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BISMUT-ZHANG THEOREM AND ANOMALY FORMULA FOR THE RAY-SINGER METRIC FOR SPACES WITH ISOLATED CONICAL SINGULARITIES

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ABSTRACT. In this article we extend to spaces with isolated conical singularities Bismut's and Zhang's generalisation of the Cheeger-Müller Theorem, *i.e.* the comparison formula between analytic torsion and Milnor torsion of a smooth compact manifold equipped with an arbitrary flat Hermitian vector bundle. We also establish anomaly formulas for all three terms appearing in our Bismut-Zhang formula for a space with isolated conical singularities, in particular we generalise Bismut's and Zhang's anomaly formula for the Ray-Singer metric to this singular context.

1. INTRODUCTION

An important comparison theorem in global analysis is the comparison of analytic (or Ray-Singer) and topological torsion for smooth compact manifolds equipped with a unitary flat vector bundle. It has been conjectured by Ray and Singer and has been independently proved by Cheeger [Che79] and Müller [Mül78]. In [Mül93] Müller extended the result to the case of odd dimensional manifolds, where only the metric on the determinant of the flat vector bundle is required to be flat, the socalled unimodular case. In the same time, in [BZ92], Bismut and Zhang combined the Witten deformation ([Wit82, HS85]) and local index techniques to generalise the result of Cheeger and Müller to arbitrary flat vector bundles with arbitrary Hermitian metrics. Bismut and Zhang compare the analytic torsion with the Milnor torsion: If the flat vector bundle is not unitary or unimodular the two torsions are no longer equal and the difference between them can be expressed in terms of the Mathai-Quillen current. In this article we refer to this most general version of the comparison theorem of torsions as the Bismut-Zhang theorem.

In the last decade a lot of progress has been made in the study of analytic torsion and the Cheeger-Müller theorem for spaces with singularities of different types. In this article we deal with spaces with conical singularities, and therefore only review the history of the Cheeger-Müller theorem for these.

The study of analytic and topological torsion on *singular spaces with conical singularities* started with work of Dar in [Dar87]. She proved that on singular spaces with *isolated conical singularities*, the analytic torsion is well-defined. She also proved that for an oriented even dimensional space with isolated conical singularities and unitary flat vector bundles, by the usual duality argument, the analytic torsion is trivial. Following an idea suggested by Lesch in [Les98, Problem 5.3], namely to study the problem of a Cheeger-Müller theorem for singular spaces via gluing formulas, several articles have computed

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and studied the analytic torsion on a truncated cone [Ver09], [MV14], [HS10], [HS11], [HS16]. The recent preprint [HS20] seems to complete this program.

Although a Cheeger-Müller theorem for spaces with isolated singularities was still missing, it was expected that the Ray-Singer metric is no longer a topological invariant in general. Only partial results on anomaly formulas for the Ray-Singer metric for spaces with isolated conical singularities exist so far: In [MV14] an asymptotic variation formula for the analytic torsion of a truncated odd dimensional cone is given.

In [MV12] Mazzeo and Vertman prove the well-definedness of analytic torsion for *incomplete edge spaces* (sometimes also called wedge spaces in the literature), *i.e.* spaces with a singular stratum of positive dimension and a cone-like metric near the singular stratum. The authors also prove topological invariance of the analytic torsion for an odd dimensional edge space with an odd dimensional singular stratum. The Cheeger-Müller theorem on an odd dimensional edge space with odd dimensional singular stratum equipped with a unimodular bundle satisfying an additional acyclicity condition has been studied in [ARS18]. The strategy in [ARS18] consists in the study of analytic torsion via degeneration of smooth metrics into conical metrics. The assumptions made in [ARS18] exclude the case of odd dimensional spaces with isolated singularities.

In [Lud20a] a generalisation of the Cheeger-Müller theorem for singular spaces with isolated conical singularities has been achieved, following yet another strategy, namely by adapting the strategy of proof of Bismut and Zhang in [BZ92] to the singular setting. The result in [Lud20a] has been proved for a Witt space, satisfying an additional spectral Witt condition, equipped with a unitary flat vector bundle. The comparison theorem in [Lud20a] establishes the equality of the Ray-Singer metric and a metric also defined in [Lud20a] and which henceforth we will call the Bismut-Zhang metric; its definition will be recalled in Section 5.2 of this paper. The Bismut-Zhang metric is defined using the gradient vector field of an anti-radial Morse function on the singular space and bears some similarity to the definition of the smooth Milnor metric: The smooth critical points of the anti-radial Morse function do give the usual contribution to the Bismut-Zhang metric, while the contribution of the singularities of the space is given by the analytic torsion of the model Witten Laplacian. Unlike the Milnor metric of a smooth compact manifold, the Bismut-Zhang metric of a singular space is not a purely topological invariant of the space, even if the flat bundle is unitary.

The aim of this article is to fully profit from the strength of the Bismut-Zhang approach to the study of torsion, and to give a Bismut-Zhang formula for spaces with isolated conical singularities, *i.e.* we establish a comparison theorem of torsions for the case where the flat vector bundle is not assumed to be unitary or unimodular. We are also able to remove the assumption made in [Lud20a], that the space satisfies the Witt and a spectral Witt condition. In the case of an odd dimensional space with isolated conical singularities, which does not satisfy the Witt condition, we thus get two comparison formulas according to the two middle perversities for intersection cohomology.

In the last part of this article we also study anomaly formulas for the Ray-Singer metric for spaces with isolated conical singularities, generalising the corresponding formulas of Bismut and Zhang [BZ92, Section IV] for smooth compact manifolds. We also describe anomaly formulas for the Bismut-Zhang metric. The study of these anomaly formulas will reveal that the variations of the three terms in the Bismut-Zhang formula are compatible with each other.

As in [BZ92] and in [Lud20a], the main technical tools of this paper are local index techniques and the Witten deformation, which are adapted to this singular setting. Further, an important role is played by the study of the small time asymptotics of the supertrace of certain operators related to the heat operator in the presence of singularities.

The main results of this paper are: the Bismut-Zhang Theorem (Theorem 5.6), the anomaly formula for the Ray-Singer metric (Theorem 7.5) and the anomaly formula for the Bismut-Zhang metric (Theorem 7.9).

The article is organised as follows: In Section 2 we recall basic definitions and facts on singular spaces with isolated conical singularities. In Section 3 we study the Berezin integral formalism as well as the Mathai-Quillen current for singular spaces and anti-radial Morse functions. In Section 4 we study the Witten deformation: We first shortly recall from [Lud17b] the Witten deformation for singular spaces with isolated conical singularities and anti-radial Morse functions in Section 4.1. In Sections 4.2-4.4, we study the local model Witten Laplacian and adapt results from [Lud20b, Lud20a] to the situation, where the space is no longer Witt and the flat bundle is no longer unitary. In Section 5 we recall the definition of the Ray-Singer and the Bismut-Zhang metric for spaces with isolated conical singularities and state one of the main results of this article, the Bismut-Zhang Theorem, comparing the two metrics. Also in Section 5 we state nine intermediate results, which are the analogues of the nine intermediate results in [BZ92, Section VII] and [Lud20a, Section 5]. Once the nine intermediate results are achieved in our more general situation, the proof of the Bismut-Zhang Theorem is completely analogous to the proof in [BZ92, Section VII] and in [Lud20a, Section 6], hence we give an outline only, omitting the details. In Section 6, we indicate how to generalise the proofs of the nine intermediate results from [Lud20a] to the present more general situation. Most of the proofs of [Lud20a] go through, once the local situation near the singular points is well understood. Some extra care is needed, in the proofs of Theorems 5.15 and 5.17, which we give in more detail. In the last section, Section 7, we discuss anomaly formulas for the Ray-Singer and the Bismut-Zhang metric of a space with isolated conical singularities, *i.e.* we study their behaviour under change of the Riemannian conical metric and of the metric on the flat bundle.

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2. Preliminaries

2.1. Singular spaces with isolated conical singularities. For a smooth manifold L and $\delta > 0$ we denote by

$$c_\delta L := ([0,\delta) \times L)\,/_{(0,x)\sim(0,y)} \tag{2.1}$$

the (open) truncated cone over L.

Let X be a connected topological space, $\mathrm{Sing}(X) \subset X$ a finite set of points, such that $X_{sm} := X \backslash \mathrm{Sing}(X)$ is a smooth manifold of dimension $n \geqslant 2$. We denote by TX (resp. by T^*X) the tangent bundle (resp. the cotangent bundle) of X_{sm}. Let g^{TX} be a Riemannian metric on X_{sm}. We assume that (X, g^{TX}) is a space with isolated conical singularities of dimension n, *i.e.*

(1) For $p \in \mathrm{Sing}(X)$, there exist an open neighbourhood $B_\delta(p)$ of p, a smooth compact connected manifold L_p of dimension

$$\dim L_p = n - 1 =: m \tag{2.2}$$

and a diffeomorphism $\varphi_p : B_\delta(p)\backslash\{p\} \simeq c_\delta L_p\backslash\{0\}$. The diffeomorphism φ_p extends to a homeomorphism, still denoted by φ_p,

$$\varphi_p : B_\delta(p) \simeq c_\delta L_p \text{ and } g^{TX}_{|B_\delta(p)\backslash\{p\}} = \varphi_p^* \left(dr^2 + r^2 g^{TL_p}\right), \tag{2.3}$$

where r is the radial coordinate and g^{TL_p} is a Riemannian metric on the manifold L_p (not depending on r).

(2) The set

$$X\backslash \left(\bigcup_{p\in\mathrm{Sing}(X)} B_\delta(p) \right)$$

is a smooth compact manifold of dimension n with boundary $\bigcup_{p\in\mathrm{Sing}(X)} L_p$.

The set $\mathrm{Sing}(X)$ is called the singular set of X. For $p \in \mathrm{Sing}(X)$, the manifold L_p is called the link of X at p. Let us emphasise that the radial coordinate r in (2.3) is fixed throughout this article.

2.2. **Flat vector bundles over** X. Let (F, ∇^F, g^F) be a flat vector bundle over X_{sm} with canonical flat connection ∇^F and (not necessarily flat) Hermitian metric g^F.

We make the following assumption: For $p \in \mathrm{Sing}(X)$, we denote by $(F_{L_p}, \nabla^{F_{L_p}}, g^{F_{L_p}})$ the restriction of (F, ∇^F, g^F) to the link L_p. We assume that the restriction of (F, ∇^F, g^F) to a punctured neighbourhood of $p \in \mathrm{Sing}(X)$ can be identified with the pull back bundle of the vector bundle $(F_{L_p}, \nabla^{F_{L_p}}, g^{F_{L_p}})$.

We denote by F^* the flat bundle dual to F, and by $F^*_{L_p}$ its restriction to L_p, $p \in \mathrm{Sing}(X)$.

Let $\omega(F, g^F)$ be the 1-form on X with values in the self-adjoint endomorphisms of F,

$$\omega(F, g^F) = (g^F)^{-1} \nabla^F g^F. \tag{2.4}$$

Note that $\omega(F, g^F) = 0$, in case g^F is flat.

Definition 2.1. We denote by $\theta(F, g^F)$ the following closed 1-form on X

$$\theta(F, g^F) = \mathrm{Tr}[\omega(F, g^F)]. \tag{2.5}$$

The cohomology class $[\theta(F, g^F)]$ measures the obstruction to the existence of a flat volume form on F. By our assumption, near $p \in \mathrm{Sing}(X)$, the form $\theta(F, g^F)$ does not depend on the radial coordinate r.

2.3. **Local model near** $p \in \mathrm{Sing}(X)$. Let $p \in \mathrm{Sing}(X)$. We denote by $cL_p := \big([0,\infty) \times L_p\big)/_{(0,x)\sim(0,y)}$ the infinite cone over L_p, by 0 the cone tip and by

$$Z_p := cL_p\backslash\{0\} \simeq \mathbb{R}_{>0} \times L_p \tag{2.6}$$

the punctured infinite cone. We write $x \in Z_p$ in its polar coordinates $x = (r, y)$, where r is the radial coordinate and y is the coordinate on the link. We equip Z_p with the conical metric $g^{TZ_p} = dr^2 + r^2 g^{TL_p}$. The flat bundle $(F_{L_p}, \nabla^{F_{L_p}}, g^{F_{L_p}})$ can be extended in a trivial way to a flat bundle $(F_{Z_p}, \nabla^{F_{Z_p}}, g^{F_{Z_p}})$ over Z_p. If no confusion can occur, we still write F for F_{Z_p}.

Let us denote by ∇^{TZ_p} the Levi-Civita connection on (TZ_p, g^{TZ_p}) and by $R^{TZ_p} = \left(\nabla^{TZ_p}\right)^2$ its curvature. We denote by ∇^{TL_p} the Levi-Civita connection on (TL_p, g^{TL_p}) and by $R^{TL_p} = \left(\nabla^{TL_p}\right)^2$ its curvature.

For convenience of the reader, we recall the following result, see e.g. [BC90, Proposition 1.2] or [O'N83, page 210],

Proposition 2.2. *Let X, Y, V be smooth vector fields on the link manifold L_p. We still denote by X, Y, V their $\mathbb{R}_+$-invariant extension to smooth vector fields on Z_p.*

(a) *The Levi-Civita connections ∇^{TZ_p} and ∇^{TL_p} are related as follows:*

$$\begin{aligned} &\nabla^{TZ_p}_X Y = \nabla^{TL_p}_X Y - r g^{TL_p}(X,Y)\tfrac{\partial}{\partial r}; &\quad &\nabla^{TZ_p}_X \tfrac{\partial}{\partial r} = \tfrac{X}{r}; \\ &\nabla^{TZ_p}_{\frac{\partial}{\partial r}} X = \tfrac{X}{r}; & &\nabla^{TZ_p}_{\frac{\partial}{\partial r}} \tfrac{\partial}{\partial r} = 0. \end{aligned} \tag{2.7}$$

(b) *The curvatures R^{TZ_p} and R^{TL_p} are related by the Gauss equation, more precisely we have:*

$$\begin{aligned} &R^{TZ_p}\left(\frac{\partial}{\partial r}, \right) = 0; \\ &R^{TZ_p}(X,Y)V = R^{TL_p}(X,Y)V - g^{TL_p}(Y,V)X + g^{TL_p}(X,V)Y. \end{aligned} \tag{2.8}$$

2.4. **Intersection cohomology.** An important topological invariant for singular spaces is the intersection homology and cohomology introduced by Goresky and MacPherson [GM80, GM83]. In this article only the intersection cohomology with lower and upper middle perversity will be of interest.

2.4.1. *Lower and upper middle perversity.* We denote by $\overline{m}$ (resp. $\overline{n}$) the lower middle (resp. upper middle) perversity in the sense of Goresky and MacPherson. A perversity $\overline{q}$ is a tuple of non negative natural numbers. For a space with isolated singularities however, the only relevant information is the last entry in this tuple. Hence by slight abuse of notation we will identify a perversity $\overline{q}$ with this last entry, more concretely,

$$\overline{q} = \begin{cases} \lfloor \frac{n}{2} \rfloor - 1 & \text{for } \overline{q} = \overline{m}, \\ \lfloor \frac{n-1}{2} \rfloor & \text{for } \overline{q} = \overline{n}. \end{cases} \tag{2.9}$$

For $\overline{q} \in \{\overline{m}, \overline{n}\}$ we denote by $IH^\bullet_{\overline{q}}(X,F)$ the intersection cohomology of X with perversity $\overline{q}$ and coefficients in the local system associated to the flat bundle F. For an even dimensional space with isolated singularities lower and upper middle perversity coincide and hence also the two intersection cohomologies are the same. More generally, if X is a Witt space, *i.e.* if $H^{\frac{n-1}{2}}(L_p, F_{L_p}) = 0$ for all $p \in \mathrm{Sing}(X)$, then by [Sie83, Theorem 3.4]

$$IH^\bullet_{\overline{m}}(X,F) \simeq IH^\bullet_{\overline{n}}(X,F). \tag{2.10}$$

In this paper, we will not assume the Witt condition, and hence in the odd dimensional case, we have to distinguish two cases.

2.4.2. *Local calculation for the intersection cohomology.* For convenience of the reader we recall the local calculation for intersection cohomology with lower and upper middle perversity, see [GM83, Section 2.4]. Let $H^\bullet(L_p, F_{L_p})$ denote the singular cohomology of the link manifold L_p. For the absolute intersection cohomology of the cone cL_p with values in the flat bundle F, denoted by $IH^{\bullet}_{\overline{q}}(cL_p, F)$ (resp. the relative intersection cohomology, denoted by $IH^{\bullet}_{\overline{q}}(cL_p, L_p, F)$), we have

$$IH^k_{\overline{q}}(cL_p, F) = \begin{cases} H^k(L_p, F_{L_p}) & \text{for } k < n - \overline{q} - 1, \\ 0 & \text{else}. \end{cases} \tag{2.11}$$

resp.

$$IH^k_{\overline{q}}(cL_p, L_p, F) = \begin{cases} H^{k-1}(L_p, F_{L_p}) & \text{for } k \geqslant n - \overline{q}, \\ 0 & \text{else}. \end{cases} \tag{2.12}$$

Note that for an odd dimensional Witt space the local calculation for the lower and upper middle perversity do clearly coincide, which lies at the heart of (2.10).

We denote by $b^k(L_p, F_{L_p}) := \dim H^k(L_p, F_{L_p})$, $k = 0, \ldots, n-1$, the Betti numbers for the link L_p. By the local calculation for intersection cohomology (2.11), (2.12), we get for the absolute Euler characteristic of the cone with coefficients in F, denoted by $I\chi_{\overline{q}}(cL_p, F)$ (resp. for the relative intersection Euler characteristic, denoted by $I\chi^{\overline{q}}(cL_p, L_p, F)$):

$$I\chi_{\overline{q}}(cL_p, F) := \sum_{k=0}^{n} (-1)^k \dim IH^k_{\overline{q}}(cL_p, F) = \sum_{k=0}^{n-\overline{q}-2} (-1)^k b^k(L_p, F_{L_p}) \tag{2.13}$$

resp.

$$I\chi_{\overline{q}}(cL_p, L_pF) := \sum_{k=0}^{n} (-1)^k \dim IH^k_{\overline{q}}(cL_p, L_p, F) = \sum_{k=n-\overline{q}}^{n} (-1)^k b^{k-1}(L_p, F_{L_p}). \tag{2.14}$$

2.5. L^2-cohomology.

2.5.1. *Maximal and minimal extension of the de Rham complex.* Throughout this article, we use the language of Hilbert complexes as introduced by Brüning and Lesch in [BL92].

We denote by $\langle\ ,\ \rangle$ the L^2-inner product on the space of sections of $\Lambda(T^*X) \otimes F$ induced from the metrics g^{TX}, g^F. We denote by $\Omega^{\bullet}_c(X, F)$ (resp. by $\mathrm{L}^2(\Lambda(T^*X) \otimes F)$) the graded vector space of smooth compactly supported sections (resp. of L^2-sections) of $\Lambda(T^*X) \otimes F$. We denote by d_c the outer differential acting on $\Omega^{\bullet}_c(X, F)$.

The de Rham complex $(\Omega^{\bullet}_c(X, F), d_c, \langle\ ,\ \rangle)$ admits several closed extensions (in the Hilbert space of L^2-forms) into a Hilbert complex, a choice of which is called an ideal boundary condition by Cheeger. Here we focus on the maximal (resp. minimal) extension, denoted by $(\mathcal{C}^{\bullet}_{\max}, d_{\max}, \langle\ ,\ \rangle)$ (resp. $(\mathcal{C}^{\bullet}_{\min}, d_{\min}, \langle\ ,\ \rangle)$), where $d_{\max}$ (resp. $d_{\min}$) denotes the maximal (resp. the minimal) closed extension of d_c in $\mathrm{L}^2(\Lambda(T^*X) \otimes F)$:

$$\begin{aligned} \operatorname{dom}(d_{\max}) &= \{\alpha \in \mathrm{L}^2(\Lambda(T^*X) \otimes F) \mid d\alpha \in \mathrm{L}^2(\Lambda(T^*X) \otimes F)\}, \\ \operatorname{dom}(d_{\min}) &= \left\{ \alpha \in \mathrm{L}^2(\Lambda(T^*X) \otimes F) \,\middle|\, \begin{array}{l} \text{there exists a sequence } \alpha_n \in \Omega^{\bullet}_c(X, F) \\ \text{s.t. } \alpha_n \xrightarrow{\mathrm{L}^2} \alpha \text{ and } d\alpha_n \text{ converges in } \mathrm{L}^2 \end{array} \right\}; \end{aligned} \tag{2.15}$$

for a form $\alpha \in \mathrm{L}^2(\Lambda(T^*X) \otimes F)$ the outer differential $d\alpha$ is meant in the sense of derivations. By [BL93, Theorems 3.7 and 3.8],

$$\operatorname{dom}(d_{\max}) / \operatorname{dom}(d_{\min}) \simeq \bigoplus_{p \in \operatorname{Sing}(X)} H^{\frac{n-1}{2}}(L_p, F_{L_p}). \tag{2.16}$$

Hence for a Witt space and in particular for an even dimensional space, the extension of the de Rham complex $(\Omega^{\bullet}_c(X, F), d_c, \langle\ ,\ \rangle)$ into a Hilbert complex is unique, $d_{\min} = d_{\max}$.

The cohomology of the maximal Hilbert complex $(\mathcal{C}^\bullet_{\max}, d_{\max}, \langle\ ,\ \rangle)$, is called the L^2-cohomology of X with values in F,

$$H^\bullet_{(2)}(X,F) := H^\bullet_{(2),\overline{m}}(X,F) := H^\bullet\big((\mathcal{C}^\bullet_{\max}, d_{\max}, \langle\ ,\ \rangle)\big). \tag{2.17}$$

We can also define the cohomology of the minimal extension $(\mathcal{C}^\bullet_{\min}, d_{\min}, \langle\ ,\ \rangle)$,

$$H^\bullet_{(2),\overline{n}}(X,F) := H^\bullet\big((\mathcal{C}^\bullet_{\min}, d_{\min}, \langle\ ,\ \rangle)\big). \tag{2.18}$$

The two cohomologies introduced here only depend on the quasi-isometry class of the Riemannian metric, in particular changing the Riemannian metric g^{TX} outside a neighbourhood of $\mathrm{Sing}(X)$, does not change $H^\bullet_{(2),\overline{q}}(X,F)$, $\overline{q} \in \{\overline{m},\overline{n}\}$. Also, changing the metrics g^{TL_p}, $p \in \mathrm{Sing}(X)$, in the normal form (2.3), does not affect the L^2-cohomologies (2.17), (2.18).

From (2.16), we have for a Witt space $H^\bullet_{(2),\overline{m}}(X,F) = H^\bullet_{(2),\overline{n}}(X,F)$.

2.5.2. L^2*-Hodge-de Rham Theorem.* By a result of Cheeger, Goresky and MacPherson [CGM82, Section 3.4], integration of L^2-forms over intersection chains induces a de Rham isomorphism

$$H^\bullet_{(2),\overline{q}}(X,F) \simeq IH^\bullet_{\overline{q}}(X,F). \tag{2.19}$$

This shows in particular, that the minimal and maximal L^2-cohomology are indeed topological invariants of X.

We denote by δ_c the (formal) adjoint of the operator d_c w.r.t. $\langle\ ,\ \rangle$ acting on compactly supported forms and by $\delta_{\min/\max}$ its minimal resp. maximal extension, which is the adjoint of $d_{\max/\min}$ w.r.t. $\langle\ ,\ \rangle$. We denote by $D^{\overline{m}}$ (resp. $D^{\overline{n}}$) the first order self-adjoint operator associated to the Hilbert complex $(\mathcal{C}^\bullet_{\max}, d_{\max}, \langle\ ,\ \rangle)$ (resp. $(\mathcal{C}^\bullet_{\min}, d_{\min}, \langle\ ,\ \rangle)$). We have

$$D^{\overline{m}} = d_{\max} + \delta_{\min} \text{ (resp. } D^{\overline{n}} = d_{\min} + \delta_{\max}). \tag{2.20}$$

For $\overline{q} \in \{\overline{m},\overline{n}\}$, we denote by $\Delta^{\overline{q}} := (D^{\overline{q}})^2$ the Laplace operators associated to the two Hilbert complexes and by $\Delta^{\overline{q},(i)}$ their restriction to i-forms. By a standard result on Hilbert complexes, the following L^2-Hodge isomorphism holds (see [Che80, Section 1 and Theorem 5.1], [BL92, Lemma 2.2 and Corollary 2.5])

$$\ker(\Delta^{\overline{q}}) =: \mathscr{H}^\bullet_{(2),\overline{q}}(X,F) \simeq H^\bullet_{(2),\overline{q}}(X,F),\ \overline{q} \in \{\overline{m},\overline{n}\}. \tag{2.21}$$

2.6. **Anti-radial Morse functions.**

Definition 2.3. A continuous function $f : X \to \mathbb{R}$ is called an anti-radial Morse function, if the following two conditions hold:

(a) The restriction $f_{sm} := f_{|X_{sm}}$ is a smooth Morse function.
(b) Near a singular point $p \in \mathrm{Sing}(X)$ the function f has the following normal form in the local coordinates (2.3):

$$f(r,y) = f(p) - \frac{1}{2}r^2. \tag{2.22}$$

For an anti-radial Morse function f, we denote by $\mathrm{Crit}(f_{sm})$ (resp. by $\mathrm{Crit}_k(f_{sm})$, $k = 0,\dots,n$) the set of critical points of f_{sm} (resp. the set of critical points of f_{sm} of index k). Set

$$\mathrm{Crit}(f) := \mathrm{Crit}(f_{sm}) \cup \mathrm{Sing}(X). \tag{2.23}$$

Let g^{TX} be a conical metric on X_{sm}. The vector field $-\nabla f := -\nabla_{g^{TX}} f$ induces a well-defined smooth flow on X_{sm}, which extends to a continuous flow $\Phi : X \times \mathbb{R} \to X$. For $p \in \mathrm{Crit}(f)$, we define the stable resp. unstable set (w.r.t. the flow Φ) by:

$$W^{s/u}(p) := \left\{ x \in X \,\middle|\, \lim_{t \to \pm\infty} \Phi(x,t) = p \right\}. \tag{2.24}$$

By condition (b) in Definition 2.3, all points in $\mathrm{Sing}(X)$ are sources for the negative gradient flow Φ, and hence there are no trajectories of Φ connecting two points in $\mathrm{Sing}(X)$. All intersections of stable and unstable sets lie in X_{sm}. For $p \in \mathrm{Crit}(f)$, $W^{s/u}(p) \cap X_{sm}$ is a smooth manifold.

Definition 2.4. We call a pair (f, g^{TX}) consisting of an anti-radial Morse function and a conical metric

(a) an anti-radial Morse-Smale pair, if the Morse-Smale transversality condition holds for $-\nabla f$, *i.e.* all stable and unstable manifolds w.r.t. the negative gradient flow Φ intersect transversally.

(b) an anti-radial standard Morse-Smale pair, if in addition, for $p \in \mathrm{Crit}_k(f_{sm})$ in local Morse coordinates $x_1, \ldots, x_n$ of an open neighbourhood $U(p)$,

$$(\nabla f)_{|U(p)} = -x_1 \frac{\partial}{\partial x_1} - \ldots - x_k \frac{\partial}{\partial x_k} + x_{k+1} \frac{\partial}{\partial x_{k+1}} + \ldots + x_n \frac{\partial}{\partial x_n}. \tag{2.25}$$

With other words, we assume that the Riemannian metric g^{TX} is the standard Euclidean metric in the Morse coordinates near $\mathrm{Crit}(f_{sm})$.

The existence of anti-radial (standard) Morse-Smale pairs is easy to prove, see [Lud20a, Section 2.8] and the references therein.

3. The Berezin integral formalism on a space with isolated conical singularities

In this section we explain the Berezin integral formalism and the Mathai-Quillen current for a space with isolated conical singularities equipped with an anti-radial Morse function. The main purpose of this study is to define the third term (the right hand side) in the Bismut-Zhang formula (Theorem 5.6) and study, in Section 3.6, its variation with respect to the metrics g^{TX} and g^F.

In Sections 3.1-3.4, for convenience of the reader, we recall basic results related to the Berezin integral formalism and the Mathai-Quillen current. In Section 3.5 we provide some explicit formulas for the Berezin integral formalism and the Mathai-Quillen current on the infinite cone. They will be used in the study of the anomaly formulas in Section 3.6 and Section 7.

3.1. **The Berezin integral.** We shortly recall the definition of the Berezin integral (see [BZ92, Section III]). Let E be an oriented Euclidean vector space of dimension n, and let V be a finite dimensional vector space. Let $e_1, \ldots, e_n$ be an oriented orthonormal basis of E, and let $e^1, \ldots, e^n$ be the corresponding dual basis of E^*. The Berezin integral $\int^B$ is the linear map $\int^B : \Lambda(V^*)\widehat{\otimes}\widehat{\Lambda(E^*)} \longrightarrow \Lambda(V^*)$ characterised by the property that for

$\alpha \in \Lambda(V^*)$ and $\beta \in \Lambda(E^*)$

$$\int^B \alpha\widehat{\beta} = \begin{cases} 0 & \text{if } \deg\beta < n, \\ \frac{(-1)^{\frac{n(n+1)}{2}}}{\pi^{n/2}}\alpha & \text{if } \beta = e^1 \wedge \ldots \wedge e^n. \end{cases} \tag{3.1}$$

In case of a non-oriented Euclidean vector space E with orientation line $o(E)$, the Berezin integral is a map $\int^B : \Lambda(V^*)\widehat{\otimes}\widehat{\Lambda(E^*)} \longrightarrow \Lambda(V^*) \otimes o(E)$.

For an antisymmetric endomorphism C of E, we identify C with an element of $\widehat{\Lambda^2(E^*)}$ given by

$$\dot{C} = \frac{1}{2}\sum_{1\leqslant i,j\leqslant n} \langle e_i, Ce_j\rangle \widehat{e^i} \wedge \widehat{e^j}. \tag{3.2}$$

3.2. **Vector bundles and the Berezin integral formalism: the Mathai-Quillen Thom forms.** Let $\pi_E : E \to X_{sm}$ be a real vector bundle of rank $\mathrm{rk}(E)$. Let g^E be a Euclidean metric on E and ∇^E a Euclidean connection on (E, g^E). We identify the curvature $R^E = (\nabla^E)^2$ with a smooth section $\dot{R}^E$ of the bundle $\Lambda^2(T^*X)\widehat{\otimes}\widehat{\Lambda^2(E^*)}$. By pullback, we get the Euclidean bundle $\pi_E^*(E, g^E)$ with Euclidean connection $\pi_E^*\nabla^E$ and curvature $\pi_E^*R^E$. The connection ∇^E defines a horizontal subspace T^HE of TE such that $TE = T^HE \oplus E$. We denote by $P^E : TE \to E$ the canonical projection and identify E with E^* by the metric g^E. Then P^E, which is a section of $T^*E \otimes E$, can be identified with a section $\dot{P}^E$ of $T^*E\widehat{\otimes}\widehat{E^*}$. Let Y be the generic element of E.

For $T \geqslant 0$, let A_T be the section of $\Lambda(T^*E)\widehat{\otimes}\pi_E^*\widehat{\Lambda(E^*)}$ on E given by

$$A_T := \frac{1}{2}\pi_E^*\dot{R}^E + \sqrt{T}\dot{P}^E + T|Y|^2. \tag{3.3}$$

The Berezin formalism applied to $V = TE$ yields a map from sections of the bundle $\Lambda(T^*E)\widehat{\otimes}\pi_E^*\widehat{\Lambda(E^*)}$ to sections of $\Lambda(T^*E) \otimes \pi_E^*o(E)$.

Definition 3.1. We define the following differential forms on E with values in $\pi_E^*o(E)$:

$$a_T := \int^B \exp(-A_T), T \geqslant 0; \quad b_T := \int^B \frac{\widehat{Y}}{2\sqrt{T}}\exp(-A_T), T > 0. \tag{3.4}$$

We denote by

$$e(E, \nabla^E) := \mathrm{Pf}\left(\frac{R^E}{2\pi}\right) := \int^B \exp\left(-\frac{\dot{R}^E}{2}\right) \tag{3.5}$$

the closed form of degree $\mathrm{rk}(E)$ on X with values in $o(E)$ representing the rational Euler class of E in Chern-Weil theory. Clearly, in case $\mathrm{rk}(E)$ odd, $e(E, \nabla^E) = 0$.

We recall a result of Mathai and Quillen [MQ86, Theorem 6.4] (see also [BZ92, Theorem 3.4]):

Theorem 3.2. *For $T \geqslant 0$ the forms a_T are closed forms of degree $\mathrm{rk}(E)$ and their cohomology class does not depend on T. For $T > 0$ the forms a_T represent the Thom class of E, such that $(\pi_E)_*a_T = 1$. Moreover $a_0 = \pi_E^*e(E, \nabla^E)$. For $T > 0$, the forms b_T have degree $\mathrm{rk}(E) - 1$, and*

$$b_T = -\frac{1}{2T}\iota_Y a_T, \quad \frac{\partial a_T}{\partial T} = -db_T. \tag{3.6}$$

The Mathai-Quillen current has been defined in [MQ86, Section 7] (see also [BZ92, Definition 3.6]):

Definition 3.3. The following well-defined current of degree $\mathrm{rk}(E)-1$ on E with values in $\pi_E^* o(E)$

$$\Psi(E,\nabla^E) := \int_0^\infty b_T dT \tag{3.7}$$

is called the Mathai-Quillen current.

