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IDEALS OF HEISENBERG TYPE AND MINIMAX ELEMENTS OF AFFINE WEYL GROUPS

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Let $\mathfrak g$ be a simple Lie algebra over $\mathbb C$. Fix a Borel subalgebra $\mathfrak b$ and a Cartan subalgebra $\mathfrak t \subset \mathfrak b$. The corresponding set of positive (resp. simple) roots is Δ^+ (resp. Π). Write θ for the highest root in Δ^+.

An ideal of $\mathfrak b$ is called ad-*nilpotent*, if it is contained in $[\mathfrak b, \mathfrak b]$. Consequently, such a subspace is fully determined by the corresponding set of positive roots, and this set is called an *ideal* (in Δ^+). Let $\mathfrak{Ad}$ denote the set of all ideals of Δ^+. We regard Δ^+ as poset with respect to the standard root order '$\preccurlyeq$' (see Section 1). Given $I \in \mathfrak{Ad}$, the minimal roots in it are said to be the *generators*. The set of generators of an ideal form an *antichain* in $(\Delta^+, \preccurlyeq)$, and this correspondence sets up a bijection between the ideals and the antichains of Δ^+. An ideal or antichain is called *strictly positive*, if it contains no simple roots. Another interesting class consists of Abelian ideals, i.e., those with the property $(I + I) \cap \Delta^+ = \varnothing$. We write $\mathfrak{Ad}_0$ (resp. $\mathfrak{Ab}$) for the set of strictly positive (resp. Abelian) ideals. Various results for $\mathfrak{Ad}$, $\mathfrak{Ad}_0$, and $\mathfrak{Ab}$ were recently obtained in [2], [3], [4], [5], [10], [11], [12], [15].

It was shown by Cellini and Papi [4] that there is a bijection between the ideals and certain elements of $\widehat W$, the affine Weyl group associated with $\mathfrak g$. Then Sommers [15] discovered a bijection between the strictly positive ideals (or antichains) and another class of elements of $\widehat W$. Following [15], the elements in these two classes are said to be *minimal* and *maximal*, respectively. Furthermore, the minimal elements of $\widehat W$ are in a bijection with the points of the coroot lattice lying in a certain simplex. The same assertion also holds for the maximal elements (with another simplex!) A different approach to (ad-nilpotent) ideals relates them with the theory of hyperplane arrangements. By a result of Shi [13], there is a bijection between $\mathfrak{Ad}$ and the dominant regions of the *Shi (*or *Catalan) arrangement*. It was then observed by Athanasiadis [2] and Panyushev [11] that the Shi bijection induces the bijection between $\mathfrak{Ad}_0$ and the bounded dominant regions. We survey these results in Section 2.

The goal of this paper is two-fold. First, we study the ideals lying inside $\mathcal H$, the set of all positive roots that are not orthogonal to θ. Such ideals are said to be *of Heisenberg type*. Second, we study the *minimax* elements of $\widehat W$, i.e., those that are simultaneously minimal and maximal. The corresponding strictly positive ideals are said to be minimax, too. Both minimal and maximal elements of $\widehat W$ are particular instances of *dominant* elements. To

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any dominant element one may attach an ideal, and we consider dominant elements of special sort, which are said to be of *Heisenberg type*, see Definition 3.2. The corresponding ideals lie in $\mathcal{H}$, hence the term. We obtain a explicit description of dominant elements of Heisenberg type and distinguish minimal and maximal ones among them. Our description says, in particular, that the dominant (resp. minimal) elements of Heisenberg type are parameterized by Δ_l (resp. $\Delta_l \setminus (-\Pi)$), where Δ_l is the set of long roots. We also prove that the minimal elements of Heisenberg type are in a bijection with the non-trivial ideals of Heisenberg type and give explicit formulae for these ideals, see Theorem 3.12. It follows that the number of non-trivial ideals inside $\mathcal{H}$ is equal to $\#(\Delta_l \setminus \Pi)$.

In Section 4, combining known results for minimal and maximal elements, we obtain a characterization of minimax elements (ideals), and establish a bijection between the minimax elements of $\widehat{W}$ and the the coroot lattice points lying in a polytope D_{mm}. If V is the real vector space spanned by Δ in $\mathfrak{t}^*$, then

$$D_{mm} := \{x \in V \mid -1 \le (\alpha, x) \le 1 \ \ \forall \alpha \in \Pi \quad \& \quad 0 \le (\theta, x) \le 2\}$$

It is the intersection of two simplices corresponding to the minimal and maximal elements of $\widehat{W}$. The geometric meaning of minimax ideals is that, under the Shi bijection, they correspond to the regions consisting of a single alcove. It would be interesting to further investigate the configuration of those dominant regions that consist of a single alcove. We also give an upper bound on the sum $\#\Gamma(I) + \#\Xi(I)$, where $\Gamma(I)$ is the set of generators (minimal elements) of I and $\Xi(I)$ is the set of maximal elements of $\Delta^+ \setminus I$. Since Π is the only antichain of cardinality $\mathrm{rk}\,\mathfrak{g}$, one readily sees that the maximal possible value of the above sum is $2\mathrm{rk}\,\mathfrak{g} - 1$, which is attained for $I = \Delta^+ \setminus \Pi$. However, for the minimax ideals this sum is at most $\mathrm{rk}\,\mathfrak{g} + 1$ (at most $\mathrm{rk}\,\mathfrak{g}$, if $\mathfrak{g} \ne \mathfrak{sl}_{2n+1}$), see Proposition 4.6.

Write $\mathfrak{Ad}_{mm}$ or $\mathfrak{Ad}_{mm}(\mathfrak{g})$ for the set of minimax ideals of $\mathfrak{g}$. In Section 5, we compute $\#\mathfrak{Ad}_{mm}(\mathfrak{g})$ for all simple Lie algebras. Using the coefficients of θ, one may form a Laurent polynomial (in x). We prove that $\#\mathfrak{Ad}_{mm}$ equals the coefficient of x in this polynomial, divided by the index of connection of Δ. It turns out that some famous integer sequences emerge in connection with the minimax ideals in classical Lie algebras. Namely,

$\#\mathfrak{Ad}_{mm}(\mathfrak{sl}_{n+1}) = M_n$, the *$n$-th Motzkin number*;

$\#\mathfrak{Ad}_{mm}(\mathfrak{sp}_{2n}) = \#\mathfrak{Ad}(\mathfrak{so}_{2n+1}) = \mathrm{dir}_n$, the number of *directed animals of size* n.

