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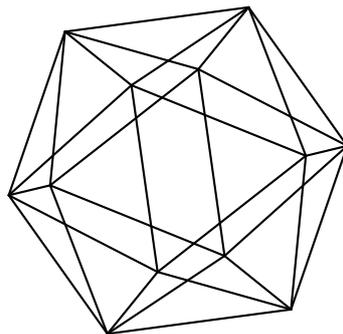
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MODEL INDEPENDENCE OF $(\infty, 2)$-CATEGORICAL NERVES

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ABSTRACT. For most models of $(\infty, 2)$-categories an embedding of the ∞-category of 2-categories into that of $(\infty, 2)$-categories has been constructed in the form of a nerve construction of some flavor. We prove that all those nerve embeddings induce equivalent functors, modulo change of model. We also show that all the nerve embeddings realize the ∞-category of 2-categories as the sub-∞-category of $(\infty, 2)$-categories that are local with respect to a certain class of maps.

CONTENTS

INTRODUCTION

It has become apparent that many phenomena of interest, such as the cobordism hypothesis, can only be properly formalized using the language of higher categories, often in the form of (∞, n)-categories for $n \geq 0$, and this paper is concerned with $(\infty, 2)$-categories. The structure of an $(\infty, 2)$-category could be summarized as a weakening of the structure present in a traditional 2-category. It consists of objects, 1- and 2-morphisms that compose suitably, as well as higher weakly invertible morphisms in dimension higher than 2 that serve as witnesses for relations between lower dimensional morphisms.

Many mathematical objects have been proposed to formalize $(\infty, 2)$-categories, each model presenting its own advantages and disadvantages. These include Barwick's 2-fold complete Segal spaces [Bar05], Verity's saturated 2-complicial sets [Ver08b, Ver17, Rie18, OR20b, RV22], Lurie's ∞-bicategories [Lur09b], Rezk's complete Segal Θ_2-spaces [Rez10],

Ara's 2-quasi-categories [Ara14], and 2-comical sets [CKM20, DKM21], as well as categories strictly enriched over a model of $(\infty, 1)$-categories [Lur09b, BR13, BR20]. In the past few years, the proof that all models are equivalent was completed, combining work by Lurie [Lur09b], Bergner–Rezk [BR13, BR20], Ara [Ara14], Gagna–Harpaz–Lanari [GHL22], Campion–Doherty–Kapulkin–Maehara [CKM20, DKM21].

It is often the case that the same construction gets implemented independently into two or more models. It is then necessary to verify that they indeed encode the same construction, modulo a change of model given by a direct comparison or a zigzag of such. In this paper, we specifically address the compatibility of several embeddings of the homotopy theory of 2-categories into that of $(\infty, 2)$-categories that have been constructed for different models.[1]

By design, the idea of an $(\infty, 2)$-category is supposed to weaken and generalize the notion of a strict 2-category. In particular, it is expected that any question about the homotopy theory of 2-categories should be equivalently addressable in the world of 2-categories or in that of $(\infty, 2)$-categories. This requirement, which is even partially axiomatized in the abstract setup by Barwick–Schommer-Pries [BSP21], could be phrased by expecting an embedding of the homotopy theory of 2-categories into that of $(\infty, 2)$-categories. Beside for providing a consistency check, the embedding of 2-categories into $(\infty, 2)$-categories is crucial in that several structural components of $(\infty, 2)$-categories, such as pasting schemes, are parametrized by strict 2-categories.

The analog question for $(\infty, 0)$-categories (a.k.a. ∞-groupoids) and $(\infty, 1)$-categories (a.k.a. ∞-categories) is equally valid although easier to address and by now fairly understood. In essentially all models for $(\infty, 0)$- and $(\infty, 1)$-categories one can easily identify or find in the literature a simple nerve construction for 0-categories (a.k.a. sets) and 1-categories and prove that this nerve construction realizes an embedding of homotopy theories into $(\infty, 0)$- and $(\infty, 1)$-categories, respectively.

For $(\infty, 2)$-categories the situation is more subtle. For instance, when one works with model categories, one technical difficulty is the fact that most models don't admit a homotopical nerve embedding that is at once fully faithful at the pointset level and a right Quillen functor at the model categorical level. In the recent years a well-behaved embedding has also been constructed in most models of $(\infty, 2)$-categories presented by model categories in the form of a homotopical functor that is homotopically fully faithful, which is in addition either right Quillen or fully faithful (but generally not both). This was achieved by the second and third author [OR21] for 2-complicial sets, by Campbell [Cam20] for 2-quasi-categories, by Gagna–Harpaz–Lanari for scaled simplicial sets [GHL22], and by the first author [Mos20] for 2-fold complete Segal spaces. For $(\infty, 2)$-categories presented by categories enriched over a model of $(\infty, 1)$-categories, this can be done by base-change along a suitable 1-dimensional nerve.

The first result of this paper, proven as Theorem 1.3 is to check that all the mentioned nerve constructions (along with a few more that we add) are compatible with each other via the known model comparisons.

Theorem A. *The aforementioned nerve embeddings of* 2-*categories into* $(\infty, 2)$-*categories constructed in different model categories are compatible with each other via known equivalences of models.*

---

[1]In the past, many ways to associate to any 2-category a classifying *space* – namely an $(\infty, 0)$-category, as opposed to an $(\infty, 2)$-category – have been provided by Street [Str87], Duskin [Dus02], Bullejos–Cegarra [BC03], Lack–Paoli [LP08]–and the equivalence of such constructions as spaces is proven in [CCG10].

At the level of ∞-categories, as part of a more general machinery Gepner–Haugseng [GH15] identified that the ∞-category of 2-categories can be understood as a localization of the ∞-category of $(\infty, 2)$-categories. More precisely, 2-categories are exactly the $(\infty, 2)$-categories that are local with respect to the 2-fold 2-point suspension of the inclusion of a point into a positive-dimensional sphere.

We prove as Theorem 1.12 that all the considered nerve embeddings induce at the level of ∞-categories precisely the inclusion of 2-categories as local objects amongst $(\infty, 2)$-categories with respect to the class of maps from the previous paragraph.

Theorem B. *The aforementioned nerve embeddings of 2-categories into $(\infty, 2)$-categories constructed in different model categories implement the embedding of 2-categories as local $(\infty, 2)$-categories.*

While overall expected, the compatibility of the nerve constructions in different models is a fundamental verification for the consistency of the theory, and a necessary ingredient in phrasing model independently many statements originally proven in a specific model.

To mention one example, in the paper [HORR21] the second and third author proved with Hackney and Riehl an $(\infty, 2)$-dimensional pasting theorem for $(\infty, 2)$-categories modeled by categories enriched over quasi-categories, and it is there explained how the compatibility of nerves which is the subject of the current paper is necessary to conclude that the pasting theorem holds in all other models.

The compatibility of nerves is expected to play a similar role in other circumstances, for instance in work in progress by the first and third author with Rasekh with the goal of developing a model independent theory of weighted limits valued in $(\infty, 2)$-categories.

Beside the novel result, we are also taking this project as an opportunity to write an accessible paper that surveys over the different nerve constructions and how they relate to each other, helping a non-specialist navigate the complex literature of $(\infty, 2)$-categories.

1. Organization of the paper

1.1. **Model categorical framework.** In this paper, we use the language of model categories to formalize the ∞-categories of $(\infty, 2)$-categories presented by different models. We refer the reader to e.g. [Hov99, Hir03] for the basic definitions from model category theory. We also assume familiarity with the basics of ∞-categories in the form of quasi-categories, see e.g. [Lur09a]. Here, we only briefly recall the key facts needed to interpret the model categorical statements as statements about homotopy theories and ∞-categories.

- Any model category $\mathcal{M}$ has an underlying ∞-category $[\mathcal{M}]_\infty$. Explicitly, the ∞-category $[\mathcal{M}]_\infty$ is obtained as the homotopy coherent nerve of a fibrant replacement of the *Hammock localization* of $\mathcal{M}$; see e.g. [DK80a, DK80b] or Appendix A for more details. For this specific model of $[\mathcal{M}]_\infty$, the set of objects is the same as the sets of objects of $\mathcal{M}$;
- Any homotopical functor $F \colon \mathcal{M} \to \mathcal{M}'$, i.e. a functor that preserves weak equivalences, induces a functor of ∞-categories $[F]_\infty \colon [\mathcal{M}]_\infty \to [\mathcal{M}']_\infty$. It can be computed on objects as $[F]_\infty(X) = F(X)$.

- Any right (resp. left) Quillen functor $F \colon \mathcal{M} \to \mathcal{M}'$ induces right (resp. left) adjoint functor of ∞-categories $[F]_\infty \colon [\mathcal{M}]_\infty \to [\mathcal{M}']_\infty$, as proven in [MG16, Thm 2.1]. It can be computed on objects as $[F]_\infty(X) \simeq F(X^{\mathrm{fib}})$ (resp. $[F]_\infty(X) \simeq F(X^{\mathrm{cof}})$). Here, X^{fib} (resp. X^{cof}) denotes a fibrant (resp. cofibrant) replacement of X in $\mathcal{M}$.
- Any right (resp. left) Quillen embedding[2] $F \colon \mathcal{M} \to \mathcal{M}'$ induces a fully faithful right adjoint (resp. left adjoint) of ∞-categories $[F]_\infty \colon [\mathcal{M}]_\infty \to [\mathcal{M}']_\infty$.
- Any left (resp. right) Quillen equivalence $F \colon \mathcal{M} \to \mathcal{M}'$ induces an equivalence of ∞-categories $[F]_\infty \colon [\mathcal{M}]_\infty \to [\mathcal{M}']_\infty$, as a consequence of what discussed in [MG16, §A.2] and [Lur18, §1.3.4]. In particular, a zigzag of Quillen equivalences induces an equivalence of ∞-categories.
- If a functor $F \colon \mathcal{M} \to \mathcal{M}'$ is such that it induces a functor $[F]_\infty \colon [\mathcal{M}]_\infty \to [\mathcal{M}']_\infty$ in more than one way, for instance it is both left and right Quillen, or it is both right Quillen and homotopical, the resulting functors are canonically equivalent.
- If functors $F \colon \mathcal{M} \to \mathcal{M}'$ and $F' \colon \mathcal{M}' \to \mathcal{M}''$ and their composite $F' \circ F \colon \mathcal{M} \to \mathcal{M}''$ induce functors of ∞-categories $[F]_\infty \colon [\mathcal{M}]_\infty \to [\mathcal{M}']_\infty$, $[F']_\infty \colon [\mathcal{M}']_\infty \to [\mathcal{M}'']_\infty$, and $[F']_\infty \circ [F]_\infty \colon [\mathcal{M}]_\infty \to [\mathcal{M}'']_\infty$ each computed using any of the rules described above, then there is a canonical equivalence $[F' \circ F]_\infty \simeq [F']_\infty \circ [F]_\infty$.

1.2. **Models of $(\infty, 2)$-categories.** We briefly recall also the main different approaches to modeling $(\infty, 2)$-categories that will be relevant for the paper. For each of the approaches, it is possible to realize the homotopy theory of $(\infty, 2)$-categories by means of a model structure in which the $(\infty, 2)$-categories are precisely the fibrant objects.

(a) *Globular models:* based on presheaves over Joyal's disk category Θ_2 [Joy97] or variants of it. They include Ara's 2-quasi-categories [Ara14] and Rezk's complete Segal Θ_2-spaces [Rez10]. The supporting model structures $\mathcal{S}et^{\Theta_2^{\mathrm{op}}}_{(\infty,2)}$ and $s\mathcal{S}et^{\Theta_2^{\mathrm{op}}}_{(\infty,2)}$ will be recalled in more detail in Theorems 2.2 and 2.4.

(b) *Bisimplicial models:* based on presheaves over $\Delta \times \Delta$. They include Barwick's 2-fold complete Segal spaces [Bar05] and Bergner–Rezk's Segal precategories [BR13]. The supporting model structures $s\mathcal{S}et^{(\Delta \times \Delta)^{\mathrm{op}}}_{(\infty,2)}$ and $P\mathcal{C}at(s\mathcal{S}et^{\Delta^{\mathrm{op}}})_{(\infty,2)}$ will be recalled in more detail in Theorem 3.3 and Section 3.4.

(c) *Enriched models:* based on categories *strictly* enriched over a model of $(\infty, 1)$-categories. They include categories enriched over Joyal's quasi-categories [Joy08a], over Rezk's complete Segal spaces [Rez01] and over Lurie's marked simplicial sets [Lur09a]. The supporting model structures $\mathcal{C}at_{s\mathcal{S}et_{(\infty,1)}}$, $\mathcal{C}at_{s\mathcal{S}et^{\Delta^{\mathrm{op}}}_{(\infty,1)}}$ and $\mathcal{C}at_{s\mathcal{S}et^+_{(\infty,1)}}$ will be recalled in more detail in Theorem 4.2.

(d) *Simplicial models:* based on presheaves over variants of the simplex category Δ. They include Verity's saturated 2-complicial sets [Ver17, OR20b, RV22], Lurie's ∞-bicategories [Lur09b], and saturated 2-precomplicial sets [OR20b] by the second and third author. The supporting models structures $ms\mathcal{S}et_{(\infty,2)}$, $s\mathcal{S}et^{sc}_{(\infty,2)}$ and $\mathcal{S}et^{t\Delta^{\mathrm{op}}}_{(\infty,2)}$ will be recalled in more detail in Theorems 5.3, 5.6 and 5.8.

---

[2]By a *right Quillen embedding* we mean a right Quillen functor in which the derived counit of any fibrant object is a weak equivalence. This is the right Quillen functor occurring in what is known in the literature as a *Quillen reflection* or *homotopy reflection* introduced in [Joy08b, §6.3]. A *left Quillen embedding* is defined dually.

(e) *Cubical models:* based on presheaves over a suitable category of cubes. The main incarnation is given by Doherty–Kapulkin–Maehara's 2-comical sets [DKM21], supported by the model structure $mc\mathcal{S}et_{(\infty,2)}$, which is a variant of a previous version by Campion–Kapulkin–Maehara [CKM20].

We know that these models of $(\infty, 2)$-categories have equivalent homotopy theories because the supporting model structures are connected by the following zigzags of Quillen equivalences[3].

$$
\begin{array}{ccc}
\mathcal{S}et_{(\infty,2)}^{\Theta_2^{\mathrm{op}}} & & \mathcal{S}et_{(\infty,2)}^{t\Delta^{\mathrm{op}}} \\
(-)_{\bullet,0} \uparrow [\mathrm{Ara14}] & & [\mathrm{OR20b}] \downarrow \mathrm{Refl} \\
s\mathcal{S}et_{(\infty,2)}^{\Theta_2^{\mathrm{op}}} & mc\mathcal{S}et_{(\infty,2)} \xrightarrow[\mathrm{[DKM21]}]{T} ms\mathcal{S}et_{(\infty,2)} \\
d_* \uparrow [\mathrm{BR20}] & [\mathrm{GHL22}] \downarrow U \\
s\mathcal{S}et_{(\infty,2)}^{(\Delta\times\Delta)^{\mathrm{op}}} & s\mathcal{S}et_{(\infty,2)}^{sc} \\
R \downarrow [\mathrm{BR20}] & [\mathrm{Lur09b}] \uparrow \mathfrak{N}^{sc} \\
P\mathcal{C}at(s\mathcal{S}et^{\Delta^{\mathrm{op}}})_{(\infty,2)} \xleftarrow[\mathfrak{R}]{[\mathrm{BR13}]} \mathcal{C}at_{s\mathcal{S}et_{(\infty,1)}^{\Delta^{\mathrm{op}}}} \xrightarrow[((-)_{\bullet,0})_*]{} \mathcal{C}at_{s\mathcal{S}et_{(\infty,1)}} \xleftarrow[U_*]{} \mathcal{C}at_{s\mathcal{S}et_{(\infty,1)}^+}
\end{array}
$$

1.3. **Models of $(\infty, 2)$-categorical nerves.** The canonical homotopy theory of strict 2-categories is presented by the following model structure, due to Lack.

Theorem 1.1 ([Lac02, Lac04]). *The category $2\mathcal{C}at$ of small 2-categories and 2-functors admits a model structure in which*

- *all 2-categories are fibrant,*
- *the weak equivalences are precisely the biequivalences, and*
- *the trivial fibrations are precisely the 2-functors that are surjective on objects, full on 1-morphisms, and fully faithful on 2-morphisms.*

Several nerve constructions for 2-categories valued in a model of $(\infty, 2)$-categories have been constructed in the form of a right Quillen embedding.[4]

(a) *Nerve into 2-quasi-categories:* A functor

$$
\mathbf{N}^{\Theta_2} \colon 2\mathcal{C}at \to s\mathcal{S}et_{(\infty,2)}^{\Theta_2^{\mathrm{op}}}
$$

was first considered by Leinster [Lei02, Def. J] and later shown by Campbell [Cam20, Rmk 5.16, Thm 5.10] to be a right Quillen embedding. This nerve and its properties will be recalled in Construction 2.6 and Theorem 2.7.

(b) *Nerve into 2-fold complete Segal spaces:* A functor

$$
\mathbf{N}^{\Delta\times\Delta} \colon 2\mathcal{C}at \to s\mathcal{S}et_{(\infty,2)}^{(\Delta\times\Delta)^{\mathrm{op}}}
$$

---

[3]Three of these Quillen equivalences are denoted R in the original sources. To distinguish them in this paper, we are using R, Refl and $\mathfrak{R}$.

[4]Given the numerous nerve constructions considered in this paper, for the sake of exposition we decided to change some of their notations to something more evocative of which model they refer to. We point out when these constructions are recalled what is the notation used in the original sources.

was constructed and shown to be a right Quillen embedding by the first author in [Mos20, §5.1, Thms 6.1.1, 6.1.3]. This nerve and its properties will be recalled in Construction 3.6 and Theorem 3.7.

(c) *Nerve into categories enriched over quasi-categories:* A functor

$$\mathbf{N}_* \colon 2\mathcal{C}at \to \mathcal{C}at_{s\mathcal{S}et_{(\infty,1)}}$$

obtained by base-change along the usual nerve functor, is used e.g. in [RV22, §1.4.2], and can be shown to be a right Quillen embedding. This nerve and its properties will be recalled in Construction 4.4.

(d) *Nerve into ∞-bicategories:* A functor

$$\mathbf{N}^{sc} \colon 2\mathcal{C}at \to ms\mathcal{S}et^{sc}_{(\infty,2)}$$

was considered by Harpaz–Nuiten–Prasma in [HNP19, §2] and shown to be a right Quillen embedding by Gagna–Harpaz–Lanari in [GHL22, Prop. 8.2, 8.3]. This nerve and its properties will be recalled in Construction 5.12 and Theorem 5.14.

(d') *Nerve into 2-precomplicial sets:* A functor

$$\mathbf{N}^{t\Delta} \colon 2\mathcal{C}at \to \mathcal{S}et^{t\Delta^{\mathrm{op}}}_{(\infty,2)}$$

was constructed and shown to be a right Quillen embedding by the second and third authors in [OR21, Thm 4.12, Cor. 4.13]. This nerve and its properties will be recalled in Construction 5.10 and Theorem 5.13.

1.4. **Equivalences of the nerve constructions.** The goal of this paper is to study how all those nerve constructions interact with the model comparison functors and prove the compatibility. In practice, this amounts to considering the following diagram of (model) categories,

(1.2)

built using some of the model comparison functors and the aforementioned nerve constructions, and show that all regions induce commutative diagrams at the level of underlying ∞-categories.

Theorem 1.3. *The diagram of underlying ∞-categories induced by* (1.2) *commutes up to equivalence.*

$$
\begin{array}{ccccccc}
[\mathcal{S}et^{\Theta_2^{\mathrm{op}}}_{(\infty,2)}]_\infty & & & & & & [\mathcal{S}et^{t\Delta^{\mathrm{op}}}_{(\infty,2)}]_\infty \\
\uparrow {\scriptstyle [(-)_{\bullet,0}]_\infty} & \xleftarrow{\ [\mathbf{N}^{\Theta_2}]_\infty\ } & & & \xrightarrow{\ [\mathbf{N}^{t\Delta}]_\infty\ } & & \downarrow {\scriptstyle [\mathrm{Refl}]_\infty} \\
[s\mathcal{S}et^{\Theta_2^{\mathrm{op}}}_{(\infty,2)}]_\infty & \quad (1) & & [2\mathcal{C}at]_\infty & & (4) & [ms\mathcal{S}et_{(\infty,2)}]_\infty \\
\uparrow {\scriptstyle [d_*]_\infty} & \xleftarrow{\ [\mathbf{N}^{\Delta\times\Delta}]_\infty\ } & & & \xrightarrow{\ [\mathbf{N}^{sc}]_\infty\ } & & \downarrow {\scriptstyle [U]_\infty} \\
[s\mathcal{S}et^{(\Delta\times\Delta)^{\mathrm{op}}}_{(\infty,2)}]_\infty & & {\scriptstyle [\mathbf{N}_*]_\infty} & & & & [s\mathcal{S}et^{sc}_{(\infty,2)}]_\infty \\
\downarrow {\scriptstyle [R]_\infty} & & (2) & & (3) & & \uparrow {\scriptstyle [\mathfrak{N}^{sc}]_\infty} \\
[P\mathcal{C}at(s\mathcal{S}et^{\Delta^{\mathrm{op}}})_{(\infty,2)}]_\infty & \xleftarrow[{[\mathfrak{R}]_\infty}]{} [\mathcal{C}at_{s\mathcal{S}et^{\Delta^{\mathrm{op}}}_{(\infty,1)}}]_\infty & \xrightarrow[{[((-)_{\bullet,0})_*]_\infty}]{} & [\mathcal{C}at_{s\mathcal{S}et_{(\infty,1)}}]_\infty & \xleftarrow[{[U_*]_\infty}]{} [\mathcal{C}at_{s\mathcal{S}et^+_{(\infty,1)}}]_\infty
\end{array}
$$

Outline of the proof. We address the commutativity of each of the regions as follows.

- The fact that the region (1) commutes is addressed as Corollary 3.16.
- The fact that the region (2) commutes is addressed as a combination of Corollaries 4.5 and 4.12.
- The fact that the region (3) commutes is addressed as a combination of Corollaries 4.5 and 5.20.
- The fact that the region (4) commutes is addressed as Corollary 5.18. $\qquad\square$

1.5. **Universal property of nerve embeddings.** In the following diagram of adjunctions of ∞-categories, Theorem 1.3 guarantees that the diagram involving the functors induced by the nerve construction functors commutes up to equivalence. Hence so does the one involving the left adjoints to the functors induced by the nerve constructions.

(1.4)

$$
\begin{array}{ccccccc}
[\mathcal{S}et^{\Theta_2^{\mathrm{op}}}_{(\infty,2)}]_\infty & & & & & & [\mathcal{S}et^{t\Delta^{\mathrm{op}}}_{(\infty,2)}]_\infty \\
\wr & \xleftarrow{\ [\mathbf{N}^{\Theta_2}]_\infty\ } & & & \xrightarrow{\ [\mathbf{N}^{t\Delta}]_\infty\ } & & \wr \\
[s\mathcal{S}et^{\Theta_2^{\mathrm{op}}}_{(\infty,2)}]_\infty & & & [2\mathcal{C}at]_\infty & & & [ms\mathcal{S}et_{(\infty,2)}]_\infty \\
\wr & \xleftarrow{\ [\mathbf{N}^{\Delta\times\Delta}]_\infty\ } & {\scriptstyle [\mathbf{N}_*]_\infty} & & {\scriptstyle [\mathbf{N}^{sc}]_\infty} & & \wr \\
[s\mathcal{S}et^{(\Delta\times\Delta)^{\mathrm{op}}}_{(\infty,2)}]_\infty & & & & & & [s\mathcal{S}et^{sc}_{(\infty,2)}]_\infty \\
\wr & & & & & & \wr \\
[P\mathcal{C}at(s\mathcal{S}et^{\Delta^{\mathrm{op}}})_{(\infty,2)}]_\infty & \simeq & [\mathcal{C}at_{s\mathcal{S}et^{\Delta^{\mathrm{op}}}_{(\infty,1)}}]_\infty & \simeq & [\mathcal{C}at_{s\mathcal{S}et_{(\infty,1)}}]_\infty & \simeq & [\mathcal{C}at_{s\mathcal{S}et^+_{(\infty,1)}}]_\infty
\end{array}
$$

Any of the ∞-categories underlying one of the model structures for $(\infty,2)$-categories from Section 1.2 can be taken to be *the* ∞-category of $(\infty,2)$-categories $\mathscr{C}at_{(\infty,2)}$, and all others are equivalent to this one – explicitly via the equivalences of ∞-categories given by the mentioned Quillen equivalences. Also, if $\mathscr{C}at_2$ denotes the ∞-category of 2-categories, then there is an equivalence of ∞-categories $\mathscr{C}at_2 \simeq [2\mathcal{C}at]_\infty$. The many models of nerves discussed in Section 1.3, all induce equivalent right adjoint functors between the ∞-category of $(\infty,2)$-categories $\mathscr{C}at_{(\infty,2)}$ and the ∞-category $[2\mathcal{C}at]_\infty$ of 2-categories:

(1.5) $$\mathscr{C}at_{(\infty,2)} \simeq [\mathcal{M}]_\infty \rightleftarrows [2\mathcal{C}at]_\infty \simeq \mathscr{C}at_2.$$

One may argue at this point that, although it was shown that those functors *do the same thing*, do they actually do *the right thing*?

To address this question, we first observe the compatibility of the embedding $\mathscr{C}at_2 \hookrightarrow \mathscr{C}at_{(\infty,2)}$ with Barwick–Schommer-Pries' framework from [BSP21].

Remark 1.6. In [BSP21, §7], Barwick–Schommer-Pries identified an axiomatic setup that guarantees that an ∞-category $\mathscr{M}$ (with extra structure) models correctly the theory of $(\infty,2)$-categories, satisfying in particular $\mathscr{M} \simeq \mathscr{C}at_{(\infty,2)}$ and deserving the name of a *model for $(\infty,2)$-categories*. Given a model of $(\infty,2)$-categories $\mathscr{M}$, the extra structure that is required is an embedding

$$(1.7) \qquad\qquad \mathscr{G}aunt_2 \hookrightarrow \mathscr{M} \simeq \mathscr{C}at_{(\infty,2)}$$

of the ∞-category $\mathscr{G}aunt_2$ of gaunt[5] 2-categories into the ∞-category $\mathscr{M}$. If we take e.g. $\mathscr{M} := [\mathcal{M}]_\infty$, for $\mathcal{M}$ any of the model categories of $(\infty,2)$-categories from Section 1.3 for which a nerve construction was described, then the embedding (1.7) can be taken to be the restriction

$$\mathscr{G}aunt_2 \hookrightarrow \mathscr{C}at_2 \simeq [2\mathcal{C}at]_\infty \hookrightarrow [\mathcal{M}]_\infty \simeq \mathscr{C}at_{(\infty,2)}$$

of the functor from (1.5), for a suitably chosen equivalence $[2\mathcal{C}at]_\infty \simeq \mathscr{C}at_2$. This could be seen by employing [BSP21, Lem. 10.2].

Next, we address how the equivalent adjunctions $\mathscr{C}at_2 \rightleftarrows \mathscr{C}at_{(\infty,2)}$ from (1.5) relate to work by Gepner–Haugseng [GH15, §6].

Remark 1.8. In [GH15, Prop. 6.1.7], Gepner–Haugseng identify a universal property that relates the ∞-category of 2-categories $\mathscr{C}at_2$ and the ∞-category $\mathscr{C}at_{(\infty,2)}$ of $(\infty,2)$-categories. More precisely, the former can be understood as a localization of the latter with respect to the class of maps

$$\Sigma^2 \Lambda := \{\Sigma^2 \Delta[0] \hookrightarrow \Sigma^2 S^k \mid k > 0\},$$

where S^k denotes the k-th sphere as an object of the ∞-category $\mathscr{S}$ of spaces, and

$$(1.9) \qquad\qquad \Sigma^2 \colon \mathscr{S} \to \mathscr{C}at_{(\infty,2)}$$

implements a suitable 2-fold 2-point suspension, constructed in [GH15, Def. 4.3.21]. From this, one deduces the existence of an adjunction

$$(1.10) \qquad\qquad \mathscr{C}at_{(\infty,2)} \rightleftarrows \mathscr{L}_{\Sigma^2 \Lambda} \mathscr{C}at_{(\infty,2)} \simeq \mathscr{C}at_2$$

with left adjoint being reflector and right adjoint being inclusion.

Our goal in Section 6 is to prove that, for compatibly chosen equivalences of ∞-categories, the incarnation

$$(1.11) \qquad [c_*]_\infty \colon \mathscr{C}at_{(\infty,2)} \simeq [\mathcal{C}at_{s\mathcal{S}et_{(\infty,1)}}]_\infty \rightleftarrows [2\mathcal{C}at]_\infty \simeq \mathscr{C}at_2 \colon [\mathbf{N}_*]_\infty$$

induces precisely the adjunction (1.10). This will show that, hence, all the adjunctions of ∞-categories (1.4) have the correct universal property and do the right thing.

Theorem 1.12. *The adjunctions of ∞-categories* (1.10) *and* (1.11) *are equivalent.*

Outline of the proof. The proof involves three steps.

---

[5]A 2-category is *gaunt* or *rigid* if it has no non-identity 2-isomorphisms and no non-identity 1-equivalences (or equivalently no non-identity 1- and 2-isomorphisms).

- First, in Remark 6.25, we will discuss why the functor between model categories

$$\Sigma^2 \colon s\mathcal{S}et_{(\infty,0)} \to \mathcal{C}at_{s\mathcal{S}et_{(\infty,1)}}$$

from Proposition 6.24 induces at the level of underlying ∞-categories the functor from (1.9), where $s\mathcal{S}et_{(\infty,0)}$ is the Kan-Quillen model structure.
- Then, in Theorem 6.36 we will show that the Quillen pair

$$c_* \colon \mathcal{C}at_{s\mathcal{S}et_{(\infty,1)}} \rightleftarrows 2\mathcal{C}at \colon \mathbf{N}_*$$

and the left Bousfield localization adjunction

$$\mathrm{Id} \colon \mathcal{C}at_{s\mathcal{S}et_{(\infty,1)}} \rightleftarrows \mathcal{L}_{\Sigma^2\Lambda}(\mathcal{C}at_{s\mathcal{S}et_{(\infty,1)}}) \colon \mathrm{Id}$$
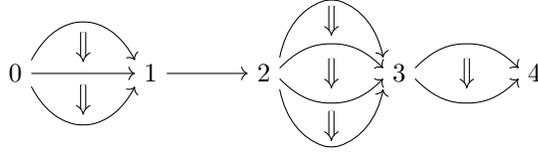
induce equivalent adjunctions at the level of underlying ∞-categories.
- Finally, in Remark 6.37 we use the previous two steps, as well as other results from the literature, to establish that the functor of ∞-categories (1.11) is indeed equivalent to (1.10), as desired. $\square$

2. Nerves in Θ_2-models

We devote this section to briefly recalling the main globular models of $(\infty, 2)$-categories, namely those based on Joyal's cell category Θ_2, and the relevant nerve constructions.

