \varnothing. \end{cases}$$

In plain words, $\mathcal{X}_0^{(2)}$ will later correspond to the situation where all elements we use are elements of $H$, and $\mathcal{X}_1^{(2)}(I,J)$ will later correspond to the situation where we use at least one element of $H'$. The subalgebra $U_{\mathcal{O}}(H)$ of $U_{\mathcal{O}}(A)$ corresponds to the subalgebra $\mathcal{Y}_0^{(2)}$ of $\mathcal{Y}^{(2)}$ generated by $\mathcal{X}_0^{(2)}$. Thus, we see that it is sufficient to prove that the twisted associative algebra $\mathcal{Y}^{(2)}$ is free as a $\mathcal{Y}_0^{(2)}$-module: the species of generators of that module, once evaluated on $(H,H')$, will give the free generators of $U_{\mathcal{O}}(A)$ as a $U_{\mathcal{O}}(H)$-module.

We shall consider the two-sorted species in the shuffle context as follows: we consider pairs of ordered sets $(I,J)$ as totally ordered sets for which all elements of $I$ are smaller than all elements of $J$. Let us examine closely a normal form from $\mathcal{N}_l^{(0)}(I,J)$. Such normal forms, which are, by definition, left combs, come in two types: those normal forms for which all leaves directly connected to the root belong to $I$, and those normal forms which have a leaf from $J$ directly connected to the root. We claim that the normal forms of the second kind are free generators of $\mathcal{Y}^{(2)}$ as a $\mathcal{Y}_0^{(2)}$-module. Indeed, every normal form can be represented as a shuffle product of several generators from $\mathcal{X}_0^{(2)}$ and a normal form of the second kind we just described, so those normal forms generate $\mathcal{Y}^{(2)}$ as a $\mathcal{Y}_0^{(2)}$-module. Freeness follows from our assumption on the leading terms: a shuffle product of a normal form from $\mathcal{Y}_0^{(2)}$ and a normal form of the second kind cannot be divisible by a leading term of the Gröbner basis. □

To conclude the proof of our theorem, we argue as follows. We just proved that for any $\mathcal{O}$-algebra $A$ with zero operations and any subalgebra $H \subset A$, the universal enveloping algebra $U_{\mathcal{O}}(A)$ is free as a $U_{\mathcal{O}}(H)$-module. Because of the

PBW property, the same is actually true for any algebra $A$ and its subalgebra $H$: we impose the usual filtration on $U_{\mathcal{O}}(A)$ and take the associated graded algebra, then the freeness holds after taking the associated graded algebras, and we may lift the free generators to the original algebra. Thus, in particular, for every subalgebra $H$ of every free algebra $A$, the universal multiplicative enveloping algebra $U_{\mathcal{O}}(A)$ is a free $U_{\mathcal{O}}(H)$-module. We also established that for every free algebra $A$ of our variety, the universal multiplicative envelope $U_{\mathcal{O}}(A)$ is a free associative algebra. According to Theorem 2.2, these two properties together imply the Nielsen–Schreier property. □

## 5. Pre-Lie algebras

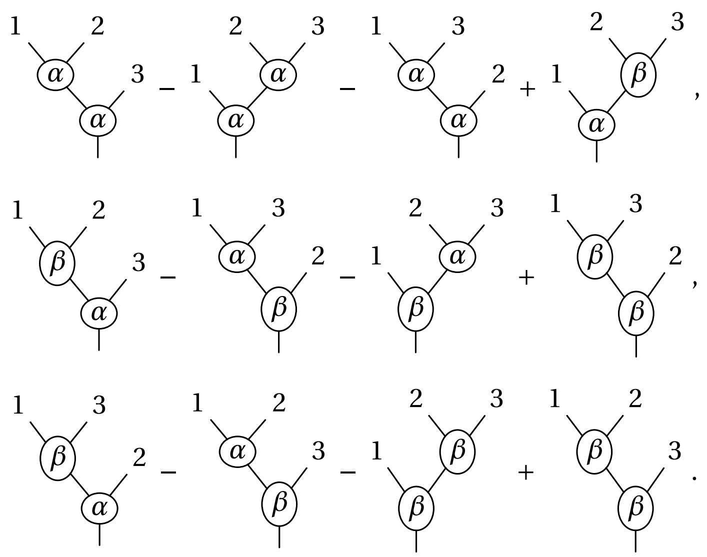
Recall that the variety of pre-Lie algebras [18], also known as right-symmetric algebras, is defined by the identity