We also show that $\#\mathfrak{Ad}_{mm}(\mathfrak{so}_{2n}) = 2\mathrm{dir}_{n-2} + \mathrm{dir}_{n-1}$. See [1], [6], [7], and [16, Ch. 6] for relevant background and numerous combinatorial interpretations of these numbers. In Section 6, an explicit matrix characterization of the minimax ideals for $\mathfrak{sl}_{n+1}$ and $\mathfrak{sp}_{2n}$ is obtained. The description for $\mathfrak{sl}_{n+1}$ can be stated as follows. Let $\{\alpha_i = \varepsilon_i - \varepsilon_{i+1} \mid i = 1, \dots, n\}$ be the standard set of simple roots. Let $\Gamma = \{\gamma_1, \dots, \gamma_k\}$ be the set of generators of an ideal I, where $\gamma_t = \alpha_{i_t} + \alpha_{i_t+1} + \ldots + \alpha_{j_t}$, $t = 1, 2, \dots, k$. Then I is minimax if and only if $j_t \ne i_s$ for all pairs (t, s). In case $t = s$, this means that γ_t cannot be a simple root. Using these descriptions, we also compute the generating function for the statistic "the number of generators" on $\mathfrak{Ad}_{mm}(\mathfrak{g})$ for $\mathfrak{g} = \mathfrak{sl}_{n+1}$ and $\mathfrak{sp}_{2n}$.

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1. Notation and other preliminaries

(1.1) Main notation. Δ is the root system of $(\mathfrak g, \mathfrak t)$ and W is the usual Weyl group. For $\alpha \in \Delta$, $\mathfrak g_\alpha$ is the corresponding root space in $\mathfrak g$.

Δ^+ is the set of positive roots and $\rho = \frac{1}{2}\sum_{\alpha\in\Delta^+}\alpha$.

$\Pi = \{\alpha_1, \dots, \alpha_p\}$ is the set of simple roots in Δ^+ and θ is the highest root in Δ^+.

$e_1, e_2, \dots, e_p$ are the exponents and h is the Coxeter number of W.

We set $V := \mathfrak t_{\mathbb R} = \oplus_{i=1}^p \mathbb R\alpha_i$ and denote by $(\ ,\ )$ a W-invariant inner product on V. As usual, $\mu^\vee = 2\mu/(\mu,\mu)$ is the coroot for $\mu \in \Delta$.

$\mathcal C = \{x \in V \mid (x,\alpha) > 0 \ \ \forall\alpha \in \Pi\}$ is the (open) fundamental Weyl chamber.

$\mathcal A = \{x \in V \mid (x,\alpha) > 0 \ \ \forall\alpha \in \Pi \ \ \& \ (x,\theta) < 1\}$ is the fundamental alcove.

$Q^+ = \{\sum_{i=1}^p n_i\alpha_i \mid n_i = 0,1,2,\dots\}$ and $Q^\vee = \oplus_{i=1}^p \mathbb Z\alpha_i^\vee \subset V$ is the coroot lattice.

Letting $\widehat V = V \oplus \mathbb R\delta \oplus \mathbb R\lambda$, we extend the inner product $(\ ,\ )$ on $\widehat V$ so that $(\delta, V) = (\lambda, V) = (\delta,\delta) = (\lambda,\lambda) = 0$ and $(\delta,\lambda) = 1$.

$\widehat\Delta = \{\Delta + k\delta \mid k \in \mathbb Z\}$ is the set of affine real roots and $\widehat W$ is the affine Weyl group.

Then $\widehat\Delta^+ = \Delta^+ \cup \{\Delta + k\delta \mid k \ge 1\}$ is the set of positive affine roots and $\widehat\Pi = \Pi \cup \{\alpha_0\}$ is the corresponding set of affine simple roots, where $\alpha_0 = \delta - \theta$. The inner product $(\ ,\ )$ on $\widehat V$ is $\widehat W$-invariant. The notation $\beta > 0$ (resp. $\beta < 0$) is a shorthand for $\beta \in \widehat\Delta^+$ (resp. $\beta \in -\widehat\Delta^+$). For α_i ($0 \le i \le p$), we let s_i denote the corresponding simple reflection in $\widehat W$. If the index of $\alpha \in \widehat\Pi$ is not specified, then we merely write s_α. The length function on $\widehat W$ with respect to $s_0, s_1, \dots, s_p$ is denoted by ℓ. For any $w \in \widehat W$, we set

$$N(w) = \{\alpha \in \widehat\Delta^+ \mid w(\alpha) \in -\widehat\Delta^+\}.$$

It is standard that $\#N(w) = \ell(w)$ and $N(w)$ is *bi-convex*. The latter means that both $N(w)$ and $\widehat\Delta^+ \setminus N(w)$ are subsets of $\widehat\Delta^+$ that are closed under addition. Furthermore, the assignment $w \mapsto N(w)$ sets up a bijection between the elements of $\widehat W$ and the finite bi-convex subsets of $\widehat\Delta^+$.

(1.2) Ideals and antichains. Recall that $\mathfrak b$ is the Borel subalgebra of $\mathfrak g$ corresponding to Δ^+ and $\mathfrak u = [\mathfrak b, \mathfrak b]$. Let $\mathfrak c \subset \mathfrak b$ be an ad-nilpotent ideal. Then $\mathfrak c = \underset{\alpha\in I}{\oplus}\mathfrak g_\alpha$ for some $I \subset \Delta^+$. This I is said to be an *ideal* (of Δ^+). More precisely, a set $I \subset \Delta^+$ is an ideal, if whenever $\gamma \in I, \mu \in \Delta^+$, and $\gamma+\mu \in \Delta$, then $\gamma+\mu \in I$. Our exposition will be mostly combinatorial, i.e., in place of ad-nilpotent ideal of $\mathfrak b$ we will deal with the respective ideals of Δ^+.

For $\mu, \gamma \in \Delta^+$, write $\mu \preccurlyeq \gamma$, if $\gamma - \mu \in Q^+$. The notation $\mu \prec \gamma$ means that $\mu \preccurlyeq \gamma$ and $\gamma \ne \mu$. We regard Δ^+ as poset under '$\preccurlyeq$'. Let $I \subset \Delta^+$ be an ideal. An element $\gamma \in I$ is called a *generator*, if $\gamma - \alpha \notin I$ for any $\alpha \in \Pi$. In other words, γ is a minimal element of I. We write $\Gamma(I)$ for the set of generators of I. It is easily seen that $\Gamma(I)$ is an *antichain* of Δ^+, i.e., $\gamma_i \not\preccurlyeq \gamma_j$ for any pair (γ_i, γ_j) in $\Gamma(I)$. Conversely, if $\Gamma \subset \Delta^+$ is an antichain, then the ideal

$$I\langle\Gamma\rangle := \{\mu \in \Delta^+ \mid \mu \succcurlyeq \gamma_i \text{ for some } \gamma_i \in \Gamma\}$$

has Γ as the set of generators. Let $\mathfrak{An}$ denote the set of all antichains in Δ^+. In view of the above bijection $\mathfrak{Ad} \xleftrightarrow{1:1} \mathfrak{An}$, we will freely switch between ideals and antichains. An ideal I is called *strictly positive*, if $I \cap \Pi = \varnothing$. The set of strictly positive ideals is denoted by $\mathfrak{Ad}_0$.

