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PUSHOUTS OF DWYER MAPS ARE $(\infty, 1)$-CATEGORICAL

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Abstract. The inclusion of 1-categories into $(\infty, 1)$-categories fails to preserve colimits in general, and pushouts in particular. In this note, we observe that if one functor in a span of categories belongs to a certain previously-identified class of functors, then the 1-categorical pushout is preserved under this inclusion. Dwyer maps, a kind of neighborhood deformation retract of categories, were used by Thomason in the construction of his model structure on 1-categories. Thomason previously observed that the nerves of such pushouts have the correct weak homotopy type. We refine this result and show that the weak homotopical equivalence is a weak categorical equivalence.

Contents

1. Introduction

Classical 1-categories define an important special case of $(\infty, 1)$-categories. The fact that $(\infty, 1)$-category theory restricts to ordinary 1-category can be understood, in part, by the observation that the inclusion of 1-categories into $(\infty, 1)$-categories is full as an inclusion of $(\infty, 2)$-categories. This full inclusion is reflective—with the left adjoint given by the functor that sends an $(\infty, 1)$-category to its quotient "homotopy category"—but not coreflective and as a consequence colimits of

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ordinary 1-categories need not be preserved by the passage to $(\infty, 1)$-categories. Indeed there are known examples of colimits of 1-categories that generate non-trivial higher-dimensional structure when the colimit as formed in the category of $(\infty, 1)$-categories.

For example, consider the span of posets:

$$\begin{tikzcd}
& & \bullet \arrow[dl] \arrow[dd] \arrow[dr] & & & & & & \bullet \arrow[dl] \arrow[dr] & & & & \\
& \bullet \arrow[dr] & & \bullet \arrow[dl] & & \hookleftarrow & & \bullet & & \bullet & & \longrightarrow & \bullet \\
& & \bullet & & & & & & & & & & \\
\bullet \arrow[uur] \arrow[urr] \arrow[rr] & & \bullet \arrow[u] & & \bullet \arrow[uul] \arrow[ull] \arrow[ll] & & \bullet \arrow[uur] \arrow[rr] & & \bullet & & \bullet \arrow[uul] \arrow[ll] & &
\end{tikzcd}$$

The pushout in 1-categories is the arrow category $\bullet \to \bullet$, while the pushout in $(\infty, 1)$-categories defines an $(\infty, 1)$-category with two objects in which the non-trivial hom-space has the homotopy type of the 2-sphere.

As a second example, let M be the monoid with five elements $e, x_{11}, x_{12}, x_{21}, x_{22}$ and multiplication rule given by $x_{ij}x_{k\ell} = x_{i\ell}$. Inverting all elements of M yields the trivial group. That is, if one considers M as a 1-category with a single object, then the pushout of the span $M \leftarrow \amalg_M \mathbb{2} \to \amalg_M \mathbb{I}$ (where $\mathbb{2}$ is the free-living arrow and $\mathbb{I}$ is the free-living isomorphism) in categories is the terminal category $\mathbb{1}$. On the other hand, the pushout of this span in $(\infty, 1)$-categories is the ∞-groupoid S^2 as follows from [Fie02, Lemma]. The results of [McD79] imply that this example is generalizable to a vast class of monoids.

More generally, the Gabriel–Zisman category of fractions $\mathcal{C}[\mathcal{W}^{-1}]$ is formed by freely inverting the morphisms in a class of arrows $\mathcal{W}$ in a 1-category $\mathcal{C}$. This can also be constructed as a pushout of 1-categories of the span

$$\mathcal{C} \longleftarrow \coprod_{w \in \mathcal{W}} \mathbb{2} \hookrightarrow \coprod_{w \in \mathcal{W}} \mathbb{I}$$

where each arrow in $\mathcal{W}$ is replaced by a free-living isomorphism. By contrast, the $(\infty, 1)$-category defined by this pushout is modelled by the Dwyer–Kan simplicial localization, which has non-trivial higher dimensional structure in many instances [DK80], [Ste17, Lemma 18], [Joy08, p. 168]. Indeed, all $(\infty, 1)$-categories arise in this way [BK12].

As the examples above show, pushouts of 1-categories in particular are problematic. Our aim in this paper is to prove that a certain class of pushout diagrams of 1-categories are guaranteed to be $(\infty, 1)$-categorical. The requirement is that one of the two maps in the span that generates the pushout belong to a class of functors between 1-categories first considered by Thomason under the name "Dwyer maps" [Tho80, Definition 4.1] that feature in a central way in the construction of the Thomason model structure on categories.

Definition 1.1 (Thomason)**.** A full sub-1-category inclusion $I\colon \mathcal{A} \hookrightarrow \mathcal{B}$ is **Dwyer map** if the following conditions hold.

(i) The category $\mathcal{A}$ is a *sieve* in $\mathcal{B}$, meaning there is a necessarily unique functor $\chi\colon \mathcal{B} \to \mathbb{2}$ with $\chi^{-1}(0) = \mathcal{A}$. We write $\mathcal{V} := \chi^{-1}(1)$ for the complementary *cosieve* of $\mathcal{A}$ in $\mathcal{B}$.

(ii) The inclusion $I\colon \mathcal{A} \hookrightarrow \mathcal{W}$ into the *minimal cosieve*[1] $\mathcal{W} \subset \mathcal{B}$ containing $\mathcal{A}$ admits a right adjoint left inverse $R\colon \mathcal{W} \to \mathcal{A}$, a right adjoint for which the unit is an identity.

Schwede describes Dwyer maps as "categorical analogs of the inclusion of a neighborhood deformation retract" [Sch19]. In fact, many examples of Dwyer maps are more like deformation retracts, in that the cosieve $\mathcal{W}$ generated by $\mathcal{A}$ is the full codomain category $\mathcal{B}$.

Example 1.2. The vertex inclusion $0\colon \mathbb{1} \to \mathbb{2}$ is a Dwyer map, with $!\colon \mathbb{2} \to \mathbb{1}$ the right adjoint left inverse. The other vertex inclusion $1\colon \mathbb{1} \to \mathbb{2}$ is not a Dwyer map.

Generalizing the previous example:

Example 1.3. If $\mathcal{A}$ is a category with a terminal object and $\mathcal{A}^{\triangleright}$ is the category which formally adds a new terminal object, then the inclusion $\mathcal{A} \hookrightarrow \mathcal{A}^{\triangleright}$ is a Dwyer map.[2]

We warn the reader that we are using the original notion of Dwyer map, not the pseudo-Dwyer maps introduced by Cisinski [Cis99], which are retracts of Dwyer maps. In particular, our Dwyer maps are not closed under retracts. Thomason observed, however, that they are stable under pushouts, as we now recall:

Lemma 1.4 ([Tho80, Proposition 4.3])**.** *Any pushout of a Dwyer map I defines a Dwyer map J:*

$$\begin{tikzcd} \mathcal{A} \arrow[r, "F"] \arrow[d, hook, "I"'] & \mathcal{C} \arrow[d, hook, "J"] \\ \mathcal{B} \arrow[r, "G"'] & \mathcal{D}. \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd}$$

Note for example, that Lemma 1.4 explains the Dwyer map of Example 1.3: if $\mathcal{A}$ has a terminal object t, then the pushout

$$\begin{tikzcd} \mathbb{1} \arrow[r, "t"] \arrow[d, hook, "0"'] & \mathcal{A} \arrow[d, hook] \\ \mathbb{2} \arrow[r] & \mathcal{A}^{\triangleright} \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd}$$

defines the category $\mathcal{A}^{\triangleright}$.

Our aim is to show that pushouts of categories involving at least one Dwyer map can also be regarded as pushouts of $(\infty,1)$-categories in the sense made precise by considering the nerve embedding from categories into quasi-categories:

Theorem 1.5. *Let*

$$\begin{tikzcd} \mathcal{A} \arrow[r, "F"] \arrow[d, hook, "I"'] & \mathcal{C} \arrow[d, hook, "J"] \\ \mathcal{B} \arrow[r, "G"'] & \mathcal{D} \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd}$$

[1]Explicitly $\mathcal{W}$ is the full subcategory of $\mathcal{B}$ containing every object that arises as the codomain of an arrow with domain in $\mathcal{A}$.

[2]If $\mathcal{A}$ does not have a terminal object, then $\mathcal{A} \to \mathcal{A}^{\triangleright}$ need not be a Dwyer map. Indeed, if $\mathcal{A} = \mathbb{1} \amalg \mathbb{1}$, the only cosieve containing $\mathcal{A}$ is $\mathcal{A}^{\triangleright}$ itself, and there cannot be a right adjoint $\mathcal{A}^{\triangleright} \to \mathcal{A}$ as $\mathcal{A}$ does not have a terminal object.

be a pushout of categories, and assume I to be a Dwyer map. Then the induced map of simplicial sets

$$N\mathcal{C} \coprod_{N\mathcal{A}} N\mathcal{B} \to N\mathcal{D}$$

is a weak categorical equivalence.

By a weak categorical equivalence, we mean a weak equivalence in Joyal's model structure for quasi-categories [JT07, §1]. Theorem 1.5 is a refinement of a similar result of Thomason [Tho80, Proposition 4.3] which proves that the same map is a weak homotopy equivalence.

In a companion paper, we give an application of Theorem 1.5 to the theory of $(\infty, 2)$-categories. There we prove:

Theorem 1.6 ([HOR$^+$21, 4.4.2]). *The space of composites of any pasting diagram in any $(\infty, 2)$-category is contractible.*

To prove this, we make use of Lurie's model structure of $(\infty, 2)$-categories as categories enriched over quasi-categories [Lur09]. In this model, a *pasting diagram* is a simplicially enriched functor out of the free simplicially enriched category defined by gluing together the objects, atomic 1-cells, and atomic 2-cells of a pasting scheme, while the composites of these cells belong to the homotopy coherent diagram indexed by the nerve of the free 2-category generated by the pasting scheme.

This pair of $(\infty, 2)$-categories has a common set of objects so the difference lies in their hom-spaces. The essential difference between the procedure of attaching an atomic 2-cell along the bottom of a pasting diagram or along the bottom of the free 2-category it generates is the difference between forming a pushout of hom-categories in the category of $(\infty, 1)$-categories or in the category of 1-categories. Since one of the functors in the span that defines the pushout under consideration is a Dwyer map, Theorem 1.5 proves that the resulting $(\infty, 2)$-categories are equivalent.

In §2, we explain how Theorem 1.5 reduces to two special cases in which we are free to make additional assumptions on the non-Dwyer functor F. In §3, we analyze 1-categorical pushouts of Dwyer maps, establish some common notation to be used in the remaining sections, and develop some terminology for the simplices in their nerves that require the greatest attention. In §4, we state and prove our first main result. Theorem 4.1 observes that the canonical comparison between the pushout of nerves of categories and the nerve of the pushout is inner anodyne, provided that one of the functors in the span is a Dwyer map and the other is an injective-on-objects faithful functor. The remaining special case, where F is instead bijective-on-objects and full, is proven in §5.

2. A reduction of the problem

We prove Theorem 1.5 in stages in which we are allowed to assume some special properties of the functor $F\colon \mathcal{A} \to \mathcal{C}$, obtained by factoring this functor using the following (weak) factorization systems on Cat.

Recall 2.1. A functor $F\colon \mathcal{A} \to \mathcal{C}$ of 1-categories may be factored as $\mathcal{A} \to \mathcal{E} \to \mathcal{C}$ by

(i) factoring the object map trivially as an identity followed by an arbitrary map:

$$\begin{tikzcd} \mathrm{ob}\mathcal{A} \arrow[rr, "F"] \arrow[dr, dashed, equal] & & \mathrm{ob}\mathcal{C} \\ & \mathrm{ob}\mathcal{E} \arrow[ur, dashed, "F"'] & \end{tikzcd}$$

(ii) then factoring each hom-set function as an epimorphism followed by a monomorphism:

$$\begin{tikzcd} \mathcal{A}(a,a') \arrow[rr, dashed, "F_{a,a'}"] \arrow[dr, two heads] & & \mathcal{C}(Fa,Fa') \\ & \mathcal{E}(a,a') \arrow[ur, dashed, tail] & \end{tikzcd}$$

This has the effect of factoring F as a bijective-on-objects full functor followed by a faithful functor.

Recall 2.2. A functor $F\colon \mathcal{A} \to \mathcal{C}$ of 1-categories may be factored as $\mathcal{A} \to \mathcal{E} \to \mathcal{C}$ by

(i) factoring the object map trivially as an identity followed by an arbitrary map:

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(ii) then factoring each hom-set function trivially as an arbitrary map followed by an identity:

$$\begin{tikzcd} \mathcal{A}(a,a') \arrow[rr, "F_{a,a'}"] \arrow[dr, dashed, "F_{a,a'}"'] & & \mathcal{C}(Fa,Fa') \\ & \mathcal{E}(a,a') \arrow[ur, dashed, equal] & \end{tikzcd}$$

This has the effect of factoring F as a bijective-on-objects functor followed by a fully-faithful functor.

Recall 2.3. A functor $F\colon \mathcal{A} \to \mathcal{C}$ of 1-categories may be factored by

(i) forming the cograph factorization:

$$\begin{tikzcd} \mathcal{A} \arrow[rr, "F"] \arrow[dr, dashed, "i_1"'] & & \mathcal{C} \\ & \mathcal{A} \amalg \mathcal{C} \arrow[ur, dashed, "\langle F,\mathrm{id}\rangle"'] & \end{tikzcd}$$

(ii) then taking the bijective-on-objects, fully-faithful factorization of the right factor defined by Recall 2.2.

This has the effect of factoring F as an injective-on-objects functor followed by a surjective-on-objects equivalence.

By combining these factorizations, we may first factor a functor $F\colon \mathcal{A} \to \mathcal{C}$ as a bijective-on-objects full functor $\mathcal{A} \to \mathcal{C}_0$ followed by a faithful functor $\mathcal{C}_0 \to \mathcal{C} = \mathcal{C}_2$. Then factor the faithful functor as an injective-on-objects functor $\mathcal{C}_0 \to \mathcal{C}_1$ followed by a surjective-on-objects equivalence $\mathcal{C}_1 \to \mathcal{C}_2$. Notice that $\mathcal{C}_0 \rightarrowtail \mathcal{C}_1$ is now both injective-on-objects and faithful.