We refer the reader to [Joy97] for the category Θ_2, which is a full subcategory of $2\mathcal{C}at$ [Ber02, MZ01]. The generic object of Θ_2 is a 2-category of the form $\theta = [i|j_1, \ldots, j_i]$ for $i \geq 0$ and $j_k \geq 0$ for $k = 1, \ldots, i$. For example, the 2-category $[4|2, 0, 3, 1]$ is the 2-category generated by the following data.

The canonical inclusion $\mathcal{S}et \hookrightarrow s\mathcal{S}et$ of sets as discrete simplicial sets induces a canonical inclusion $\mathcal{S}et^{\Theta_2^{\mathrm{op}}} \hookrightarrow s\mathcal{S}et^{\Theta_2^{\mathrm{op}}}$, which preserves limits and colimits. In particular, we often regard Θ_2-sets as discrete Θ_2-spaces without further specification.

For any object θ in Θ_2, we denote by $\Theta_2[\theta]$ the Θ_2-set represented by θ via the Yoneda embedding $\Theta_2 \hookrightarrow \mathcal{S}et^{\Theta_2^{\mathrm{op}}}$.

2.1. **The models.** The following mathematical object was identified by Rezk [Rez10] as a model for $(\infty, 2)$-categories. We recall the definition for completeness, but it will not be needed in this paper.

Definition 2.1. A *complete Segal Θ_2-space* is a Θ_2-space $X \colon \Theta_2^{\mathrm{op}} \to s\mathcal{S}et$ that is local[6] with respect to the class of maps

---

[6]Given any small category $\mathcal{C}$, there are well-defined derived mapping spaces $\mathrm{Map}^h_{s\mathcal{S}et^{\mathcal{C}^{\mathrm{op}}}}(B, X)$ with respect to the homotopical structure on $s\mathcal{S}et^{\mathcal{C}^{\mathrm{op}}}$ given by levelwise weak equivalences in $s\mathcal{S}et_{(\infty,0)}$. For an explicit construction see e.g. [Rez10, §2.8]. We then say that a presheaf $X \colon \mathcal{C}^{\mathrm{op}} \to s\mathcal{S}et_{(\infty,0)}$ is *local* with respect to a set of maps S of $s\mathcal{S}et^{\mathcal{C}^{\mathrm{op}}}$ if for every $f \colon A \to B$ in S the induced map on derived mapping spaces $\mathrm{Map}^h_{s\mathcal{S}et^{\mathcal{C}^{\mathrm{op}}}}(f, X) \colon \mathrm{Map}^h_{s\mathcal{S}et^{\mathcal{C}^{\mathrm{op}}}}(B, X) \to \mathrm{Map}^h_{s\mathcal{S}et^{\mathcal{C}^{\mathrm{op}}}}(A, X)$ with respect to levelwise weak homotopy equivalences is a weak equivalence of Kan complexes.

(1) for all $i, j_1, \ldots, j_i \geq 0$, the *horizontal Segality extension*

$$\Theta_2[1|j_1] \underset{\Theta_2[0]}{\amalg} \cdots \underset{\Theta_2[0]}{\amalg} \Theta_2[1|j_i] \to \Theta_2[i|j_1, \ldots, j_i]$$

induced by the inclusions $[\langle s-1, s\rangle|j_s] \colon [1|j_s] \to [i|j_1, \ldots, j_i]$ where $\langle s-1, s\rangle \colon [1] \to [i]$ sends $0 \mapsto s-1$ and $1 \mapsto s$, for $1 \leq s \leq i$;

(2) for all $j \geq 0$ the *vertical Segality extension*

$$\Theta_2[1|1] \underset{\Theta_2[1|0]}{\amalg} \cdots \underset{\Theta_2[1|0]}{\amalg} \Theta_2[1|1] \to \Theta_2[1|j]$$

induced by the inclusions $[1|\langle t-1, t\rangle] \colon [1|1] \to [1|j]$ where $\langle t-1, t\rangle \colon [1] \to [j]$ sends $0 \mapsto t-1$ and $1 \mapsto t$, for $1 \leq t \leq j$;

(3) the *horizontal completeness extension*

$$\Theta_2[0] \to \Theta_2[0] \underset{\Theta_2[1|0]}{\amalg} \Theta[3|0, 0, 0] \underset{\Theta_2[1|0]}{\amalg} \Theta_2[0],$$

where the right-hand side is the colimit of the diagram

$$\Theta_2[0] \xleftarrow{\;\;!\;\;} \Theta_2[1|0] \xrightarrow{\langle 0, 2\rangle} \Theta_2[3|0, 0, 0] \xleftarrow{\langle 1, 3\rangle} \Theta_2[1|0] \xrightarrow{\;\;!\;\;} \Theta_2[0]$$

and the map is induced by the inclusion $\langle 0\rangle \colon [0] \to [3]$;

(4) the *vertical completeness extension*[7]

$$\Theta_2[1|0] \to \Theta_2[1|0] \underset{\Theta_2[1|1]}{\amalg} \Theta[1|3] \underset{\Theta_2[1|1]}{\amalg} \Theta_2[1|0],$$

where the right-hand side is the colimit of the diagram

$$\Theta_2[1|0] \xleftarrow{[1|!]} \Theta_2[1|1] \xrightarrow{[1|\langle 0, 2\rangle]} \Theta_2[1|3] \xleftarrow{[1|\langle 1, 3\rangle]} \Theta_2[1|0] \xrightarrow{[1|!]} \Theta_2[1|0]$$

and the map is induced by the inclusion $[1|\langle 0\rangle] \colon [1|0] \to [1|3]$.

The following model structure is obtained as a left Bousfield localization of the injective model structure on $s\mathcal{S}et^{\Theta_2^{\mathrm{op}}}$.

Theorem 2.2 ([Rez10, Thm 8.1]). *The category $s\mathcal{S}et^{\Theta_2^{\mathrm{op}}}$ of Θ_2-spaces admits a model structure, denoted $s\mathcal{S}et_{(\infty,2)}^{\Theta_2^{\mathrm{op}}}$, in which*

- *the fibrant objects are the injectively fibrant complete Segal Θ_2-spaces, and*
- *the cofibrations are the monomorphisms, and in particular every object is cofibrant.*

The following mathematical object was envisioned by Joyal [Joy97] and formalized by Ara [Ara14, §5] as a model for $(\infty, 2)$-categories.

Definition 2.3. A *2-quasi-category* is a Θ_2-set $X \colon \Theta_2^{\mathrm{op}} \to \mathcal{S}et$ that has the right lifting property with respect to the class of maps (1)-(4) from Definition 2.1.

Theorem 2.4 ([Ara14, §5.17]). *The category $\mathcal{S}et^{\Theta_2^{\mathrm{op}}}$ of Θ_2-sets admits a model structure, denoted $\mathcal{S}et_{(\infty,2)}^{\Theta_2^{\mathrm{op}}}$, in which*

- *the fibrant objects are the 2-quasi-categories, and*
- *the cofibrations are the monomorphisms, and in particular every object is cofibrant.*

---

[7]While the completeness conditions are not the same as in [Rez10, §11.4], one can use [Rez10, §4.4,§10] to see that the two descriptions localizations are defining the same model structure.

Ara showed as [Ara14, Thm 8.4] that the functor $(-)_0 \colon s\mathcal{S}et \to \mathcal{S}et$, which extracts the 0-th component, induces a right Quillen equivalence

$$(2.5) \qquad\qquad (-)_{\bullet,0} \colon s\mathcal{S}et_{(\infty,2)}^{\Theta_2^{\mathrm{op}}} \to \mathcal{S}et_{(\infty,2)}^{\Theta_2^{\mathrm{op}}}.$$

2.2. **The nerve.** A nerve construction $\mathbf{N}^{\Theta_2}\mathcal{D}$ for any 2-category $\mathcal{D}$ was identified by Leinster [Lei02, Def. J] and further studied by Campbell[8] [Cam20]. Its construction is based on the notion of a normal pseudofunctor, which we recall later as Definition C.1. Roughly speaking, those are maps between 2-categories that preserve identities strictly and preserve compositions up to coherent isomorphism.

Construction 2.6 ([Lei02, Def. J])**.** Let $\mathcal{D}$ be a 2-category. The nerve $\mathbf{N}^{\Theta_2}\mathcal{D}$ is the Θ_2-set given for any $\theta \in \Theta_2$ by the set of normal pseudofunctors from θ to $\mathcal{D}$

$$\mathbf{N}_\theta^{\Theta_2}\mathcal{D} \coloneqq (\mathbf{N}^{\Theta_2}\mathcal{D})_\theta \coloneqq 2\mathcal{C}at_{\mathrm{nps}}(\theta, \mathcal{D}).$$

The assignment extends to a functor $\mathbf{N}^{\Theta_2} \colon 2\mathcal{C}at \to \mathcal{S}et^{\Theta_2^{\mathrm{op}}}$.

The homotopical properties of these nerve constructions follow from a combination of work by Campbell [Cam20] and Lack [Lac04], as explained in [Cam20, Rmk 5.16].

Theorem 2.7. *The functor* $\mathbf{N}^{\Theta_2} \colon 2\mathcal{C}at \to \mathcal{S}et_{(\infty,2)}^{\Theta_2^{\mathrm{op}}}$ *is a right Quillen embedding, and in particular a right Quillen and homotopical functor.*

3. Nerves in bisimplicial models

We devote this section to briefly recalling the main bisimplicial models of $(\infty, 2)$-categories, and the relevant nerve constructions and model comparisons with the material from the previous section. We also prove the compatibility with the nerve construction from the previous section.

The canonical inclusion $\mathcal{S}et \hookrightarrow s\mathcal{S}et$ of sets as discrete simplicial sets induces a canonical inclusion $\mathcal{S}et^{(\Delta\times\Delta)^{\mathrm{op}}} \hookrightarrow s\mathcal{S}et^{(\Delta\times\Delta)^{\mathrm{op}}}$, which preserves limits and colimits. In particular, we often regard bisimplicial sets as discrete bisimplicial spaces without further specification.

For any $[i, j]$ in $\Delta \times \Delta$, we denote by $\Delta[i, j]$ the bisimplicial set represented by $[i, j]$ via the Yoneda embedding $\Delta \times \Delta \hookrightarrow \mathcal{S}et^{(\Delta\times\Delta)^{\mathrm{op}}}$.

3.1. **The first model.** The following mathematical object was identified by Barwick [Bar05, §2.3] as a model of $(\infty, 2)$-categories. It was also further studied with slightly different presentations, by Lurie [Lur09b, Def. 1.3.6], Johnson-Freyd–Scheimbauer [JFS17, §2] and Bergner–Rezk [BR20, Def. 5.3]. See also [Hau13, §2.2.2] and [Mos20, §4.2]. We recall the definition for completeness, but it will not be needed in this paper.

Definition 3.1. A 2-*fold complete Segal space* is a bisimplicial space $X \colon (\Delta\times\Delta)^{\mathrm{op}} \to s\mathcal{S}et$ that is local with respect to the all maps of the following types:

(1) for all $i, j \geq 0$, the *horizontal Segality extension*

$$\Delta[1, j] \underset{\Delta[0,j]}{\amalg} \cdots \underset{\Delta[0,j]}{\amalg} \Delta[1, j] \to \Delta[i, j]$$

induced by the inclusions $\langle s - 1, s \rangle \colon [1] \to [i]$ sending $0 \mapsto s - 1, 1 \mapsto s$, for $1 \leq s \leq i$;

---

[8]In the original source, the nerve is denoted $N\mathcal{D}$, as opposed to $\mathbf{N}^{\Theta_2}\mathcal{D}$.

(2) for all $i, j \geq 0$, the *vertical Segality extension*

$$\Delta[i,1] \underset{\Delta[i,0]}{\amalg} \cdots \underset{\Delta[i,0]}{\amalg} \Delta[i,1] \to \Delta[i,j]$$

induced by the inclusions $\langle t-1, t \rangle \colon [1] \to [j]$ sending $0 \mapsto t-1$, $1 \mapsto t$, for $1 \leq t \leq j$;

(3) for all $j \geq 0$ the *horizontal completeness extension*

$$\Delta[0,j] \to \Delta[0,j] \underset{\Delta[1,j]}{\amalg} \Delta[3,j] \underset{\Delta[1,j]}{\amalg} \Delta[0,j],$$

where the right-hand side is defined similarly to Definition 2.1(3);

(4) for all $i \geq 0$ the *vertical completeness extension*

$$\Delta[i,0] \to \Delta[i,0] \underset{\Delta[i,1]}{\amalg} \Delta[i,3] \underset{\Delta[i,1]}{\amalg} \Delta[i,0],$$

where the right-hand side is defined similarly to Definition 2.1(3);

(5) for all $j \geq 0$, the *vertical homotopical constantness extension*

$$\Delta[0,0] \to \Delta[0,j]$$

induced by the inclusion $\langle 0 \rangle \colon [0] \to [j]$.

Remark 3.2. If X is a 2-fold complete Segal space, the fact that X is local with respect to maps of the form 1 implies that for any $i, j \geq 0$ the simplicial space $X_{\bullet, j}$ is local with respect to the map

$$\Delta[1] \underset{\Delta[0]}{\amalg} \cdots \underset{\Delta[0]}{\amalg} \Delta[1] \to \Delta[i]$$

induced by the inclusion $\langle s-1, s \rangle \colon [1] \to [i]$ for $1 \leq s \leq i$. In particular, $X_{\bullet, j}$ is a Segal space.

Theorem 3.3 ([Bar05, Ch. 2]). *The category* $s\mathcal{S}et^{(\Delta \times \Delta)^{\mathrm{op}}}$ *of bisimplicial spaces admits a model structure, denoted* $s\mathcal{S}et^{(\Delta \times \Delta)^{\mathrm{op}}}_{(\infty, 2)}$, *in which*

- *the fibrant objects are the injectively fibrant* 2-*fold complete Segal spaces, and*
- *the cofibrations are the monomorphisms, and in particular every object is cofibrant.*

Precomposition with the functor $d \colon \Delta \times \Delta \to \Theta_2$, given on objects by $[i, j] \mapsto [i | j, \ldots, j]$, induces a functor $d^* \colon s\mathcal{S}et^{\Theta_2^{\mathrm{op}}} \to s\mathcal{S}et^{(\Delta \times \Delta)^{\mathrm{op}}}$. Explicitly,

$$(d^* X)_{i,j} \coloneqq X_{d[i,j]} = X_{[i | j, \ldots, j]}.$$

This functor admits a right adjoint d_*, which was proven by Bergner–Rezk as [BR20, Cor. 7.1] to be a right Quillen equivalence

$$(3.4) \qquad\qquad d_* \colon s\mathcal{S}et^{(\Delta \times \Delta)^{\mathrm{op}}}_{(\infty, 2)} \to s\mathcal{S}et^{\Theta_2^{\mathrm{op}}}_{(\infty, 2)}.$$

3.2. **The nerve.** A homotopically well-behaved nerve construction for bisimplicial models was studied by the first author in [Mos20], relying on the language of double categories. In this paper we aim at giving a presentation that is self-contained in the 2-categorical world, so we take a slightly different viewpoint in recalling the necessary ingredients to describe the aforementioned nerve construction.

Three functors involving the category $\mathcal{D}bl\mathcal{C}at$ of double categories, namely the functors $L, L^\simeq \colon \mathcal{D}bl\mathcal{C}at \to 2\mathcal{C}at$ and $\mathbb{C} \colon s\mathcal{S}et^{\Delta \times \Delta^{\mathrm{op}}} \to \mathcal{D}bl\mathcal{C}at$, are considered in [Mos20, §§2,5,6]. The composite functors $L\mathbb{C}, L^\simeq\mathbb{C} \colon s\mathcal{S}et^{(\Delta \times \Delta)^{\mathrm{op}}} \cong \mathcal{S}et^{(\Delta \times \Delta \times \Delta)^{\mathrm{op}}} \to 2\mathcal{C}at$ are described more explicitly in [Mos20, Descr. 6.3.1, 6.3.2]. The following relation between the two functors is discussed in the proof of [Mos20, Thm 6.2.5].

Proposition 3.5. *For $i, j, k \geq 0$, there is a natural biequivalence of 2-categories*

$$L^{\simeq}\mathbb{C}\Delta[i, j, k] \to L\mathbb{C}\Delta[i, j, k].$$

The following nerve construction $\mathbf{N}^{\Delta \times \Delta}\mathcal{D}$ for any 2-category $\mathcal{D}$ was constructed by the first author[9] [Mos20, §5.1] using the functor $L^{\simeq}\mathbb{C}$.

Construction 3.6 ([Mos20, §5.1]). Let $\mathcal{D}$ be a 2-category. The nerve $\mathbf{N}^{\Delta \times \Delta}\mathcal{D}$ is the bisimplicial space given for any $i, j, k \geq 0$ by

$$\mathbf{N}^{\Delta \times \Delta}_{i,j,k}\mathcal{D} \coloneqq (\mathbf{N}^{\Delta \times \Delta}\mathcal{D})_{i,j,k} \coloneqq 2\mathcal{C}at(L^{\simeq}\mathbb{C}\Delta[i, j, k], \mathcal{D}).$$

The assignment extends to a functor $\mathbf{N}^{\Delta \times \Delta} \colon 2\mathcal{C}at \to s\mathcal{S}et^{(\Delta \times \Delta)^{\mathrm{op}}}$, which is the right adjoint to the functor $L^{\simeq}\mathbb{C} \colon s\mathcal{S}et^{(\Delta \times \Delta)^{\mathrm{op}}} \to 2\mathcal{C}at$.

Theorem 3.7 ([Mos20, Thms 6.1.1, 6.1.3]). *The functor $\mathbf{N}^{\Delta \times \Delta} \colon 2\mathcal{C}at \to s\mathcal{S}et^{(\Delta \times \Delta)^{\mathrm{op}}}_{(\infty,2)}$ is a right Quillen embedding, and in particular a homotopical and right Quillen functor.*

Although the functor $L^{\simeq}\mathbb{C}$ is the one actually featuring in the definition of the nerve $\mathbf{N}^{\Delta \times \Delta}$ from Construction 3.6, for the purpose of this paper it will be sufficient to have an explicit description of its easier version, the functor $L\mathbb{C}$. In order to give such a description, which is achieved in Proposition 3.13, we first need to discuss preliminary material, including the 2-categories $\mathcal{O}_2^{\sim}[i]$ and $\widetilde{\mathcal{O}_2[k]}$, and several flavors of tensor products between 2-categories.

We denote by $\Sigma[1] = [1|1]$ the free-living 2-cell, by $\Sigma\mathcal{I}$ the free-living 2-isomorphism, and by $\mathcal{O}_2[i]$ the i-th 2-truncated oriental; see e.g. [Mos20, Def. 5.1.1] for an explicit description of this 2-category.[10]

Notation 3.8 ([Mos20, Def. 5.1.1]). For $i \geq 0$, let $\mathcal{O}_2^{\sim}[i]$ be the 2-category obtained by gluing an invertible 2-cell $\Sigma\mathcal{I}$ on each generating 2-cell $\Sigma[1]$ of the 2-truncated i-oriental $\mathcal{O}_2[i]$; it can be expressed as the pushout of 2-categories

$$
\begin{array}{ccc}
\coprod\limits_{\binom{[i]}{[2]}} \Sigma[1] & \longrightarrow & \mathcal{O}_2[i] \\
\downarrow & & \downarrow \\
\coprod\limits_{\binom{[i]}{[2]}} \Sigma\mathcal{I} & \longrightarrow & \mathcal{O}_2^{\sim}[i]
\end{array}
$$

where the coproducts are indexed over the set of generating 2-cells of $\mathcal{O}_2[i]$.

We denote by $[1]$ the free-living 1-cell, and by $\mathcal{E}$ the free adjoint equivalence; see e.g. [Lac04, §6] or [OR21, Not. 1.9] for an explicit description of this 2-category.

Notation 3.9 ([Mos20, Def. 5.1.1]). For $k \geq 0$, let $\widetilde{\mathcal{O}_2[k]}$ be the 2-category obtained by gluing an adjoint equivalence $\mathcal{E}$ on each generating 1-cell $[1]$ of the 2-category $\mathcal{O}_2^{\sim}[k]$; it can be expressed as the pushout of 2-categories

---

[9]In the original source, the nerve is denoted $\mathbb{NH}^{\simeq}\mathcal{D}$, as opposed to $\mathbf{N}^{\Delta \times \Delta}\mathcal{D}$.

[10]As simplicial categories, there is an isomorphism $N_*\mathcal{O}_2[i] \cong \mathfrak{C}\Delta[i]$, where $\mathfrak{C} \colon s\mathcal{S}et \to s\mathcal{C}at$ is the left adjoint to the homotopy coherent nerve functor.

$$
\begin{array}{ccc}
\coprod\limits_{\binom{[k]}{[1]}} [1] & \longrightarrow & \mathcal{O}_2^{\sim}[k] \\
\downarrow & & \downarrow \\
\coprod\limits_{\binom{[k]}{[1]}} \mathcal{E} & \longrightarrow & \widetilde{\mathcal{O}_2[k]}
\end{array}
$$

where the coproducts are indexed over the set of generating 1-cells of $\mathcal{O}_2^{\sim}[k]$.

Next, we explore choices of tensor products for 2-categories.

Remark 3.10. We consider several choices to form a 2-category of 2-functors between two 2-categories $\mathcal{B}$ and $\mathcal{D}$, which all have 2-functors from $\mathcal{B}$ to $\mathcal{D}$ as objects, and modifications as 2-cells, but differ in the 1-cells.

(0) The 2-category $[\mathcal{B}, \mathcal{D}]$ consists of 2-functors, (strict) natural transformations, and modifications.
(1) The 2-category $[\mathcal{B}, \mathcal{D}]_{\mathrm{ps}}$ consists of 2-functors, pseudonatural transformations, and modifications.
(2) The 2-category $[\mathcal{B}, \mathcal{D}]_{\mathrm{lax}}$ consists of 2-functors, lax[11] natural transformations, and modifications.
(3) The 2-category $[\mathcal{B}, \mathcal{D}]_{\mathrm{ic}}$ consists of 2-functors, icons, and modifications. We recall that an *icon* is a lax natural transformation for which each component is an identity.

The first three notions were first discussed by Gray in [Gra74, §I.2], and the last one by Lack in [Lac10]. We refer the reader to the recent paper by Johnson–Yau [JY21, Ch. 4] for explicit definitions. The definition of the different kinds of natural transformations appears as [JY21, Def. 4.2.1], where they call pseudonatural transformations "strong", the one of icons as [JY21, Def. 4.6.2], and the one of modifications as [JY21, Def. 4.4.1].

There are canonical and natural maps of 2-categories

$$[\mathcal{B}, \mathcal{D}] \hookrightarrow [\mathcal{B}, \mathcal{D}]_{\mathrm{ps}} \hookrightarrow [\mathcal{B}, \mathcal{D}]_{\mathrm{lax}} \hookleftarrow [\mathcal{B}, \mathcal{D}]_{\mathrm{ic}}.$$

Those constructions define functors

$$[-,-], [-,-]_{\mathrm{ps}}, [-,-]_{\mathrm{lax}}, [-,-]_{\mathrm{ic}} \colon 2\mathcal{C}at^{\mathrm{op}} \times 2\mathcal{C}at \to 2\mathcal{C}at$$

Each of those constructions is the internal hom functor for a corresponding tensor product which is part of a two-variable adjunction $2\mathcal{C}at \times 2\mathcal{C}at \to 2\mathcal{C}at$, some of which are discussed in [Gra74, Thm I.4.9, Thm I.4.14, Cor. I.4.17], and [Gur13, Thm 3.16].

The corresponding tensor products

$$\times, \otimes_{\mathrm{ps}}, \otimes, \otimes_{\mathrm{ic}} \colon 2\mathcal{C}at \times 2\mathcal{C}at \to 2\mathcal{C}at.$$

are, respectively:

(0) the *cartesian product* of 2-categories $\mathcal{A} \times \mathcal{B}$;
(1) the *pseudo Gray tensor product* of 2-categories $\mathcal{A} \otimes_{\mathrm{ps}} \mathcal{B}$;
(2) the *lax Gray tensor product* of 2-categories $\mathcal{A} \otimes \mathcal{B}$; and
(3) a construction that we may call the *icon tensor product* of 2-categories $\mathcal{A} \otimes_{\mathrm{ic}} \mathcal{B}$.

---

[11]There are different conventions in the literature for the meaning of the word *lax* (as opposed to *oplax* or *colax*), with equivalent resulting theories. The convention that we follow in this paper is consistent with the one used in e.g. in [Lac10, JFS17, Hau21], and it is opposite to the conventions of e.g. [Gur13, AL20, AM20].

They are related via canonical and natural maps of 2-categories

$$\mathcal{A} \times \mathcal{B} \leftarrow \mathcal{A} \otimes_{\mathrm{ps}} \mathcal{B} \leftarrow \mathcal{A} \otimes \mathcal{B} \rightarrow \mathcal{A} \otimes_{\mathrm{ic}} \mathcal{B}.$$

Here, the left-pointing maps are classical (see e.g. [Gra74, §I.4.24] and [Gur13, Cor. 3.22]), and the right-pointing map is a consequence of Lemma 3.11.

To highlight the difference between the four flavors, the four tensor products of the category $[1]$ with itself, or equivalently the corresponding naturality square, look as follows.

$$
\begin{array}{cccc}
(0) \quad \begin{array}{c} \bullet \longrightarrow \bullet \\ \downarrow \ /\!\!/ \ \downarrow \\ \bullet \longrightarrow \bullet \end{array} &
(1) \quad \begin{array}{c} \bullet \longrightarrow \bullet \\ \downarrow \ \stackrel{\cong}{/\!\!/} \ \downarrow \\ \bullet \longrightarrow \bullet \end{array} &
(2) \quad \begin{array}{c} \bullet \longrightarrow \bullet \\ \downarrow \ /\!\!/ \ \downarrow \\ \bullet \longrightarrow \bullet \end{array} &
(3) \quad \begin{array}{c} \bullet \longrightarrow \bullet \\ \| \ /\!\!/ \ \| \\ \bullet \longrightarrow \bullet \end{array} \\
[1] \times [1] & [1] \otimes_{\mathrm{ps}} [1] & [1] \otimes [1] & [1] \otimes_{\mathrm{ic}} [1]
\end{array}
$$

Recall that the inclusion functor $\mathcal{S}et \hookrightarrow \mathcal{C}at$ that regards every set as a discrete category admits left and right adjoint functors $\pi_0, \mathrm{Ob}\colon \mathcal{C}at \to \mathcal{S}et$. They send a category $\mathcal{D}$ to the set $\pi_0 \mathcal{D}$ of equivalence classes of its objects modulo the relation of being connected by a zigzag of 1-morphisms, and to the set of objects $\mathrm{Ob}\,\mathcal{D}$ of $\mathcal{D}$, respectively. The functor $\mathrm{Ob}\colon \mathcal{C}at \to \mathcal{S}et$ also admits a right adjoint functor $\mathrm{ch}\colon \mathcal{S}et \to \mathcal{C}at$, which sends a set S to the 1-category $\mathrm{ch}\,S$ whose set of objects is S and which has exactly one morphism between any pair of objects.

These functors induce by base-change functors $(\pi_0)_*, \mathrm{Ob}_*\colon 2\mathcal{C}at \to \mathcal{C}at$ which are left and right adjoint to the inclusion functor $\mathcal{C}at \hookrightarrow 2\mathcal{C}at$ that regards every category as a discrete 2-category. They send a 2-category $\mathcal{D}$ to the category $(\pi_0)_*\mathcal{D}$ with the same objects as $\mathcal{D}$ and hom-sets between two objects c, d in $\mathcal{D}$ given by $\pi_0(\mathcal{D}(c, d))$, and to its underlying category $\mathrm{Ob}_*\,\mathcal{D}$ obtained by forgetting the 2-morphisms, respectively.[12] By base-change, we also get a right adjoint $\mathrm{ch}_*\colon \mathcal{C}at \to 2\mathcal{C}at$ for the functor $\mathrm{Ob}_*\colon 2\mathcal{C}at \to \mathcal{C}at$, which sends a category $\mathcal{C}$ to the 2-category $\mathrm{ch}_*\,\mathcal{C}$ whose underlying category is $\mathcal{C}$ and which has exactly one 2-cell between any pair of parallel 1-cells.

Lemma 3.11. *For any* 2-*categories* $\mathcal{A}$ *and* $\mathcal{B}$ *there is a pushout of* 2-*categories*

$$
\begin{array}{ccc}
\mathrm{Ob}_*\,\mathcal{A} \otimes \mathrm{Ob}\,\mathcal{B} & \longrightarrow & \mathcal{A} \otimes \mathcal{B} \\
\downarrow & & \downarrow \\
\pi_0(\mathrm{Ob}_*\,\mathcal{A}) \otimes \mathrm{Ob}\,\mathcal{B} & \rightarrow & \mathcal{A} \otimes_{\mathrm{ic}} \mathcal{B}
\end{array}
$$

Proof. The statement follows formally from the fact that for any 2-categories $\mathcal{B}$ and $\mathcal{D}$ the commutative square

$$
\begin{array}{ccc}
[\mathcal{B}, \mathcal{D}]_{\mathrm{ic}} & \longrightarrow & [\mathcal{B}, \mathcal{D}]_{\mathrm{lax}} \\
\downarrow & & \downarrow \\
\prod_{\mathrm{Ob}\,\mathcal{B}} \mathrm{Ob}\,\mathcal{D} & \longrightarrow & \prod_{\mathrm{Ob}\,\mathcal{B}} \mathrm{ch}_*\,\mathrm{Ob}_*\,\mathcal{D}
\end{array}
$$

is a pullback of 2-categories. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

The following lemma can be understood as a special instance of [AL20, Prop. 4.5].

---

[12] The constructions $\mathrm{Ob}\,\mathcal{C}$, $\pi_0\mathcal{C}$, $\mathrm{Ob}_*\,\mathcal{D}$ and $(\pi_0)_*\mathcal{D}$ correspond to $\tau_{\leq 0}^{\mathrm{b}}\mathcal{C}$, $\tau_{\leq 0}^{\mathrm{i}}\mathcal{C}$, $\tau_{\leq 1}^{\mathrm{b}}\mathcal{D}$, and $\tau_{\leq 1}^{\mathrm{i}}\mathcal{D}$, respectively, following [AM20, §1.2], for a category $\mathcal{C}$ and a 2-category $\mathcal{D}$.