Given $I \in \mathfrak{Ad}$, write $\Xi(I)$ for the set of maximal elements of $\Delta^+ \setminus I$.

2. Minimal and maximal elements in affine Weyl groups and dominant regions of the Catalan arrangement

This section is a review of known results. As is well known, $\widehat{W}$ is isomorphic to a semi-direct product of W and $Q^\vee$ [9]. Given $w \in \widehat{W}$, there is a unique decomposition

$$w = v{\cdot}t_r \ , \tag{2.1}$$

where $v \in W$ and t_r is the translation corresponding to $r \in Q^\vee$. Then $w^{-1} = v^{-1}{\cdot}t_{-v(r)}$. The word "translation" means the following. The group $\widehat{W}$ has two natural actions:

(a) the linear action on $\widehat{V} = V \oplus \mathbb{R}\delta \oplus \mathbb{R}\lambda$;

(b) the affine-linear action on V.

We use '$\ast$' to denote the second action. For the linear action, we have $w^{-1}(x) = v^{-1}(x) + (x, v(r))\delta$ for any $x \in V \oplus \mathbb{R}\delta$. In particular,

$$\begin{aligned} w^{-1}(\alpha_i) &= v^{-1}(\alpha_i) + (\alpha_i, v(r))\delta, \quad i \ge 1, \\ w^{-1}(\alpha_0) &= -v^{-1}(\theta) + (1 - (\theta, v(r)))\delta \ . \end{aligned} \tag{2.2}$$

While the affine-linear action is given by $w^{-1}\ast x = v^{-1}(x) - r$ for $x \in V$. In particular, $t_r \ast y = y + r$, so that t_r is a true translation for the $\ast$-action on V.

Let us say that $w \in \widehat{W}$ is *dominant*, if $w(\alpha) > 0$ for all $\alpha \in \Pi$. Obviously, w is dominant if and only if $N(w) \subset \cup_{k\ge1}(k\delta - \Delta^+)$. It also follows from [4, 1.1] that w is dominant if and only if $w^{-1}\ast \mathcal{A} \subset \mathcal{C}$. Write $\widehat{W}_{dom}$ for the set of dominant elements.

2.3 Proposition [12].

(i) *If $w = v{\cdot}t_r \in \widehat{W}_{dom}$, then $r \in -\overline{\mathcal{C}}$;*

(ii) *The mapping $\widehat{W}_{dom} \to Q^\vee$ given by $w = v{\cdot}t_r \mapsto v(r)$ is a bijection.*

Letting $\delta - I := N(w) \cap (\delta - \Delta^+)$, we easily deduce that I is an ideal, if $w \in \widehat{W}_{dom}$. We say $\delta - I$ is the *first layer* of $N(w)$ and $I = I_w$ is the *first layer ideal* of w. However, an ideal I may well arise from different dominant elements. To obtain a bijection, one has to impose further constraints on dominant elements. One may attempt to consider either maximal or minimal bi-convex subsets with first layer $\delta - I$. This naturally leads to notions of 'minimal' and 'maximal' elements. This terminology suggested in [15] is also explained by a relationship between these elements and dominant regions of the Shi arrangement, see Theorem 2.19.

2.4 Definition. $w \in \widehat{W}$ is called *minimal*, if

(i) w is dominant;
(ii) if $\alpha \in \widehat{\Pi}$ and $w^{-1}(\alpha) = k\delta + \mu$ for some $\mu \in \Delta$, then $k \geq -1$.

Using (i), condition (ii) can be made more precise. If $k \in \{-1, 0\}$, then $\mu \in \Delta^+$.
The set of minimal elements is denoted by $\widehat{W}_{min}$. For any $\gamma \in I$, define the number $l(\gamma, I)$ as follows

$$l(\gamma, I) = \max\{m \mid \gamma = \sum_{i=1}^{m} \varkappa_i, \ \ \varkappa_i \in I\} .$$

2.5 Proposition [4, Prop. 2.12]. *There is a bijection between $\widehat{W}_{min}$ and $\mathfrak{Ad}$. Namely,*

- *given $w \in \widehat{W}_{min}$, the corresponding ideal is $\{\mu \in \Delta^+ \mid \delta - \mu \in N(w)\}$;*
- *given $I \in \mathfrak{Ad}$, the corresponding minimal element is determined by the finite bi-convex set*

$$\bigcup_{k \geq 1} (k\delta - I^k) = \{m\delta - \gamma \mid \gamma \in I \ \ \& \ \ 1 \leq m \leq l(\gamma, I)\} \subset \widehat{\Delta}^+ .$$

Here I^k is defined inductively by $I^k = (I^{k-1} + I) \cap \Delta^+$.

The minimal element corresponding to $I \in \mathfrak{Ad}$ is denoted by $w_{min}(I)$. Conversely, the first layer ideal of $w \in \widehat{W}_{min}$ (= the ideal corresponding to w) is denoted by I_w.
If $N \subset \widehat{\Delta}^+$ is an arbitrary finite convex subset, containing $\delta - I$, then it must also contain $N(w_{min}(I)) = \cup_{i \geq 1}(k\delta - I^k)$. So, the latter is the minimal bi-convex subset containing $\delta - I$.
We have also the following

2.6 Corollary. *For $w \in \widehat{W}_{min}$, we have $\ell(w) = \sum_{k \geq 1} \#((I_w)^k)$.*

In terms of minimal elements, one can give an explicit description of the generators of an ideal.

2.7 Proposition [11, Theorem 2.2] [15, Cor. 6.3(1)].
If $w \in \widehat{W}_{min}$, then $\Gamma(I_w) = \{\gamma \in \Delta^+ \mid w(\delta - \gamma) \in -\widehat{\Pi}\}$.

A geometric description of the minimal elements relates them to the integral points in a certain simplex.

2.8 Proposition [5, Prop. 2 & 3]. *Set $D_{min} = \{x \in V \mid (x, \alpha) \geq -1 \ \forall \alpha \in \Pi \ \ \& \ \ (x, \theta) \leq 2\}$. Then*

1. $w = v{\cdot}t_r \in \widehat{W}_{min} \iff \begin{cases} w \text{ is dominant}, \\ v(r) \in D_{min} \cap Q^\vee . \end{cases}$
2. *The mapping $\widehat{W}_{min} \to D_{min} \cap Q^\vee$, $w = v{\cdot}t_r \mapsto v(r)$, is a bijection.*

It follows that $\#(\mathfrak{Ad})$ equals the number of integral points in D_{min}. (Unless otherwise stated, an 'integral point' is a point lying in $Q^\vee$.) A pleasant feature of this situation is that there is an element of $\widehat{W}$ that takes D_{min} to a dilated closed fundamental alcove. Namely, $w(D_{min}) = (h+1)\overline{\mathcal{A}}$ for some $w \in \widehat{W}$, see [5, Thm. 1]. Write θ as a linear combination of simple roots: $\theta = \sum_i c_i\alpha_i$. The integers c_i are said to be the *coordinates of* θ. By a result of M. Haiman [8, 7.4], the number of integral points in $t\overline{\mathcal{A}}$ is equal to

$$\prod_{i=1}^{p} \frac{t+e_i}{1+e_i} \tag{2.9}$$

whenever t is relatively prime with all the coordinates of θ. Since this condition is satisfied for $t = h+1$, one obtains, see [5],

$$\#\widehat{W}_{min} = \#\mathfrak{Ad} = \prod_{i=1}^{p} \frac{h+e_i+1}{e_i+1}\ . \tag{2.10}$$

Maximal elements of $\widehat{W}$ are introduced by E. Sommers in [15]. He also obtained main results for these elements.