A Dwyer map $I\colon \mathcal{A} \hookrightarrow \mathcal{B}$ is in particular injective-on-objects and faithful. Form the following pushouts in Cat:

$$\begin{tikzcd}
\mathcal{A} \arrow[r, "\text{full}", "\text{bij ob}"'] \arrow[d, hook] & \mathcal{C}_0 \arrow[r, tail, "\text{inj ob}", "\text{faith}"'] \arrow[d, hook] & \mathcal{C}_1 \arrow[r, two heads, "\sim"] \arrow[d, hook] & \mathcal{C}_2 \arrow[d, hook] \\
\mathcal{B} \arrow[r] & \mathcal{D}_0 \arrow[ul, phantom, "\ulcorner", very near start] \arrow[r, tail] & \mathcal{D}_1 \arrow[ul, phantom, "\ulcorner", very near start] \arrow[r, "\sim"] & \mathcal{D}_2 \arrow[ul, phantom, "\ulcorner", very near start]
\end{tikzcd}$$

The canonical model structure on Cat, whose cofibrations are injective-on-objects functors, whose fibrations are isofibrations, and whose weak equivalences are equivalences of categories, is left proper. Thus, $\mathcal{D}_1 \to \mathcal{D}_2$ is an equivalence since Cat is left proper.

Applying nerves and then take iterated pushouts gives rise to the following diagram of simplicial sets:

$$\begin{tikzcd}
N\mathcal{A} \arrow[r] \arrow[d, tail] & N\mathcal{C}_0 \arrow[r, tail] \arrow[d, tail] & N\mathcal{C}_1 \arrow[r, two heads, "\sim"] \arrow[d, tail] & N\mathcal{C}_2 \arrow[d, tail] \\
N\mathcal{B} \arrow[r] & P_0 \arrow[ul, phantom, "\ulcorner", very near start] \arrow[r, tail] \arrow[d] & P_1 \arrow[ul, phantom, "\ulcorner", very near start] \arrow[r, "\sim"] \arrow[d] & P_2 \arrow[ul, phantom, "\ulcorner", very near start] \arrow[d] \\
& N\mathcal{D}_0 \arrow[r, tail] & Q_1 \arrow[ul, phantom, "\ulcorner", very near start] \arrow[r] \arrow[d] & Q_2 \arrow[ul, phantom, "\ulcorner", very near start] \arrow[d] \\
& & N\mathcal{D}_1 \arrow[r] \arrow[dr, bend right, "\sim"'] & R_2 \arrow[ul, phantom, "\ulcorner", very near start] \arrow[d] \\
& & & N\mathcal{D}_2
\end{tikzcd}$$

The nerve functor carries injective-on-objects faithful functors to monomorphisms of simplicial sets, which are the cofibrations in the Joyal model structure. Thus, the maps decorated with a tail are cofibrations. The nerve functor also carries equivalences of categories to weak equivalences in the Joyal model structure. Since this model structure is left proper, the maps decorated with a tilde are weak equivalences.

Using this diagram we may reduce Theorem 1.5 to two special cases:

Lemma 2.4. *Consider a pushout diagram of categories in which I is a Dwyer map:*

$$\begin{tikzcd}
\mathcal{A} \arrow[r, "F"] \arrow[d, hook, "I"'] & \mathcal{C} \arrow[d, hook, "J"] \\
\mathcal{B} \arrow[r, "G"'] & \mathcal{D} \arrow[ul, phantom, "\ulcorner", very near start]
\end{tikzcd}$$

If the induced map of simplicial sets $N\mathcal{C} \amalg_{N\mathcal{A}} N\mathcal{B} \to N\mathcal{D}$ is a weak categorical equivalence whenever

(i) the functor F is injective-on-objects and faithful, or
(ii) the functor F is bijective-on-objects and full,

then this map is a weak categorical equivalence for any functor F.

Proof. The second of these stated assumptions tells us that the map $P_0 \to N\mathcal{D}_0$ is a weak equivalence while the first assumption tells us that the map $Q_1 \to N\mathcal{D}_1$ is a weak equivalence, as indicated in the following diagram

$$\begin{tikzcd}
N\mathcal{A} \arrow[r] \arrow[d, tail] & N\mathcal{C}_0 \arrow[r, tail] \arrow[d, tail] & N\mathcal{C}_1 \arrow[r, two heads, "\sim"] \arrow[d, tail] & N\mathcal{C}_2 \arrow[d, tail] \\
N\mathcal{B} \arrow[r] \arrow[ur, phantom, "\ulcorner", very near end] & P_0 \arrow[r, tail] \arrow[d, "\wr"] \arrow[ur, phantom, "\ulcorner", very near end] & P_1 \arrow[r, "\sim"] \arrow[d, "\wr"] \arrow[ur, phantom, "\ulcorner", very near end] & P_2 \arrow[d] \arrow[ddd, bend left, "\wr"] \\
 & N\mathcal{D}_0 \arrow[r, tail] \arrow[ur, phantom, "\ulcorner", very near end] & Q_1 \arrow[r] \arrow[d, "\wr"] \arrow[ur, phantom, "\ulcorner", very near end] & Q_2 \arrow[d] \\
 & & N\mathcal{D}_1 \arrow[r] \arrow[dr, bend right, "\sim"'] \arrow[ur, phantom, "\ulcorner", very near end] & R_2 \arrow[d] \\
 & & & N\mathcal{D}_2
\end{tikzcd}$$

Since the Joyal model structure is left proper, the map $P_1 \to Q_1$ is a weak equivalence. Hence, the map $P_2 \to N\mathcal{D}_2$ is a weak equivalence by two-of-three, which is the general case of the Dwyer map theorem. $\square$

Thus, to prove Theorem 1.5 it suffices to prove the two special cases (i) and (ii), which appear as Theorem 4.1 and Theorem 5.1 below.

3. Dwyer pushouts and their nerves

We now establish some notation that we will freely reference in the remainder of this paper. By Definition 1.1, a Dwyer map $I\colon \mathcal{A} \hookrightarrow \mathcal{B}$ uniquely determines a functor $\chi\colon \mathcal{B} \to \mathbb{2}$ that classifies the sieve $\mathcal{A} := \chi^{-1}(0)$ and its complementary cosieve $\mathcal{V} := \chi^{-1}(1)$

$$\begin{tikzcd}
\mathcal{V} \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{B} \arrow[d, "\chi"] & \mathcal{A} \arrow[l, hook'] \arrow[d] \arrow[dl, phantom, "\llcorner", very near start] \\
\mathbb{1} \arrow[r, hook, "1"'] & \mathbb{2} & \mathbb{1} \arrow[l, hook', "0"]
\end{tikzcd} \tag{3.1}$$

as well as a right adjoint left inverse adjunction $(I \dashv R, \varepsilon\colon IR \Rightarrow \mathrm{id}_{\mathcal{W}})$ associated to the inclusion of $\mathcal{A}$ into the minimal cosieve $\mathcal{A} \subset \mathcal{W} \subset \mathcal{B}$. This data may be summarized by the diagram

$$\begin{tikzcd}
 & & \varnothing \arrow[dl] \arrow[dr] \arrow[d, phantom, "\vee"] & \\
 & \mathcal{U} \arrow[dl, hook'] \arrow[dr] \arrow[d, phantom, "\vee"] & {} & \mathcal{A} \arrow[dl, hook', "\perp"] \arrow[d] \arrow[dl, phantom, "\llcorner", very near start] \\
\mathcal{V} \arrow[d] \arrow[dr, hook] \arrow[dr, phantom, "\lrcorner", very near start] & {} & \mathcal{W} \arrow[dl, hook'] \arrow[ur, dashed, bend left, "R"] & \mathbb{1} \arrow[ddll, hook', "0"] \\
\mathbb{1} \arrow[dr, hook, "1"'] & \mathcal{B} \arrow[d] & & \\
 & \mathbb{2} & &
\end{tikzcd} \tag{3.2}$$

in which $\mathcal{U} := \mathcal{W} \cap \mathcal{V} \cong \mathcal{W} \backslash \mathcal{A}$. By inspection, the codomain category of a Dwyer map admits the following description:

Lemma 3.3. *For any Dwyer map $I\colon \mathcal{A} \hookrightarrow \mathcal{B}$ as in (3.2), there is a partitioning*

$$\mathrm{ob}\mathcal{A} \amalg \mathrm{ob}\mathcal{V} \xrightarrow{\cong} \mathrm{ob}\mathcal{B}$$

of the objects of the codomain category while the hom-sets are given by

$$\mathcal{A}(a,a') \cong \mathcal{B}(a,a') \qquad \mathcal{V}(v,v') \cong \mathcal{B}(v,v') \qquad \mathcal{A}(a,Ru) \cong \mathcal{B}(a,u)$$

for all $a,a' \in \mathcal{A}$, $v,v' \in \mathcal{V}$, and $u \in \mathcal{U}$, and are empty otherwise. For objects $a,a' \in \mathcal{A}$ and $u,u' \in \mathcal{U}$, the composition map

$$\begin{tikzcd}
\mathcal{B}(u,u') \times \mathcal{B}(a,u) \times \mathcal{B}(a',a) \arrow[rr, "\circ"] & & \mathcal{B}(a',u') \\
\mathcal{B}(u,u') \times \mathcal{A}(a,Ru) \times \mathcal{A}(a',a) \arrow[u, "\cong"'] \arrow[d, "R\times\mathrm{id}"'] & & \\
\mathcal{A}(Ru,Ru') \times \mathcal{A}(a,Ru) \times \mathcal{A}(a',a) \arrow[rr, "\circ"] & & \mathcal{A}(a',Ru') \arrow[uu, "\cong"']
\end{tikzcd}$$

is the unique map making the diagram commute. □

In particular, a sequence of composable morphisms in $\mathcal{B}$ may start with morphisms in the full subcategory $\mathcal{A}$ and may then leave this subcategory by means of a **bridging morphism** $f\colon a \to u$ from $a \in \mathcal{A}$ to $u \in \mathcal{U}$ but if it does so it will necessarily continue in, and never leave, the full subcategory $\mathcal{U}$. This gives us a natural partitioning of the simplices in the nerve of $\mathcal{B}$.

Definition 3.4. Each n-simplex σ of $N\mathcal{B}$ defines a composite map

$$\Delta^n \xrightarrow{\sigma} N\mathcal{B} \xrightarrow{\chi} \Delta^1$$

the data of which is determined by the **bridge index** $r \in \{+, n, \ldots, 1, -\}$ of the n-simplex σ. When the composite map $\chi\sigma\colon \Delta^n \to \Delta^1$ is surjective, the bridge index is an integer that corresponds to the minimal vertex of Δ^n that maps to the vertex 1. When $\chi\sigma\colon \Delta^n \to \Delta^1$ is constant at 0 or 1, the bridge index is $+$ or $-$, respectively.

The simplices with bridge index $+$ or $-$ are exactly those that lie in the images of the full inclusions $N\mathcal{A} \hookrightarrow N\mathcal{B}$ and $N\mathcal{V} \hookrightarrow N\mathcal{B}$ induced by (3.1), respectively.

Definition 3.5. By Lemma 3.3, an n-simplex with integer bridge index r has the form

$$a_0 \xrightarrow{f_1} \cdots \xrightarrow{f_{r-1}} a_{r-1} \xrightarrow{\hat{f}} u_r \xrightarrow{h_{r+1}} \cdots \xrightarrow{h_n} u_n \tag{3.6}$$

where $f_i\colon a_{i-1} \to a_i \in \mathcal{A}$ for $1 \leq i \leq r-1$, $h_j\colon u_{j-1} \to u_j \in \mathcal{U}$ for $r < j \leq n$, and $f\colon a_{r-1} \to Ru_r \in \mathcal{C}$ with adjoint transpose $\hat{f}\colon a_{r-1} \to u_r$ is the unique bridging morphism in the sequence. We refer to the simplices with integer bridge index as **bridging simplices**.

Definition 3.7. A bridging simplex is **bascule**[3] if its bridging morphism $\hat{f}\colon a \to u$ is a component of the counit of the adjunction

$$\begin{tikzcd} \mathcal{A} \arrow[r, hook, bend left, "I"] \arrow[r, phantom, "\perp"] & \mathcal{W} \subset \mathcal{B} \arrow[l, bend left, "R"] \end{tikzcd}$$

i.e., if $\hat{f} = \varepsilon_u\colon Ru \to u \in \mathcal{B}$ is the transpose of $\mathrm{id}_{Ru}\colon Ru \to Ru \in \mathcal{A}$, in which case $a = Ru$.

Lemma 3.8. *Let σ be a bascule n-simplex in $N\mathcal{B}$ of bridge index r. Then the kth face $\sigma \cdot \delta^k$ is*

- *a non-bridging simplex if $k = r - 1 = 0$ or if $k = r = n$,*
- *a non-bascule simplex of bridge index $r - 1$ if $k = r - 1 > 0$ and the edge from the $(r-2)$nd vertex to the $(r-1)$st vertex is not an identity,*
- *a simplex of bridge index r if $k = r < n$, or*
- *a bascule simplex otherwise.*

Proof. We prove the case $k = r - 1 > 0$, which is the most subtle and most important, and leave the others to the reader.