Lemma 3.12. *Given any 2-category $\mathcal{A}$ in which any 1-morphism is an equivalence, and any 2-category $\mathcal{B}$, the canonical map is an isomorphism of 2-categories*

$$\mathcal{A} \otimes \mathcal{B} \xrightarrow{\cong} \mathcal{A} \otimes_{\mathrm{ps}} \mathcal{B}.$$

Proof. First, observe we have the following commutative diagram of 2-categories.

$$
\begin{array}{ccccc}
\coprod\limits_{\mathrm{Mor}\,\mathcal{A} \times \mathrm{Mor}\,\mathcal{B}} \Sigma[1] & \longrightarrow & \coprod\limits_{\mathrm{Mor}\,\mathcal{A} \times \mathrm{Mor}\,\mathcal{B}} [1] \otimes [1] & \longrightarrow & \mathcal{A} \otimes \mathcal{B} \\
\downarrow & & \llcorner\ \downarrow & & \llcorner\ \downarrow \\
\coprod\limits_{\mathrm{Mor}\,\mathcal{A} \times \mathrm{Mor}\,\mathcal{B}} \Sigma\mathcal{I} & \longrightarrow & \coprod\limits_{\mathrm{Mor}\,\mathcal{A} \times \mathrm{Mor}\,\mathcal{B}} [1] \otimes_{\mathrm{ps}} [1] & \longrightarrow & \mathcal{A} \otimes_{\mathrm{ps}} \mathcal{B}
\end{array}
$$

Here, the external and left-hand commutative squares are pushouts, so the right-hand one is too. If $\mathcal{B}$ meets the assumptions of the lemma, the map $\coprod[1] \otimes [1] \to \mathcal{A} \otimes \mathcal{B}$ factors through the canonical inclusion $\coprod[1] \otimes [1] \to \coprod \mathcal{E} \otimes [1]$ and we obtain a lift in the above right-hand square, constructed as follows.

$$
\begin{array}{ccccc}
\coprod\limits_{\mathrm{Mor}\,\mathcal{A} \times \mathrm{Mor}\,\mathcal{B}} [1] \otimes [1] & \longrightarrow & \coprod\limits_{\mathrm{Mor}\,\mathcal{A} \times \mathrm{Mor}\,\mathcal{B}} \mathcal{E} \otimes [1] & \dashrightarrow & \mathcal{A} \otimes \mathcal{B} \\
\downarrow & & \downarrow & & \downarrow \\
\coprod\limits_{\mathrm{Mor}\,\mathcal{A} \times \mathrm{Mor}\,\mathcal{B}} [1] \otimes_{\mathrm{ps}} [1] & \longrightarrow & \coprod\limits_{\mathrm{Mor}\,\mathcal{A} \times \mathrm{Mor}\,\mathcal{B}} \mathcal{E} \otimes_{\mathrm{ps}} [1] & \longrightarrow & \mathcal{A} \otimes_{\mathrm{ps}} \mathcal{B}
\end{array}
$$

It follows from the universal property of pushouts that in any pushout square that admits a diagonal lift the right vertical map is an isomorphism of 2-categories, which concludes the proof. $\qquad\square$

With the following proposition, we can now give an explicit description of the functor L on representable presheaves $\Delta[i, j, k]$. It could be taken as a definition by the reader who encounters it for the first time, or as a statement for the reader who is familiar with the double categorical framework, whose necessary ingredients we recall in the proof.

Proposition 3.13. *For $i, j, k \geq 0$ there is a natural isomorphism of 2-categories*

$$L\mathbb{C}\Delta[i, j, k] \cong \mathcal{O}_2^{\sim}[j] \otimes_{\mathrm{ic}} (\mathcal{O}_2^{\sim}[i] \otimes_{\mathrm{ps}} \widetilde{\mathcal{O}_2[k]}).$$

Proof. First, we recall the relevant constructions and definitions from [Mos20] needed to prove the desired claim.

(Recall 1) The *horizontal and vertical embeddings* $\mathbb{H}, \mathbb{V} \colon 2\mathcal{C}at \to \mathcal{D}bl\mathcal{C}at$, which regard any 2-category $\mathcal{D}$ as a *horizontal* and *vertical double category*, recalled as [Mos20, Def. 2.1.7, Rmk 2.1.10].

(Recall 2) Their respective right adjoint functors $\mathbf{H}, \mathbf{V} \colon \mathcal{D}bl\mathcal{C}at \to 2\mathcal{C}at$, namely the *underlying horizontal* and *vertical 2-category* functors, are discussed in [Mos20, Def. 2.1.8, Rmk 2.1.10].

(Recall 3) The functor $\mathbb{C} \colon \mathcal{S}et^{(\Delta \times \Delta \times \Delta)^{\mathrm{op}}} \to \mathcal{D}bl\mathcal{C}at$ from [Mos20, Prop. 5.1.4].

(Recall 4) The left adjoint functor $L \colon \mathcal{D}bl\mathcal{C}at \to 2\mathcal{C}at$ of $\mathbb{H}$, discussed in [MSV22, §6].

(Recall 5) The pseudo hom double category $[\![-, -]\!] \colon \mathcal{D}bl\mathcal{C}at^{\mathrm{op}} \times \mathcal{D}bl\mathcal{C}at \to \mathcal{D}bl\mathcal{C}at$ from [Böh20, §2.2].

(Recall 6) The corresponding pseudo Gray tensor product of double categories of [Böh20]
$$\otimes_{\mathrm{ps}}^{\mathrm{dbl}} \colon \mathcal{DblCat} \times \mathcal{DblCat} \to \mathcal{DblCat}.$$

Next, we collect a few important facts that we will use.

(Obs. 1) For any 2-categories $\mathcal{B}$ and $\mathcal{D}$, there is a natural isomorphism of 2-categories
$$\mathbf{V}[\![\mathbb{H}\mathcal{B}, \mathbb{H}\mathcal{D}]\!] \cong [\mathcal{B}, \mathcal{D}]_{\mathrm{ic}}.$$

This can be deduced from a careful analysis of the involved 2-categories.

(Obs. 2) For $i, j, k \geq 0$, by [Mos20, Def. 2.2.4, Def. 5.1.3] the value of $\mathbb{C}$ at $\Delta[i, j, k]$ is given by the 2-category
$$\mathbb{X}_{i,j,k} = \mathbb{C}\Delta[i, j, k] = \mathbb{V}\mathcal{O}_2^\sim[j] \otimes_{\mathrm{ps}}^{\mathrm{dbl}} \mathbb{H}\mathcal{O}_2^\sim[i] \otimes_{\mathrm{ps}}^{\mathrm{dbl}} \mathbb{H}\widetilde{\mathcal{O}_2[k]}.$$

(Obs. 3) By [MSV22, Lem. 7.8], for any 2-categories $\mathcal{A}$ and $\mathcal{B}$, there is an isomorphism of double categories
$$\mathbb{H}\mathcal{A} \otimes_{\mathrm{ps}}^{\mathrm{dbl}} \mathbb{H}\mathcal{B} \cong \mathbb{H}(\mathcal{A} \otimes_{\mathrm{ps}} \mathcal{B}).$$

Now, for any 2-category $\mathcal{D}$ and $i, j, k \geq 0$, we obtain natural bijections

$$
\begin{aligned}
2\mathcal{C}at(L\mathbb{C}\Delta[i, j, k], \mathcal{D}) &\cong \mathcal{DblCat}(\mathbb{C}\Delta[i, j, k], \mathbb{H}\mathcal{D}) && \text{(Recall 4)} \\
&\cong \mathcal{DblCat}(\mathbb{V}\mathcal{O}_2^\sim[j] \otimes_{\mathrm{ps}}^{\mathrm{dbl}} \mathbb{H}\mathcal{O}_2^\sim[i] \otimes_{\mathrm{ps}}^{\mathrm{dbl}} \mathbb{H}\widetilde{\mathcal{O}_2[k]}, \mathbb{H}\mathcal{D}) && \text{(Obs. 2)} \\
&\cong \mathcal{DblCat}(\mathbb{V}\mathcal{O}_2^\sim[j] \otimes_{\mathrm{ps}}^{\mathrm{dbl}} \mathbb{H}(\mathcal{O}_2^\sim[i] \otimes_{\mathrm{ps}} \widetilde{\mathcal{O}_2[k]}), \mathbb{H}\mathcal{D}) && \text{(Obs. 3)} \\
&\cong \mathcal{DblCat}(\mathbb{V}\mathcal{O}_2^\sim[j], [\![\mathbb{H}(\mathcal{O}_2^\sim[i] \otimes_{\mathrm{ps}} \widetilde{\mathcal{O}_2[k]}), \mathbb{H}\mathcal{D}]\!]) && \text{(Recall 5)} \\
&\cong 2\mathcal{C}at(\mathcal{O}_2^\sim[j], \mathbf{V}[\![\mathbb{H}(\mathcal{O}_2^\sim[i] \otimes_{\mathrm{ps}} \widetilde{\mathcal{O}_2[k]}), \mathbb{H}\mathcal{D}]\!]) && \text{(Recall 2)} \\
&\cong 2\mathcal{C}at(\mathcal{O}_2^\sim[j], [\mathcal{O}_2^\sim[i] \otimes_{\mathrm{ps}} \widetilde{\mathcal{O}_2[k]}, \mathcal{D}]_{\mathrm{ic}}) && \text{(Obs. 1)} \\
&\cong 2\mathcal{C}at(\mathcal{O}_2^\sim[j] \otimes_{\mathrm{ic}} (\mathcal{O}_2^\sim[i] \otimes_{\mathrm{ps}} \widetilde{\mathcal{O}_2[k]}), \mathcal{D}). && \text{Rmk 3.10}
\end{aligned}
$$

The claim then follows from the Yoneda lemma. $\square$

3.3. Nerve comparison.

We study the compatibility between $\mathbf{N}^{\Delta \times \Delta}$ and $\mathbf{N}^{\Theta_2}$.

Lemma 3.14. *For any 2-category θ in Θ_2 there is an isomorphism of bisimplicial sets*
$$d^* \Theta_2[\theta] = d^* \mathbf{N}^{\Theta_2} \theta \cong \mathbf{N}_{i,j,0}^{\Delta \times \Delta} \theta.$$

Proof. As a preliminary observation, there is a canonical map
$$\mathcal{O}_2^\sim[j] \otimes_{\mathrm{ic}} \mathcal{O}_2^\sim[i] \to [j] \otimes_{\mathrm{ic}} [i] \cong [i | j, \ldots, j],$$

that can be seen by inspection to be a biequivalence.

Given that the 2-category θ is gaunt, namely it does not have any non-identity 2-isomorphisms and any non-identity 1-equivalences, the functor $2\mathcal{C}at(-, \theta)$ sends biequivalences to bijections, and the functors $2\mathcal{C}at(-, \theta)$ and $2\mathcal{C}at_{\mathrm{nps}}(-, \theta)$ are isomorphic. Hence, for any $i, j \geq 0$ we find natural bijections

$$
\begin{aligned}
(d^* \mathbf{N}^{\Theta_2} \theta)_{i,j} = \mathbf{N}_{d[i,j]}^{\Theta_2} \theta = \mathbf{N}_{[i|j,\ldots,j]}^{\Theta_2} \theta &\cong 2\mathcal{C}at_{\mathrm{nps}}([i | j, \ldots, j], \theta) \\
&\cong 2\mathcal{C}at([i | j, \ldots, j], \theta) \cong 2\mathcal{C}at(\mathcal{O}_2^\sim[j] \otimes_{\mathrm{ic}} \mathcal{O}_2^\sim[i], \theta) \\
&\cong 2\mathcal{C}at(L\mathbb{C}\Delta[i, j, 0], \theta) \cong \mathbf{N}_{i,j,0}^{\Delta \times \Delta} \theta,
\end{aligned}
$$

as desired. $\square$

Theorem 3.15. *For any 2-category $\mathcal{D}$ there is a natural isomorphism of Θ_2-sets*

$$d_* \mathbf{N}^{\Delta \times \Delta}_{\bullet, 0} \mathcal{D} \cong \mathbf{N}^{\Theta_2} \mathcal{D}.$$

The crucial technical computation occurring in the proof is proven later as Proposition C.8.

Proof of Theorem 3.15. For any 2-category $\mathcal{D}$ and any object θ in Θ_2 there is a natural bijection

$$
\begin{aligned}
d_* \mathbf{N}^{\Delta \times \Delta}_{\theta, 0} \mathcal{D} &\cong s\mathcal{S}et^{\Theta_2^{\mathrm{op}}}(\Theta_2[\theta], d_* \mathbf{N}^{\Delta \times \Delta} \mathcal{D}) \\
&\cong s\mathcal{S}et^{(\Delta \times \Delta)^{\mathrm{op}}}(d^* \Theta_2[\theta], \mathbf{N}^{\Delta \times \Delta} \mathcal{D}) && d^* \dashv d_* \\
&\cong s\mathcal{S}et^{(\Delta \times \Delta)^{\mathrm{op}}}(d^* \mathbf{N}^{\Theta_2} \theta, \mathbf{N}^{\Delta \times \Delta} \mathcal{D}) && \text{Lemma 3.14} \\
&\cong s\mathcal{S}et^{(\Delta \times \Delta)^{\mathrm{op}}}(\mathbf{N}^{\Delta \times \Delta} \theta, \mathbf{N}^{\Delta \times \Delta} \mathcal{D}) && \text{Lemma 3.14} \\
&\cong 2\mathcal{C}at_{\mathrm{nps}}(\theta, \mathcal{D}) && \text{Proposition C.8} \\
&\cong \mathbf{N}^{\Theta_2}_{\theta} \mathcal{D},
\end{aligned}
$$

as desired. $\square$

Recall the right Quillen equivalences from (2.5) and (3.4) and the nerve constructions from Constructions 2.6 and 3.6.

Corollary 3.16. *The diagram of ∞-categories*

$$
\begin{array}{ccc}
& [2\mathcal{C}at]_\infty & \\
{\scriptstyle [\mathbf{N}^{\Delta \times \Delta}]_\infty} \swarrow & & \searrow {\scriptstyle [\mathbf{N}^{\Theta_2}]_\infty} \\
[s\mathcal{S}et^{(\Delta \times \Delta)^{\mathrm{op}}}_{(\infty,2)}]_\infty \xrightarrow[{[d_*]_\infty}]{} & [s\mathcal{S}et^{\Theta_2^{\mathrm{op}}}_{(\infty,2)}]_\infty \xrightarrow[{[(-)_{\bullet,0}]_\infty}]{} & [\mathcal{S}et^{\Theta_2^{\mathrm{op}}}_{(\infty,2)}]_\infty
\end{array}
$$

commutes up to equivalence.

Proof. The corollary is an application of the "right Quillen" version of Lemma A.1 to the diagram

$$
\begin{array}{ccc}
& 2\mathcal{C}at & \\
{\scriptstyle \mathbf{N}^{\Delta \times \Delta}} \swarrow & & \searrow {\scriptstyle \mathbf{N}^{\Theta_2}} \\
s\mathcal{S}et^{(\Delta \times \Delta)^{\mathrm{op}}}_{(\infty,2)} \xrightarrow[{d_*}]{} & s\mathcal{S}et^{\Theta_2^{\mathrm{op}}}_{(\infty,2)} \xrightarrow[{(-)_{\bullet,0}}]{} & \mathcal{S}et^{\Theta_2^{\mathrm{op}}}_{(\infty,2)}
\end{array} .
$$

The fact that all the assumptions of the lemma are met is from Theorems 2.7, 3.7 and 3.15.
$\square$

3.4. **A further model.** Alternative models of $(\infty, 2)$-categories, due to Bergner–Rezk, arises as the class of fibrant objects of two model structures on the category $P\mathcal{C}at(s\mathcal{S}et^{\Delta^{\mathrm{op}}})$, see [BR13, §6.7, §6.11]. Here, $P\mathcal{C}at(s\mathcal{S}et^{\Delta^{\mathrm{op}}})$ denotes the full subcategory of $s\mathcal{S}et^{(\Delta \times \Delta)^{\mathrm{op}}}$ spanned by the bisimplicial spaces X for which X_0 is a set. We refer to those as *precategory objects in simplicial spaces.*[13] One is referred to as the *injective-like* model structure on

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[13] In the original source those are referred to as *Segal precategories in simplicial spaces.*

$P\mathcal{C}at(s\mathcal{S}et^{\Delta^{\mathrm{op}}})$, and one as the *projective-like*. In this paper, we make use of the projective-like, which we denote $P\mathcal{C}at(s\mathcal{S}et^{\Delta^{\mathrm{op}}})_{(\infty,2)}$. We will never need an explicit description of this model structure, and we only use the fact that it comes with two Quillen equivalences, which will be recalled as (3.17) and (4.6).

The canonical inclusion functor $I\colon P\mathcal{C}at(s\mathcal{S}et^{\Delta^{\mathrm{op}}}) \hookrightarrow s\mathcal{S}et^{(\Delta\times\Delta)^{\mathrm{op}}}$ admits a right adjoint R, which was proven by Bergner–Rezk as [BR20, Prop. 9.5, Thm 9.6] to be a right Quillen equivalence

$$(3.17) \qquad R\colon s\mathcal{S}et^{(\Delta\times\Delta)^{\mathrm{op}}}_{(\infty,2)} \to P\mathcal{C}at(s\mathcal{S}et^{\Delta^{\mathrm{op}}})_{(\infty,2)}.$$

The functor $I\colon P\mathcal{C}at(s\mathcal{S}et^{\Delta^{\mathrm{op}}})_{(\infty,2)} \to s\mathcal{S}et^{(\Delta\times\Delta)^{\mathrm{op}}}_{(\infty,2)}$ reflects weak equivalences between precategories, in the sense of the following lemma.

Lemma 3.18. *The functor $I\colon P\mathcal{C}at(s\mathcal{S}et^{\Delta^{\mathrm{op}}})_{(\infty,2)} \to s\mathcal{S}et^{(\Delta\times\Delta)^{\mathrm{op}}}_{(\infty,2)}$ reflects weak equivalences. That is, if $f\colon X \to Y$ is a map in $P\mathcal{C}at(s\mathcal{S}et^{\Delta^{\mathrm{op}}})_{(\infty,2)}$ such that its image $If\colon IX \to IY$ is a weak equivalence in $s\mathcal{S}et^{(\Delta\times\Delta)^{\mathrm{op}}}_{(\infty,2)}$, then $f\colon X \to Y$ is a weak equivalence in $P\mathcal{C}at(s\mathcal{S}et^{\Delta^{\mathrm{op}}})_{(\infty,2)}$.*

This statement already occurs in the proof of [BR20, Thm 9.6], recalling from [BR13, §6.7, §6.11] that the weak equivalences of the two model structures from [BR13, §6] considered on $P\mathcal{C}at(s\mathcal{S}et^{\Delta^{\mathrm{op}}})$ coincide. We recollect an outline of the argument here for the reader's convenience.

Proof. Bergner–Rezk introduce a functor $L\colon P\mathcal{C}at(s\mathcal{S}et^{\Delta^{\mathrm{op}}}) \to P\mathcal{C}at(s\mathcal{S}et^{\Delta^{\mathrm{op}}})$ and natural weak equivalence $X \xrightarrow{\simeq} LX$ in $s\mathcal{S}et^{(\Delta\times\Delta)^{\mathrm{op}}}_{(\infty,2)}$ for every X in $P\mathcal{C}at(s\mathcal{S}et^{\Delta^{\mathrm{op}}})$, in [BR13, §6.7]. By construction, LX is an injectively fibrant Segal space. It is discussed in [BR13, §6.7, §6.11] that the functor L detects weak equivalences of $P\mathcal{C}at(s\mathcal{S}et^{\Delta^{\mathrm{op}}})_{(\infty,2)}$ in the following sense: a map $f\colon X \to Y$ in $P\mathcal{C}at(s\mathcal{S}et^{\Delta^{\mathrm{op}}})$ is a weak equivalence in $P\mathcal{C}at(s\mathcal{S}et^{\Delta^{\mathrm{op}}})_{(\infty,2)}$ if and only if the induced map $Lf\colon LX \to LY$ is a Dwyer–Kan equivalence in the sense of [BR20, Def. 8.2].

Now assume a map $f\colon X \to Y$ in $P\mathcal{C}at(s\mathcal{S}et^{\Delta^{\mathrm{op}}})$ is a weak equivalence viewed as $If\colon IX \to IY$ in $s\mathcal{S}et^{(\Delta\times\Delta)^{\mathrm{op}}}_{(\infty,2)}$. We consider the commutative diagram

$$\begin{array}{ccc} X & \longrightarrow & Y \\ \downarrow & & \downarrow \\ LX & \longrightarrow & LY \end{array}.$$

By assumption and by construction of L, all but possibly the lower horizontal map are weak equivalences in $s\mathcal{S}et^{(\Delta\times\Delta)^{\mathrm{op}}}_{(\infty,2)}$. By 2-out-of-3, the lower horizontal map must also be. Now once again by construction, its source and target are (injectively fibrant) Segal spaces. By [BR20, Thm 8.18], this being a weak equivalence in $s\mathcal{S}et^{(\Delta\times\Delta)^{\mathrm{op}}}_{(\infty,2)}$ is equivalent to being a Dwyer–Kan equivalence, thus showing that f is a weak equivalence in $P\mathcal{C}at(s\mathcal{S}et^{\Delta^{\mathrm{op}}})_{(\infty,2)}$, as desired. $\qquad\square$

From [BR20, §9], for any bisimplicial space X, the value RX of the right adjoint R to the inclusion of $P\mathcal{C}at(s\mathcal{S}et^{\Delta^{\mathrm{op}}})$ into trisimplicial sets can be understood as the following pullback in trisimplicial sets.

$$
\begin{array}{ccc}
RX & \longrightarrow & \mathrm{cosk}_0(X_{0,0,0}) \\
\downarrow & \lrcorner & \downarrow \\
X & \longrightarrow & \mathrm{cosk}_0(X_{0,\bullet})
\end{array}
$$

(3.19)

Here, $\mathrm{cosk}_0\colon s\mathcal{S}et^{\Delta^{\mathrm{op}}} \to s\mathcal{S}et^{\Delta\times\Delta^{\mathrm{op}}}$ denotes the 0-th coskeleton functor used in [BR20, §9]. We give an alternative description of RX.

Remark 3.20. For $i,j,k \geq 0$, let $\overline{\Delta}[i,j,k]$ denote the following pushout of bisimplicial spaces.

$$
\begin{array}{ccc}
\coprod\limits_{i+1} \Delta[0,j,k] & \longrightarrow & \Delta[i,j,k] \\
\downarrow & & \ulcorner\ \downarrow \\
\coprod\limits_{i+1} \Delta[0,0,0] & \longrightarrow & \overline{\Delta}[i,j,k]
\end{array}
$$

Notice that, although not all objects occurring in the span belong to $P\mathcal{C}at(s\mathcal{S}et^{\Delta^{\mathrm{op}}})$, the pushout $\overline{\Delta}[i,j,k]$ does in fact belong to $P\mathcal{C}at(s\mathcal{S}et^{\Delta^{\mathrm{op}}})$.

Given the pullback (3.19), we deduce that for any bisimplicial space X there is a natural bijection

$$
(RX)_{i,j,k} \cong s\mathcal{S}et^{(\Delta\times\Delta)^{\mathrm{op}}}(\overline{\Delta}[i,j,k], X).
$$

Remark 3.21. For any $j,k \geq 0$ there is an isomorphism of bisimplicial spaces

$$
\overline{\Delta}[0,j,k] \cong \Delta[0,0,0].
$$

3.5. **The nerve.** In the remainder of this subsection, we study the bisimplicial space $R\mathbf{N}^{\Delta\times\Delta}\mathcal{D}$, which will be relevant in addressing the compatibility of the nerve constructions for bisimplicial and enriched models.

Remark 3.22. Observe that for any $i,j,k \geq 0$ and $\mathcal{D}$ a 2-category there is a natural bijection

$$
(R\mathbf{N}^{\Delta\times\Delta}\mathcal{D})_{i,j,k} \cong s\mathcal{S}et^{(\Delta\times\Delta)^{\mathrm{op}}}(\overline{\Delta}[i,j,k], \mathbf{N}^{\Delta\times\Delta}\mathcal{D}) \cong 2\mathcal{C}at(L^{\simeq}\mathbb{C}\overline{\Delta}[i,j,k], \mathcal{D}).
$$

Proposition 3.23. *For any $i,j,k \geq 0$ there is a natural isomorphism of 2-categories*

$$
L^{\simeq}\mathbb{C}\overline{\Delta}[i,j,k] \cong L\mathbb{C}\overline{\Delta}[i,j,k].
$$

In particular, for any 2-category $\mathcal{D}$ and $i,j,k \geq 0$ there is a natural bijection

$$
(R\mathbf{N}^{\Delta\times\Delta}\mathcal{D})_{i,j,k} \cong 2\mathcal{C}at(L^{\simeq}\mathbb{C}\overline{\Delta}[i,j,k], \mathcal{D}) \cong 2\mathcal{C}at(L\mathbb{C}\overline{\Delta}[i,j,k], \mathcal{D}).
$$

The proof relies on the following lemma.

Lemma 3.24. *For any $i,j,k \geq 0$ there is a pushout of 2-categories*

$$
\begin{array}{ccc}
\coprod\limits_{i+1}\coprod\limits_{k+1} L^{\simeq}\mathbb{C}\Delta[0,j,0] & \longrightarrow & L^{\simeq}\mathbb{C}\Delta[i,j,k] \\
\downarrow & & \ulcorner\ \downarrow \\
\coprod\limits_{i+1}\coprod\limits_{k+1} [0] & \longrightarrow & L\mathbb{C}\Delta[i,j,k] \ .
\end{array}
$$

Proof. Let $\mathcal{P}$ be the pushout of the span of 2-categories

$$\coprod_{i+1} \coprod_{k+1} [0] \longleftarrow \coprod_{i+1} \coprod_{k+1} L^{\simeq}\mathbb{C}\Delta[0,j,0] \longrightarrow L^{\simeq}\mathbb{C}\Delta[i,j,k] \,.$$

Using the naturality of the map in Proposition 3.5, we get an induced commutative diagram of 2-categories.

In the diagram, the left vertical map is a coproduct of the biequivalence

$$L^{\simeq}\mathbb{C}\Delta[0,j,0] \to L\mathbb{C}\Delta[0,j,0] \cong \mathcal{O}_2^{\sim}[j] \otimes_{\mathrm{ic}} (\mathcal{O}_2^{\sim}[0] \otimes_{\mathrm{ps}} \widetilde{\mathcal{O}_2[0]}) \cong [0],$$

built using Propositions 3.5 and 3.13, so it is a biequivalence itself.

Also, the top horizontal map is obtained by applying the composite left Quillen functor $L^{\simeq}\mathbb{C}\colon s\mathcal{S}et^{(\Delta \times \Delta)^{\mathrm{op}}} \to 2\mathcal{C}at$ from Theorem 3.7 to the cofibration

$$\coprod_{i+1} \coprod_{k+1} \Delta[0,j,0] \to \Delta[i,j,k],$$

so it is a cofibration itself.

Since the model structure on $2\mathcal{C}at$ is left proper by [Lac02, Thm 6.3] (see also [Lac04, §2]), it follows that the bottom horizontal map

$$L^{\simeq}\mathbb{C}\Delta[i,j,k] \to \mathcal{P}$$

is also a biequivalence. Since the map $L^{\simeq}\mathbb{C}\Delta[i,j,k] \to L\mathbb{C}\Delta[i,j,k]$ is a biequivalence by Proposition 3.5, then by 2-out-of-3 the comparison map

$$\mathcal{P} \to L\mathbb{C}\Delta[i,j,k]$$

is also a biequivalence. Using the explicit description from [Mos20, Desc. 6.3.1], we now see that this comparison map is an isomorphism on underlying 1-categories, which is sufficient to conclude that it must in fact be an isomorphism of 2-categories, as biequivalences are in particular isomorphisms on 2-morphisms. $\square$

We can now prove the proposition.

Proof of Proposition 3.23. We argue that for any $i,j,k \geq 0$ there is an isomorphism of 2-categories

(3.25) $$L^{\simeq}\mathbb{C}\overline{\Delta}[i,j,k] \cong L\mathbb{C}\overline{\Delta}[i,j,k]$$

that is natural in i,j,k.

To this end, we consider the following commutative diagram in of 2-categories.

$$
\begin{array}{ccccc}
\coprod_{i+1}\coprod_{k+1}[0] & =\!\!=\!\!= & \coprod_{i+1}\coprod_{k+1}[0] & =\!\!=\!\!= & \coprod_{i+1}\coprod_{k+1}[0] \\[4pt]
\big\| & & \big\uparrow & & \big\uparrow \\[4pt]
\coprod_{i+1}\coprod_{k+1}[0] & \longleftarrow & \coprod_{i+1}\coprod_{k+1}L^{\simeq}\mathbb{C}\Delta[0,j,0] & =\!\!=\!\!= & \coprod_{i+1}\coprod_{k+1}L^{\simeq}\mathbb{C}\Delta[0,j,0] \\[4pt]
\big\downarrow & & \big\downarrow & & \big\downarrow \\[4pt]
\coprod_{i+1}[0] & \longleftarrow & \coprod_{i+1}L^{\simeq}\mathbb{C}\Delta[0,j,k] & \longrightarrow & L^{\simeq}\mathbb{C}\Delta[i,j,k]
\end{array}
$$

The colimit of this diagram can be equivalently computed by either taking the colimit of the colimit of each row, or by taking the colimit of the colimit of each column.

On the one hand, by Remark 3.20 and using the fact that $L^{\simeq}$ is a left adjoint functor, the colimit of each row produces the span of 2-categories

$$
\begin{array}{c}
\coprod_{i+1}\coprod_{k+1}[0] \\[6pt]
\big\| \\[6pt]
\coprod_{i+1}\coprod_{k+1}[0] \\[6pt]
\big\downarrow \\[6pt]
L^{\simeq}\mathbb{C}\overline{\Delta}[i,j,k]
\end{array}
$$

whose pushout is $L^{\simeq}\mathbb{C}\overline{\Delta}[i,j,k]$.

On the other hand, by Lemma 3.24 we see that the pushout of each column produces the following span of 2-categories

$$
\coprod_{i+1}[0] \longleftarrow \coprod_{i+1}L\mathbb{C}\Delta[0,j,k] \longrightarrow L\mathbb{C}\Delta[i,j,k]
$$

and by Remark 3.20 and using the fact that L and $\mathbb{C}$ are left adjoint functors, its colimit is $L\mathbb{C}\overline{\Delta}[i,j,k]$.

Hence, the isomorphism (3.25) follows. $\qquad\square$

Remark 3.26. Given any 2-category $\mathcal{D}$, combining Theorem 3.7 and (3.17) we know that $R\mathbf{N}^{\Delta\times\Delta}\mathcal{D}$ is fibrant in $P\mathcal{C}at(s\mathcal{S}et^{\Delta^{\mathrm{op}}})_{(\infty,2)}$. By Remark 3.2, for any $j \geq 0$ we know that $(R\mathbf{N}^{\Delta\times\Delta}\mathcal{D})_{\bullet,j}$ is a Segal space. It follows that for any $i,j \geq 0$ we have a weak equivalence of spaces

$$
(R\mathbf{N}^{\Delta\times\Delta}\mathcal{D})_{i,j} \simeq (R\mathbf{N}^{\Delta\times\Delta}\mathcal{D})_{1,j} \underset{(R\mathbf{N}^{\Delta\times\Delta}\mathcal{D})_{0,j}}{\times} \cdots \underset{(R\mathbf{N}^{\Delta\times\Delta}\mathcal{D})_{0,j}}{\times} (R\mathbf{N}^{\Delta\times\Delta}\mathcal{D})_{1,j}.
$$

This motivates us to understand better the sets $(R\mathbf{N}^{\Delta\times\Delta}\mathcal{D})_{0,j,k}$ and $(R\mathbf{N}^{\Delta\times\Delta}\mathcal{D})_{1,j,k}$, which we achieve in Propositions 3.27 and 3.30.