3.3. **The Berezin integral formalism and the Mathai-Quillen current on** TX**.** We will mostly consider the Berezin integral formalism and the Mathai-Quillen current for the tangent space, *i.e.* we choose, with the notation of the previous section, $E = TX$ equipped with the conical Riemannian metric g^{TX} and its Levi-Civita connection ∇^{TX}. We denote by $o(TX)$ the orientation bundle of X. Let $e_1,\dots,e_n$ be an orthonormal basis of TX, and let $e^1,\dots,e^n$ be the corresponding dual basis of T^*X.

We identify the curvature $R^{TX} = (\nabla^{TX})^2$ with a smooth section $\dot{R}^{TX}$ of the bundle $\Lambda^2(T^*X)\widehat{\otimes}\widehat{\Lambda^2(T^*X)}$,

$$\dot{R}^{TX} = \frac{1}{4}\sum_{ijkl}\langle e_k, R^{TX}(e_i,e_j)e_l\rangle e^i\wedge e^j\wedge \widehat{e^k}\wedge\widehat{e^l}. \tag{3.8}$$

Let $f: X\to\mathbb{R}$ be an anti-radial Morse function. We define the following section of $\Lambda(T^*X)\widehat{\otimes}\widehat{\Lambda(T^*X)}$, see [BZ92, Proposition 3.10],

$$B_T := (\nabla f)^* A_T = \frac{\dot{R}^{TX}}{2} + \sqrt{T}\sum_{i=1}^n e^i\wedge \widehat{\nabla^{TX}_{e_i}\nabla} f + T|df|^2. \tag{3.9}$$

The Berezin integral formalism defines a map from smooth sections of $\Lambda(T^*X)\widehat{\otimes}\widehat{\Lambda(T^*X)}$ to smooth sections of $\Lambda(T^*X)\otimes o(TX)$. By [BZ92, Remark 3.8],

$$(\nabla f)^*\Psi(TX,\nabla^{TX}) = \int_0^\infty\left(\int^B \frac{\widehat{df}}{2\sqrt{T}}\exp(-B_T)\right)dT \tag{3.10}$$

is a well-defined locally integrable current on X_{sm} with values in $o(TX)$, smooth on $X\backslash\mathrm{Crit}(f)$. Moreover, by [BZ92, (6.1)], we have the following identity of currents on X_{sm}

$$d(\nabla f)^*\Psi(TX,\nabla^{TX}) = e(TX,\nabla^{TX}) - \sum_{p\in\mathrm{Crit}(f_{sm})}(-1)^{\mathrm{ind}(p)}\delta_p. \tag{3.11}$$

3.4. **Secondary Euler class.** Let g^{TX}, g'^{TX} be two conical metrics on TX. Let $\mathbb{R}\ni l\mapsto g_l^{TX}$ be a family of conical metrics connecting $g^{TX}=g_0^{TX}$ and $g'^{TX}=g_1^{TX}$. We assume that near $p\in\mathrm{Sing}(X)$ the metric g_l^{TX}, $l\in\mathbb{R}$, is of the form $g^{TX}_{l|B_\delta(p)} = dr^2 + r^2 g_l^{TL_p}$, where $\mathbb{R}\ni l\mapsto g_l^{TL_p}$ is a family of Riemannian metrics on L_p and r is the radial coordinate. Let ∇_l^{TX} denote the Levi-Civita connection on (TX,∇_l^{TX}), and R_l^{TX} the curvature of ∇_l^{TX}. Let $\rho: X\times\mathbb{R}\to X$ be the canonical projection. Let $g^{TX,\mathrm{tot}}$ be the metric on ρ^*TX which coincides with g_l^{TX} over $X\times\{l\}$. Let $\nabla^{TX,\mathrm{tot}}$ be the connection over ρ^*TX,

$$\nabla^{TX,\mathrm{tot}} = \rho^*\nabla_l^{TX} + dl\left(\frac{\partial}{\partial l} + \frac{1}{2}(g_l^{TX})^{-1}\frac{\partial g_l^{TX}}{\partial l}\right). \tag{3.12}$$

Then $\nabla^{TX,\mathrm{tot}}$ preserves the metric $g^{TX,\mathrm{tot}}$. The curvature $R^{TX,\mathrm{tot}} = (\nabla^{TX,\mathrm{tot}})^2$ is given by (see [BZ92, (4.51)])

$$R^{TX,\mathrm{tot}} = \rho^* R^{TX}_l + dl\left(\frac{\partial}{\partial l}\nabla^{TX}_l - \frac{1}{2}\left[\nabla^{TX}_l, (g^{TX}_l)^{-1}\frac{\partial g^{TX}_l}{\partial l}\right]\right). \tag{3.13}$$

We denote by

$$\widetilde{e}(TX, \nabla^{TX}_l) := \int_0^1 dl\,\iota_{\partial_l} e(\rho^* TX, \nabla^{TX,\mathrm{tot}}) \in \Omega^{n-1}(X, o(TX)). \tag{3.14}$$

Since the Euler form $e(\rho^*TX, \nabla^{TX,\mathrm{tot}})$ is a closed form on $X \times \mathbb{R}$, we get

$$d\widetilde{e}(TX, \nabla^{TX}_l) = e(TX, \nabla'^{TX}) - e(TX, \nabla^{TX}), \tag{3.15}$$

hence $\widetilde{e}(TX, \nabla^{TX}_l)$ is a secondary Euler class in the sense of Chern-Simons.

By [BZ92, (3.34)] the following identity holds modulo exact currents

$$\Psi(TX, \nabla'^{TX}) - \Psi(TX, \nabla^{TX}) = \pi^*\widetilde{e}(TX, \nabla^{TX}_l), \tag{3.16}$$

where $\pi : TX \to X$ is the canonical projection.

3.5. **The Berezin integral formalism and the Mathai-Quillen current on the infinite cone.** In this section we study in more detail the notions introduced in Sections 3.3-3.4 locally near a singular point $p \in \mathrm{Sing}(X)$, *i.e.* on the punctured infinite cone Z_p equipped with the conical metric $g^{TZ_p} = dr^2 + r^2 g^{TL_p}$. We denote by $o(TZ_p)$ the orientation bundle of Z_p.

Let $e_1, \dots, e_n$ be an orthonormal basis of TZ_p with $e_1 = e_r := \frac{\partial}{\partial r}$. Let $e^1, \dots, e^n$ be the corresponding dual basis of T^*Z_p. We denote by ∇^{TZ_p} the Levi-Civita connection on (TZ_p, g^{TZ_p}). Similarly to (3.8), we identify the curvature $R^{TZ_p} = (\nabla^{TZ_p})^2$ with a smooth section $\dot{R}^{TZ_p}$ of $\Lambda^2(T^*Z_p)\widehat{\otimes}\Lambda^2(\widehat{T^*Z_p})$.

We denote by ∇^{sp} the Levi-Civita connection on $(TZ_p, g^{sp} = dr^2 + g^{TL_p})$. We denote by $e(TZ_p, \nabla^{TZ_p})$ (resp. by $e(TZ_p, \nabla^{sp})$) the Euler form associated with (TZ_p, ∇^{TZ_p}) (resp. with (TZ_p, ∇^{sp})). We denote by $\widetilde{e}(TZ_p, \nabla^{TZ_p}, \nabla^{sp})$ the Chern-Simons class of smooth forms on Z_p with values in $o(TZ_p)$ of degree $n-1$, which is defined modulo exact forms, such that

$$d\widetilde{e}(TZ_p, \nabla^{TZ_p}, \nabla^{sp}) = e(TZ_p, \nabla^{sp}) - e(TZ_p, \nabla^{TZ_p}). \tag{3.17}$$

In case n odd, clearly,

$$e(TZ_p, \nabla^{TZ_p}) = 0,\ e(TZ_p, \nabla^{sp}) = 0 \ \text{ and } \ \widetilde{e}(TZ_p, \nabla^{TZ_p}, \nabla^{sp}) = 0. \tag{3.18}$$

In case n even, due to the flat radial direction on the cone (more precisely from (2.8)) resp. since ∇^{sp} is the product connection, we also have

$$e(TZ_p, \nabla^{TZ_p}) = 0,\ e(TZ_p, \nabla^{sp}) = 0, \tag{3.19}$$

and hence $\widetilde{e}(TZ_p, \nabla^{TZ_p}, \nabla^{sp})$ is a closed form.

We denote by $f^p : Z_p \to \mathbb{R}$, $f^p(r, y) = f(p) - \frac{1}{2}r^2$, the model anti-radial Morse function on the infinite cone Z_p. For $T \geqslant 0$, as in (3.9), we have the following smooth section of $\Lambda(T^*Z_p)\widehat{\otimes}\Lambda(\widehat{T^*Z_p})$ over Z_p,

$$B^p_T := (\nabla f^p)^* A^p_T = \frac{\dot{R}^{TZ_p}}{2} - \sqrt{T}\sum_{i=1}^n e^i \wedge \widehat{e^i} + Tr^2. \tag{3.20}$$

We can define the Mathai-Quillen current on the infinite cone $\Psi(TZ_p, \nabla^{TZ_p})$ as in (3.7). The Berezin formalism gives a map $\int^{B,p}$ from smooth sections of $\Lambda(T^*Z_p)\widehat{\otimes}\widehat{\Lambda(T^*Z_p)}$ to smooth sections of $\Lambda(T^*Z_p) \otimes o(TZ_p)$. Hence

$$(3.21)\qquad \begin{aligned}(\nabla f^p)^*\Psi(TZ_p, \nabla^{TZ_p}) &= \int_0^\infty \int^{B,p} \left(\frac{\widehat{df^p}}{2\sqrt{T}} \exp(-B_T^p) \right) dT \\ &= -\int_0^\infty \int^{B,p} \left(\frac{r\widehat{e^r}}{2\sqrt{T}} \exp(-B_T^p) \right) dT\end{aligned}$$

Since $TZ_p \simeq \mathbb{R} \times TL_p$ and fixing the orientation by $e^r := \frac{\partial}{\partial r}$ on the first factor, we get an identification of the orientation lines $o(TZ_p)$ and $o(TL_p)$. We denote by $e(L_p, \nabla^{TL_p}) \in \Omega^{n-1}(L_p, o(TL_p))$ the Euler form of (TL_p, ∇^{TL_p}), which by the above identification can be seen as a form in $\Omega^{n-1}(Z_p, o(TZ_p))$.

Proposition 3.4. *Let $p \in \mathrm{Sing}(X)$.*

(a) *The form $(\nabla f^p)^*\Psi(TZ_p, \nabla^{TZ_p})$ is an $(n-1)$-form not depending on the radial coordinate. Moreover the following identity holds modulo exact currents,*

$$(3.22)\qquad \eta_p := (\nabla f^p)^*\Psi(TZ_p, \nabla^{TZ_p}) = \begin{cases} -\widetilde{e}(TZ_p, \nabla^{TZ_p}, \nabla^{sp}) & \text{if } n \text{ is even,} \\ \frac{1}{2} e(TL_p, \nabla^{TL_p}) & \text{if } n \text{ is odd.} \end{cases}$$

(b) *Let $h : Z_p \to \mathbb{R}, (r, y) \mapsto h(y)$, be a function not depending on the radial coordinate r. Then the following integral is well-defined and does not depend on $T > 0$:*

$$(3.23)\qquad \int_{Z_p} h(y) \int^{B,p} \exp(-B_T^p).$$

Moreover

$$(3.24)\qquad \int_{Z_p} h(y) \int^{B,p} \exp(-B_T^p) = -\int_{L_p} h(y)\eta_p.$$

(c) *The following integral is well-defined and does not depend on $T > 0$:*

$$(3.25)\qquad \alpha_p := \int_{Z_p} \int^{B,p} \exp(-B_T^p).$$

Moreover

$$(3.26)\qquad \alpha_p = -\int_{L_p} \eta_p = \begin{cases} \int_{L_p} \widetilde{e}(TZ_p, \nabla^{TZ_p}, \nabla^{sp}) & \text{for } n \text{ even,} \\ -\frac{1}{2}\chi(L_p, \mathbb{C}) & \text{for } n \text{ odd.} \end{cases}$$

Proof. For $a > 0$, let h_a be the radial scaling $r \mapsto ar$. We have the following scaling properties (see [Lud20b, Lemma 6.5]):

$$(3.27)\qquad \begin{aligned} h_a^* \int^{B,p} \left(\frac{\widehat{df^p}}{2\sqrt{T}} \exp(-B_T^p) \right) &= a^2 \int^{B,p} \left(\frac{\widehat{df^p}}{2a\sqrt{T}} \exp(-B_{a^2T}^p) \right), \\ h_a^* \int^{B,p} \exp(-B_T^p) &= \int^{B,p} \exp(-B_{a^2T}^p). \end{aligned}$$

(a) Using (3.27) and the change of variables $T \rightsquigarrow Tr^2$ in the integral (3.21) we get the first claim. The second claim has been proved in [Lud20b, Proposition 6.6].

(b) The fact that the integral (3.23) does not depend on $T > 0$ follows from the scaling property (3.27). Well-definedness of the integral at $r = \infty$ follows due to the Gaussian factor $\exp(-Tr^2)$ in $\int^{B,p} \exp(-B_T^p)$; well-definedness at $r = 0$ follows using (2.8), (3.20). For the proof of (3.24), we use a transgression argument very similar to [Lud20b, Theorem 6.7]: By (3.6), for $T > 0$,

$$\frac{\partial}{\partial T}\left(\int^{B,p} \exp(-B_T^p)\right) = -d\left(\int^{B,p}\left(\frac{\widehat{df^p}}{2\sqrt{T}}\exp(-B_T^p)\right)\right). \tag{3.28}$$

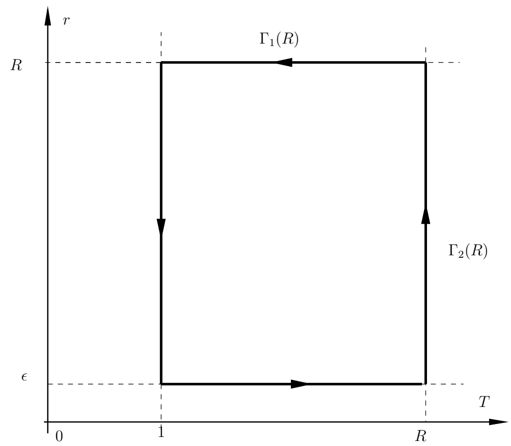
By Theorem 3.2, (3.20) and (3.28), the following n-form on $\mathbb{R}_{>0} \times Z_p = \mathbb{R}_{>0} \times \mathbb{R}_{>0} \times L_p$,

$$\omega := \int^{B,p} \exp(-B_T^p)(r,y) - dT \wedge \left(\int^{B,p}\left(\frac{\widehat{df^p}}{2\sqrt{T}}\exp(-B_T^p)\right)\right)(r,y) \tag{3.29}$$

is closed. Note moreover, that we have

$$dh \wedge \omega = 0, \tag{3.30}$$

since it is a form of top degree $n+1$ on $\mathbb{R}_{>0} \times Z_p$ and its summands do not contain dT resp. $e^r = dr$. We denote by $\Gamma_R \subset \mathbb{R}_{>0} \times \mathbb{R}_{>0}$ the closed contour depicted below, and by $\widetilde{\Gamma_R} := \Gamma_R \times L_p \subset \mathbb{R}_{>0} \times Z_p$.



By Stokes' Theorem, closedness of the form ω, and (3.30) we have

$$\begin{aligned}
0 &= \int_{\mathrm{int}(\widetilde{\Gamma_R})} (hd\omega + dh \wedge \omega) = \int_{\mathrm{int}(\widetilde{\Gamma_R})} d(h\omega) = \int_{\widetilde{\Gamma_R}} h(y)\omega \\
&= -\int_{[1,R]\times\{\epsilon\}\times L_p} h(y) dT \wedge \left(\int^{B,p}\left(\frac{\widehat{df^p}}{2\sqrt{T}}\exp(-B_T^p)\right)\right)(\epsilon, y) \\
&\quad - \int_{\{1\}\times[\epsilon,R]\times L_p} h(y)\int^{B,p} \exp(-B_1^p)(r,y) + \int_{(\Gamma_1(R)\cup\Gamma_2(R))\times L_p} h(y)\omega.
\end{aligned} \tag{3.31}$$

Note that for $\epsilon > 0$ resp. $T > 0$, due to the Gaussian factor $\exp(-Tr^2)$ in $\exp(-B_T^p)$, the first two integrals on the right hand side of (3.31) are well-defined as $R \to \infty$. Moreover $\omega_{|\Gamma_1(R)\times L_p} = \mathcal{O}(\exp(-R^2))$ and $\omega_{|\Gamma_2(R)\times L_p} = \mathcal{O}(\exp(-\epsilon^2 R))$. Taking the limit $R \to \infty$ in

(3.31) gives

$$\begin{aligned}\int_{[1,\infty)\times\{\epsilon\}\times L_p} dT\wedge h(y)\left(\int^{B,p}\left(\frac{\widehat{df^p}}{2\sqrt{T}}\exp(-B^p_T)\right)\right)(\epsilon,y)\\ = -\int_{\{1\}\times[\epsilon,\infty)\times L_p} h(y)\int^{B,p}\exp(-B^p_1)(r,y).\end{aligned} \tag{3.32}$$

Using the scaling property (3.27) one has

$$\begin{aligned}\int_{[1,\infty)\times\{\epsilon\}\times L_p} dT\wedge h(y)\left(\int^{B,p}\left(\frac{\widehat{df^p}}{2\sqrt{T}}\exp(-B^p_T)\right)\right)(\epsilon,y)\\ = \int_{[\epsilon^2,\infty)\times\{1\}\times L_p} dT\wedge h(y)\left(\int^{B,p}\left(\frac{\widehat{df^p}}{2\sqrt{T}}\exp(-B^p_T)\right)\right)(1,y).\end{aligned} \tag{3.33}$$

Using (3.32) and (3.33) and taking the limit $\epsilon\to 0$, we get (3.24). (c) The statement is a consequence of part (a) and (b). □

Remark 3.5. Note that, for $p\in \mathrm{Crit}(f_{sm})$, we get

$$\alpha_p = (-1)^{\mathrm{ind}(p)}. \tag{3.34}$$

Starting with two conical metrics g^{TZ_p}, g'^{TZ_p} and a family $\mathbb{R}\ni l\mapsto g_l^{TZ_p}$ connecting them, we can define all the notions introduced in Section 3.4: $\rho_p: Z_p\times\mathbb{R}\to Z_p$, $g^{TZ_p,\mathrm{tot}}$, $\nabla^{TZ_p,\mathrm{tot}}$, $\widetilde{e}(TZ_p,\nabla_l^{TZ_p})$, etc. We hereby apply the formalism introduced in Section 3.2 to the Euclidean vector bundle $(\rho_p^*T^*Z_p, g^{TZ_p,\mathrm{tot}})$ over $Z_p\times\mathbb{R}$, with Euclidean connection $\nabla^{TZ_p,\mathrm{tot}}$.

We denote by $\widetilde{\nabla f^p} = -r\frac{\partial}{\partial r}$ the section on $\rho_p^*TZ_p$ induced from ∇f^p. Set

$$\widetilde{e}_\Psi(Z_p,\nabla_l^{TZ_p}) := \int_0^1 dl\,\iota_{\partial_l}(\widetilde{\nabla f^p})^*\Psi(\rho^*TZ_p,\nabla^{TZ_p,\mathrm{tot}})\in\Omega^{n-2}(Z_p,o(TZ_p)). \tag{3.35}$$

We define, for $T\geqslant 0$, the following smooth section of $\Lambda(T^*(Z_p\times\mathbb{R}))\widehat{\otimes}\Lambda(\widehat{\rho_p^*T^*Z_p})$,

$$\widetilde{B}^p_T := (\widetilde{\nabla f^p})^*A^p_T = \frac{\dot{R}^{TZ_p,\mathrm{tot}}}{2} - \sqrt{T}\sum_{i=1}^n e^i\wedge\widehat{e^i} + Tr^2, \tag{3.36}$$

and

$$e_T(\rho_p^*TZ_p,\nabla^{TZ_p,\mathrm{tot}}) := \int^{B,p}\exp\left(-\widetilde{B}^p_T\right)\in\Omega^n(Z_p\times\mathbb{R}, o(\rho_p^*TZ_p)); \tag{3.37}$$

in particular $e_0(\rho_p^*TZ_p,\nabla^{TZ_p,\mathrm{tot}}) = e(\rho_p^*TZ_p,\nabla^{TZ_p,\mathrm{tot}})$.

Proposition 3.6.

(a) *We have the following refinement of* (3.16)*, locally near* p,

$$(\nabla f^p)^*\Psi(TZ_p,\nabla^{TZ_p}) - (\nabla f^p)^*\Psi(TZ_p,\nabla'^{TZ_p}) = -\widetilde{e}(TZ_p,\nabla_l^{TZ_p}) - d\widetilde{e}_\Psi(Z_p,\nabla_l^{TZ_p}). \tag{3.38}$$

(b) *The form*

$$(\widetilde{\nabla f^p}^*)\Psi(\rho^*TZ_p,\nabla^{TZ_p,\mathrm{tot}}) \tag{3.39}$$

is an $(n-1)$-form on $Z_p \times \mathbb{R}$ not depending on the radial coordinate r. Moreover, we have the following identity

(3.40)
$$\int_{Z_p} \theta(F, g^F) \wedge \iota_{\partial_l} e_1(\rho^* TZ_p, \nabla^{TZ,\mathrm{tot}}) = -\int_{L_p} \theta(F, g^F) \wedge \iota_{\partial_l} (\widetilde{\nabla f^p})^* \Psi(\rho^* TZ_p, \nabla^{TZ_p,\mathrm{tot}}).$$

Proof. (a) Since $\widetilde{\nabla f^p}$ is nowhere vanishing on $Z_p \times \mathbb{R}$, we get from [BZ92, Remark 3.8]

$$d^{Z_p\times\mathbb{R}} (\widetilde{\nabla f^p})^* \Psi(\rho_p^* TZ_p, \nabla^{TZ_p,\mathrm{tot}}) = e(\rho_p^* TZ_p, \nabla^{TZ_p,\mathrm{tot}}). \tag{3.41}$$

By comparing the coefficients of dl in (3.41), and integrating over $l \in [0,1]$, we get (3.38).

(b) Using (2.7), (2.8) and (3.13) we have

$$R^{TZ_p,\mathrm{tot}}(e_r, \ ) = 0. \tag{3.42}$$

For $a > 0$, let again h_a be the radial scaling $r \to ar$; then using (3.42), we have $h_a^* \dot{R}^{TZ_p,\mathrm{tot}} = \dot{R}^{TZ_p,\mathrm{tot}}$. Hence, we get the following scaling property

$$h_a^* \int^{B,p} \left(\frac{\widehat{\widetilde{df^p}}}{2\sqrt{T}} \exp(-\widetilde{B}_T^p) \right) = a^2 \int^{B,p} \left(\frac{\widehat{\widetilde{df^p}}}{2a\sqrt{T}} \exp(-\widetilde{B}_{a^2T}^p) \right). \tag{3.43}$$

Using (3.43) we can now argue as in the proof of Proposition 3.4 (a) to prove the first claim.

In view of (3.42) and the definition of the Berezin integral, the n-form on $Z_p \times \mathbb{R}$

$$e(\rho_p^* TZ_p, \nabla^{TZ_p,\mathrm{tot}}) = \int^{B,p} \exp\left(-\frac{\dot{R}^{TZ_p,\mathrm{tot}}}{2} \right) \tag{3.44}$$

does not contain e^r. Since, as seen in Section 2.2, the form $\theta(F, g^F)$ on Z_p does not depend on the radial coordinate, the form $\theta(F, g^F) \wedge \iota_{\partial l} e(\rho_p^* TZ_p, \nabla^{TZ_p,\mathrm{tot}})$ is an n-form on Z_p (depending on the parameter l) not containing e^r.

Hence

$$\theta(F, g^F) \wedge \iota_{\partial l} e(\rho_p^* TZ_p, \nabla^{TZ_p,\mathrm{tot}}) = 0. \tag{3.45}$$

From (3.6) we get

$$\partial_T e_T(\rho_p^* TZ_p, \nabla^{TZ_p,\mathrm{tot}}) = d^{Z_p\times\mathbb{R}} \int^{B,p} \frac{r\widehat{e^r}}{2\sqrt{T}} \exp\left(-\widetilde{B}_T^p\right). \tag{3.46}$$

Using (3.43) and (3.46) we get

$$\begin{aligned} e_1(\rho_p^* TZ_p, \nabla^{TZ_p,\mathrm{tot}}) &- e(\rho_p^* TZ_p, \nabla^{TZ_p,\mathrm{tot}}) \\ &= d^{Z_p\times\mathbb{R}} \left(\int_0^1 dT \left(\int^{B,p} \frac{r\widehat{e^r}}{2\sqrt{T}} \exp\left(-\widetilde{B}_T^p\right) \right) (r, y) \right) \\ &= d^{Z_p\times\mathbb{R}} \left(\int_0^{r^2} dT \left(\int^{B,p} \frac{r\widehat{e^r}}{2\sqrt{T}} \exp\left(-\widetilde{B}_T^p\right) \right) (1, y) \right) \\ &=: d^{Z_p\times\mathbb{R}} (dl \wedge \gamma_1(l, r, y) + \gamma_2(l, r, y)). \end{aligned} \tag{3.47}$$

Using (3.42) and the definition of the Berezin integral, one sees that γ_1 (resp. γ_2) is a form of degree $n-2$ (resp. of degree $n-1$) on $Z_p \times \mathbb{R}$ not containing e^r. Therefore

$$\iota_{\partial_l} d^{Z_p\times\mathbb{R}} \left(\int_0^1 dT \left(\int^{B,p} \frac{r\widehat{e^r}}{2\sqrt{T}} \exp\left(-\widetilde{B}^p_T\right)\right)(r,y)\right) = -d^{L_p}\gamma_1 - e^r \wedge \frac{\partial \gamma_1}{\partial r} + \frac{\partial \gamma_2}{\partial l}. \tag{3.48}$$

Since $\theta(F, g^F)$ does not depend on the radial coordinate and is a closed form, we get using (3.43), (3.45), (3.47), (3.48) and Stokes' Theorem

$$\begin{aligned}\int_{Z_p} \theta(F,g^F) \wedge \iota_{\partial_l} e_1(\rho_p^* TZ_p, \nabla^{TZ_p,\mathrm{tot}}) &= -\int_{Z_p} \theta(F,g^F) \wedge e^r \wedge \frac{\partial\gamma_1}{\partial r} \\ &= -\int_{L_p} \theta(F,g^F) \wedge \iota_{\partial_l} (\widetilde{\nabla f^p})^* \Psi(\rho^* TZ_p, \nabla^{TZ_p,\mathrm{tot}}).\end{aligned} \tag{3.49}$$

□

3.6. **Variation formula for the integral** $-\int_X \theta(F,g^F)(\nabla f)^*\Psi(TX,\nabla^{TX})$**.** The current $(\nabla f)^*\Psi(TX,\nabla^{TX})$ has been defined in Section 3.3 and is a locally integrable current with values in $o(TX)$, smooth on $X\backslash \mathrm{Crit}(f)$. We have

$$\theta(F,g^F)(\nabla f)^*\Psi(TX,\nabla^{TX}) = 0 \text{ near } \mathrm{Sing}(X), \tag{3.50}$$

since it is a form of top degree not containing e^r. The right hand side in the Bismut-Zhang formula in Theorem 5.6 is given by

$$-\int_X \theta(F,g^F)(\nabla f)^*\Psi(TX,\nabla^{TX}). \tag{3.51}$$

The aim of this section is to study the dependence of the integral (3.51) with respect to the metrics g^F and g^{TX}.

We denote by $g^{'TX}$ a second conical metric on X, and by $g^{'F}$ a second metric on F satisfying the assumption explained in Section 2.2. The Levi-Civita connection associated to $g^{'TX}$ is denoted by $\nabla^{'TX}$ and we denote by $\nabla' f$ the gradient vector field of f with respect to the conical metric $g^{'TX}$. Let $\mathbb{R} \ni l \mapsto g_l^{TX}$ be a family of conical metrics on X connecting g^{TX}, $g^{'TX}$ as explained in Section 3.4.

For $\epsilon > 0$ small enough, we denote by $X_\epsilon := X\backslash(\cup_{p\in\mathrm{Sing}(X)} B_\epsilon(p))$. We identify the orientation bundle of X_ϵ and the orientation bundle of ∂X_ϵ using the Stokes convention. Note that the Stokes convention and the convention on orientation at the beginning of Proposition 3.4 differ by a sign.

The next proposition generalises [BZ92, Theorem 6.1] to our singular setting:

Proposition 3.7. *The following identity holds:*

$$\int_X \theta(F,g^F)(\nabla f)^*\Psi(TX,\nabla^{TX}) = \int_X \theta(F,g^F)(\nabla' f)^*\Psi(TX,\nabla^{TX}). \tag{3.52}$$

Proof. We denote by $\nabla_l f$ the gradient of f w.r.t. the metric g_l^{TX}; we have for $l \in [0,1]$,

$$\nabla_l f = -r\partial_r \text{ loc. near } \mathrm{Sing}(X). \tag{3.53}$$

We define the homotopy

$$H : X \times [0,1] \to TX, \quad H_l = \nabla_l f. \tag{3.54}$$

Then

$$\begin{aligned}(\nabla f)^*\Psi(TX,\nabla^{TX}) - (\nabla' f)^*\Psi(TX,\nabla^{TX})\\ = d\left(\int_0^1 dl\iota_{\partial_l}H^*\Psi(TX,\nabla^{TX})\right) + \int_0^1 dl\iota_{\partial_l}H^*d\Psi(TX,\nabla^{TX})\end{aligned} \tag{3.55}$$

By (3.11) and (3.53), the last term on the right hand side of (3.55) vanishes. By (3.53), we also have $\iota_{\partial_l}H^*\Psi(TX,\nabla^{TX}) = 0$ near $\mathrm{Sing}(X)$. Therefore we get from (3.55), that $(\nabla f)^*\Psi(X,\nabla^{TX}) - (\nabla' f)^*\Psi(X,\nabla^{TX})$ is an exact current and can be written as

$$(\nabla f)^*\Psi(X,\nabla^{TX}) - (\nabla' f)^*\Psi(X,\nabla^{TX}) = d\sigma, \tag{3.56}$$

for a form σ which vanishes near $\mathrm{Sing}(X)$. Since $\theta(F,g^F)$ is closed, we get using (3.56) and Stokes' Theorem, for $\epsilon > 0$ small enough,

$$\begin{aligned}\int_{X_\epsilon}\theta(F,g^F)(\nabla f)^*\Psi(TX,\nabla^{TX}) - \int_{X_\epsilon}\theta(F,g^F)(\nabla' f)^*\Psi(TX,\nabla^{TX}) =\\ = \int_{X_\epsilon}\theta(F,g^F)\wedge d\sigma = -\int_{\partial X_\epsilon}\theta(F,g^F)\wedge\sigma = 0.\end{aligned} \tag{3.57}$$

The claim of the proposition follows by taking the limit $\epsilon \searrow 0$ in (3.57). □

Recall that η_p, $\widetilde{e}_\Psi(Z_p,\nabla_l^{TZ_p})$, $p \in \mathrm{Sing}(X)$, have been defined in (3.22), (3.35). The following theorem generalises [BZ92, Theorem 6.3] to our situation

Theorem 3.8. *The following identity holds*

$$\begin{aligned}&\int_X \theta(F,g^F)(\nabla f)^*\Psi(TX,\nabla^{TX}) - \int_X \theta(F,g'^F)(\nabla' f)^*\Psi(TX,\nabla'^{TX})\\ &= \int_X \log\left(\frac{\|\ \|'^2_{\det F}}{\|\ \|^2_{\det F}}\right)e(TX,\nabla^{TX}) - \sum_{p\in\mathrm{Crit}(f_{sm})}(-1)^{\mathrm{ind}(p)}\log\left(\frac{\|\ \|'^2_{\det F_p}}{\|\ \|^2_{\det F_p}}\right)\\ &- \int_X \theta(F,g'^F)\widetilde{e}(TX,\nabla_l^{TX})\\ &+ \sum_{p\in\mathrm{Sing}(X)}\left(\int_{L_p}\log\left(\frac{\|\ \|'^2_{\det F}}{\|\ \|^2_{\det F}}\right)\wedge\eta_p - \int_{L_p}\theta(F,g'^F)\wedge\widetilde{e}_\Psi(Z_p,\nabla_l^{TZ_p})\right).\end{aligned} \tag{3.58}$$

Remark 3.9. (a) The only difference between (3.58) and the corresponding smooth formula in [BZ92, Theorem 6.3] are the last two terms on the right hand side, which are the contribution of the singularities of X.
(b) Note that the two integrals over X on the right hand side of (3.58) are well-defined: The first integrand vanishes near $\mathrm{Sing}(X)$ by (3.18), (3.19), the second integrand vanishes near $\mathrm{Sing}(X)$ by (3.14), (3.45).