$$(a_1a_2)a_3 - a_1(a_2a_3) = (a_1a_3)a_2 - a_1(a_3a_2).$$

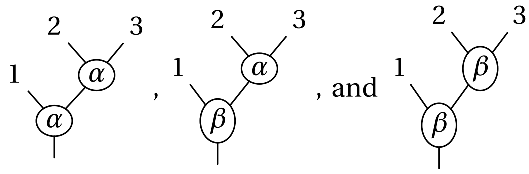
Existing results about pre-Lie algebras suggest that this variety might be Nielsen–Schreier. For instance, according to a result of Kozybaev, Makar-Limanov and the second author [41], two-generated subalgebras of free pre-Lie algebras are free. Moreover, in the context of our general result, it is worth recalling the result of Kozybaev and the second author [40] (see also [23, 38]) that the underlying vector space of the universal multiplicative enveloping algebra of a pre-Lie algebra $L$ is isomorphic to $T(L) \otimes S(L)$, meaning that a PBW type theorem holds for universal multiplicative envelopes. We shall now show how to use the operad theory approach in this case.

**Theorem 5.1.** *The variety of pre-Lie algebras has the Nielsen–Schreier property.*

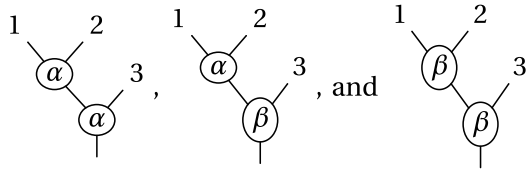
*Proof.* Let us consider the operations $\alpha(a_1, a_2) = a_1a_2$ and $\beta(a_1, a_2) = a_2a_1$ which generate the operad of right-symmetric algebras as a shuffle operad. In terms of these operations, the right-symmetric identities correspond to vanishing of the elements



For the reverse graded path-lexicographic ordering corresponding to the ordering $\alpha > \beta$ of generators, the defining relations form a Gröbner basis with the leading terms



satisfying the first combinatorial condition of Theorem 4.1. For the graded path-lexicographic ordering corresponding to the ordering $\beta > \alpha$ of generators, the defining relations form a Gröbner basis with the leading terms



satisfying the second combinatorial condition of Theorem 4.1. Both of these statements are easy to verify using the following observation outlined in [21, Corollary 1]. For a shuffle operad with quadratic relations $G$, the shuffle tree monomials for which each quadratic divisor is a leading term of an element of $G$ span the Koszul dual operad, and $G$ is a quadratic Gröbner basis if and only if the number of such shuffle tree monomials with $n$ leaves is equal to the dimension of the arity $n$ component of the Koszul dual operad. Combining them together completes the proof. □

In [41], it is shown that automorphisms of two-generated free pre-Lie algebras are tame. Theorem 5.1, combined with the result of Lewin mentioned in Section 2.1, immediately implies the following generalization.

**Corollary 5.2.** *Automorphisms of finitely generated free pre-Lie algebras are tame.*

Let us remark that in [39] it is claimed that the variety of right-symmetric algebras does not have the Nielsen–Schreier property. Unfortunately, there is an issue with the two main proofs of that paper that rely on highly intricate computations. We studied the arguments of [39] in detail, and we believe that we identified the problematic parts. First, the claimed polynomial relation between particular five elements of the free two-generated algebra does not hold (we checked this using the `albert` software for computations in nonassociative algebras [33]). Second, the proof of non-freeness of the multiplicative universal envelope of the free one-generated right-symmetric algebra seems to start with a correct identity but then makes a claim on algebraic independence that is false.