Proposition 3.27. *For any 2-category $\mathcal{D}$ and $j,k \geq 0$ there is a natural bijection*

$$
(R\mathbf{N}^{\Delta\times\Delta}\mathcal{D})_{0,j,k} \cong \mathrm{Ob}\,\mathcal{D}.
$$

Proof. By Remarks 3.20 and 3.21, for any $j, k \geq 0$, we have a natural bijection

$$(R\mathbf{N}^{\Delta \times \Delta} \mathcal{D})_{0,j,k} \cong s\mathcal{S}et^{(\Delta \times \Delta)^{\mathrm{op}}}(\overline{\Delta}[0, j, k], \mathbf{N}^{\Delta \times \Delta} \mathcal{D})$$

$$\cong s\mathcal{S}et^{(\Delta \times \Delta)^{\mathrm{op}}}(\Delta[0, 0, 0], \mathbf{N}^{\Delta \times \Delta} \mathcal{D})$$

$$\cong \mathbf{N}_{0,0,0}^{\Delta \times \Delta} \mathcal{D} \cong 2\mathcal{C}at([0], \mathcal{D}) \cong \mathrm{Ob}\, \mathcal{D},$$

as desired. $\square$

We now proceed to describing $(R\mathbf{N}^{\Delta \times \Delta} \mathcal{D})_{1,j,k}$, which requires some extra work.

Given any category $\mathcal{A}$, we denote by $\Sigma \mathcal{A}$ the 2-point suspension of $\mathcal{A}$, which consists of two distinct objects and a single interesting hom-category given by $\mathcal{A}$. The construction extends to a left adjoint functor $\Sigma \colon \mathcal{C}at \to 2\mathcal{C}at_{*,*}$.

Lemma 3.28. *For any 2-category $\mathcal{A}$ there is a pushout of 2-categories*

$$
\begin{array}{ccc}
\mathcal{A} \amalg \mathcal{A} & \longrightarrow & [0] \amalg [0] \\
\downarrow & & \downarrow \\
\mathcal{A} \otimes [1] & \longrightarrow & \Sigma(\pi_0)_* \mathcal{A}
\end{array}
$$

Proof. If we denote by $\mathcal{P}\mathcal{A}$ the following pushout of 2-categories,

$$
\begin{array}{ccc}
\mathcal{A} \amalg \mathcal{A} & \longrightarrow & [0] \amalg [0] \\
\downarrow & & \downarrow \\
\mathcal{A} \otimes [1] & \longrightarrow & \mathcal{P}\mathcal{A}
\end{array}
$$

this construction can also be regarded as a left adjoint functor $\mathcal{P} \colon 2\mathcal{C}at \to 2\mathcal{C}at_{*,*}$. At the same time, also $\Sigma(\pi_0)_*$ defines a left adjoint functor $\Sigma(\pi_0)_* \colon 2\mathcal{C}at \to 2\mathcal{C}at_{*,*}$. One can now prove by direct inspection that for any i-cell $\Sigma^i[1]$ for $i = 0, 1, 2$, there is a natural isomorphism of bipointed 2-categories

$$\mathcal{P}\Sigma^i[1] \cong \Sigma(\pi_0)_* \Sigma^i[1].$$

It follows by cocontinuity that for every 2-category $\mathcal{A}$ there is an isomorphism of (bipointed) 2-categories

$$\mathcal{P}\mathcal{A} \cong \Sigma(\pi_0)_* \mathcal{A},$$

concluding the proof. $\square$

For $j, k \geq 0$, we let $\widetilde{[k]}$ denote the unique contractible groupoid with $k + 1$ objects, namely the category with $k + 1$ objects and a unique morphism between any two objects, and $\Sigma([j] \times \widetilde{[k]})$ the 2-point suspension of the 1-category $[j] \times \widetilde{[k]}$. This is the 2-category with two objects and a single interesting hom-category given by $[j] \times \widetilde{[k]}$.

Lemma 3.29. *For any $j, k \geq 0$ there is a pushout of 2-categories*

$$
\begin{array}{ccc}
L\mathbb{C}\Delta[0, j, k] \amalg L\mathbb{C}\Delta[0, j, k] & \longrightarrow & [0] \amalg [0] \\
\downarrow & & \downarrow \\
L\mathbb{C}\Delta[1, j, k] & \longrightarrow & \Sigma([j] \times \widetilde{[k]})
\end{array}
$$

Proof. Denote by $\mathcal{P}$ the following pushout of 2-categories.

$$
\begin{array}{ccc}
L\mathbb{C}\Delta[0,j,k] \amalg L\mathbb{C}\Delta[0,j,k] & \longrightarrow & [0] \amalg [0] \\
\downarrow & & \downarrow \\
L\mathbb{C}\Delta[1,j,k] & \longrightarrow & \mathcal{P}
\end{array}
$$

Consider the following commutative diagram of 2-categories.

$$
\begin{array}{ccccc}
\pi_0(\mathrm{Ob}_* \mathcal{O}_2^\sim[j]) \otimes \mathrm{Ob}([1] \otimes_{\mathrm{ps}} \widetilde{\mathcal{O}_2[k]}) & \overset{\cong}{\longleftarrow} & \coprod_2 \pi_0(\mathrm{Ob}_* \mathcal{O}_2^\sim[j]) \otimes \mathrm{Ob}\,\widetilde{\mathcal{O}_2[k]} & \longrightarrow & \coprod_2 [0] \\
\uparrow & & \uparrow & & \| \\
\mathrm{Ob}_* \mathcal{O}_2^\sim[j] \otimes \mathrm{Ob}([1] \otimes_{\mathrm{ps}} \widetilde{\mathcal{O}_2[k]}) & \overset{\cong}{\longleftarrow} & \coprod_2 \mathrm{Ob}_* \mathcal{O}_2^\sim[j] \otimes \mathrm{Ob}\,\widetilde{\mathcal{O}_2[k]} & \longrightarrow & \coprod_2 [0] \\
\downarrow & & \downarrow & & \| \\
\mathcal{O}_2^\sim[j] \otimes ([1] \otimes_{\mathrm{ps}} \widetilde{\mathcal{O}_2[k]}) & \longleftarrow & \coprod_2 \mathcal{O}_2^\sim[j] \otimes \widetilde{\mathcal{O}_2[k]} & \longrightarrow & \coprod_2 [0]
\end{array}
$$

The colimit of this diagram can be equivalently computed by either taking the colimit of the colimit of each row, or by taking the colimit of the colimit of each column.

By doing pushouts of each column first, we get using Lemma 3.11 the pushout of the span

$$
\mathcal{O}_2^\sim[j] \otimes_{\mathrm{ic}} ([1] \otimes_{\mathrm{ps}} \widetilde{\mathcal{O}_2[k]}) \longleftarrow \coprod_2 \mathcal{O}_2^\sim[j] \otimes_{\mathrm{ic}} \widetilde{\mathcal{O}_2[k]} \longrightarrow \coprod_2 [0] .
$$

By Proposition 3.13, we identify the pushout to be computed as the pushout of the span

$$
L\mathbb{C}\Delta[1,j,k] \longleftarrow \coprod_2 L\mathbb{C}\Delta[0,j,k] \longrightarrow \coprod_2 [0]
$$

which gives precisely $\mathcal{P}$.

Now note that there are natural isomorphisms of 2-categories

$$
\begin{aligned}
\mathcal{O}_2^\sim[j] \otimes ([1] \otimes_{\mathrm{ps}} \widetilde{\mathcal{O}_2[k]}) &\cong \mathcal{O}_2^\sim[j] \otimes (\widetilde{\mathcal{O}_2[k]} \otimes_{\mathrm{ps}} [1]) && \text{symmetry of } \otimes_{\mathrm{ps}} \\
&\cong \mathcal{O}_2^\sim[j] \otimes (\widetilde{\mathcal{O}_2[k]} \otimes [1]) && \text{Lemma 3.12} \\
&\cong (\mathcal{O}_2^\sim[j] \otimes \widetilde{\mathcal{O}_2[k]}) \otimes [1]. && \text{associativity of } \otimes
\end{aligned}
$$

By doing pushouts of each row, we get using the above isomorphism and Lemma 3.28 applied to $\mathcal{A} = \mathcal{O}_2^\sim[j] \otimes \widetilde{\mathcal{O}_2[k]}$ the pushout of the span

$$\coprod_{i+1} [0]$$

$$\|$$

$$\coprod_{i+1} [0]$$

$$\downarrow$$

$$\Sigma(\pi_0)_*(\mathcal{O}_2^{\sim}[j] \otimes \widetilde{\mathcal{O}_2[k]})$$

which gives precisely $\Sigma(\pi_0)_*(\mathcal{O}_2^{\sim}[j] \otimes \widetilde{\mathcal{O}_2[k]})$. Combining [AM20, Prop. A.27, §A.31], remembering that $(\pi_0)_*\mathcal{D} \cong \tau_{\leq 1}^i \mathcal{D}$, we obtain that for any $j, k \geq 0$, there are natural isomorphisms of 2-categories

$$\Sigma(\pi_0)_*(\mathcal{O}_2^{\sim}[j] \otimes \widetilde{\mathcal{O}_2[k]}) \cong \Sigma(((\pi_0)_*\mathcal{O}_2^{\sim}[j]) \times ((\pi_0)_*\widetilde{\mathcal{O}_2[k]}))$$

$$\cong \Sigma([j] \times \widetilde{[k]}).$$

So the desired isomorphism follows. $\qquad\square$

We can now describe $(R\mathbf{N}^{\Delta \times \Delta}\mathcal{D})_{1,j,k}$.

Proposition 3.30. *For any $j, k \geq 0$ there is a natural isomorphism of 2-categories*

$$L\mathbb{C}\overline{\Delta}[1, j, k] \cong \Sigma([j] \times \widetilde{[k]}).$$

In particular, for any 2-category $\mathcal{D}$ and $j, k \geq 0$ there is a natural bijection

$$(R\mathbf{N}^{\Delta \times \Delta}\mathcal{D})_{1,j,k} \cong 2\mathcal{C}at(L^{\simeq}\mathbb{C}\overline{\Delta}[1, j, k], \mathcal{D}) \cong 2\mathcal{C}at(\Sigma([j] \times \widetilde{[k]}), \mathcal{D}).$$

Proof. We show that for $j, k \geq 0$ there is a natural isomorphism of 2-categories

(3.31) $$L\mathbb{C}\overline{\Delta}[1, j, k] \cong \Sigma([j] \times \widetilde{[k]}).$$

By Remark 3.20, we know that $\overline{\Delta}[1, j, k]$ is the pushout of the span

$$\Delta[0, 0, 0] \amalg \Delta[0, 0, 0] \longleftarrow \Delta[0, j, k] \amalg \Delta[0, j, k] \longrightarrow \Delta[1, j, k] .$$

Since L and $\mathbb{C}$ are left adjoint functors, we obtain that $L\mathbb{C}\overline{\Delta}[1, j, k]$ is the pushout of the span

$$[0] \amalg [0] \longleftarrow L\mathbb{C}\Delta[0, j, k] \amalg L\mathbb{C}\Delta[0, j, k] \longrightarrow L\mathbb{C}\Delta[1, j, k] .$$

By Lemma 3.29, its pushout is $\Sigma([j] \times \widetilde{[k]})$. Hence, the isomorphism (3.31) follows.

The second part of the statement is a consequence of the above isomorphism and Proposition 3.23. $\qquad\square$

4. Nerves in categories enriched over $(\infty, 1)$-categories

We refer the reader to [Lur09a, Def. A.3.2.16] for the definition of an excellent monoidal model category. The following cases are relevant in this paper.

(0) Let $\mathcal{V} = \mathcal{C}at$ be the canonical model structure on the category $\mathcal{C}at$ of small categories (see e.g. [Rez96]), which is seen to be excellent using the fact that the ordinary nerve functor $\mathbf{N}: \mathcal{C}at \to s\mathcal{S}et_{(\infty,1)}$ creates weak equivalences and commutes with filtered colimits.

(1) Let $\mathcal{V} = s\mathcal{S}et_{(\infty,1)}$ be the Joyal model structure on the category $s\mathcal{S}et$ of simplicial sets from [Joy08b, Thm 6.12], which is excellent by [Lur09a, Ex. A.3.2.23].
(2) Let $\mathcal{V} = s\mathcal{S}et_{(\infty,1)}^{\Delta^{\mathrm{op}}}$ being the Rezk model structure from [Rez01, Thm 7.2] on the category $s\mathcal{S}et^{\Delta^{\mathrm{op}}}$ of simplicial spaces, which is discussed to be excellent in [BR13, Thm 3.11].
(3) Let $\mathcal{V} = s\mathcal{S}et_{(\infty,1)}^{+}$ be the Lurie model structure on the category $s\mathcal{S}et^{+}$ of marked simplicial sets from [Lur09a, Prop. 3.1.3.7], which is excellent by [Lur09a, Ex. A.3.2.22].

4.1. **The models.** All enriched models of $(\infty,2)$-categories will be a special case of the following.

Definition 4.1. Let $\mathcal{V}$ be an excellent monoidal model category. A *locally fibrant $\mathcal{V}$-category* is a $\mathcal{V}$-category $\mathcal{D}$ for which for any pair of objects c, d in $\mathcal{D}$ the hom-object $\mathcal{D}(c,d)$ is fibrant in $\mathcal{V}$.

Theorem 4.2 ([Lur09a, Thm A.3.2.24]). *Let $\mathcal{V}$ be an excellent monoidal model category. The category of small categories enriched over $\mathcal{V}$ admits a model structure in which*

- *the fibrant objects are the* locally fibrant $\mathcal{V}$-categories*, and*
- *the trivial fibrations are precisely the $\mathcal{V}$-functors that are surjective on objects, and locally a trivial fibration in $\mathcal{V}$.*

We denote this model structure by $\mathcal{C}at_{\mathcal{V}}$.

We specialize this construction to the following cartesian model categories.

(0) Let $\mathcal{V} = \mathcal{C}at$ be the canonical model structure. We then obtain precisely the model category $\mathcal{C}at_{\mathcal{C}at} = 2\mathcal{C}at$ from Theorem 1.1, as discussed in [BM13, Ex. 1.8], in which every object is fibrant.
(1) Let $\mathcal{V} = s\mathcal{S}et_{(\infty,1)}$ be the Joyal model structure. We then obtain the model category $\mathcal{C}at_{s\mathcal{S}et_{(\infty,1)}}$, in which the fibrant objects are the categories enriched over quasi-categories.
(2) Let $\mathcal{V} = s\mathcal{S}et_{(\infty,1)}^{\Delta^{\mathrm{op}}}$ being the Rezk model structure. We then obtain the model category $\mathcal{C}at_{s\mathcal{S}et_{(\infty,1)}^{\Delta^{\mathrm{op}}}}$, in which the fibrant objects are the categories enriched over complete Segal spaces.
(3) Let $\mathcal{V} = s\mathcal{S}et_{(\infty,1)}^{+}$ be the Lurie model structure on the category $s\mathcal{S}et^{+}$. We then obtain the model category $\mathcal{C}at_{s\mathcal{S}et_{(\infty,1)}^{+}}$, in which the fibrant objects are the categories enriched over naturally marked quasi-categories.

We recall from [Cru09, Thm 4.2.4] or [EK66] that any lax monoidal functor $F\colon \mathcal{V} \to \mathcal{V}'$ induces a base-change functor $F_*\colon \mathcal{C}at_{\mathcal{V}} \to \mathcal{C}at_{\mathcal{V}'}$. This is in particular the case when F is (strong) monoidal. For any $\mathcal{V}$-category $\mathcal{D}$, the $\mathcal{V}'$-category $F_*\mathcal{D}$ has the same set of objects as $\mathcal{D}$, and for any two objects c, d in $\mathcal{D}$ the hom-categories are defined by $(F_*\mathcal{D})(c,d) \coloneqq F(\mathcal{D}(c,d))$. If $F\colon \mathcal{V} \to \mathcal{V}'$ is a right adjoint functor with a monoidal left adjoint functor $L\colon \mathcal{V}' \to \mathcal{V}$, then L_* is the left adjoint of F_*.

Proposition 4.3. *Let $\mathcal{V}$, $\mathcal{V}'$ be excellent monoidal model categories, and $F\colon \mathcal{V} \to \mathcal{V}'$ a right adjoint functor whose left adjoint functor is monoidal. Denote by $F_*\colon \mathcal{C}at_{\mathcal{V}} \to \mathcal{C}at_{\mathcal{V}'}$ the induced base-change functor.*

(1) If F is a right Quillen functor, then F_* is a right Quillen functor.
(2) If F is a right Quillen embedding, then F_* is a right Quillen embedding.
(3) If F is a Quillen equivalence, then F_* is a Quillen equivalence.

Proof. Parts (1) and (3) are treated in [Lur09a, Rmk. A.3.2.6], while Part (2) can easily be verified as a variant of (3). ☐

As special cases, we obtain the following model comparison functors.

(a) The functor $(-)_{\bullet,0} \colon s\mathcal{S}et^{\Delta^{\mathrm{op}}} \to s\mathcal{S}et$ is shown to be a right Quillen equivalence in [JT07, §4] and its left adjoint is product-preserving because it is a right adjoint itself, as discussed e.g. in [JT07, §2]. We then obtain a right Quillen equivalence

$$((-)_{\bullet,0})_* \colon \mathcal{C}at_{s\mathcal{S}et^{\Delta^{\mathrm{op}}}_{(\infty,1)}} \to \mathcal{C}at_{s\mathcal{S}et_{(\infty,1)}}.$$

(b) The underlying simplicial set functor $U \colon s\mathcal{S}et^+ \to s\mathcal{S}et$ is a right Quillen equivalence by [Lur09a, Thm 3.1.5.1] and its left adjoint, given by the functor $(-)^\flat \colon s\mathcal{S}et \to s\mathcal{S}et^+$ which marks a simplicial set minimally, preserves finite products. We then obtain a right Quillen equivalence

$$U_* \colon \mathcal{C}at_{s\mathcal{S}et^+_{(\infty,1)}} \to \mathcal{C}at_{s\mathcal{S}et_{(\infty,1)}}.$$

4.2. The nerves.
The proposition can also be used to produce valuable nerve constructions.

Construction 4.4. All the following base-change functors are special instances of Proposition 4.3.

(1) The ordinary nerve functor $\mathbf{N} \colon \mathcal{C}at \to s\mathcal{S}et$ is a right Quillen embedding and its left adjoint functor preserves finite products by [Joy08b, Prop. B.0.15], there attributed to Gabriel–Zisman. We then obtain a right Quillen embedding

$$\mathbf{N}_* \colon 2\mathcal{C}at \to \mathcal{C}at_{s\mathcal{S}et_{(\infty,1)}}.$$

(2) The natural nerve functor [14] $\mathbf{N}^\natural \colon \mathcal{C}at \to s\mathcal{S}et^+$ from [GHL22, Formula (1.1)] is a right Quillen embedding by [GHL22, Lem. 1.9] and its left adjoint preserves finite products by [GHL22, §1.1]. We then obtain a right Quillen embedding

$$\mathbf{N}^\natural_* \colon 2\mathcal{C}at \to \mathcal{C}at_{s\mathcal{S}et^+_{(\infty,1)}}.$$

(3) The Rezk nerve functor[15] $\mathbf{N}^R \colon \mathcal{C}at \to s\mathcal{S}et^{\Delta^{\mathrm{op}}}$ from [Rez01, §3.5] and recalled in Appendix B is a right Quillen embedding by Proposition B.3 and we verify that its left adjoint preserves finite products in Lemma B.2. We then obtain a right Quillen embedding

$$\mathbf{N}^R_* \colon 2\mathcal{C}at \to \mathcal{C}at_{s\mathcal{S}et^{\Delta^{\mathrm{op}}}_{(\infty,1)}}.$$

The three nerve constructions are compatible with each other, as the next corollary shows.

Corollary 4.5. *The diagram of ∞-categories*

---

[14] In the original source, $\mathbf{N}^\natural \mathcal{D}$ is obtained as the value of a composite functor $\mathrm{N}^+ \iota \mathcal{D}$.

[15] In the original source, $\mathbf{N}^R \mathcal{C}$ is the *classifying diagram* of $\mathcal{C}$, denoted $N\mathcal{D}$.

commutes up to equivalence.

Proof. The corollary is an application of the "right Quillen" version of Lemma A.1 to the following diagram.

$$
\begin{array}{ccc}
 & 2\mathcal{C}at & \\
\mathbf{N}_*^R \swarrow & \downarrow \mathbf{N}_* & \searrow \mathbf{N}_*^{\natural} \\
\mathcal{C}at_{s\mathcal{S}et^{\Delta^{\mathrm{op}}}_{(\infty,1)}} \xrightarrow[((-)_{\bullet,0})_*]{} \mathcal{C}at_{s\mathcal{S}et_{(\infty,1)}} & \xleftarrow[U_*]{} & \mathcal{C}at_{s\mathcal{S}et^+_{(\infty,1)}}
\end{array}
$$

The fact that the diagram commutes up to isomorphism is a consequence of the fact that the diagram

$$
\begin{array}{ccc}
 & \mathcal{C}at & \\
\mathbf{N}^R \swarrow & \downarrow \mathbf{N} & \searrow \mathbf{N}^{\natural} \\
s\mathcal{S}et^{\Delta^{\mathrm{op}}}_{(\infty,1)} \xrightarrow[((-)_{\bullet,0})]{} s\mathcal{S}et_{(\infty,1)} & \xleftarrow[U]{} & s\mathcal{S}et^+_{(\infty,1)}
\end{array}
$$

commutes up to isomorphism. $\square$

4.3. **Nerve comparison.** Bergner–Rezk consider an enriched nerve functor in [BR13, Def. 7.3], obtained by regarding a bisimplicial category as a simplicial object in simplicial spaces, and show that it defines a right Quillen equivalence

$$(4.6) \qquad\qquad \mathfrak{R}\colon \mathcal{C}at_{s\mathcal{S}et^{\Delta^{\mathrm{op}}}_{(\infty,1)}} \to P\mathcal{C}at(s\mathcal{S}et^{\Delta^{\mathrm{op}}})_{(\infty,2)}.$$

If $\mathcal{Q}$ is a category enriched over simplicial spaces with object set $\mathcal{Q}_0$, and $\mathcal{Q}_1$ denotes the simplicial space

$$\mathcal{Q}_1 = \coprod_{a,b\in\mathcal{Q}_0} \mathcal{Q}(a,b),$$

by definition of $\mathfrak{R}$ (as given in [BR13, Def. 7.3]) there are isomorphisms of bisimplicial sets

$$(\mathfrak{R}\mathcal{Q})_0 \cong \mathcal{Q}_0 \text{ and } (\mathfrak{R}\mathcal{Q})_1 \cong \mathcal{Q}_1,$$

and for any $i \geq 0$

$$(4.7) \qquad\qquad (\mathfrak{R}\mathcal{Q})_i \cong \underbrace{\mathcal{Q}_1 \underset{\mathcal{Q}_0}{\times} \mathcal{Q}_1 \underset{\mathcal{Q}_0}{\times} \ldots \underset{\mathcal{Q}_0}{\times} \mathcal{Q}_1}_{i}.$$

First, we aim at giving an explicit description for $(\mathfrak{R}\mathbf{N}_*^R \mathcal{D})_{i,j,k}$, which we achieve in Proposition 4.9.

Given any category $\mathcal{A}$ and $i \geq 0$, we define inductively a 2-category $\Sigma_i\mathcal{A}$, called the $(i+1)$-*point suspension* of $\mathcal{A}$. We set $\Sigma_0\mathcal{A} \coloneqq [0]$, and for $i \geq 1$ the 2-category $\Sigma_i\mathcal{A}$ can be understood as the pushout of 2-categories

$$
\begin{array}{ccc}
[0] & \longrightarrow & \Sigma_{i-1}\mathcal{A} \\
\downarrow & & \downarrow \\
\Sigma\mathcal{A} & \longrightarrow & \Sigma_i\mathcal{A} \,.
\end{array}
$$

The construction extends to a functor $\Sigma_i\colon \mathcal{C}at \to 2\mathcal{C}at_{*,*}$.

Lemma 4.8. *Given a category $\mathcal{A}$ and $i \geq 1$ there is a pushout of 2-categories*

$$
\begin{array}{ccc}
\coprod_{i+1} \mathcal{A} & \longrightarrow & \coprod_{i+1} [0] \\
\downarrow & & \downarrow \\
\mathcal{A} \otimes [i] & \longrightarrow & \Sigma_i (\pi_0)_* \mathcal{A}
\end{array}
$$

Proof. The statement can be proven by induction on $i \geq 1$. The basis of the induction, namely the case $i = 1$, is precisely Lemma 3.28, and we now show the inductive step.

For $i > 1$, denote by $\mathcal{P}$ the following pushout.

$$
\begin{array}{ccc}
\coprod_{i+1} \mathcal{A} & \longrightarrow & \coprod_{i+1} [0] \\
\downarrow & & \downarrow \\
\mathcal{A} \otimes [i] & \longrightarrow & \mathcal{P}
\end{array}
$$

Consider the following commutative diagram of 2-categories.

$$
\begin{array}{ccccc}
\mathcal{A} \otimes [1] & \longleftarrow & \mathcal{A} \amalg \mathcal{A} & \longrightarrow & [0] \amalg [0] \\
\uparrow & & \uparrow & & \uparrow \\
\mathcal{A} & = & \mathcal{A} & \longrightarrow & [0] \\
\downarrow & & \downarrow & & \downarrow \\
\mathcal{A} \otimes [i-1] & \longleftarrow & \coprod_i \mathcal{A} & \longrightarrow & \coprod_i [0]
\end{array}
$$

The colimit of this diagram can be equivalently computed by either taking the colimit of the colimits of each row, or by taking the colimit of the colimits of each column. Following the first procedure, the resulting 2-category is the pushout of the span

$$
\mathcal{A} \otimes [i] \longleftarrow \coprod_{i+1} \mathcal{A} \longrightarrow \coprod_{i+1} [0]
$$

which gives precisely $\mathcal{P}$.

Instead, following the second procedure, the resulting 2-category is by induction hypothesis the pushout of the span

$$
\begin{array}{c}
\Sigma(\pi_0)_* \mathcal{A} \\
\uparrow \\
[0] \\
\downarrow \\
\Sigma_{i-1}(\pi_0)_* \mathcal{A}
\end{array}
$$

which is $\Sigma_i(\pi_0)_*\mathcal{A}$. So the desired isomorphism follows. $\qquad\square$

For $i, j, k \geq 0$, let $\Sigma_i([j] \times \widetilde{[k]})$ denote the $(i+1)$-point suspension of $[j] \times \widetilde{[k]}$, which is obtained by gluing i consecutive copies of $\Sigma([j] \times \widetilde{[k]})$.

Proposition 4.9. *For any 2-category $\mathcal{D}$ and $i, j, k \geq 0$ we have a natural bijection*

$$(\mathfrak{R}\mathbf{N}_*^R\mathcal{D})_{i,j,k} \cong 2\mathcal{C}at(\Sigma_i([j] \times \widetilde{[k]}), \mathcal{D}).$$

Proof. For any $i \geq 0$ we have a natural isomorphism of bisimplicial spaces

$$\left(\mathfrak{R}\mathbf{N}_*^R\mathcal{D}\right)_i \cong (\mathbf{N}_*^R\mathcal{D})_1 \underset{(\mathbf{N}_*^R\mathcal{D})_0}{\times} (\mathbf{N}_*^R\mathcal{D})_1 \underset{(\mathbf{N}_*^R\mathcal{D})_0}{\times} \cdots \underset{(\mathbf{N}_*^R\mathcal{D})_0}{\times} (\mathbf{N}_*^R\mathcal{D})_1$$

$$\cong (\mathbf{N}_*^R\mathcal{D})_1 \underset{\mathrm{Ob}\,\mathcal{D}}{\times} (\mathbf{N}_*^R\mathcal{D})_1 \underset{\mathrm{Ob}\,\mathcal{D}}{\times} \cdots \underset{\mathrm{Ob}\,\mathcal{D}}{\times} (\mathbf{N}_*^R\mathcal{D})_1$$

$$\cong \coprod_{d_0,\ldots,d_i \in \mathrm{Ob}\,\mathcal{D}} \mathbf{N}^R\mathcal{D}(d_0, d_1) \times \mathbf{N}^R\mathcal{D}(d_1, d_2) \times \ldots \times \mathbf{N}^R\mathcal{D}(d_{i-1}, d_i)$$

induced by the Segal maps. So for any $j, k \geq 0$ we get a natural bijection

$$\left(\mathfrak{R}\mathbf{N}_*^R\mathcal{D}\right)_{i,j,k} \cong \coprod_{d_0,\ldots,d_i \in \mathrm{Ob}\,\mathcal{D}} \mathbf{N}^R\mathcal{D}(d_0, d_1)_{j,k} \times \ldots \times \mathbf{N}^R\mathcal{D}(d_{i-1}, d_i)_{j,k}$$

$$\cong \coprod_{d_0,\ldots,d_i \in \mathrm{Ob}\,\mathcal{D}} \mathcal{C}at([j] \times \widetilde{[k]}, \mathcal{D}(d_0, d_1)) \times \ldots \times \mathcal{C}at([j] \times \widetilde{[k]}, \mathcal{D}(d_{i-1}, d_i))$$

$$\cong 2\mathcal{C}at(\Sigma_i([j] \times \widetilde{[k]}), \mathcal{D}),$$

as desired. $\qquad\square$

Next, we show the comparison between $\mathfrak{R}\mathbf{N}_*^R\mathcal{D}$ and $R\mathbf{N}^{\Delta \times \Delta}\mathcal{D}$.

Theorem 4.10. *For any 2-category $\mathcal{D}$ there is a natural map of bisimplicial spaces*

$$\mathfrak{R}\mathbf{N}_*^R\mathcal{D} \to R\mathbf{N}^{\Delta \times \Delta}\mathcal{D}$$

that is a weak equivalence in $s\mathcal{S}et^{(\Delta \times \Delta)^{\mathrm{op}}}_{(\infty,2)}$ and in $P\mathcal{C}at(s\mathcal{S}et^{\Delta^{\mathrm{op}}})_{(\infty,2)}$.

First, we give a more general version of Lemma 3.29.

Lemma 4.11. *For any $i, j, k \geq 0$ there is a pushout of 2-categories*

$$
\begin{array}{ccc}
\coprod\limits_{i+1} \mathcal{O}_2^\sim[j] \otimes_{\mathrm{ic}} \widetilde{\mathcal{O}_2[k]} & \longrightarrow & \coprod\limits_{i+1} [0] \\
\downarrow & & \downarrow \\
\mathcal{O}_2^\sim[j] \otimes_{\mathrm{ic}} ([i] \otimes_{\mathrm{ps}} \widetilde{\mathcal{O}_2[k]}) & \longrightarrow & \Sigma_i([j] \times \widetilde{[k]})\,.
\end{array}
$$

Proof. The proof is similar to Lemma 3.29 replacing $[1]$ with $[i]$ and using Lemma 4.8. $\qquad\square$

We can now prove the theorem.