The face $\sigma \cdot \delta^{r-1}$ of a bascule simplex is formed by composing the indicated morphisms

$$\begin{tikzcd} & & & Ru_r \arrow[dr, "\varepsilon_{u_r}"] & & & \\ a_0 \arrow[r] & \cdots \arrow[r] & a_{r-2} \arrow[ur, "f_{r-1}"] \arrow[rr, "\widehat{f_{r-1}}"'] & & u_r \arrow[r] & \cdots \arrow[r] & u_n \end{tikzcd}$$

The bridging morphism $\widehat{f_{r-1}}\colon a_{r-2} \to u_r$ in $\sigma \cdot \delta^{r-1}$ is a counit component if and only if its transpose f_{r-1} is the identity. Thus $\sigma \cdot \delta^{r-1}$ is a non-bascule simplex except in this case. □

Reversing this process, we see that any bridging n-simplex of bridge index r arises as the rth face of a unique bascule $(n+1)$-simplex of bridge index $r + 1$—its **bascule lift**—obtained by factoring the bridging morphism $\hat{f}\colon a \to u \in \mathcal{W} \subset \mathcal{B}$ through its adjoint transpose $f\colon a \to Ru \in \mathcal{A}$:

$$\begin{tikzcd} & Ru \arrow[dr, "\varepsilon_u"] & \\ a \arrow[ur, "f"] \arrow[rr, "\hat{f}"'] & & u. \end{tikzcd}$$

Consequently:

Corollary 3.9. *Fix a positive integer n. For any $1 \le r \le n$, the rth face map defines a bijection between*

- *the bascule $(n+1)$-simplices of bridge index $r + 1$ and*
- *the bridging n-simplices of bridge index r*

that restricts to a bijection between

- *the non-degenerate bascule $(n+1)$-simplices of bridge index $r + 1$ and*
- *the non-degenerate non-bascule bridging n-simplices of bridge index r.* □

[3]A bascule bridge is also known as a drawbridge. The idea, as explained by Corollary 3.9 is that the relation between a bascule simplex of bridge index r and its $(r-1)$st face is analogous to that between a bascule bridge that is raised to allow boat traffic and a bascule bridge that is flat for car traffic.

Now consider the pushout of a Dwyer map along an arbitrary functor $F\colon \mathcal{A} \to \mathcal{C}$.

$$\begin{tikzcd} & & \mathcal{A} \arrow[rr, "F"] \arrow[dl, "I"'] \arrow[dd] \arrow[dr, phantom, "\ulcorner"] & & \mathcal{C} \arrow[dl, "J"] \arrow[dd] \\ \mathcal{V} \arrow[r, hook] \arrow[dd] & \mathcal{B} \arrow[rr, "G" near start] \arrow[dd, "\chi"'] & & \mathcal{D} \arrow[dd, dashed, "\pi" near start] & \\ & & \mathbb{1} \arrow[rr, equal] \arrow[dl, "0"'] & & \mathbb{1} \arrow[dl, "0"] \\ \mathbb{1} \arrow[r, hook, "1"] & \mathbb{2} \arrow[rr, equal] & & \mathbb{2} & \end{tikzcd} \tag{3.10}$$

The induced functor $\pi\colon \mathcal{D} \to \mathbb{2}$ partitions the objects of $\mathcal{D}$ into the two fibers $\mathrm{ob}(\pi^{-1}(0)) \cong \mathrm{ob}\mathcal{C}$ and $\mathrm{ob}(\pi^{-1}(1)) \cong \mathrm{ob}\mathcal{V}$ and prohibits any morphisms from the latter to the former.

The right adjoint left inverse adjunction $(I \dashv R, \varepsilon\colon IR \Rightarrow \mathrm{id}_{\mathcal{W}})$ associated to the inclusion of $\mathcal{A}$ into the minimal cosieve $\mathcal{A} \subset \mathcal{W} \subset \mathcal{B}$ pushes out to define a right adjoint left inverse $(J \dashv S, \nu\colon JS \Rightarrow \mathrm{id}_{\mathcal{Y}})$ to the inclusion of $\mathcal{C}$ into the minimal cosieve $\mathcal{C} \subset \mathcal{Y} \subset \mathcal{D}$.

$$\begin{tikzcd} \mathcal{A} \arrow[r, "F"] \arrow[d, hook, bend right, "I"'] \arrow[d, phantom, "\dashv"] & \mathcal{C} \arrow[d, hook, bend right, "J"'] \arrow[d, phantom, "\dashv"] \\ \mathcal{W} \arrow[u, bend right, "R"'] \arrow[r, "G"'] \arrow[d, hook] \arrow[ur, phantom, "\ulcorner" near end] & \mathcal{Y} \arrow[u, bend right, "S"'] \arrow[d, hook] \\ \mathcal{B} \arrow[r, "G"'] \arrow[ur, phantom, "\ulcorner" near end] & \mathcal{D} \end{tikzcd} \qquad \begin{tikzcd} \mathcal{A} \arrow[r, "F"] \arrow[d, hook, "I"'] & \mathcal{C} \arrow[d, hook, "J"] \arrow[ddr, equal, bend left] & \\ \mathcal{W} \arrow[r, "G"'] \arrow[dr, "R"'] \arrow[ur, phantom, "\ulcorner" near end] & \mathcal{Y} \arrow[dr, dashed, "S"] & \\ & \mathcal{A} \arrow[r, "F"'] & \mathcal{C} \end{tikzcd} \qquad \begin{tikzcd} \mathcal{A} \arrow[r, "F"] \arrow[d, hook, "I"'] & \mathcal{C} \arrow[d, hook, "J"] \arrow[dr, "\Delta"] & \\ \mathcal{W} \arrow[r, "G"'] \arrow[dr, "\varepsilon"'] \arrow[ur, phantom, "\ulcorner" near end] & \mathcal{Y} \arrow[dr, dashed, "\nu"] & \mathcal{C}^{\mathbb{2}} \arrow[d, "J^{\mathbb{2}}"] \\ & \mathcal{W}^{\mathbb{2}} \arrow[r, "G^{\mathbb{2}}"'] & \mathcal{Y}^{\mathbb{2}} \end{tikzcd}$$

These observations explain the closure of Dwyer maps under pushout and furthermore can be used to explicitly describe the structure of the category $\mathcal{D}$ defined by the pushout of a Dwyer map, as proven in [BMO+15, Proof of Lemma 2.5]; cf. also [Sch19, Construction 1.2] and [AM14, §7.1].

Proposition 3.11. *The objects in the pushout category $\mathcal{D}$ are given by*

$$\mathrm{ob}\mathcal{C} \amalg \mathrm{ob}\mathcal{V} \xrightarrow{\cong} \mathrm{ob}\mathcal{D}$$

while the hom-sets are given by

$$\mathcal{C}(c,c') \cong \mathcal{D}(c,c') \qquad \mathcal{V}(v,v') \cong \mathcal{B}(v,v') \cong \mathcal{D}(v,v') \qquad \begin{array}{rcl} \mathcal{C}(c,Su) & \xrightarrow[\cong]{\nu_u \circ (-)} & \mathcal{D}(c,u) \\ f & \longmapsto & \hat{f} \end{array}$$

for all $c, c' \in \mathcal{C}$, $v, v' \in \mathcal{V}$, and $u \in \mathcal{U}$, and are empty otherwise. Functoriality of the inclusions J and G defines the composition on the image of $\mathcal{C}$ and $\mathcal{V}$. For objects $c, c' \in \mathcal{C}$ and $u, u' \in \mathcal{U}$, the composition map

$$\begin{tikzcd} \mathcal{D}(u,u') \times \mathcal{D}(c,u) \times \mathcal{D}(c',c) \arrow[r, "\circ"] & \mathcal{D}(c',u') \\ \mathcal{D}(u,u') \times \mathcal{C}(c,Su) \times \mathcal{C}(c',c) \arrow[u, "\wr\Vert"] \arrow[d, "S\times\mathrm{id}"'] & \\ \mathcal{C}(Su,Su') \times \mathcal{C}(c,Su) \times \mathcal{C}(c',c) \arrow[r, "\circ"] & \mathcal{C}(c',Su') \arrow[uu, "\Vert\wr"'] \end{tikzcd}$$

is the unique map making the diagram commute.[4]

To summarize, J and G define fully-faithful inclusions

$$\begin{tikzcd} \mathcal{V} \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{D} \arrow[d, "\pi"] & \mathcal{C} \arrow[l, hook'] \arrow[d] \arrow[dl, phantom, "\llcorner", very near start] \\ \mathbb{1} \arrow[r, hook, "1"] & \mathbb{2} & \mathbb{1} \arrow[l, hook', "0"'] \end{tikzcd} \tag{3.12}$$

that are jointly surjective on objects. In particular, we may identify $\mathcal{V}$ with the complementary cosieve of $\mathcal{C}$ in $\mathcal{D}$. Adopting the terminology introduced above, the category $\mathcal{D}$ contains additional **bridging morphisms** $\hat{f}\colon c \to u$ from the fiber over 0 to the fiber over 1 but only with codomain in $\mathcal{U} = \mathcal{V} \cap \mathcal{W} = \mathcal{V} \cap \mathcal{Y}$ and these all arise as adjoint transposes under $J \dashv S$ of some morphism $f\colon c \to Su$ in $\mathcal{C}$.

Remark 3.13. Under the identifications of Proposition 3.11, the action of the functor $G\colon \mathcal{B} \to \mathcal{D}$ on hom-sets is described by

$$\begin{tikzcd} \mathcal{B}(a,a') \arrow[rr, "G"] \arrow[d, phantom, "\wr\|"] & & \mathcal{D}(Ga,Ga') \arrow[d, phantom, "\|"] \\ \mathcal{A}(a,a') \arrow[r, "F"] & \mathcal{C}(Fa,Fa') \arrow[r, "J", "\cong"'] & \mathcal{D}(Fa,Fa') \end{tikzcd}$$

$$\begin{tikzcd} \mathcal{B}(v,v') \arrow[r, "G", "\cong"'] & \mathcal{D}(v,v') \end{tikzcd}$$

$$\begin{tikzcd} \mathcal{A}(a,Ru) \arrow[r, "F"] \arrow[d, "\varepsilon_u\circ(-)"', "\wr"] & \mathcal{C}(Fa,FRu) \arrow[r, equal] & \mathcal{C}(Fa,Su) \arrow[d, "\nu_u\circ(-)", "\wr"'] \\ \mathcal{B}(a,u) \arrow[r, "G"] & \mathcal{D}(Ga,u) \arrow[r, equal] & \mathcal{D}(Fa,u) \end{tikzcd} \tag{3.14}$$

for all $a, a' \in \mathcal{A}$, $v, v' \in \mathcal{V}$, and $u \in \mathcal{U}$. Note these three cases are disjoint and cover all the non-empty hom-sets in $\mathcal{B}$ and $\mathcal{D}$.

In particular, a bridging morphism $\hat{f}\colon c \to u$ in $\mathcal{D}$ lies in the image of G if and only if the transpose $f\colon c \to Su = FRu$ in $\mathcal{C}$ equals Fg for some $g\colon a \to Ru$ in $\mathcal{A}$. In this case, G carries the transpose $\hat{g}\colon a \to u$ in $\mathcal{B}$ of g to $\hat{f}\colon Ga = Fc \to u$ in $\mathcal{D}$.

Since the pushout $J\colon \mathcal{C} \hookrightarrow \mathcal{D}$ is a Dwyer map classified by $\pi\colon \mathcal{D} \to \mathbb{2}$, the terminology introduced in Definitions 3.4, 3.5 and 3.7 also apply to simplices in $N\mathcal{D}$, as do the results proven in Lemma 3.8 and Corollary 3.9. Moreover, the pair of functors F and G define a strict adjunction morphism from the adjunction $I \dashv R$ to the adjunction $J \dashv S$, meaning that the action on homs commutes with adjoint transposition in the sense expressed by the commutative diagram (3.14). Consequently:

Lemma 3.15. *The functor $NG\colon N\mathcal{B} \to N\mathcal{D}$ preserves bascule simplices and bascule lifts.*

Proof. A bridging simplex in $N\mathcal{B}$ is bascule if its bridging morphism is a component ε_u of the counit ε. The image of this simplex in $N\mathcal{D}$ then has $G\varepsilon_u = \nu_{Gu}$ as the bridging morphism, which is a component of the counit ν. Thus, NG preserves bascule simplices. As a map of simplicial sets, NG commutes with taking the rth face map of bascule simplices of bridge index $r+1$. Thus, it commutes with bascule lifts, which can be understood by Corollary 3.9 as the inverse of this operation. □

[4] Note if $u \in \mathcal{U}$ and $v \in \mathcal{V}\backslash\mathcal{U}$, then $\mathcal{B}(u,v) = \varnothing$.

Remark 3.16. While any simplex in $N\mathcal{B}$ that maps to a bridging simplex in $N\mathcal{D}$ is bridging, when $G\colon \mathcal{B} \to \mathcal{D}$ is not injective, basculeness is not necessarily reflected. For instance, if there are any objects $u \in \mathcal{U}$ so that $\mathrm{id}_{Su} \in \mathcal{C}$ has a non-identity preimage in $\mathcal{A}$, then the transpose of this map defines a bridging non-bascule 1-simplex in $\mathcal{B}$ that maps to a bascule 1-simplex in $\mathcal{D}$.

Now let $P = N\mathcal{C} \amalg_{N\mathcal{A}} N\mathcal{B}$ be the simplicial set defined as the pushout of the nerves. Via the composite map $P \to N\mathcal{D} \xrightarrow{\pi} \Delta^1$, its simplices each are assigned a bridge index, as in Definition 3.4. The set of n-simplices may be partitioned as follows

$$P_n \cong N\mathcal{C}_n \amalg (N\mathcal{A} \overset{n}{|} N\mathcal{U})_n \amalg \cdots \amalg (N\mathcal{A} \overset{1}{|} N\mathcal{U})_n \amalg N\mathcal{V}_n$$

where $N\mathcal{C}_n$ and $N\mathcal{V}_n$ are identified with the subsets of bridge indices $+$ and $-$, respectively, and $(N\mathcal{A}|^k N\mathcal{U})_n \subset N\mathcal{B}_n$ is the subset of bridging n-simplices in $\mathcal{B}$ of bridge index k. The natural comparison map

$$\begin{array}{ccc}
P_n \cong N\mathcal{C}_n \amalg (N\mathcal{A} \overset{n}{|} N\mathcal{U})_n \amalg \cdots \amalg (N\mathcal{A} \overset{1}{|} N\mathcal{U})_n \amalg N\mathcal{V}_n \\
\downarrow \qquad\qquad \downarrow \\
N\mathcal{D}_n \cong N\mathcal{C}_n \amalg (N\mathcal{C} \overset{n}{|} N\mathcal{U})_n \amalg \cdots \amalg (N\mathcal{C} \overset{1}{|} N\mathcal{U})_n \amalg N\mathcal{V}_n \\
\searrow \qquad \downarrow \\
\{+, n, \ldots, 1, -\}
\end{array}$$

is bijective on the fibers over $-$ and $+$ but need neither be injective nor surjective on the other fibers.