Proof. We have, by definition of $\theta(F,g^F)$,

$$\theta(F,g^F) - \theta(F,g'^F) = -d\log\left(\frac{\|\ \|'^2_{\det F}}{\|\ \|^2_{\det F}}\right). \tag{3.59}$$

Using the identity of currents (3.11), as well as (3.18), (3.19), (3.22), (3.50), (3.59) and Stokes' Theorem, we get

$$
\begin{aligned}
&\int_X (\theta(F,g^F)-\theta(F,g'^F))(\nabla f)^*\Psi(TX,\nabla^{TX}) \\
&\quad= \int_{X_\epsilon} (\theta(F,g^F)-\theta(F,g'^F))(\nabla f)^*\Psi(TX,\nabla^{TX}) \\
&\quad= \int_{X_\epsilon} \log\left(\frac{\|\ \|'^2_{\det F}}{\|\ \|^2_{\det F}}\right) e(TX,\nabla^{TX}) - \sum_{p\in \mathrm{Crit}(f_{sm})} (-1)^{\mathrm{ind}(p)} \log\left(\frac{\|\ \|'^2_{\det F_p}}{\|\ \|^2_{\det F_p}}\right) \\
&\qquad - \int_{\partial X_\epsilon} \log\left(\frac{\|\ \|'^2_{\det F}}{\|\ \|^2_{\det F}}\right) \wedge (\nabla f)^*\Psi(TX,\nabla^{TX}) \\
&\quad= \int_X \log\left(\frac{\|\ \|'^2_{\det F}}{\|\ \|^2_{\det F}}\right) e(TX,\nabla^{TX}) - \sum_{p\in \mathrm{Crit}(f_{sm})} (-1)^{\mathrm{ind}(p)} \log\left(\frac{\|\ \|'^2_{\det F_p}}{\|\ \|^2_{\det F_p}}\right) \\
&\qquad + \sum_{p\in \mathrm{Sing}(X)} \int_{L_p} \log\left(\frac{\|\ \|'^2_{\det F}}{\|\ \|^2_{\det F}}\right) \wedge \eta_p.
\end{aligned} \tag{3.60}
$$

Using Proposition 3.6 (a), (3.16), $d\theta(F,g^F)=0$ and Stokes' Theorem, we get

$$
\begin{aligned}
&\int_X \theta(F,g'^F)\Big((\nabla f)^*\Psi(TX,\nabla^{TX}) - (\nabla f)^*\Psi(TX,\nabla'^{TX})\Big) \\
&\quad= -\lim_{\epsilon\to 0}\int_{X_\epsilon} \theta(F,g'^F)\widetilde{e}(TX,\nabla_l^{TX}) - \sum_{p\in\mathrm{Sing}(X)} \int_{L_p} \theta(F,g'^F)\wedge \widetilde{e}_\Psi(Z_p,\nabla_l^{TZ_p}) \\
&\quad= -\int_X \theta(F,g'^F)\widetilde{e}(TX,\nabla_l^{TX}) - \sum_{p\in\mathrm{Sing}(X)} \int_{L_p} \theta(F,g'^F)\wedge \widetilde{e}_\Psi(Z_p,\nabla_l^{TZ_p}).
\end{aligned} \tag{3.61}
$$

The claim of Theorem 3.8 follows from Proposition 3.7, (3.60) and (3.61). □

4. Witten deformation for singular spaces with isolated conical singularities using anti-radial Morse functions

An important role in the extension of the Cheeger-Müller theorem by Bismut and Zhang [BZ92] is played by the Witten deformation. The Witten deformation is a technique proposed by [Wit82]. Rigorous proofs have been given by Helffer and Sjöstrand in [HS85] using semi-classical analysis. In [BZ94, Section 6] Bismut and Zhang gave a different proof of the hard part of Witten's program using a result of Laudenbach [Lau92]. In [ÁC17] and in [Lud17b] the easy part of the Witten deformation has been generalised to singular spaces with iterated conical singularities and anti-radial Morse functions. The hard part of the Witten deformation, which is a crucial ingredient here, has been generalised in [Lud17b] to spaces with isolated conical singularities and anti-radial Morse functions.

Section 4 is organised as follows: In Section 4.1 we recall from [Lud17b] the main ideas of the Witten deformation for singular spaces with isolated conical singularities and anti-radial Morse functions. Most of the proofs of the intermediate results of Section 5.6 consist of two steps: localisation and a local computation near $p \in \mathrm{Crit}(f)$. In Sections 4.2–4.4 we give all ingrediences needed for the second step for $p \in \mathrm{Sing}(X)$. To this purpose, we generalise the study of the spectral properties and the local index techniques for the local model Witten Laplacian in [Lud20b] and [Lud20a, Section 4] to the case where the space X does no longer satisfy the Witt and spectral Witt condition assumed in [Lud20a] and the bundle F is not necessarily unitary.

4.1. **Witten deformation. Spectral Gap Theorem.** Let $f : X \to \mathbb{R}$ be an anti-radial Morse function and $T \geqslant 0$. The de Rham complex $(\Omega_c^\bullet(X,F), d_c, \langle\ ,\ \rangle)$ can be deformed by deforming the differential d_c via

$$d_{T,c} := e^{-Tf} d_c e^{Tf}. \tag{4.1}$$

The complex $(\Omega_c^\bullet(X,F), d_{T,c}, \langle\ ,\ \rangle)$ admits a maximal and a minimal extension denoted by $(\widetilde{\mathcal{C}}^\bullet_{T,\max/\min}, d_{T,\max/\min}, \langle\ ,\ \rangle)$. The Hilbert complex $(\widetilde{\mathcal{C}}^\bullet_{T,\max/\min}, d_{T,\max/\min}, \langle\ ,\ \rangle)$ still computes $H^\bullet_{(2),\overline{m}}(X,F)$ (resp. $H^\bullet_{(2),\overline{n}}(X,F)$). We denote by $\delta_{T,c}$ the (formal) adjoint of $d_{T,c}$ w.r.t. the L^2-inner product $\langle\ ,\ \rangle$. We again have the first order operator $\widetilde{D}_T^{\overline{m}} := d_{T,\max} + \delta_{T,\min}$ (resp. $\widetilde{D}_T^{\overline{n}} := d_{T,\min} + \delta_{T,\max}$) associated to the Hilbert complex $(\widetilde{\mathcal{C}}^\bullet_{T,\max}, d_{T,\max}, \langle\ ,\ \rangle)$ (resp. $(\widetilde{\mathcal{C}}^\bullet_{T,\min}, d_{T,\min}, \langle\ ,\ \rangle)$). We denote by $\widetilde{\Delta}_T^{\overline{q}} = (\widetilde{D}_T^{\overline{q}})^2$, $\overline{q} \in \{\overline{m}, \overline{n}\}$, the second order operators associated to the two Hilbert complexes.

We denote by $\widehat{c}(\nabla f)$ the Clifford multiplication acting on sections α of $\Lambda(T^*X)\otimes F$ by

$$\widehat{c}(\nabla f)\alpha = df \wedge \alpha \ + \nabla f \lrcorner\, \alpha. \tag{4.2}$$

We denote by $L_{\nabla f}$ the Lie derivative in direction of ∇f and by $L^*_{\nabla f}$ its adjoint with respect to the L^2-inner product $\langle\ ,\ \rangle$. The operator $L_{\nabla f} + L^*_{\nabla f}$ acting on $\mathrm{L}^2(\Lambda(T^*X)\otimes F)$ is a bounded operator of order 0. This follows from the smooth theory combined with the fact that on L^2-sections of $\Lambda(T^*X)\otimes F$ with support in a neighbourhood of $\mathrm{Sing}(X)$, the operator $L_{\nabla f} + L^*_{\nabla f}$ acts simply as the number operator $(n-2N)$, where N denotes the multiplication by the form degree, see Section 4.2.2 for more details. Also, from the normal form (2.22) of f near $\mathrm{Sing}(X)$, we have that the gradient ∇f and the Hessian of f are bounded.

We have, for $\overline{q} \in \{\overline{m}, \overline{n}\}$ (see [BZ92, Proposition 5.5], [Lud17b, Propositions 3.8 and 5.1])

$$\begin{aligned} \widetilde{D}_T^{\overline{q}} &= D^{\overline{q}} + T\widehat{c}(\nabla f), \quad \mathrm{dom}(\widetilde{D}_T^{\overline{q}}) = \mathrm{dom}(D^{\overline{q}}), \\ \widetilde{\Delta}_T^{\overline{q}} &= \Delta^{\overline{q}} + T(L_{\nabla f} + L^*_{\nabla f}) + T^2|\nabla f|^2, \quad \mathrm{dom}(\widetilde{\Delta}_T^{\overline{q}}) = \mathrm{dom}(\Delta^{\overline{q}}). \end{aligned} \tag{4.3}$$

Using the local model form of the Witten Laplacian $\widetilde{\Delta}_T^{\overline{q}}$ near $p \in \mathrm{Sing}(X)$ (see (4.21)), one can show inductively, that for $\overline{q} \in \{\overline{m}, \overline{n}\}$, $l \in \mathbb{N}$, $T \geqslant 0$:

$$\mathrm{dom}((\widetilde{\Delta}_T^{\overline{q}})^l) = \mathrm{dom}((\Delta^{\overline{q}})^l). \tag{4.4}$$

There is a second, equivalent way of describing the Witten deformation: The de Rham complex $(\Omega_c^\bullet(X,F), d_c, \langle\ ,\ \rangle)$ can also be deformed by deforming the L^2-inner product $\langle\ ,\ \rangle$ via

$$\langle \alpha, \beta\rangle_T := \int_X \langle \alpha, \beta\rangle_{\Lambda(T^*X)\otimes F}(x) e^{-2Tf(x)} d\mathrm{vol}_X(x); \tag{4.5}$$

here $d\mathrm{vol}_X$ denotes the Riemannian volume form on (X, g^{TX}).

The deformed complex $(\Omega_c^\bullet(X,F), d_c, \langle\ ,\ \rangle_T)$ also admits a maximal and minimal extension into a Hilbert complex $(\mathcal{C}^\bullet_{T,\max/\min}, d_{\max/\min}, \langle\ ,\ \rangle_T)$. We denote by $\delta'_{T,c}$ the (formal) adjoint of d_c w.r.t. the twisted L^2-inner product $\langle\ ,\ \rangle_T$ and by $\delta'_{T,\max/\min}$ its maximal resp. its minimal extension. The first order operator associated to the Hilbert complex $(\mathcal{C}^\bullet_{T,\max/\min}, d_{\max/\min}, \langle\ ,\ \rangle_T)$ (resp. $(\mathcal{C}^\bullet_{T,\max/\min}, d_{\max/\min}, \langle\ ,\ \rangle_T)$) is given by

$$D_T^{\overline{m}} = d_{\max} + \delta'_{T,\min} \text{ (resp. } D_T^{\overline{n}} = d_{\min} + \delta'_{T,\max}). \tag{4.6}$$

We denote by $\Delta_T^{\overline{q}} := (D_T^{\overline{q}})^2$, $\overline{q} \in \{\overline{m}, \overline{n}\}$ the second order self-adjoint operator associated to the two Hilbert complexes.

There are natural isomorphisms of Hilbert complexes

$$(\widetilde{\mathcal{C}}^\bullet_{T,\max/\min}, d_{T,\max/\min}, \langle\ ,\ \rangle) \longrightarrow (\mathcal{C}^\bullet_{T,\max/\min}, d_{\max/\min}, \langle\ ,\ \rangle_T),\ \omega \mapsto e^{Tf}\omega. \tag{4.7}$$

Hence, for $\overline{q} \in \{\overline{m}, \overline{n}\}$,

$$\widetilde{D}_T^{\overline{q}} = e^{-Tf} D_T^{\overline{q}} e^{Tf} \text{ and } \widetilde{\Delta}_T^{\overline{q}} = e^{-Tf} \Delta_T^{\overline{q}} e^{Tf}. \tag{4.8}$$

We denote by $(\mathcal{S}^\bullet_{T,\max/\min}, d_{\max/\min}, \langle\ ,\ \rangle_T)$ the Witten complex, *i.e.* the subcomplex of $(\mathcal{C}^\bullet_{T,\max/\min}, d_{\max/\min}, \langle\ ,\ \rangle_T)$ generated by the eigenforms of $\Delta_T^{\overline{m}/\overline{n}}$ to eigenvalues in $[0,1]$. For $k = 0, \dots, n$, we denote by $c_k(f_{sm}) := \#\mathrm{Crit}_k(f_{sm})$ and by

$$\begin{aligned} c_k^{\overline{q}}(f) &:= c_k^{\overline{q}}(f, F) := \mathrm{rk}(F) \cdot c_k(f_{sm}) + \sum_{p \in \mathrm{Sing}(X)} IH^k_{\overline{q}}(cL_p, L_p, F) \\ &= \begin{cases} \mathrm{rk}(F) \cdot c_k(f_{sm}) + \displaystyle\sum_{p \in \mathrm{Sing}(X)} b^{k-1}(L_p, F_{L_p}) & \text{for } k \geqslant n - \overline{q}, \\ \mathrm{rk}(F) \cdot c_k(f_{sm}) & \text{else.} \end{cases} \end{aligned} \tag{4.9}$$

In [Lud17b, Theorem I] the following Spectral Gap Theorem has been proved, which we recall here for convenience of the reader:

Theorem 4.1. (Spectral Gap Theorem) *Let (X, g^{TX}) be a space with isolated conical singularities and $\overline{q} \in \{\overline{m}, \overline{n}\}$. Let $f : X \to \mathbb{R}$ be an anti-radial Morse function on X. We assume that the Riemannian metric g^{TX} is Euclidean in the Morse coordinates in an open neighbourhood of* $\mathrm{Crit}(f_{sm})$.

(a) *There exist $C_1 > 0$, $C_2 > 0$, $C_3 > 0$ and $T_0 > 0$, such that for $T \geqslant T_0$:*

$$\mathrm{Spec}(\widetilde{\Delta}_T^{\overline{q}}) \cap (C_1 e^{-C_2 T}, C_3 T) = \varnothing. \tag{4.10}$$

(b) *For $k = 0, \dots, n$, and $T \geqslant T_0$,*

$$\dim \mathcal{S}^k_{T,\max} = c_k^{\overline{m}}(f) \text{ and } \dim \mathcal{S}^k_{T,\min} = c_k^{\overline{n}}(f). \tag{4.11}$$

4.2. **The model Witten Laplacian $\Delta_T^{p,\overline{q}}$, $p \in \mathrm{Sing}(X)$, $\overline{q} \in \{\overline{m}, \overline{n}\}$, $T \geqslant 0$.** We now study the situation locally near $p \in \mathrm{Sing}(X)$, *i.e.* on the punctured infinite cone Z_p. As in Section 3.5, we denote by $f^p : cL_p \longrightarrow \mathbb{R}, (r, y) \mapsto f(p) - \frac{1}{2}r^2$ the model anti-radial Morse function on the infinite cone.

4.2.1. *A useful unitary transformation.* Let $\pi : Z_p \simeq \mathbb{R}_{>0} \times L_p \to L_p$ be the projection into the second factor. We denote by $\mathrm{L}^2(\Lambda^k(T^*L_p) \otimes F_{L_p})$ the space of L^2-sections of $\Lambda^k(T^*L_p) \otimes F_{L_p}$ with respect to the L^2-metric induced from g^{TL_p} and $g^{F_{L_p}}$. We denote by $\mathrm{L}^2(\Lambda^k(T^*Z_p) \otimes F)$ the space of L^2-sections of $\Lambda^k(T^*Z_p) \otimes F$ with respect to the L^2-metric $\langle\ ,\ \rangle$ induced from g^{TZ_p} and g^F.

For $k = 0, \ldots n$, the bijective maps

$$\begin{aligned} U_k : \ & C_0^\infty(\mathbb{R}_+, \Omega^{k-1}(L_p, F_{L_p}) \oplus \Omega^k(L_p, F_{L_p})) \longrightarrow \Omega_0^k(Z_p, F) \\ & (\phi_{k-1}, \phi_k) \mapsto r^{k-1-m/2}\pi^*\phi_{k-1} \wedge dr + r^{k-m/2}\pi^*\phi_k, \end{aligned} \tag{4.12}$$

extend to unitary maps

$$U_k : \mathrm{L}^2\big(\mathbb{R}_+, \mathrm{L}^2((\Lambda^{k-1}(T^*L_p) \oplus \Lambda^k(T^*L_p)) \otimes F_{L_p})\big) \longrightarrow \mathrm{L}^2\big(\Lambda^k(T^*Z_p) \otimes F\big). \tag{4.13}$$

We denote by d_{L_p} the exterior derivative on $\Omega^\bullet(L_p, F_{L_p})$, and by δ_{L_p} its adjoint with respect to the L^2-metric on $\Omega^\bullet(L_p, F_{L_p})$ induced from the metrics g^{TL_p} and $g^{F_{L_p}}$. We denote by S_p the following self-adjoint elliptic operator on the link L_p:

$$S_p := \begin{pmatrix} c_0 & \delta_{L_p} & 0 & \ldots & 0 \\ d_{L_p} & c_1 & & \ddots & \vdots \\ 0 & & & & 0 \\ \vdots & \ddots & & c_{m-1} & \delta_L \\ 0 & \ldots & 0 & d_{L_p} & c_m \end{pmatrix} \text{ where } c_k := (-1)^k \left(k - \frac{m}{2}\right). \tag{4.14}$$

For the Laplace operator $\boldsymbol{\Delta}^{p,\mathrm{ev/odd}}$ on the infinite cone acting on compactly supported even (resp. odd) forms we have (see e.g. [BS87, Section 5]):

$$U^{-1}\boldsymbol{\Delta}^{p,\mathrm{ev/odd}}U = -\frac{\partial^2}{\partial r^2} + r^{-2}\left[\left(S \pm \frac{1}{2}\right)^2 - \frac{1}{4}\right]. \tag{4.15}$$

We denote by $\boldsymbol{\Delta}^p_{\max}$ the maximal extension of $\boldsymbol{\Delta}^p$. We denote by $\boldsymbol{\Delta}^{p,\overline{q}}$ the self-adjoint extensions of $\boldsymbol{\Delta}^p$ with domains

$$\mathrm{dom}(\boldsymbol{\Delta}^{p,\overline{m}}) = \{\omega \in \mathrm{dom}(\boldsymbol{\Delta}^p_{\max}) \mid \omega \in \mathrm{dom}\left(\delta_{\min} d_{\max}\right) \cap \mathrm{dom}\left(d_{\max}\delta_{\min}\right) \text{ as } r \to 0\} \tag{4.16}$$

resp.

$$\mathrm{dom}(\boldsymbol{\Delta}^{p,\overline{n}}) = \{\omega \in \mathrm{dom}(\boldsymbol{\Delta}^p_{\max}) \mid \omega \in \mathrm{dom}\left(\delta_{\max} d_{\min}\right) \cap \mathrm{dom}\left(d_{\min}\delta_{\max}\right) \text{ as } r \to 0\}. \tag{4.17}$$

The boundary conditions in (4.16), (4.17) at $r = 0$ are inherited from the boundary condition near $p \in \mathrm{Sing}(X)$ for the operator $\Delta^{\overline{q}}$. Note that the cone is complete at $r = \infty$ and hence we do not need to specify boundary conditions there. Recall that, for n even, lower and upper middle perversity coincide, $\overline{m} = \overline{n}$. On the other hand, for n even, we also have $d_{\max} = d_{\min}$.

For $\overline{q} \in \{\overline{m}, \overline{n}\}$ the extensions $\boldsymbol{\Delta}^{p,\overline{q}}$ of $\boldsymbol{\Delta}^p$ are scale invariant: For $a > 0$, let h_a be the operator acting on sections w of $\Lambda(T^*Z_p) \otimes F$ by $h_a w(r, y) = w(ar, y)$; then

$$h_a^{-1}\boldsymbol{\Delta}^{p,\overline{q}}h_a = a^2\boldsymbol{\Delta}^{p,\overline{q}}. \tag{4.18}$$

Note that (4.18) also says that the domains of $\boldsymbol{\Delta}^{p,\overline{q}}$ are invariant under radial scaling.

4.2.2. *Definition of the model Witten Laplacian* $\boldsymbol{\Delta}_T^{p,\overline{q}}$*,* $p \in \mathrm{Sing}(X)$*,* $\overline{q} \in \{\overline{m}, \overline{n}\}$. We denote by $e_1, \ldots, e_n$ an orthonormal basis of TZ_p, by $e^1, \ldots, e^n$ the dual basis of T^*Z_p. We denote by

$$c(e_k) := e^k - \iota_{e_k}, \widehat{c}(e_k) := e^k + \iota_{e_k}, k = 1, \ldots, n, \tag{4.19}$$

the Clifford operators. Let us denote again by N the number operator acting by multiplication with the form degree. We have (see [BZ92, (11.1)])

(4.20) $$N = \frac{1}{2}\sum_{i=1}^{n} c(e_i)\widehat{c}(e_i) + \frac{n}{2}.$$

The action of the Witten Laplacian $\widetilde{\Delta}^{\overline{q}}_T$, $T > 0$, on forms with support in a neighbourhood of $p \in \mathrm{Sing}(X)$ can be identified with the action of the model Witten Laplacian $\mathbf{\Delta}^{p,\overline{q}}_T$ on the infinite cone Z_p, which we now define: Let $\mathbf{\Delta}^p_T$ denote the following operator acting on compactly supported forms on Z_p with values in the bundle F:

(4.21) $$\mathbf{\Delta}^p_T := \mathbf{\Delta}^p + T(n-2N) + T^2r^2 = \mathbf{\Delta}^p - T\sum_{i=1}^{n} c(e_i)\widehat{c}(e_i) + T^2r^2,$$

where for the last identity we have used (4.20).

We denote by $(\Omega^\bullet_c(Z_p,F), d_{T,c})$, where $d_{T,c}\omega := d_c\omega + Tdf^p \wedge \omega = d_c\omega - Trdr \wedge \omega$, the deformed de Rham complex of smooth compactly supported forms on the infinite cone Z_p. We consider the maximal extension $(\mathcal{C}^\bullet_{T,\max}(Z_p,F), d_{T,\max}, \langle\ ,\ \rangle)$ resp. the minimal extension $(\mathcal{C}^\bullet_{T,\min}(Z_p,F), d_{T,\min}, \langle\ ,\ \rangle)$ of the deformed de Rham complex on the infinite cone. The model Witten Laplacian $\mathbf{\Delta}^{p,\overline{m}}_T$ (resp. $\mathbf{\Delta}^{p,\overline{n}}_T$) is the closed selfadjoint extension of the operator $\mathbf{\Delta}^p_T$ associated to the Hilbert complex $(\mathcal{C}^\bullet_{T,\max}(Z_p,F), d_{T,\max}, \langle\ ,\ \rangle)$ (resp. $(\mathcal{C}^\bullet_{T,\min}(Z_p,F), d_{T,\min}, \langle\ ,\ \rangle)$).

We denote by $\mathbf{\Delta}^p_{T,\max}$ the maximal extension of $\mathbf{\Delta}^p_T$. The domain of the model Witten Laplacian can be described as follows:

(4.22)
$$\mathrm{dom}(\mathbf{\Delta}^{p,\overline{m}}_T) = \{\omega \in \mathrm{dom}(\mathbf{\Delta}^p_{T,\max}) \mid \omega \in \mathrm{dom}\left(\delta_{T,\min}d_{T,\max}\right) \cap \mathrm{dom}\left(d_{T,\max}\delta_{T,\min}\right) \text{ as } r \to 0\}$$

resp.

(4.23)
$$\mathrm{dom}(\mathbf{\Delta}^{p,\overline{n}}) = \{\omega \in \mathrm{dom}(\mathbf{\Delta}^p_{T,\max}) \mid \omega \in \mathrm{dom}\left(\delta_{T,\max}d_{T,\min}\right) \cap \mathrm{dom}\left(d_{T,\min}\delta_{T,\max}\right) \text{ as } r \to 0\}.$$

The boundary conditions in (4.22), (4.23) at $r = 0$ are inherited from the boundary condition near $p \in \mathrm{Sing}(X)$ for the Witten Laplacian $\widetilde{\Delta}^{\overline{q}}_T$. Again by completeness, at $r = \infty$ we do not need to specify boundary conditions.

For $\overline{q} \in \{\overline{m}, \overline{n}\}$, $T \geqslant 0$ from (4.18), (4.22) and (4.23) we get the following scaling property for the model Witten Laplacian

(4.24) $$h_a^{-1}\mathbf{\Delta}^{p,\overline{q}}_T h_a = a^2\mathbf{\Delta}^{p,\overline{q}}_{T/a^2}.$$

From (4.24), we get, for $T > 0$,

(4.25) $$\mathrm{Spec}\left(\mathbf{\Delta}^{p,\overline{q}}_T\right) = T\,\mathrm{Spec}\left(\mathbf{\Delta}^{p,\overline{q}}_1\right).$$

4.2.3. *Spectral data for the model Witten Laplacian* $\mathbf{\Delta}^{p,\overline{q}}_T$. The separation of variables for the Laplacian $\mathbf{\Delta}^{p,\overline{q}}$ on the infinite cone has first been used by Cheeger (see e.g. [Che83, Section 3]) to split the computations of the spectral data of $\mathbf{\Delta}^{p,\overline{q}}$ according to the spectrum of the transversal Laplacian Δ_{L_p}. In [Ver09] the analytic torsion of a truncated cone has been computed also using the separation of variables trick. In view of (4.15), (4.21) we can apply here the same principle to study the spectral properties of $\mathbf{\Delta}^{p,\overline{q}}_T$.

We denote by $\operatorname{Spec}\big(\Delta^{(k)}_{L_p,ccl}\big)$, $k = 0,\dots,n-2$, the co-closed spectrum of $\Delta^{(k)}_{L_p}$. For $\mu \in \operatorname{Spec}\big(\Delta^{(k)}_{L_p,ccl}\big)$ we denote by

$$\mathcal{H}^k_{\mu,ccl}(L_p,F_{L_p}) := \{\psi \in \Omega^k(L_p,F_{L_p}) \mid \Delta^{(k)}_{L_p}\psi = \mu\psi, \delta_{L_p}\psi = 0\} \tag{4.26}$$

the space of co-closed eigenforms of the operator $\Delta^{(k)}_{L_p}$ to the eigenvalue μ. In particular $\mathcal{H}^k_{0,ccl}(L_p,F_{L_p}) = \mathcal{H}^k(L_p,F_{L_p})$ is the space of harmonic k-forms on the link L_p. We have the following orthogonal decomposition

$$\begin{aligned} &\mathrm{L}^2\left(\Lambda(T^*L_p)\otimes F_{L_p}\right)\\ &= \Big(\bigoplus_{k=0}^{n-1}\mathcal{H}^k(L_p,F_{L_p})\Big)\bigoplus\Big(\bigoplus_{\substack{0\leqslant k\leqslant n-2\\ \mu\in\operatorname{Spec}\left(\Delta^{(k)}_{L_p,ccl}\right)\setminus\{0\}}}\Big(\mathcal{H}^k_{\mu,ccl}(L_p,F_{L_p})\oplus d_{L_p}\mathcal{H}^k_{\mu,ccl}(L_p,F_{L_p})\Big)\Big). \end{aligned} \tag{4.27}$$

For $k = -1,\dots,n-2$, set

$$\alpha_k := (k+1-n/2). \tag{4.28}$$

For $k = 0,\dots,n-2$, and $\mu \in \operatorname{Spec}\left(\Delta^{(k)}_{L_p,ccl}\right)\setminus\{0\}$, set

$$\beta(\mu) := \beta_k(\mu) := \sqrt{\alpha_k^2+\mu}. \tag{4.29}$$

We can split all spectral computations for $\mathbf{\Delta}^{p,\bar{q}}_T$ into computations on subcomplexes of the Hilbert complexes $(C^\bullet_{T,\max/\min}(Z_p,F), d_{T,\max/\min}, \langle\ ,\ \rangle)$.

__Subcomplex of type 1:__ Let $\mu \in \operatorname{Spec}\left(\Delta^{(k)}_{L_p,ccl}\right)\setminus\{0\}$. For $0 \neq \psi \in \mathcal{H}^k_{\mu,ccl}(L_p,F_{L_p})$, we denote by

$$\begin{aligned} \xi_1 = \xi_1(\psi) &:= (0,\psi) \in \Omega^{k-1}(L_p,F_{L_p})\oplus\Omega^k(L_p,F_{L_p}),\\ \xi_2 = \xi_2(\psi) &:= \big(\psi,0\big) \in \Omega^{k}(L_p,F_{L_p})\oplus\Omega^{k+1}(L_p,F_{L_p}),\\ \xi_3 = \xi_3(\psi) &:= \big(0,\mu^{-1/2}d_{L_p}\psi\big) \in \Omega^{k}(L_p,F_{L_p})\oplus\Omega^{k+1}(L_p,F_{L_p}),\\ \xi_4 = \xi_4(\psi) &:= (\mu^{-1/2}d_{L_p}\psi,0) \in \Omega^{k+1}(L_p,F_{L_p})\oplus\Omega^{k+2}(L_p,F_{L_p}). \end{aligned} \tag{4.30}$$

We still denote by $\xi_1 \in C^\infty(\mathbb{R}_+,\Omega^{k-1}(L_p,F_{L_p})\oplus\Omega^k(L_p,F_{L_p}))$ the constant function with value ξ_1. Similarly for $\xi_2,\xi_3 \in C^\infty(\mathbb{R}_+,\Omega^{k}(L_p,F_{L_p})\oplus\Omega^{k+1}(L_p,F_{L_p}))$ as well as for $\xi_4 \in C^\infty(\mathbb{R}_+,\Omega^{k+1}(L_p,F_{L_p})\oplus\Omega^{k+2}(L_p,F_{L_p}))$. The __subcomplex of type 1__ associated to $0 \neq \psi \in \mathcal{H}^k_{\mu,ccl}(L_p,F_{L_p})$ is the subcomplex:

$$0 \to \langle U_k(\xi_1)\rangle \xrightarrow{d_T} \langle U_{k+1}(\xi_2),U_{k+1}(\xi_3)\rangle \xrightarrow{d_T} \langle U_{k+2}(\xi_4)\rangle \to 0. \tag{4.31}$$

By the proof of [Lud17b, Theorem 4.2] it is known already that the subcomplex (4.31) does not yield any contribution to $\ker(\mathbf{\Delta}^{p,\bar{q}}_T)$. Therefore, by the Hodge theorem, to study the eigenequation

$$\mathbf{\Delta}^{p,\bar{q}}_T\omega = \lambda\omega,\ \lambda \neq 0, \tag{4.32}$$

on the subcomplex (4.31) it is sufficient to study the eigenequation (4.32) on $\langle U_k(\xi_1)\rangle$ and on $\langle U_{k+2}(\xi_4)\rangle$. On $\langle U_k(\xi_1),U_{k+2}(\xi_4)\rangle$, using the unitary transformation (4.12), the action of the model Witten Laplacian can be identified with the action of the following regular singular operator on $\mathrm{L}^2(\mathbb{R}_{>0})$:

$$L_\mu := -\partial_r^2 + r^{-2}\left(\beta_k(\mu)^2 - \frac{1}{4}\right) + T(n-2N) + T^2r^2, \tag{4.33}$$

where N is the degree operator on $\operatorname{span}\langle U(\xi_i(\psi))\rangle$, $i = 1, 4$. The operator L_μ is in the limit point case at ∞. Moreover, at $r = 0$, the operator L_μ is in the limit point case iff $\beta_k(\mu)^2 \geqslant 1$. If $0 < \beta_k(\mu)^2 < 1$ however, one has to choose boundary conditions. Hence, we have to study the eigenequation on the half-line $\mathbb{R}_{>0}$:

$$L_\mu g = \left(-\partial_r^2 + r^{-2}\left(\beta_k(\mu)^2 - \frac{1}{4}\right) + T(n-2N) + T^2r^2\right) g = \lambda g, \tag{4.34}$$

imposing appropriate boundary conditions at $r = 0$, induced from the boundary conditions (4.22), (4.23) for $\mathbf{\Delta}_T^{p,\bar{q}}$. Arguing as in [Ver09, Section 4.1], the boundary conditions (4.22), (4.23) can both be translated into the following boundary condition for g:

$$g(r) = \mathcal{O}(r^{1/2}) \text{ as } r \to 0, \text{ and } g(r) \in \mathrm{L}^2(\mathbb{R}_{>0}). \tag{4.35}$$

It is at this stage of the computation, where we profit from the fact that we have to study the eigenequation (4.32) only on $\langle U_k(\xi_1), U_{k+2}(\xi_4)\rangle$, where one can translate the boundary conditions (4.22), (4.23) easily, namely into (4.35).

We denote by L_j^β the Laguerre polynomial. The subcomplex of type 1 corresponding to $0 \neq \psi \in \mathcal{H}^k_{\mu,ccl}(L_p, F_{L_p})$, $\mu \neq 0$, yields the following eigenvalues and eigenforms of the model Witten Laplacian $\mathbf{\Delta}_T^{p,\bar{q}}$ (each of multiplicity 1), $j \in \mathbb{N}_0$:

eigenvalue of $\mathbf{\Delta}_T^{p,\bar{q}}$	eigenform of $\mathbf{\Delta}_T^{p,\bar{q}}$
$(4j+n-2k+2\beta+2)T$	$\phi_1 := r^{\beta+\frac{1}{2}}\exp\left(\frac{-Tr^2}{2}\right) L_j^\beta(Tr^2)\cdot U_k(\xi_1) \in \Omega^k(Z_p, F)$
$(4j+n-2k+2\beta-2)T$	$\phi_4 := r^{\beta+\frac{1}{2}}\exp\left(\frac{-Tr^2}{2}\right) L_j^\beta(Tr^2)\cdot U_{k+2}(\xi_4) \in \Omega^{k+2}(Z_p, F)$
$(4j+n-2k+2\beta+2)T$ $(4j+n-2k+2\beta-2)T$	$d_T\phi_1 \in \langle U_{k+1}(\xi_2), U_{k+1}(\xi_3)\rangle \subset \Omega^{k+1}(Z_p, F)$ $\delta_T\phi_4 \in \langle U_{k+1}(\xi_2), U_{k+1}(\xi_3)\rangle \subset \Omega^{k+1}(Z_p, F)$

Subcomplex of type 2: Let $0 \neq \eta \in \mathcal{H}^k(L_p, F_{L_p})$, $k = 0, \dots, n-1$, be a harmonic form on L_p. As before $(0,\eta) \in C^\infty(\mathbb{R}_+, \Omega^{k-1}(L_p, F_{L_p}) \oplus \Omega^k(L_p, F_{L_p}))$ denotes the constant function with value $(0,\eta)$, and similarly for $(\eta, 0) \in C^\infty(\mathbb{R}_+, \Omega^k(L_p, F_{L_p}) \oplus \Omega^{k+1}(L_p, F_{L_p}))$. We have the following subcomplex of type 2:

$$0 \to \langle U_k(0,\eta)\rangle \xrightarrow{d_T} \langle U_{k+1}(\eta, 0)\rangle \to 0. \tag{4.36}$$

Using the unitary transformation (4.12) we get the following eigenequation on $\langle(0,\eta)\rangle$ (resp. on $\langle(\eta,0)\rangle$)

$$(-\partial_r^2 + r^{-2}(c_k^2 \pm (-1)^k c_k) + T(n-2N) + T^2r^2)g = \lambda g, \tag{4.37}$$

with the boundary conditions induced from (4.22), (4.23).