Proof of Theorem 4.10. We first build the desired map. To this end, consider the following map of spans.

$$\begin{array}{ccccc}
\mathcal{O}_2^{\sim}[j] \otimes_{\mathrm{ic}} (\mathcal{O}_2^{\sim}[i] \otimes_{\mathrm{ps}} \widetilde{\mathcal{O}_2[k]}) & \longleftarrow & \coprod_{i+1} \mathcal{O}_2^{\sim}[j] \otimes_{\mathrm{ic}} \widetilde{\mathcal{O}_2[k]} & \longrightarrow & \coprod_{i+1} [0] \\
\downarrow & & \| & & \| \\
\mathcal{O}_2^{\sim}[j] \otimes_{\mathrm{ic}} ([i] \otimes_{\mathrm{ps}} \widetilde{\mathcal{O}_2[k]}) & \longleftarrow & \coprod_{i+1} \mathcal{O}_2^{\sim}[j] \otimes_{\mathrm{ic}} \widetilde{\mathcal{O}_2[k]} & \longrightarrow & \coprod_{i+1} [0]
\end{array}$$

By Proposition 3.13, the top row is given by the span

$$L\mathbb{C}\Delta[i, j, k] \longleftarrow \coprod_{i+1} L\mathbb{C}\Delta[0, j, k] \longrightarrow \coprod_{i+1} L\mathbb{C}\Delta[0, 0, 0]$$

and using Remark 3.20 and the fact that $L\mathbb{C}$ commutes with colimits, its pushout is precisely $L\mathbb{C}\overline{\Delta}[i, j, k]$. By Lemma 4.11 the pushout of the bottom row is $\Sigma_i([j] \times \widetilde{[k]})$. Hence the map of spans yields the unique induced map of pushouts

$$L\mathbb{C}\overline{\Delta}[i, j, k] \to \Sigma_i([j] \times \widetilde{[k]}).$$

Composing with the map in Proposition 3.5, we get a map

$$L^{\simeq}\mathbb{C}\overline{\Delta}[i, j, k] \to L\mathbb{C}\overline{\Delta}[i, j, k] \to \Sigma_i([j] \times \widetilde{[k]})$$

which induces by Propositions 3.23 and 4.9 a map of sets

$$(I\mathfrak{R}\mathbf{N}_*^R\mathcal{D})_{i,j,k} = (\mathfrak{R}\mathbf{N}_*^R\mathcal{D})_{i,j,k} \to \mathbf{N}_{i,j,k}^{\Delta \times \Delta}\mathcal{D}$$

which induces a map of bisimplicial spaces

$$I\mathfrak{R}\mathbf{N}_*^R\mathcal{D} \to \mathbf{N}^{\Delta \times \Delta}\mathcal{D}$$

which induces a map in $P\mathcal{C}at(s\mathcal{S}et^{\Delta^{\mathrm{op}}})$

$$\mathfrak{R}\mathbf{N}_*^R\mathcal{D} \to R\mathbf{N}^{\Delta \times \Delta}\mathcal{D}$$

as desired.

We now argue this map is a levelwise weak equivalence. By Propositions 3.27, 3.30 and 4.9, it induces isomorphisms in $s\mathcal{S}et$ for $i = 0, 1$ and $j \geq 0$

$$(\mathbf{N}_*^R\mathcal{D})_{0,j} = (\mathfrak{R}\mathbf{N}_*^R\mathcal{D})_{0,j} \xrightarrow{\cong} (R\mathbf{N}^{\Delta \times \Delta}\mathcal{D})_{0,j}, \quad (\mathbf{N}_*^R\mathcal{D})_{0,1} = (\mathfrak{R}\mathbf{N}_*^R\mathcal{D})_{1,j} \xrightarrow{\cong} (R\mathbf{N}^{\Delta \times \Delta}\mathcal{D})_{1,j}.$$

Using the fact that $R\mathbf{N}^{\Delta \times \Delta}\mathcal{D}$ and $\mathfrak{R}\mathbf{N}_*^R\mathcal{D}$ are Segal objects by Remark 3.26 and (4.7), it follows that for any $i, j \geq 0$ it induces a weak equivalence in $s\mathcal{S}et_{(\infty,0)}$

$$(\mathfrak{R}\mathbf{N}_*^R\mathcal{D})_{i,j} \to (R\mathbf{N}^{\Delta \times \Delta}\mathcal{D})_{i,j},$$

showing that the desired map is a weak equivalence in $s\mathcal{S}et_{(\infty,2)}^{(\Delta \times \Delta)^{\mathrm{op}}}$.

Finally, the fact that the desired map is a weak equivalence in $P\mathcal{C}at(s\mathcal{S}et^{\Delta^{\mathrm{op}}})_{(\infty,2)}$ is a consequence of Lemma 3.18. $\qquad\square$

Finally, we compare the nerves from Construction 4.4.

Corollary 4.12. *The diagram of ∞-categories*

$$[2\mathcal{C}at]_\infty$$

$$[\mathbf{N}^{\Delta\times\Delta}]_\infty \qquad [R\mathbf{N}^{\Delta\times\Delta}]_\infty \qquad [\mathbf{N}_*^R]_\infty$$

$$[s\mathcal{S}et^{(\Delta\times\Delta)^{\mathrm{op}}}_{(\infty,2)}]_\infty \xrightarrow[{[R]_\infty}]{} [P\mathcal{C}at(s\mathcal{S}et^{\Delta^{\mathrm{op}}})_{(\infty,2)}]_\infty \xleftarrow[{[\mathfrak{R}]_\infty}]{} [\mathcal{C}at_{s\mathcal{S}et^{\Delta^{\mathrm{op}}}_{(\infty,1)}}]_\infty$$

commutes up to equivalence.

Proof. The corollary follows from applying twice the "right Quillen" version of Lemma A.1 to the following diagram.

$$2\mathcal{C}at$$

$$\mathbf{N}^{\Delta\times\Delta} \qquad R\mathbf{N}^{\Delta\times\Delta} \qquad \mathbf{N}_*^R$$

$$s\mathcal{S}et^{(\Delta\times\Delta)^{\mathrm{op}}}_{(\infty,2)} \xrightarrow[R]{} P\mathcal{C}at(s\mathcal{S}et^{\Delta^{\mathrm{op}}})_{(\infty,2)} \xleftarrow[\mathfrak{R}]{} \mathcal{C}at_{s\mathcal{S}et^{\Delta^{\mathrm{op}}}_{(\infty,1)}}$$

The fact that all the assumptions of the lemma are met are from Theorems 3.7 and 4.10 and Construction 4.4. $\square$

5. NERVES IN SIMPLICIAL MODELS

5.1. **The models.** Verity envisioned a model of $(\infty, 2)$-categories (part of a family of (∞, n)-categories for general n) based on simplicial sets endowed with a subset of distinguished simplices.

Definition 5.1. A *simplicial set with marking*[16] is a simplicial set with a set of distinguished simplices – called *marked* – in positive dimension and containing degenerate simplices.

Amongst all simplicial sets with marking, the following identify those that are $(\infty, 2)$-categories. The following mathematical object was identified by Verity [Ver17] as a model for $(\infty, 2)$-categories, and was further studied in [Rie18, §3.3], [OR20b, §1.3] and [RV22, App. D].

Definition 5.2. A *saturated 2-complicial set*[17] is a simplicial set that has the right lifting property with respect to all maps of the following kinds:

(1) for $m > 1$ and $0 < k < m$, the *complicial inner horn extension*

$$\Lambda^k[m] \to \Delta^k[m];$$

   here, $\Delta^k[m]$ is the standard m-simplex in which a non-degenerate simplex is marked if and only if it contains the vertices $\{k-1, k, k+1\} \cap [m]$, and $\Lambda^k[m]$ is the regular sub-simplicial set with marking of $\Delta^k[m]$ whose simplicial set is the k-horn $\Lambda^k[m]$;

(2) for $m \geq 2$ and $0 < k < m$, the *complicial thinness extension*

$$\Delta^k[m]' \to \Delta^k[m]'';$$

   here, $\Delta^k[m]'$ is the standard m-simplex with marking obtained from $\Delta^k[m]$ by additionally marking the $(k-1)$-st and $(k+1)$-st face of $\Delta[m]$, and $\Delta^k[m]''$ is the standard

---

[16]Originally referred to as *stratified simplicial set* e.g. in [Ver08a, Def. 96], *simplicial sets with normality* [Str82] and *hollow simplicial sets* [Str87].

[17]Sometimes for brevity referred to as *2-complicial set*.

m-simplex with marking obtained from $\Delta^k[m]'$ by additionally marking the k-th face of $\Delta[m]$;

(3) for $m > 2$, the *triviality extension*

$$\Delta[m] \to \Delta[m]_t;$$

here, $\Delta[m]$ is the minimally marked m-simplex, and $\Delta[m]_t$ is the thin m-simplex in which the only non-degenerate simplex marked is the unique m-simplex;

(4) for $m \geq -1$, the *complicial saturation extension*

$$\Delta[3]_{\mathrm{eq}} \star \Delta[m] \to \Delta[3]_\sharp \star \Delta[m];$$

here, $\Delta[3]_{\mathrm{eq}}$ is the standard 3-simplex with marking given by all simplices in dimension at least 2, as well as the 1-simplices $[0, 2]$ and $[1, 3]$, and $\Delta[3]_\sharp$ is the standard 3-simplex with the maximal marking.

See e.g. [OR20b, Def. 1.19] for more details. We refer the reader to [Ver08a] for the join $\star \colon ms\mathcal{S}et \times ms\mathcal{S}et \to ms\mathcal{S}et$ of marked simplicial sets.

The following model structure is obtained as an application of Verity's machinery from [Ver08b, Thm 100], and was further studied in [Rie18, §4.3], and [OR20b, Thm 1.25].

Theorem 5.3. *The category $ms\mathcal{S}et$ of simplicial sets with marking admits a model structure, denoted $ms\mathcal{S}et_{(\infty,2)}$, in which*

- *the fibrant objects are the saturated 2-complicial sets, and*
- *the cofibrations are the monomorphisms (of underlying simplicial sets), and in particular every object is cofibrant.*

Lurie proposed a simplified variant of this idea that focuses on the study of $(\infty, 2)$-categories (as opposed to (∞, n)-categories for general n), based on simplicial sets with marking only in dimension 2.

Definition 5.4 ([Lur09b, Def. 3.1.1])**.** A *scaled simplicial set* is a simplicial set with a *scaling*, namely a set of distinguished 2-simplices – called *marked* or *thin* – containing degenerate 2-simplices.

Amongst all scaled simplicial sets, the following identify those that are $(\infty, 2)$-categories. To recall this definition, we use the author's original convention that we denote a simplicial set with marking by listing a pair (X, T) where X is the underlying simplicial set, and T is the set of non-degenerate scaled simplices.

Definition 5.5 ([Lur09b, Def. 4.1.1])**.** An ∞-*bicategory*[18] is a simplicial set that has the right lifting property with respect to all maps indicated in [Lur09b, Def. 3.1.3], namely

(1) for $m \geq 2$ and $0 < k < m$ the *scaled inner horn extension*

$$(\Lambda^k[m], \{[k-1, k, k+1]\}) \to (\Delta[m], \{[k-1, k, k+1]\});$$

(2) for $n \geq 3$ the *scaled outer horn extension*

$$(\Lambda^0[m] \underset{\Delta[1]}{\amalg} \Delta[0], \{[0, 1, n]\}) \to (\Delta[m] \underset{\Delta[1]}{\amalg} \Delta[0], \{[0, 1, n]\}),$$

where the pushouts are induced by the map $\langle 0, 1 \rangle \colon \Delta[1] \to \Delta[m]$;

---

[18]This was originally referred to as a *weak ∞-bicategory*, but was shown by Gagna–Harpaz–Lanari in [GHL22, Thm 5.1] to agree with the original definition of ∞-*bicategory* from [Lur09b, Def. 4.2.8]

(3) the *scaled saturation extension*

$$(\Delta[4], T) \to (\Delta[4], \{T \cup \{[0,3,4], [0,1,4]\}\}),$$

where $T = \{[0,2,4], [1,2,3], [0,1,3], [1,3,4], [0,1,2]\}$.

The following model structure is obtained as an application of Smith Theorem.

Theorem 5.6 ([Lur09b, Thm 4.2.7]). *The category $sSet^{sc}$ of scaled simplicial sets admits a model structure, denoted $sSet^{sc}_{(\infty,2)}$, in which*

- *the fibrant objects are the ∞-bicategories, and*
- *the cofibrations are the monomorphisms (of underlying simplicial sets), and in particular every object is cofibrant.*

Gagna–Harpaz–Lanari prove in [GHL22, Thm 7.9] that the canonical forgetful functor defines a right Quillen equivalence

$$U \colon msSet_{(\infty,2)} \to sSet^{sc}_{(\infty,2)},$$

A further variant of Verity's original framework is given by working with $t\Delta$-*sets*, where $t\Delta$ is an enlargement of the ordinary simplex category Δ. More precisely, the category $t\Delta$ contains Δ as a non-full subcategory, and in addition to the objects $[n]$ for $n \geq 0$ it also contains objects of the form $[n]_t$ together with a map $[n] \to [n]_t$ for each $n \geq 1$. We refer the reader to [OR20b, Not. 1.1] or [RV22, Not. D.1.4] for more details on the category $t\Delta$.

Any $t\Delta$-set $X \colon t\Delta^{\mathrm{op}} \to Set$ can be seen as a *simplicial set with multiple marking*. The underlying simplicial set of X is the restriction of X along the inclusion $\Delta^{\mathrm{op}} \to t\Delta^{\mathrm{op}}$, so $X([n]) = X_n$ is the set of n-simplices, while $X([n]_t)$ is the set of marked n-simplices; by definition, there is a structure map $X([n]_t) \to X([n]) = X_n$ for every $n \geq 1$, that remembers which simplex each marking belongs to. Notice that an n-simplex can be marked multiple times, namely, multiple elements of $X([n]_t)$ can map to the same element in X_n. According to this interpretation, simplicial sets with marking are precisely the $t\Delta$-sets for which all structure maps $X([n]_t) \to X([n]) = X_n$ are monomorphisms[19], and there is an inclusion $msSet \hookrightarrow Set^{t\Delta^{\mathrm{op}}}$.

Definition 5.7 ([OR20b, Def. 1.23]). A *2-precomplicial set*[20] is a $t\Delta$-set that has the right lifting property with respect to the maps of the kinds (1)-(4) from Definition 5.2.

The following model structure is an application of Cisinski's machinery from [Cis06, §1.3].

Theorem 5.8 ([OR20b, Thm 1.28]). *The category $Set^{t\Delta^{\mathrm{op}}}$ of simplicial sets with multiple marking admits a model structure, denoted $Set^{t\Delta^{\mathrm{op}}}_{(\infty,2)}$, in which*

- *the fibrant objects are the saturated 2-precomplicial sets, and*
- *the cofibrations are the monomorphisms (of underlying simplicial sets), and in particular every object is cofibrant.*

The inclusion $msSet \hookrightarrow Set^{t\Delta^{\mathrm{op}}}$ admits a left adjoint Refl, which was proven by the second and third author as [OR20b, Prop. 1.31] to be a left Quillen equivalence

$$(5.9) \qquad \qquad \mathrm{Refl} \colon Set^{t\Delta^{\mathrm{op}}}_{(\infty,2)} \to msSet.$$

---

[19]This approach looks more complicated at first glance, but offers certain technical advantages because, unlike the category of simplicial sets with marking, the category of $t\Delta$-sets is a category of presheaves.

[20]We warn the reader that the same terminology is also used in [Ver08a, §6] to mean something unrelated.

Given a $t\Delta$-set X, the functor Refl preserves the underlying simplicial set, so that we have $(\mathrm{Refl}X)_n = X_n = X([n])$, and the set of marked n-simplices $(\mathrm{Refl}X)([n]_t)$ is determined by the epi-mono factorization of the structure map

$$X([n]_t) \twoheadrightarrow (\mathrm{Refl}X)([n]_t) \hookrightarrow X([n]).$$

This means that an n-simplex is marked in $\mathrm{Refl}X$ if and only if it has at least one marking in X.

5.2. **The nerves.** Nerve constructions have been identified for the three discussed simplicial models of $(\infty, 2)$-categories, and they are all based on the same underlying simplicial set: the Duskin nerve[21] $\mathbf{N}^D\mathcal{D}$ of a 2-category $\mathcal{D}$ from [Dus02, §6].

The *Duskin nerve* $\mathbf{N}^D\mathcal{D}$ of a 2-category $\mathcal{D}$ is the (3-coskeletal) simplicial set in which the set of n-simplices is given by

$$(\mathbf{N}^D\mathcal{D})_n := 2\mathcal{C}at(\mathcal{O}_2[n], \mathcal{D}).$$

The assignment extends to a functor $\mathbf{N}^D \colon 2\mathcal{C}at \to s\mathcal{S}et$. In particular,

(0) a 0-simplex consists of an object x of $\mathcal{D}$;
(1) a 1-simplex consists of a 1-morphism $a \colon x \to y$ of $\mathcal{D}$;
(2) a 2-simplex consists of a 2-cell $\varphi \colon c \Rightarrow b \circ a$ of $\mathcal{D}$ of the form

(3) a 3-simplex consists of four 2-cells of $\mathcal{D}$ that satisfy the following pasting equality.

The face maps can be read off from the pictures.

Construction 5.10 ([OR21, Const. 4.8]). Let $\mathcal{D}$ be a 2-category. The nerve $\mathbf{N}^{t\Delta}\mathcal{D}$ is the simplicial set $\mathbf{N}^D\mathcal{D}$ with marking given by the following:

(1) all 1-simplices inhabited by equivalences, each marked as many times as ways of completing the equivalence to an adjoint equivalence;
(2) all 2-simplices inhabited by isomorphisms, each marked uniquely;
(3) all simplices in dimension higher than 2, each marked uniquely.

This assignment extends to a functor $\mathbf{N}^{t\Delta} \colon 2\mathcal{C}at \to \mathcal{S}et^{t\Delta^{\mathrm{op}}}$.

Remark 5.11. Given $\mathcal{D}$ a 2-category, $\mathrm{Refl}\,\mathbf{N}^{t\Delta}\mathcal{D}$ is the simplicial set $\mathbf{N}^D\mathcal{D}$ endowed with the marking described in [Rie18, Prop. 3.1.10]. Essentially, the difference between $\mathbf{N}^{t\Delta}\mathcal{D}$ and $\mathrm{Refl}\,\mathbf{N}^{t\Delta}\mathcal{D}$ is that in the former each 1-equivalence is marked many times, while in the latter it is marked only once (without remembering the data of any specific adjoint equivalence).[22]

---

[21]In the original source, $\mathbf{N}^D\mathcal{D}$ is denoted $\mathbf{Ner}\mathcal{D}$.

[22]Another marking on $\mathbf{N}^{t\Delta}\mathcal{D}$ considered in the literature is the Roberts–Street nerve from e.g. [Ver08a], for which the marked simplices are those inhabited by an identity cell. This nerve has important properties, but is not homotopically well-behaved, and does not play a role in this paper.

Construction 5.12 ([GHL22, Def. 8.1])**.** Let $\mathcal{D}$ be a 2-category. The nerve[23] $\mathbf{N}^{sc}\mathcal{D}$ is the simplicial set $\mathbf{N}^{D}\mathcal{D}$ with scaling given by the set of all 2-simplices inhabited by isomorphisms. The assignment extends to a functor $\mathbf{N}^{sc}\colon 2\mathcal{C}at \to s\mathcal{S}et^{sc}$.

These nerve constructions are well behaved homotopically.

Theorem 5.13 ([OR21, Thms 4.10,4.12])**.** *The functor* $\mathbf{N}^{t\Delta}\colon 2\mathcal{C}at \to s\mathcal{S}et^{t\Delta^{\mathrm{op}}}_{(\infty,2)}$ *is a right Quillen embedding, and in particular a homotopical and right Quillen functor.*

Theorem 5.14 ([GHL22, Prop. 8.2, 8.3])**.** *The functor* $\mathbf{N}^{sc}\colon 2\mathcal{C}at \to s\mathcal{S}et^{sc}_{(\infty,2)}$ *is a right Quillen embedding, and in particular a homotopical and right Quillen functor.*

Remark 5.15. The functor $\mathrm{Refl}\,\mathbf{N}^{t\Delta}\colon 2\mathcal{C}at \to ms\mathcal{S}et$ is *not* a right adjoint functor. Indeed, if it admitted a left adjoint $L\colon ms\mathcal{S}et \to 2\mathcal{C}at$, then we would have a natural bijection for any 2-category $\mathcal{D}$

$$2\mathcal{C}at(L\Delta[1]_t, \mathcal{D}) \cong ms\mathcal{S}et(\Delta[1]_t, \mathrm{Refl}\,\mathbf{N}^{t\Delta}\mathcal{D}) \cong (\mathrm{Refl}\,\mathbf{N}^{t\Delta}\mathcal{D})([1]_t) \cong \mathrm{eq}\mathcal{D},$$

where $\mathrm{eq}\mathcal{D}$ denotes the set of equivalences in $\mathcal{D}$. However, one can use e.g. [Rie17, Prop. 2.4.8] to see that the functor $\mathrm{eq}\colon 2\mathcal{C}at \to \mathcal{S}et$ given by $\mathcal{D} \mapsto \mathrm{eq}\mathcal{D}$ is not corepresentable, obtaining a contradiction.

Proposition 5.16. *The functor* $\mathrm{Refl}\,\mathbf{N}^{t\Delta}\colon 2\mathcal{C}at \to ms\mathcal{S}et_{(\infty,2)}$ *is homotopical and induces a fully faithful functor at the level of ∞-categories.*

Proof. The functor $\mathrm{Refl}\,\mathbf{N}^{t\Delta}\colon 2\mathcal{C}at \to ms\mathcal{S}et_{(\infty,2)}$ is the composite of the right Quillen functor $\mathbf{N}^{t\Delta}\colon 2\mathcal{C}at \to \mathcal{S}et^{t\Delta^{\mathrm{op}}}_{(\infty,2)}$ from Theorem 5.13, followed by the left Quillen functor $\mathrm{Refl}\colon \mathcal{S}et^{t\Delta^{\mathrm{op}}}_{(\infty,2)} \to ms\mathcal{S}et_{(\infty,2)}$ from (5.9), which are both in particular homotopical and homotopically fully faithful. Hence, $\mathrm{Refl}\,\mathbf{N}^{t\Delta}$ is homotopical and homotopically fully faithful. $\square$

5.3. **Nerve comparisons.** The nerve constructions are compatible with each other as follows.

Proposition 5.17. *For any 2-category $\mathcal{D}$ there is an isomorphism of scaled simplicial sets*

$$\mathbf{N}^{sc}\mathcal{D} \cong U\mathrm{Refl}\,\mathbf{N}^{t\Delta}\mathcal{D}.$$

Proof. The two scaled simplicial sets $\mathbf{N}^{sc}\mathcal{D}$ and $U\mathrm{Refl}\,\mathbf{N}^{t\Delta}\mathcal{D}$ have the same underlying simplicial set, given by the Duskin nerve $\mathbf{N}^{D}\mathcal{D}$, and by reading through the relevant definitions and the explicit description of the reflector one can see that the marked 2-simplices are precisely those inhabited by a 2-isomorphism of $\mathcal{D}$. $\square$

Corollary 5.18. *The diagram of ∞-categories*

$$
\begin{array}{ccc}
& [2\mathcal{C}at]_\infty & \\
{\scriptstyle [\mathbf{N}^{t\Delta}]_\infty} \swarrow & {\scriptstyle \downarrow [\mathrm{Refl}\,\mathbf{N}^{t\Delta}]_\infty} & \searrow {\scriptstyle [\mathbf{N}^{sc}]_\infty} \\
[\mathcal{S}et^{t\Delta^{\mathrm{op}}}_{(\infty,2)}]_\infty \xrightarrow[{[\mathrm{Refl}]_\infty}]{} & [ms\mathcal{S}et_{(\infty,2)}]_\infty \xrightarrow[{[U]_\infty}]{} & [s\mathcal{S}et^{sc}_{(\infty,2)}]_\infty
\end{array}
$$

commutes up to equivalence.

---

[23]In the original source, $\mathbf{N}^{sc}\mathcal{D}$ is denoted $\mathcal{N}_2\mathcal{D}$.

Proof. The commutativity of the left triangle is an application of the "left Quillen" version of Lemma A.1 to the diagram

$$
\begin{array}{ccc}
 & & 2\mathcal{C}at \\
 & \overset{\mathbf{N}^{t\Delta}}{\swarrow} & \downarrow \text{Refl}\,\mathbf{N}^{t\Delta} \\
\mathcal{S}et^{t\Delta^{\mathrm{op}}}_{(\infty,2)} & \xrightarrow[\text{Refl}]{} & ms\mathcal{S}et_{(\infty,2)}
\end{array}
$$

where the assumptions of the lemma are met by Theorem 5.13 and Proposition 5.16. Then, the commutativity of the right triangle is an application of the "right Quillen" version of Lemma A.1 to the diagram

$$
\begin{array}{ccc}
2\mathcal{C}at & & \\
\text{Refl}\,\mathbf{N}^{t\Delta} \downarrow & \overset{\mathbf{N}^{sc}}{\searrow} & \\
ms\mathcal{S}et_{(\infty,2)} & \xrightarrow[U]{} & s\mathcal{S}et^{sc}_{(\infty,2)}
\end{array}
$$

where the assumptions of the lemma are met by Theorem 5.14 and Propositions 5.16 and 5.17. $\qquad\square$

We now discuss how the nerve constructions of simplicial models compare with those from the enriched models. Lurie showed as [Lur09b, Thm 0.0.3] that the *scaled homotopy coherent nerve*[24] functor introduced as [Lur09b, Def. 3.1.10] defines a right Quillen equivalence

$$
\mathfrak{N}^{sc} \colon \mathcal{C}at_{s\mathcal{S}et^{+}{}_{(\infty,1)}} \to s\mathcal{S}et^{sc}_{(\infty,2)}.
$$

Proposition 5.19 ([GHL22, Prop. 8.2]). *For any 2-category $\mathcal{D}$ there is an isomorphism of scaled simplicial sets*

$$
\mathfrak{N}^{sc}\mathbf{N}^{\natural}_{*}\mathcal{D} \cong \mathbf{N}^{sc}\mathcal{D}.
$$

Corollary 5.20. *The diagram of ∞-categories*

$$
\begin{array}{ccc}
 & [2\mathcal{C}at]_{\infty} & \\
\overset{[\mathbf{N}^{\natural}_{*}]_{\infty}}{\swarrow} & & \overset{[\mathbf{N}^{sc}]_{\infty}}{\searrow} \\
[\mathcal{C}at_{s\mathcal{S}et^{+}_{(\infty,1)}}]_{\infty} & \xrightarrow[{[\mathfrak{N}^{sc}]_{\infty}}]{} & [s\mathcal{S}et^{sc}_{(\infty,2)}]_{\infty}
\end{array}
$$

commutes up to equivalence.

Proof. The corollary is an application of the "right Quillen" version of Lemma A.1 to the following diagram.

$$
\begin{array}{ccc}
 & 2\mathcal{C}at & \\
\overset{\mathbf{N}^{\natural}_{*}}{\swarrow} & & \overset{\mathbf{N}^{sc}}{\searrow} \\
\mathcal{C}at_{s\mathcal{S}et^{+}_{(\infty,1)}} & \xrightarrow[{\mathfrak{N}^{sc}}]{} & s\mathcal{S}et^{sc}_{(\infty,2)}
\end{array}
$$

The fact that all the assumptions of the lemma are met are from Construction 4.4, Proposition 5.19, and Theorem 5.14. $\qquad\square$

---

[24]In the original source, $\mathfrak{N}^{sc}\mathcal{D}$ is denoted $N^{sc}\mathcal{D}$.

6. Nerves of 2-categories as local $(\infty, 2)$-categories

The goal of this subsection is to prove Theorem 1.12, which will be completed in Section 6.3. The ingredients for the proof are Remarks 6.25 and 6.37 and Theorem 6.36. We also use some of their 0-dimensional analogs – Proposition 6.4 and Remark 6.6 – and 1-dimensional analogs – Theorem 6.18 and Remark 6.19 – which are treated in Sections 6.1 and 6.2, respectively.

6.1. **The 0-dimensional case.** The goal of this subsection is to show that the Quillen pair

$$\pi_0 \colon s\mathcal{S}et_{(\infty,0)} \rightleftarrows \mathcal{S}et \colon \mathrm{disc}$$

is equivalent to the left Bousfield localization of the Kan–Quillen model structure $s\mathcal{S}et_{(\infty,0)}$ with respect to a set Λ of maps. We also discuss in Remark 6.6 that this entails that the discrete embedding realizes sets as local $(\infty, 0)$-categories with respect to the set of maps Λ.

Recall from e.g. [AC22] that there is a canonical model structure on $\mathcal{S}et$ in which the weak equivalences are the bijections, and every object is fibrant and cofibrant. Recall from [Qui67] that the category $s\mathcal{S}et$ of simplicial sets admits the Kan–Quillen model structure $s\mathcal{S}et_{(\infty,0)}$, in which the weak equivalences are the weak homotopy equivalences, everything is cofibrant and the fibrant objects are precisely the Kan complexes.

The functor $\mathrm{disc} \colon \mathcal{S}et \to s\mathcal{S}et$ that regards each set as a discrete simplicial set admits a left adjoint given by the functor $\pi_0 \colon s\mathcal{S}et \to \mathcal{S}et$ that takes a simplicial set to its set of connected components. The following is a straightforward verification.

Proposition 6.1. *The functor* $\mathrm{disc} \colon \mathcal{S}et \to s\mathcal{S}et_{(\infty,0)}$ *is a right Quillen embedding.*

In particular, we have a Quillen reflection pair

$$\pi_0 \colon s\mathcal{S}et_{(\infty,0)} \rightleftarrows \mathcal{S}et \colon \mathrm{disc}.$$

Remark 6.2. The essential image of the functor $[\mathrm{disc}]_\infty \colon [\mathcal{S}et]_\infty \to [s\mathcal{S}et_{(\infty,0)}]_\infty$ is the full sub-∞-category of $[s\mathcal{S}et_{(\infty,0)}]_\infty$ generated by the homotopically discrete $(\infty, 0)$-categories.

For $k > 0$, let $S^k \coloneqq \partial\Delta[k]$ denote the simplicial k-sphere. Since the model structure $s\mathcal{S}et_{(\infty,0)}$ is combinatorial and left proper, the following model structure exists.

Proposition 6.3. *The category $s\mathcal{S}et$ admits the left Bousfield localization $\mathcal{L}_\Lambda s\mathcal{S}et_{(\infty,0)}$ of the model structure $s\mathcal{S}et_{(\infty,0)}$ with respect to the set Λ of maps of the form*

$$\Delta[0] \hookrightarrow S^k, \quad \textit{for } k > 0.$$

In particular, there is a Quillen reflection pair

$$\mathrm{Id} \colon s\mathcal{S}et_{(\infty,0)} \rightleftarrows \mathcal{L}_\Lambda s\mathcal{S}et_{(\infty,0)} \colon \mathrm{Id}.$$

The following is a straightforward verification.

Proposition 6.4. *The functor* $\mathrm{disc} \colon \mathcal{S}et \to \mathcal{L}_\Lambda s\mathcal{S}et_{(\infty,0)}$ *defines a right Quillen equivalence.*

The following relates two approaches to localizations of ∞-categories and is classical, but it is described e.g. in the proof of [Lur09a, Prop. A.3.7.8].

We refer the reader to [Lur09a, Def. 5.2.7.2, Prop. 5.5.4.15] for a discussion on the localization $\mathcal{L}_S \mathcal{Q}$ of a quasi-category $\mathcal{Q}$ with respect to a set of edges S, and to [Hir03, Ch. 3] for the left Bousfield localization $\mathcal{L}_S \mathcal{M}$ of a model category $\mathcal{M}$ with respect to a set of morphisms S, namely the localization in the context of model categories.