We refer to a bridging n-simplex in P as **bascule**, if it is bascule when identified with a simplex in $\amalg_k (N\mathcal{A}|^k N\mathcal{U})_n \subset N\mathcal{B}_n$.

4. Dwyer pushouts along injective functors

We begin by considering the case where $F\colon \mathcal{A} \to \mathcal{C}$ is injective-on-objects and faithful, in which case we are able to strengthen our conclusion and prove that the canonical comparison map is inner anodyne.

Theorem 4.1. *Let*

$$\begin{tikzcd}
\mathcal{A} \arrow[r, "F"] \arrow[d, hook, "I"'] & \mathcal{C} \arrow[d, hook, "J"] \\
\mathcal{B} \arrow[r, "G"'] \arrow[ur, phantom, "\ulcorner"] & \mathcal{D}
\end{tikzcd}$$

be a pushout of categories, in which I is a Dwyer map and F is faithful and injective on objects. Then the induced inclusion of simplicial sets

$$N\mathcal{C} \coprod_{N\mathcal{A}} N\mathcal{B} \hookrightarrow N\mathcal{D}$$

is inner anodyne and in particular a weak categorical equivalence.

Proof. Observe in this case that $N\mathcal{C} \amalg_{N\mathcal{A}} N\mathcal{B} \hookrightarrow N(\mathcal{C} \amalg_{\mathcal{A}} \mathcal{B})$ is an inclusion. We will filter this inclusion through a series of subcomplexes

$$N\mathcal{C} \coprod_{N\mathcal{A}} N\mathcal{B} = K^{0,\infty} \hookrightarrow \cdots \hookrightarrow K^{m-1,\infty} \hookrightarrow K^{m,\infty} \hookrightarrow \cdots \hookrightarrow N(\mathcal{C} \coprod_{\mathcal{A}} \mathcal{B}) = N\mathcal{D}$$

and then further filter each step as follows

$$K^{m-1,\infty} = K^{m,0} \hookrightarrow \cdots \hookrightarrow K^{m,t-1} \hookrightarrow K^{m,t} \hookrightarrow \cdots \hookrightarrow K^{m,\infty}.$$

We will then express each $K^{m,t-1} \hookrightarrow K^{m,t}$ as a pushout of inner horn inclusions.

For m and t non-negative integers, let $K^{m,t} \subset N\mathcal{D}$ be the smallest simplicial subset of $N\mathcal{D}$ containing:

- $N\mathcal{C} \amalg_{N\mathcal{A}} N\mathcal{B}$,
- all simplices of dimension strictly less than m,
- all bridging m-simplices of bridge index at least $m-t+1$,
- all bascule m-simplices, and
- all bascule $(m+1)$-simplices of bridge index at least $(m+1)-t+1$.

Note that if $m < m'$ and t, t' are arbitrary, then $K^{m,t} \subset K^{m',t'}$. If $t < t'$, then $K^{m,t} \subset K^{m,t'}$. The simplicial subsets $K^{m,0}$ can be described without explicit reference to bridge index, as the indicated bridge indices are out of range. It is immediate that $K^{m,m+1} = K^{m,m+2} = K^{m,m+3} = \cdots = K^{m+1,0}$ since integral bridge index is always positive. We will observe below that we can do better, and the $K^{m,t}$ sequence for fixed m actually stabilizes at $t = m$, rather than $m+1$. We write $K^{m,\infty}$ for this stable value.

To warm up, let us start with low dimensions beginning with $m = 0$. Every 0-simplex is non-bridging, so in particular is non-bascule, hence $K^{0,0} = N\mathcal{C} \amalg_{N\mathcal{A}} N\mathcal{B}$. The simplicial subset $K^{0,1}$ adds the bascule 1-simplices of bridge index 1, that is, all bascule 1-simplices. These $\nu_u \colon Su \to u$ are indexed by the objects $u \in \mathcal{U} \subset \mathcal{V}$, that is, those objects which are in $\mathcal{W}$ but not in $\mathcal{A}$. Since $\nu_G = G\varepsilon$, such a 1-simplex is the image under $NG \colon N\mathcal{B} \to N\mathcal{D}$ of the 1-simplex $\varepsilon_u \colon Ru \to u$, hence was already present in $K^{0,0} = N\mathcal{C} \amalg_{N\mathcal{A}} N\mathcal{B}$. Therefore

$$N\mathcal{C} \coprod_{N\mathcal{A}} N\mathcal{B} = K^{0,0} = K^{0,1} = \cdots = K^{0,\infty} = K^{1,0}.$$

We now turn to the case of $m = 1$. The simplicial subset $K^{1,1}$ adds 1-simplices of bridge index 1 and bascule 2-simplices of bridge index 2. But some of these simplices are already in $K^{1,0}$, namely

- the bascule 1-simplices,
- the degenerate bascule 2-simplices of bridge index 2,
- the non-bascule 1-simplices of bridge index 1 which are in the image of $N\mathcal{B}$, and
- the non-degenerate bascule 2-simplices of bridge index 2 which are in the image of $N\mathcal{B}$.

The last two are in bijection via Corollary 3.9. But the simplices that are *actually* added to $K^{1,1}$ are

- the *non-bascule* 1-simplices of bridge index 1 which are *not* in the image of $N\mathcal{B}$ and
- the *non-degenerate* bascule 2-simplices of bridge index 2 which are *not* in the image of $N\mathcal{B}$.

Corollary 3.9 tells us these two sets are in bijection via the first face map, so it suffices to add the latter set of simplices. If $\sigma = (c_0 \to c_1 \xrightarrow{\nu_{u_2}} u_2)$ is a bascule 2-simplex of bridge index 2 which is not in the image of $N\mathcal{B}$, then $\sigma \cdot \delta^0$ is a bascule 1-simplex, hence is in $K^{1,0}$, while $\sigma \cdot \delta^2$ is a 1-simplex in $N\mathcal{C} \subset K^{1,0}$. We thus can form $K^{1,1}$ via the following pushout

$$\begin{tikzcd} \coprod_{\sigma\in B_{1,1}} \Lambda^2_1 \arrow[r, hook] \arrow[d] & \coprod_{\sigma\in B_{1,1}} \Delta^2 \arrow[d] \\ K^{1,0} \arrow[r, hook] & K^{1,1} \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd}$$

where the coproducts are indexed by the set $B_{1,1}$ of non-degenerate bascule 2-simplices of bridge index 2. This shows that $K^{1,0} \hookrightarrow K^{1,1}$ is inner anodyne.

To build $K^{1,2}$, we are meant to add bascule 2-simplices of bridge index 1. But such a bascule 2-simplex $c_0 \xrightarrow{\nu_{u_1}} u_1 \to u_2$ has that $c_0 = Su_1 = FRu_1$, and is actually just $NG(Ru_1 \xrightarrow{\varepsilon_{u_1}} u_1 \to u_2)$, hence already in $K^{0,0} \subset K^{1,1}$. Thus $K^{1,1} = K^{1,2}$.

This last argument generalizes: a bascule $(m{+}1)$-simplex σ of bridge index 1

$$c_0 \xrightarrow{\nu_{u_1}} u_1 \to u_2 \to \cdots \to u_{m+1}$$

is in the image of G, hence is contained in $K^{0,0} \subset K^{m,0} \subset K^{m,m}$. But these are the only elements that could be added in passing from $K^{m,m}$ to $K^{m,m+1}$, which is why we always have $K^{m,m} = K^{m,m+1}$ for any m.

Let's now pass to the general case. For $1 \le t \le m < \infty$, let $B_{m,t}$ be the set of non-degenerate bascule $(m{+}1)$-simplices in $N\mathcal{D}$ of bridge index $m+2-t$ which are not in the image of $N\mathcal{B}$. Then the $(m{+}1{-}t)$th face map is a bijection from the set $B_{m,t}$ to the set of non-degenerate, non-bascule m-simplices in $N\mathcal{D}$ of bridge index $m+1-t$ which are not in the image of $N\mathcal{B}$, since $N\mathcal{B} \hookrightarrow N\mathcal{D}$, as an injective map of simplicial sets, respects the bijections of Corollary 3.9. These two sets are precisely what is new that must be added to go from $K^{m,t-1}$ to $K^{m,t}$.

We will attach these simplices to $K^{m,t-1}$ via a horn $\Lambda^{m+1}_{m+1-t} \hookrightarrow \Delta^{m+1}$ for each $\sigma \in B_{m,t}$. To do so, we must argue that all of the other faces of such simplices σ belong to $K^{m,t-1}$ already. By Lemma 3.8, the $(m{+}2{-}t)$th face map takes elements of $B_{m,t}$ to m-simplices of bridge index $m+2-t$ when $t > 1$, or to non-bridging simplices when $t = 1$; in both cases, the $(m{+}2{-}t)$th face is contained in $K^{m,t-1}$. By Lemma 3.8, for $\sigma \in B_{m,t}$ and all other $k \neq m+1-t, m+2-t$, the face $\sigma \cdot \delta^k$ is either bascule or non-bridging, and hence is contained in $K^{m,0} \subset K^{m,t-1}$. We can thus form $K^{m,t}$ as the pushout

$$\begin{tikzcd} \coprod_{\sigma\in B_{m,t}} \Lambda^{m+1}_{m+1-t} \arrow[r, hook] \arrow[d] & \coprod_{\sigma\in B_{m,t}} \Delta^{m+1} \arrow[d] \\ K^{m,t-1} \arrow[r, hook] & K^{m,t}. \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd}$$

Since $1 \le t \le m$, we have $m \ge m+1-t \ge 1$, so this is an inner anodyne extension. $\square$

5. Dwyer pushouts along bijective-on-objects and full functors

Suppose $F\colon \mathcal{A} \to \mathcal{C}$ is a bijective-on-objects and full functor and $I\colon \mathcal{A} \hookrightarrow \mathcal{B}$ is a Dwyer map. In this case, the object bijection of Proposition 3.11 restricts to define a bijection

$$\mathrm{ob}\mathcal{B} \xleftarrow[\approx]{} \mathrm{ob}\mathcal{A} \amalg \mathrm{ob}\mathcal{V} \xrightarrow[\approx]{(F,\mathrm{id})} \mathrm{ob}\mathcal{C} \amalg \mathrm{ob}\mathcal{V} \xrightarrow[\approx]{} \mathrm{ob}\mathcal{D}$$

which is the action on objects of the functor $G : \mathrm{ob}\mathcal{B} \cong \mathrm{ob}\mathcal{D}$. While pushouts of full functors need not be full in general, in this case, by Remark 3.13, $G\colon \mathcal{B} \to \mathcal{D}$ is a bijective-on-objects and full functor whose actions on hom-sets are given by

$$\begin{tikzcd} \mathcal{B}(Ia, Ia') \arrow[rr, two heads, "G"] & & \mathcal{D}(GIa, GIa') \\ \mathcal{A}(a, a') \arrow[u, phantom, "\wr\|"] \arrow[r, two heads, "F"] & \mathcal{C}(Fa, Fa') \arrow[r, "J", "\approx"'] & \mathcal{D}(JFa, JFa') \arrow[u, phantom, "\|"] \end{tikzcd}$$

$$\mathcal{B}(v, v') \xrightarrow[\approx]{G} \mathcal{D}(Gv, Gv')$$

$$\begin{tikzcd} \mathcal{B}(Ia, u) \arrow[rr, two heads, "G"] & & \mathcal{D}(GIa, Gu) \\ \mathcal{A}(a, Ru) \arrow[u, phantom, "\wr\|"] \arrow[r, two heads, "F"] & \mathcal{C}(Fa, FRu) = \mathcal{C}(Fa, SGu) \arrow[r, "\nu_{Gu}\circ J(-)", "\approx"'] & \mathcal{D}(JFa, Gu) \arrow[u, phantom, "\|"] \end{tikzcd}$$

for all $a, a' \in \mathcal{A}$, $v, v' \in \mathcal{V}$, and $u \in \mathcal{U}$. Note these three cases are disjoint and cover all the non-empty hom-sets in $\mathcal{B}$ and $\mathcal{D}$. The functor $G\colon \mathcal{B} \to \mathcal{D}$ need not be faithful, however: it will identify any parallel morphisms from $\mathcal{A}$ that are identified in $\mathcal{C}$ by F and will also identify parallel morphisms from an object of $\mathcal{A}$ to an object of $\mathcal{U}$ whenever the transposes in $\mathcal{A}$ are identified in $\mathcal{C}$ by F.

Theorem 5.1. *Let*

$$\begin{tikzcd} \mathcal{A} \arrow[r, "F"] \arrow[d, hook, "I"'] & \mathcal{C} \arrow[d, hook, "J"] \\ \mathcal{B} \arrow[r, "G"'] \arrow[ur, phantom, "\ulcorner", very near end] & \mathcal{D} \end{tikzcd}$$

be a pushout of categories, in which I is a Dwyer map and F is bijective on objects and full. Then the induced map of simplicial sets

$$N\mathcal{C} \coprod_{N\mathcal{A}} N\mathcal{B} \to N\mathcal{D}$$

is a weak categorical equivalence.

Write $P = N\mathcal{C} \amalg_{N\mathcal{A}} N\mathcal{B}$ for the pushout of the nerves, and $N\mathcal{D}$ for the nerve of the pushout. The map $NG\colon N\mathcal{B} \twoheadrightarrow N\mathcal{D}$ factors as a pair of epimorphisms $N\mathcal{B} \twoheadrightarrow P$ and $P \twoheadrightarrow N\mathcal{D}$. The first map only identifies simplices that are in the simplicial subset $N\mathcal{A} \subset N\mathcal{B}$ and which become identified under $NF\colon N\mathcal{A} \twoheadrightarrow N\mathcal{C}$ — the first case just mentioned — while the job of the second map is to identify bridging simplices in $N\mathcal{B}$ — the second case just described.

$$\begin{tikzcd}
N\mathcal{B}_n \cong N\mathcal{A}_n \amalg (N\mathcal{A}\overset{n}{|}N\mathcal{U})_n \amalg \cdots \amalg (N\mathcal{A}\overset{1}{|}N\mathcal{U})_n \amalg N\mathcal{V}_n \arrow[d, two heads] \\
P_n \cong N\mathcal{C}_n \amalg (N\mathcal{A}\overset{n}{|}N\mathcal{U})_n \amalg \cdots \amalg (N\mathcal{A}\overset{1}{|}N\mathcal{U})_n \amalg N\mathcal{V}_n \arrow[d, two heads] \\
N\mathcal{D}_n \cong N\mathcal{C}_n \amalg (N\mathcal{C}\overset{n}{|}N\mathcal{U})_n \amalg \cdots \amalg (N\mathcal{C}\overset{1}{|}N\mathcal{U})_n \amalg N\mathcal{V}_n \arrow[d] \\
\{+, n, \ldots, 1, -\}
\end{tikzcd}$$

Notation 5.2. Write $B\mathcal{B}_n \subset N\mathcal{B}_n$ and $B\mathcal{D}_n \subset N\mathcal{D}_n$ for the subsets of bascule n-simplices and $B^r\mathcal{B}_n$ and $B^r\mathcal{D}_n$ for the bascule n-simplices with bridge index r.