## 6. ALGEBRAS WITH TWO COMPATIBLE LIE BRACKETS

Recall that an algebra with two compatible Lie brackets is a vector space $V$ equipped with two operations $a_1, a_2 \mapsto [a_1, a_2]$ and $a_1, a_2 \mapsto \{a_1, a_2\}$ which are skew-symmetric, satisfy the Jacobi identity individually, and additionally their sum also satisfies the Jacobi identity. The latter condition is equivalent to the identity

$$[\{a_1,a_2\},a_3]-[\{a_1,a_3\},a_2]-[a_1,\{a_2,a_3\}]+ \\ +\{[a_1,a_2],a_3\}-\{[a_1,a_3],a_2\}-\{a_1,[a_2,a_3]\}=0.$$

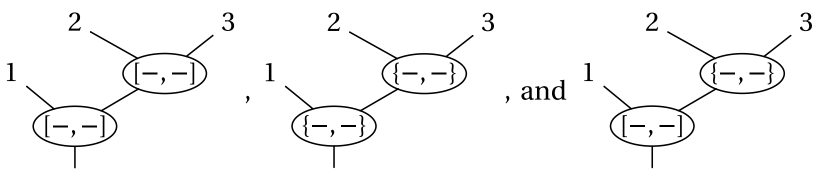
Since it is known [20] that the dimension of the $n$-th component of the operad of two compatible Lie brackets is equal to $n^{n-1}$, which is also the dimension of the $n$-th component of the operad of right-symmetric algebras [18], the result of Theorem 5.1 suggests that the variety of algebras with two compatible Lie brackets might have the Nielsen–Schreier property. We shall now show that it is indeed the case.

**Theorem 6.1.** *The variety of algebras with two compatible Lie brackets has the Nielsen–Schreier property.*

*Proof.* For the reverse graded path-lexicographic ordering corresponding to the ordering

$$[-,-]>\{-,-\}$$

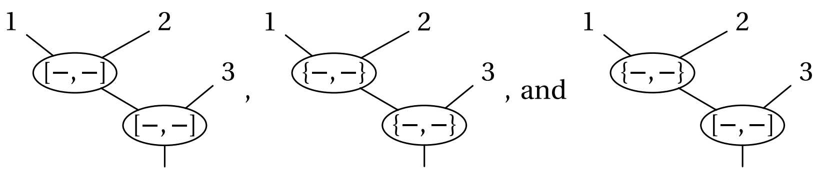
of generators, the defining relations form a Gröbner basis with the leading terms



satisfying the first combinatorial condition of Theorem 4.1. For the graded path-lexicographic ordering corresponding to the ordering

$$[-,-]>\{-,-\}$$

of generators, the defining relations form a Gröbner basis with the leading terms



satisfying the second combinatorial condition of Theorem 4.1. (Both of these statements easily follow from the observation quoted above, combined with the known fact that the arity $n$ component of the Koszul dual operad is equal to $n$.) Combining these observations completes the proof. □

An almost identical proof works for Lie algebras with several compatible Lie brackets [66].

## 7. VARIETIES WHOSE IDENTITIES DO NOT USE SUBSTITUTIONS OF OPERATIONS

In this section we record a generalization of the results of Kurosh and Polin mentioned in the introduction.