Proposition 6.5. *Given a combinatorial left proper model category $\mathcal{M}$ and a set of maps S, denote by $\mathcal{L}_S\mathcal{M}$ the left Bousfield localization and by $\mathscr{L}_S[\mathcal{M}]_\infty$ the localization in the sense of ∞-categories. Then there is a diagram of ∞-categories*

$$[\mathcal{L}_S\mathcal{M}]_\infty \xrightarrow{\simeq} \mathscr{L}_S[\mathcal{M}]_\infty$$
$$[\mathrm{Id}_\mathcal{M}]_\infty \searrow \qquad \swarrow$$
$$[\mathcal{M}]_\infty$$

that commutes up to equivalence.

With the following remark we verify that the map of ∞-categories induced by Proposition 6.4 does implement the inclusion of the ∞-category of sets into the ∞-category of spaces considered by Gepner–Haugseng in [GH15, §6].

Remark 6.6. We know – and it is also mentioned in [GH15, §6] – that the underlying ∞-category of the Kan–Quillen model structure $s\mathcal{S}et_{(\infty,0)}$ models the established ∞-category $\mathscr{S}$ of spaces, meaning there exists an equivalence of ∞-categories

$$(6.7) \qquad\qquad\qquad\qquad [s\mathcal{S}et_{(\infty,0)}]_\infty \simeq \mathscr{S}.$$

Any such equivalence can be used to construct a specific equivalence of ∞-categories

$$(6.8)\qquad \begin{aligned} [\mathcal{S}et]_\infty &\simeq [\mathcal{L}_\Lambda s\mathcal{S}et_{(\infty,0)}]_\infty & &\text{Proposition 6.4}\\ &\simeq \mathscr{L}_\Lambda[s\mathcal{S}et_{(\infty,0)}]_\infty & &\text{Proposition 6.5}\\ &\simeq \mathscr{L}_\Lambda\mathscr{S} & &(6.7)\\ &\simeq \mathscr{S}et & &\text{[GH15, Lem. 6.1.6(1)]} \end{aligned}$$

between the $(\infty\text{-})$category of sets $\mathscr{S}et$ and the underlying $(\infty\text{-})$category of the model structure $\mathcal{S}et$ on sets. Via the chosen identifications (6.7) and (6.8), we see that the functor $[\mathrm{disc}]_\infty\colon [\mathcal{S}et]_\infty \to [s\mathcal{S}et_{(\infty,0)}]_\infty$ and the canonical inclusion $\mathscr{S}et \hookrightarrow \mathscr{S}$ from [GH15, Def. 6.1.6(i)] are equivalent. Indeed, this is witnessed by the following diagram of ∞-categories

$$\begin{array}{ccccccccc} [\mathcal{S}et]_\infty & \xrightarrow{\simeq} & [\mathcal{L}_\Lambda s\mathcal{S}et_{(\infty,0)}]_\infty & \xrightarrow{\simeq} & \mathscr{L}_\Lambda[s\mathcal{S}et_{(\infty,0)}]_\infty & \xrightarrow{\simeq} & \mathscr{L}_\Lambda\mathscr{S} & \xrightarrow{\simeq} & \mathscr{S}et \\ [\mathrm{disc}]_\infty \big\uparrow\big\downarrow & & \big\uparrow\big\downarrow & & \big\uparrow\big\downarrow & & \big\uparrow\big\downarrow & & \big\uparrow\big\downarrow \\ [s\mathcal{S}et_{(\infty,0)}]_\infty & = & [s\mathcal{S}et_{(\infty,0)}]_\infty & = & [s\mathcal{S}et_{(\infty,0)}]_\infty & \xrightarrow{\simeq} & \mathscr{S} & = & \mathscr{S} \end{array}$$

which commutes up to equivalence, using Proposition 6.5 and [GH15, Lem. 6.1.6(i)].

6.2. **The 1-dimensional case.** The goal of this subsection is to show that the Quillen pair given by the ordinary nerve–categorification adjunction

$$c\colon s\mathcal{S}et_{(\infty,1)} \rightleftarrows \mathcal{C}at \colon \mathbf{N}$$

is equivalent to the left Bousfield localization of the Joyal model structure $s\mathcal{S}et_{(\infty,1)}$ with respect to a set $\Sigma\Lambda$ of maps. We also discuss in Remark 6.19 that this entails that the nerve embedding realizes 1-categories as local $(\infty,1)$-categories with respect to the set of maps $\Sigma\Lambda$.

The following is a well-known fact, and of straightforward verification.

Proposition 6.9. *The functor $\mathbf{N}\colon \mathcal{C}at \to s\mathcal{S}et_{(\infty,1)}$ is a right Quillen embedding.*

In particular, we have a Quillen reflection pair

$$c \colon s\mathcal{S}et_{(\infty,1)} \rightleftarrows \mathcal{C}at \colon \mathbf{N}.$$

Remark 6.10. The essential image of the functor $[\mathbf{N}]_\infty \colon [\mathcal{C}at]_\infty \to [s\mathcal{S}et_{(\infty,1)}]_\infty$ is the full sub-∞-category of $[s\mathcal{S}et_{(\infty,1)}]_\infty$ generated by the locally homotopically discrete $(\infty,1)$-categories.

Recall that the (right-sided) suspension of simplicial sets defines a left adjoint functor $\Sigma \colon s\mathcal{S}et \to s\mathcal{S}et_{*,*}$. Given a simplicial set X, the suspension can be understood as the following pushout of simplicial sets.

$$
\begin{array}{ccc}
X & \longrightarrow & X \star \Delta[0] \\
\downarrow & & \downarrow \\
\Delta[0] & \longrightarrow & \Sigma X
\end{array}
$$

Recall from [Hir21] that given any model category $\mathcal{M}$, there is a model category $\mathcal{M}_{*,*}$ of bipointed objects in $\mathcal{M}$, in which fibrations, cofibrations, and weak equivalences are created by the forgetful functor $\mathcal{M}_{*,*} \to \mathcal{M}$.

The proof of the following could be adapted from [OR20a, Lemma 2.7], using ideas from [Joy08b, Prop. 6.29].

Proposition 6.11. *The suspension functor* $\Sigma \colon s\mathcal{S}et_{(\infty,0)} \to (s\mathcal{S}et_{(\infty,1)})_{*,*}$ *is a left Quillen functor.*

Since the model structure $s\mathcal{S}et_{(\infty,1)}$ is combinatorial and left proper, the following model structure exists.

Proposition 6.12. *The category* $s\mathcal{S}et$ *admits the left Bousfield localization* $\mathcal{L}_{\Sigma\Lambda} s\mathcal{S}et_{(\infty,1)}$ *of the Joyal model structure* $s\mathcal{S}et_{(\infty,1)}$ *with respect to the set* $\Sigma\Lambda$ *of maps of the form*

(6.13) $\Sigma\Delta[0] \hookrightarrow \Sigma S^k, \quad for\ k > 0.$

So there is a Quillen reflection pair

$$\mathrm{Id} \colon s\mathcal{S}et_{(\infty,1)} \rightleftarrows \mathcal{L}_{\Sigma\Lambda} s\mathcal{S}et_{(\infty,1)} \colon \mathrm{Id}.$$

To prove the desired result, we will show that the nerve functor induces a right Quillen equivalence $\mathbf{N} \colon \mathcal{C}at \to \mathcal{L}_{\Sigma\Lambda} s\mathcal{S}et_{(\infty,1)}$.

Proposition 6.14. *The nerve functor* $\mathbf{N} \colon \mathcal{C}at \to \mathcal{L}_{\Sigma\Lambda} s\mathcal{S}et_{(\infty,1)}$ *defines a right Quillen embedding.*

Remark 6.15. For every simplicial set X there is a natural isomorphism of categories

$$c\Sigma X \cong \Sigma \pi_0 cX \cong \Sigma \pi_0 X.$$

Proof of Proposition 6.14. By [Hir03, Prop. 3.3.18] and Proposition 6.9, it is sufficient to show that c sends all maps from (6.13) to (weak) equivalences in $\mathcal{C}at$.

Let $k > 0$. The functor c sends the map

$$\Sigma\Delta[0] \hookrightarrow \Sigma S^k$$

to the map

$$c\Sigma\Delta[0] \hookrightarrow c\Sigma S^k,$$

which is by Remark 6.15

$$\Sigma\pi_0\Delta[0] \hookrightarrow \Sigma\pi_0 S^k,$$

which is the identity isomorphism at $\Sigma[0]$. This concludes the proof that the desired functor is right Quillen.

The fact that it is a right Quillen embedding follows directly from Proposition 6.9 as the derived counits of $\mathbf{N}\colon \mathcal{C}at \to s\mathcal{S}et_{(\infty,1)}$ and $\mathbf{N}\colon \mathcal{C}at \to \mathcal{L}_{\Sigma\Lambda}s\mathcal{S}et_{(\infty,1)}$ coincide at a fibrant object in $\mathcal{L}_{\Sigma\Lambda}s\mathcal{S}et_{(\infty,1)}$. $\qquad\square$

Proposition 6.16. *The suspension functor* $\Sigma\colon \mathcal{L}_\Lambda s\mathcal{S}et_{(\infty,0)} \to (\mathcal{L}_{\Sigma\Lambda}(s\mathcal{S}et_{(\infty,1)}))_{*,*}$ *is a left Quillen functor.*

Proof. As an instance of [Hir03, Theorem 3.3.20] combined with the fact that every object is cofibration in $s\mathcal{S}et_{(\infty,0)}$, we know that

$$\Sigma\colon \mathcal{L}_\Lambda s\mathcal{S}et_{(\infty,0)} \to \mathcal{L}_{\Sigma\Lambda}((s\mathcal{S}et_{(\infty,1)})_{*,*})$$

is a left Quillen functor. Further, since left Bousfield localizations commute with taking bipointed model structures, the model structures

$$\mathcal{L}_{\Sigma\Lambda}((s\mathcal{S}et_{(\infty,1)})_{*,*}) = (\mathcal{L}_{\Sigma\Lambda}(s\mathcal{S}et_{(\infty,1)}))_{*,*}$$

are equal. This concludes the proof. $\qquad\square$

The functor $\Sigma\colon s\mathcal{S}et \to s\mathcal{S}et_{*,*}$ admits a right adjoint $\mathrm{Hom}^R\colon s\mathcal{S}et_{*,*} \to s\mathcal{S}et$, used e.g. in [Lur09a, §1.2.2]. For any simplicial set X with given vertices x and y we write $X(x,y) := \mathrm{Hom}_X^R(x,y)$.

Remark 6.17. The following facts are of straightforward verifications. The first one uses the explicit description from e.g. [BV73, Prop. 4.12] of the category cX in the case of X being a quasi-category; see also [Joy08b, Prop. 1.11].

(1) For any quasi-category X with vertices x and y there is a bijection

$$\pi_0(X(x,y)) \cong (cX)(x,y).$$

(2) For any category $\mathcal{C}$ there is an isomorphism of simplicial sets

$$\mathrm{disc}(\mathcal{C}(x,y)) \cong (\mathbf{N}\mathcal{C})(x,y).$$

Theorem 6.18. *The nerve functor* $\mathbf{N}\colon \mathcal{C}at \to \mathcal{L}_{\Sigma\Lambda}s\mathcal{S}et_{(\infty,1)}$ *defines a right Quillen equivalence.*

Proof. By Proposition 6.14, it remains to prove that the component of the derived unit at every object X in $\mathcal{L}_{\Sigma\Lambda}s\mathcal{S}et_{(\infty,1)}$ is a weak equivalence. We do this by first proving it in the case of X being fibrant in $\mathcal{L}_{\Sigma\Lambda}s\mathcal{S}et_{(\infty,1)}$, and then treating the general case.

Assume that X is fibrant in $\mathcal{L}_{\Sigma\Lambda}s\mathcal{S}et_{(\infty,1)}$. Then for any vertices x and y in X, the tuple (X,x,y) is fibrant in $(\mathcal{L}_{\Sigma\Lambda}(s\mathcal{S}et_{(\infty,1)}))_{*,*}$ so $X(x,y)$ is fibrant in $\mathcal{L}_\Lambda s\mathcal{S}et_{(\infty,0)}$ by Proposition 6.16.

By Proposition 6.4, the (derived) unit at $X(x,y)$ is a weak equivalence in $\mathcal{L}_\Lambda s\mathcal{S}et_{(\infty,0)}$

$$
\begin{aligned}
X(x,y) &\simeq \mathrm{disc}(\pi_0(X(x,y)) \\
&\cong \mathrm{disc}((cX)(x,y)) && \text{Remark 6.17(1)} \\
&\cong (\mathbf{N}cX)(x,y). && \text{Remark 6.17(2)}
\end{aligned}
$$

between fibrant objects. Hence, it is already a weak equivalence in $s\mathcal{S}et_{(\infty,0)}$.

This weak equivalence

$$X(x, y) \to (\mathbf{N}cX)(x, y)$$

is precisely the map obtained by taking Hom^R of the (derived) unit of (X, x, y). This means that the (derived) unit of X

$$X \to \mathbf{N}cX$$

is locally a weak equivalence of simplicial sets, as well as a bijection on objects. By the fundamental theorem of $(\infty, 1)$-categories, originally due to Joyal [Joy08b] and recalled e.g. in [Cis19, Thm 3.9.7], we deduce that the (derived) unit is then a weak equivalence in $s\mathcal{S}et_{(\infty,1)}$, so in particular in the localization $\mathcal{L}_{\Sigma\Lambda} s\mathcal{S}et_{(\infty,1)}$ as desired.

Now if X is more generally any (cofibrant) simplicial set, we consider a fibrant replacement X^{fib} in $\mathcal{L}_{\Sigma\Lambda} s\mathcal{S}et_{(\infty,1)}$ and the following naturality diagram.

$$
\begin{array}{ccc}
X & \longrightarrow & \mathbf{N}cX \\
\downarrow & & \downarrow \\
X^{\mathrm{fib}} & \longrightarrow & \mathbf{N}c(X^{\mathrm{fib}})
\end{array}
$$

Here, the left vertical map is a weak equivalence in $\mathcal{L}_{\Sigma\Lambda} s\mathcal{S}et_{(\infty,1)}$ by construction, the right vertical map is a weak equivalence because both $\mathbf{N}$ and c are homotopical, and the bottom horizontal arrow is a weak equivalence by the case that we already treated. It follows by 2-out-of-3 that the top horizontal map, which is the (derived) unit of X, is a weak equivalence, as desired. □

Remark 6.19. We know – and it is also mentioned in [Lur09a, Ch. 3] – that the underlying ∞-category of the Joyal model structure $s\mathcal{S}et_{(\infty,1)}$ models the established ∞-category $\mathscr{C}at_{(\infty,1)}$ of ∞-categories, so there exists an equivalence of ∞-categories

$$(6.20) \qquad\qquad [s\mathcal{S}et_{(\infty,1)})]_\infty \simeq \mathscr{C}at_{(\infty,1)}.$$

Any such equivalence can be used to construct a specific equivalence of ∞-categories

$$
(6.21) \quad
\begin{aligned}
[\mathcal{C}at]_\infty &\simeq [\mathcal{L}_{\Sigma\Lambda} s\mathcal{S}et_{(\infty,1)}]_\infty && \text{Theorem 6.18} \\
&\simeq \mathscr{L}_{\Sigma\Lambda}[s\mathcal{S}et_{(\infty,1)}]_\infty && \text{Proposition 6.5} \\
&\simeq \mathscr{L}_{\Sigma\Lambda}\mathscr{C}at_{(\infty,1)} && (6.20) \\
&\simeq \mathscr{C}at_1 && [\text{GH15, Lem. 6.1.7(v)}]
\end{aligned}
$$

between the established ∞-category $\mathscr{C}at_1$ of categories and the underlying ∞-category of the model structure $\mathcal{C}at$ on categories. Via the chosen identifications (6.20) and (6.21), we see that the functor $[\mathbf{N}]_\infty \colon [\mathcal{C}at]_\infty \to [s\mathcal{S}et_{(\infty,1)}]_\infty$ and the canonical inclusion functor $\mathscr{C}at_1 \hookrightarrow \mathscr{C}at_{(\infty,1)}$ from [GH15, Lem. 6.1.7(v)] – used with $n = 1$ – are equivalent. Indeed, this is witnessed by the following diagram of ∞-categories

$$
\begin{array}{ccccccccc}
[\mathcal{C}at]_\infty & \xrightarrow{\simeq} & [\mathcal{L}_{\Sigma\Lambda} s\mathcal{S}et_{(\infty,1)}]_\infty & \xrightarrow{\simeq} & \mathscr{L}_{\Sigma\Lambda}[s\mathcal{S}et_{(\infty,1)}]_\infty & \xrightarrow{\simeq} & \mathscr{L}_{\Sigma\Lambda}\mathscr{C}at_{(\infty,1)} & \xrightarrow{\simeq} & \mathscr{C}at_1 \\
{\scriptstyle [\mathbf{N}]_\infty}\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
[s\mathcal{S}et_{(\infty,1)}]_\infty & = & [s\mathcal{S}et_{(\infty,1)}]_\infty & = & [s\mathcal{S}et_{(\infty,1)}]_\infty & \xrightarrow{\simeq} & \mathscr{C}at_{(\infty,1)} & = & \mathscr{C}at_{(\infty,1)}
\end{array}
$$

which commutes up to equivalence, using Proposition 6.5, [GH15, Lem. 6.1.9], and [GH15, Lem. 6.1.7(i)].

6.3. **The 2-dimensional case.** The goal of this subsection is to show that the Quillen reflection pair from Construction 4.4(1)

$$c_* \colon \mathcal{C}at_{s\mathcal{S}et_{(\infty,1)}} \rightleftarrows 2\mathcal{C}at \colon \mathbf{N}_*.$$

is equivalent to the left Bousfield localization of $\mathcal{C}at_{s\mathcal{S}et_{(\infty,1)}}$ with respect to a set $\Sigma^2\Lambda$ of maps. We also discuss in Remark 6.37 that this entails that the nerve embedding realizes 2-categories as local $(\infty, 2)$-categories with respect to the set of maps $\Sigma^2\Lambda$.

Remark 6.22. The essential image of the functor $[\mathbf{N}_*]_\infty \colon [2\mathcal{C}at]_\infty \to [\mathcal{C}at_{s\mathcal{S}et_{(\infty,1)}}]_\infty$ is the full sub-∞-category of $[\mathcal{C}at_{s\mathcal{S}et_{(\infty,1)}}]_\infty$ generated by the $(\infty, 2)$-categories that are locally equivalent to 1-categories.

Recall that there is a suspension functor $\Sigma \colon s\mathcal{S}et \to (\mathcal{C}at_{s\mathcal{S}et})_{*,*}$ which is a left adjoint. Given a simplicial set X, the simplicial category ΣX has two objects and a single non-trivial hom-simplicial set given by X. The following is briefly discussed e.g. as [HORR21, Lem. 4.1.5].

Proposition 6.23. *The suspension functor* $\Sigma \colon s\mathcal{S}et_{(\infty,1)} \to (\mathcal{C}at_{s\mathcal{S}et_{(\infty,1)}})_{*,*}$ *is a left Quillen functor.*

We consider the composite functor

$$\Sigma^2 \colon s\mathcal{S}et \xrightarrow{\Sigma} s\mathcal{S}et_{*,*} \xrightarrow{U} s\mathcal{S}et \xrightarrow{\Sigma} (\mathcal{C}at_{s\mathcal{S}et})_{*,*}.$$

Proposition 6.24. *The 2-fold suspension functor* $\Sigma^2 \colon s\mathcal{S}et_{(\infty,0)} \to (\mathcal{C}at_{s\mathcal{S}et_{(\infty,1)}})_{*,*}$ *is a left Quillen functor.*

Proof. It is a composite of the left Quillen (hence homotopical) functor

$$\Sigma \colon s\mathcal{S}et_{(\infty,0)} \to (s\mathcal{S}et_{(\infty,1)})_{*,*}$$

from Proposition 6.11 with the homotopical functor

$$U \colon (s\mathcal{S}et_{(\infty,1)})_{*,*} \to s\mathcal{S}et_{(\infty,1)},$$

which just forgets the two base points, and with the left Quillen (hence homotopical) functor

$$\Sigma \colon s\mathcal{S}et_{(\infty,1)} \to (\mathcal{C}at_{s\mathcal{S}et_{(\infty,1)}})_{*,*}$$

from Proposition 6.23. $\qquad\square$

Remark 6.25. Let $\mathcal{V} = s\mathcal{S}et_{(\infty,1)}$, so that in particular $\mathscr{V} = [\mathcal{V}]_\infty = [s\mathcal{S}et_{(\infty,1)}]_\infty \simeq \mathscr{C}at_\infty$. The suspension functor from Proposition 6.23 is a left Quillen functor, and induces a functor of ∞-categories

$$(6.26) \qquad\qquad [\Sigma]_\infty \colon [\mathcal{V}]_\infty \to [(\mathcal{C}at_\mathcal{V})_{*,*}]_\infty.$$

In [GH15, Def. 4.3.21] Gepner–Haugseng consider a functor

$$(6.27) \qquad\qquad\qquad \mathscr{V} \to \mathscr{C}at_{\mathscr{V}}^{\{0,1\}}.$$

Here, $\mathscr{C}at_{\mathscr{V}}^{\{0,1\}}$ denotes the ∞-category of ∞-categories enriched over $\mathscr{V}$ with fixed set of objects $\{0, 1\}$, as defined in [GH15, Def. 5.4.3]. As shown in [Hau15, §5], this ∞-category can be realized as the underlying ∞-category $[\mathcal{C}at_{\mathcal{V}}^{\{0,1\}}]_\infty \simeq \mathscr{C}at_{\mathscr{V}}^{\{0,1\}}$ of the model category $\mathcal{C}at_{\mathcal{V}}^{\{0,1\}}$ of $\mathcal{V}$-categories with set of objects $\{0, 1\}$, considered in [Hau15, Lemma 3.20].

Via the canonical map

$$(6.28) \qquad\qquad [(\mathcal{C}at_\mathcal{V})_{*,*}]_\infty \to [\mathcal{C}at_{\mathcal{V}}^{\{0,1\}}]_\infty \simeq \mathscr{C}at_{\mathscr{V}}^{\{0,1\}}$$

we will see in Proposition 6.29 that the two functors (6.26) and (6.27) are compatible, as they fit in a diagram of ∞-categories that commutes up to equivalence.

Proposition 6.29. *There is a diagram of ∞-categories*

$$
\begin{array}{ccc}
\mathscr{V} & \longrightarrow & \mathscr{C}at_{\mathscr{V}}^{\{0,1\}} \\
\| & & \downarrow \\
[\mathcal{V}]_\infty & \xrightarrow{[\Sigma]_\infty} & [(\mathcal{C}at_\mathcal{V})_{*,*}]_\infty
\end{array}
$$

that commutes up to equivalence, where the functors involved are those of (6.26), (6.27), *and* (6.28).

Proof. Each of the functors of ∞-categories involved in the diagram admits a right adjoint. We prove that the diagram of right adjoints commutes up to equivalence:

$$
\begin{array}{ccc}
[(\mathcal{C}at_\mathcal{V})_{*,*}]_\infty & \xrightarrow{[\mathrm{Hom}]_\infty} & [\mathcal{V}]_\infty \\
\downarrow & & \| \\
\mathscr{C}at_{\mathscr{V}}^{\{0,1\}} & \xrightarrow{\mathscr{H}om} & \mathscr{V}
\end{array}
$$

Building the desired commutative diagram out of smaller ones requires several ingredients, for which we provide references for the interested reader. The diagram is:

The ∞-categories featuring in the diagram are the following:

- $(\mathcal{C}at_\mathcal{V})_{*,*}$ is the bipointed model structure, obtained as an instance of [Hir21] applied to the model structure from Theorem 4.2.
- $\mathcal{C}at_\mathcal{V}^{\{0,1\}}$ is the model category of $\mathcal{V}$-categories with set of objects $\{0,1\}$, considered in [Hau15, Lem. 3.20].

- $\mathcal{V}^{\{0,1\} \times \{0,1\}}$ is the category of functors endowed with the injective model structure.
- $\mathrm{Alg}_{\Delta^{\mathrm{op}}_{\{0,1\}}}(\mathscr{V})$ is an instance of [GH15, §1.2] with the non-symmetric ∞-operad $\Delta^{\mathrm{op}}_{\{0,1\}}$ from [Hau15, Def. 2.8].
- $\mathrm{Alg}_{(\Delta^{\mathrm{op}}_{\{0,1\}})_{\mathrm{triv}}}(\mathscr{V})$ is an instance of [GH15, §1.2] with the non-symmetric ∞-operad $(\Delta^{\mathrm{op}}_{\{0,1\}})_{\mathrm{triv}}$ from [GH15, Def. 3.4.1].
- $(\Delta^{\mathrm{op}}_{\{0,1\}})_{[1]}$ is the fiber at $[1]$, which is an object of Δ^{op} of the map $\Delta^{\mathrm{op}}_{\{0,1\}} \to \Delta^{\mathrm{op}}$.

The functors of ∞-categories featuring in the diagram are the following:

$\diamond$ The functor $\mathrm{ev}_{(0,1)}$ is given by evaluation at the object $(0,1) \in \{0,1\} \times \{0,1\}$.
$\diamond$ The functor V is from [GH15, Proof of Prop. 5.2].
$\diamond$ The functor $\tau^*_{\Delta^{\mathrm{op}}_{\{0,1\}}}$ is the one considered in [GH15, §A.4, §3.4].
$\diamond$ The functor η is constructed on the level of model categories in [Hau15, Proof of Prop. 5.2], and the functor induced at the level of ∞-categories is further described in [GH15, Def. 4.3.1, Prop. 5.4.4].

We address the commutativity of each of the labeled regions as follows.

$\star$ The fact that the region (1) commutes is addressed as [Hau15, Proof of Prop. 5.2].
$\star$ The fact that the region (2) commutes is addressed as a combination of [Hau15, Lem. 3.20], [GH15, §3.4] and [GH15, §A.4].
$\star$ The fact that the region (3) commutes is addressed in [GH15, §A.4, A.5].

This concludes the proof. $\qquad\square$

Proposition 6.30. *The category $\mathcal{C}at_{s\mathcal{S}et}$ has the left Bousfield localization $\mathcal{L}_{\Sigma^2\Lambda}\mathcal{C}at_{s\mathcal{S}et_{(\infty,1)}}$ of the model structure $\mathcal{C}at_{s\mathcal{S}et_{(\infty,1)}}$ with respect to the set $\Sigma^2\Lambda$ of maps of the form*

$$(6.31) \qquad \Sigma^2\Delta[0] \hookrightarrow \Sigma^2 S^k, \quad \text{for } k > 0.$$

Proof. The Bousfield localization exists because the model category $s\mathcal{S}et_{(\infty,1)}$ is combinatorial and left proper by [Lur09a, Prop. A.3.2.4]. $\qquad\square$

So there is a Quillen reflection pair

$$\mathrm{Id}\colon \mathcal{C}at_{s\mathcal{S}et_{(\infty,1)}} \rightleftarrows \mathcal{L}_{\Sigma^2\Lambda}\mathcal{C}at_{s\mathcal{S}et_{(\infty,1)}} \colon \mathrm{Id}.$$

To prove the desired result, we will show that the nerve functor induces a right Quillen equivalence $\mathbf{N}_*\colon 2\mathcal{C}at \to \mathcal{L}_{\Sigma^2\Lambda}\mathcal{C}at_{s\mathcal{S}et_{(\infty,1)}}$. First, we prove the following.

Proposition 6.32. *The nerve functor $\mathbf{N}_*\colon 2\mathcal{C}at \to \mathcal{L}_{\Sigma^2\Lambda}\mathcal{C}at_{s\mathcal{S}et_{(\infty,1)}}$ defines a right Quillen embedding.*

Proof. By [Hir03, Prop. 3.3.18] and Construction 4.4(1), it is sufficient to show that c_* sends all elementary maps from (6.31) to biequivalences of 2-categories.

Let $k > 0$. The functor c_* sends the map

$$\Sigma^2\Delta[0] \hookrightarrow \Sigma^2 S^k$$

to the map

$$c_*\Sigma^2\Delta[0] \hookrightarrow c_*\Sigma^2 S^k,$$

which is the map

$$\Sigma c\Sigma\Delta[0] \hookrightarrow \Sigma c\Sigma S^k,$$

which is the map

$$\Sigma^2\pi_0\Delta[0] \hookrightarrow \Sigma^2\pi_0 S^k,$$

which is the identity at $\Sigma^2[0]$. This concludes the proof that the desired functor is right Quillen.

The fact that it is a right Quillen embedding follows directly from Construction 4.4(1) as the derived counits of $\mathbf{N}_*\colon 2\mathcal{C}at \to \mathcal{C}at_{s\mathcal{S}et_{(\infty,1)}}$ and $\mathbf{N}_*\colon 2\mathcal{C}at \to \mathcal{L}_{\Sigma^2\Lambda}\mathcal{C}at_{s\mathcal{S}et_{(\infty,1)}}$ coincide at a fibrant object in $\mathcal{L}_{\Sigma^2\Lambda}\mathcal{C}at_{s\mathcal{S}et_{(\infty,1)}}$. $\qquad\square$

Proposition 6.33. *The suspension functor* $\Sigma\colon \mathcal{L}_{\Sigma\Lambda}s\mathcal{S}et_{(\infty,1)} \to (\mathcal{L}_{\Sigma^2\Lambda}(\mathcal{C}at_{s\mathcal{S}et_{(\infty,1)}}))_{*,*}$ *is a left Quillen functor.*

Proof. As an instance of [Hir03, Theorem 3.3.20] combined with the fact that every object is cofibration in $s\mathcal{S}et_{(\infty,1)}$, we know that

$$\Sigma\colon \mathcal{L}_{\Sigma\Lambda}s\mathcal{S}et_{(\infty,1)} \to \mathcal{L}_{\Sigma^2\Lambda}((\mathcal{C}at_{s\mathcal{S}et_{(\infty,1)}})_{*,*})$$

is a left Quillen functor. Further, since left Bousfield localizations commute with taking bipointed model structures, the model structures

$$\mathcal{L}_{\Sigma^2\Lambda}((\mathcal{C}at_{s\mathcal{S}et_{(\infty,1)}})_{*,*}) = (\mathcal{L}_{\Sigma^2\Lambda}(\mathcal{C}at_{s\mathcal{S}et_{(\infty,1)}}))_{*,*}$$

are equal. This concludes the proof. $\qquad\square$

Lemma 6.34. *The functor* $c_*\colon \mathcal{C}at_{s\mathcal{S}et_{(\infty,1)}} \to 2\mathcal{C}at$ *is homotopical.*

Remark 6.35. A map $f\colon \mathcal{Q} \to \mathcal{Q}'$ is a weak equivalence in $\mathcal{C}at_{s\mathcal{S}et_{(\infty,1)}}$ if and only if the following are satisfied.