When G is bijective on objects and full, the epimorphism $NG\colon N\mathcal{B}_n \twoheadrightarrow N\mathcal{D}_n$ restricts to an epimorphism $NG\colon B^r\mathcal{B}_n \twoheadrightarrow B^r\mathcal{D}_n$ for each $1 \leq r \leq n$.

We define an 'anti-filtration'

$$\begin{tikzcd}
P =: Q^1 \arrow[r, two heads] \arrow[drrr, two heads] & \cdots \arrow[r, two heads] & Q^{n-1} \arrow[r, two heads] \arrow[dr, two heads] & Q^n \arrow[r, two heads] \arrow[d, two heads] & \cdots \arrow[r, two heads] & \mathrm{colim}_n Q^n \arrow[dll, dashed, "\approx"] \\
& & & N\mathcal{D} & &
\end{tikzcd}$$

with the aim of proving that each $Q^{n-1} \twoheadrightarrow Q^n$ is a weak categorical equivalence.

Construction 5.3. For each $n \geq 0$, the simplicial set Q^n is defined by the pushout:

$$\begin{tikzcd}
& N\mathcal{A} \arrow[r] \arrow[d] & N\mathcal{C} \arrow[d] \arrow[ddr, bend left] & \\
\coprod\limits_{k=1}^{n} \coprod\limits_{B\mathcal{B}_k} \times\Delta^k \arrow[r] \arrow[d] & N\mathcal{B} \arrow[r] & P \arrow[d] \arrow[ul, phantom, "\ulcorner" very near start] & \\
\coprod\limits_{k=1}^{n} \coprod\limits_{B\mathcal{D}_k} \times\Delta^k \arrow[rr] \arrow[drrr, bend right] & & Q^n \arrow[dr, dashed] \arrow[ul, phantom, "\ulcorner" very near start] & \\
& & & N\mathcal{D}
\end{tikzcd}
\quad\rightsquigarrow\quad
\begin{tikzcd}
\coprod\limits_{B\mathcal{B}_n} \Delta^n \arrow[r] \arrow[d, two heads] & Q^{n-1} \arrow[d, two heads] \\
\coprod\limits_{B\mathcal{D}_n} \Delta^n \arrow[r] & Q^n \arrow[ul, phantom, "\ulcorner" very near start]
\end{tikzcd}$$

Since the bascule 1-simplices of $N\mathcal{B}$ and $N\mathcal{D}$ are identical—both are in bijection with the objects in $\mathcal{U}$—$P \cong Q^1$.

Lemma 5.4. *By construction, the map* $\mathrm{sk}_{n-1}Q^n \to \mathrm{sk}_{n-1}N\mathcal{D}$ *is an isomorphism. Consequently* $\mathrm{colim}_n Q^n \cong N\mathcal{D}$*, and* $Q^{n-1} \twoheadrightarrow Q^n$ *is bijective on* $(n-2)$*-skeleta.*

Proof. By construction of the quotient Q^n of P, each bascule simplex in $N\mathcal{B}$ of dimension up to and including n is identified with its image in $N\mathcal{D}$. Since, by Corollary 3.9, each simplex of dimension at most $n-1$ in $N\mathcal{D}$ arises as a face of such a simplex, the map $Q^n \to N\mathcal{D}$ is bijective on n-skeleta. Thus, the anti-filtration converges to $N\mathcal{D}$. By the 2-of-3 property of isomorphisms it follows that $Q^{n-1} \to Q^n$ is bijective on $(n-2)$-skeleta. $\square$

In order to show that $Q^{n-1} \to Q^n$ is a weak categorical equivalence, we factor it as a finite composite

$$Q^{n-1} = Q^{n,n+1} \twoheadrightarrow Q^{n,n} \twoheadrightarrow \cdots \twoheadrightarrow Q^{n,1} = Q^n$$

using the partition of the bascule simplices according to their bridge index:

$$B\mathcal{B}_n = B^n\mathcal{B}_n \amalg \cdots \amalg B^1\mathcal{B}_n.$$

Construction 5.5. For each $n \geq 0$, and $r = n, \ldots, 1$, we form the pushout below-left, which factors as below-right:

$$\begin{tikzcd} \coprod_{t\geq r} B^t\mathcal{B}_n \times \Delta^n \arrow[r] \arrow[d, two heads] & Q^{n-1} \arrow[d] \arrow[ddr, bend left, two heads] \\ \coprod_{t\geq r} B^t\mathcal{D}_n \times \Delta^n \arrow[r] \arrow[drr, bend right] \arrow[ur, phantom, "\ulcorner" very near end] & Q^{n,r} \arrow[dr, dashed, two heads] \\ & & Q^n \end{tikzcd} \qquad \begin{tikzcd} \coprod_{B^r\mathcal{B}_n} \Delta^n \arrow[r] \arrow[d, two heads] & Q^{n,r+1} \arrow[d] \\ \coprod_{B^r\mathcal{D}_n} \Delta^n \arrow[r] \arrow[ur, phantom, "\ulcorner" very near end] & Q^{n,r} \end{tikzcd}$$

By construction $Q^{n,1} = Q^n$. In fact:

Lemma 5.6. *The quotient map $Q^{n,2} \twoheadrightarrow Q^{n,1}$ is the identity.*

Proof. Since the functor $G\colon \mathcal{B} \to \mathcal{D}$ is bijective on objects, the map $NG\colon N\mathcal{B} \to N\mathcal{D}$ is bijective on bascule simplices of bridge index 1. Thus, no quotienting happens in the last step. ☐

Thus, we assume that $r > 1$ henceforth and seek to analyze the quotient maps $Q^{n,r+1} \to Q^{n,r}$. Our aim, finally achieved in Lemma 5.22, is to express this quotient map as a pushout of a weak categorical equivalence along a monomorphism, allowing us to conclude that $Q^{n,r+1} \to Q^{n,r}$ is itself a weak categorical equivalence.

As a first step towards achieving this, we define simplicial sets that capture the shape of the data in $Q^{n,r+1}$ that will be quotiented. For $r > 1$ and $\sigma \in B^r\mathcal{D}_n$, consider the set $B_\sigma \subset B^r\mathcal{B}_n$ of *bascule* preimages of σ, i.e., the fiber of the map $B^r\mathcal{B}_n \to B^r\mathcal{D}_n$ over σ. We next give a simplicial set H_σ which can be given as a colimit of a star-shaped diagram whose legs are indexed by B_σ.

Construction 5.7. Let $\sigma \in B^r\mathcal{D}_n$ and define H_σ to be the simplicial set formed as the pushout

$$\begin{tikzcd} \coprod_{\tau\in B_\sigma} \Lambda^n_{r-1} \arrow[r, tail, "\sim"] \arrow[d, two heads] & \coprod_{\tau\in B_\sigma} \Delta^n \arrow[d, two heads] \arrow[ddr, bend left, two heads] \\ \Lambda^n_{r-1} \arrow[r, tail, "\sim"] \arrow[drr, bend right, tail, "\sim"'] \arrow[ur, phantom, "\ulcorner" very near end] & H_\sigma \arrow[dr, dashed, two heads, "\sim"] \\ & & \Delta^n \end{tikzcd}$$

which comes equipped with a weak equivalence to Δ^n.

Lemma 5.8. *If $\sigma \in B^r\mathcal{D}_n$ is a bascule n-simplex of bridge index r, then the square*

$$\begin{tikzcd} \coprod_{\tau \in B_\sigma} \Delta^n \arrow[rr] \arrow[dd, two heads] & & N\mathcal{B} \arrow[d, two heads] \\ & & Q^{n,r+1} \arrow[d, two heads] \\ H_\sigma \arrow[r, "\sim"] \arrow[urr, dashed] & \Delta^n \arrow[r, "\sigma"] & N\mathcal{D} \end{tikzcd}$$

admits a unique lift through $Q^{n,r+1}$.

Proof. Suppose τ_1 and τ_2 are two elements of B_σ. We want to show that $\tau_1 \cdot \delta^k$ and $\tau_2 \cdot \delta^k$ map to the same element in $Q^{n,r+1}$ for $k \neq r-1$.

For $k \neq r-1, r$, the faces $\sigma \cdot \delta^k$, $\tau_1 \cdot \delta^k$, and $\tau_2 \cdot \delta^k$ are bascule $(n-1)$-simplices (of bridge index $r-1$ or r) by Lemma 3.8, with $\tau_1 \cdot \delta^k, \tau_2 \cdot \delta^k \in B_{\sigma \cdot \delta^k}$. This implies that $\tau_1 \cdot \delta^k$ and $\tau_2 \cdot \delta^k$ are identified in Q^{n-1}.

The (possibly non-bascule) face $\sigma \cdot \delta^r$ is non-bridging if $r = n$ and otherwise has bridge index r. In the former case, $\sigma \cdot \delta^r$ has a unique lift to P so we suppose $r < n$. In this case, the bascule lift σ' of $\sigma \cdot \delta^r$ has bridge index $r+1$ and satisfies $\sigma' \cdot \delta^{r+1} = \sigma \cdot \delta^r$. For $\tau = \tau_1$ or τ_2, we can likewise form the bascule lifts τ' of $\tau \cdot \delta^r$, which again have bridge index $r+1$ and satisfy $\tau' \cdot \delta^{r+1} = \tau \cdot \delta^r$. These bascule lifts can be described explicitly: for $\sigma \cdot \delta^r$, the bascule lift (see Corollary 3.9) takes the form

$$a_0 \xrightarrow{f_1} \cdots \xrightarrow{f_{r-1}} a_{r-1} \xrightarrow{Sh_{r+1}} Su_{r+1} \xrightarrow{\nu_{u_{r+1}}} u_{r+1} \xrightarrow{h_{r+2}} \cdots \xrightarrow{h_n} u_n$$

where $a_{r-1} = Ru_r$. In other words, the edge from the $(r-1)$st vertex to the rth vertex is obtained by applying the functor S to the edge from the rth vertex to the $(r+1)$st vertex of σ. The bascule lifts of τ_1 and τ_2 may be described similarly using the functor R. Thus, by Lemma 3.15, the bascule lifts τ_1' and τ_2' are bascule preimages of σ'. Hence $\tau_1', \tau_2' \in B_{\sigma'}$ are identified in $Q^{n,r+1}$, so their faces must be identified as well. $\square$

Using the lift constructed in Lemma 5.8, we have a commutative square whose left-hand leg is a weak categorical equivalence:

$$\begin{tikzcd} H_\sigma \arrow[r, dashed] \arrow[d, two heads, "\wr"'] & Q^{n,r+1} \arrow[d, two heads] \\ \Delta^n \arrow[r, "\sigma"'] & N\mathcal{D} \end{tikzcd}$$

However, the map $H_\sigma \to Q^{n,r+1}$ is not necessarily injective, so this weak categorical equivalence would not necessarily be preserved under pushout. Thus, in two steps, we will introduce further quotients of H_σ in order to obtain a simplicial set that maps injectively into $Q^{n,r+1}$.

The first replacement of H_σ, the simplicial set U_σ of Construction 5.9, takes care of the fact that when σ is degenerate, some of the n-simplices of H_σ arise from degenerate simplices of $N\mathcal{B}$. In the second step, we also take into account that some faces of the n-simplices of U_σ might be identified in $N\mathcal{D}$. We will start with some preparations for these constructions.

Given some $\tau \in B_\sigma \subseteq N\mathcal{B}_n$, we use the Eilenberg–Zilber property to produce a unique pair (s_τ, y_τ) with $s_\tau \colon \Delta^n \twoheadrightarrow \Delta^{m_\tau}$ a degeneracy operator and y_τ a non-degenerate m_τ-simplex of $N\mathcal{B}$, such that $y_\tau \cdot s_\tau = \tau$. Likewise, we can form $(s_\sigma \colon \Delta^n \twoheadrightarrow \Delta^{m_\sigma}, y_\sigma \in N\mathcal{D}_{m_\sigma})$ with y_σ non-degenerate in $N\mathcal{D}$ and $y_\sigma \cdot s_\sigma = \sigma$.

We have factorizations of s_σ:

$$\begin{tikzcd}
\Delta^n \arrow[rr, two heads, "s_\sigma"] \arrow[dr, two heads, "s_\tau"'] & & \Delta^{m_\sigma} \\
& \Delta^{m_\tau} \arrow[ur, two heads, "s'_\tau"'] &
\end{tikzcd}$$

Further, let m_σ be the minimal value of m_τ amongst all bascule preimages τ of σ. Note that $m_\sigma < n$ if and only if σ is degenerate.