**Proposition 7.1.** *Suppose that all identities of the variety $\mathfrak{M}$ are combinations of structure operations (no substitutions are used). Then the variety $\mathfrak{M}$ has the Nielsen–Schreier property.*

*Proof.* In the language of operads, we are talking about free operads. Indeed, each structure operation with $k$ arguments *a priori* generates the regular representation of the group $S_k$, and identities that are combinations of structure operations give a collection of submodules in the regular modules which have to be quotiented out. What remains is certain collection of representations of symmetric groups that generates our operad freely. In particular, Theorem 4.1 applies tautologically, since there are no relations to consider. □

It turns out that this proposition implies the Nielsen–Schreier property for the variety of Akivis algebras [3, 63], first proved in [64]. Recall that an Akivis algebra is an algebra with one skew-symmetric binary operation $[-,-]$ and one ternary operation $(-,-,-)$ satisfying the identitiy

$$[[a_1,a_2],a_3]+[[a_2,a_3],a_1]+[[a_3,a_1],a_2]=\sum_{\sigma\in S_3}(-1)^{\sigma}(a_{\sigma(1)},a_{\sigma(2)},a_{\sigma(3)}).$$

**Corollary 7.2.** *The variety of Akivis algebras has the Nielsen–Schreier property.*

*Proof.* The six-dimensional space of ternary generators of the corresponding operad is the regular representation of $S_3$ generated by $(-,-,-)$; as such, it splits into a direct sum of one copy of the trivial representation, one copy of the sign representation, and two copies of the two-dimensional irreducible representation. We note that the element

$$\sum_{\sigma\in S_3}(-1)^{\sigma}(a_{\sigma(1)},a_{\sigma(2)},a_{\sigma(3)}).$$

found in the right hand side of the defining identity of the Akivis algebras is precisely the generator corresponding to the copy of the sign representation, and the Akivis identity allows one to eliminate this element, replacing it by the Jacobiator

$$[[a_1,a_2],a_3]+[[a_2,a_3],a_1]+[[a_3,a_1],a_2].$$

This elimination implements an isomorphism between the operad of the Akivis algebras and the free operad generated by one skew-symmetric binary operation $[-,-]$ and a five-dimensional space of ternary operations where the $S_3$-action is the direct sum of the trivial representation and two copies of the two-dimensional irreducible representation, so it has the Nielsen–Schreier property. □

## 8. Intersection of Nielsen–Schreier varieties

Let us record a simple general observation that allows one to construct new Nielsen–Schreier varieties from known ones.

**Proposition 8.1.** *Suppose that two varieties with disjoint sets of structure operations both satisfy the combinatorial criterion of Theorem 4.1. THen the intersection of those varieties satisfy this condition as well. In particular, that intersection has the Nielsen–Schreier property.*

*Proof.* In the language of operads, the intersection of two varieties with disjoint sets of structure operations corresponds to the categorical coproduct, also known as the free product, of the corresponding operads. The Gröbner basis of such operad is the union of the two Gröbner bases, and so Theorem 4.1 applies. □

Our first observation is that this result applies to Lie-admissible algebras [5]. Recall that a Lie-admissible algebra is an algebra with one binary operation satisfying the identitiy

$$\sum_{\sigma\in S_3}(-1)^\sigma\big((a_{\sigma(1)}a_{\sigma(2)})a_{\sigma(3)}-a_{\sigma(1)}(a_{\sigma(2)}a_{\sigma(3)})\big)=0.$$

**Corollary 8.2.** *The variety of Lie-admissible algebras has the Nielsen–Schreier property.*

*Proof.* In terms of the operations $a_1\circ a_2=a_1a_2+a_2a_1$ and $[a_1,a_2]=a_1a_2-a_2a_1$, the Lie admissibility relation becomes the Jacobi identity for the second operation (this observation goes back to [48]). Thus, the variety of Lie-admissible algebras is the intersection of the variety of Lie algebras and the variety of all commutative algebras, both of which satisfy the combinatorial conditions of Theorem 4.1. ☐

## 9. Parametric families of Nielsen–Schreier varieties

9.1. **Deformation of right-normed third power nil identity.** Our methods lead to the following rather striking generalization of a result of the second author who proved that the variety of algebras satisfying the identity $xx^2=0$ has the Nielsen–Schreier property.