2.11 Definition. $w \in \widehat{W}$ is called *maximal*, if

(i) w is dominant;
(ii) if $\alpha \in \widehat{\Pi}$ and $w^{-1}(\alpha) = k\delta + \mu$ for some $\mu \in \Delta$, then $k \le 1$.

Using (i), condition (ii) can be made more precise. If $k = 1$, then $\mu \in -\Delta^+$; if $k = 0$, then $\mu \in \Delta^+$ The set of maximal elements is denoted by $\widehat{W}_{max}$.

If $I \in \mathfrak{Ad}_0$, then for any $\mu \in \Delta^+$ we define the number $k(\mu, I)$ as follows:

$$k(\mu, I) = \min\{n \mid \mu = \sum_{i=1}^{n} \nu_i, \quad \nu_i \in \Delta^+ \setminus I\}\ .$$

Notice that this definition makes sense only for strictly positive ideals.

2.12 Proposition [15, Sect. 5]. *There is a bijection between $\widehat{W}_{max}$ and $\mathfrak{Ad}_0$. Namely,*

- *given $w \in \widehat{W}_{min}$, the corresponding strictly positive ideal is $\{\mu \in \Delta^+ \mid \delta - \mu \in N(w)\}$;*
- *given $I \in \mathfrak{Ad}_0$, the corresponding maximal element is determined by the finite bi-convex set*

$$\{m\delta - \gamma \mid \gamma \in I \quad \& \quad 1 \le m \le k(\gamma, I) - 1\}. \tag{$\diamond$}$$

The maximal element corresponding to $I \in \mathfrak{Ad}_0$ is denoted by $w_{max}(I)$. Accordingly, the strictly positive ideal corresponding to $w \in \widehat{W}_{max}$ (the first layer ideal of w) is I_w. It follows from the previous discussion that to any $I \in \mathfrak{Ad}_0$ one can assign two elements of $\widehat{W}$; namely, $w_{min}(I)$ and $w_{max}(I)$.

Let $w \in \widehat{W}$ be any dominant element with first layer ideal $I \in \mathfrak{Ad}_0$. Since $\widehat{\Delta}^+ \setminus N(w)$ is convex and contains $\delta - (\Delta^+ \setminus I)$, it follows from the very definition of numbers $k(\gamma, I)$

that $m\delta-\gamma \in \widehat{\Delta}^+ \setminus N(w)$ for all $m \ge k(\gamma,I)$. Hence $N(w)$ is contained in the finite set given by Eq. 2.12($\diamond$). This shows that $N(w_{max}(I))$ is the maximal bi-convex subset with first layer $\delta - I$. Remember also that $N(w_{min}(I))$ is the minimal bi-convex subset with first layer $\delta - I$. Hence $N(w_{min}(I)) \subset N(w_{max}(I))$ and therefore

$$l(\gamma,I) \le k(\gamma,I)-1 \quad \text{for any } \gamma \in I\ . \tag{2.13}$$

Recall that $\Xi(I)$ is the set of maximal elements of $\Delta^+ \setminus I$. If $I \in \mathfrak{Ad}_0$, then a description of $\Xi(I)$ can be given in terms of $w_{max}(I)$:

2.14 Proposition [15, 6.3(2)]. *If* $w \in \widehat{W}_{max}$*, then* $\Xi(I_w) = \{\gamma \in \Delta^+ \mid w(\delta-\gamma) \in \widehat{\Pi}\}$.

Next, we recall a geometric characterization of the maximal elements.

2.15 Proposition (cf. [15, Prop. 5.6]). *Set* $D_{max} = \{x \in V \mid (x,\alpha) \le 1\ \forall \alpha \in \Pi\ \&\ (x,\theta) \ge 0\}$*. Then*

1. $w = v{\cdot}t_r \in \widehat{W}_{max} \iff \begin{cases} w \text{ is dominant}, \\ v(r) \in D_{max} \cap Q^\vee\ . \end{cases}$
2. *The mapping* $\widehat{W}_{max} \to D_{max} \cap Q^\vee$*,* $w = v{\cdot}t_r \mapsto v(r)$*, is a bijection.*

In order to compute $\#(D_{max} \cap Q^\vee)$, we replace D_{max} with another simplex. Let $\{\varpi_i^\vee\}_{i=1}^p$ denote the dual basis of V for $\{\alpha_i\}_{i=1}^p$. Set $\rho^\vee = \sum_{i=1}^p \varpi_i^\vee$. Since the sum of the coordinates of θ equals $h-1$, the translation $x \mapsto t_{-\rho^\vee} * x = x - \sum_{i=1}^p \varpi_i^\vee$ takes D_{max} to the negative dilated fundamental alcove

$$-(h-1)\overline{\mathcal{A}} = \{x \in V \mid (x,\alpha) \le 0 \quad \forall\, \alpha \in \Pi;\ (x,\theta) \ge 1-h\}.$$

It may happen that $\rho^\vee$ does not belong to $Q^\vee$, so that this translation, which is in the extended affine Weyl group, does not belong to $\widehat{W}$, while we wish to have a transformation from $\widehat{W}$. Nevertheless, since $h-1$ is relatively prime with the index of connection of Δ, it follows from [5, Lemma 1] that there is an element of $\widehat{W}$ that takes D_{max} to $(1-h)\overline{\mathcal{A}}$.

Using again the above-mentioned result of Haiman, see Eq. (2.9), one obtains, see [2], [11], [15],

$$\#\widehat{W}_{max} = \#(\mathfrak{Ad}_0) = \prod_{i=1}^{p} \frac{h+e_i-1}{e_i+1}\ . \tag{2.16}$$

Recall a bijection between $\mathfrak{Ad}$ and the dominant regions of the Catalan arrangement, which is due to Shi [14, Theorem 1.4].