Case 1: Let us first consider the case, where either n is even, or n is odd and $k \neq \lfloor n/2 \rfloor$.

By [Ver09, Proposition 7.1], we have to study the equation (4.37) with the boundary condition

$$g = O(r^{1/2}) \text{ as } r \to 0, \text{ and } g(r) \in \mathrm{L}^2(\mathbb{R}_{>0}). \tag{4.38}$$

Hence, the same computation as in [Lud20b, Section 4] shows, that the subcomplex of type 2 corresponding to $0 \neq \eta \in \mathscr{H}^k(L_p, F_{L_p})$, yields the following eigenvalues and eigenforms of the model Witten Laplacian (each of multiplicity 1), $j \in \mathbb{N}_0$:

eigenvalue of $\mathbf{\Delta}_T^{p,\overline{n}}$	eigenform of $\mathbf{\Delta}_T^{p,\overline{n}}$
$\big(4j - 2\alpha_k + 4 + 2\lvert\alpha_k\rvert\big)T$	$r^{\lvert\alpha_k\rvert+\frac{1}{2}} \exp\left(\frac{-Tr^2}{2}\right) L_j^{\lvert\alpha_k\rvert}(Tr^2)U_k((0,\eta)) \in \Omega^k(Z_p, F)$
$\big(4j - 2\alpha_k + 2(\lvert\alpha_{k-1}\rvert + 1)\big)T$	$r^{\lvert\alpha_{k-1}\rvert+\frac{1}{2}} \exp\left(\frac{-Tr^2}{2}\right) L_j^{\lvert\alpha_{k-1}\rvert}(Tr^2)U_{k+1}((\eta,0)) \in \Omega^{k+1}(Z_p, F)$

Case 2: For $n = 2\nu + 1$ odd and $k = \nu$, the two boundary conditions in (4.22), (4.23) do differ on a subcomplex of type 2.

Case 2 (a): Let $\overline{q} = \overline{n}$. For $k = \nu$, on $\langle(0,\eta)\rangle$, by (4.37) and [Ver09, Propositions 7.1 and 7.3], we study the eigenequation on $\mathrm{L}^2(\mathbb{R}_{>0})$

$$(-\partial_r^2 + T + T^2r^2)g = \lambda g, \tag{4.39}$$

with boundary condition

$$g = O(r^{1/2}) \text{ and } (\partial_r - Tr)g = O(1) \text{ as } r \to 0;\ g(r) \in \mathrm{L}^2(\mathbb{R}_{>0}). \tag{4.40}$$

The only L^2-solutions of (4.39) occur for $\lambda_j = 2jT, j \in \mathbb{N}$, with eigenfunction $g_{\lambda_j} = \exp(-Tr^2/2)H_{j-1}(\sqrt{T}r)$, where H_j denotes the Hermite polynomial. The following reccurence relation of Hermite polynomials holds $H_j' = 2rH_j - H_{j+1}$. Moreover, as $r \to 0$, we have $H_j(r) = O(1)$ if j is even, and $H_j(r) = O(r)$ if j is odd. Hence, taking into account the boundary conditions at $r \to 0$ in (4.40), only the eigenvalues $\lambda_j = 2jT, j \in \mathbb{N}$ with j even appear.

On $\langle(\eta,0)\rangle$ we study the eigenequation

$$(-\partial_r^2 - T + T^2r^2)g = \lambda g, \tag{4.41}$$

with boundary condition

$$g = O(1) \text{ and } (\partial_r - Tr)g = O(r^{1/2}) \text{ as } r \to 0;\ g(r) \in \mathrm{L}^2(\mathbb{R}_{>0}), \tag{4.42}$$

and argue similarly. The above arguments show that the subcomplex of type 2 corresponding to $0 \neq \eta \in \mathscr{H}^\nu(L_p, F_{L_p})$, yields the following eigenvalues and eigenforms of the model Witten Laplacian (each of multiplicity 1):

eigenvalue of $\mathbf{\Delta}_T^{p,\overline{q}}$	eigenform of $\mathbf{\Delta}_T^{p,\overline{q}}$
$4jT, j \in \mathbb{N}$	$\exp\left(\frac{-Tr^2}{2}\right) H_{2j-1}(\sqrt{T}r)U_\nu((0,\eta)) \in \Omega^\nu(Z_p, F)$ $= \exp\left(\frac{-Tr^2}{2}\right) rL_{j-1}^{1/2}(Tr^2)U_\nu((0,\eta))$
$4jT, j \in \mathbb{N}_0$	$\exp\left(\frac{-Tr^2}{2}\right) H_{2j}(\sqrt{T}r)U_{\nu+1}((\eta,0)) \in \Omega^{\nu+1}(Z_p, F)$ $= \exp\left(\frac{-Tr^2}{2}\right) L_j^{-1/2}(Tr^2)U_{\nu+1}((\eta,0))$

Case 2 (b): Let $\overline{q} = \overline{m}$. This time we have the boundary condition (4.42) on $\langle(0,\eta)\rangle$ (resp. (4.40) on $\langle(\eta,0)\rangle$).

Arguing similarly as in Case 2 (a), this time we get

eigenvalue of $\boldsymbol{\Delta}_T^{p,\overline{m}}$	eigenform of $\boldsymbol{\Delta}_T^{p,\overline{m}}$
$(4j+2)T, j \in \mathbb{N}_0$	$\exp\left(\frac{-Tr^2}{2}\right) H_{2j}(\sqrt{T}r)U_\nu((0,\eta)) \in \Omega^\nu(Z_p,F)$ $= \exp\left(\frac{-Tr^2}{2}\right) L_j^{-1/2}(Tr^2)U_\nu((0,\eta))$
$(4j+2)T, j \in \mathbb{N}_0$	$\exp\left(\frac{-Tr^2}{2}\right) H_{2j+1}(\sqrt{T}r)U_{\nu+1}((\eta,0)) \in \Omega^{\nu+1}(Z_p,F)$ $= \exp\left(\frac{-Tr^2}{2}\right) rL_j^{1/2}(Tr^2)U_{\nu+1}((\eta,0))$

Remark 4.2. (a) From the above computation together with the local calculation for the relative intersection cohomology of a cone (recalled in (2.12)), we get

$$\ker \boldsymbol{\Delta}_T^{p,\overline{q},(k)} \simeq IH_{\overline{q}}^k(cL_p, L_p, F). \tag{4.43}$$

Note that only harmonic forms on the link L_p do contribute to $\ker \boldsymbol{\Delta}_T^{p,\overline{q}}$.

(b) Note that for n odd, the only difference in the spectral data of $\boldsymbol{\Delta}_T^{p,\overline{m}}$ and $\boldsymbol{\Delta}_T^{p,\overline{n}}$ stems from the harmonic forms on L_p of degree $\lfloor n/2 \rfloor$.

(c) The above computation generalises the computations in [Lud20b, Section 4], where the Witt and a spectral Witt condition are assumed. Under the assumptions made in [Lud20b], the model Witten Laplacian is the Friedrichs extension of (4.21) and the boundary condition at $r \to 0$ always translates into (4.38). Also, for an odd dimensional Witt space, case 2 in the above discussion does not appear.

4.2.4. *Heat kernel of the model Witten Laplacian.*

Definition 4.3. Let $\overline{q} \in \{\overline{m}, \overline{n}\}$ the lower middle (resp. upper middle) perversity.

(a) For $(t,T) \in \mathbb{R}_{>0} \times \mathbb{R}_{\geqslant 0}$, we denote by $Q_{t,T}^{p,\overline{q}}(x,x')$, $x,x' \in Z_p$, the kernel of the operator $\exp\big(-t\boldsymbol{\Delta}_T^{p,\overline{q}}\big)$ with respect to $d\mathrm{vol}_{Z_p}$. Set $Q_t^{p,\overline{q}}(x,x') := Q_{t,0}^{p,\overline{q}}(x,x')$, $x,x' \in Z_p$. We denote by $Q_{t,T}^{p,\overline{q},(k)}(x,x')$, $x,x' \in Z_p$, the kernel of the operator $\exp\big(-t\boldsymbol{\Delta}_T^{p,\overline{q}}\big)$ restricted to k-forms, $k \in \{0,\dots,n\}$.

(b) For $(t,T) \in \mathbb{R}_{>0} \times \mathbb{R}_{\geqslant 0}$, we denote by $U_{t,T}^{p,\overline{q}}(x,x')$, $x,x' \in Z_p$, the kernel of the operator $\exp\big(-t^2\boldsymbol{\Delta}_{T/t}^{p,\overline{q}}\big)$ with respect to $d\mathrm{vol}_{Z_p}$.

Note that, for $(t,T) \in \mathbb{R}_{>0} \times \mathbb{R}_{\geqslant 0}$, we have directly from Definition 4.3,

$$U_{t,T}^{p,\overline{q}}(x,x') = Q_{t^2,T/t}^{p,\overline{q}}(x,x'),\ x,x' \in Z_p. \tag{4.44}$$

Using the computation of the spectral data for $\boldsymbol{\Delta}_T^{p,\overline{q}}$ we can generalise [Lud20a, Proposition 4.5]. The heat kernel of the model operator for $T>0$ and for $T=0$ are related as follows:

Proposition 4.4. *Let $(t,T) \in \mathbb{R}_{>0}^2$, $(r,y),(r',y') \in Z_p$. Then*

$$\begin{aligned} &Q_{t,T}^{p,\overline{q},(k)}\big((r,y),(r',y')\big) \\ &= e^{-(n-2k)tT}T^{n/2}\exp\left(-T\tanh(tT)\frac{r^2+r'^2}{2}\right) Q_{\sinh(2tT)/2}^{p,\overline{q},(k)}\Big(\big(\sqrt{T}r,y\big),\big(\sqrt{T}r',y'\big)\Big). \end{aligned} \tag{4.45}$$

4.2.5. *The asymptotics of* $\mathrm{Tr_s}\left[f^p \exp\left(-t^2\boldsymbol{\Delta}_{T/t}^{p,\bar{q}}\right)\right]$ *in the range* $t \in]0,1]$, $T \in [0, d/t]$. We denote by $\mathrm{Tr_s} := \mathrm{Tr}_{|\Omega^{even}} - \mathrm{Tr}_{|\Omega^{odd}}$ the supertrace of an operator.

Using the spectral data of $\boldsymbol{\Delta}_T^{p,\bar{q}}$ computed in Section 4.2.3 and proceeding as in [Lud20a, Sections 4.3.3-4.3.5] one can study the asymptotics of the supertraces $\mathrm{Tr_s}\left[N\exp\left(-t\boldsymbol{\Delta}_T^{p,\bar{q}}\right)\right]$, $\mathrm{Tr_s}\left[f^p\exp\left(-t\boldsymbol{\Delta}_T^{p,\bar{q}}\right)\right]$ in the different zones of parameters (t,T).

Here we only give the generalisation of [Lud20a, Proposition 4.12], where the non-flatness of the metric g^F comes into play:

Proposition 4.5. *Let* $p \in \mathrm{Sing}(X)$ *and* $\epsilon > 0$, $d > 0$. *There exists* $C > 0$, *such that for* $t \in]0,1]$, $T \in [0, d/t]$*:*

$$\Big| \int_{\{0\leqslant r\leqslant\epsilon\}\times L_p} f^p\left(\mathrm{Tr_s}\left[U_{t,T}^{p,\bar{q}}((r,y),(r,y))\right]d\mathrm{vol}_{Z_p} - \mathrm{rk}(F)\int^{B,p}\exp(-B_{T^2}^p)\right) - f(p)\gamma_p^{\bar{q}}(F)$$
$$+\frac{t}{2}\int_{\{0\leqslant r\leqslant\epsilon\}\times L_p}\theta(F,g^F)d\left(\int^{B,p}\widehat{df^p}\exp(-B_{T^2}^p)\right)\Big| \leqslant Ct^2.$$

Proof. The proof of [Lud20a, Proposition 4.12] can be generalised directly to this situation. □

Remark 4.6. Note that

$$\theta(F,g^F)\int^{B,p}\widehat{df^p}\exp(-B_{T^2}^p) = 0, \tag{4.46}$$

since it is an n-form on Z_p not containing e^r.

4.3. **Zeta function and Ray-Singer torsion for the model Witten Laplacian** $\boldsymbol{\Delta}_T^{p,\bar{q}}$**.** We denote by $\boldsymbol{\Delta}_T^{p,\bar{q},\perp}$ the restriction of $\boldsymbol{\Delta}_T^{p,\bar{q}}$ to $(\ker\boldsymbol{\Delta}_T^{p,\bar{q}})^\perp$. Using the spectral properties of $\boldsymbol{\Delta}_T^{p,\bar{q}}$, computed in Section 4.2.3, one can show that, for $\Re(s) >> 0$, the zeta function

$$s \mapsto \zeta_T^{p,\bar{q}}(s) := -\mathrm{Tr_s}\left[N\left(\boldsymbol{\Delta}_T^{p,\bar{q},\perp}\right)^{-s}\right] \tag{4.47}$$

is a well-defined holomorphic function. From (4.25) and (4.47),

$$\zeta_T^{p,\bar{q}}(s) = T^{-s}\zeta_1^{p,\bar{q}}(s), \tag{4.48}$$

and it is therefore enough to study the zeta function for $T = 1$.

Arguing as in the corresponding statement for a Witt space equipped with a unitary bundle in [Lud20a, Proposition 4.14], we get

Proposition 4.7. *The function* $\zeta_T^{p,\bar{q}}$, $T > 0$, *extends to a meromorphic function on* $\mathbb{C}$, *which is holomorphic at* $s = 0$.

For a complex vector space V of dimension 1 we denote by V^{-1} its dual. For a vector space V we denote by $\det V$ the maximal exterior power of V. We denote by

$$\det IH_{\bar{q}}^\bullet(cL_p, L_p, F) = \bigotimes_{k=0}^{n}\left(\det IH_{\bar{q}}^k(cL_p, L_p, F)\right)^{(-1)^k}. \tag{4.49}$$

The L^2-metric on sections of the vector bundle $\Lambda(T^*Z_p)\otimes F$ restricts to a metric on $\ker\boldsymbol{\Delta}_T^{p,\bar{q}} \simeq IH_{\bar{q}}^\bullet(cL_p, L_p, F)$. We denote the induced metric on the line $\det IH_{\bar{q}}^\bullet(cL_p, L_p, F)$

by $|\ |^{RS}_{\det IH^\bullet_{\overline{q}}(cL_p,L_p,F),T}$. We denote by

$$\|\ \|^{RS}_{\det IH^\bullet_{\overline{q}}(cL_p,L_p,F),T} := |\ |^{RS}_{\det IH^\bullet_{\overline{q}}(cL_p,L_p,F),T} \exp\left(\frac{1}{2}(\zeta_T^{p,\overline{q}})'(0)\right). \tag{4.50}$$

Set $\zeta_p^{\overline{q}} := \zeta_1^{\overline{q},p}$.

Definition 4.8. For $p \in \mathrm{Sing}(X)$, the Ray-Singer metric on the line $\det IH^\bullet_{\overline{q}}(cL_p, L_p, F)$ is defined as

$$\|\ \|^{RS}_{\det IH^\bullet_{\overline{q}}(cL_p,L_p,F)} := \|\ \|^{RS}_{\det IH^\bullet_{\overline{q}}(cL_p,L_p,F),1} = |\ |^{RS}_{\det IH^\bullet_{\overline{q}}(cL_p,L_p,F),1} \exp\left(\frac{1}{2}(\zeta_p^{\overline{q}})'(0)\right). \tag{4.51}$$

Remark 4.9. In Definition 4.8 we have defined the Ray-Singer metric on the line $\det IH^\bullet_{\overline{q}}(cL_p, L_p, F)$ by choosing the parameter $T = 1$ in (4.50). Proceeding as in the proof of [Lud20a, Proposition 4.15], one can prove that the definition is independent of the choice of $T > 0$.

4.4. **Local index techniques on the infinite cone.**

4.4.1. *Definition of the Cheeger invariant.* The Cheeger invariant of X at $p \in \mathrm{Sing}(X)$ is the contribution of the singularity p to the Chern-Gauss-Bonnet theorem for spaces with isolated conical singularities [Che83, Theorem 5.1]. In this section, we recall its definition in some detail, since similar arguments will play an important role in Section 6.2 as well as in the study of the anomaly formulas in Section 7.

Definition/Lemma 4.10. Let $\overline{q} \in \{\overline{m}, \overline{n}\}$. The following integral expression

$$\begin{aligned}\gamma_p^{\overline{q}}(F) &:= \int_{Z_p} \mathrm{Tr_s}\left[Q_t^{p,\overline{q}}((r,y),(r,y))\right] d\mathrm{vol}_{Z_p} \\ &= \frac{1}{2}\int_0^\infty u^{-1}\int_{L_p} \mathrm{Tr_s}\left[Q_u^{p,\overline{q}}((1,y),(1,y))\right] d\mathrm{vol}_{L_p} du\end{aligned} \tag{4.52}$$

is well-defined and will be called the Cheeger invariant of X at p.

Proof. In this proof we follow arguments in [Che83, Section 2], [BC90, Section 1(f)], [Les97, Lemma 2.2.4], [Lud20b, Section 7]. We first prove the equality of the two integral expressions in (4.52), for $t > 0$: The scaling properties (4.18) of $\boldsymbol{\Delta}^{p,\overline{q}}$ on the infinite cone imply (see e.g. [Che80, Section 2] and [BC90, Proposition 1.7])

$$Q_t^{p,\overline{q}}((r,y),(r,y)) = \frac{1}{r^n} Q_{t/r^2}^{p,\overline{q}}((1,y),(1,y)). \tag{4.53}$$

The equality of the two integrals in (4.52) follows from (4.53) using the change of variables $u = t/r^2$.

To finish the proof it is now enough to prove the well-definedness of the second integral in (4.52). Using local index techniques we get the following asymptotic expansion, as $u \searrow 0$,

$$\mathrm{Tr_s}\left[Q_u^{p,\overline{q}}((r,y),(r,y))\right] d\mathrm{vol}_{Z_p} = \mathrm{rk}(F)\cdot e(TZ_p, \nabla^{TZ_p}) + \mathcal{O}(\sqrt{u}); \tag{4.54}$$

the expansion is uniform on compact sets of Z_p. By (3.18), (3.19) the constant term in the asymptotic expansion (4.54) vanishes and therefore, as $u \searrow 0$,

$$\mathrm{Tr_s}\left[Q_u^{p,\overline{q}}\big((1,y),(1,y)\big)\right] = \mathcal{O}(\sqrt{u}). \tag{4.55}$$

This gives well-definedness of the second integral in (4.52) at 0.

Using the characterisation of $\mathrm{dom}(\boldsymbol{\Delta}^{p,\overline{q}})$ near the cone point (see e.g. [Les97, Lemma 2.2.4]), there exists $a > 0$ such that, as $r \searrow 0$,

$$Q_1^{p,\overline{q}}\big((r,y),(r,y)\big) \sim r^{2a-n}. \tag{4.56}$$

From (4.53), (4.56) and setting $u = 1/r^2$, as $u \to \infty$,

$$Q_u^{p,\overline{q}}\big((1,y),(1,y)\big) \sim u^{-a}. \tag{4.57}$$

This gives well-definedness of the second integral in (4.52) at ∞. □

4.4.2. *An index formula for the infinite cone.* The invariant α_p has been defined in (3.25) and studied further in Section 3.5. The following theorem generalises the index theorem for the infinite cone [Lud20b, Theorem II] to the current situation. We give the proof in some detail, since the same local index techniques will be used in Sections 6.2 and 7.3.

Theorem 4.11. *For* $(t,T) \in \mathbb{R}^2_{>0}$,

$$\begin{aligned} I\chi^{\overline{q}}(cL_p, L_p, F) &= \mathrm{Tr_s}\left[\exp(-t\boldsymbol{\Delta}_T^{p,\overline{q}})\right] = \int_{Z_p} \mathrm{Tr_s}\left[Q_{t,T}^{p,\overline{q}}\big((r,y),(r,y)\big)\right] d\mathrm{vol}_{Z_p} \\ &= \mathrm{rk}(F)\int_{Z_p}\int^{B,p} \exp(-B^p_{T^2}) + \gamma_p^{\overline{q}}(F) = \mathrm{rk}(F)\alpha_p + \gamma_p^{\overline{q}}(F). \end{aligned} \tag{4.58}$$

Remark 4.12. (a) By the definition of the Cheeger invariant $\gamma_p^{\overline{q}}(F)$ in (4.52) and since $\int^{B,p} \exp(-B_0^p) = e(TZ_p, \nabla^{TZ_p}) = 0$, the third equality in (4.58) also holds for $T = 0$.

(b) On the right hand side of (4.58) two terms appear: the interior contribution α_p does not depend on the perversity $\overline{q} \in \{\overline{m}, \overline{n}\}$, while the contribution of the singular point $\gamma_p^{\overline{q}}(F)$ does. This is a general phenomenon for spaces with conical singularities, appearing e.g. also in the Chern-Gauss-Bonnet Theorem [Che83, Theorem 5.1] (recalled in (6.12)) as well as in the anomaly formulas in Section 7.

Proof. The trace class property as well as the second identity in (4.58) follow from the spectral properties of the model Witten Laplacian $\boldsymbol{\Delta}_T^{p,\overline{q}}$. As explained in Section 4.2.2, the model Witten Laplacian $\boldsymbol{\Delta}_T^{p,\overline{q}}$ is the Laplacian associated to the minimal resp. maximal deformed complex of L^2-forms on the infinite cone. We therefore get the first identity in (4.58) by a McKean-Singer argument applied to these complexes and (4.43).

Since the first two identities in (4.58) hold for all $(t,T) \in \mathbb{R}^2_{>0}$, it is enough to show the third identity in (4.58) as $t \searrow 0$ and T fixed.

We start by splitting the integral into two parts

$$\begin{aligned} &\mathrm{Tr_s}\left[\exp(-t\boldsymbol{\Delta}_T^{p,\overline{q}})\right] = \int_{Z_p} \mathrm{Tr_s}\left[Q_{t,T}^{p,\overline{q}}\big((r,y),(r,y)\big)\right] d\mathrm{vol}_{Z_p} \\ &= \int_{\{0\leqslant r\leqslant 1\}\times L_p} \mathrm{Tr_s}\left[Q_{t,T}^{p,\overline{q}}\big((r,y),(r,y)\big)\right] d\mathrm{vol}_{Z_p} + \int_{\{r\geqslant 1\}\times L_p} \mathrm{Tr_s}\left[Q_{t,T}^{p,\overline{q}}\big((r,y),(r,y)\big)\right] d\mathrm{vol}_{Z_p}. \end{aligned} \tag{4.59}$$

In Step 1 (resp. Step 2) we treat the first (resp. the second) integral on the right hand side of (4.59) as $t \searrow 0$.

Step 1: We study the first integral on the right hand side of (4.59) *as* $t \searrow 0$: Let us denote by $h_{\sqrt{t}}$ the operator acting on sections w of $\Lambda(T^*Z_p) \otimes F$ by

$$h_{\sqrt{t}}w(r,y) = w\left(\sqrt{t}r, y\right). \tag{4.60}$$

From (4.15) and (4.21) we have

$$h_{\sqrt{t}}(t\mathbf{\Delta}_T^{p,\overline{q}})h_{\sqrt{t}}^{-1} = \mathbf{\Delta}_{tT}^{p,\overline{q}}; \tag{4.61}$$

We denote by $P_{t,T}^{p,\overline{q}}((r,y),(r',y'))$ the kernel of the operator $\exp(-h_{\sqrt{t}}(t\mathbf{\Delta}_T^{p,\overline{q}})h_{\sqrt{t}}^{-1})$ with respect to $d\mathrm{vol}_{Z_p}$. From (4.61),

$$Q_{t,T}^{p,\overline{q}}\big((r,y),(r',y')\big) = \frac{1}{t^{n/2}}P_{t,T}^{p,\overline{q}}\left(\left(\frac{r}{\sqrt{t}}, y\right), \left(\frac{r'}{\sqrt{t}}, y'\right)\right) \text{ for } (r,y),(r',y') \in Z_p. \tag{4.62}$$

From (4.61) we have that $h_{\sqrt{t}}(t\mathbf{\Delta}_T^{p,\overline{q}})h_{\sqrt{t}}^{-1}$ is the self-adjoint closed extension of the operator

$$h_{\sqrt{t}}(t\mathbf{\Delta}_T^{p})h_{\sqrt{t}}^{-1} = \mathbf{\Delta}^p + tT(n-2N) + t^2T^2r^2, \tag{4.63}$$

with domain defined by (4.22), (4.23). Note that for $r \to 0$ the boundary conditions (4.16), (4.17) of the undeformed Laplacian $\mathbf{\Delta}^{p,\overline{q}}$ and the boundary conditions (4.22), (4.23) for the model Witten Laplacian $\mathbf{\Delta}_{tT}^{p,\overline{q}}$ coincide. From (4.61), (4.63), as $t \searrow 0$ and for $r \in [0, 1/\sqrt{t}]$, the coefficients of the operator $h_{\sqrt{t}}(t\mathbf{\Delta}_T^{p,\overline{q}})h_{\sqrt{t}}^{-1}$ converge uniformly together with their derivatives of any order to the coefficients of the non-deformed Laplacian $\mathbf{\Delta}^{p,\overline{q}}$.

Using (4.53), (4.61), (4.62) and (4.63) and making the change of variables $u = \frac{1}{r^2}$, we get for the first integral on the right hand side of (4.59), as $t \searrow 0$,

$$\begin{aligned} \int_{\{0\leqslant r\leqslant 1\}\times L_p} &\mathrm{Tr_s}\left[Q_{t,T}^{p,\overline{q}}\big((r,y),(r,y)\big)\right] r^m dr d\mathrm{vol}_{L_p} = \\ &= \int_{\{r\leqslant 1/\sqrt{t}\}\times L_p} \mathrm{Tr_s}\left[P_{t,T}^{p,\overline{q}}\big((r,y),(r,y)\big)\right] r^m dr d\mathrm{vol}_{L_p} \\ &= \int_{\{r\leqslant 1/\sqrt{t}\}\times L_p} \left\{\mathrm{Tr_s}\left[Q_1^{p,\overline{q}}((r,y),(r,y))\right]\right\} r^m dr d\mathrm{vol}_{L_p} + \mathcal{O}(t) \\ &= \frac{1}{2}\int_t^\infty \frac{du}{u}\int_{L_p} \mathrm{Tr_s}\left[Q_u^{p,\overline{q}}((1,y),(1,y))\right] d\mathrm{vol}_{L_p} + \mathcal{O}(t) \\ &\xrightarrow{t\searrow 0} \frac{1}{2}\int_0^\infty \frac{du}{u}\int_{L_p} \mathrm{Tr_s}\left[Q_u^{p,\overline{q}}((1,y),(1,y))\right] d\mathrm{vol}_{L_p} = \gamma_p^{\overline{q}}(F). \end{aligned} \tag{4.64}$$

The second identity in (4.64) follows from (4.61) using a Duhamel principle argument as in e.g. [Roe98, Section 7], where the elliptic estimates have to be replaced by the singular elliptic estimates of [Les97, Section 1.4] for the operator $\mathbf{\Delta}^{p,\overline{q}}$. Alternatively it can also be derived using the explicit comparison formula for the heat kernels in (4.45).

Step 2: We study the second integral on the right hand side of (4.59) *as* $t \searrow 0$.

Let us denote by $h_{1/\sqrt{t}}$ the operator acting on sections w of $\Lambda(T^*Z_p) \otimes F$ by

$$h_{1/\sqrt{t}}w(r,y) = w\left(\frac{r}{\sqrt{t}}, y\right). \tag{4.65}$$

Set

$$N_{t,T}^{p,\bar q} := h_{1/\sqrt{t}}(t\mathbf{\Delta}_T^{p,\bar q})h_{1/\sqrt{t}}^{-1} = t^2\mathbf{\Delta}_{T/t}^{p,\bar q}, \tag{4.66}$$

i.e. $N_{t,T}^{p,\bar q}$ is a self-adjoint closed extension of

$$\begin{aligned} h_{1/\sqrt{t}}(t\mathbf{\Delta}_T^{p})h_{1/\sqrt{t}}^{-1} &= t^2\left(\mathbf{\Delta}^p + \frac{T}{t}(n-2N) + \left(\frac{T}{t}\right)^2 r^2\right) \\ &= t^2\left(\mathbf{\Delta}^p - \frac{T}{t}\sum_{i=1}^n c(e_i)\widehat{c}(e_i) + \left(\frac{T}{t}\right)^2 r^2\right). \end{aligned} \tag{4.67}$$

We denote by $R_{t,T}^{p,\bar q}((r,y),(r',y'))$ the kernel of the operator $\exp(-N_{t,T}^{p,\bar q})$ with respect to $d\mathrm{vol}_{Z_p}$. From (4.66),

$$Q_{t,T}^{p,\bar q}\big((r,y),(r',y')\big) = t^{n/2}\cdot R_{t,T}^{p,\bar q}\left(\left(\sqrt{t}r,y\right),\left(\sqrt{t}r',y'\right)\right) \text{ for } (r,y),(r',y')\in Z_p. \tag{4.68}$$

We now proceed by applying local index techniques as in [BZ92, Theorem 13.4] and in [Lud20b, Theorem II]: We fix a point $z=(r,y)\in Z_p$. For $\epsilon>0$ small enough, we identify $B_\epsilon^{T_zZ_p}(0)$ with $B_\epsilon(z)$ using geodesic coordinates centred at z. For $x\in B_\epsilon(z)\simeq B_\epsilon^{T_zZ_p}(0)$, we identify T_xZ_p, F_x with T_zZ_p, F_z by parallel transport along the geodesic $t\in[0,1]\to tx$ with respect to the connections ∇^{TZ_p}, ∇^F. The operator $N_{t,T}^{p,\bar q}$ is now seen as an operator acting on sections of $(\Lambda(T^*Z_p)\otimes F)_z$ over $B_\epsilon^{T_zZ_p}(0)$. We consider the operator we get from $N_{t,T}^{p,\bar q}$ by rescaling via $x\to x/t$ and then replacing the Clifford operators $c(e_k)$, $\widehat{c}(e_k)$, defined in (4.19), with

$$c_t(e_k) := \frac{e^k}{\sqrt{t}} - \sqrt{t}i_{e_k}, \quad \widehat{c}_t(e_k) := \frac{\widehat{e^k}}{\sqrt{t}} + \sqrt{t}i_{\widehat{e_k}}, k=1,\dots,n. \tag{4.69}$$

As in [BZ92, Theorem 13.4], we get, uniformly on compact sets,

(4.70)
$$\begin{aligned} &\mathrm{Tr_s}\left[R_{t,T}^{p,\bar q}((r,y),(r,y))\right]d\mathrm{vol}_{Z_p} \\ &\quad = \mathrm{rk}(F)\int^{B,p}\exp\left(-B_{T^2}^p\right) + t\left(\int^{B,p}\left(\frac{1}{2}\nabla^{TZ_p} + \iota_{T\widehat{\nabla f}}\right)\widehat{\theta}(F,\nabla^F)\exp(-B_{T^2}^p)\right) + \mathcal{O}(t^2) \\ &\quad = \mathrm{rk}(F)\int^{B,p}\exp\left(-B_{T^2}^p\right) + \frac{1}{2}td\left(\int^{B,p}\widehat{\theta}(F,\nabla^F)\exp(-B_{T^2}^p)\right) + \mathcal{O}(t^2). \end{aligned}$$

Using (4.68) and (4.70), we get for the third integral on the right hand side of (4.59):

$$\begin{aligned} \int_{\{r\geqslant 1\}\times L_p}\mathrm{Tr_s}\left[Q_{t,T}^{p,\bar q}\big((r,y),(r,y)\big)\right]d\mathrm{vol}_{Z_p} &= \int_{\{r\geqslant\sqrt{t}\}\times L_p}\mathrm{Tr_s}\left[R_{t,T}^{p,\bar q}\big((r,y),(r,y)\big)\right]d\mathrm{vol}_{Z_p} \\ &\xrightarrow[t\searrow 0]{} \mathrm{rk}(F)\int_{Z_p}\int^{B,p}\exp(-B_{T^2}^p) = \mathrm{rk}(F)\alpha_p. \end{aligned} \tag{4.71}$$

Inserting (4.64) and (4.71) into (4.59) completes the proof of the third identity in (4.58). □

5. The Bismut-Zhang Theorem

In this section we recall the definition of the Ray-Singer (resp. the Bismut-Zhang metric) for a space with isolated conical singularities in Section 5.1 (resp. Section 5.2). One of the main results of this paper is the compairison theorem between this two metrics, which is stated in Theorem 5.6. Sections 5.4-5.7 give an outline of the proof of the Bismut-Zhang Theorem.