Construction 5.9. We create a new star-shaped diagram indexed over B_σ so that the τ leg is of the form

$$\Lambda^n_{r-1} \longrightarrow \Delta^n \xrightarrow{s_\tau} \Delta^{m_\tau}$$

and define U_σ to be the pushout:

$$\begin{tikzcd}
\coprod\limits_{\tau \in B_\sigma} \Lambda^n_{r-1} \arrow[r] \arrow[d, two heads] & \coprod\limits_{\tau \in B_\sigma} \Delta^{m_\tau} \arrow[d, two heads] \\
\Lambda^n_{r-1} \arrow[r] \arrow[ur, phantom, "\ulcorner", very near end] & U_\sigma
\end{tikzcd}$$

The simplicial sets H_σ and U_σ fit into the following diagram:

$$\begin{tikzcd}
\coprod\limits_{\tau \in B_\sigma} \Lambda^n_{r-1} \arrow[r, tail, "\sim"] \arrow[d] & \coprod\limits_{\tau \in B_\sigma} \Delta^n \arrow[r, two heads, "s_\tau"] \arrow[d] & \coprod\limits_{\tau \in B_\sigma} \Delta^{m_\tau} \arrow[r] \arrow[d] & N\mathcal{B} \arrow[d, two heads] \\
\Lambda^n_{r-1} \arrow[r, tail, "\sim"] \arrow[ur, phantom, "\ulcorner", very near end] & H_\sigma \arrow[r, two heads] \arrow[d, "\wr"] \arrow[ur, phantom, "\ulcorner", very near end] & U_\sigma \arrow[r] \arrow[d] & Q^{n,r+1} \arrow[d] \\
& \Delta^n \arrow[r, two heads, "s_\sigma"'] & \Delta^{m_\sigma} \arrow[r, "y_\sigma"'] & N\mathcal{D}
\end{tikzcd} \tag{5.10}$$

Since $H_\sigma \twoheadrightarrow U_\sigma$ is a pushout of the degeneracy maps s_τ associated to $\tau \in B_\sigma$, the canonical map $H_\sigma \to Q^{n,r+1}$ of Lemma 5.8 factors uniquely through U_σ as displayed above.

Lemma 5.11. *For each $\sigma \in B^r\mathcal{D}_n$ a bascule n-simplex of bridge index r, the square*

$$\begin{tikzcd}
H_\sigma \arrow[r] \arrow[d, "\wr"] & U_\sigma \arrow[d] \\
\Delta^n \arrow[r, "s_\sigma"] & \Delta^{m_\sigma}
\end{tikzcd}$$

is a pushout.

Proof. By (5.10), it suffices to show that the interior commutative square below is a pushout, so we consider a cone with nadir Z under the span:

$$\begin{tikzcd}
\coprod\limits_{\tau} \Delta^n \arrow[r, "s_\tau"] \arrow[d, "\nabla"'] & \coprod\limits_{\tau} \Delta^{m_\tau} \arrow[d, "s'_\tau"] \arrow[ddr, bend left, "z_\tau"] & \\
\Delta^n \arrow[r, "s_\sigma"'] \arrow[drr, bend right, "z"'] \arrow[ur, phantom, "\ulcorner" very near end] & \Delta^{m_\sigma} \arrow[dr, dashed, "z_{\tau_0}"] & \\
& & Z
\end{tikzcd}$$

We know that there is some τ_0 for which $s'_{\tau_0} = \mathrm{id}$. Thus, if this cone factors through the claimed pushout it must factor via the map z_{τ_0}. Since $s_\sigma = s_{\tau_0}$, the bottom triangle commutes. It remains to argue that the rightmost triangle commutes, which we may show one component at a time.

To verify that $z_\tau = z_{\tau_0} \cdot s'_\tau$ it suffices to verify this commutativity after precomposing with the epimorphism s_τ. Now we have

$$z_\tau \cdot s_\tau = z = z_{\tau_0} \cdot s_{\tau_0} = z_{\tau_0} \cdot s_\sigma = z_{\tau_0} \cdot s'_\tau \cdot s_\tau,$$

as desired. $\square$

Lemma 5.12. *Let $\sigma \in B^r\mathcal{D}_n$. If σ is degenerate, then we can exhibit U_σ as the colimit of the smaller star-shaped diagram*

$$\begin{tikzcd}
\coprod\limits_{\mathrm{nd}B_\sigma} \Lambda^n_{r-1} \arrow[r, tail, "\sim"] \arrow[d] & \coprod\limits_{\mathrm{nd}B_\sigma} \Delta^n \arrow[d] \\
\Lambda^n_{r-1} \arrow[r, tail, "\sim"] \arrow[d, two heads, "s_\sigma"'] \arrow[ur, phantom, "\ulcorner" very near end] & \bullet \arrow[d, two heads] \\
\Delta^{m_\sigma} \arrow[r, tail, "\sim"] \arrow[ur, phantom, "\ulcorner" very near end] & U_\sigma
\end{tikzcd} \tag{5.13}$$

where all but one of the legs is indexed by the set $\mathrm{nd}B_\sigma$ of non-degenerate elements of B_σ, and the final leg being $\Lambda^n_{r-1} \hookrightarrow \Delta^n \xrightarrow{s_\sigma} \Delta^{m_\sigma}$.

Proof. We will exhibit a natural isomorphism between cones under the diagram (5.13) and cones under the defining pushout of Construction 5.9.

We can choose and fix a simplex ρ of dimension $m_\rho = m_\sigma$ in the defining diagram for U_σ (and then in particular we necessarily have $s_\rho = s_\sigma$). Sending the last leg to ρ, we can view the smaller diagram as a subdiagram of the defining diagram for U_σ. Every cone under the latter gives in particular a cone under the former by restriction. Now assume we are given a cone under the smaller diagram. We will show that it extends uniquely to a cone under the larger diagram, thus proving the claim.

Let τ be one of the simplices in the definition of U_σ so that $m_\tau < n$ and $\tau \neq \rho$. Since $s_\tau \colon \Delta^n \to \Delta^{m_\tau}$ is not the identity, it has sections $\Delta^{m_\tau} \to \Delta^n$ of which at least one has an image contained in Λ^n_{r-1}. Choose and fix one such section d_τ. Define the map on the summand Δ^{m_τ} as

$$\Delta^{m_\tau} \xrightarrow{d_\tau} \Lambda^n_{r-1} \to \Delta^n \xrightarrow{s_\rho} \Delta^{m_\sigma}.$$

We observe that the composite

$$\Lambda^n_{r-1} \to \Delta^n \xrightarrow{s_\tau} \Delta^{m_\tau} \xrightarrow{d_\tau} \Lambda^n_{r-1} \to \Delta^n \xrightarrow{s_\rho} \Delta^{m_\sigma}$$

equals

$$\Lambda^n_{r-1} \to \Delta^n \xrightarrow{s_\rho} \Delta^{m_\sigma}$$

since we have $s_\rho = s_\sigma = s'_\tau s_\tau$. Thus, the newly defined maps indeed extend the cone under the small diagram into a cone under the large one. As for uniqueness, we observe that the morphism

$$\Lambda^n_{r-1} \to \Delta^n \xrightarrow{s_\tau} \Delta^{m_\tau}$$

is an epimorphism since it has a section, implying the desired uniqueness.

In total, this proves the bijection between the cones under the respective diagrams, yielding the claim. □

By the 2-of-3 property, the canonical retraction to the map $\Delta^{m_\sigma} \xrightarrow{\sim} U_\sigma$ of (5.13) is an equivalence:

Corollary 5.14. *Let $\sigma \in B^r\mathcal{D}_n$. The map $U_\sigma \twoheadrightarrow \Delta^{m_\sigma}$ is an equivalence.* □

We now have a pushout

$$\begin{tikzcd} H_\sigma \arrow[r, two heads] \arrow[d, two heads, "\wr"'] & U_\sigma \arrow[r] \arrow[d, two heads, "\wr"'] & Q^{n,r+1} \arrow[d, two heads] \\ \Delta^n \arrow[r, two heads] \arrow[ur, phantom, "\ulcorner"] & \Delta^{m_\sigma} \arrow[r, "y_\sigma"'] & N\mathcal{D} \end{tikzcd}$$

Unfortunately, the map $U_\sigma \to Q^{n,r+1}$ may still fail to be injective. Over the next sequence of lemmas, we will identify the image of the map

$$\coprod_{\sigma \in B^r\mathcal{D}_n} U_\sigma \to Q^{n,r+1}.$$

Construction 5.15. For $\sigma \in B^r\mathcal{D}_n$, let $\mathrm{core}(\sigma) \subseteq U_\sigma$ be the image

$$\begin{tikzcd} \Lambda^n_{r-1} \arrow[r, tail] \arrow[dr, two heads] & H_\sigma \arrow[r] & U_\sigma \\ & \mathrm{core}(\sigma) \arrow[ur, tail] & \end{tikzcd}$$

of the canonical map $\Lambda^n_{r-1} \to U_\sigma$.

Lemma 5.16. *Let $\sigma \in B^r\mathcal{D}_n$. If σ is non-degenerate, then $\Lambda^n_{r-1} \to \mathrm{core}(\sigma)$ is an isomorphism. If σ is degenerate, then the composite*

$$\mathrm{core}(\sigma) \rightarrowtail U_\sigma \twoheadrightarrow \Delta^{m_\sigma}$$

is an isomorphism.

Proof. If σ is non-degenerate, then every element in B_σ is also non-degenerate. This implies $s_\tau = \mathrm{id}_{[n]}$ for all $\tau \in B_\sigma$, so $H_\sigma \to U_\sigma$ is an isomorphism, and the monomorphism $\Lambda^n_{r-1} \to H_\sigma$ identifies Λ^n_{r-1} with its image.

If σ is degenerate, the map $\Lambda^{n-1} \to H_\sigma \to U_\sigma$ is identified with the diagonal composite in the lower square of (5.13). That square then displays the image factorization. □

Construction 5.17. Define a simplicial set $I^{n,r}$ as the image factorization of the composite map $\coprod_{\sigma \in B^r\mathcal{D}_n} \mathrm{core}(\sigma) \to N\mathcal{D}$ given by

$$\begin{tikzcd}
\coprod\limits_{\sigma \in B^r\mathcal{D}_n} \mathrm{core}(\sigma) \arrow[r, tail] \arrow[dd, dashed, two heads] & \coprod\limits_{\sigma \in B^r\mathcal{D}_n} U_\sigma \arrow[r] \arrow[d, two heads, "\wr"'] & Q^{n,r+1} \arrow[dd, two heads] \\
& \coprod\limits_{\sigma \in B^r\mathcal{D}_n} \Delta^{m_\sigma} \arrow[dr] & \\
I^{n,r} \arrow[rr, dashed, tail] & & N\mathcal{D}
\end{tikzcd}$$

where the sums are over all bascule n-simplices σ of bridge index r, for fixed $r > 1$.

This gives the diagram

$$\begin{tikzcd}
I^{n,r} \arrow[d, equal] & \coprod\limits_{\sigma \in B^r\mathcal{D}_n} \mathrm{core}(\sigma) \arrow[l, two heads] \arrow[r, tail] \arrow[d, equal] & \coprod\limits_{\sigma \in B^r\mathcal{D}_n} U_\sigma \arrow[d, "\wr"] & I^{n,r} \amalg_{(\coprod_\sigma \mathrm{core}(\sigma))} (\coprod_\sigma U_\sigma) \arrow[d, dashed, "\wr"] \\
I^{n,r} & \coprod\limits_{\sigma \in B^r\mathcal{D}_n} \mathrm{core}(\sigma) \arrow[l, two heads] \arrow[r, tail] & \coprod\limits_{\sigma \in B^r\mathcal{D}_n} \Delta^{m_\sigma} & I^{n,r} \amalg_{(\coprod_\sigma \mathrm{core}(\sigma))} (\coprod_\sigma \Delta^{m_\sigma})
\end{tikzcd}$$

Taking pushouts of each row yields the map above-right, which is an equivalence since both the top and bottom pushouts are homotopy pushouts. Note this map is the pushout of coproduct of the weak equivalences $U_\sigma \to \Delta^{m_\sigma}$ as displayed below-left. We will demonstrate in Proposition 5.18 that the map $\coprod_\sigma \mathrm{core}(\sigma) \to \coprod_\sigma U_\sigma \to Q^{n,r+1}$ factors through $I^{n,r}$, allowing us to form the square

$$\begin{tikzcd}
\coprod U_\sigma \arrow[r] \arrow[d, "\wr"] \arrow[dr, phantom, "\ulcorner" very near end] & I^{n,r} \amalg_{(\coprod_\sigma \mathrm{core}(\sigma))} (\coprod_\sigma U_\sigma) \arrow[r, dashed] \arrow[d, "\wr"] & Q^{n,r+1} \arrow[d] \\
\coprod \Delta^{m_\sigma} \arrow[r] & I^{n,r} \amalg_{(\coprod_\sigma \mathrm{core}(\sigma))} (\coprod_\sigma \Delta^{m_\sigma}) \arrow[r] & N\mathcal{D}
\end{tikzcd}$$

Then, we will show in Lemma 5.22 that, when we form the pushout in the right-hand square, we recover the simplicial set $Q^{n,r}$.

$$\begin{tikzcd}
\coprod_\sigma U_\sigma \arrow[r] \arrow[d, "\wr"] \arrow[dr, phantom, "\ulcorner" very near end] & I^{n,r} \amalg_{(\coprod_\sigma \mathrm{core}(\sigma))} (\coprod_\sigma U_\sigma) \arrow[r, dashed] \arrow[d, "\wr"] \arrow[dr, phantom, "\ulcorner" very near end] & Q^{n,r+1} \arrow[d] \\
\coprod_\sigma \Delta^{m_\sigma} \arrow[r] & I^{n,r} \amalg_{(\coprod_\sigma \mathrm{core}(\sigma))} (\coprod_\sigma \Delta^{m_\sigma}) \arrow[r] & Q^{n,r}.
\end{tikzcd}$$

Finally, we will demonstrate in Proposition 5.23 that the map $I^{n,r} \amalg_{(\coprod_\sigma \mathrm{core}(\sigma))} (\coprod_\sigma U_\sigma) \to Q^{n,r+1}$ is injective. It follows that $Q^{n,r+1} \to Q^{n,r}$ is a weak equivalence, as desired.

Proposition 5.18. *Let $\sigma \in B^r\mathcal{D}_n$. The dashed lift exists in the following diagram*

$$\begin{tikzcd} \coprod_{\sigma\in B^r\mathcal{D}_n} \mathrm{core}(\sigma) \arrow[r] \arrow[d, two heads] & \coprod_{\sigma\in B^r\mathcal{D}_n} U_\sigma \arrow[r] & Q^{n,r+1} \arrow[d, "\pi"] \\ I^{n,r} \arrow[rru, dashed] \arrow[rr, tail] & & N\mathcal{D} \end{tikzcd}$$

Proof. The quotient map $\coprod_{\sigma\in B^r\mathcal{D}_n} \mathrm{core}(\sigma) \twoheadrightarrow I^{n,r}$ may:

- identify simplices in $\mathrm{core}(\sigma)$ for a fixed σ, and may also
- identify simplices in $\mathrm{core}(\sigma)$ with simplices in $\mathrm{core}(\sigma')$ for $\sigma \neq \sigma'$.