For $\mu \in \Delta^+$ and $k \in \mathbb{Z}$, define the hyperplane $\mathcal{H}_{\mu,k}$ in V as $\{x \in V \mid (x,\mu) = k\}$. The *Catalan arrangement*, $\mathrm{Cat}(\Delta)$, is the collection of hyperplanes $\mathcal{H}_{\mu,k}$, where $\mu \in \Delta^+$ and $k = -1,0,1$. The *regions* of an arrangement are the connected components of the complement in V of the union of all its hyperplanes. Any region lying in $\mathcal{C}$ is said to be *dominant*. Obviously, the dominant regions of $\mathrm{Cat}(\Delta)$ are the same as those for the *Shi arrangement* $\mathrm{Shi}(\Delta)$. The latter is the collection of hyperplanes $\mathcal{H}_{\mu,k}$, where $\mu \in \Delta^+$ and $k = 0, 1$. But, it is sometimes more convenient to deal with the arrangement $\mathrm{Cat}(\Delta)$, since it is W-invariant.

The Shi bijection takes an ideal $I \subset \Delta^+$ to the dominant region

$$R_I = \{x \in \mathcal{C} \mid (x, \gamma) > 1, \text{ if } \gamma \in I \quad \& \quad (x, \gamma) < 1, \text{ if } \gamma \notin I\} . \tag{2.17}$$

A region of an arrangement is called *bounded*, if it is contained in a sphere about the origin.

2.18 Proposition [2],[11]. *$I \in \mathfrak{Ad}(\mathfrak{g})_0$ if and only if the region R_I is bounded.*

A relationship between the theory of minimal/maximal elements and regions of the Catalan arrangement is as follows.

2.19 Theorem.
(i) [5] *Suppose $I \in \mathfrak{Ad}$ and $w = w_{min}(I)$. Then $w^{-1} * A$ is the alcove closest to the origin in R_I;*
(ii) [15] *Suppose $I \in \mathfrak{Ad}_0$ and $w = w_{max}(I)$. Then $w^{-1} * A$ is the alcove most distant from the origin in R_I.*

3. Dominant elements and ideals of Heisenberg type

Given $w \in \widehat{W}$, write $w(\alpha_0) = -m\delta + \nu$, where $m \in \mathbb{Z}$ and $\nu \in \Delta$. Since $(\theta, \theta) = (\alpha_0, \alpha_0) = (\nu, \nu)$, the root ν is long. The root ν is called the *rootlet* of w, denoted $\mathrm{rt}(w)$. In what follows, Δ_l stands for the set of long roots. In general, the rootlet can be any root in Δ_l; but, for various classes of elements of $\widehat{W}$, different constraints emerge.

3.1 Lemma.
- (i) *If $1 \neq w \in \widehat{W}_{dom}$, then $m \geq 1$;*
- (ii) *If $w \in \widehat{W}_{min}$, then $\mathrm{rt}(w) \notin -\Pi$;*
- (iii) *If $w \in \widehat{W}_{max}$, then $\mathrm{rt}(w) \notin \Pi$.*

Proof. (i) Recall the "affine-linear decomposition" $w = \bar{w} \cdot t_r$, where $\bar{w} \in W$ and $r \in -\overline{\mathcal{C}}$. If $w \neq 1$, then $r \neq 0$, see Prop. 2.3. Hence $(\theta, r) < 0$. Furthermore, $-\overline{\mathcal{C}}$ contains no elements $x \in Q^\vee$ such that $(x, \theta) = -1$. For, Haiman's formula (2.9) shows that $\overline{A}$ contains a unique element of $Q^\vee$, namely, the origin. Hence $(r, \theta) \leq -2$. It remains to observe that $w(\alpha_0) = -\bar{w}(\theta) + (1 + (\theta, r))\delta$. Thus, $-m = 1 + (\theta, r) \leq -1$.
(ii) If $w(\alpha_0) = -m\delta - \alpha$, $\alpha \in \Pi$, then $w^{-1}(\alpha) = -(m+1)\delta + \theta$ and $-(m+1) \leq -2$. Hence $w \notin \widehat{W}_{min}$.
(iii) If $w(\alpha_0) = -m\delta + \alpha$, $\alpha \in \Pi$, then $w^{-1}(\alpha) = (m+1)\delta - \theta$ and $m + 1 \geq 2$. Hence $w \notin \widehat{W}_{max}$. □

Remark. Notice that the formulas for $w(\alpha_0)$ shows that $\mathrm{rt}(w) = -\bar{w}(\theta)$.

Because there is a bijection between the minimal elements of $\widehat{W}$ and the ideals of Δ^+, one may define the rootlet for any ideal. That is, given $w \in \widehat{W}_{min}$, we set $\mathrm{rt}(I_w) := \mathrm{rt}(w)$. It should be noticed that if $I \in \mathfrak{Ad}_0$, then the roots $\mathrm{rt}(w_{min}(I))$ and $\mathrm{rt}(w_{max}(I))$ are usually different. So, if we wish to get an unambiguous notion of the rootlet for any ideal, then the minimal elements have to be exploited.

Amongst all ideals of Δ^+, we distinguish the ideal consisting of all roots that are not orthogonal to θ. Set

$$\mathcal{H} = \{\gamma \in \Delta^+ \mid (\gamma, \theta) > 0\} .$$

The corresponding subspace of $[\mathfrak{b}, \mathfrak{b}]$ is the standard Heisenberg subalgebra, so that we say that $\mathcal{H}$ is the *Heisenberg ideal*. Notice that $\mathcal{H}^2 = \{\theta\}$ and $\mathcal{H}^3 = \varnothing$.

3.2 Definition. We say that $w \in \widehat{W}$ is *of Heisenberg type*, if $w = vs_0$ for some $v \in W$. An ideal is said to be *of Heisenberg type*, if it is contained in $\mathcal{H}$.

The term for elements of $\widehat{W}$ is explained by the following

3.3 Lemma. *If $w \in \widehat{W}_{dom}$ is of Heisenberg type, then the first layer ideal of w lies in $\mathcal{H}$.*

Proof. Recall that the first layer ideal of w consists of all $\gamma \in \Delta^+$ such that $w(\delta - \gamma) < 0$. If w is of the form vs_0 $(v \in W)$ and $(\gamma, \theta) = 0$, then $w(\delta - \gamma) = \delta - v(\gamma) > 0$. □

The main goal of this section is to give a characterization of the dominant elements of Heisenberg type, and then to describe all ideals in $\mathcal{H}$.