(1) The map f is *essentially surjective up to equivalence*; namely it induces an essentially surjective functor
$$\tau_* f\colon \tau_*\mathcal{Q} \to \tau_*\mathcal{Q}',$$
where $\tau_*\colon \mathcal{C}at_{s\mathcal{S}et} \to \mathcal{C}at$ is the base-change functor along Joyal's functor $\tau\colon s\mathcal{S}et \to \mathcal{S}et$ from [Joy08b, §1] given by the composite
$$s\mathcal{S}et \xrightarrow{c} \mathcal{C}at \xrightarrow{\mathrm{core}} \mathcal{G}pd \xrightarrow{\pi_0} \mathcal{S}et.$$

(2) The map f is *a local weak equivalence*; namely it induces a weak equivalence in $s\mathcal{S}et_{(\infty,1)}$
$$f\colon \mathcal{Q}(x,y) \to \mathcal{Q}'(f(x),f(y))$$
for any objects x and y in $\mathcal{Q}$.

Proof of Lemma 6.34. Given a weak equivalence $f\colon \mathcal{Q} \to \mathcal{Q}'$ in $\mathcal{C}at_{s\mathcal{S}et_{(\infty,1)}}$, we have a weak equivalence in $s\mathcal{S}et_{(\infty,1)}$
$$\mathcal{Q}(x,y) \to \mathcal{Q}'(x,y),$$
for any objects x and y in $\mathcal{Q}$, by Remark 6.35(2). Then, since c is homotopical, there is an induced equivalence of categories
$$(c_*\mathcal{Q})(x,y) = c\mathcal{Q}(x,y) \to c\mathcal{Q}'(x,y) = (c_*\mathcal{Q}')(x,y).$$
Moreover, by Remark 6.35(1) the functor
$$(\pi_0)_*(\mathrm{core})_* c_*\mathcal{Q} = \tau_*\mathcal{Q} \to \tau_*\mathcal{Q}' = (\pi_0)_*(\mathrm{core})_* c_*\mathcal{Q}'$$
is essentially surjective on objects. Hence we obtain that the 2-functor
$$c_*\mathcal{Q} \to c_*\mathcal{Q}'$$
is a weak equivalence in $2\mathcal{C}at$, as desired. $\qquad\square$

Theorem 6.36. *The nerve functor* $\mathbf{N}_*\colon 2\mathcal{C}at \to \mathcal{L}_{\Sigma^2\Lambda}\mathcal{C}at_{s\mathcal{S}et_{(\infty,1)}}$ *defines a right Quillen equivalence.*

Proof. By Proposition 6.32, it remains to prove that the component of the derived unit at every object $\mathcal{Q}$ in $\mathcal{L}_{\Sigma^2\Lambda}\mathcal{C}at_{s\mathcal{S}et_{(\infty,1)}}$ is a weak equivalence. We do this by first proving it in the case of $\mathcal{Q}$ being fibrant in $\mathcal{L}_{\Sigma^2\Lambda}\mathcal{C}at_{s\mathcal{S}et_{(\infty,1)}}$, and then treat the general case.

Assume that $\mathcal{Q}$ is fibrant in $\mathcal{L}_{\Sigma^2\Lambda}\mathcal{C}at_{s\mathcal{S}et_{(\infty,1)}}$. For any vertices x and y the tuple $(\mathcal{Q}, x, y)$ is fibrant in $(\mathcal{L}_{\Sigma^2\Lambda}\mathcal{C}at_{s\mathcal{S}et_{(\infty,1)}})_{*,*}$ so $\mathcal{Q}(x, y)$ is fibrant in $\mathcal{L}_{\Sigma\Lambda}s\mathcal{S}et_{(\infty,1)}$ by Proposition 6.33.

By Theorem 6.18, the (derived) unit at $\mathcal{Q}(x, y)$ is a weak equivalence in $\mathcal{L}_{\Lambda}s\mathcal{S}et_{(\infty,1)}$

$$\begin{aligned}\mathcal{Q}(x, y) &\simeq \mathbf{N}(c(\mathcal{Q}(x, y)) \\ &\cong \mathbf{N}_*((c_*\mathcal{Q})(x, y)) &&\text{Remark 6.17(1)} \\ &\cong (\mathbf{N}_*c_*\mathcal{Q})(x, y). &&\text{Remark 6.17(2)}\end{aligned}$$

between fibrant objects. Hence, it is already a weak equivalence in $s\mathcal{S}et_{(\infty,1)}$.

This weak equivalence

$$\mathcal{Q}(x, y) \to (\mathbf{N}_*c_*\mathcal{Q})(x, y)$$

is precisely the one obtained by taking Hom of the (derived) unit of $(\mathcal{Q}, x, y)$. This means that the (derived) unit of $\mathcal{Q}$

$$\mathcal{Q} \to \mathbf{N}_*c_*\mathcal{Q}$$

is locally a weak equivalence in $s\mathcal{S}et_{(\infty,1)}$, as well as a bijection on objects. By Remark 6.35 we deduce that the (derived) unit is then a weak equivalence in $\mathcal{C}at_{s\mathcal{S}et_{(\infty,1)}}$, so in particular in the localization $\mathcal{L}_{\Sigma^2\Lambda}\mathcal{C}at_{s\mathcal{S}et_{(\infty,1)}}$ as desired.

Now if $\mathcal{Q}$ is more generally any (cofibrant) simplicial set, we consider a fibrant replacement $\mathcal{Q}^{\mathrm{fib}}$ in $\mathcal{L}_{\Sigma^2\Lambda}\mathcal{C}at_{s\mathcal{S}et_{(\infty,1)}}$ and the following naturality diagram.

$$\begin{array}{ccc} \mathcal{Q} & \longrightarrow & \mathbf{N}_*c_*\mathcal{Q} \\ \downarrow & & \downarrow \\ \mathcal{Q}^{\mathrm{fib}} & \longrightarrow & \mathbf{N}_*c_*(\mathcal{Q}^{\mathrm{fib}}) \end{array}$$

Here, the left vertical map is a weak equivalence in $\mathcal{L}_{\Sigma^2\Lambda}\mathcal{C}at_{s\mathcal{S}et_{(\infty,1)}}$ by construction, the right vertical map is a weak equivalence because both $\mathbf{N}_*$ and c_* are homotopical by Lemma 6.34 and Proposition 6.32, and the bottom horizontal arrow is a weak equivalence by the case that we already treated. It follows by 2-out-of-3 that the top horizontal map, which is the (derived) unit of $\mathcal{Q}$, is a weak equivalence, as desired. $\square$

Remark 6.37. By [Lur09b, Rmk 0.0.4], we know that the underlying ∞-category of the model structure $\mathcal{C}at_{s\mathcal{S}et_{(\infty,1)}}$ models the established ∞-category $\mathscr{C}at_{(\infty,2)}$ of $(\infty, 2)$-categories, so there exists an equivalence of ∞-categories

(6.38) $$[\mathcal{C}at_{s\mathcal{S}et_{(\infty,1)}}]_\infty \simeq \mathscr{C}at_{(\infty,2)}.$$

Any such equivalence can be used to construct a specific equivalence of ∞-categories

(6.39) $$\begin{aligned}[2\mathcal{C}at]_\infty &\simeq [\mathcal{L}_{\Sigma^2\Lambda}\mathcal{C}at_{s\mathcal{S}et_{(\infty,1)}}]_\infty &&\text{Theorem 6.36} \\ &\simeq \mathscr{L}_{\Sigma^2\Lambda}[\mathcal{C}at_{s\mathcal{S}et_{(\infty,1)}}]_\infty &&\text{Proposition 6.5} \\ &\simeq \mathscr{L}_{\Sigma^2\Lambda}\mathscr{C}at_{(\infty,2)} &&(6.38), \text{Remark 6.25} \\ &\simeq \mathscr{C}at_2 &&[\text{GH15, Lem. 6.1.6(1)}]\end{aligned}$$

between the established ∞-category of 2-categories $\mathscr{C}at_2$ and the underlying ∞-category of the model structure $2\mathcal{C}at$ on 2-categories. Via the chosen identifications (6.38) and (6.39), we see that the functor $[\mathbf{N}_*]_\infty\colon [2\mathcal{C}at]_\infty \to [\mathcal{C}at_{s\mathcal{S}et_{(\infty,1)}}]_\infty$ and the canonical inclusion $\mathscr{C}at_2 \hookrightarrow \mathscr{C}at_{(\infty,2)}$ from [GH15, Lem. 6.1.6(v)] – for $n = 2$ – are equivalent. Indeed, this is witnessed by the following diagram of ∞-categories

$$
\begin{array}{ccccccccc}
[2\mathcal{C}at]_\infty & \xrightarrow{\simeq} & [\mathcal{L}_{\Sigma^2\Lambda}\mathcal{C}at_{s\mathcal{S}et_{(\infty,1)}}]_\infty & \xrightarrow{\simeq} & \mathscr{L}_{\Sigma^2\Lambda}[\mathcal{C}at_{s\mathcal{S}et_{(\infty,1)}}]_\infty & \xrightarrow{\simeq} & \mathscr{L}_{\Sigma^2\Lambda}\mathscr{C}at_{(\infty,1)} & \xrightarrow{\simeq} & \mathscr{C}at_2 \\
{\scriptstyle [\mathbf{N}_*]_\infty}\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
[\mathcal{C}at_{s\mathcal{S}et_{(\infty,1)}}]_\infty & = & [\mathcal{C}at_{s\mathcal{S}et_{(\infty,1)}}]_\infty & = & [\mathcal{C}at_{s\mathcal{S}et_{(\infty,1)}}]_\infty & \xrightarrow{\simeq} & \mathscr{C}at_{(\infty,2)} & = & \mathscr{C}at_{(\infty,2)}
\end{array}
$$

which commutes up to equivalence, using Proposition 6.5 and Remark 6.25 and [GH15, Def. 6.1.7(v)].

Appendix A. The nerve comparison lemma

To assert the commutativity at the level of ∞-categories of each of the regions in the diagram from Theorem 1.3, we will make use of the following lemma.

Lemma A.1 (Nerve comparison lemma). *Let $\mathcal{M}$ and $\mathcal{M}'$ be two model categories. Suppose we are given the following:*

- *a left Quillen functor, resp. right Quillen functor, $F\colon \mathcal{M} \to \mathcal{M}'$;*
- *a homotopical functor $H\colon 2\mathcal{C}at \to \mathcal{M}$ that takes values in the subcategory of cofibrant, resp. fibrant, objects in $\mathcal{M}$;*
- *a homotopical functor $H'\colon 2\mathcal{C}at \to \mathcal{M}'$; and*
- *a natural weak equivalence $FH \overset{\simeq}{\Longrightarrow} H'$.*

Then, the diagram of categories on the left

$$
\begin{array}{ccc}
& 2\mathcal{C}at & \\
{\scriptstyle H}\swarrow & & \searrow{\scriptstyle H'} \\
\mathcal{M} & \xrightarrow{\quad F \quad} & \mathcal{M}'
\end{array}
\qquad \rightsquigarrow \qquad
\begin{array}{ccc}
& [2\mathcal{C}at]_\infty & \\
{\scriptstyle [H]_\infty}\swarrow & & \searrow{\scriptstyle [H']_\infty} \\
[\mathcal{M}]_\infty & \xrightarrow{\quad [F]_\infty \quad} & [\mathcal{M}']_\infty
\end{array}
$$

induces a diagram of ∞-categories that commutes up to equivalence.[25]

Remark A.2. The second (resp. third) condition of Lemma A.1 is automatically satisfied when $H\colon 2\mathcal{C}at \to \mathcal{M}$ (resp. $H'\colon 2\mathcal{C}at \to \mathcal{M}'$) is right Quillen.

We choose to work with the following model of $[\mathcal{M}]_\infty$ for a model category $\mathcal{M}$, regarded as a relative category $(\mathcal{M}, \mathcal{W})$ when equipped with its class of weak equivalences $\mathcal{W}$.

Following e.g. [BSP21, Const. 15.1], given a relative category $(\mathcal{C}, \mathcal{W})$, the underlying ∞-category is

$$[\mathcal{C}]_\infty := \mathfrak{N}((L_H(\mathcal{C}, \mathcal{W}))^{\mathrm{fib}}).$$

Here, the functor $\mathfrak{N}\colon \mathcal{C}at_{s\mathcal{S}et_{(\infty,0)}} \to s\mathcal{S}et_{(\infty,1)}$ denotes the homotopy coherent nerve functor defined by [Cor82] and which is a right Quillen functor by [Lur09a, Thm 2.2.5.1], while $(-)^{\mathrm{fib}}\colon \mathcal{C}at_{s\mathcal{S}et_{(\infty,0)}} \to \mathcal{C}at_{s\mathcal{S}et_{(\infty,0)}}$ denotes any functorial fibrant replacement in the Bergner model structure $\mathcal{C}at_{s\mathcal{S}et_{(\infty,0)}}$ from [Lur09a, Thm 3.2.4, Ex. 3.2.23]; for instance, one could take $(\mathrm{Ex}^\infty)_*\colon \mathcal{C}at_{s\mathcal{S}et_{(\infty,0)}} \to \mathcal{C}at_{s\mathcal{S}et_{(\infty,0)}}$.

---

[25]Meaning that the two functors are equivalent in the ∞-category of functors from $[2\mathcal{C}at]_\infty \to [\mathcal{M}']_\infty$.

The following fact is essentially discussed in [MG16, §A.3.1], following [DK80a, Prop. 3.3, 3.5].

Proposition A.3. *Let $G, G'\colon (\mathcal{C}, \mathcal{W}) \to (\mathcal{C}', \mathcal{W}')$ be homotopical functors of relative categories, and let $\alpha\colon G \xRightarrow{\simeq} G'$ a natural weak equivalence. Then G and G' induce equivalent functors of quasi-categories*

$$[G]_\infty \simeq [G']_\infty \colon [\mathcal{C}]_\infty = \mathfrak{N}((L_H(\mathcal{C}, \mathcal{W}))^{\mathrm{fib}}) \to \mathfrak{N}((L_H(\mathcal{C}', \mathcal{W}'))^{\mathrm{fib}}) = [\mathcal{C}']_\infty.$$

We can now prove the lemma.

Proof of Lemma A.1. The lemma follows from Proposition A.3 by taking $G = FH$ and $G' = H'$. Indeed, we have equivalences of functors

$$[H']_\infty \simeq [FH]_\infty \simeq [F]_\infty \circ [H]_\infty,$$

which concludes the proof. $\qquad\square$

Appendix B. Complements on the Rezk nerve of categories

We collect in this appendix a series of elementary properties of the Rezk nerve that we did not find in the literature. We denote by $\widetilde{[k]}$ the contractible groupoid with $k + 1$ objects.

Construction B.1 ([Rez01, §3.5]). Let $\mathcal{C}$ be a category. The *Rezk nerve* $\mathbf{N}^R \mathcal{C}$ is the simplicial space given for any $j, k \geq 0$ by

$$\mathbf{N}^R_{j,k} \mathcal{C} := \mathcal{C}at([j] \times \widetilde{[k]}, \mathcal{C}).$$

The assignment extends to a functor $\mathbf{N}^R\colon \mathcal{C}at \to s\mathcal{S}et^{\Delta^{\mathrm{op}}}$.

Recall from [Rez01, Rmk 5.6] that the Rezk nerve has a left adjoint $c^R\colon s\mathcal{S}et^{\Delta^{\mathrm{op}}} \to \mathcal{C}at$.

Lemma B.2. *The left adjoint $c^R\colon s\mathcal{S}et^{\Delta^{\mathrm{op}}} \to \mathcal{C}at$ preserves finite products.*

Proof. Since both $\mathcal{C}at$ and $s\mathcal{S}et^{\Delta^{\mathrm{op}}}$ are cartesian closed, products commute with colimits, hence it suffices to prove that for any $j, k, j', k' \geq 0$ we have an isomorphism of bisimplicial sets

$$c^R(\Delta[j, k] \times \Delta[j', k']) \cong c^R(\Delta[j, k]) \times c^R(\Delta[j', k']).$$

We will prove that both sides are isomorphic to $[j] \times [j'] \times (\widetilde{[k] \times [k']})$. For the right-hand side, we have

$$
\begin{aligned}
c^R(\Delta[j, k]) \times c^R(\Delta[j', k']) &\cong [j] \times \widetilde{[k]} \times [j'] \times \widetilde{[k']} \\
&\cong [j] \times [j'] \times \widetilde{[k]} \times \widetilde{[k']} \\
&\cong [j] \times [j'] \times \widetilde{[k] \times [k']}.
\end{aligned}
$$

For the left-hand side, we need the following observations.

(1) For all $j, k \geq 0$ there is an isomorphism of bisimplicial sets

$$\Delta[j, k] \cong \Delta[j, 0] \times \Delta[0, k].$$

(2) The functor $\widetilde{(-)}\colon \mathcal{C}at \to \mathcal{G}pd$ is left adjoint to the inclusion functor $\mathcal{G}pd \hookrightarrow \mathcal{C}at$; in particular, the functor $\widetilde{(-)}$ preserves colimits.

(3) The left adjoint $c\colon s\mathcal{S}et \to \mathcal{C}at$ of the ordinary nerve functor preserves colimits, and it also preserves finite products by [Joy08b, Prop. B.0.15], there attributed to Gabriel–Zisman. Then, for any $j, j' \geq 0$ we obtain an isomorphism of categories

$$[j] \times [j'] \cong c\Delta[j] \times c\Delta[j'] \cong c(\Delta[j] \times \Delta[j'])$$
$$\cong c(\operatorname*{colim}_{\Delta \downarrow \Delta[j] \times \Delta[j']} \Delta[a]) \cong \operatorname*{colim}_{\Delta \downarrow \Delta[j] \times \Delta[j']} c\Delta[a]$$
$$\cong \operatorname*{colim}_{\Delta \downarrow \Delta[j] \times \Delta[j']} [a].$$

We then have the following isomorphisms of categories

$$c^R(\Delta[j,k] \times \Delta[j',k']) \cong c^R(\Delta[j,0] \times \Delta[j',0] \times \Delta[0,k] \times \Delta[0,k']) \qquad \text{Obs. (1)}$$
$$\cong c^R(\operatorname*{colim}_{\Delta \downarrow \Delta[j] \times \Delta[j']} \Delta[a,0] \times \operatorname*{colim}_{\Delta \downarrow \Delta[k] \times \Delta[k']} \Delta[0,b])$$
$$\cong c^R(\operatorname*{colim}_{\Delta \downarrow \Delta[j] \times \Delta[j']} \operatorname*{colim}_{\Delta \downarrow \Delta[k] \times \Delta[k']} (\Delta[a,0] \times \Delta[0,b]))$$
$$\cong \operatorname*{colim}_{\Delta \downarrow \Delta[j] \times \Delta[j']} \operatorname*{colim}_{\Delta \downarrow \Delta[k] \times \Delta[k']} c^R(\Delta[a,0] \times \Delta[0,b]) \qquad c^R \text{ left adjoint}$$
$$\cong \operatorname*{colim}_{\Delta \downarrow \Delta[j] \times \Delta[j']} \operatorname*{colim}_{\Delta \downarrow \Delta[k] \times \Delta[k']} c^R(\Delta[a,b]) \qquad \text{Obs. (1)}$$
$$\cong \operatorname*{colim}_{\Delta \downarrow \Delta[j] \times \Delta[j']} \operatorname*{colim}_{\Delta \downarrow \Delta[k] \times \Delta[k']} [a] \times \widetilde{[b]}$$
$$\cong \operatorname*{colim}_{\Delta \downarrow \Delta[j] \times \Delta[j']} [a] \times \operatorname*{colim}_{\Delta \downarrow \Delta[k] \times \Delta[k']} \widetilde{[b]}$$
$$\cong \operatorname*{colim}_{\Delta \downarrow \Delta[j] \times \Delta[j']} [a] \times (\widetilde{\operatorname*{colim}_{\Delta \downarrow \Delta[k] \times \Delta[k']} [b]}) \qquad \text{Obs. (2)}$$
$$\cong [j] \times [j'] \times \widetilde{[k] \times [k']}, \qquad \text{Obs. (3)}$$

as desired. $\qquad\qquad\square$

Proposition B.3. *The Rezk nerve* $\mathbf{N}^R\colon \mathcal{C}at \to s\mathcal{S}et_{(\infty,1)}^{\Delta^{\mathrm{op}}}$ *is a right Quillen embedding, and in particular a right Quillen and homotopical functor.*

Proof. We argue that the functor $\mathbf{N}^R\colon \mathcal{C}at \to s\mathcal{S}et_{(\infty,1)}^{\Delta^{\mathrm{op}}}$ can be understood as the composite of the ordinary nerve $\mathbf{N}\colon \mathcal{C}at \to s\mathcal{S}et_{(\infty,1)}$ of categories into simplicial sets, which is easily seen to be a right Quillen embedding and the functor $t^!\colon s\mathcal{S}et_{(\infty,1)} \to s\mathcal{S}et_{(\infty,1)}^{\Delta^{\mathrm{op}}}$ from [JT07, §4], which is shown to be a right Quillen equivalence. It will then follow that $\mathbf{N}^R$ is a right Quillen embedding.

In order to prove the claim, we observe that for any category $\mathcal{C}$ and $j, k \geq 0$ there is a natural bijection

$$(t^!\mathbf{N}\mathcal{C})_{j,k} \cong s\mathcal{S}et^{\Delta^{\mathrm{op}}}(\Delta[j,k], t^!\mathbf{N}\mathcal{C}) \cong s\mathcal{S}et(t_!\Delta[j,k], \mathbf{N}\mathcal{C})$$
$$\cong s\mathcal{S}et(\Delta[j] \times \mathbf{N}\widetilde{[k]}, \mathbf{N}\mathcal{C}) \cong s\mathcal{S}et(\mathbf{N}([j] \times \widetilde{[k]}), \mathbf{N}\mathcal{C})$$
$$\cong \mathcal{C}at([j] \times \widetilde{[k]}, \mathcal{C}) \cong \mathbf{N}_{j,k}^R\mathcal{C},$$

as desired. $\qquad\qquad\square$

Appendix C. Complements on the bisimplicial nerve of 2-categories

The homotopically correct nerve of 2-categories into 2-quasi-categories is based on the notion of *normal pseudofunctor*, also referred to as *normalized* or *strictly unital pseudofunctor* or *homomorphism*, or *weak functor*. Roughly speaking, a normal pseudofunctor is a map between 2-categories that preserves identities strictly and preserves composition up to coherent isomorphism. We now recall the main aspects of the definitions, referring the reader to other sources, see e.g. Bénabou [Bén67, Rmk 4.2], Street [Str96, Ex. 9.7] or Johnson–Yau [JY19, Def. 4.1], for a more detailed treatment.

Given a 2-category $\mathcal{A}$, we denote by $\operatorname{Ob}\mathcal{A}$, $\operatorname{Mor}\mathcal{A}$, and $2\operatorname{Mor}\mathcal{A}$ the sets of objects, 1-morphisms, and 2-morphisms in $\mathcal{A}$, respectively. We denote by s, t, i, and c the source, target, identity, and composition maps for 1-morphisms, and by s, t, i, c_h, and c_v the source, target, identity, horizontal composition, and vertical composition maps for 2-morphisms.

We denote by $\operatorname{Comp}\mathcal{A} := 2\mathcal{C}at(\mathcal{O}_2^\sim[2], \mathcal{A})$, the set of 2-isomorphisms in $\mathcal{A}$ of the form

$$
\begin{array}{ccc}
& y & \\
f \nearrow & \Uparrow \cong & \searrow g \\
x & \xrightarrow{\quad h \quad} & z \,,
\end{array}
$$

which comes with three maps $d_0, d_1, d_2 \colon \operatorname{Comp}\mathcal{A} \to \operatorname{Mor}\mathcal{A}$ picking each of the boundary of the 2-isomorphisms, and two maps $s_0, s_1 \colon \operatorname{Mor}\mathcal{A} \to \operatorname{Comp}\mathcal{A}$ sending a 1-morphism to its identity 2-morphism in the two usual ways.

Finally, we denote by $2\operatorname{Iso}\mathcal{A}$, the set of 2-isomorphisms in $\mathcal{A}$. Note that there is a map $e \colon \operatorname{Comp}\mathcal{A} \to 2\operatorname{Iso}\mathcal{A}$, which extracts the 2-isomorphism component, e.g. it sends the above picture to the corresponding 2-isomorphism $h \cong gf$.

Definition C.1. A *normal pseudofunctor* $F \colon \mathcal{A} \to \mathcal{B}$ between two 2-categories $\mathcal{A}$ and $\mathcal{B}$ consists of the following data

(0) an assignment on objects, namely a function $F_0 \colon \operatorname{Ob}\mathcal{A} \to \operatorname{Ob}\mathcal{B}$;
(1) an assignment on 1-morphisms, namely a function $F_1 \colon \operatorname{Mor}\mathcal{A} \to \operatorname{Mor}\mathcal{B}$;
(2) an assignment on 2-morphisms, namely a function $F_2 \colon 2\operatorname{Mor}\mathcal{A} \to 2\operatorname{Mor}\mathcal{B}$;
(3) a *compositor* of F, namely a function $\widetilde{F} \colon \operatorname{Mor}\mathcal{A} \times_{\operatorname{Ob}\mathcal{A}} \operatorname{Mor}\mathcal{A} \to \operatorname{Comp}\mathcal{B}$;

with the requirement that the following axioms be satisfied.

(a) The assignments of F on objects, 1- and 2-morphisms commute with source, target, and identities:

$$
\begin{array}{ccc}
\operatorname{Ob}\mathcal{A} \overset{s}{\underset{t}{\rightleftarrows}} \xrightarrow{i} \operatorname{Mor}\mathcal{A} & \overset{s}{\underset{t}{\rightleftarrows}} \xrightarrow{i} & 2\operatorname{Mor}\mathcal{A} \\
\Big\downarrow{F_0} \qquad \Big\downarrow{F_1} & & \Big\downarrow{F_2} \\
\operatorname{Ob}\mathcal{B} \overset{s}{\underset{t}{\rightleftarrows}} \xrightarrow{i} \operatorname{Mor}\mathcal{B} & \overset{s}{\underset{t}{\rightleftarrows}} \xrightarrow{i} & 2\operatorname{Mor}\mathcal{B}
\end{array}
$$

This gives that the images under F of a 1-morphism $f \colon x \to y$ and a 2-morphism $\alpha \colon f \Rightarrow g$ are of the form $Ff \colon Fx \to Fy$ and $F\alpha \colon Ff \Rightarrow Fg$, respectively, and that $F(\operatorname{id}_x) = \operatorname{id}_{Fx}$ and $F(\operatorname{id}_f) = \operatorname{id}_{Ff}$ for any object x and any 1-morphism f.

(b) The boundaries of $\widetilde{F}$ is determined by the following commutative diagram:

$$\begin{array}{ccc}
\operatorname{Mor}\mathcal{A} \xleftarrow{\;c\;} \operatorname{Mor}\mathcal{A} \underset{\operatorname{Ob}\mathcal{A}}{\times} \operatorname{Mor}\mathcal{A} =\!=\!=\!= \operatorname{Mor}\mathcal{A} \underset{\operatorname{Ob}\mathcal{A}}{\times} \operatorname{Mor}\mathcal{A} \\
{\scriptstyle F_1}\downarrow \qquad\qquad {\scriptstyle \widetilde{F}}\downarrow \qquad\qquad\qquad \downarrow{\scriptstyle F_1\times F_1} \\
\operatorname{Mor}\mathcal{B} \xleftarrow{\;d_1\;} \operatorname{Comp}\mathcal{B} \xrightarrow{\;(d_2,d_0)\;} \operatorname{Mor}\mathcal{B} \underset{\operatorname{Ob}\mathcal{B}}{\times} \operatorname{Mor}\mathcal{B}
\end{array}$$

When evaluated at an element $(f\colon x \to y, g\colon y \to z)$ this gives a 2-isomorphism $\widetilde{F}_{f,g}$ of the following form.

$$\begin{array}{ccc}
 & Fy & \\
{\scriptstyle Ff}\nearrow & \Uparrow \widetilde{F}_{f,g} & \searrow{\scriptstyle Fg} \\
Fx & \xrightarrow{\;F(gf)\;} & Fz
\end{array}$$

(c) The compositor $\widetilde{F}$ is compatible with identities in the sense that the following diagram commutes:

$$\begin{array}{ccc}
\operatorname{Mor}\mathcal{A} \xrightarrow{\;(\mathrm{id},\,it)\;} \operatorname{Mor}\mathcal{A} \underset{\operatorname{Ob}\mathcal{A}}{\times} \operatorname{Mor}\mathcal{A} \xleftarrow{\;(is,\,\mathrm{id})\;} \operatorname{Mor}\mathcal{A} \\
{\scriptstyle F_1}\downarrow \qquad\qquad {\scriptstyle \widetilde{F}}\downarrow \qquad\qquad\qquad \downarrow{\scriptstyle F_1} \\
\operatorname{Mor}\mathcal{B} \xrightarrow{\;s_1\;} \operatorname{Comp}\mathcal{B} \xleftarrow{\;s_0\;} \operatorname{Mor}\mathcal{B}
\end{array}$$

When evaluated at an element $f\colon x \to y$, this gives that the 2-isomorphisms $\widetilde{F}_{f,\mathrm{id}_y}$ and $\widetilde{F}_{\mathrm{id}_x,f}$ are both the identity 2-morphism at f.

(d) The assignment of F on 2-morphisms commutes with vertical composition of 2-morphisms:

$$\begin{array}{ccc}
2\operatorname{Mor}\mathcal{A} \underset{\operatorname{Mor}\mathcal{A}}{\times} 2\operatorname{Mor}\mathcal{A} \xrightarrow{\;F_2\times F_2\;} 2\operatorname{Mor}\mathcal{B} \underset{\operatorname{Mor}\mathcal{B}}{\times} 2\operatorname{Mor}\mathcal{B} \\
{\scriptstyle c_v}\downarrow \qquad\qquad\qquad\qquad \downarrow{\scriptstyle c_v} \\
2\operatorname{Mor}\mathcal{A} \xrightarrow{\;F_2\;} 2\operatorname{Mor}\mathcal{B}
\end{array}$$

When evaluated at an element $(\alpha\colon f \Rightarrow g\colon x \to y, \beta\colon g \Rightarrow h\colon x \to y)$, this gives that $F(\beta\alpha) = (F\beta)(F\alpha)$.

(e) The compositor $\widetilde{F}$ is 2-natural in the sense that the following diagram commutes:

$$\begin{array}{ccc}
2\operatorname{Mor}\mathcal{A} \underset{\operatorname{Ob}\mathcal{A}}{\times} 2\operatorname{Mor}\mathcal{A} \xrightarrow{\;(\widetilde{F}(s\times s),\,F_2\times F_2)\;} \operatorname{Comp}\mathcal{B} \underset{\operatorname{Mor}\mathcal{B}\underset{\operatorname{Ob}\mathcal{B}}{\times}\operatorname{Mor}\mathcal{B}}{\times} \left(2\operatorname{Mor}\mathcal{B} \underset{\operatorname{Ob}\mathcal{B}}{\times} 2\operatorname{Mor}\mathcal{B}\right) \\
\| \qquad\qquad\qquad\qquad\qquad\qquad \downarrow{\scriptstyle c_v(\mathrm{id}\times c_h)} \\
2\operatorname{Mor}\mathcal{A} \underset{\operatorname{Ob}\mathcal{A}}{\times} 2\operatorname{Mor}\mathcal{A} \qquad\qquad\qquad\qquad 2\operatorname{Mor}\mathcal{B} \\
{\scriptstyle (c_h,\,t\times t)}\downarrow \qquad\qquad\qquad\qquad\qquad \uparrow{\scriptstyle c_v} \\
2\operatorname{Mor}\mathcal{A} \underset{\operatorname{Mor}\mathcal{A}}{\times} \left(\operatorname{Mor}\mathcal{A} \underset{\operatorname{Ob}\mathcal{A}}{\times} \operatorname{Mor}\mathcal{A}\right) \xrightarrow{\;F_2\times\widetilde{F}\;} 2\operatorname{Mor}\mathcal{B} \underset{\operatorname{Mor}\mathcal{B}}{\times} \operatorname{Comp}\mathcal{B}
\end{array}$$

When evaluated at an element $(\alpha\colon f \Rightarrow f'\colon x \to y, \beta\colon g \Rightarrow g'\colon y \to z)$, this gives the following pasting equality.