**Theorem 9.1.** *For every $\alpha\neq 1$, the variety of algebras satisfying the identity*

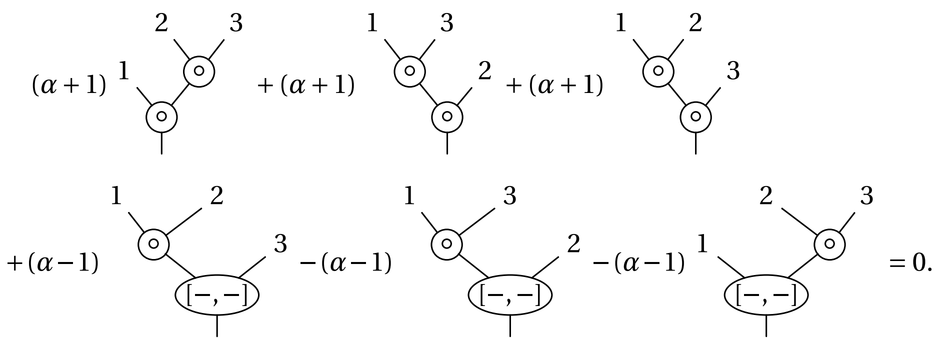
$$xx^2+\alpha x^2x=0$$

*has the Nielsen–Schreier property.*

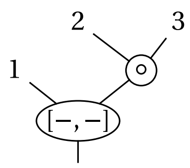
*Proof.* This identity is equivalent to the multilinear one

$$\sum_{\sigma\in S_3}\big(a_{\sigma(1)}(a_{\sigma(2)}a_{\sigma(3)})+\alpha(a_{\sigma(1)}a_{\sigma(2)})a_{\sigma(3)}\big)=0.$$

It will be convenient to present our variety via the symmetrized and the skew-symmetrized operations $a_1\circ a_2=a_1a_2+a_2a_1$ and $[a_1,a_2]=a_1a_2-a_2a_1$. In the language of shuffle operads, our identity becomes



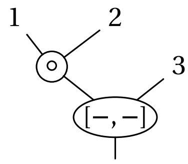
If $\alpha\neq 1$, then this relation has the leading term



for the reverse graded path-lexicographic ordering corresponding to the ordering

$$[-,-] > (-\circ -)$$

of generators, and the leading term




for the graded path-lexicographic ordering corresponding to the ordering

$$[-,-] > (-\circ -)$$

of generators. Since these monomials have no self-overlaps, the given relation forms a Gröbner basis in both cases. We note that the combinatorial conditions of Theorem 4.1 are satisfied, completing the proof. □

In particular, setting $\alpha = -1$, we see that the variety of algebras satisfying the identity $xx^2 = x^2x$, that is the variety of third power associative algebras, has the Nielsen–Schreier property. This obviously fails for the variety of all power associative algebras, which is defined, by a remarkable result of Albert [4, 5] by the above identity together with just one extra identity $(x^2x)x = x^2x^2$.

9.2. **Alia and one-sided alia algebras.** Recall that alia (anti-Lie-admissible) algebras [26] are the algebras with the following identity for the symmetrized and the skew-symmetrized operations:

$$[a_1,a_2]\circ a_3 + [a_2,a_3]\circ a_1 + [a_3,a_1]\circ a_2 = 0.$$

In the same paper one finds the definition of a left alia algebra as the algebra satisfying the identity

$$[a_1,a_2]a_3 + [a_2,a_3]a_1 + [a_3,a_1]a_2 = 0,$$

and the "opposite" definition of a right alia algebra as the algebra satisfying the identity

$$a_3[a_1,a_2] + a_1[a_2,a_3] + a_2[a_3,a_1] = 0.$$

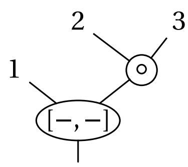
We shall show that the corresponding varieties have the Nielsen–Schreier property, proving the following general result.