3.4 Proposition. *Let $w = vs_0 \in \widehat{W}$ be of Heisenberg type. Then*

(i) $w \in \widehat{W}_{dom} \Leftrightarrow v(\alpha) > 0$ *for any* $\alpha \in \Pi$ *such that* $(\alpha, \theta) = 0$*;*
(ii) $\mathrm{rt}(w) = v(\theta)$*;*
(iii) $w \in \widehat{W}_{min} \Leftrightarrow w \in \widehat{W}_{dom}$ *and* $v(\theta) \notin -\Pi$*;*
(iv) $w \in \widehat{W}_{max} \Leftrightarrow w \in \widehat{W}_{dom}$*,* $(v(\theta), \theta) \geq 0$*, and* $v(\theta) \notin \Pi$*.*

Proof. (i) The property of being dominant means $w(\alpha) > 0$ for all $\alpha \in \Pi$. If $(\alpha, \theta) > 0$, then $w(\alpha) = v(\delta - (\theta - \alpha)) = \delta - v(\theta - \alpha) > 0$, i.e., it is always satisfied. If $(\alpha, \theta) = 0$, then $w(\alpha) = v(\alpha)$. Hence the condition.

(ii) $vs_0(\alpha_0) = v(\theta) - \delta$.

(iii) & (iv). The conditions of minimality and maximality impose constraints on the coefficient of δ in $w^{-1}(\alpha)$, $\alpha \in \widehat{\Pi}$.

For $\alpha \in \Pi$, we have $w^{-1}(\alpha) = s_0 \cdot v^{-1}(\alpha)$. If $v^{-1}(\alpha) = \pm\theta$, then $s_0(\pm\theta) = \pm(2\delta - \theta)$, which is bad. More precisely, w is not maximal, if $v(\theta) = \alpha$; and w is not minimal, if $v(\theta) = -\alpha$. These are the only bad possibilities.

For α_0, we have $w^{-1}(\alpha_0) = s_0(\delta - v^{-1}(\theta)) = \delta - v^{-1}(\theta) - (\theta, v(\theta)^\vee)(\delta - \theta)$. It is easily seen that the only bad possibility is $(\theta, v(\theta)) < 0$, in which case w fails to be maximal. □

Thus, the conditions of maximality and minimality for the Heisenberg type elements are stated in terms of the rootlet.

Recall the following standard fact:

(3.5) *Suppose* $u, v \in W$*. Then* $\ell(u)+\ell(v)=\ell(uv)$ *if and only if* $N(u)\cap N(v^{-1})=\varnothing$*. If these conditions are satisfied, then* $N(uv)=N(v)\sqcup v^{-1}(N(u))$*.*

If $\ell(u)+\ell(v)=\ell(uv)$, then we say that uv is a *reduced decomposition* for this product. We will also repeatedly use the following result, see [10, Theorem 4.1]:

(3.6) *Given* $\nu\in\Delta_l^+$*, there is a unique shortest element in* W *taking* θ *to* ν*, denoted by* w_ν*. We have* $N(w_\nu^{-1})=\{\gamma\in\Delta^+ \mid (\gamma,\nu^\vee)=-1\}$*.*

Given $\nu\in\Delta^+$, write s_ν for the corresponding reflection in W.

3.7 Lemma. *Suppose* $\nu\in\Delta_l^+$*. Then*

(i) $\ell(s_\nu)=2(\rho,\nu^\vee)-1$*;*
(ii) $\ell(s_\nu w_\nu)=\ell(s_\nu)+\ell(w_\nu)=(\rho,\theta^\vee+\nu^\vee)-1$*.*
(iii) $N(s_\nu)=\{\nu\}\cup\{\gamma\in\Delta^+ \mid (\gamma,\nu^\vee)=1\ \ \&\ \ \gamma\prec\nu\}$*.*

Proof. (i) We have $s_\nu(-\nu)=\nu$. Recall that $(\rho,\alpha^\vee)=1$ for all $\alpha\in\Pi$. We will argue by induction on $(\rho,\nu^\vee)$. If $\nu\in\Pi$, then $\ell(s_\nu)=1$, and the claim is true.
A straightforward calculation shows that $(\rho,s_\alpha(\mu)^\vee)=(\rho,\mu^\vee)-(\alpha,\mu^\vee)$ for $\mu\in\Delta$ and $\alpha\in\Pi$. If μ is long, we have $(\alpha,\mu^\vee)\ge -1$ unless $\mu=-\alpha$. Hence $(\rho,s_\alpha(\mu)^\vee)\le(\rho,\mu^\vee)+1$, if $\mu\ne-\alpha$, and $(\rho,s_\alpha(-\alpha)^\vee)=(\rho,-\alpha^\vee)+2$. Let $s_\nu=s_{i_1}\dots s_{i_k}$ be a reduced decomposition. Consider the corresponding sequence of roots:

$$(-\nu)\overset{s_{i_k}}{\to}\circ\overset{s_{i_{k-1}}}{\to}\circ\dots\overset{s_{i_2}}{\to}\circ\overset{s_{i_1}}{\to}\nu\ .$$

The level function $\mu\mapsto(\rho,\mu^\vee)$ attains $2(\rho,\nu^\vee)$ values between $-(\rho,\nu^\vee)$ and $(\rho,\nu^\vee)$, because the zero level is missing. Moreover, it follows from the previous inequalities that at each step one can go up at most to the next existing level. Hence, $\ell(s_\nu)\ge 2(\rho,\nu^\vee)-1$. On the other hand, $s_\nu=s_\alpha s_\gamma s_\alpha$, where $\gamma=s_\alpha(\nu)$. Because $\nu\notin\Pi$, one can find an $\alpha\in\Pi$ so that $(\alpha,\nu^\vee)=1$. Then we obtain $(\rho,\gamma^\vee)=(\rho,\nu^\vee)-1$ and by the induction assumption $\ell(s_\nu)\le 2+\ell(s_\gamma)=2(\rho,\nu^\vee)-1$. This completes the inductive step.

(ii) The first equality follows from Eq. (3.5). Indeed, $N(w_\nu^{-1})=\{\gamma\mid(\gamma,\nu^\vee)=-1\}$ and for such γ we have $s_\nu(\gamma)=\gamma+\nu>0$. Hence $N(w_\nu^{-1})\cap N(s_\nu)=\varnothing$. The second equality follows from (i) and the fact that $\ell(w_\nu)=(\rho,\theta^\vee-\nu^\vee)$, see [10, Theorem 4.2].

(iii) Obvious. □

In what follows, we write $N(s_\nu)^0$ for $\{\gamma\in\Delta^+ \mid (\gamma,\nu^\vee)=1\ \ \&\ \ \gamma\prec\nu\}$.