5.1. **The Ray-Singer metric on** $\det IH^{\bullet}_{\overline{q}}(X,F)$**.** We denote by $\Delta^{\overline{q},\perp}$ the restriction of $\Delta^{\overline{q}}$ to $\left(\ker \Delta^{\overline{q}}\right)^{\perp}$. We denote by N the number operator acting on sections of the bundle $\Lambda(T^*X)\otimes F$ by multiplication with the form degree. For $s\in\mathbb{C}$, $\Re(s)>\frac{n}{2}$, set

$$\zeta_{\overline{q}}(s) := -\operatorname{Tr_s}\left[N(\Delta^{\overline{q},\perp})^{-s}\right]. \tag{5.1}$$

By a result of A. Dar [Dar87, Section 4], the function $\zeta_{\overline{q}}$ extends to a meromorphic function on the whole complex plane, which is holomorphic at $s=0$. The result in [Dar87, Section 4] has been proved in the case of unitary flat vector bundles, but the same proof also works in the current situation. Incidentally the holomorphicity of $\zeta_{\overline{q}}$ at 0 also follows by using the Mellin transform and Theorem 5.15 below.

We denote by $\det IH^{\bullet}_{\overline{q}}(X,F)$ the complex line

$$\det IH^{\bullet}_{\overline{q}}(X,F) := \bigotimes_{k=0}^{n}\left(\det IH^{k}_{\overline{q}}(X,F)\right)^{(-1)^k}. \tag{5.2}$$

The L^2-metric on sections of $\Lambda(T^*X)\otimes F$ restricts to a metric on the space of L^2-harmonic forms $\mathcal{H}^{\bullet}_{(2),\overline{q}}(X,F)$. Using the isomorphisms (2.19) and (2.21) we get an induced metric on the line $\det IH^{\bullet}_{\overline{q}}(X,F)$, which we denote by $|\ \ |^{RS}_{\det IH^{\bullet}_{\overline{q}}(X,F)}$.

Definition 5.1. The Ray-Singer metric $\|\ \ \|^{RS}_{\det IH^{\bullet}_{\overline{q}}(X,F)}$ on the line $\det IH^{\bullet}_{\overline{q}}(X,F)$ is defined as

$$\|\ \ \|^{RS}_{\det IH^{\bullet}_{\overline{q}}(X,F)} := |\ \ |^{RS}_{\det IH^{\bullet}_{\overline{q}}(X,F)}\exp\left(\frac{1}{2}\zeta'_{\overline{q}}(0)\right). \tag{5.3}$$

We will discuss the anomaly formulas for $\|\ \ \|^{RS}_{\det IH^{\bullet}_{\overline{q}}(X,F)}$, *i.e.* the dependence of the Ray-Singer metric $\|\ \ \|^{RS}_{\det IH^{\bullet}_{\overline{q}}(X,F)}$ on the choices of the metrics g^{TX}, g^F, in Section 7.

5.2. **The Bismut-Zhang metric on** $\det IH^{\bullet}_{\overline{q}}(X,F)$**.** In [Lud20a, Section 2.10] the Bismut-Zhang metric has been defined for a Witt space equipped with a unitary bundle (*i.e.* g^F is flat). The definition can be generalised easily to the current situation. We start by recalling the singular Morse-Thom-Smale complex defined in [Lud17b, Section 6] and [Lud17a]. For a singular point $p\in\operatorname{Sing}(X)$, we denote by $o(TL_p)$ the orientation bundle of L_p. Moreover, we have the following notation: $\Omega^{\bullet}(L_p,F^*_{L_p}\otimes o(TL_p))$ is the space of smooth de Rham forms on L_p, $H^{\bullet}(L_p,F^*_{L_p}\otimes o(TL_p))$ is the cohomology of L_p. Let

$$\Xi^k_p \subset \Omega^k(L_p,F^*_{L_p}\otimes o(TL_p)) \tag{5.4}$$

be a set of closed forms on L_p, whose cohomology classes form a basis of $H^k(L_p,F^*_{L_p}\otimes o(TL_p))$, $\operatorname{span}(\Xi^k_p)\simeq H^k(L_p,F^*_{L_p}\otimes o(TL_p))$.

To a given anti-radial Morse-Smale pair (f, g^{TX}) and the sets Ξ_p^{n-k}, $p \in \mathrm{Sing}(X)$, $k \geqslant n-\overline{q}$, one can associate a geometric complex, which computes the intersection homology of X with values in F^* and perversity $\overline{q}$, $IH_\bullet^{\overline{q}}(X, F^*)$.

We equip the unstable manifolds of points in $\mathrm{Crit}(f_{sm})$ with an orientation, this induces co-orientations on all stable manifolds.

Let $p \in \mathrm{Sing}(X)$ and $q \in \mathrm{Crit}_{k-1}(f_{sm})$. Let $\varphi_p : B_\delta(p) \to c_\delta L_p$ be the chart in (2.3). In the following we make the identification $\varphi_p^{-1}(r = a) \simeq L_p$, $0 < a < \delta$. Since $W^s(q)$ is contractible, we can canonically trivialise the bundle F^* along $W^s(q)$ using the flat connection; hence for flat sections h of F^* we identify $h_{|W^s(q)\cap L_p}$ with an element in the fibre F_q^*. Using the Morse-Smale transversality and the flow, one can show that for a closed form $\xi \in \Omega^{n-k}(L_p, F_{L_p}^* \otimes o(TL_p))$, the following integral is well-defined and does not depend on $0 < a < \delta$:

$$\left(\int_{W^s(q)\cap L_p} \xi\right) \in F_q^*. \tag{5.5}$$

For $p, q \in \mathrm{Crit}(f_{sm})$ with $\mathrm{ind}(p) - \mathrm{ind}(q) = 1$, by Morse-Smale transversality, the space of trajectories of Φ starting in p and ending in q is a finite set, which we denote by $\Gamma(p, q)$. The trajectory space $\Gamma(p, q)$ carries an orientation induced from the orientation of the unstable cells and the flow Φ (see [Lau92, Section (c)] for more details). With other words, to each trajectory $\gamma \in \Gamma(p, q)$ we can assign $n_\gamma(p, q) \in \{\pm 1\}$. We denote by $\tau_\gamma : F_p^* \to F_q^*$ the map induced from parallel transport w.r.t. the flat connection along the trajectory γ.

Definition/Proposition 5.2. We denote by $(C_\bullet^{\overline{q}}(X, f, g^{TX}, F^*), \partial_\bullet)$ the following complex:

$$\begin{aligned} &C_k^{\overline{q}}(X, f, g^{TX}, F^*) \\ &= \begin{cases} \left(\bigoplus_{p\in \mathrm{Crit}_k(f_{sm})} \big\langle [W^u(p)] \big\rangle \otimes F_p^*\right) \bigoplus \left(\bigoplus_{p \in \mathrm{Sing}(X)} \mathrm{span}(\Xi_p^{n-k})\right) & \text{if } k \geqslant n - \overline{q}, \\ \bigoplus_{p\in \mathrm{Crit}_k(f_{sm})} \big\langle [W^u(p)] \big\rangle \otimes F_p^* & \text{else.} \end{cases} \end{aligned} \tag{5.6}$$

The boundary operator $\partial_\bullet$ is defined as follows: For $p \in \mathrm{Crit}_k(f_{sm})$, $h \in F_p^*$:

(5.7)
$$\partial_k\left([W^u(p)] \otimes h\right) = \sum_{q \in \mathrm{Crit}_{k-1}(f_{sm})} \sum_{\gamma\in\Gamma(p,q)} n_\gamma(p, q) \cdot [W^u(q)] \otimes \tau_\gamma(h) \in C_{k-1}^{\overline{q}}(X, f, g^{TX}, F^*).$$

For $p \in \mathrm{Sing}(X)$, $k \geqslant n - \overline{q}$, $\xi_p^{n-k} \in \Xi_p^{n-k}$:

$$\partial_k[\xi_p^{n-k}] = \sum_{q\in \mathrm{Crit}_{k-1}(f_{sm})} \left(\int_{W^s(q)\cap L_p} \xi_p^{n-k}\right) \cdot [W^u(q)] \in C_{k-1}^{\overline{q}}(X, f, g^{TX}, F^*). \tag{5.8}$$

The complex $\big(C_\bullet^{\overline{q}}(X, f, g^{TX}, F^*), \partial_\bullet\big)$ is well-defined, *i.e.* $\partial_\bullet^2 = 0$.

Remark 5.3. In [Lud17a] the well-definedness of $(C_\bullet^{\overline{q}}(X, f, g^{TX}, F^*), \partial_\bullet)$ has been proved for the case, where the pair (f, g^{TX}) is an anti-radial standard Morse-Smale pair. Using a perturbation argument in [HS85, Proposition 5.1], we can extend the result to the case of an anti-radial Morse-Smale pair (f, g^{TX}), which is not necessarily standard, as follows:

For $\epsilon > 0$ one can construct an anti-radial Morse function f_ϵ and a Riemannian metric g_ϵ^{TX} on X_{sm} with the following properties

- $\mathrm{Crit}(f) = \mathrm{Crit}(f_\epsilon)$.
- The pair $(f_\epsilon, g_\epsilon^{TX})$ coincides with (f, g^{TX}) on $X \backslash (\cup_{p \in \mathrm{Crit}(f_{sm})} B_\epsilon(p))$.
- In a neighbourhood $B_{\epsilon/2}(p)$, $p \in \mathrm{Crit}(f_{sm})$, the metric g_ϵ^{TX} is the Euclidean metric in the Morse coordinates for f_ϵ near p.
- The pair $(f_\epsilon, g_\epsilon^{TX})$ is an anti-radial standard Morse-Smale pair and the complexes $(C_\bullet^{\overline{q}}(X, f, g^{TX}, F^*), \partial_\bullet)$ and $(C_\bullet^{\overline{q}}(X, f_\epsilon, g_\epsilon^{TX}, F^*), \partial_\bullet)$ coincide. More precisely the perturbation does not affect the stable and unstable sets.

The above complex computes the intersection homology of X:

$$H_\bullet\left(C_\bullet^{\overline{q}}(X, f, g^{TX}, F^*), \partial_\bullet\right) \simeq IH_\bullet^{\overline{q}}(X, F^*). \tag{5.9}$$

For an anti-radial standard Morse-Smale pair (f, g^{TX}) this has been proved in [Lud17a, Theorem 6.2]. By Remark 5.3 the isomorphism (5.9) also holds if the pair is not standard.

Let $(C_\bullet^{sm}, \partial_\bullet)$ denote the subcomplex of $(C_\bullet^{\overline{q}}(X, f, g^{TX}, F^*), \partial_\bullet)$ generated by $\mathrm{Crit}(f_{sm})$. There is an exact sequence of complexes

$$0 \to (C_\bullet^{sm}, \partial_\bullet) \to (C_\bullet^{\overline{q}}(X, f, g^{TX}, F^*), \partial_\bullet) \to ((C^{\overline{q}}(X, f, g^{TX}, F^*)/C^{sm})_\bullet, \partial_\bullet) \to 0. \tag{5.10}$$

Note that the boundary map in the quotient complex $((C^{\overline{q}}(X, f, g^{TX}, F^*)/C^{sm})_\bullet, \partial_\bullet)$ is trivial. By the local calculation for intersection homology (2.12) and Poincaré duality on the link manifold L_p, we have

$$H_\bullet((C^{\overline{q}}(X, f, g^{TX}, F^*)/C^{sm})_\bullet, \partial_\bullet) \simeq \bigoplus_{p \in \mathrm{Sing}(X)} IH_\bullet^{\overline{q}}(cL_p, L_p, F^*). \tag{5.11}$$

From (5.9) and (5.10) we get a natural isomorphism

(5.12)

$$\begin{aligned} \det H_\bullet(C_\bullet^{sm}, \partial_\bullet) \otimes \det(\oplus_{p \in \mathrm{Sing}(X)} IH_\bullet^{\overline{q}}(cL_p, L_p, F^*)) &\simeq \det H_\bullet(C_\bullet^{\overline{q}}(X, f, g^{TX}, F^*), \partial_\bullet) \\ &\simeq \det IH_\bullet^{\overline{q}}(X, F^*) \simeq (\det IH_{\overline{q}}^\bullet(X, F))^{-1}. \end{aligned}$$

The metrics g^{F_p} on the fibre F_p, $p \in \mathrm{Crit}(f_{sm})$, induce a metric on $(C_\bullet^{sm}, \partial_\bullet)$. We get an induced metric on $\det H_\bullet(\mathrm{Hom}((C_\bullet^{sm}, \partial_\bullet), \mathbb{C}))$ (see [Mil66], [BZ92, Section I (d)]). Recall that $\| \ \|^{RS}_{\det IH_{\overline{q}}^\bullet(cL_p, L_p, F)}$, $p \in \mathrm{Sing}(X)$, has been defined in Definition 4.8.

Definition 5.4. We denote by $\| \ \|^{\nabla f, g^{TX}, g^F}_{\det IH_{\overline{q}}^\bullet(X,F)}$ the metric on the line $\det IH_{\overline{q}}^\bullet(X, F)$ induced via the natural isomorphisms (5.12) from the metric on $\det H^\bullet(\mathrm{Hom}((C_\bullet^{sm}, \partial_\bullet), \mathbb{C}))$ and the metrics $\| \ \|^{RS}_{\det IH_{\overline{q}}^\bullet(cL_p, L_p, F)}$, $p \in \mathrm{Sing}(X)$. We call $\| \ \|^{\nabla f, g^{TX}, g^F}_{\det IH_{\overline{q}}^\bullet(X,F)}$ the Bismut-Zhang metric associated to ∇f and the pair of metrics g^{TX}, g^F.

Remark 5.5.

(a) The superscripts g^{TX}, g^F indicate that, by construction, the metric $\| \ \|^{\nabla f, g^{TX}, g^F}_{\det IH_{\overline{q}}^\bullet(X,F)}$ a priori does depend on the metrics $(g^{TL_p}, g^{F_{L_p}})$, $p \in \mathrm{Sing}(X)$, and on $g^F_{|\mathrm{Crit}(f_{sm})}$. Unlike in the smooth situation, even in case g^F flat, $\| \ \|^{\nabla f, g^{TX}, g^F}_{\det IH_{\overline{q}}^\bullet(X,F)}$ is not a purely topological invariant in general. Anomaly formulas for the Bismut-Zhang metric will be discussed in Section 7.3.

(b) In case that L_p is the standard sphere S^{n-1}, *i.e.* p is a smooth point, g^{TX} is Euclidean near p and g^F is flat near p, then the metric $\| \quad \|^{RS}_{\det IH^\bullet_{\bar{q}}(cL_p, L_p, F)} = \| \quad \|^{RS}_{\det H^\bullet(D^n, S^{n-1}, F)}$ is trivial, see [Lud20a, Remark 2.11 (c)]. Then the Bismut-Zhang metric is equal to the Milnor metric defined in [BZ92, Definition 1.9].

5.3. Statement of the Bismut-Zhang Theorem for spaces with isolated conical singularities.

Theorem 5.6. *Let (X, g^{TX}) be a space with isolated conical singularities. Let (F, ∇^F, g^F) be a flat vector bundle over X_{sm} as in Section 2.2. Let $f : X \to \mathbb{R}$ be an anti-radial Morse function, g_0^{TX} a Riemannian metric on X, coinciding with g^{TX} in a neighbourhood of $\operatorname{Sing}(X)$ and such that the pair (f, g_0^{TX}) is Morse-Smale. Set $Y := \nabla_{g_0} f$. Then:*

$$\log\left(\frac{\| \quad \|^{RS}_{\det IH^\bullet_{\bar{q}}(X,F)}}{\| \quad \|^{Y, g^{TX}, g^F}_{\det IH^\bullet_{\bar{q}}(X,F)}}\right)^2 = -\int_X \theta(F, g^F) Y^* \Psi(TX, \nabla^{TX}). \tag{5.13}$$

Remark 5.7. (a) Let us again emphasise that it is important, that the radial coordinate r in (2.3) and (2.22) is the same.

(b) In the case of a smooth compact manifold, by Remark 5.5 (b), the statement of Theorem 5.6 reduces to the extension of the Cheeger-Müller theorem given by Bismut and Zhang in [BZ92, Theorem 0.2].

(c) In case that (F, ∇^F, g^F) is a unitary flat vector bundle, we have $\theta(F, \nabla^F) = 0$. Assuming the Witt and a spectral Witt condition, the formula has been proved in [Lud20a].

5.4. Simplifying assumption. Using the anomaly formulas of Theorems 3.8, 7.5, 7.9 as well as Proposition 3.7 and Remark 5.3, it is clear that to establish the Theorem 5.6 in full generality, it is enough to establish it for any pair g^{TX}, g^F of metrics on X, F. We can therefore assume in particular, that $g^{TX} = g_0^{TX}$, and that g^F is flat near $\operatorname{Crit}(f_{sm})$ and the pair (f, g^{TX}) is an anti-radial standard Morse-Smale pair.

5.5. A contour integral. *For the rest of this Section and for Section 6, we assume that the pair (f, g^{TX}) is an anti-radial standard Morse-Smale pair and that g^F is flat near $\operatorname{Crit}(f_{sm})$.*

Definition 5.8. Let ω be the 1-form on $(T, t) \in \mathbb{R}_{\geqslant 0} \times \mathbb{R}_{>0}$

$$\omega = \frac{dt}{2t} \operatorname{Tr_s}\left[N \exp(-t(D^{\bar{q}}_T)^2)\right] - dT \operatorname{Tr_s}\left[f \exp(-t(D^{\bar{q}}_T)^2)\right]. \tag{5.14}$$

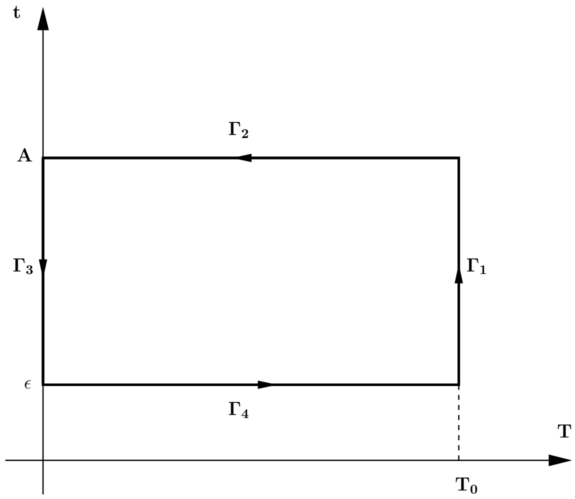
Proposition 5.9. *The form ω is closed and moreover*

$$\omega = \frac{dt}{2t} \operatorname{Tr_s}\left[N \exp\left(-t(\widetilde{D}^{\bar{q}}_T)^2\right)\right] - dT \operatorname{Tr_s}\left[f \exp\left(-t(\widetilde{D}^{\bar{q}}_T)^2\right)\right]. \tag{5.15}$$

Proof. Since, for $l \in \mathbb{N}$ and $T \geqslant 0$, $\operatorname{dom}((D^{\bar{q}}_T)^l) = \operatorname{dom}((D^{\bar{q}})^l)$ (see (4.4)), the first claim can be proved as in [BZ92, Theorem 5.6]. The second claim follows using (4.8). □

Let $\epsilon, A, T_0 \in \mathbb{R}$ with $0 < \epsilon < 1 < A < \infty$ and $0 < T_0 < \infty$. Denote by $\Gamma = \cup_{i=1}^4 \Gamma_i$ the oriented contour in $\mathbb{R}_{\geqslant 0} \times \mathbb{R}_{>0}$ depicted below. For $k = 1, \ldots, 4$, set

$$I_k^0 := \int_{\Gamma_k} \omega. \tag{5.16}$$



As a corollary of Proposition 5.9 we get

Corollary 5.10.

$$\sum_{k=1}^{4} I_k^0 = 0. \tag{5.17}$$

5.6. **Nine intermediate results.** We denote the intersection Euler characteristic of X with perversity $\overline{q}$ and coefficients in F by

$$I\chi_{\overline{q}} := I\chi_{\overline{q}}(X,F) := \sum_{k=0}^{n}(-1)^k \dim IH^k_{\overline{q}}(X,F). \tag{5.18}$$

The "number of critical points of index k" of the anti-radial Morse function f, $c_k^{\overline{q}}(f)$ has been defined in (4.9). From the Spectral Gap Theorem (Theorem 4.1), we get the following Poincaré-Hopf formula

$$I\chi_{\overline{q}} = \mathrm{rk}(F)\cdot\sum_{k=0}^{n}(-1)^k c_k(f_{sm}) + \sum_{p\in \mathrm{Sing}(X)} I\chi_{\overline{q}}(cL_p, L_p, F) = \sum_{k=0}^{n}(-1)^k c_k^{\overline{q}}(f). \tag{5.19}$$

We denote by

$$\begin{aligned} &I\chi'_{\overline{q}} := I\chi'_{\overline{q}}(X,F) := \sum_{k=0}^{n}(-1)^k k \dim IH^k_{\overline{q}}(X,F),\\ &I\widetilde{\chi}'_{\overline{q}} := I\widetilde{\chi}'_{\overline{q}}(X,F) := \sum_{k=0}^{n}(-1)^k k c_k^{\overline{q}}(f),\\ &\mathrm{Tr_s}[f,F,\overline{q}] := \mathrm{rk}(F)\sum_{p\in \mathrm{Crit}(f_{sm})}(-1)^{\mathrm{ind}(p)} f(p) + \sum_{p\in\mathrm{Sing}(X)} f(p)\cdot I\chi_{\overline{q}}(cL_p,L_p,F). \end{aligned} \tag{5.20}$$

Moreover

$$\chi_{sm} := \mathrm{rk}(F)\sum_{k=0}^{n}(-1)^k c_k(f_{sm}), \text{ and } \widetilde{\chi}'_{sm} := \mathrm{rk}(F)\sum_{k=0}^{n}(-1)^k k c_k(f_{sm}). \tag{5.21}$$

Let $T \geqslant 0$. We denote by $P_T^{\overline{q},\,]1,\infty[}$ the orthogonal projection w.r.t. $\langle\ ,\ \rangle_T$ to the space of eigenforms of the Laplacian $\Delta_T^{\overline{q}} = (D_T^{\overline{q}})^2$ to eigenvalues in $]1,\infty[$. We denote by $D_T^{\overline{q},2,\,]0,1]}$ (resp. $D_T^{\overline{q},2,[0,1]}$) the restriction of $\Delta_T^{\overline{q}}$ to the eigenspace of $\Delta_T^{\overline{q}}$ associated to eigenvalues in the interval $]0,1]$ (resp. in the interval $[0,1]$). By the Hodge theorem for the complex $(\mathcal{C}^\bullet_{T,\max/\min}, d_{\max/\min}, \langle\ ,\ \rangle_T)$ and the Cheeger-Goresky-MacPherson theorem (2.19), we have canonical isomorphisms

$$\ker \Delta_T^{\overline{q}} \simeq H^\bullet(\mathcal{C}^\bullet_{T,\max/\min}, d_{\max/\min}, \langle\ ,\ \rangle_T) \simeq IH^\bullet_{\overline{q}}(X,F). \tag{5.22}$$

The twisted L^2-metric $\langle\ ,\ \rangle_T$ restricted to $\ker \Delta_T^{\overline{q}} \simeq IH^\bullet_{\overline{q}}(X,F)$ induces a metric on the line $\det IH^\bullet_{\overline{q}}(X,F)$, which we denote by $|\ \ |^{RS}_{\det IH^\bullet_{\overline{q}}(X,F),T}$.

The following intermediate results are the analogues of [BZ92, Theorems 7.6–7.14] resp. [Lud20a, Theorems 5.4–5.12].

Theorem 5.11. *The following identity holds, for $T \to \infty$,*

(5.23)
$$\mathrm{Tr_s}\left[N \log\left(D_T^{\overline{q},2,]0,1]}\right)\right] - \log\left(\frac{\|\ \ \|^{\nabla f, g^{TX}, g^F}_{\det IH^\bullet_{\overline{q}}(X,F)}}{|\ \ |^{RS}_{\det IH^\bullet_{\overline{q}}(X,F),T}}\right)^2 + 2T\mathrm{Tr_s}[f,F,\overline{q}]$$
$$+\left(\frac{n}{2} I\chi_{\overline{q}} - I\widetilde{\chi}'_{\overline{q}}\right)\log(T) - \left(\frac{n}{2}\chi_{sm} - \widetilde{\chi}'_{sm}\right)\log(\pi) + \sum_{p \in \mathrm{Sing}(X)} \left(\zeta_p^{\overline{q}}\right)'(0) = \mathcal{O}(\exp(-cT)).$$

Theorem 5.12. *Given ϵ, A with $0 < \epsilon < A < \infty$, there exists $C > 0$ such that, for $t \in [\epsilon, A]$, $T \geqslant 1$,*

$$\left|\mathrm{Tr_s}[N \exp(-t(D_T^{\overline{q}})^2)] - I\widetilde{\chi}'_{\overline{q}}\right| \leqslant \frac{C}{\sqrt{T}}. \tag{5.24}$$

Theorem 5.13. *For any $t > 0$,*

$$\lim_{T\to\infty} \mathrm{Tr_s}\left[N \exp(-t(D_T^{\overline{q}})^2) P_T^{\overline{q},]1,\infty[}\right] = 0. \tag{5.25}$$

Moreover there exist $c > 0$, $C > 0$ such that, for $t \geqslant 1$, $T \geqslant 0$,

$$\left|\mathrm{Tr_s}\left[N \exp(-t(D_T^{\overline{q}})^2) P^{]1,\infty[}_{\overline{q},T}\right]\right| \leqslant C\exp(-ct). \tag{5.26}$$

Theorem 5.14. *For $T > 0$ large enough and $k = 0, \dots, n$,*

$$\dim \mathcal{S}^k_{T,\max} = c_k^{\overline{m}}(f), \quad \dim \mathcal{S}^k_{T,\min} = c_k^{\overline{n}}(f). \tag{5.27}$$

Moreover $\lim_{T\to\infty} \mathrm{Tr}\left[D_T^{\overline{q},2,[0,1]}\right] = 0.$

Let $e_1, \dots, e_n$ be an orthonormal basis of TX, $e^1, \dots, e^n$ the dual basis of T^*X. Let W be the smooth section of $\Lambda(T^*X)\widehat{\otimes}\widehat{\Lambda(T^*X)}$ defined by

$$W := \frac{1}{2}\sum_{i=1}^n e^i \wedge \widehat{e^i}. \tag{5.28}$$

Note that W does not depend on the choice of orthonormal basis $e_1, \dots, e_n$.

Theorem 5.15. *The following asymptotic expansion holds, as $t \searrow 0$,*

(5.29)
$$\mathrm{Tr_s}\left[N\exp(-t(D^{\overline{q}})^2)\right] = \mathrm{rk}(F)\int_X \int^B W \exp\left(-\frac{\dot{R}^{TX}}{2}\right)\sqrt{t}^{-1} + \frac{n}{2}\cdot I\chi_{\overline{q}}(X,F) + \mathcal{O}(\sqrt{t}).$$

Note, that the leading coefficient in the expansion (5.29) vanishes in case n even. This is due to the fact, that the integrand in the Berezin integral $\int_X \int^B W \exp\left(-\frac{\dot{R}^{TX}}{2}\right)$ is a sum of forms of degree (k,k), with k odd.

Theorem 5.16. *Let $0 < t < 1$ be small enough. Then there exists $c > 0$ such that, as $T \to \infty$,*

$$\operatorname{Tr_s}\left[f \exp(-t(D^{\overline{q}}_T)^2)\right] = \operatorname{Tr_s}[f, F, \overline{q}] + \left(\frac{n}{4} I\chi_{\overline{q}} - \frac{1}{2} I\widetilde{\chi}'_{\overline{q}}\right)\frac{1}{T} + \mathcal{O}(\exp(-cT)). \tag{5.30}$$

The Cheeger invariant $\gamma^{\overline{q}}_p(F)$, $p \in \operatorname{Sing}(X)$, has been defined in (4.52).

Theorem 5.17. *For any $d > 0$, there exists $C > 0$ such that, for $0 < t \leqslant 1$, $0 \leqslant T \leqslant \frac{d}{t}$,*

$$\Bigg|\operatorname{Tr_s}\left[f \exp\Big(-\big(tD^{\overline{q}} + T\widehat{c}(\nabla f)\big)^2\Big)\right] - \operatorname{rk}(F)\int_X f \int^B \exp(-B_{T^2})$$
$$+ \frac{t}{2}\int_X \theta(F, g^F)\int^B \widehat{df}\exp(-B_{T^2}) - \sum_{p \in \operatorname{Sing}(X)} f(p)\gamma^{\overline{q}}_p(F)\Bigg| \leqslant Ct^2.$$

Theorem 5.18. *For any $T > 0$, the following identity holds,*

$$\lim_{t\to 0}\frac{1}{t^2}\left(\operatorname{Tr_s}\left[f \exp\left(-\left(tD^{\overline{q}} + \frac{T}{t}\widehat{c}(\nabla f)\right)^2\right)\right] - \operatorname{Tr_s}[f, F, \overline{q}]\right) =$$
$$= \left(\frac{n}{4}\chi_{sm} - \frac{1}{2}\widetilde{\chi}'_{sm}\right)\frac{1}{T\tanh(T)}$$
$$-\frac{1}{2T}\sum_{p \in \operatorname{Sing}(X)}\left(\operatorname{Tr_s}[N\exp(-\boldsymbol{\Delta}^{p,\overline{q},\perp}_T)] - \sum_{k \geqslant n-\overline{q}}(-1)^k\left(\frac{n}{2} - k\right)b^{k-1}(L_p, F_{L_p})\right).$$

Theorem 5.19. *There exist $t_0 > 0$, $c > 0$, $C > 0$, such that, for $t \in]0, t_0]$ and $T \geqslant 1$,*

$$\left|\frac{1}{t^2}\left(\operatorname{Tr_s}\left[f \exp\left(-\left(tD^{\overline{q}} + \frac{T}{t}\widehat{c}(\nabla f)\right)^2\right)\right] - \operatorname{Tr_s}[f, F, \overline{q}] - \frac{t^2}{T}\left(\frac{n}{4}I\chi_{\overline{q}} - \frac{1}{2}I\widetilde{\chi}'_{\overline{q}}\right)\right)\right|$$
$$\leqslant C\exp(-cT).$$

5.7. **Outline of the proof of the Bismut-Zhang Theorem.** As in [BZ92, Section VII] and in [Lud20a, Section 6], the strategy of proof is the following: First, using the nine intermediate results, one studies each I^0_k, $k = 1, \ldots, 4$, individually, by taking in succession the limits $A \to \infty$, $T_0 \to \infty$ and $\epsilon \to 0$. Each term I^0_k, $k = 1, \ldots, 4$, will diverge at one or several stages. Using the fundamental identity in Corollary 5.10, the Berezin integral formalism (in particular [Lud20a, Theorem 3.4]) and Theorem 5.11, one can prove that the divergences match up and the statement of the Bismut-Zhang Theorem follows. Since the proof is completely analogous to [BZ92, Section VII] and [Lud20a, Section 6] we omit the details here.

6. Proof of the nine intermediate results

In this section we give some comments on the proofs of the nine intermediate results of Section 5.6. Most of the proofs are direct generalisations of the proofs in [Lud20a, Section7], where the case of a Witt space equipped with a unitary bundle has been treated. For most of the proofs of the intermediate results we will just recall the main arguments of the proofs in [Lud20a, Section 7] and explain, why they carry through to the general case. More details are given for the proofs of the Theorems 5.15 and 5.17, where more care is needed for the generalisation of the arguments in [Lud20a]. It is precisely in the proof of Theorem 5.17, that the non flatness of g^F takes its effect and the right hand side of the formula in the Bismut-Zhang Theorem (Theorem 5.6) appears through local index techniques.

In Section 6.1 we give a sketch of the proofs of Theorems 5.11–5.14. Section 6.2 is devoted to the proof of Theorem 5.15. In Section 6.3 we comment on the proofs of Theorems 5.16-5.19 and give the proof of Theorem 5.17 in some detail.

6.1. **Proof of Theorems 5.11-5.14.** The proofs of Theorems 5.11–5.14 rely on the Witten deformation.

Proof of Theorems 5.12-5.14. Using the Witten deformation for the anti-radial Morse function $f : X \to \mathbb{R}$, more precisely [Lud17b, Section 5], one can proceed as in the proof of [BZ92, Theorems 7.7 and 7.8] to get the claims of Theorems 5.12 and 5.13. The proofs in [Lud17b, Section 5] are only given for the lower middle perversity $\overline{m}$, but for deformation parameter $T \in \mathbb{R}$. The Hodge star operator $*^F : \Lambda^k(T^*X) \otimes F \to \Lambda^{n-k}(T^*X) \otimes F^* \otimes o(TX)$ induces isomorphisms of Hilbert complexes $(\widetilde{\mathcal{C}}(X,F)^\bullet_{T,\max/\min}, d_{T,\max/\min}, \langle\,,\,\rangle) \to (\widetilde{\mathcal{C}}^\bullet_{-T,\min/\max}(X, F^*\otimes o(TX)), d_{-T,\min/\max}, \langle\,,\,\rangle)$. Using this duality, the easy part of the Witten deformation (in particular the Spectral Gap Theorem) hold for the perversity $\overline{n}$ as well.