(f) The compositor $\widetilde{F}$ is compatible with composition of 1-morphisms in the sense that the following diagram commutes:

where φ and ψ compute the total composite of the pasting diagrams. When evaluated at an element $(f\colon x \to y, g\colon y \to z, h\colon z \to w)$, this gives the following pasting equality.

The following can be deduced from [LP08, §3] or [Gur13, §2.3.3].

Remark C.2. If $F\colon \mathcal{A} \to \mathcal{B}$ is a normal pseudofunctor, the compositor $\widetilde{F}$ can be seen as a map into $2\operatorname{Iso}\mathcal{B}$ by post-composing with $e\colon \operatorname{Comp}\mathcal{B} \to 2\operatorname{Iso}\mathcal{B}$. Then, using (c) and (f), this map can be uniquely extended to a function

$$\operatorname{Mor}\mathcal{A} \underset{\operatorname{Ob}\mathcal{A}}{\times} \ldots \underset{\operatorname{Ob}\mathcal{A}}{\times} \operatorname{Mor}\mathcal{A} \to 2\operatorname{Iso}\mathcal{B},$$

which sends a sequence of composable 1-morphisms $f_1, \ldots, f_k$ in $\mathcal{A}$ to a 2-isomorphism in $\mathcal{B}$ of the form

$$F(f_k) \circ \ldots \circ F(f_1) \cong F(f_k \circ \ldots \circ f_1).$$

Lemma C.3. *Let $\mathcal{T}$ be a cofibrant 2-category, namely a 2-category whose underlying 1-category $\operatorname{Ob}_* \mathcal{T}$ is free. Then any normal pseudofunctor $F\colon \mathcal{A} \to \mathcal{B}$ induces a function*

$$F_*\colon 2\mathcal{C}at(\mathcal{T}, \mathcal{A}) \to 2\mathcal{C}at(\mathcal{T}, \mathcal{B}).$$

which is natural in $\mathcal{A}$, $\mathcal{B}$ and cofibrant $\mathcal{T}$ with respect to strict 2-functors.

Proof. Let $G\colon \mathcal{T} \to \mathcal{A}$ be a 2-functor. Consider the following data:

(0) $(F_*G)_0\colon \mathrm{Ob}\,\mathcal{T} \to \mathrm{Ob}\,\mathcal{B}$ defined as $F_*G(x) \coloneqq F(G(x))$ on an object x in $\mathcal{T}$;

(1) $(F_*G)_1\colon \mathrm{Mor}\,\mathcal{T} \to \mathrm{Mor}\,\mathcal{B}$ defined as $F_*G(f) \coloneqq F(G(f))$ on a generating 1-morphism f in $\mathcal{T}$, and extended appropriately to obtain a functor $\mathrm{Ob}_*\,\mathcal{T} \to \mathrm{Ob}_*\,\mathcal{A}$, taking advantage of the fact that $\mathrm{Ob}_*\,\mathcal{T}$ is a free 1-category;

(2) $(F_*G)_2\colon 2\,\mathrm{Mor}\,\mathcal{T} \to 2\,\mathrm{Mor}\,\mathcal{B}$ with $F_*G(\alpha)$ defined on a 2-cell $\alpha\colon f_k\circ\ldots\circ f_1 \Rightarrow g_l\circ\ldots\circ g_1$ in $\mathcal{T}$ as the composite

$$
\begin{array}{ccc}
F(G(f_k)) \circ \ldots \circ F(G(f_1)) & \xrightarrow{\ F_*G(\alpha)\ } & F(G(g_l)) \circ \ldots \circ F(G(g_1)) \\
\|\wr & & \|\wr \\
F(G(f_k) \circ \ldots \circ G(f_1)) & & F(G(g_l) \circ \ldots \circ (g_1)) \\
\| & & \| \\
F(G(f_k \circ \ldots \circ f_1)) & \xrightarrow[\ F(G(\alpha))\ ]{} & F(G(g_l \circ \ldots \circ g_1))
\end{array}
$$

which involves the 2-isomorphisms for F from Remark C.2, and the fact that G preserves compositions strictly.

It remains to see that this does indeed define a 2-functor F_*G. It is clear by construction that F_*G preserves compositions of 1-morphisms. Then, it preserves horizontal compositions of 2-morphisms by 2-naturality of $\widetilde{F}$, and vertical compositions of 2-morphisms since both F and G preserve those strictly. Note that F_*G preserves 1- and 2-identities since both F and G preserve them strictly.

The desired naturality follows from the definitions. $\qquad\qquad\square$

Recall from e.g. [Rez10, §11] or [Ara14, §7.1] (resp. [BSP21, Def. 3.1]) that a 2-category is said to be *rigid* (resp. *gaunt*) if it has no non-identity invertible 1- and 2-morphisms. Examples of gaunt 2-categories to which we apply the following lemma in this paper are the 2-categories θ which are objects of Θ_2.

Throughout this section, we follow the notational convention that

$$[\alpha^*, \beta^*]\colon \mathbf{N}^{\Delta\times\Delta}_{i',j',0}\mathcal{A} \to \mathbf{N}^{\Delta\times\Delta}_{i,j,0}\mathcal{B}$$

denotes the simplicial map induced by the simplicial operators $\alpha\colon [i] \to [i']$, $\beta\colon [j] \to [j']$, and $\mathrm{id}\colon [0] \to [0]$.

Proposition C.4. *For any gaunt 2-category $\mathcal{A}$ and any 2-category $\mathcal{B}$ there is a natural function*

$$\mathbf{N}^{\Delta\times\Delta}\colon 2\mathcal{C}at_{\mathrm{nps}}(\mathcal{A}, \mathcal{B}) \to s\mathcal{S}et^{(\Delta\times\Delta)^{\mathrm{op}}}(\mathbf{N}^{\Delta\times\Delta}\mathcal{A}, \mathbf{N}^{\Delta\times\Delta}\mathcal{B}).$$

Proof. This follows directly from Lemma C.3 with $\mathcal{T} = L^{\simeq}\mathbb{C}\Delta[i,j,k]$, using that $L^{\simeq}\mathbb{C}$ is a left Quillen functor by [Mos20, Thm 6.1.1] and hence that every 2-category in its image is cofibrant. $\qquad\qquad\square$

Remark C.5. For a 2-category $\mathcal{B}$, we give explicit relations between the sets $\mathbf{N}^{\Delta\times\Delta}_{i,j,k}\mathcal{B}$ for low values of $i, j, k \geq 0$ with the structural data of $\mathcal{B}$.

- For $(i, j, k) = (0, 0, 0)$ there is a bijection

$$\mathbf{N}^{\Delta\times\Delta}_{0,0,0}\mathcal{B} \cong 2\mathcal{C}at([0], \mathcal{B}) = \mathrm{Ob}\,\mathcal{B},$$

- For $(i, j, k) = (1, 0, 0)$ there a bijection

$$\mathbf{N}^{\Delta\times\Delta}_{1,0,0}\mathcal{B} \cong 2\mathcal{C}at([1], \mathcal{B}) = \mathrm{Mor}\,\mathcal{B},$$

- For $(i, j, k) = (1, 1, 0)$ there is an inclusion

$$\mathbf{N}^{\Delta \times \Delta}_{1,1,0} \mathcal{B} \leftarrow 2\mathcal{C}at(\mathcal{O}^{\sim}_2[1] \otimes_{\mathrm{ic}} \mathcal{O}^{\sim}_2[1], \mathcal{B}) \cong 2\mathcal{C}at(\Sigma[1], \mathcal{B}) \cong 2\operatorname{Mor}\mathcal{B}$$

induced by the map $L^{\simeq}\mathbb{C}\Delta[1, 1, 0] \to L\mathbb{C}[1, 1, 0]$ from Proposition 3.5; note that this inclusion can also be obtained as the pullback

$$
\begin{array}{ccc}
2\operatorname{Mor}\mathcal{B} & \longrightarrow & \mathbf{N}^{\Delta \times \Delta}_{0,0,0}\mathcal{B} \times \mathbf{N}^{\Delta \times \Delta}_{0,0,0}\mathcal{B} \\
\downarrow \quad \lrcorner & & \downarrow {\scriptstyle [s_0, \mathrm{id}] \times [s_0, \mathrm{id}]} \\
\mathbf{N}^{\Delta \times \Delta}_{1,1,0}\mathcal{B} & \xrightarrow[([d_1, \mathrm{id}], [d_0, \mathrm{id}])]{} & \mathbf{N}^{\Delta \times \Delta}_{0,1,0}\mathcal{B} \times \mathbf{N}^{\Delta \times \Delta}_{0,1,0}\mathcal{B}
\end{array}
$$

which only makes use of the simplicial structure of $\mathbf{N}^{\Delta \times \Delta}\mathcal{B}$.

- For $(i, j, k) = (2, 0, 0)$ there is a bijection

$$\mathbf{N}^{\Delta \times \Delta}_{2,0,0} \mathcal{B} \cong 2\mathcal{C}at(\mathcal{O}^{\sim}_2[2], \mathcal{B}) = \operatorname{Comp}\mathcal{B}.$$

Remark C.6. If $\mathcal{A}$ is a gaunt 2-category, the following relations hold.

- For $(i, j, k) = (1, 1, 0)$ there is a bijection

$$\mathbf{N}^{\Delta \times \Delta}_{1,1,0} \mathcal{A} \cong 2\operatorname{Mor}\mathcal{A},$$

- For $(i, j, k) = (2, 0, 0)$ there is a bijection

$$\mathbf{N}^{\Delta \times \Delta}_{2,0,0} \mathcal{A} \cong \operatorname{Mor}\mathcal{A} \underset{\mathrm{Ob}\,\mathcal{A}}{\times} \operatorname{Mor}\mathcal{A},$$

- For $(i, j, k) = (2, 1, 0)$ there is a bijection

$$\mathbf{N}^{\Delta \times \Delta}_{2,1,0} \mathcal{A} \cong 2\operatorname{Mor}\mathcal{A} \underset{\mathrm{Ob}\,\mathcal{A}}{\times} 2\operatorname{Mor}\mathcal{A},$$

- For $(i, j, k) = (3, 0, 0)$ there is a bijection

$$\mathbf{N}^{\Delta \times \Delta}_{3,0,0} \mathcal{A} \cong \operatorname{Mor}\mathcal{A} \underset{\mathrm{Ob}\,\mathcal{A}}{\times} \operatorname{Mor}\mathcal{A} \underset{\mathrm{Ob}\,\mathcal{A}}{\times} \operatorname{Mor}\mathcal{A},$$

- For $(i, j, k) = (1, 2, 0)$ there is a bijection

$$\mathbf{N}^{\Delta \times \Delta}_{1,2,0} \mathcal{A} \cong 2\operatorname{Mor}\mathcal{A} \underset{\operatorname{Mor}\mathcal{A}}{\times} 2\operatorname{Mor}\mathcal{A}.$$

Proposition C.7. *For any gaunt 2-category $\mathcal{A}$ and any 2-category $\mathcal{B}$ there is a natural function*

$$\gamma \colon s\mathcal{S}et^{(\Delta \times \Delta)^{\mathrm{op}}}(\mathbf{N}^{\Delta \times \Delta}\mathcal{A}, \mathbf{N}^{\Delta \times \Delta}\mathcal{B}) \to 2\mathcal{C}at_{\mathrm{nps}}(\mathcal{A}, \mathcal{B}).$$

Proof. Given a map $f \colon \mathbf{N}^{\Delta \times \Delta}\mathcal{A} \to \mathbf{N}^{\Delta \times \Delta}\mathcal{B}$ in $s\mathcal{S}et^{(\Delta \times \Delta)^{\mathrm{op}}}$, we produce a normal pseudo-functor $\gamma f \colon \mathcal{A} \to \mathcal{B}$ as follows:

(0) the assignment on objects, $(\gamma f)_0 \colon \operatorname{Ob}\mathcal{A} \to \operatorname{Ob}\mathcal{B}$, is given by

$$f_{0,0,0} \colon \mathbf{N}^{\Delta \times \Delta}_{0,0,0}\mathcal{A} \to \mathbf{N}^{\Delta \times \Delta}_{0,0,0}\mathcal{B};$$

(1) the assignment on 1-morphisms, $(\gamma f)_1 \colon \operatorname{Mor}\mathcal{A} \to \operatorname{Mor}\mathcal{B}$, is given by

$$f_{1,0,0} \colon \mathbf{N}^{\Delta \times \Delta}_{1,0,0}\mathcal{A} \to \mathbf{N}^{\Delta \times \Delta}_{1,0,0}\mathcal{B};$$

(2) the assignment on 2-morphisms, $(\gamma f)_2 \colon 2\operatorname{Mor}\mathcal{A} \to 2\operatorname{Mor}\mathcal{B}$, is induced by

$$f_{1,1,0} \colon \mathbf{N}^{\Delta \times \Delta}_{1,1,0}\mathcal{A} \to \mathbf{N}^{\Delta \times \Delta}_{1,1,0}\mathcal{B}$$

by requesting that $(\gamma f)_2$ is the unique map that fits into the following commutative diagram:

$$
\begin{array}{ccc}
2\operatorname{Mor}\mathcal{A} & \overset{(\gamma f)_2}{\dashrightarrow} & 2\operatorname{Mor}\mathcal{B}\\
\cong\downarrow & & \downarrow\\
\mathbf{N}^{\Delta\times\Delta}_{1,1,0}\mathcal{A} & \underset{f_{1,1,0}}{\longrightarrow} & \mathbf{N}^{\Delta\times\Delta}_{1,1,0}\mathcal{B}
\end{array}
$$

(3) the compositor $\widetilde{\gamma f}\colon \operatorname{Mor}\mathcal{A}\underset{\operatorname{Ob}\mathcal{A}}{\times}\operatorname{Mor}\mathcal{A}\to\operatorname{Comp}\mathcal{B}$ is induced by

$$
f_{2,0,0}\colon \mathbf{N}^{\Delta\times\Delta}_{2,0,0}\mathcal{A}\to\mathbf{N}^{\Delta\times\Delta}_{2,0,0}\mathcal{B}
$$

by requesting that $\widetilde{\gamma f}$ is the unique map that fits into the following commutative diagram:

$$
\begin{array}{ccc}
\operatorname{Mor}\mathcal{A}\underset{\operatorname{Ob}\mathcal{A}}{\times}\operatorname{Mor}\mathcal{A} & \overset{\widetilde{\gamma f}}{\dashrightarrow} & \operatorname{Comp}\mathcal{B}\\
\cong\downarrow & & \downarrow\cong\\
\mathbf{N}^{\Delta\times\Delta}_{2,0,0}\mathcal{A} & \underset{f_{2,0,0}}{\longrightarrow} & \mathbf{N}^{\Delta\times\Delta}_{2,0,0}\mathcal{B}
\end{array}
$$

We verify that γf does indeed define a normal pseudofunctor.

(a) The compatibility of γf with source, target, and identities follows from the commutativity of the following diagrams:

$$
\begin{array}{ccccc}
\mathbf{N}^{\Delta\times\Delta}_{0,0,0}\mathcal{A} & \underset{\xleftarrow{[s_0,\,\mathrm{id}]}}{\overset{[d_1,\,\mathrm{id}]}{\underset{[d_0,\,\mathrm{id}]}{\leftleftarrows}}} & \mathbf{N}^{\Delta\times\Delta}_{1,0,0}\mathcal{A} & \underset{\xleftarrow{[\mathrm{id},s_0]}}{\overset{[\mathrm{id},d_1]}{\underset{[\mathrm{id},d_0]}{\leftleftarrows}}} & \mathbf{N}^{\Delta\times\Delta}_{1,1,0}\mathcal{A}\\
f_{0,0,0}\downarrow & & f_{1,0,0}\downarrow & & \downarrow f_{1,1,0}\\
\mathbf{N}^{\Delta\times\Delta}_{0,0,0}\mathcal{B} & \underset{\xleftarrow{[s_0,\,\mathrm{id}]}}{\overset{[d_1,\,\mathrm{id}]}{\underset{[d_0,\,\mathrm{id}]}{\leftleftarrows}}} & \mathbf{N}^{\Delta\times\Delta}_{1,0,0}\mathcal{B} & \underset{\xleftarrow{[\mathrm{id},s_0]}}{\overset{[\mathrm{id},d_1]}{\underset{[\mathrm{id},d_0]}{\leftleftarrows}}} & \mathbf{N}^{\Delta\times\Delta}_{1,1,0}\mathcal{B}
\end{array}
$$

(b) The boundaries of $\widetilde{\gamma f}$ satisfy the required condition because of the commutativity of the following diagram:

$$
\begin{array}{ccccc}
\mathbf{N}^{\Delta\times\Delta}_{1,0,0}\mathcal{A} & \xleftarrow{[d_1,\,\mathrm{id}]} & \mathbf{N}^{\Delta\times\Delta}_{2,0,0}\mathcal{A} & \overset{[(d_2,d_0),\,\mathrm{id}]}{\underset{\cong}{\longrightarrow}} & \mathbf{N}^{\Delta\times\Delta}_{1,0,0}\mathcal{A}\underset{\mathbf{N}^{\Delta\times\Delta}_{0,0,0}\mathcal{A}}{\times}\mathbf{N}^{\Delta\times\Delta}_{1,0,0}\mathcal{A}\\
f_{1,0,0}\downarrow & & f_{2,0,0}\downarrow & & \downarrow f_{1,0,0}\times f_{1,0,0}\\
\mathbf{N}^{\Delta\times\Delta}_{1,0,0}\mathcal{B} & \underset{[d_1,\,\mathrm{id}]}{\longleftarrow} & \mathbf{N}^{\Delta\times\Delta}_{2,0,0}\mathcal{B} & \underset{[(d_2,d_0),\,\mathrm{id}]}{\longrightarrow} & \mathbf{N}^{\Delta\times\Delta}_{1,0,0}\mathcal{B}\underset{\mathbf{N}^{\Delta\times\Delta}_{0,0,0}\mathcal{B}}{\times}\mathbf{N}^{\Delta\times\Delta}_{1,0,0}\mathcal{B}
\end{array}
$$

(c) The compatibility of $\widetilde{\gamma f}$ with identities follows from the commutativity of the following diagram:

$$
\begin{array}{ccccc}
\mathbf{N}^{\Delta\times\Delta}_{1,0,0}\mathcal{A} & \xrightarrow{[s_1,\,\mathrm{id}]} & \mathbf{N}^{\Delta\times\Delta}_{2,0,0}\mathcal{A} & \xleftarrow{[s_0,\,\mathrm{id}]} & \mathbf{N}^{\Delta\times\Delta}_{1,0,0}\mathcal{A}\\
f_{1,0,0}\downarrow & & f_{2,0,0}\downarrow & & \downarrow f_{1,0,0}\\
\mathbf{N}^{\Delta\times\Delta}_{1,0,0}\mathcal{B} & \underset{[s_1,\,\mathrm{id}]}{\longrightarrow} & \mathbf{N}^{\Delta\times\Delta}_{2,0,0}\mathcal{B} & \underset{[s_0,\,\mathrm{id}]}{\longleftarrow} & \mathbf{N}^{\Delta\times\Delta}_{1,0,0}\mathcal{B}
\end{array}
$$

(d) The fact that γf preserves vertical composition of 2-morphisms strictly follows from the commutativity of the following diagram:

$$
\begin{array}{ccc}
\mathbf{N}_{1,1,0}^{\Delta \times \Delta}\mathcal{A} \underset{\mathbf{N}_{1,0,0}^{\Delta \times \Delta}\mathcal{A}}{\times} \mathbf{N}_{1,1,0}^{\Delta \times \Delta}\mathcal{A} & \xrightarrow{f_{1,1,0} \times f_{1,1,0}} & \mathbf{N}_{1,1,0}^{\Delta \times \Delta}\mathcal{B} \underset{\mathbf{N}_{1,0,0}^{\Delta \times \Delta}\mathcal{B}}{\times} \mathbf{N}_{1,1,0}^{\Delta \times \Delta}\mathcal{B} \\
{\scriptstyle[\mathrm{id},(d_2,d_0)]}\uparrow\,\cong & & \uparrow{\scriptstyle[\mathrm{id},(d_2,d_0)]} \\
\mathbf{N}_{1,2,0}^{\Delta \times \Delta}\mathcal{A} & \xrightarrow{\quad f_{1,2,0}\quad} & \mathbf{N}_{1,2,0}^{\Delta \times \Delta}\mathcal{B} \\
{\scriptstyle[\mathrm{id},d_1]}\downarrow & & \downarrow{\scriptstyle[\mathrm{id},d_1]} \\
\mathbf{N}_{1,1,0}^{\Delta \times \Delta}\mathcal{A} & \xrightarrow{\quad f_{1,1,0}\quad} & \mathbf{N}_{1,1,0}^{\Delta \times \Delta}\mathcal{B}
\end{array}
$$

(e) The 2-naturality of $\widetilde{\gamma f}$ follows from the commutativity of the following diagram:

$$
\begin{array}{ccc}
\mathbf{N}_{2,0,0}^{\Delta \times \Delta}\mathcal{A} \underset{X}{\times} (\mathbf{N}_{1,1,0}^{\Delta \times \Delta}\mathcal{A} \underset{\mathbf{N}_{0,0,0}^{\Delta \times \Delta}\mathcal{A}}{\times} \mathbf{N}_{1,1,0}^{\Delta \times \Delta}\mathcal{A}) & \xrightarrow[(f_{1,1,0} \times f_{1,1,0})]{f_{2,0,0} \times} & \mathbf{N}_{2,0,0}^{\Delta \times \Delta}\mathcal{B} \underset{Y}{\times} (\mathbf{N}_{1,1,0}^{\Delta \times \Delta}\mathcal{B} \underset{\mathbf{N}_{0,0,0}^{\Delta \times \Delta}\mathcal{B}}{\times} \mathbf{N}_{1,1,0}^{\Delta \times \Delta}\mathcal{B}) \\
{\scriptstyle([\mathrm{id},d_1],[(d_2,d_0),\mathrm{id}])}\uparrow\,\cong & & \uparrow{\scriptstyle([\mathrm{id},d_1],[(d_2,d_0),\mathrm{id}])} \\
\mathbf{N}_{2,1,0}^{\Delta \times \Delta}\mathcal{A} & \xrightarrow{\quad f_{2,1,0}\quad} & \mathbf{N}_{2,1,0}^{\Delta \times \Delta}\mathcal{B} \\
{\scriptstyle([d_1,\mathrm{id}],[\mathrm{id},d_0])}\downarrow & & \downarrow{\scriptstyle([d_1,\mathrm{id}],[\mathrm{id},d_0])} \\
\mathbf{N}_{1,1,0}^{\Delta \times \Delta}\mathcal{A} \underset{\mathbf{N}_{1,0,0}^{\Delta \times \Delta}\mathcal{A}}{\times} \mathbf{N}_{2,0,0}^{\Delta \times \Delta}\mathcal{A} & \xrightarrow{\; f_{1,1,0} \times f_{2,0,0}\;} & \mathbf{N}_{1,1,0}^{\Delta \times \Delta}\mathcal{B} \underset{\mathbf{N}_{1,0,0}^{\Delta \times \Delta}\mathcal{B}}{\times} \mathbf{N}_{2,0,0}^{\Delta \times \Delta}\mathcal{B}
\end{array}
$$

where

$$
X = \mathbf{N}_{1,0,0}^{\Delta \times \Delta}\mathcal{A} \underset{\mathbf{N}_{0,0,0}^{\Delta \times \Delta}\mathcal{A}}{\times} \mathbf{N}_{1,0,0}^{\Delta \times \Delta} \quad \text{and} \quad Y = \mathbf{N}_{1,0,0}^{\Delta \times \Delta}\mathcal{B} \underset{\mathbf{N}_{0,0,0}^{\Delta \times \Delta}\mathcal{B}}{\times} \mathbf{N}_{1,0,0}^{\Delta \times \Delta}\mathcal{B}.
$$

The fact that we retrieve the diagram of Definition C.1(e) comes from the fact that $\mathbf{N}_{2,1,0}^{\Delta \times \Delta}\mathcal{B}$ is the following pullback

$$
\begin{array}{ccc}
\mathbf{N}_{2,1,0}^{\Delta \times \Delta}\mathcal{B} & \xrightarrow{([\mathrm{id},d_1],[(d_2,d_0),\mathrm{id}])} & \mathbf{N}_{2,0,0}^{\Delta \times \Delta}\mathcal{B} \underset{Y}{\times} (\mathbf{N}_{1,1,0}^{\Delta \times \Delta}\mathcal{B} \underset{\mathbf{N}_{0,0,0}^{\Delta \times \Delta}\mathcal{B}}{\times} \mathbf{N}_{1,1,0}^{\Delta \times \Delta}\mathcal{B}) \\
{\scriptstyle([d_1,\mathrm{id}],[\mathrm{id},d_0])}\downarrow & \lrcorner & \downarrow{\scriptstyle\Phi} \\
\mathbf{N}_{1,1,0}^{\Delta \times \Delta}\mathcal{B} \underset{\mathbf{N}_{1,0,0}^{\Delta \times \Delta}\mathcal{B}}{\times} \mathbf{N}_{2,0,0}^{\Delta \times \Delta}\mathcal{B} & \xrightarrow{\quad \Psi \quad} & 2\,\mathrm{Mor}\,\mathcal{B}
\end{array}
$$

where Φ and Ψ compute the total composite of the pasting diagrams.

(f) The compatibility of $\widetilde{\gamma f}$ with respect to composition of 1-morphisms follows from the commutativity of the following diagram:

$$\begin{CD}
\mathbf{N}_{2,0,0}^{\Delta\times\Delta}\mathcal{A} \underset{\mathbf{N}_{1,0,0}^{\Delta\times\Delta}\mathcal{A}}{\times} \mathbf{N}_{2,0,0}^{\Delta\times\Delta}\mathcal{A} @>{f_{2,0,0}\times f_{2,0,0}}>> \mathbf{N}_{2,0,0}^{\Delta\times\Delta}\mathcal{B} \underset{\mathbf{N}_{1,0,0}^{\Delta\times\Delta}\mathcal{B}}{\times} \mathbf{N}_{2,0,0}^{\Delta\times\Delta}\mathcal{B}
\end{CD}$$

with vertical maps $[(d_2,d_0),\mathrm{id}]\cong$ and $[(d_2,d_0),\mathrm{id}]$

$$\mathbf{N}_{3,0,0}^{\Delta\times\Delta}\mathcal{A} \xrightarrow{\;f_{3,0,0}\;} \mathbf{N}_{3,0,0}^{\Delta\times\Delta}\mathcal{B}$$

with vertical maps $[(d_3,d_1),\mathrm{id}]\cong$ and $[(d_3,d_1),\mathrm{id}]$

$$\begin{CD}
\mathbf{N}_{2,0,0}^{\Delta\times\Delta}\mathcal{A} \underset{\mathbf{N}_{1,0,0}^{\Delta\times\Delta}\mathcal{A}}{\times} \mathbf{N}_{2,0,0}^{\Delta\times\Delta}\mathcal{A} @>{f_{2,0,0}\times f_{2,0,0}}>> \mathbf{N}_{2,0,0}^{\Delta\times\Delta}\mathcal{B} \underset{\mathbf{N}_{1,0,0}^{\Delta\times\Delta}\mathcal{B}}{\times} \mathbf{N}_{2,0,0}^{\Delta\times\Delta}\mathcal{B}
\end{CD}$$

The fact that we retrieve the diagram of Definition C.1(f) comes from the fact that $\mathbf{N}_{3,0,0}^{\Delta\times\Delta}\mathcal{B}$ is the following pullback

$$\begin{CD}
\mathbf{N}_{3,0,0}^{\Delta\times\Delta}\mathcal{B} @>{[(d_2,d_0),\mathrm{id}]}>> \mathbf{N}_{2,0,0}^{\Delta\times\Delta}\mathcal{B} \underset{\mathbf{N}_{1,0,0}^{\Delta\times\Delta}\mathcal{B}}{\times} \mathbf{N}_{2,0,0}^{\Delta\times\Delta}\mathcal{B} \\
@V{[(d_3,d_1),\mathrm{id}]}VV @VV{\varphi}V \\
\mathbf{N}_{2,0,0}^{\Delta\times\Delta}\mathcal{B} \underset{\mathbf{N}_{1,0,0}^{\Delta\times\Delta}\mathcal{B}}{\times} \mathbf{N}_{2,0,0}^{\Delta\times\Delta}\mathcal{B} @>{\psi}>> 2\operatorname{Mor}\mathcal{B}
\end{CD}$$

where φ and ψ compute the total composite of the pasting diagrams.
The desired naturality follows from the definitions. $\qquad\square$

We will need the following auxiliary fact, asserting a type of fully faithfulness for $\mathbf{N}^{\Delta\times\Delta}$ when restricted to certain 2-categories.

Proposition C.8. *For any gaunt 2-category $\mathcal{A}$ and any 2-category $\mathcal{B}$ there is a natural bijection*

$$\mathbf{N}^{\Delta\times\Delta}\colon 2\mathcal{C}at_{\mathrm{nps}}(\mathcal{A},\mathcal{B}) \cong s\mathcal{S}et^{(\Delta\times\Delta)^{\mathrm{op}}}(\mathbf{N}^{\Delta\times\Delta}\mathcal{A},\mathbf{N}^{\Delta\times\Delta}\mathcal{B}).$$

Proof. We now argue that given a map $f\colon \mathbf{N}^{\Delta\times\Delta}\mathcal{A} \to \mathbf{N}^{\Delta\times\Delta}\mathcal{B}$ in $s\mathcal{S}et^{(\Delta\times\Delta)^{\mathrm{op}}}$ we have

$$\mathbf{N}^{\Delta\times\Delta}(\gamma f) = f.$$

Since $\mathbf{N}^{\Delta\times\Delta}\mathcal{A}$ is 3-coskeletal[26] and $\mathbf{N}_{i,j,k}^{\Delta\times\Delta}\mathcal{A} = \mathbf{N}_{i,j,0}^{\Delta\times\Delta}\mathcal{A}$ for all $i,j,k \geq 0$, it is enough to check that $\mathbf{N}_{i,j,0}^{\Delta\times\Delta}(\gamma f) = f_{i,j,0}$ for any $i,j \geq 0$ with $i+j \leq 2$, which we see by direct inspection.

We now argue that

$$\gamma(\mathbf{N}^{\Delta\times\Delta}F) = F.$$

For this, it is enough to observe that by definition F and $\gamma(\mathbf{N}^{\Delta\times\Delta}F)$ agree on objects, 1- and 2-morphisms, and on the compositors. $\qquad\square$

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