3.8 Proposition. *Suppose* $\nu\in\Delta_l^+$*. Then*

(i) $w_\nu s_0$ *is a minimal element with rootlet* ν*, and* $I_{w_\nu s_0}$ *is Abelian. The element* $w_\nu s_0$ *is also maximal, if* $\nu\notin\Pi$*.*
(ii) $s_\nu w_\nu s_0$ *is dominant and* $\mathrm{rt}(s_\nu w_\nu s_0)=-\nu$*. Next,* $s_\nu w_\nu s_0\in\widehat{W}_{min}$ *if and only if* $\nu\notin\Pi$*;* $s_\nu w_\nu s_0\in\widehat{W}_{max}$ *if and only if* $(\theta,\nu)=0$*.*
(iii) *If* $\nu\notin\Pi$*, then the ideal* $I_{s_\nu w_\nu s_0}$ *is not Abelian.*
(iv) *If* $\nu\in\Pi$*, then* $I_{s_\nu w_\nu s_0}=I_{w_\nu s_0}$ *is Abelian.*

Proof. (i) Since $w_\nu s_0(\alpha_0)+\delta=\nu$, we obtain $\mathrm{rt}(w_\nu s_0)=\nu$. That $w_\nu s_0$ is minimal (in particular, dominant) and the ideal $I_{w_\nu s_0}$ is Abelian is shown in [10, Theorem 4.2]. The assertion on maximality stems from Proposition 3.4.

(ii) Set $w=s_\nu w_\nu s_0$. From the very definition of w_ν, it follows that $w(\alpha_0)+\delta=-\nu$. Hence the rootlet is as required.

In order to prove that w is dominant, we apply Proposition 3.4(i). Here $v=s_\nu w_\nu$. If $\alpha\in\Pi$ and $(\theta,\alpha)=0$, then $s_\nu w_\nu(\alpha)=w_\nu(\alpha)-(w_\nu(\alpha),\nu^\vee)\nu=w_\nu(\alpha)>0$. The last inequality follows from the fact that, by part (i), $w_\nu s_0$ is dominant. It is also not hard to give a direct argument. (Indeed, if $w_\nu(\alpha)=-\mu<0$, then $\mu\in N(w_\nu^{-1})$. Therefore $(\mu,\nu^\vee)=-1$ by Eq. (3.6). That is, $1=(w_\nu(\alpha),\nu^\vee)=(\alpha,\theta^\vee)$, a contradiction!

The assertions on maximality and minimality follows from Proposition 3.4(iii),(iv).

(iii) Notice that $s_\nu w_\nu s_0(2\delta-\theta)=-\nu<0$. Since $s_\nu w_\nu s_0$ is minimal for $\nu\not\in\Pi$, it follows from Proposition 2.5 that $\theta\in I^2$.

(iv) By Eq. (3.5) and Lemma 3.7(ii), we have $N(s_\nu w_\nu s_0)=N(w_\nu s_0)\cup s_0 w_\nu^{-1}(N(s_\nu))$. If $\nu\in\Pi$, then $N(s_\nu)=\{\nu\}$, and the second set equals $\{2\delta-\theta\}$. Thus, $N(s_\nu w_\nu s_0)$ and $N(w_\nu s_0)$ have the same first layers. Since $I_{w_\nu s_0}$ is Abelian and $N(w_\nu s_0)=\delta-I_{w_\nu s_0}$, we are done. □

The next assertion yields a converse to Proposition 3.8, and completes our characterization of the dominant elements of Heisenberg type.

3.9 Proposition. *Suppose $w=vs_0\in\widehat{W}_{dom}$, where $v\in W$. Set* $\nu=\begin{cases} v(\theta), & \textit{if } v(\theta)>0\\ -v(\theta), & \textit{if } v(\theta)<0\end{cases}$. *Then $v=w_\nu$, if $v(\theta)>0$; and $v=s_\nu w_\nu$, if $v(\theta)=-\nu<0$.*

Proof. 1. Suppose $\nu=v(\theta)>0$. Then $(\gamma,\nu^\vee)=(v^{-1}(\gamma),\theta^\vee)$. Therefore, if $(\gamma,\nu^\vee)=-1$, then $v^{-1}(\gamma)<0$. In view of Eq. (3.6), this means that $N(v^{-1})\supset N(w_\nu^{-1})$. It follows that there is a "reduced decomposition" $v^{-1}=\kappa w_\nu^{-1}$ (i.e., $\ell(v)=\ell(\kappa)+\ell(w_\nu)$). Then $v=w_\nu\kappa^{-1}$ and $\kappa^{-1}(\theta)=\theta$. This means that κ^{-1} takes the (possibly reducible) root system $\Delta\cap\langle\theta\rangle^\perp$ to itself. Therefore, if $\kappa\ne\mathrm{id}$, then there is an $\alpha\in\Pi\cap\langle\theta\rangle^\perp$ such that $\kappa^{-1}(\alpha)<0$. Since $v=w_\nu\kappa^{-1}$ is a reduced decomposition and therefore $N(v)\supset N(\kappa^{-1})$, we have $v(\alpha)<0$ as well. But Proposition 3.4(i) says that $v(\alpha)$ must be positive here. This contradiction shows that $\kappa=\mathrm{id}$, and we are done.

2. Suppose $-\nu=v(\theta)<0$. We use the same idea as in part 1. If $N(v^{-1})\supset N(w_\nu^{-1}s_\nu)$, then we will obtain a reduced decomposition of the form $v^{-1}=\kappa w_\nu^{-1}s_\nu$, with $\kappa^{-1}(\theta)=\theta$, and eventually prove that $\kappa=\mathrm{id}$. So, it suffices to establish the above containment. By Lemma 3.7(ii) and Eq.(3.5), we have

$$N(w_\nu^{-1}s_\nu)=N(s_\nu)\cup s_\nu{\cdot}N(w_\nu^{-1})=\{\nu\}\cup\{\gamma\in\Delta^+\mid(\gamma,\nu^\vee)=1\ \&\ \gamma\prec\nu\}\cup s_\nu{\cdot}N(w_\nu^{-1})\,.$$

Let us check that the three sets in the RHS are contained in $N(v^{-1})$.

- $v^{-1}(\nu)=-\theta<0$;
- If $(\gamma,\nu^\vee)=1$, then $1=(v^{-1}(\gamma),-\theta^\vee)$. Hence $v^{-1}(\gamma)<0$;

- If $\gamma \in N(w_\nu^{-1})$, then $v^{-1}s_\nu(\gamma),\theta^\vee) = (s_\nu(\gamma, -\nu^\vee) = (\gamma,\nu^\vee) = -1$. Hence $v^{-1}s_\nu(\gamma) < 0$.

□

Combining Propositions 3.8 and 3.9, we obtain

3.10 Theorem. *Suppose $w = vs_0$, where $v \in W$. Then $w \in \widehat{W}_{dom}$ if and only if v is either w_ν or $s_\nu w_\nu$ for some $\nu \in \Delta_l^+$. Furthermore, $\mathrm{rt}(w)$ is ν (resp. $-\nu$) in the first (resp. second) case.*

It follows that the number of dominant elements of Heisenberg type is equal to $\#(\Delta_l)$. Using Proposition 3.8(ii), we also conclude that the number of minimal elements of Heisenberg type is equal to $\#(\Delta_l \setminus \Pi)$, and the number of maximal elements of Heisenberg type is equal to $\#(\Delta_l^+ \setminus \Pi) + \#(\{\gamma \in \Delta_l^+ \mid (\gamma,\theta) = 0\})$. However, it is not yet clear that if I is a non-trivial ideal (resp. strictly positive ideal) in $\mathcal{H}$, then the corresponding minimal (resp. maximal) element is necessarily of Heisenberg type. Actually, this appears to be true for minimal elements, but not for maximal.