The claim of Theorem 5.14 follows from the Spectral Gap Theorem (Theorem 4.1) and (4.8). □

Proof of Theorem 5.11. The main ingredient of the proof of Theorem 5.11 is the hard part of the Witten deformation, *i.e.* the comparison result of the Witten complex $(\mathcal{S}_{T,\max/\min}, d_{\max,\min}, \langle\,,\,\rangle_T)$ and the Morse-Thom-Smale complex $(C^{\overline{q}}_\bullet(X, f, g^{TX}, F^*), \partial_\bullet)$, defined in Definition 5.2. This comparison result has been established in [Lud17b, Theorem II], without assuming the Witt condition, for the lower middle perversity $\overline{m}$, the case of the upper middle perversity $\overline{n}$ being similar. The proof of Theorem 5.11 follows precisely the proof of the corresponding statement in [Lud20a, Theorem 5.6]. For n odd and the upper middle perversity $\overline{n}$ we have only to additionally take into account in all computations the contribution of $H^{(n-1)/2}(L_p, F_{L_p})$, $p \in \mathrm{Sing}(X)$. □

6.2. **Proof of Theorem 5.15: The asymptotics of** $\mathrm{Tr_s}[N\exp(-t\Delta)]$ **as** $t \searrow 0$**.** The proof of Theorem 5.15 in case n odd works precisely as in the case where the flat vector bundle (F, ∇^F, g^F) is unitary, see [Lud20a, Theorem 5.8].

In this section we give a proof of Theorem 5.15 in case n even. For the rest of this section we therefore always assume that n is even. Recall that for n even, $\overline{q} = \overline{m} = \overline{n}$. We will therefore omit the sub- and superscipt $\overline{q}$.

Definition 6.1. Let $t > 0$.

(a) We denote by $S_t(x, x')$, $x, x' \in X_{sm}$, the kernel of the operator $N\exp(-t\Delta)$ with respect to $d\mathrm{vol}_X$.

(b) Let $p \in \mathrm{Sing}(X)$. For $T \geqslant 0$, we denote by $S^p_{t,T}(x, x')$, $x, x' \in Z_p$, the kernel of the operator $N\exp(-t\boldsymbol{\Delta}^p_T)$ with respect to $d\mathrm{vol}_{Z_p}$. For $T = 0$, we also write $S^p_t(x, x') := S^p_{t,0}(x, x')$.

Proof of Theorem 5.15 in case n even.

Using local index techniques as in the proofs of [BZ92, Theorem 4.20 and Theorem 7.10] and (4.20), one has the following pointwise asymptotic expansion, as $t \searrow 0$,

$$\begin{aligned} &\mathrm{Tr_s}[S_t(x,x)]d\mathrm{vol}_X \\ &= \left(\frac{n}{2}\mathrm{rk}(F)e(TX,\nabla^{TX}) + \frac{1}{2}\int^B W\exp\left(-\frac{\dot{R}^{TX}}{2}\right)\nabla^{TX}\widehat{\theta}(F,g^F)\right)(x) + \mathcal{O}(t) \\ &=: a_0(x)d\mathrm{vol}_X + \mathcal{O}(t). \end{aligned} \tag{6.1}$$

The asymptotic expansion (6.1) is uniform on compact sets, the coefficients depend only on local geometrical data of X_{sm}.

Since ∇^{TX} is torsionfree, we have $\nabla^{TX}W = 0$. Hence, using [BGV04, Proposition 1.50] and the Bianchi identity,

$$\begin{aligned} &\int^B W\exp\left(-\frac{\dot{R}^{TX}}{2}\right)\nabla^{TX}\widehat{\theta}(F,\nabla^F) \\ &= d\left(\int^B W\exp\left(-\frac{\dot{R}^{TX}}{2}\right)\widehat{\theta}(F,\nabla^F)\right) - \int^B \left(\nabla^{TX}W\right)\exp\left(-\frac{\dot{R}^{TX}}{2}\right)\widehat{\theta}(F,\nabla^F) \\ &= d\left(\int^B W\exp\left(-\frac{\dot{R}^{TX}}{2}\right)\widehat{\theta}(F,g^F)\right). \end{aligned} \tag{6.2}$$

Recall that, near $\mathrm{Sing}(X)$, the form $\theta(F,g^F)$ does not depend on the radial coordinate. Using (2.8), (3.19), (5.28) and (6.2) we get that locally near $p \in \mathrm{Sing}(X)$:

$$a_0(r,y)d\mathrm{vol}_X = \frac{1}{2}d\left(\int^B W\exp\left(-\frac{\dot{R}^{TX}}{2}\right)\widehat{\theta}(F,g^F)\right) = d_{L_p}\tau_p \wedge r^{-1}dr \tag{6.3}$$

with $\tau_p \in \Omega^{n-2}(L_p, F_{L_p})$ a smooth form not depending on the radial coordinate r.

Unlike in the case of a smooth manifold, it is not enough to just integrate the pointwise asymptotic expansion (6.1) to get $\mathrm{Tr_s}[N\exp(-t\Delta)]$; indeed the integrals over the local coefficients are not always defined and we have to take the finite part of these integrals instead. For $0 < \epsilon < \delta$, denote by $X_\epsilon := X\backslash(\cup_{p\in\mathrm{Sing}(X)}B_\epsilon(p))$. The finite part of the integral over the constant coefficient in the expansion (6.1) is just given by

$$\int_{X_\epsilon} a_0(x)d\mathrm{vol}_X. \tag{6.4}$$

Using (3.19), (6.1)–(6.4) and Stokes' Theorem we get

$$\int_{X_\epsilon} a_0(x)d\mathrm{vol}_X = \frac{n}{2}\mathrm{rk}(F)\int_X e(TX,\nabla^{TX}). \tag{6.5}$$

There will also be contributions to $\mathrm{Tr_s}[N\exp(-t\Delta)]$ coming from the singularities $p \in \mathrm{Sing}(X)$, which we now explain: From the pointwise asymptotic expansion (6.1) and from (6.3), we get for the kernel of the operator $N\exp(-t\boldsymbol{\Delta}^p)$ on Z_p, as $u \searrow 0$,

$$\int_{L_p}\mathrm{Tr_s}[S^p_u((1,y),(1,y))]d\mathrm{vol}_{L_p} = \int_{L_p} a_0(1,y)d\mathrm{vol}_{L_p} + \mathcal{O}(u) = \mathcal{O}(u). \tag{6.6}$$

Using the scaling property (4.18),

$$S_t^p((r,y),(r,y)) = \frac{1}{r^n} S^p_{t/r^2}((1,y),(1,y)). \tag{6.7}$$

Using (6.6), (6.7) and arguing as in Section 4.4.1 the following integral is well defined:

$$\gamma_p^{tors}(F) := \frac{1}{2}\int_0^\infty \frac{1}{u}\int_{L_p} \mathrm{Tr_s}[S_u^p((1,y),(1,y))]d\mathrm{vol}_{L_p}du. \tag{6.8}$$

From (6.6), (6.7), (6.8) and using the change of variable $u = t/r^2$, as $t \searrow 0$,

$$\begin{aligned}\int_{\{0\leqslant r\leqslant \epsilon\}\times L_p} \mathrm{Tr_s}[S_t^p((r,y),(r,y))]d\mathrm{vol}_{Z_p} &= \frac{1}{2}\int_{\epsilon t}^\infty \frac{1}{u}\int_{L_p} \mathrm{Tr_s}[S_u^p((1,y),(1,y))]d\mathrm{vol}_{L_p}du \\ &= \gamma_p^{tors}(F) + \mathcal{O}(t).\end{aligned} \tag{6.9}$$

Using Duhamel's principle and the singular elliptic estimates of Lesch [Les97, Section 1.4], one has that, for $N > 0$, there exists an L^2-integrable function $\rho : X \to \mathbb{R}$, such that for $x \in Z_{p,\epsilon} \simeq B_\epsilon(p)$,

$$|S_t(x,x) - S_t^p(x,x)| \leqslant \rho^2(x)t^N, \tag{6.10}$$

see [Che83, Section 1], [Les97, Theorem 1.4.11]. Using (6.1), (6.5), (6.9) and (6.10) we get, as $t \searrow 0$,

$$\begin{aligned}\mathrm{Tr_s}[N\exp(-t\Delta)] &= \int_{X_\epsilon} \mathrm{Tr_s}[S_t(x,x)]d\mathrm{vol}_X + \sum_{p\in \mathrm{Sing}(X)}\int_{B_\epsilon(p)} \mathrm{Tr_s}[S_t(x,x)]d\mathrm{vol}_X \\ &= \int_{X_\epsilon} \mathrm{Tr_s}[S_t(x,x)]d\mathrm{vol}_X + \sum_{p\in \mathrm{Sing}(X)}\int_{\{0\leqslant r\leqslant \epsilon\}\times L_p} \mathrm{Tr_s}[S_t^p(x,x)]d\mathrm{vol}_{Z_p} + \mathcal{O}(t) \\ &= \int_{X_\epsilon} a_0(x)d\mathrm{vol}_X + \gamma_p^{tors}(F) + \mathcal{O}(t) \\ &= \frac{n}{2}\mathrm{rk}(F)\int_X e(TX,\nabla^{TX}) + \gamma_p^{tors}(F) + \mathcal{O}(t).\end{aligned} \tag{6.11}$$

Cheeger's Chern-Gauss-Bonnet Theorem for spaces with isolated conical singularities [Che83, Theorem 5.1] states that

$$I\chi(X,F) = \mathrm{rk}(F)\int_X e(TX,\nabla^{TX}) + \gamma_p(F), \tag{6.12}$$

where the Cheeger invariant $\gamma_p(F)$ has been defined in (4.52). Comparing (6.11) and (6.12) and using the below Proposition 6.3 (b) we get the claim.

□

Remark 6.2. (a) In the asymptotic expansions, as $t \searrow 0$, of $\mathrm{Tr}[\exp(-t\Delta^{(k)})]$, $k = 0,\dots,n$, there are logarithmic terms $\log(t)$ appearing, which by taking the alternating weighted sum do cancel out. More precisely, the logarithmic term in the expansion (6.11) of $\mathrm{Tr_s}[N\exp(-t\Delta)]$ is

$$\sum_{p\in \mathrm{Sing}(X)}\int_{L_p} a_0(1,y)d\mathrm{vol}_{L_p}, \tag{6.13}$$

which vanishes by (6.3).

(b) Note that using the asymptotic expansion in Theorem 5.15 and the Mellin transform one can show that the torsion zeta function (5.1) extends to a meromorphic function on $\mathbb{C}$, which is holomorphic at 0. For this latter result, it is crucial that no logarithmic term appears in the asymptotic expansion of $\mathrm{Tr_s}[N\exp(-t\Delta)]$.

Proposition 6.3.

(a) *We have as $t \searrow 0$,*

(6.14)
$$\left|\mathrm{Tr_s}[N\exp(-t\mathbf{\Delta}^p_1)] - t^{-1}\mathrm{rk}(F)\int_{Z_p}\int^{B,p} W\exp(-B^p_1) - \gamma^{tors}_p(F) - \frac{n}{2}\mathrm{rk}(F)\alpha_p\right| \to 0.$$

(b) *We have the following relation between the Cheeger invariant and the torsion Cheeger invariant:*

$$\gamma^{tors}_p(F) = \frac{n}{2}\gamma_p(F). \tag{6.15}$$

Proof. (a) Let $T > 0$ be fixed. We proceed as in the proof of Theorem 4.11. First, we split the integral into two parts

(6.16)
$$\begin{aligned}\mathrm{Tr_s}\left[N\exp(-t\mathbf{\Delta}^p_T)\right] &= \int_{Z_p}\mathrm{Tr_s}\left[S^p_{t,T}\big((r,y),(r,y)\big)\right]d\mathrm{vol}_{Z_p}\\ &= \int_{\{0\leqslant r\leqslant t^{1/4}\}\times L_p}\mathrm{Tr_s}\left[S^p_{t,T}\big((r,y),(r,y)\big)\right]d\mathrm{vol}_{Z_p} + \int_{\{r\geqslant t^{1/4}\}\times L_p}\mathrm{Tr_s}\left[S^p_{t,T}\big((r,y),(r,y)\big)\right]d\mathrm{vol}_{Z_p}.\end{aligned}$$

Proceeding as in Step 1 of the proof of Theorem 4.11, we get for the first integral on the right hand side of (6.16), as $t \searrow 0$,

$$\begin{aligned}\int_{\{0\leqslant r\leqslant t^{1/4}\}\times L_p}&\mathrm{Tr_s}\left[S^p_{t,T}\big((r,y),(r,y)\big)\right]d\mathrm{vol}_{Z_p}\\ &= \int_{\{0\leqslant r\leqslant 1/t^{1/4}\}\times L_p}\mathrm{Tr_s}\left[S^p_{1,tT}\big((r,y),(r,y)\big)\right]d\mathrm{vol}_{Z_p}\\ &= \int_{\{0\leqslant r\leqslant 1/t^{1/4}\}\times L_p}\mathrm{Tr_s}\left[S^p_{1}\big((r,y),(r,y)\big)\right]d\mathrm{vol}_{Z_p} + \mathcal{O}(t)\\ &= \frac{1}{2}\int_{\sqrt{t}}^{\infty}\frac{du}{u}\int_{L_p}\mathrm{Tr_s}\left[S^p_{u}\big((1,y),(1,y)\big)\right]d\mathrm{vol}_{L_p} + \mathcal{O}(t)\\ &\xrightarrow{t\searrow 0}\frac{1}{2}\int_0^{\infty}\frac{du}{u}\int_{L_p}\mathrm{Tr_s}\left[S^p_u((1,y),(1,y))\right]d\mathrm{vol}_{L_p} = \gamma^{tors}_p(F).\end{aligned} \tag{6.17}$$

We now use local index techniques as in Step 2 of Theorem 4.11; the notations are as in loc. cit.: We first scale the operator $t\mathbf{\Delta}^p_T$ via the radial scaling $h_{1/\sqrt{t}} : r \to r/\sqrt{t}$. We get the operator

$$N^p_{t,T} = t^2\mathbf{\Delta}^p_{T/t}. \tag{6.18}$$

Let us denote by $\widetilde{S}^p_{t,T}(x,y) = S^p_{t^2,T/t}(x,y)$, $x,y \in Z_p$, the kernel of $N\exp(-N^p_{t,T})$ w.r.t. $d\mathrm{vol}_{Z_p}$. We hence get for the second integral in (6.16):

(6.19)
$$\int_{\{r\geqslant t^{1/4}\}\times L_p} \mathrm{Tr_s}\left[S^p_{t,T}\big((r,y),(r,y)\big)\right] d\mathrm{vol}_{Z_p} = \int_{\{r\geqslant t^{3/4}\}\times L_p} \mathrm{Tr_s}\left[\widetilde{S}^p_{t,T}\big((r,y),(r,y)\big)\right] d\mathrm{vol}_{Z_p}.$$

As in Step 2 of the proof of Theorem 4.11 we apply to $N^p_{t,T}$ the (local) scaling $x \to tx$ and replace the Clifford variables $c(e_k), \widehat{c}(e_k)$ by $c_t(e_k), \widehat{c}_t(e_k)$ (see (4.69)).

We denote by C_t the operator we get from $\frac{1}{2}\sum c(e_i)\widehat{c}(e_i)$ by the above scaling. We have

$$tC_t \xrightarrow{t\searrow 0} W = \frac{1}{2}\sum_{i=1}^n e^i \wedge \widehat{e}^i. \tag{6.20}$$

Using (4.20), (6.20) and proceeding as in [BZ92, Theorem 13.4], we get the following asymptotics as $t \searrow 0$, uniformly on compact sets,

$$\begin{aligned}
&\mathrm{Tr_s}\left[\widetilde{S}^p_{t,T}\big((r,y),(r,y)\big)\right] d\mathrm{vol}_{Z_p}\\
&= \frac{1}{t}\mathrm{rk}(F)\int^{B,p} W\exp(-B^p_{T^2}) + \int^{B,p} W\left(\frac{1}{2}\nabla^{TZ_p} + \iota_{T\widehat{\nabla f}}\right)\widehat{\theta}(F,g^F)\exp(-B^p_{T^2})\\
&\quad + \frac{n}{2}\mathrm{rk}(F)\int^{B,p}\exp(-B^p_{T^2}) + \mathcal{O}(t)\\
&= \frac{1}{t}\mathrm{rk}(F)\int^{B,p} W\exp(-B^p_{T^2}) + \frac{1}{2}d\left(\int^{B,p} W\widehat{\theta}(F,g^F)\exp(-B^p_{T^2})\right)\\
&\quad + \frac{n}{2}\mathrm{rk}(F)\int^{B,p}\exp(-B^p_{T^2}) + \mathcal{O}(t)\\
&=: \frac{1}{t}a_{-1}(T,(r,y))dr + a_0(T,(r,y))dr + \mathcal{O}(t),
\end{aligned} \tag{6.21}$$

where for the last equality we have used $\nabla^{TZ_p}W = 0$ and [BZ92, (3.23)]. The $(n-1)$-form $\int^{B,p} W\widehat{\theta}(F,g^F)\exp(-B^p_{T^2})$ contains e^r, hence we have

$$d\left(\int^{B,p} W\widehat{\theta}(F,g^F)\exp(-B^p_{T^2})\right) = e^r \wedge d\beta(r,T), \tag{6.22}$$

for an $(n-2)$-form β on the link L_p.

To compensate the non compactness of Z_p, we will use in addition to the pointwise asymptotic expansion (6.21), which is uniform on compact sets only, the homogeneity of the cone: For $a > 0$, we denote by h_a the radial scaling $r \to ar$. We have the following scaling properties

$$\begin{aligned}
&h_a^*\int^{B,p} W\exp(-B_{T^2}) = a\int^{B,p} W\exp(-B_{a^2T^2}),\\
&h_a^*\int^{B,p} W\widehat{\theta}(F,g^F)\exp(-B^p_{T^2}) = \int^{B,p} W\widehat{\theta}(F,g^F)\exp(-B^p_{a^2T^2}),\\
&h_a(N^{p,\overline{q}}_{t,T})h_a^{-1} = N^{p,\overline{q}}_{t/a,aT},\ \ \widetilde{S}^p_{t,T}\big((r,y),(r,y)\big) = a^{-n}\widetilde{S}^p_{t/a,aT}\big((r/a,y),(r/a,y)\big).
\end{aligned} \tag{6.23}$$

Arguing as in [BZ92, Theorem 13.5], the point-wise asymptotic expansion (6.21) is uniform on compact sets for $T \in [0, 1/t]$, as $t \searrow 0$. The Berezin integrals in (6.21) have exponential decay as $r \to \infty$. For $r \geqslant t^{3/4}$ we have $u := t/r \to 0$ as $t \to 0$. Using (6.21), (6.22), (6.23) and the change of variables $u = t/r$, we get

(6.24)
$$\begin{aligned}
&\int_{\{r\geqslant t^{3/4}\}\times L_p} \mathrm{Tr_s}\left[\widetilde{S}^p_{t,T}\big((r,y),(r,y)\big)\right] d\mathrm{vol}_{Z_p}\\
&= \int_0^{t^{1/4}} \frac{1}{u}\int_{L_p} \mathrm{Tr_s}\left[\widetilde{S}^p_{u,\frac{Tt}{u}}\big((1,y),(1,y)\big)\right] d\mathrm{vol}_{L_p} du\\
&= \int_0^{t^{1/4}} \frac{1}{u}\int_{L_p} \left\{u^{-1}a_{-1}\left(\frac{Tt}{u},(1,y)\right) + a_0\left(\frac{Tt}{u},(1,y)\right) + \mathcal{O}(u)\right\} du\\
&= \frac{1}{t}\mathrm{rk}(F)\int_{\{r\geqslant t^{3/4}\}\times L_p}\int^{B,p} W\exp(-B^p_{T^2}) + \frac{n}{2}\mathrm{rk}(F)\int_{\{r\geqslant t^{3/4}\}\times L_p}\int^{B,p}\exp(-B^p_{T^2}) + \mathcal{O}(t^{1/4}).
\end{aligned}$$

We have, using the properties of the involved Berezin integrals,

$$\int_{\{r\leqslant t^{3/4}\}\times L_p}\int^{B,p} W\exp(-B^p_{T^2}) = \mathcal{O}(t^{3/2}), \int_{\{r\leqslant t^{3/4}\}\times L_p}\int^{B,p}\exp(-B^p_{T^2}) = \mathcal{O}(t^{3/4}). \tag{6.25}$$

From (3.25), (6.19), (6.24) and (6.25), we get for the second integral on the right hand side of (6.16) as $t \searrow 0$,

(6.26)
$$\begin{aligned}
&\int_{\{r\geqslant t^{1/4}\}\times L_p} \mathrm{Tr_s}\left[S^p_{t,T}\big((r,y),(r,y)\big)\right] d\mathrm{vol}_{Z_p}\\
&\qquad= \int_{\{r\geqslant t^{3/4}\}\times L_p} \mathrm{Tr_s}\left[\widetilde{S}^p_{t,T}\big((r,y),(r,y)\big)\right] d\mathrm{vol}_{Z_p}\\
&\qquad= \frac{1}{t}\mathrm{rk}(F)\int_{Z_p}\int^{B,p} W\exp(-B^p_{T^2}) + \frac{n}{2}\mathrm{rk}(F)\int_{Z_p}\int^{B,p}\exp(-B^p_{T^2}) + \mathcal{O}(t^{1/4})\\
&\qquad= \frac{1}{t}\mathrm{rk}(F)\int_{Z_p}\int^{B,p} W\exp(-B^p_{T^2}) + \frac{n}{2}\mathrm{rk}(F)\alpha_p + \mathcal{O}(t^{1/4}).
\end{aligned}$$

Using (6.16), (6.17), (6.26) and setting $T = 1$, we get the claim.

(b) In [Lud20a, Proposition 4.13] we have proved another asymptotic expansion of $\mathrm{Tr_s}\left[N\exp\left(-t\mathbf{\Delta}^p_1\right)\right]$ as $t \searrow 0$:

$$\left|\mathrm{Tr_s}[N\exp(-t\mathbf{\Delta}^p_1)] - \frac{\mathrm{rk}(F)}{t}\int_{Z_p} r^2\int^{B,p}\exp(-B^p_1) - \frac{n}{2}I\chi(cL_p, L_p, F)\right| \to 0. \tag{6.27}$$

Using Theorem 4.11 and comparing the constant terms in the asymptotic expansions (6.14) and (6.27), we get the claim. (Equality of the leading coefficients is a consequence of (3.6)).

□

6.3. **Proof of Theorems 5.16-5.19.** The proofs of Theorems 5.16-5.19 consist of two steps: a localisation argument and an explicit computation using the local model operator near $\mathrm{Crit}(f)$; the latter has been taken care of in Section 4.2. The localisation argument in the proofs of Theorems 5.16-5.19 do mostly rely on the singular elliptic estimates of Lesch [Les97, Section 1.4] for $\Delta^{\overline{q}}$, the Spectral Gap Theorem, Theorem 4.1, for the operator $\widetilde{\Delta}^{\overline{q}}_T$, and the fact that $\mathrm{dom}(\Delta^{\overline{q}})^l = \mathrm{dom}(\widetilde{\Delta}^{\overline{q}}_T)^l$, $l \in \mathbb{N}$ (see (4.4)). All these ingredients hold without assuming the Witt condition and without assuming that g^F is flat; the singular elliptic estimates of Lesch are available for every closed extension of a symmetric elliptic differential operator of Fuchs type.

Hence the proofs of Theorems 5.16-5.19 follow by a direct generalisation of the proofs of the corresponding statements in [Lud20a]. Here we only give the proof of Theorem 5.17 in some detail; it is at this stage, that the non-flatness of g^F comes into play and the term on the right hand side of the Bismut-Zhang formula appears.

Let $\delta > 0$ be as in Section 2.1(1). For the whole section, we choose $\epsilon < \delta/2$ small enough such that

- the balls $B_{2\epsilon}(p)$, $p \in \mathrm{Crit}(f)$, do not intersect each other,
- the restriction of the Riemannian metric g^{TX} to $B_{2\epsilon}(p)$, $p \in \mathrm{Crit}(f_{sm})$, is Euclidean in the Morse coordinates,
- the restriction of g^F to $B_{2\epsilon}(p)$, $p \in \mathrm{Crit}(f_{sm})$, is flat.

We identify a 2ϵ-neighbourhood $B_{2\epsilon}(p)$ of a critical point $p \in \mathrm{Crit}(f_{sm})$ (resp. $p \in \mathrm{Sing}(X)$) with a neighbourhood in $\mathbb{R}^n$ (resp. in cL_p).

The action of the Witten Laplacian $\widetilde{\Delta}^{\overline{q}}_T$ on forms with compact support near $p \in \mathrm{Crit}(f_{sm})$ can be identified with the action of the model Witten Laplacian $\mathbf{\Delta}^p_T$, which is an operator acting on sections of $\Lambda((\mathbb{R}^n)^*) \otimes F_p$, see [BZ92, Proposition 8.2] for more details. Let us emphasise that the model Witten Laplacian $\mathbf{\Delta}^p_T$, $p \in \mathrm{Crit}(f_{sm})$, does of course not depend on the perversity $\overline{q}$, *i.e.* on the chosen extension $\widetilde{\Delta}^{\overline{q}}_T$.

For $p \in \mathrm{Sing}(X)$, $(t,T) \in \mathbb{R}_{>0} \times \mathbb{R}_{\geqslant 0}$ we have denoted by $U^{p,\overline{q}}_{t,T}(x,x')$, $x, x' \in Z_p$, the kernel of the operator $\exp\big(-t^2 \mathbf{\Delta}^{p,\overline{q}}_{T/t}\big)$ with respect to $d\mathrm{vol}_{Z_p}$ (see Definition 4.3).

Definition 6.4. Let $(t,T) \in \mathbb{R}_{>0} \times \mathbb{R}_{\geqslant 0}$.

(a) Let $p \in \mathrm{Crit}(f_{sm})$. We denote by $U^p_{t,T}(x,x') = U^{p,\overline{q}}_{t,T}(x,x')$, $x, x' \in \mathbb{R}^n \simeq T_pX$, the kernel of the operator $\exp\big(-t^2 \mathbf{\Delta}^p_{T/t}\big)$ with respect to $d\mathrm{vol}_{\mathbb{R}^n}$. (The superscript $\overline{q}$ is just introduced to unify notations, there is no dependence on the perversity at this point).

(b) We denote by $U^{\overline{q}}_{t,T}(x,x')$, $x, x' \in X_{sm}$, the kernel of the operator $\exp(-(tD^{\overline{q}} + T\widehat{c}(\nabla f))^2)$ with respect to $d\mathrm{vol}_X$.

Proof of Theorem 5.17: An estimate of $\mathrm{Tr_s}\Big[f\exp\Big(-\big(tD^{\overline{q}} + T\widehat{c}(\nabla f)\big)^2\Big)\Big]$ in the range $0 < t \leqslant 1$, $0 \leqslant T \leqslant \frac{d}{t}$.

Set $X'_\epsilon := X \backslash \big(\cup_{p \in \mathrm{Crit}(f)} B_\epsilon(p)\big)$. Using local index techniques as in [BZ92, Theorems 13.4 and 13.5] one can prove that there exists $C > 0$ such that for $t \in]0,1]$, $T \in [0, d/t]$ and $x \in X'_\epsilon$,

(6.28)
$$\left| \mathrm{Tr_s}\left[U^{\overline{q}}_{t,T}(x,x)\right] d\mathrm{vol}_X - \mathrm{rk}(F)\int^B \exp(-B_{T^2}) - \frac{t}{2} d\left(\int^B \widehat{\theta}(F,g^F)\exp(-B_{T^2})\right)\right| \leqslant Ct^2.$$

Let $p \in \mathrm{Crit}(f)$. Using the singular elliptic estimates of Lesch for $\Delta^{\overline{q}}$, the Spectral Gap Theorem for $\widetilde{\Delta}^{\overline{q}}_T$ and a finite propagation speed argument, and proceeding as in [Lud20a, Proposition 7.8], we can prove, that there exist $c > 0$ and a continuous L^2-function $\rho : X_{sm} \to \mathbb{R}_{\geqslant 0}$ such that, for $t \in]0,1]$, $T \in [0, d/t]$ and $x \in B_\epsilon(p)$,

$$\left|U^{\overline{q}}_{t,T}(x,x) - U^{p,\overline{q}}_{t,T}(x,x)\right| \leqslant \rho^2(x) \exp\left(-\frac{c}{t^2}\right). \tag{6.29}$$

On $B_{2\epsilon}(p)$, $p \in \mathrm{Sing}(X)$, we have by [BZ92, Theorem 3.13] and (4.46),

$$\theta(F, g^F) \int^B \widehat{df} \exp(-B_{T^2}) = df \int^B \widehat{\theta}(F, g^F) \exp(-B_{T^2}) = 0. \tag{6.30}$$

The form $\int^B \widehat{\theta}(F, g^F) \exp(-B_{T^2})$ is an $(n-1)$-form containing e^r, hence

$$\left(\int^B \widehat{\theta}(F, g^F) \exp(-B_{T^2})\right)_{|\partial B_\epsilon(p)} = 0. \tag{6.31}$$

The identities (6.30) and (6.31) also hold for $p \in \mathrm{Crit}(f_{sm})$, since by our assumption g^F is flat, hence $\theta(F, g^F) = 0$, on $B_{2\epsilon}(p)$.

By [BZ92, Theorem 3.13], (6.31) and Stokes' Theorem, we have

$$\begin{aligned}
\int_{X'_\epsilon} \theta(F, g^F) \int^B \widehat{df} \exp(-B_{T^2}) &= \int_{X'_\epsilon} df \int^B \widehat{\theta}(F, g^F) \exp(-B_{T^2}) \\
&= \int_{X'_\epsilon} d\left(f(x) \int^B \widehat{\theta}(F, g^F) \exp(-B_{T^2})\right) \\
&\quad - \int_{X'_\epsilon} f(x) d\left(\int^B \widehat{\theta}(F, g^F) \exp(-B_{T^2})\right) \\
&= \int_{\partial X'_\epsilon} f(x) \int^B \widehat{\theta}(F, g^F) \exp(-B_{T^2}) - \int_{X'_\epsilon} f(x) d\left(\int^B \widehat{\theta}(F, g^F) \exp(-B_{T^2})\right) \\
&= - \int_{X'_\epsilon} f(x) d\left(\int^B \widehat{\theta}(F, g^F) \exp(-B_{T^2})\right).
\end{aligned} \tag{6.32}$$

By (6.28) and (6.32), for $t \in]0,1]$, $T \in [0, d/t]$:

(6.33)

$$\begin{aligned}
&\int_{X'_\epsilon} \left\{ f(x) \left(\mathrm{Tr_s}[U^{\overline{q}}_{t,T}(x,x)] d\mathrm{vol}_X - \mathrm{rk}(F) \int^B \exp(-B_{T^2}) \right) + \frac{t}{2} \theta(F, g^F) \int^B \widehat{df} \exp(-B_{T^2}) \right\} \\
&\quad = \int_{X'_\epsilon} f(x) \left\{ \mathrm{Tr_s}[U^{\overline{q}}_{t,T}(x,x)] d\mathrm{vol}_X - \mathrm{rk}(F) \int^B \exp(-B_{T^2}) \right\} \\
&\qquad\qquad - \frac{t}{2} \int_{X'_\epsilon} f(x) d\left(\int^B \widehat{\theta}(F, g^F) \exp(-B_{T^2})\right) = \mathcal{O}(t^2).
\end{aligned}$$

By (6.29), (6.30) and (6.33) for $t \in]0,1]$, $T \in [0, d/t]$:

(6.34)

$$\Big| \mathrm{Tr_s}\left[f \exp\left(-(tD^{\overline{q}} + T\widehat{c}(\nabla f))^2\right)\right] - \mathrm{rk}(F) \int_X f(x) \int^B \exp(-B_{T^2}) - \sum_{p \in \mathrm{Sing}(X)} f(p) \gamma^{\overline{q}}_p(F)$$

$$
\begin{aligned}
&\quad + \frac{t}{2}\int_X \theta(F,g^F)\int^B \widehat{df}\exp(-B_{T^2})\Big| \\
&= \Big|\int_X f(x)\left\{\mathrm{Tr_s}[U^{\overline{q}}_{t,T}(x,x)]d\mathrm{vol}_X - \mathrm{rk}(F)\int^B \exp(-B_{T^2})\right\} - \sum_{p\in \mathrm{Sing}(X)} f(p)\gamma^{\overline{q}}_p(F) \\
&\quad + \frac{t}{2}\int_X \theta(F,g^F)\int^B \widehat{df}\exp(-B_{T^2})\Big| \\
&\leqslant \sum_{p\in\mathrm{Sing}(X)}\left|\int_{B_\epsilon(p)} f(x)\left\{\mathrm{Tr_s}\left[U^{p,\overline{q}}_{t,T}(x,x)\right]d\mathrm{vol}_{B_\epsilon(p)} - \mathrm{rk}(F)\int^B \exp(-B_{T^2})\right\} - f(p)\gamma^{\overline{q}}_p(F)\right| \\
&\quad + \sum_{p\in\mathrm{Crit}(f_{sm})}\left|\int_{B_\epsilon(p)} f(x)\left\{\mathrm{Tr_s}\left[U^{p}_{t,T}(x,x)\right]d\mathrm{vol}_{B_\epsilon(p)} - \mathrm{rk}(F)\int^B \exp(-B_{T^2})\right\}\right| \\
&\quad + Ct^2.
\end{aligned}
$$

Let $p \in \mathrm{Crit}(f_{sm})$. By [BZ92, (13.62)], we have for $t \in]0,1]$, $T \in [0, d/t]$,

$$
\left|\int_{B_\epsilon(p)} f(x)\Big(\mathrm{Tr_s}\left[U^{p}_{t,T}(x,x)\right]d\mathrm{vol}_{B_\epsilon(p)} - \mathrm{rk}(F)\int^B \exp(-B_{T^2})\Big)\right| \leqslant Ct^2. \tag{6.35}
$$

The claim of Theorem 5.17 follows from Proposition 4.5, (6.30), (6.34) and (6.35). □

7. Anomaly formulas

In this section we will always consider a family of metrics $l \in \mathbb{R} \to (g^{TX}_l, g^F_l)$ on TX, F depending smoothly on the parameter l. We assume moreover that the metrics g^{TX}_l are conical, *i.e.* for $p \in \mathrm{Sing}(X)$, there is a family of Riemannian metrics $l \in \mathbb{R} \to g^{TL_p}_l$, such that $g^{TX}_l = dr^2 + r^2 g^{TL_p}_l$ near p. Similarly, we assume that near $p \in \mathrm{Sing}(X)$, the metric g^F_l is of the form explained in Section 2.2.