We argue that for both sorts of identifications in $N\mathcal{D}$ these identifications lift to $Q^{n,r+1}$, and in fact treat both cases simultaneously by considering a pair of simplices $\sigma, \sigma' \in B^r\mathcal{D}_n$ which may or may not be distinct. Since $\mathrm{sk}_{n-2}Q^{n,r+1} \to \mathrm{sk}_{n-2}N\mathcal{D}$ is an isomorphism, the only quotienting that occurs is in higher dimensions. Since each $\mathrm{core}(\sigma)$ and thus $I^{n,r}$ is $(n-1)$-skeletal, it suffices to consider identifications of $(n-1)$-simplices, at least one of which is non-degenerate. We will argue that any $(n-1)$-simplices $\gamma \in \mathrm{core}(\sigma)$ and $\gamma' \in \mathrm{core}(\sigma')$ that are identified in $N\mathcal{D}$ are also identified in $Q^{n,r+1}$. Write $\bar{\gamma}, \bar{\gamma}' \in Q^{n,r+1}$ for their images.

We introduce some further notation to better describe these images. We write the bascule n-simplex σ as

$$a_0 \xrightarrow{f_1} \cdots \xrightarrow{f_{r-1}} a_{r-1} \xrightarrow{\varepsilon} u_r \xrightarrow{h_{r+1}} u_{r+1} \xrightarrow{h_{r+2}} \cdots \xrightarrow{h_n} u_n$$

and let $\tau \in N\mathcal{B}$ be a fixed bascule preimage of σ

$$a_0 \xrightarrow{p_1} \cdots \xrightarrow{p_{r-1}} a_{r-1} \xrightarrow{\varepsilon} u_r \xrightarrow{h_{r+1}} u_{r+1} \xrightarrow{h_{r+2}} \cdots \xrightarrow{h_n} u_n$$

that is "maximally degenerate," meaning it is chosen with the property that p_k an identity if and only if $F(p_k) = f_k$ is an identity. Note there is a map μ as displayed below

$$\begin{tikzcd} & & \Delta^n \arrow[r, "\tau"] \arrow[d] & N\mathcal{B} \arrow[d, "\chi"] \\ \Delta^{n-1} \arrow[r, "\gamma"] \arrow[rrr, bend right, "\bar{\gamma}"'] & \mathrm{core}(\sigma) \arrow[r] \arrow[ur, dashed, "\mu"] & U_\sigma \arrow[r] & Q^{n,r+1} \arrow[d, "\pi"] \\ & & & N\mathcal{D} \end{tikzcd}$$

that lands in $\Lambda^n_{r-1} \subset \Delta^n$. The map μ is either the inclusion $\Lambda^n_{r-1} \to \Delta^n$ if σ is non-degenerate, or is a section to the degeneracy operator involved in σ. Thus we may use our lift τ of σ to describe the image of $\mathrm{core}(\sigma)$ in $Q^{n,r+1}$: in particular $\bar{\gamma} = \chi\tau\mu\gamma$. Define τ' and μ' analogously for σ' so that $\bar{\gamma}' = \chi\tau'\mu'\gamma'$.

Our goal is to show that if $\pi\bar{\gamma} = \pi\bar{\gamma}'$, then $\bar{\gamma} = \bar{\gamma}'$. As noted above, we may assume without loss of generality that γ is a non-degenerate $(n-1)$-simplex, which presents two possibilities for σ:

(i) If σ is non-degenerate, then $\mathrm{core}(\sigma) = \Lambda^n_{r-1}$ and $\mu\gamma = \delta^k \colon \Delta^{n-1} \to \Delta^n$ for some $k \neq r-1$. In this case, $\bar{\gamma} = \chi\tau\delta^k$.

(ii) If σ is degenerate, then the only way for $\mathrm{core}(\sigma) = \Delta^{m_\sigma}$ to have a non-degenerate $(n-1)$-simplex is if $m_\sigma = n-1$. That is, $\sigma = \tilde{\sigma} \cdot s^\ell$ for $\ell \neq r-1$ and $\tilde{\sigma}$ non-degenerate. In this case, μ is given by δ^ℓ, so $\bar{\gamma} = \chi\tau\delta^\ell$.

Thus in either case $\bar{\gamma} = \chi\tau\delta^k$ for $k \neq r-1$. Similarly, if γ' is non-degenerate, then $\bar{\gamma}' = \chi\tau'\delta^{k'}$ for some $k' \neq r-1$. If both γ and γ' are non-degenerate, then we have $\bar{\gamma} = \bar{\gamma}'$ by Lemma 5.19 below.

If γ' is degenerate, then we will show that $\bar{\gamma} = \bar{\gamma}'$ in Lemma 5.20 below. This uses the following description of $\bar{\gamma}'$.

(iii) If γ' is degenerate, then $\mu'\gamma' \colon \Delta^{n-1} \to \Delta^n$ is of the form

$$\Delta^{n-1} \xrightarrow{s} \mathrel{\mkern-14mu}\rightarrow \Delta^\ell \rightarrowtail^{d} \Delta^{n-1} \xrightarrow{\delta^{k'}} \Delta^n$$

where $\ell < n-1$ and $k' \neq r-1$ and $s \neq \mathrm{id}$.

Thus after we have proved Lemmas 5.19 and 5.20 below, the current proposition will be established. $\square$

Lemma 5.19. *Using the notation from the proof of Proposition 5.18, suppose that $\gamma \in \mathrm{core}(\sigma)$ and $\gamma' \in \mathrm{core}(\sigma')$ are non-degenerate $(n-1)$-simplices. If $\pi\bar{\gamma} = \pi\bar{\gamma}'$, then $\bar{\gamma} = \bar{\gamma}' \in Q^{n,r+1}$.*

Proof. As we saw in the proof of Proposition 5.18, the non-degeneracy assumption implies $\pi\bar{\gamma} = \pi\chi\tau\delta^k = \sigma\delta^k$ and $\pi\bar{\gamma}' = \pi\chi\tau'\delta^{k'} = \sigma'\delta^{k'}$ with $k, k' \neq r-1$. If the simplices $\tau\delta^k$, $\tau'\delta^{k'}$, and $\sigma\delta^k = \sigma'\delta^{k'}$ are non-bridging, then $\tau\delta^k$ and $\tau'\delta^{k'}$ are identified in P, hence in $Q^{n,r+1}$. Below we assume these simplices are bridging simplices.

Suppose that both $\tau\delta^k$ and $\tau'\delta^{k'}$ are bascule. Then their common image $\sigma\delta^k = \sigma'\delta^{k'}$ is bascule as well, and since we have already identified all bascule preimages of $(n-1)$-simplices by the stage Q^{n-1}, we know that the bascule preimages $\tau\delta^k$ and $\tau'\delta^{k'}$ are identified in $Q^{n,r+1}$.

We must still consider the case when one or both of $\tau\delta^k$ or $\tau'\delta^{k'}$ are non-bascule bridging simplices. Without loss of generality, suppose that $\tau\delta^k$ is non-bascule. By Lemma 3.8 this means that $k = r$; since this is a bridging simplex, we have that it is of bridge index r. Since $\tau\delta^r = \tau\delta^k$ and $\tau'\delta^{k'}$ both map to the same element in $N\mathcal{D}$, they have the same bridge index.

Write $\tilde{\tau}$ and $\tilde{\tau}'$ for the bascule lifts of $\tau\delta^r = \tau\delta^k$ and $\tau'\delta^{k'}$. The bascule lift $\tilde{\tau}$ of $\tau\delta^r$ is pictured below, but note that $\tilde{\tau}'$ may not be of the same form if $k' > r$:

$$a_0 \xrightarrow{p_1} \cdots \xrightarrow{p_{r-1}} a_{r-1} \xrightarrow{Rh_{r+1}} Ru_{r+1} \xrightarrow{\varepsilon} u_{r+1} \xrightarrow{h_{r+2}} \cdots \xrightarrow{h_n} u_n$$

We know that $NG(\tau\delta^k) = NG(\tau'\delta^{k'})$, hence NG identifies their bascule lifts $NG(\tilde{\tau}) = NG(\tilde{\tau}')$. Since $\tilde{\tau}$ and $\tilde{\tau}'$ are bascule n-simplices of bridge index $r+1$, they are identified in $Q^{n,r+1}$. Thus their faces $\tilde{\tau}\delta^r = \tau\delta^k$ and $\tilde{\tau}'\delta^r = \tau'\delta^{k'}$ are identified there as well. $\square$

Lemma 5.20. *Using the notation from the proof of Proposition 5.18, suppose that $\gamma \in \mathrm{core}(\sigma)$ is non-degenerate, $\gamma' \in \mathrm{core}(\sigma')$ is degenerate, and $\pi\bar{\gamma} = \pi\bar{\gamma}'$. Then $\bar{\gamma} = \bar{\gamma}' \in Q^{n,r+1}$.*

Proof. By (i) and (ii) from the proof of Proposition 5.18, we have $\bar{\gamma} = \chi\tau\delta^k$ and $k \neq r-1$. By (iii) we have $\bar{\gamma}' = \chi\tau'\delta^{k'}ds$, where s is not the identity and $k' \neq r-1$. Our assumption is that

$$\sigma\delta^k = \sigma'\delta^{k'}ds \tag{5.21}$$

which is a degenerate simplex in $N\mathcal{D}$.

We first argue that we can rule out the case where $\sigma = \tilde{\sigma} s^\ell$ is degenerate as in (ii). In this case, we have $k = \ell$ and $\sigma\delta^\ell = \tilde{\sigma} s^\ell \delta^\ell = \tilde{\sigma}$. Thus, the left-hand side of (5.21) is non-degenerate, while the right-hand side is degenerate, a contradiction.

Thus we must be in the situation of (i) meaning that σ is non-degenerate. Thus, for its face $\sigma\delta^k$ to be degenerate means that composing at the kth vertex produces an identity, which can only happen if $k \neq r-1, r$. Thus one of the following holds:

$$\begin{array}{llll} f_{k+1}f_k = \mathrm{id} & \text{and} & a_{k-1} = a_{k+1} & \text{if} \quad 1 \leq k < r-1 \\ h_{k+1}h_k = \mathrm{id} & \text{and} & u_{k-1} = u_{k+1} & \text{if} \quad r+1 \leq k < n. \end{array}$$

Since σ is non-degenerate, this is the only identity that appears in this simplex $\sigma\delta^k$. Also note, since $k \neq r-1, r$, Lemma 3.8 tells us that $\sigma\delta^k$ is bascule. The element $\tau\delta^k$ is a bascule preimage of the bascule $(n-1)$-simplex $\sigma\delta^k$.

Using this information, we will define an $(n-2)$-simplex $\tilde{\tau}$ of $N\mathcal{B}$ so that $\pi(\tilde{\tau}s^{k-1}) = \pi(\tau\delta^k)$. Define $\tilde{\tau}$ to be one of the following two $(n-2)$-simplices in $N\mathcal{B}$, depending on whether $k < r-1$ or $k > r$:[5]

$$a_0 \xrightarrow{p_1} \cdots \xrightarrow{p_{k-1}} a_{k+1} \xrightarrow{p_{k+2}} \cdots \xrightarrow{p_{r-1}} a_{r-1} \xrightarrow{\varepsilon} u_r \xrightarrow{h_{r+1}} u_{r+1} \xrightarrow{h_{r+2}} \cdots \xrightarrow{h_n} u_n$$

$$a_0 \xrightarrow{p_1} \cdots \xrightarrow{p_{r-1}} a_{r-1} \xrightarrow{\varepsilon} u_r \xrightarrow{h_{r+1}} u_{r+1} \xrightarrow{h_{r+2}} \cdots \xrightarrow{h_{k-1}} u_{k+1} \xrightarrow{h_{k+2}} \cdots \xrightarrow{h_n} u_n$$

Let $\tilde{\sigma}$ be the image of $\tilde{\tau}$ in $N\mathcal{D}$. We have

$$\tilde{\sigma} s^{k-1} = \sigma\delta^k = \sigma'\delta^{k'} ds$$

with $\tilde{\sigma}$ non-degenerate and s non-identity, so it follows by the Eilenberg–Zilber Lemma that $\sigma'\delta^{k'}d = \tilde{\sigma}$ and $s = s^{k-1}$ and $\ell = n-2$. We have $\mathrm{sk}_{n-2}Q^{n,r+1} \to \mathrm{sk}_{n-2}N\mathcal{D}$ is an isomorphism. This implies that the $(n-2)$-simplices $\tilde{\tau}$ and $\tau'\delta^{k'}d$ of $N\mathcal{B}$ are identified in $Q^{n,r+1}$, hence we know $\chi\tilde{\tau}s^{k-1} = \chi\tau'\delta^{k'}ds^{k-1} = \bar{\gamma}'$.

But $\tau\delta^k$ and $\tilde{\tau}s^{k-1}$ are both bascule $(n-1)$-simplices with common image in $N\mathcal{D}$, hence become equal in $Q^{n,r+1}$. Thus $\bar{\gamma} = \chi\tau\delta^k = \chi\tilde{\tau}s^{k-1} = \bar{\gamma}'$. $\square$

Lemma 5.22. *The square*

$$\begin{tikzcd} I^{n,r} \coprod_{(\coprod_\sigma \mathrm{core}(\sigma))} (\coprod_\sigma U_\sigma) \arrow[r, dashed] \arrow[d, "\simeq"'] & Q^{n,r+1} \arrow[d] \\ I^{n,r} \coprod_{(\coprod_\sigma \mathrm{core}(\sigma))} (\coprod_\sigma \Delta^{m_\sigma}) \arrow[r] \arrow[ur, phantom, "\ulcorner"] & Q^{n,r} \end{tikzcd}$$

is a pushout.