**Theorem 9.2.** *For any $\alpha$, the variety of algebras satisfying the identity*

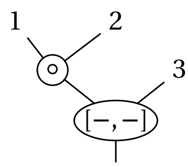
$$a_3[a_1,a_2] + a_1[a_2,a_3] + a_2[a_3,a_1] + \alpha([a_1,a_2]a_3 + [a_2,a_3]a_1 + [a_3,a_1]a_2) = 0$$

*has the Nielsen–Schreier property.*

*Proof.* For $\alpha = -1$, we obtain the variety of Lie-admissible algebras, so Corollary 8.2 applies. Suppose that $\alpha \neq -1$. Analogous to the proof of Theorem 9.1: if we write everything in terms of the symmetrized and the skew-symmetrized operations, we obtain, for the two orderings of interest, the leading terms




and



respectively. In each case, there are no self-overlaps, so we obtain a Gröbner basis, and Theorem 4.1 applies. □

## 10. Increasing degrees of identities

One may also adapt the proof of Theorem 9.1 to establish the following result.

**Theorem 10.1.** *For every degree $n \geq 1$, the variety of algebras satisfying the right nil identity*

$$x(x(\cdots(xx^2))) = 0$$

*has the Nielsen–Schreier property.*

*Proof.* This identity is equivalent to the multilinear one

$$\sum_{\sigma\in S_n} a_{\sigma(1)}(a_{\sigma(2)}(\cdots(a_{\sigma(n-2)}(a_{\sigma(n-1)}a_{\sigma(n)})))) = 0.$$

Let us use, once again, the symmetrized and the skew-symmetrized operations $a_1 \circ a_2 = a_1a_2 + a_2a_1$ and $[a_1, a_2] = a_1a_2 - a_2a_1$. For the reverse graded path-lexicographic ordering corresponding to the ordering

$$[-,-] > (- \circ -)$$

of generators, this identity has as the leading term the only right comb with all the vertices but the one at the top labelled $[-,-]$, and the top vertex labelled $- \circ -$. For the graded path-lexicographic ordering corresponding to the ordering

$$[-,-] > (- \circ -)$$

of generators, this identity has as the leading term the left comb with all the vertices but the one at the top labelled $[-,-]$, the top vertex labelled $- \circ -$, and the leaves labelled $1,\ldots, n$ in the planar order. In each case, there are no self-overlaps, so we obtain a Gröbner basis, and Theorem 4.1 applies. □

## Acknowledgements

We wish to dedicate this paper to the memory of Vyacheslav Alexandrovich Artamonov who passed away in June 2021. Not only had he been interested in Nielsen–Schreier varieties of algebras throughout his mathematical life [7, 8, 9, 10], but also his mathematical heritage is strongly connected with the operad theory: while operads were first defined by J. P. May in 1971 in his work on iterated loop spaces [37, 49], the same notion seems to have been first introduced under a much more technical name of a "clone of multilinear operations" in Artamonov's 1969 paper [6].

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INSTITUT DE RECHERCHE MATHÉMATIQUE AVANCÉE, UMR 7501, UNIVERSITÉ DE STRASBOURG ET CNRS, 7 RUE RENÉ-DESCARTES, 67000 STRASBOURG CEDEX, FRANCE

*Email address*: vdotsenko@unistra.fr

DEPARTMENT OF MATHEMATICS, WAYNE STATE UNIVERSITY, DETROIT, MI 48202, USA; DEPARTMENT OF MATHEMATICS, AL-FARABI KAZAKH NATIONAL UNIVERSITY, ALMATY, 050040, KAZAKHSTAN; INSTITUTE OF MATHEMATICS AND MATHEMATICAL MODELING, ALMATY, 050010, KAZAKHSTAN,

*Email address*: umirbaev@wayne.edu