In the following we use a sub- resp. a superscript l to characterise operators associated to the pair (g^{TX}_l, g^F_l).

7.1. Spectral gap condition.

In this section we explain an additional assumption on the metrics (g^{TX}, g^F), which will be in place for most of the results in Section 7. It is used in particular in the proof of Proposition 7.3 to deal with the boundary terms appearing near $p \in \mathrm{Sing}(X)$ when applying Stokes' Theorem.

For $p \in \mathrm{Sing}(X)$, the following spectral gap condition for the first order differential operator S_p on the link L_p, defined in (4.14), will be assumed:

$$
\mathrm{Spec}(S_p) \cap (-1/2, 1/2) = \{0\}. \tag{7.1}
$$

By [BL93, Corollary 2.3] condition (7.1) is equivalent to the following spectral gap condition for the transversal Laplacian Δ_{L_p} on the link L_p:

(7.2)

$$
\begin{cases}
\mathrm{Spec}(\Delta^{(\nu-1)}_{L_p,ccl}) \cap (0,1) = \varnothing & \text{if } n = 2\nu \text{ is even}, \\
\Big(\mathrm{Spec}(\Delta^{(\nu-1)}_{L_p,ccl}) \cap (0,3/4)\Big) \cup \Big(\mathrm{Spec}(\Delta^{(\nu)}_{L_p,ccl}) \cap (0,3/4)\Big) = \varnothing & \text{if } n = 2\nu+1 \text{ is odd}.
\end{cases}
$$

Condition (7.1) (and hence (7.2)) can be achieved by a rescaling of the metric g^{TL_p} into $c^2 g^{TL_p}$ with $c > 0$ sufficiently small.

We denote by $D_{\min}$ (resp. $D_{\max}$) the minimal (resp. the maximal) closed extension of the first order operator $D_c = d_c + \delta_c$. By [BS88, Theorem 3.2 and Lemma 3.2] we have

$$\operatorname{dom}(D_{\min}) = \left\{ \omega \in \operatorname{dom}(D_{\max}) \,\middle|\, \begin{array}{c} \|U^{-1}(\omega)\|_{\mathrm{L}^2(\Lambda(TL_p)\otimes F)} = o(r^{1/2}|\log r|^{1/2}) \\ \text{locally near } p \in \operatorname{Sing}(X) \end{array} \right\}, \tag{7.3}$$

where U is the unitary transformation defined in Section 4.2.1. Also by [BS88, Theorem 3.2 and Lemma 3.2], in case n even, assuming the spectral gap condition (7.1) the operator D_c is essentially self-adjoint, hence

$$D^{\overline{m}} = D^{\overline{n}} = D_{\min} = D_{\max}. \tag{7.4}$$

In case $n = 2\nu + 1$ odd, assuming (7.1), we have

$$\operatorname{dom}(D^{\mathrm{ev}}_{\max})/\operatorname{dom}(D^{\mathrm{ev}}_{\min}) \simeq \bigoplus_{p\in\operatorname{Sing}(X)} \mathscr{H}^\nu(L_p, F_{L_p}). \tag{7.5}$$

By [BS88, Lemma 3.2], [ALMP18, Section 5], there are continuous linear functionals $a, b : \operatorname{dom}(D_{\max}) \to \bigoplus_{p\in\operatorname{Sing}(X)} \mathscr{H}^\nu(L_p, F_{L_p})$ such that, for $\omega \in \operatorname{dom}(D_{\max})$, locally near $p \in \operatorname{Sing}(X)$,

$$\omega - a(\omega) - b(\omega) \wedge dr \in \operatorname{dom}(D_{\min}). \tag{7.6}$$

Moreover using [ALMP18, Lemma 5.2] we can characterise the extensions $D^{\overline{q}}$, $\overline{q} \in \{\overline{m}, \overline{n}\}$, by

$$\begin{aligned} \operatorname{dom}(D^{\overline{m}}) &= \{\omega \in \operatorname{dom}(D_{\max}) \mid b(\omega) = 0\}, \\ \operatorname{dom}(D^{\overline{n}}) &= \{\omega \in \operatorname{dom}(D_{\max}) \mid a(\omega) = 0\}. \end{aligned} \tag{7.7}$$

7.2. **Anomaly formula for the Ray-Singer metric $\|\ \ \|^{RS}_{\det IH^\bullet_{\overline{q}}(X,F)}$.** Let $e_1, \dots, e_n$ be an ONB of (TX, g_l^{TX}). We denote by $*_l$, $*_l^F$ the Hodge star operators associated to the metrics g_l^{TX}, (g_l^{TX}, g_l^F). By [BZ92, Proposition 4.15], we have

$$*_l^{-1}\frac{\partial *_l}{\partial l} = -\frac{1}{2}\sum_{1\leqslant i,j\leqslant n}\left\langle (g_l^{TX})^{-1}\frac{\partial g_l^{TX}}{\partial l}e_i, e_j\right\rangle_{g_l^{TX}} c(e_i)\widehat{c}(e_j). \tag{7.8}$$

Set

$$\dot{\omega}_l^X := -\frac{1}{2}\sum_{1\leqslant i,j\leqslant n}\left\langle (g_l^{TX})^{-1}\frac{\partial g_l^{TX}}{\partial l}e_i, e_j\right\rangle_{g_l^{TX}} e^i \wedge \widehat{e}^j. \tag{7.9}$$

Similarly to (7.9) we define, for $p \in \operatorname{Sing}(X)$,

$$\begin{aligned} \dot{\omega}_l^{Z_p} &:= -\frac{1}{2}\sum_{1\leqslant i,j\leqslant n}\left\langle (g_l^{TZ_p})^{-1}\frac{\partial g_l^{TZ_p}}{\partial l}e_i, e_j\right\rangle_{g_l^{TZ_p}} e^i \wedge \widehat{e}^j \\ &= -\frac{1}{2}\sum_{2\leqslant i,j\leqslant n}\left\langle (g_l^{TZ_p})^{-1}\frac{\partial g_l^{TZ_p}}{\partial l}e_i, e_j\right\rangle_{g_l^{TZ_p}} e^i \wedge \widehat{e}^j. \end{aligned} \tag{7.10}$$

Note that the coefficients in the above sum do not depend on the radial coordinate.

We denote by

$$\sigma_l := (*_l^F)^{-1}\frac{\partial *_l^F}{\partial l} = *_l^{-1}\frac{\partial *_l}{\partial l} + (g_l^F)^{-1}\frac{\partial g_l^F}{\partial l}. \tag{7.11}$$

The characteristic class $e(\rho^* TX, \nabla^{TX,\mathrm{tot}})$ associated to the family of conical Riemannian metrics $(g_l^{TX})_l$ has been defined in Section 3.4.

Theorem 7.1. *Let $\mathbb{R} \ni l \to (g_l^{TX}, g_l^F)$ be a family of metrics on TX, F as explained at the beginning of Section 7. Then we have the following asymptotic expansion as $t \searrow 0$:*

$$\begin{aligned}
&\mathrm{Tr_s}\left[\left(*_l^{-1}\frac{\partial *_l}{\partial l} + (g_l^F)^{-1}\frac{\partial g_l^F}{\partial l}\right)\exp(-t(D_l^{\overline{q}})^2)\right]\\
&= \int_X \mathrm{Tr}\left[(g_l^F)^{-1}\frac{\partial g_l^F}{\partial l}\right] e(TX, \nabla_l^{TX})\\
&+ \frac{1}{\sqrt{t}}\mathrm{rk}(F)\int_X \int^B \dot{\omega}_l^X \exp\left(-\frac{1}{2}\dot{R}_l^{TX}\right) + \int_X \iota_{\partial_l} e(\rho^* TX, \nabla^{TX,\mathrm{tot}})\theta(F, g_l^F)\\
&+ \sum_{p\in \mathrm{Sing}(X)} (c_{p,l}^{\overline{q}} + \tilde{c}_{p,l}^{\overline{q}}) + \mathcal{O}(t^{1/2}),
\end{aligned} \tag{7.12}$$

where the contributions of the singularities $c_{p,l}^{\overline{q}}$, $\tilde{c}_{p,l}^{\overline{q}}$, $p \in \mathrm{Sing}(X)$, are given by the following well-defined integrals:

$$\begin{aligned}
c_{p,l}^{\overline{q}} &:= \frac{1}{2}\int_0^\infty u^{-1}\int_{L_p} \mathrm{Tr_s}\left[(g_l^F)^{-1}\frac{\partial g_l^F}{\partial l} Q_u^{p,l,\overline{q}}((1,y),(1,y))\right] d\mathrm{vol}_{L_p} du,\\
\tilde{c}_{p,l}^{\overline{q}} &:= \frac{1}{2}\int_0^\infty u^{-1}\int_{L_p} \mathrm{Tr_s}\left[*_l^{-1}\frac{\partial *_l}{\partial l} Q_u^{p,l,\overline{q}}((1,y),(1,y))\right] d\mathrm{vol}_{L_p} du.
\end{aligned} \tag{7.13}$$

Remark 7.2.
(a) The first three terms on the right hand side of the formula (7.12) are the interiour contribution, familiar from the anomaly formula for the Ray-Singer metric on a smooth compact manifold [BZ92, Theorem 4.14, Theorem 4.20]. They do not depend on the chosen extension of $D_{l,c} = d_c + \delta_{l,c}$. The contributions of the singularities of X to the formula (7.12), $c_{p,l}^{\overline{q}}$, $\tilde{c}_{p,l}^{\overline{q}}$, $p \in \mathrm{Sing}(X)$, do depend on the chosen extension $D_l^{\overline{q}}$.
(b) One can establish a corresponding formula for every other closed self-adjoint extension of the Laplacian, which is invariant under radial scaling near $\mathrm{Sing}(X)$.
(c) Note the following vanishing properties for the coefficients in (7.12): If n is even, $\int^B \dot{\omega}_l^X \exp\left(-\frac{1}{2}\dot{R}_l^{TX}\right) = 0$ since the integrand is a sum of forms of type (k,k), k odd. If n is odd, clearly, $e(TX, \nabla_l^{TX}) = 0$, $e(\rho^* TX, \nabla^{TX,\mathrm{tot}}) = 0$.

Proof. We denote by $\exp(-t(D_l^{\overline{q}})^2)(x,y)$, $x, y \in X_{sm}$, $t > 0$, the kernel of the heat operator $\exp(-t(D_l^{\overline{q}})^2)$. Using local index techniques as in [BZ92, Theorem 4.20] (more precisely [BZ92, (4.61)]), we get the following pointwise asymptotic expansion as $t \searrow 0$:

$$\begin{aligned}
&\mathrm{Tr_s}\left[(g_l^F)^{-1}\frac{\partial g_l^F}{\partial l}\exp(-t(D_l^{\overline{q}})^2)(x,x)\right] d\mathrm{vol}_X(x)\\
&= \left(\mathrm{Tr}\left[(g_l^F)^{-1}\frac{\partial g_l^F}{\partial l}\right]\int^B \exp\left(-\frac{1}{2}\dot{R}_l^{TX}\right)\right)(x) + \mathcal{O}(t^{1/2})\\
&= \left(\mathrm{Tr}\left[(g_l^F)^{-1}\frac{\partial g_l^F}{\partial l}\right] e(TX, \nabla_l^{TX})\right)(x) + \mathcal{O}(t^{1/2}).
\end{aligned} \tag{7.14}$$

The expansion (7.14) is uniform on compact sets; the coefficients do only depend on local geometric data and do not depend on the chosen extension of $D_{l,c} := d_c + \delta_{l,c}$. Since by (3.18), (3.19), $e(TX,\nabla_l^{TX})$ vanishes near $p\in\mathrm{Sing}(X)$, the coefficient of t^0 in the above pointwise asymptotic expansion (7.14) vanishes near the singularities of X.

For $p\in\mathrm{Sing}(X)$, the following integral expression is well-defined:

$$
\begin{aligned}
c_{p,l}^{\overline{q}} &:= \lim_{t\to 0}\int_{\{0\leqslant r\leqslant \epsilon\}\times L_p}\mathrm{Tr_s}\left[(g_l^F)^{-1}\frac{\partial g_l^F}{\partial l}Q_t^{p,l,\overline{q}}((r,y),(r,y))\right]d\mathrm{vol}_{Z_p}\\
&=\lim_{t\to 0}\frac{1}{2}\int_t^{\infty}u^{-1}\int_{L_p}\mathrm{Tr_s}\left[(g_l^F)^{-1}\frac{\partial g_l^F}{\partial l}Q_u^{p,l,\overline{q}}((1,y),(1,y))\right]d\mathrm{vol}_{L_p}du\\
&=\frac{1}{2}\int_0^{\infty}u^{-1}\int_{L_p}\mathrm{Tr_s}\left[(g_l^F)^{-1}\frac{\partial g_l^F}{\partial l}Q_u^{p,l,\overline{q}}((1,y),(1,y))\right]d\mathrm{vol}_{L_p}du.
\end{aligned}\tag{7.15}
$$

To prove the well-definedness of the integrals in (7.15) we use the same arguments as in the proof of the well-definedness of the Cheeger invariant $\gamma_p^{\overline{q}}(F)$ in Section 4.4.1: For the first identity in (7.15) we have used the change of variables $u=t/r^2$, the scaling property (4.53) for the heat kernel on the infinite cone and the fact that, on Z_p, the operator $\left((g_l^F)^{-1}\frac{\partial g_l^F}{\partial l}\right)$ does not depend on the radial coordinate. From (3.18), (3.19) and (7.14) we have, as $u\searrow 0$,

$$
\mathrm{Tr_s}\left[(g_l^F)^{-1}\frac{\partial g_l^F}{\partial l}Q_u^{p,l,\overline{q}}((1,y),(1,y))\right]\sim \mathcal{O}(u^{1/2}),\tag{7.16}
$$

which shows the well-definedness of the last integral in (7.15) at $u=0$. The well-definedness at $u=\infty$ follows using the characterisation of $\mathrm{dom}(\boldsymbol{\Delta}^{p,l,\overline{q}})$, see (4.56), (4.57).

Proceeding with Cheeger's strategy [Che83, Section 2] (which has already been used in the proof of Theorem 5.15 in Section 6.2), we get from (7.14) and (7.15), as $t\searrow 0$:

$$
\begin{aligned}
&\mathrm{Tr_s}\left[(g_l^F)^{-1}\frac{\partial g_l^F}{\partial l}\exp(-t(D_l^{\overline{q}})^2)\right]\\
&=\int_X\mathrm{Tr}\left[(g_l^F)^{-1}\frac{\partial g_l^F}{\partial l}\right]e(TX,\nabla_l^{TX})+\sum_{p\in\mathrm{Sing}(X)}c_{p,l}^{\overline{q}}+\mathcal{O}(t^{1/2}).
\end{aligned}\tag{7.17}
$$

Note that due to the vanishing of the Euler form $e(TX,\nabla_l^{TX})$ near $\mathrm{Sing}(X)$, the first integral on the right hand side of (7.17) is well-defined and moreover no logarithmic term $\log(t)$ appears in (7.17).

Using local index techniques as in [BZ92, Theorem 4.20] as $t\searrow 0$

$$
\begin{aligned}
&\mathrm{Tr_s}\left[*_l^{-1}\frac{\partial *_l}{\partial l}\exp(-t(D_l^{\overline{q}})^2)(x,x)\right]d\mathrm{vol}_X=\\
&=\begin{cases}\frac{1}{2}\left(\int^B\exp\left(-\frac{1}{2}\dot{R}_l^{TX}\right)\dot{\omega}_l^X\nabla_l^{TX}\widehat{\theta}(F,g_l^F)\right)(x)+\mathcal{O}(t) & \text{if } n \text{ is even},\\[2ex]
\frac{1}{\sqrt{t}}\mathrm{rk}(F)\left(\int^B\dot{\omega}_l^X\exp\left(-\frac{1}{2}\dot{R}_l^{TX}\right)\right)(x)+\mathcal{O}(t^{1/2}) & \text{if } n \text{ is odd}.\end{cases}
\end{aligned}\tag{7.18}
$$

Indeed the above asymptotics has been worked out in [BZ92, Theorem 4.20] for n even. As remarked in [BM06, (4.23b)], proceeding as in [BZ92, (4.55)-(4.63)] one gets (7.18) for the case n odd as well. Note that the leading coefficients in the above expansions (7.18) vanish near $\mathrm{Sing}(X)$: Using (2.7), (2.8) and the fact that $\theta(F,g^F)$ does not depend on the radial coordinate, one has that the integrands in the Berezin integrals appearing in (7.18), near $\mathrm{Sing}(X)$, are a sum of summands not containing either e^r or $\widehat{e}^r$.

The $(n-1)$-form $\int^B \exp\left(-\frac{1}{2}\dot{R}_l^{TX}\right)\dot{\omega}_l^X\widehat{\theta}(F,g_l^F)$ vanishes near $\mathrm{Sing}(X)$, since by (2.8), (7.10) and the fact that the form $\theta(F,g_l^F)$ does not depend on the radial coordinate, the integrand in the Berezin integral does not contain $\widehat{e}^r$. Using Stokes' Theorem, the Bianchi identity and [BZ92, (4.74)-(4.86)] we have, for $\epsilon>0$ small enough,

$$\begin{aligned}
&\frac{1}{2}\int_X\int^B \exp\left(-\frac{1}{2}\dot{R}_l^{TX}\right)\dot{\omega}_l^X\nabla_l^{TX}\widehat{\theta}(F,g_l^F)\\
&=\frac{1}{2}\int_{X_\epsilon}\int^B \exp\left(-\frac{1}{2}\dot{R}_l^{TX}\right)\dot{\omega}_l^X\nabla_l^{TX}\widehat{\theta}(F,g_l^F)\\
&=-\frac{1}{2}\int_{X_\epsilon}\int^B \exp\left(-\frac{1}{2}\dot{R}_l^{TX}\right)(\nabla_l^{TX}\dot{\omega}_l^X)\widehat{\theta}(F,g_l^F)\\
&\quad+\frac{1}{2}\int_{\partial X_\epsilon}\int^B \exp\left(-\frac{1}{2}\dot{R}_l^{TX}\right)\dot{\omega}_l^X\widehat{\theta}(F,g_l^F)\\
&=-\frac{1}{2}\int_{X}\int^B \exp\left(-\frac{1}{2}\dot{R}_l^{TX}\right)(\nabla_l^{TX}\dot{\omega}_l^X)\widehat{\theta}(F,g_l^F)\\
&=\int_X \iota_{\partial_l}e(\rho^*TX,\nabla^{TX,\mathrm{tot}})\theta(F,g_l^F).
\end{aligned}\tag{7.19}$$

Recall that, in (3.45), we have seen that the integrand on the right hand side of (7.19) vanishes near $\mathrm{Sing}(X)$.

We now define

$$\begin{aligned}
\widetilde{c}_{p,l}^{\overline{q}} &:= \lim_{t\to 0}\int_{\{0\leqslant r\leqslant 1\}\times L_p}\mathrm{Tr_s}\left[*_l^{-1}\frac{\partial *_l}{\partial l}Q_t^{p,l,\overline{q}}((r,y),(r,y))\right]d\mathrm{vol}_{Z_p}\\
&=\lim_{t\to 0}\frac{1}{2}\int_t^\infty u^{-1}\int_{L_p}\mathrm{Tr_s}\left[*_l^{-1}\frac{\partial *_l}{\partial l}Q_u^{p,l,\overline{q}}((1,y),(1,y))\right]d\mathrm{vol}_{L_p}du\\
&=\frac{1}{2}\int_0^\infty u^{-1}\int_{L_p}\mathrm{Tr_s}\left[*_l^{-1}\frac{\partial *_l}{\partial l}Q_u^{p,l,\overline{q}}((1,y),(1,y))\right]d\mathrm{vol}_{L_p}du.
\end{aligned}\tag{7.20}$$

The well-definedness of the term (7.20) follows with analogous arguments as for the well-definedness of the integral in (7.15). To get well-definedness of the integral on the right hand side of (7.20) at $r=0$ the vanishing of the leading term in the asymptotic expansion (7.18) near $\mathrm{Sing}(X)$ is crucial.

Using (7.18), (7.19) and (7.20) and proceeding with Cheeger's strategy as in Section 6.2, we get

$$\begin{aligned}
&\mathrm{Tr_s}\left[*_l^{-1}\frac{\partial *_l}{\partial l}\exp(-t(D_l^{\overline{q}})^2)\right]\\
&=\frac{1}{\sqrt{t}}\mathrm{rk}(F)\int_X\int^B \dot{\omega}_l^X\exp\left(-\frac{1}{2}\dot{R}_l^{TX}\right)+\int_X\iota_{\partial_l}e(\rho^*TX,\nabla^{TX,\mathrm{tot}})\theta(F,g_l^F)\\
&\quad+\sum_{p\in\mathrm{Sing}(X)}\widetilde{c}_{p,l}^{\overline{q}}+\mathcal{O}(t^{1/2}).
\end{aligned}\tag{7.21}$$

Again there is no term in $\log(t)$ appearing in (7.21), since the coefficient of t^0 in the pointwise asymptotic expansion (7.18) vanishes near $\mathrm{Sing}(X)$. All integrals in (7.21) are well-defined by the proceeding discussion.

□

In the next proposition we adapt a trick explained by Cheeger in [Che79, Theorem 3.10] for manifolds with boundary (and absolute or relative boundary conditions at the boundary) to our situation. We decompose the action of the Laplacian according to the Hodge decomposition for the complex $(\mathcal{C}_{\max/\min}, d_{\max/\min}, \langle\ ,\ \rangle)$ into its action on exact, coexact and harmonic forms

$$\Delta^{\overline{q}} = \Delta^{\overline{q}}_{ex} + \Delta^{\overline{q}}_{cex} + \Delta^{\overline{q}}_{harm}. \tag{7.22}$$

Proposition 7.3. *Let $\mathbb{R} \ni l \to (g^{TX}_l, g^F_l)$ be a family of metrics on TX, F as explained at the beginning of Section 7 and such that the spectral gap condition* (7.1) *is satisfied.*

(a) *The following holds:*

$$\begin{aligned}
&\frac{d}{dl}\mathrm{Tr}[\exp(-t\Delta^{\overline{q},(k)}_l)]\\
&= - t\Big\{\mathrm{Tr}[\Delta^{\overline{q},(k+1)}_l \exp(-t\Delta^{\overline{q},(k+1)}_{l,ex})\sigma_l] - \mathrm{Tr}[\Delta^{\overline{q},(k)}_l \exp(-t\Delta^{\overline{q},(k)}_{l,cex})\sigma_l]\\
&\qquad +\mathrm{Tr}[\Delta^{\overline{q},(k)}_l \exp(-t\Delta^{\overline{q},(k)}_{l,ex})\sigma_l] - \mathrm{Tr}[\Delta^{\overline{q},(k-1)}_l \exp(-t\Delta^{\overline{q},(k-1)}_{l,cex})\sigma_l]\Big\}\\
&=t\frac{d}{dt}\Big\{\mathrm{Tr}[\exp(-t\Delta^{\overline{q},(k+1)}_{l,ex})\sigma_l] - \mathrm{Tr}[\exp(-t\Delta^{\overline{q},(k)}_{l,cex})\sigma_l]\\
&\qquad +\mathrm{Tr}[\exp(-t\Delta^{\overline{q},(k)}_{l,ex})\sigma_l] - \mathrm{Tr}[\exp(-t\Delta^{\overline{q},(k-1)}_{l,cex})\sigma_l]\Big\}.
\end{aligned} \tag{7.23}$$

(b) *The following holds:*

$$\frac{d}{dl}\mathrm{Tr_s}[N\exp(-t\Delta^{\overline{q}}_l)] = -t\frac{d}{dt}\mathrm{Tr_s}[\exp(-t\Delta^{\overline{q}}_l)\sigma_l]. \tag{7.24}$$

Remark 7.4. We have

$$\dot{\Delta} = d\dot{\delta} + \dot{\delta}d, \quad \dot{\delta} = -\sigma\delta + \delta\sigma. \tag{7.25}$$

For a smooth compact manifold, since the operators d and δ_l commute with $\exp(-t\Delta^{(k)}_l)$, the first identity in (7.23) is equivalent to

$$\frac{d}{dl}\mathrm{Tr}[\exp(-t\Delta^{(k)}_l)] = -t\mathrm{Tr}[\dot{\Delta}^{(k)}_l\exp(-t\Delta^{(k)}_l)]. \tag{7.26}$$

In the presence of singularities the commutation property only holds on the domain of the Laplacian $\Delta^{\overline{q}}_l$, which however is not invariant under σ_l.

Proof. (a) Denote by $\pi_{1,2} : X \times X \to X$ the two projections. We denote by $\Box^{\overline{q}}_l = \partial_t + \Delta^{\overline{q}}_l$ the heat operator on X. We denote by $P^{l,\overline{q}}_t(x,y)$, $t > 0$, the fundamental solution for the heat equation associated to $\Delta^{\overline{q},(k)}_l$. The fundamental solution $P^{l,\overline{q}}_t(x,y)$ is a smooth double form in $\pi_1^*(\Lambda^k(T^*X)\otimes F)\otimes\pi_2^*(\Lambda^k(T^*X)\otimes F)$ satisfying the heat equation in each variable. We denote simply by $P^{\overline{q}}_t(x,y)$, etc. the operators associated to $l = 0$.

For $\alpha > 0$, we denote by $X_\alpha := X\setminus \cup_{p\in\mathrm{Sing}(X)} B_\alpha(p)$. In the following we use the following abbreviating notation, for two double forms ω, ω':

$$\langle\omega(x,z), \omega'(z,x)\rangle_\alpha := \int_{X_\alpha}\omega(x,z)\wedge *^F_z\omega'(z,x). \tag{7.27}$$

In the following all operations are applied to the variable z and correspond to $l = 0$. We have

$$\begin{aligned}
&\langle P_\epsilon^{l,\overline{q}}(x,z), P_{t-\epsilon}^{\overline{q}}(z,x)\rangle_\alpha - \langle P_{t-\epsilon}^{l,\overline{q}}(x,z), P_\epsilon^{\overline{q}}(z,x)\rangle_\alpha \\
&= \int_\epsilon^{t-\epsilon} \partial_s(\langle P_{t-s}^{l,\overline{q}}(x,z), P_s^{\overline{q}}(z,x)\rangle_\alpha) ds \\
&= \int_\epsilon^{t-\epsilon} \langle \partial_s P_{t-s}^{l,\overline{q}}(x,z), P_s^{\overline{q}}(z,x)\rangle_\alpha ds + \int_\epsilon^{t-\epsilon} \langle P_{t-s}^{l,\overline{q}}(x,z), \partial_s P_s^{\overline{q}}(z,x)\rangle_\alpha ds \\
&= -\int_\epsilon^{t-\epsilon} \langle \square^{\overline{q}} P_{t-s}^{l,\overline{q}}(x,z), P_s^{\overline{q}}(z,x)\rangle_\alpha ds + \int_\epsilon^{t-\epsilon} \langle \Delta^{\overline{q}} P_{t-s}^{l,\overline{q}}(x,z), P_s^{\overline{q}}(z,x)\rangle_\alpha ds \\
&\quad + \int_\epsilon^{t-\epsilon} \langle P_{t-s}^{l,\overline{q}}(x,z), \square^{\overline{q}} P_s^{\overline{q}}(z,x)\rangle_\alpha ds - \int_\epsilon^{t-\epsilon} \langle P_{t-s}^{l,\overline{q}}(x,z), \Delta^{\overline{q}} P_s^{\overline{q}}(z,x)\rangle_\alpha ds.
\end{aligned} \tag{7.28}$$

Applying Stokes' Theorem and using $\square^{\overline{q}} P_t^{\overline{q}} = 0$ in (7.28), we have

$$\begin{aligned}
&\langle P_{t-\epsilon}^{l,\overline{q}}(x,z), P_\epsilon^{\overline{q}}(z,x)\rangle_\alpha - \langle P_\epsilon^{l,\overline{q}}(x,z), P_{t-\epsilon}^{\overline{q}}(z,x)\rangle_\alpha \\
&= \int_\epsilon^{t-\epsilon} \langle \square^{\overline{q}} P_{t-s}^{l,\overline{q}}(x,z), P_s^{\overline{q}}(z,x)\rangle_\alpha ds \\
&\quad - \int_\epsilon^{t-\epsilon} \left\{ \int_{\partial X_\alpha} \delta P_{t-s}^{l,\overline{q}}(x,z) \wedge *^F P_s^{\overline{q}}(z,x) \right\} ds \\
&\quad \pm \int_\epsilon^{t-\epsilon} \left\{ \int_{\partial X_\alpha} *^F dP_{t-s}^{l,\overline{q}}(x,z) \wedge P_s^{\overline{q}}(z,x) \right\} ds \\
&\quad \pm \int_\epsilon^{t-\epsilon} \left\{ \int_{\partial X_\alpha} *^F P_{t-s}^{l,\overline{q}}(x,z) \wedge \delta P_s^{\overline{q}}(z,x) \right\} ds \\
&\quad - \int_\epsilon^{t-\epsilon} \left\{ \int_{\partial X_\alpha} P_{t-s}^{l,\overline{q}}(x,z) \wedge * dP_s^{\overline{q}}(z,x) \right\} ds.
\end{aligned} \tag{7.29}$$

We now consider the second boundary integral in (7.29): Let n be odd. We have $dP_{t-s}^{l,\overline{q}}(x,-) \in \mathrm{dom}(D^{l,\overline{q}})$ and $P_s(-,x) \in \mathrm{dom}(D^{\overline{q}})$. Therefore, by (7.6) and (7.7), locally near $p \in \mathrm{Sing}(X)$, we have asymptotic expansions

$$dP_{t-s}^{l,\overline{q}} = a_l + b_l \wedge dr + \omega_l, \quad P_s^{\overline{q}} = a + b \wedge dr + \omega, \tag{7.30}$$

with a, b, a_l, b_l as in (7.7) and $\omega \in \mathrm{dom}(D_{\min})$, $\omega_l \in \mathrm{dom}(D_{l,\min})$; a_l, b_l, ω_l depending smoothly on the parameter l. For the leading term of $[*^F dP_{t-s}^{l,\overline{q}}(x,z) \wedge P_s^{\overline{q}}(z,x)]_{|\partial X_\alpha}$ we hence have, using (7.7) and (7.30),

$$[*^F(a_l + b_l \wedge dr) \wedge (a + b \wedge dr)]_{|\partial X_\alpha} = \tilde{*}^F b_l \wedge a = 0, \tag{7.31}$$

where $\tilde{*}^F$ denotes the Hodge star operator on the link L_p associated to the metrics $g^{TL_p}, g^{F_{L_p}}$.

From (7.3), (7.4), (7.30) and (7.31) we get, for both n even or odd,

$$\int_{\partial X_\alpha} *^F dP_{t-s}^{l,\overline{q}}(x,z) \wedge P_s^{\overline{q}}(z,x) = o(\alpha^{1/2}|\log \alpha|^{1/2}). \tag{7.32}$$

By similar arguments, the third and fourth boundary integral in (7.29) are also $o(\alpha^{1/2}|\log\alpha|^{1/2})$. Note that, since $\mathrm{dom}\,\delta_{\min/\max} \neq \mathrm{dom}\,\delta_{l,\min/\max}$, we can not argue in the same fashion for the first boundary integral in (7.29), but we will treat this term later in (7.36).