Proof. Recall that $Q^{n,r}$ is built from $Q^{n,r+1}$ via the pushout

$$\begin{tikzcd} \coprod_{\sigma \in B^r\mathcal{D}_n} \coprod_{\tau \in B_\sigma} \Delta^n \arrow[r] \arrow[d, two heads] & Q^{n,r+1} \arrow[d, two heads] \\ \coprod_{\sigma \in B^r\mathcal{D}_n} \Delta^n \arrow[r] \arrow[ur, phantom, "\ulcorner"] & Q^{n,r} \end{tikzcd}$$

[5]In the first case, we have $\tilde{\tau}s^{k-1}$ may be distinct from $\tau\delta^k$, since we don't know that $p_{k+1}p_{k_1} =$ id holds, only that it is in the F-preimage of the identity. But in the second case, $h_{k+1}h_k = \mathrm{id}$ so $\tilde{\tau}s^{k-1} = \tau\delta^k$ already holds in $N\mathcal{B}$.

which factors through the left-hand pushout square by (5.10) and Lemma 5.11

$$\begin{tikzcd}
\coprod\limits_{\sigma\in B^r\mathcal{D}_n}\coprod\limits_{\tau\in B_\sigma}\Delta^n \arrow[r] \arrow[d, two heads] & \coprod\limits_{\sigma\in B^r\mathcal{D}_n}\coprod\limits_{\tau\in B_\sigma}\Delta^{m_\tau} \arrow[r] \arrow[d, two heads] & Q^{n,r+1} \arrow[d, two heads] \\
\coprod\limits_{\sigma\in B^r\mathcal{D}_n}\Delta^n \arrow[r] & \coprod\limits_{\sigma\in B^r\mathcal{D}_n}\Delta^{m_\sigma} \arrow[r] \arrow[ul, phantom, "\ulcorner", very near start] & Q^{n,r} \arrow[ul, phantom, "\ulcorner", very near start]
\end{tikzcd}$$

By (5.10), the right-hand pushout square factors as

$$\begin{tikzcd}
\coprod\limits_{\sigma\in B^r\mathcal{D}_n}\coprod\limits_{\tau\in B_\sigma}\Delta^{m_\tau} \arrow[dr, two heads] \arrow[drr, bend left] \arrow[ddr, bend right] & & \\
& \coprod\limits_{\sigma\in B^r\mathcal{D}_n}U_\sigma \arrow[r] \arrow[d, two heads] & Q^{n,r+1} \arrow[d, two heads] \\
& \coprod\limits_{\sigma\in B^r\mathcal{D}_n}\Delta^{m_\sigma} \arrow[r] & Q^{n,r} \arrow[ul, phantom, "\ulcorner", very near start]
\end{tikzcd}$$

and since the map $\coprod_\sigma\coprod_\tau \Delta^{m_\tau} \to \coprod_\sigma U_\sigma$ is an epimorphism, this smaller square is still a pushout square. Finally, this smaller square factors through the left-hand pushout

$$\begin{tikzcd}
\coprod_{\sigma\in B^r\mathcal{D}_n} U_\sigma \arrow[r] \arrow[d, "\wr"'] & I^{n,r}\underset{(\coprod_\sigma \mathrm{core}(\sigma))}{\amalg}(\coprod_\sigma U_\sigma) \arrow[r, dashed] \arrow[d, "\wr"] & Q^{n,r+1} \arrow[d] \\
\coprod_{\sigma\in B^r\mathcal{D}_n} \Delta^{m_\sigma} \arrow[r] & I^{n,r}\underset{(\coprod_\sigma \mathrm{core}(\sigma))}{\amalg}(\coprod_\sigma \Delta^{m_\sigma}) \arrow[r] \arrow[ul, phantom, "\ulcorner", very near start] & Q^{n,r}
\end{tikzcd}$$

so we may conclude that the right-hand square is a pushout as claimed. □

Proposition 5.23. *The dashed map in the diagram is injective:*

$$\begin{tikzcd}
\coprod\limits_{\sigma\in B^r\mathcal{D}_n}\mathrm{core}(\sigma) \arrow[r, tail] \arrow[d, two heads] & \coprod\limits_{\sigma\in B^r\mathcal{D}_n}U_\sigma \arrow[d] \arrow[ddr, bend left] & \\
I^{n,r} \arrow[r] \arrow[drr, tail, bend right] & \bullet \arrow[dr, dashed] \arrow[ul, phantom, "\ulcorner", very near start] & \\
& & Q^{n,r+1}
\end{tikzcd}$$

By its construction as a lift of the monomorphism $I^{n,r} \rightarrowtail N\mathcal{D}$, the map $I^{n,r} \rightarrowtail Q^{n,r+1}$ constructed in Proposition 5.18 is automatically a monomorphism. So it remains only to consider simplices in the complement of the image of $I^{n,r}$ in the pushout, or equivalently in the complement of the image of $\coprod \mathrm{core}\,\sigma \rightarrowtail \coprod U_\sigma$, and argue that any two such are neither identified with each other nor with something in the image of $I^{n,r}$. As everything in the pushout part of the diagram is n-skeletal, it is enough, by [Cis06, 8.1.28], to restrict attention to simplices of dimension at most n. To argue this, we make use of the following definition.

There are two types of n-simplices in U_σ that are not in the image of $\mathrm{core}(\sigma) \subset U_\sigma$:

- The n-simplices τ associated to some non-degenerate bascule preimage of σ.

- Degenerate n-simplices on the face $\tau \cdot \delta^{r-1}$ for some τ as above.

Note both cases are canonically identified with simplices in $N\mathcal{B}$. Our aim is to show that both varieties of n-simplex are not identified with any others under the canonical projection $N\mathcal{B} \to Q^{n,r+1}$.

Lemma 5.24. *Suppose that $\tau \in N\mathcal{B}$ is a non-degenerate bascule n-simplex of bridge index r, z is one of the simplices τ, $\tau\delta^{r-1}$, $\tau\delta^{r-1}s^i$, and $\chi\colon N\mathcal{B} \to Q^{n,r+1}$ is the defining projection map. Then $\chi^{-1}\chi(z) = \{z\}$.*

Proof. By construction, $Q^{n,r+1}$ is a quotient of $N\mathcal{B}$ identifying certain bridging simplices that become identified in $N\mathcal{D}$, namely those with dimension less than n and with bridge index greater than r. Thus, to see that these elements are not identified with any others, it is enough to show that if $\tau'\colon \Delta^{n'} \to N\mathcal{B}$ is a bascule simplex

- with dimension $n' < n$, or
- with dimension n and bridge index $r' > r$,

then $\tau\delta^{r-1}$, $\tau\delta^{r-1}s^i$, and τ are not in the image of $\tau'\colon \Delta^{n'} \to N\mathcal{B}$. Notice that if either τ or $\tau\delta^{r-1}s^i$ is in the image of such a map, then so is $\tau\delta^{r-1}$, hence it is enough to show that $\tau\delta^{r-1}$ is not in such an image.

Assume first that τ' is of dimension $n' < n$. Since $\tau\delta^{r-1}$ is a non-degenerate simplex of dimension $n-1$, this necessarily implies $n' = n-1$. This would in turn mean $\tau\delta^{r-1} = \tau'$, which is a contradiction because $\tau\delta^{r-1}$ is non-bascule and τ' is bascule by assumption.

Suppose τ' has dimension n and bridge index $r' > r$ and that $r > 1$. If $\tau'\delta^k = \tau\delta^{r-1}$, then $\tau'\delta^k$ is non-bascule and has bridge index $r-1$. By Lemma 3.8 this means $k = r'-1 = r-1$ or $k = r' = r-1$. Neither of these can happen since $r' > r$.

Suppose τ' has dimension n and bridge index $r' > r$ and $r = 1$. If $\tau'\delta^k = \tau\delta^0$ which is non-bridging, we either have $k = r'-1 = 0$ or $k = r' = n$ by Lemma 3.8. The first case means that $r' = r$, which can't happen. The second case can't happen, since all vertices of $\tau'\delta^n$ are in $\mathcal{A}$ and all of the vertices of $\tau\delta^0$ are in $\mathcal{V}$. □

Lemma 5.25. *Let $\tau, \tau' \in B^r\mathcal{B}_n$, and suppose that $x, x' \in \Delta^n$ are two elements of dimension $n-1$ or n so that $\tau(x), \tau'(x') \in N\mathcal{B}$ map to the same element of $Q^{n,r+1}$. If $x \notin \Lambda^n_{r-1}$ and τ is non-degenerate, then $\tau = \tau'$ and $x = x'$.*

Proof. Since $x \notin \Lambda^n_{r-1}$, we either have $x = \mathrm{id}$ or $x = \delta^{r-1}$ or $x = \delta^{r-1}s^i$. We have $\tau x = \tau' x'$ by Lemma 5.24. Letting $d = \mathrm{id}$ in the first two cases or $d = \delta^i$ in the last case, we have $\tau = \tau' x' d$ or $\tau\delta^{r-1} = \tau' x' d$. Since these elements of $N\mathcal{B}$ are non-degenerate, $x'd$ is either an identity or a coface map.

We can conclude that $\tau = \tau'$. In the first case, this is because $\tau = \tau' x' d = \tau' \mathrm{id}$. For the second case, we have that $\tau\delta^{r-1} = \tau'\delta$ for some codimension one coface map δ. Since τ and τ' are bascule simplices of bridge index r and $\tau\delta^{r-1} = \tau'\delta$ is non-bascule of bridge index $r-1$ (or non-integral bridge index $-$ when $r = 1$), it follows from Lemma 3.8 (or observation when $r = 1$) that $\delta = \delta^{r-1}$. Since $(-)\cdot\delta^{r-1}$ is injective on bascule simplices of bridge index r (for $r > 1$ this is Corollary 3.9 while for $r = 1$ this is just observation), we have $\tau = \tau'$ in the second case as well.

Our final goal is to show that $x = x'$. If $x'd = \mathrm{id}$ then by dimension reason we have $x' = \mathrm{id}$, hence $x = x'$. If $x'd$ is a coface, with $\tau\delta^{r-1} = \tau' x' d = \tau x' d$, we must have $x'd = \delta^{r-1}$ by Lemma 3.8. If $x = \delta^{r-1}$ then $d = \mathrm{id}$ so $x' = x'd = \delta^{r-1}$. If

$x = \delta^{r-1}s^i$, then $d = \delta^i$ and $x' \colon \Delta^n \to \Delta^n$ factors as $\delta^j s^k \colon \Delta^n \to \Delta^{n-1} \to \Delta^n$. Since $\delta^j s^k \delta^i = \delta^{r-1}$ we have $s^k\delta^i = \mathrm{id}$ and so $j = r-1$. But then $\tau\delta^{r-1}s^i = \tau x = \tau x' = \tau\delta^{r-1}s^k$, and since $\tau\delta^{r-1}$ is non-degenerate we have $i = k$. Thus $x = \delta^{r-1}s^i = x'$. □

Proof of Proposition 5.23. Note that the pushout $I \cup (\coprod U_\sigma)$ is n-skeletal, so by [Cis06, Lemme 8.1.28] it is enough to show that $(I \cup (\coprod U_\sigma))_k \to Q^{n,r+1}_k$ is injective for $k \le n$. To do so, we will show that if two elements in $\coprod U_\sigma$ of dimension at most n map to the same element of $Q^{n,r+1}$, then either they are equal or they are both in $\coprod \mathrm{core}(\sigma)$. We assume that at least one of the elements is not in $\coprod \mathrm{core}(\sigma)$; this implies that both elements have dimension at least $n-1$ since $\mathrm{sk}_{n-2}\,\mathrm{core}(\sigma) = \mathrm{sk}_{n-2}U_\sigma$ for all $\sigma \in B^r\mathcal{D}_n$.

Suppose $\bar{x} \in U_\sigma$ is a simplex of dimension $n-1$ or n which does not lie in $\mathrm{core}(\sigma)$. Then there is a non-degenerate simplex $\tau \in \mathrm{nd}B_\sigma \subseteq B^r\mathcal{B}_n$ along with $x \in \Delta^n$ so that $\bar{x}$ is represented by x in the τ-leg defining U_σ. Since $\bar{x} \notin \mathrm{core}(\sigma)$, we have $x \notin \Lambda^n_{r-1}$. We can also represent another element $\bar{x}' \in U_{\sigma'} \subseteq \coprod U_\sigma$ by an element $x' \in \Delta^n$ and $\tau' \in B_{\sigma'}$ since the map $s_{\tau'}$ in (5.10) above Lemma 5.11 admits a section.

Again writing $\chi \colon N\mathcal{B} \to Q^{n,r+1}$, the element $\bar{x}$ of $\coprod U_\sigma$ maps to the element $\chi(\tau(x))$ of $Q^{n,r+1}$, while $\bar{x}'$ maps to $\chi(\tau'(x'))$. If these images in $Q^{n,r+1}$ coincide, then $\tau = \tau'$ and $x = x'$ by Lemma 5.25, which implies that $\bar{x} = \bar{x}'$. □

Corollary 5.26. *The map $Q^{n,r+1} \to Q^{n,r}$ is a weak equivalence.*

Proof. By Lemma 5.22 and Proposition 5.23, the map $Q^{n,r+1} \to Q^{n,r}$ is a pushout of a weak equivalence along a monomorphism

$$\begin{tikzcd}
I^{n,r} \underset{(\coprod_\sigma \mathrm{core}(\sigma))}{\amalg} (\coprod_\sigma U_\sigma) \arrow[r, tail] \arrow[d, "\wr"'] & Q^{n,r+1} \arrow[d, "\wr"] \\
I^{n,r} \underset{(\coprod_\sigma \mathrm{core}(\sigma))}{\amalg} (\coprod_\sigma \Delta^{m_\sigma}) \arrow[r, tail] \arrow[ur, phantom, "\ulcorner", very near end] & Q^{n,r}
\end{tikzcd}$$

and thus is a weak equivalence by left properness of the Joyal model structure. □

Proof of Theorem 5.1. The previous corollary implies that $Q^{n-1} \to Q^n$ is a weak categorical equivalence for all n. As weak categorical equivalences are closed under filtered colimits [Cis19, Corollary 3.9.8], this implies that $P \to \mathrm{colim}_n Q^n \cong N\mathcal{D}$ is a weak categorical equivalence as well. □

Since we have established Theorem 4.1 and Theorem 5.1, our main result, Theorem 1.5, now follows from Lemma 2.4.

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