Differentiating $\Box_l P_t^{l,\overline{q}} = 0$ in l, we get

$$\dot{\Box}_l^{\overline{q}} P_t^{l,\overline{q}} + \Box_l^{\overline{q}} \dot{P}_t^{l,\overline{q}} = 0. \tag{7.33}$$

Thus differentiating (7.29) in l, setting $l = 0$ and using (7.32) and (7.33), we get

$$\begin{aligned} &\langle \dot{P}_{t-\epsilon}^{\overline{q}}(x,z), P_\epsilon^{\overline{q}}(z,x)\rangle_\alpha - \langle \dot{P}_\epsilon^{\overline{q}}(x,z), P_{t-\epsilon}^{\overline{q}}(z,x)\rangle_\alpha + o(\alpha^{1/2}|\log\alpha|^{1/2}) \\ &= -\int_\epsilon^{t-\epsilon} \langle \dot{\Box}^{\overline{q}} P_{t-s}^{\overline{q}}(x,z), P_s^{\overline{q}}(z,x)\rangle_\alpha ds - \int_\epsilon^{t-\epsilon} \left\{ \int_{\partial X_\alpha} \delta \dot{P}_{t-s}^{\overline{q}}(x,z) \wedge *^F P_s^{\overline{q}}(z,x) \right\} ds. \end{aligned} \tag{7.34}$$

Using (7.25) and applying Stokes' Theorem we get from (7.34)

$$\begin{aligned} &\langle \dot{P}_{t-\epsilon}^{\overline{q}}(x,z), P_\epsilon^{\overline{q}}(z,x)\rangle_\alpha - \langle \dot{P}_\epsilon^{\overline{q}}(x,z), P_{t-\epsilon}^{\overline{q}}(z,x)\rangle_\alpha + o(\alpha^{1/2}|\log\alpha|^{1/2}) \\ &= -\int_\epsilon^{t-\epsilon} \left\{ \langle \dot{\delta} d P_{t-s}^{\overline{q}}(x,z), P_s^{\overline{q}}(z,x)\rangle_\alpha + \langle \dot{\delta} P_{t-s}^{\overline{q}}(x,z), \delta P_s^{\overline{q}}(z,x)\rangle_\alpha \right\} ds \\ &\quad - \int_\epsilon^{t-\epsilon} \left\{ \int_{\partial X_\alpha} (\delta \dot{P}_{t-s}^{\overline{q}}(x,z) + \dot{\delta} P_{t-s}^{\overline{q}}(x,z)) \wedge *^F P_s^{\overline{q}}(z,x) \right\} ds \\ &= \int_\epsilon^{t-\epsilon} \left\{ \langle \sigma \delta d P_{t-s}^{\overline{q}}(x,z), P_s^{\overline{q}}(z,x)\rangle_\alpha - \langle \sigma d P_{t-s}^{\overline{q}}(x,z), d P_s^{\overline{q}}(z,x)\rangle_\alpha \right\} ds \\ &\quad + \int_\epsilon^{t-\epsilon} \left\{ \langle \sigma \delta P_{t-s}^{\overline{q}}(x,z), \delta P_s^{\overline{q}}(z,x)\rangle_\alpha - \langle \sigma P_{t-s}^{\overline{q}}(x,z), d\delta P_s^{\overline{q}}(z,x)\rangle_\alpha \right\} ds \\ &\quad - \int_\epsilon^{t-\epsilon} \left\{ \int_{\partial X_\alpha} (\delta \dot{P}_{t-s}^{\overline{q}}(x,z) + \dot{\delta} P_{t-s}^{\overline{q}}(x,z)) \wedge *^F P_s^{\overline{q}}(z,x) \right\} ds \\ &\quad \pm \int_\epsilon^{t-\epsilon} \left\{ \int_{\partial X_\alpha} *^F \sigma d P_{t-s}^{\overline{q}}(x,z) \wedge P_s^{\overline{q}}(z,x) \right\} ds \\ &\quad \pm \int_\epsilon^{t-\epsilon} \left\{ \int_{\partial X_\alpha} *^F \sigma P_{t-s}^{\overline{q}}(x,z) \wedge \delta P_s^{\overline{q}}(z,x) \right\} ds \end{aligned} \tag{7.35}$$

We now treat the first boundary integral on the right hand side of (7.35): Since $\delta_l P^{l,\overline{q}} \in \mathrm{dom}(D_l^{\overline{q}})$ and arguing as in (7.30)-(7.32), we have

$$\int_{\partial X_\alpha} \delta_l P_{t-s}^{l,\overline{q}}(x,z) \wedge *^F P_s^{\overline{q}}(z,x) = o(\alpha^{1/2}|\log\alpha|^{1/2}). \tag{7.36}$$

Differentiating (7.36) and setting $l = 0$, we get

$$\int_{\partial X_\alpha} (\delta \dot{P}_{t-s}^{\overline{q}}(x,z) + \dot{\delta} P_{t-s}^{\overline{q}}(x,z)) \wedge *^F P_s^{\overline{q}}(z,x) = o(\alpha^{1/2}|\log\alpha|^{1/2}). \tag{7.37}$$

We now treat the second boundary integral in (7.35): Let $n = 2\nu + 1$ be odd. Since $dP_{t-s}^{\overline{q}}, P_s^{\overline{q}} \in \mathrm{dom}(D^{\overline{q}})$ we have, locally near $p \in \mathrm{Sing}(X)$,

$$P_s^{\overline{q}} - (a + b \wedge dr),\ dP_{t-s}^{\overline{q}} - (a' + b' \wedge dr) \in \mathrm{dom}(D^{\min}). \tag{7.38}$$

with a, a', b, b' as in (7.7). By the assumption on the metrics explained at the beginning of Section 7, the operator σ is an operator on the link (not depending on r). Hence from (7.3), (7.38) we get that

$$\|U^{-1}(\sigma d P_{t-s}^{\overline{q}} - (f(a') + f(b') \wedge dr))\|_{L^2(\Lambda(TL_p)\otimes F_{L_p})} = o(r^{1/2}|\log r|^{1/2}), \tag{7.39}$$

where $f : \mathcal{H}^\nu(L_p, F_{L_p}) \to \Omega^\nu(L_p, F_{L_p})$ is a $\mathbb{C}$-linear map. Using (7.7), (7.38), (7.39) the leading term in the expansion of $[*^F \sigma d P^{\overline{q}}_{t-s}(x,z) \wedge P^{\overline{q}}_s(z,x)]_{|\partial X_\alpha}$ is

$$[*^F(f(a') + f(b') \wedge dr) \wedge (a + b \wedge dr)]_{|\partial X_\alpha} = \widetilde{*}^F f(b') \wedge a = 0. \tag{7.40}$$

Hence from (7.3), (7.4), (7.38), (7.39) and (7.40) we get, for both n even or odd,

$$\int_{\partial X_\alpha} *^F \sigma d P^{\overline{q}}_{t-s}(x,z) \wedge P^{\overline{q}}_s(z,x) = o(\alpha^{1/2} |\log \alpha|^{1/2}). \tag{7.41}$$

We can argue similarly for the third boundary term in (7.35).

Using (7.37) and (7.41), by taking the limit $\alpha \to 0$ in (7.35) we get:

$$\begin{aligned} &\langle \dot{P}^{\overline{q}}_{t-\epsilon}(x,z), P^{\overline{q}}_\epsilon(z,x)\rangle - \langle \dot{P}^{\overline{q}}_\epsilon(x,z), P^{\overline{q}}_{t-\epsilon}(z,x)\rangle \\ &= \int_\epsilon^{t-\epsilon} \left\{\langle \sigma\delta d P^{\overline{q}}_{t-s}(x,z), P^{\overline{q}}_s(z,x)\rangle - \langle \sigma d P^{\overline{q}}_{t-s}(x,z), d P^{\overline{q}}_s(z,x)\rangle\right\} ds \\ &\quad + \int_\epsilon^{t-\epsilon} \left\{\langle \sigma\delta P^{\overline{q}}_{t-s}(x,z), \delta P^{\overline{q}}_s(z,x)\rangle - \langle \sigma P^{\overline{q}}_{t-s}(x,z), d\delta P^{\overline{q}}_s(z,x)\rangle\right\} ds \end{aligned} \tag{7.42}$$

Taking the trace with respect to x and the limit $\epsilon \to 0$ on the left hand side of (7.42), and using the semi-group property for $\exp(-t\Delta^{\overline{q}})$, we get the left hand side of (7.23). On the right hand side of (7.42) we do reverse the order of integration (w.r.t. x and z) and take the limit $\epsilon \to 0$; we get the right hand side of (7.23).

(b) The statement follows from (a) by taking the alternating weighted sum. □

Theorem 7.5. *Let $\mathbb{R} \ni l \to (g^{TX}_l, g^F_l)$ be a family of metrics on TX, F as explained at the beginning of Section 7 and such that the spectral gap condition* (7.1) *is satisfied. Then, the variation*

$$\partial_l \log\left(\left(\|\ \ \|^{RS}_{\det IH^\bullet_{\overline{q}}(X,F),l}\right)^2\right) \tag{7.43}$$

is given by the coefficient of t^0 in the asymptotic expansion for $t \searrow 0$ of

$$\mathrm{Tr_s}\left[\left(*_l^{-1}\frac{\partial *_l}{\partial l} + (g^F_l)^{-1}\frac{\partial g^F_l}{\partial l}\right)\exp(-(D^{\overline{q}}_l)^2)\right]. \tag{7.44}$$

Hence

$$\begin{aligned} &\partial_l \log\left(\left(\|\ \ \|^{RS}_{\det IH^\bullet_{\overline{q}}(X,F),l}\right)^2\right) \\ &= \int_X \mathrm{Tr}\left[(g^F_l)^{-1}\frac{\partial g^F_l}{\partial l}\right] e(TX, \nabla^{TX}_l) + \int_X \iota_{\partial_l} e(\rho^* TX, \nabla^{TX,\mathrm{tot}})\theta(F, g^F_l) \\ &\quad + \sum_{p \in \mathrm{Sing}(X)} (c^{\overline{q}}_{p,l} + \widetilde{c}^{\overline{q}}_{p,l}), \end{aligned} \tag{7.45}$$

where the contributions of the singularities $c^{\overline{q}}_{p,l}$, $\widetilde{c}^{\overline{q}}_{p,l}$, $p \in \mathrm{Sing}(X)$, are as defined in (7.13).

Proof. Using Proposition 7.3 (b) we can proceed as in the smooth situation (see [BGS88, (1.114)–(1.122)]) to get the first claim. The formula (7.45) then follows from Theorem 7.1. □

Remark 7.6. Let X be an even dimensional oriented space with isolated conical singularities and (F, ∇^F, g^F) a unitary flat vector bundle on X. It has been proved in [Dar87], by the usual Poincaré duality argument, that in this case the Ray-Singer metric is trivial. Let $\mathbb{R} \ni l \to g_l^{TX}$ be a family of conical metrics on TX. Then, clearly by Dar's result

$$\partial_l \log \left(\left(\| \quad \|^{RS}_{\det IH^\bullet_{\overline{q}}(X,F),l} \right)^2 \right) = 0. \tag{7.46}$$

The result (7.46) can be recovered using Theorem 7.5, since again by a duality argument,

$$\mathrm{Tr_s} \left[*_l^{-1} \frac{\partial *_l}{\partial l} \exp(-(D_l^{\overline{q}})^2) \right] = 0. \tag{7.47}$$

7.3. **Anomaly formula for the Bismut-Zhang metric** $\| \quad \|^{Y,g^{TX},g^F}_{\det IH^\bullet_{\overline{q}}(X,F)}$. The aim of this section is the study of anomaly formulas for the Bismut-Zhang metric. The next two theorems give anomaly formulas for the metric $\| \quad \|^{RS}_{\det IH^\bullet_{\overline{q}}(cL_p,L_p,F)}$, $p \in \mathrm{Sing}(X)$, which has been introduced in Definition 4.8 and is the contribution of the singular points of X to the Bismut-Zhang metric.

Theorem 7.7. *Let $p \in \mathrm{Sing}(X)$. Let $l \in \mathbb{R} \to (g^{TZ_p}, g_l^F)$ be a family of metrics on TZ_p, F_{Z_p} as explained at the beginning of Section 7, $g^{TZ_p} = dr^2 + r^2 g^{TL_p}$ being a fixed conical metric. We assume that the spectral gap condition* (7.1) *is satisfied. Then*

$$\begin{aligned} \partial_l \log \left(\left(\| \quad \|^{RS}_{\det IH^\bullet_{\overline{q}}(cL_p,L_p,F),l} \right)^2 \right) &= \lim_{t \to 0} \mathrm{Tr_s} \left[(g_l^F)^{-1} \frac{\partial g_l^F}{\partial l} \exp(-t \mathbf{\Delta}_1^{l,\overline{q}}) \right] \\ &= c^{\overline{q}}_{p,l} - \int_{L_p} \mathrm{Tr} \left[(g_l^F)^{-1} \frac{\partial g_l^F}{\partial l} \right] \wedge \eta_p, \end{aligned} \tag{7.48}$$

where η_p (resp. $c^{\overline{q}}_{p,l}$) is as defined in (3.22) *(resp.* (7.13)*).*

Proof. The operator $(g_l^F)^{-1} \frac{\partial g_l^F}{\partial l}$ on the infinite cone Z_p does not depend on the radial coordinate.

We proceed as in the proof of Theorem 4.11 and split the integral into two parts

$$\begin{aligned} \mathrm{Tr_s} \left[(g_l^F)^{-1} \frac{\partial g_l^F}{\partial l} \exp(-t \mathbf{\Delta}_1^{p,l,\overline{q}}) \right] &= \int_{Z_p} \mathrm{Tr_s} \left[(g_l^F)^{-1} \frac{\partial g_l^F}{\partial l} Q^{p,l,\overline{q}}_{t,1}\big((r,y),(r,y)\big) \right] d\mathrm{vol}_{Z_p} \\ = \int_{\{0 \leqslant r \leqslant 1\} \times L_p} & \mathrm{Tr_s} \left[(g_l^F)^{-1} \frac{\partial g_l^F}{\partial l} Q^{p,l,\overline{q}}_{t,1}\big((r,y),(r,y)\big) \right] d\mathrm{vol}_{Z_p} \\ + \int_{\{r \geqslant 1\} \times L_p} & \mathrm{Tr_s} \left[(g_l^F)^{-1} \frac{\partial g_l^F}{\partial l} Q^{p,l,\overline{q}}_{t,1}\big((r,y),(r,y)\big) \right] d\mathrm{vol}_{Z_p}. \end{aligned} \tag{7.49}$$

Proceeding as in Step 1 of the proof of Theorem 4.11, we get for the first integral on the right hand side of (7.49), as $t \searrow 0$,

$$\begin{aligned} &\int_{\{0 \leqslant r \leqslant 1\} \times L_p} \mathrm{Tr_s} \left[(g_l^F)^{-1} \frac{\partial g_l^F}{\partial l} Q^{p,l,\overline{q}}_{t,1}\big((r,y),(r,y)\big) \right] d\mathrm{vol}_{Z_p} \\ &\qquad \xrightarrow{t \searrow 0} \frac{1}{2} \int_0^\infty \frac{du}{u} \int_{L_p} \mathrm{Tr_s} \left[(g_l^F)^{-1} \frac{\partial g_l^F}{\partial l} Q^{p,l,\overline{q}}_u((1,y),(1,y)) \right] d\mathrm{vol}_{L_p} = c^{\overline{q}}_{p,l}. \end{aligned} \tag{7.50}$$

Proceeding as in Step 2 of the proof of Theorem 4.11 and using Proposition 3.4 (b), we get for the second integral in (7.49) as $t \searrow 0$,

$$\begin{aligned}(7.51)\quad &\int_{\{r\geqslant 1\}\times L_p} \mathrm{Tr_s}\left[\left((g_l^F)^{-1}\frac{\partial g_l^F}{\partial l}\right) Q_{t,1}^{p,l,\bar q}\big((r,y),(r,y)\big)\right] d\mathrm{vol}_{Z_p} \\ &\xrightarrow[t\searrow 0]{} \int_{Z_p} \mathrm{Tr}\left[(g_l^F)^{-1}\frac{\partial g_l^F}{\partial l}\right]\int^{B,p}\exp(-B_1^p) = -\int_{L_p}\mathrm{Tr}\left[(g_l^F)^{-1}\frac{\partial g_l^F}{\partial l}\right]\wedge \eta_p.\end{aligned}$$

Using (7.49)-(7.51) we get the second identity in (7.48). We get the first identity in (7.48), by proceeding as in the proof of Proposition 7.3 and Theorem 7.5. □

Theorem 7.8. *Let $p \in \mathrm{Sing}(X)$. Let $l \in \mathbb{R} \to (g_l^{TZ_p} = dr^2 + r^2 g_l^{TL_p}, g^F)$ be a family of conical metrics on the infinite cone Z_p; the metric g^F on the flat bundle F_{Z_p} is fixed. We assume that the spectral gap condition* (7.1) *holds. Then*

$$(7.52)\quad \partial_l \log\left(\left(\|\ \ \|^{RS}_{\det IH^\bullet_{\bar q}(cL_p,L_p,F),l}\right)^2\right) = \tilde c^{\bar q}_{p,l} + \int_{L_p}\theta(F,\nabla^F)\wedge \iota_{\partial_l}(\widetilde{\nabla f^p})^*\Psi(\rho^*TZ_p,\nabla^{TZ_p,\mathrm{tot}}),$$

with $\tilde c^{\bar q}_{p,l}$ as defined in (7.13).

Proof. The operator $*_l^{-1}\frac{\partial *_l}{\partial l}$ on the infinite cone Z_p does not depend on the radial coordinate. Proceeding as in the proof of Theorem 7.5 we can prove that

$$(7.53)\quad \partial_l \log\left(\left(\|\ \ \|^{RS}_{\det IH^\bullet_{\bar q}(cL_p,L_p,F),l}\right)^2\right)$$

is given by the coefficient of t^0 in the asymptotic expansion of $\mathrm{Tr_s}\left[*_l^{-1}\frac{\partial *_l}{\partial l}\exp(-t\boldsymbol{\Delta}_1^{p,l,\bar q})\right]$ as $t \searrow 0$. Hence, to complete the proof we have to determine this coefficient.

Let $T > 0$ be fixed. First, we split the integral into two parts

$$\begin{aligned}(7.54)\quad \mathrm{Tr_s}\left[*_l^{-1}\frac{\partial *_l}{\partial l}\exp(-t\boldsymbol{\Delta}_T^{p,l,\bar q})\right] &= \int_{Z_p}\mathrm{Tr_s}\left[*_l^{-1}\frac{\partial *_l}{\partial l}Q_{t,T}^{p,l,\bar q}\big((r,y),(r,y)\big)\right]d\mathrm{vol}_{Z_p} \\ &= \int_{\{0\leqslant r\leqslant t^{1/4}\}\times L_p}\mathrm{Tr_s}\left[*_l^{-1}\frac{\partial *_l}{\partial l}Q_{t,T}^{p,l,\bar q}\big((r,y),(r,y)\big)\right]d\mathrm{vol}_{Z_p} \\ &\quad + \int_{\{r\geqslant t^{1/4}\}\times L_p}\mathrm{Tr_s}\left[*_l^{-1}\frac{\partial *_l}{\partial l}Q_{t,T}^{p,l,\bar q}\big((r,y),(r,y)\big)\right]d\mathrm{vol}_{Z_p}.\end{aligned}$$

Proceeding as in Step 1 of the proof of Theorem 4.11, we get for the first integral on the right hand side of (7.54), as $t \searrow 0$,

$$\begin{aligned}(7.55)\quad &\int_{\{0\leqslant r\leqslant t^{1/4}\}\times L_p}\mathrm{Tr_s}\left[*_l^{-1}\frac{\partial *_l}{\partial l}Q_{t,T}^{p,l,\bar q}\big((r,y),(r,y)\big)\right]d\mathrm{vol}_{Z_p} \\ &\xrightarrow{t\searrow 0} \frac{1}{2}\int_0^\infty \frac{du}{u}\int_{L_p}\mathrm{Tr_s}\left[*_l^{-1}\frac{\partial *_l}{\partial l}Q_u^{p,l,\bar q}((1,y),(1,y))\right]d\mathrm{vol}_{L_p} = \tilde c^{\bar q}_{p,l}.\end{aligned}$$

We now use local index techniques as in Step 2 of Theorem 4.11; the notations are as in loc. cit.: We first scale the operator $t\boldsymbol{\Delta}_T^{p,l,\bar q}$ via the radial scaling $h_{1/\sqrt{t}} : r \to r/\sqrt{t}$. We get the operator

$$(7.56)\quad N_{t,T}^{p,l,\bar q} = t^2\boldsymbol{\Delta}_{T/t}^{p,l,\bar q},$$

with heat kernel $R^{p,l,\bar{q}}_{t,T}$. We hence get for the second integral in (7.54):

$$\begin{aligned}\int_{\{r\geqslant t^{1/4}\}\times L_p}&\operatorname{Tr}_s\left[*_l^{-1}\frac{\partial *_l}{\partial l}Q^{p,l,\bar{q}}_{t,T}\big((r,y),(r,y)\big)\right]d\mathrm{vol}_{Z_p}\\&=\int_{\{r\geqslant t^{3/4}\}\times L_p}\operatorname{Tr}_s\left[*_l^{-1}\frac{\partial *_l}{\partial l}R^{p,l,\bar{q}}_{t,T}\big((r,y),(r,y)\big)\right]d\mathrm{vol}_{Z_p}.\end{aligned}\tag{7.57}$$

As in Step 2 of the proof of Theorem 4.11 we apply to $N^{p,l,\bar{q}}_{t,T}$ the (local) scaling $x\to tx$ and replace the Clifford variables $c(e_k),\widehat{c}(e_k)$ by $c_t(e_k),\widehat{c}_t(e_k)$. We denote by C_t the operator we get from $*_l^{-1}\frac{\partial *_l}{\partial l}$ by the above scaling. We have

$$tC_t\xrightarrow{t\searrow 0}\dot{\omega}^{Z_p},\tag{7.58}$$

where $\dot{\omega}^{Z_p}$ has been defined in (7.10). Using (7.58) and prodeeding as in [BZ92, Theorem 13.4], we get the following asymptotics as $t\searrow 0$, uniformly on compact sets,

$$\begin{aligned}&\operatorname{Tr}_s\left[*_l^{-1}\frac{\partial *_l}{\partial l}R^{p,l,\bar{q}}_{t,T}\big((r,y),(r,y)\big)\right]d\mathrm{vol}_{Z_p}\\&=\frac{1}{t}\int^{B,p}\dot{\omega}_l^{Z_p}\exp(-B^p_{T^2})+\int^{B,p}\dot{\omega}_l^{Z_p}\left(\left(\frac{1}{2}\nabla_l^{TZ_p}+\iota_{\widehat{T\nabla f^p}}\right)\widehat{\theta}(F,g^F)\right)\exp(-B^p_{T^2})+\mathcal{O}(t).\end{aligned}\tag{7.59}$$

Recall that the form $e_T(\rho^*TZ_p,\nabla^{TZ_p,\mathrm{tot}})$ has been defined in (3.37). From (7.10) and $\nabla_{g_l^{TZ_p}}f^p=-r\partial_r$, we have $\iota_{\widehat{T\nabla f}}\dot{\omega}^{Z_p}=0$. Hence, using [BZ92, Theorem 3.2], [BZ92, Theorem 3.13] and proceeding as in [BZ92, (4.74)-(4.86)] we get

$$\begin{aligned}&\int^{B,p}\dot{\omega}_l^{Z_p}\left(\left(\frac{1}{2}\nabla_l^{TZ_p}+\iota_{\widehat{T\nabla f^p}}\right)\widehat{\theta}(F,g^F)\right)\exp(-B^p_{T^2})\\&=\frac{1}{2}d\left(\int^{B,p}\dot{\omega}_l^{Z_p}\widehat{\theta}(F,g^F)\exp(-B^p_{T^2})\right)\\&\quad-\int^{B,p}\left(\left(\frac{1}{2}\nabla_l^{TZ_p}+\iota_{\widehat{T\nabla f^p}}\right)\dot{\omega}_l^{Z_p}\right)\widehat{\theta}(F,g^F)\exp(-B^p_{T^2})\\&=\frac{1}{2}d\left(\int^{B,p}\dot{\omega}_l^{Z_p}\widehat{\theta}(F,g^F)\exp(-B^p_{T^2})\right)-\int^{B,p}\left(\frac{1}{2}\nabla_l^{TZ_p}\dot{\omega}_l^{Z_p}\right)\widehat{\theta}(F,g^F)\exp(-B^p_{T^2})\\&=\frac{1}{2}d\left(\int^{B,p}\dot{\omega}_l^{Z_p}\widehat{\theta}(F,g^F)\exp(-B^p_{T^2})\right)-\int^{B,p}\theta(F,g^F)\left(-\frac{1}{2}\widehat{\nabla_l^{TZ_p}\omega_l^{Z_p}}\right)\exp(-B^p_{T^2})\\&=\frac{1}{2}d\left(\int^{B,p}\dot{\omega}_l^{Z_p}\widehat{\theta}(F,g^F)\exp(-B^p_{T^2})\right)-\int^{B,p}\theta(F,g^F)\iota_{\partial l}e_{T^2}(\rho^*TZ_p,\nabla^{TZ_p,\mathrm{tot}}).\end{aligned}\tag{7.60}$$

The form $\int^{B,p}\dot{\omega}_l^{Z_p}\widehat{\theta}(F,g^F)\exp(-B^p_{T^2})$ is an $(n-1)$-form containing e^r. Hence

$$d\left(\int^{B,p}\dot{\omega}_l^{Z_p}\widehat{\theta}(F,g^F)\exp(-B^p_{T^2})\right)=e^r\wedge d\beta(r,T),\tag{7.61}$$

for an $(n-2)$-form β on L_p.

The coefficients in the asymptotic expansion on the left hand side of (7.59), enjoy scaling properties analogous to those described in (6.23). Proceeding as in (6.24), we get from (7.59), (7.60) and (7.61),

$$\begin{aligned}\int_{\{r\geqslant t^{3/4}\}\times L_p} \mathrm{Tr_s}\Big[R^{p,l,\bar q}_{t,T}\big((r,y),(r,y)\big)\Big]\,d\mathrm{vol}_{Z_p} &= \frac{1}{t}\int_{\{r\geqslant t^{3/4}\}\times L_p}\int^{B,p} \dot\omega_l^{TZ_p}\exp(-B^p_{T^2})\\ &- \int_{\{r\geqslant t^{3/4}\}\times L_p}\int^{B,p}\theta(F,g^F)\iota_{\partial l}e_{T^2}(\rho^*TZ,\nabla^{TZ,\mathrm{tot}}) + \mathcal{O}(t^{1/4}).\end{aligned} \tag{7.62}$$

Moreover, as $t\searrow 0$,

$$\begin{aligned}&\int_{\{r\leqslant t^{3/4}\}\times L_p}\int^{B,p}\dot\omega^{TZ_p}\exp(-B^p_{T^2}) = \mathcal{O}(t^{3/2}),\\ &\int_{\{r\leqslant t^{3/4}\}\times L_p}\int^{B,p}\theta(F,g^F)\iota_{\partial l}e_{T^2}(\rho^*TZ,\nabla^{TZ,\mathrm{tot}}) = \mathcal{O}(t^{3/4}).\end{aligned} \tag{7.63}$$

Using Proposition 3.6(b), (7.57), (7.62) and (7.63), setting $T=1$, we get for the second integral on the right hand side of (7.54) as $t\searrow 0$,

$$\begin{aligned}&\int_{\{r\geqslant t^{1/4}\}\times L_p} \mathrm{Tr_s}\Big[*_l^{-1}\frac{\partial *_l}{\partial l}Q^{p,l,\bar q}_{t,1}\big((r,y),(r,y)\big)\Big]\,d\mathrm{vol}_{Z_p}\\ &= \int_{\{r\geqslant t^{3/4}\}\times L_p} \mathrm{Tr_s}\Big[*_l^{-1}\frac{\partial *_l}{\partial l}R^{p,l,\bar q}_{t,1}\big((r,y),(r,y)\big)\Big]\,d\mathrm{vol}_{Z_p}\\ &= \frac{1}{t}\int_{Z_p}\int^{B,p}\dot\omega_l^{Z_p}\exp(-B^p_1) - \int_{Z_p}\int^{B,p}\theta(F,g^F)\iota_{\partial l}e_1(\rho^*TZ_p,\nabla^{TZ_p,\mathrm{tot}}) + \mathcal{O}(t^{1/4})\\ &= \frac{1}{t}\int_{Z_p}\int^{B,p}\dot\omega_l^{Z_p}\exp(-B^p_1) + \int_{L_p}\theta(F,g^F)\wedge\iota_{\partial_l}(\widetilde{\nabla f^p})^*\Psi(\rho^*TZ_p,\nabla^{TZ_p,\mathrm{tot}})\\ &\quad + \mathcal{O}(t^{1/4}).\end{aligned} \tag{7.64}$$

Using (7.54), (7.55) and (7.64) we get that the coefficient of t^0 in the asymptotic expansion of $\mathrm{Tr_s}\Big[*_l^{-1}\frac{\partial *_l}{\partial l}\exp(-t\mathbf{\Delta}_1^{p,l,\bar q})\Big]$, as $t\searrow 0$, is given by

$$\widetilde{c}^{\bar q}_{p,l} + \int_{L_p}\theta(F,g^F)\wedge\iota_{\partial_l}(\widetilde{\nabla f^p})^*\Psi(\rho^*TZ_p,\nabla^{TZ_p,\mathrm{tot}}). \tag{7.65}$$

□

Let $l\in\mathbb{R}\to(g_l^{TX},g_l^F)$ be a family of metrics on TX, F as at the beginning of Section 7. We denote by $\|\ \|^{Y,g_l^{TX},g_l^F}_{\det IH^\bullet_{\bar q}(X,F)}$ the associated Bismut-Zhang metric.

Theorem 7.9. *Let $\mathbb{R} \ni l \to (g_l^{TX}, g_l^F)$ be a family of metrics on TX, F as explained at the beginning of Section 7 and satisfying the spectral gap condition* (7.1). *Then*

(7.66)
$$\partial_l \log\left((\| \ \|^{Y,g_l^{TX},g_l^F}_{\det IH^\bullet_{\overline{q}}(X,F)})^2\right) = \sum_{p\in \mathrm{Crit}(f_{sm})} (-1)^{\mathrm{ind}(p)}\partial_l \log(\| \ \|^2_{\det F_p,l}) + \sum_{p\in\mathrm{Sing}(X)}\left(-\int_{L_p}\mathrm{Tr}\left[(g_l^F)^{-1}\frac{\partial g_l^F}{\partial l}\right]\wedge \eta_{p,l} + c^{\overline{q}}_{p,l} + \tilde{c}^{\overline{q}}_{p,l}\right) + \sum_{p\in\mathrm{Sing}(X)}\int_{L_p}\theta(F,g_l^F)\wedge \iota_{\partial_l}(\widetilde{\nabla f^p})^*\Psi(\rho^*TZ_p,\nabla^{TZ_p,\mathrm{tot}}),$$

where $\eta_{p,l}$ (resp. $c^{\overline{q}}_{p,l}$ and $\tilde{c}^{\overline{q}}_{p,l}$) is as defined in (3.22) *(resp.* (7.13)*).*

Proof. The proof follows from the definition of the Bismut-Zhang metric (Definition 5.4), and Theorems 7.7 and 7.8. □

Remark 7.10. Putting together Theorems 7.5 and 7.9, we have

(7.67)
$$\partial_l \log\left(\frac{\| \ \|^{RS}_{\det IH^\bullet_{\overline{q}}(X,F),l}}{\| \ \|^{Y,g_l^{TX},g_l^F}_{\det IH^\bullet_{\overline{q}}(X,F)}}\right)^2 = \int_X \mathrm{Tr}\left[(g_l^F)^{-1}\frac{\partial g_l^F}{\partial l}\right] e(TX,\nabla_l^{TX}) + \int_X \iota_{\partial_l} e(\rho^*TX,\nabla^{TX,\mathrm{tot}})\theta(F,g_l^F) - \sum_{p\in \mathrm{Crit}(f_{sm})} (-1)^{\mathrm{ind}(p)}\partial_l \log(\| \ \|^2_{\det F_p,l}) + \sum_{p\in\mathrm{Sing}(X)}\int_{L_p}\mathrm{Tr}\left[(g_l^F)^{-1}\frac{\partial g_l^F}{\partial l}\right]\wedge\eta_{p,l} - \sum_{p\in\mathrm{Sing}(X)}\int_{L_p}\theta(F,g_l^F)\wedge \iota_{\partial_l}(\widetilde{\nabla f^p})^*\Psi(\rho^*TZ_p,\nabla^{TZ_p,\mathrm{tot}}).$$

Integrating (7.67) over $l \in [0,1]$ and comparing with (3.58) we have

(7.68)
$$\left(\frac{\| \ \|'^{RS}_{\det IH^\bullet_{\overline{q}}(X,F)}}{\| \ \|^{Y,g'^{TX},g'^F}_{\det IH^\bullet_{\overline{q}}(X,F)}}\right)^2 - \left(\frac{\| \ \|^{RS}_{\det IH^\bullet_{\overline{q}}(X,F)}}{\| \ \|^{Y,g^{TX},g^F}_{\det IH^\bullet_{\overline{q}}(X,F)}}\right)^2 = = -\int_X \theta(F,g'^F)(\nabla' f)^*\Psi(TX,\nabla'^{TX}) + \int_X \theta(F,g^F)(\nabla f)^*\Psi(TX,\nabla^{TX}),$$

which shows that the variations of the three terms in the Bismut-Zhang formula w.r.t. the two metrics (g^{TX}, g^F) are compatible.

The result in [BZ92, Theorem 16.1] can also be generalised to this setting: Let us fix a flat Hermitian vector bundle (F,∇^F,g^F). Let (f,g_0^{TX}), $(f', g_0'^{TX})$ be anti-radial Morse-Smale pairs, we assume that the conical metrics g_0^{TX}, $g_0'^{TX}$ coincide in an open neighbourhood of $\mathrm{Sing}(X)$. We denote by $Y = \nabla_{g_0^{TX}} f$, $Y' = \nabla_{g_0'^{TX}} f'$ the gradient vector fields. Let $\| \ \|^{Y',g_0'^{TX},g^F}_{\det IH^\bullet_{\overline{q}}(X,F)}$, $\| \ \|^{Y,g_0^{TX},g^F}_{\det IH^\bullet_{\overline{q}}(X,F)}$ denote the associated Bismut-Zhang metrics on $\det IH^\bullet_{\overline{q}}(X,F)$.

Let g^{TX} be a further arbitray conical Riemannian metric on X, which does also coincide with the conical metrics g_0^{TX}, $g_0^{\prime TX}$ in an open neighbourhood of $\mathrm{Sing}(X)$; we denote by ∇^{TX} the Levi-Civita connection of (TX, g^{TX}).

Theorem 7.11. *In the situation described above we have*

(7.69)

$$\log\left(\frac{\| \quad \|^{Y', g_0^{\prime TX}, g^F}_{\det IH^\bullet_{\bar{q}}(X,F)}}{\| \quad \|^{Y, g_0^{TX}, g^F}_{\det IH^\bullet_{\bar{q}}(X,F)}}\right)^2 = \int_X \theta(F, g^F)(Y')^*\Psi(TX, \nabla^{TX}) - \int_X \theta(F, g^F) Y^*\Psi(TX, \nabla^{TX}).$$

Proof. The theorem is a consequence of the Bismut-Zhang theorem, Theorem (5.6). It can be proved independently by an easy generalisation of [BZ92, Section XVI]. □

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