3.11 Proposition. *If $w \in \widehat{W}_{min}$ and $I_w \subset \mathcal{H}$, then w is of Heisenberg type.*

Proof. It is straightforward to verify that $s_\theta s_0 \in \widehat{W}_{min}$ and $I_{s_\theta s_0} = \mathcal{H}$ (cf. [11, Example 2.6]). It follows that $N(w) \subset N(s_\theta s_0)$. Therefore $s_\theta s_0 = \tilde{w}w$ is a reduced decomposition. Since $w = w's_0$ for some $w' \in \widehat{W}$, we conclude that $s_\theta = \tilde{w}w'$ is a reduced decomposition for s_θ. Whence $\tilde{w}, w' \in W$. □

Remark. It happens quite often that $w \in \widehat{W}_{max}$ and $I_w \subset \mathcal{H}$, but w is not of Heisenberg type. A "uniform" example can be described as follows. Suppose θ is a fundamental weight and ν is the unique simple root such that $(\theta,\nu) > 0$. According to Proposition 3.8(ii), $s_\nu w_\nu s_0$ is dominant but neither maximal nor minimal. Letting $I = I_{s_\nu w_\nu s_0}$, one can show that $w_{min}(I) = w_\nu s_0$ (this follows from Proposition 3.8(iv)) and $w_{max}(I) = s_0 s_\nu w_\nu s_0$ (a straightforward computation). That is, $w_{max}(I)$ is not of Heisenberg type.

Thus, we have obtained a bijection between

- the roots in $\Delta_l^+ \setminus (-\Pi)$;
- the minimal elements of Heisenberg type;
- the non-trivial ideals lying in $\mathcal{H}$.

Under this bijection, the roots in $\Delta_l^+ \setminus (-\Pi)$ are obtained as the rootlets of the corresponding ideals. In particular, different ideals of Heisenberg type have different rootlets.

An explicit construction of the Heisenberg-type elements is given in Proposition 3.8. It is also possible to explicitly describe the corresponding ideals.

3.12 Theorem. *Suppose $\nu \in \Delta_l^+$. Then*

(i) $I_{w_\nu s_0} = \{\theta\} \cup \{\theta + w_\nu^{-1}(N(w_\nu^{-1})\} = \{\theta\} \cup \{\theta - N(w_\nu)\}$;

(ii) $I_{s_\nu w_\nu s_0} = \{\theta\} \cup \{\theta - N(w_\nu)\} \cup \{\theta - w_\nu^{-1}(N(s_\nu)^0)\} = I_{w_\nu s_0} \cup \{\theta - w_\nu^{-1}(N(s_\nu)^0)\}$, *if* $\nu \notin \Pi$.

Proof. (i) Since $I_{w_\nu s_0}$ is Abelian, $\#(I_{w_\nu s_0}) = \ell(w_\nu s_0) = (\rho, \theta^\vee - \nu^\vee) + 1$. Since the set in the right-hand side has the required cardinality, it suffices to verify that it is contained in $I_{w_\nu s_0}$. Observe that $(\gamma, \theta^\vee) = (w_\nu(\gamma), \nu^\vee) = -1$ for any $\gamma \in w_\nu^{-1}(N(w_\nu^{-1})$. Therefore we know how s_0 acts on $w_\nu^{-1}(N(w_\nu^{-1}))$. Then

$$w_\nu s_0(\delta - \theta - w_\nu^{-1}(N(w_\nu^{-1})) = w_\nu(-w_\nu^{-1}(N(w_\nu^{-1})) = -N(w_\nu^{-1}) \subset -\Delta^+ .$$

Also, $w_\nu s_0(\delta - \theta) = \nu - \delta < 0$.

(ii) Remember that $\mathcal{H}^2 = \{\theta\}$. Since $I := I_{s_\nu w_\nu s_0} \subset \mathcal{H}$ and is not Abelian by Proposition 3.8(iii), we have $I^2 = \{\theta\}$ as well. Therefore, using Corollary 2.6, we obtain $\#(I) = \ell(s_\nu w_\nu s_0) - 1 = \ell(s_\nu w_\nu) = (\rho, \theta^\vee + \nu^\vee) - 1$. Here, again, the set in the right-hand side has the prescribed cardinality, Hence it remains to show that it is contained in I. First, it follows from (3.5) that $I_{w_\nu s_0} \subset I_{s_\nu w_\nu s_0}$. Next, we have $(w_\nu^{-1}(N(s_\nu)^0), \theta^\vee) = (N(s_\nu)^0, \nu^\vee) = 1$. Therefore, we know how s_0 acts on $w_\nu^{-1}(N(s_\nu)^0)$, and hence we can compute that all roots in $s_\nu w_\nu s_0(\delta - \theta + w_\nu^{-1}(N(s_\nu)^0))$ are negative. □

Remark. In principle, it is harmless to omit the hypothesis in part (ii) that $\nu \notin \Pi$. For, $N(s_\nu)^0 = \varnothing$, if $\nu \in \Pi$. Then part (ii) would assert in this case that $I_{s_\nu w_\nu s_0} = I_{w_\nu s_0}$, which was already proved in Proposition 3.8(iv).

4. Minimax elements and ideals: general properties

An element $w \in \widehat{W}$ is called *minimax*, if it is simultaneously minimal and maximal. Combining the corresponding definitions of Section 2, we obtain the following:

4.1 Definition. $w \in \widehat{W}$ is called *minimax*, if

(i) w is dominant;
(ii) if $\alpha \in \widehat{\Pi}$ and $w^{-1}(\alpha) = k\delta + \mu$ for some $\mu \in \Delta$, then $-1 \le k \le 1$.

The corresponding (strictly positive) ideal is said to be *minimax*, too. That is, $I \in \mathfrak{Ad}_0$ is minimax, if $w_{min}(I) = w_{max}(I)$. Therefore we merely write $w(I)$, if I is minimax. The set of all minimax elements, which is clearly equal to $\widehat{W}_{min} \cap \widehat{W}_{max}$, is denoted by $\widehat{W}_{mm}$. Accordingly, $\mathfrak{Ad}_{mm}$ is the set of minimax ideals in Δ^+.

Combining geometric descriptions of minimal and maximal elements given in Section 2, we obtain the following.

4.2 Proposition.
Set $D_{mm} = D_{min} \cap D_{max} = \{x \in V \mid -1 \le (x, \alpha) \le 1 \ \ \forall \alpha \in \Pi \ \ \& \ \ 0 \le (x, \theta) \le 2\}$. *Then*

1. $w = v{\cdot}t_r \in \widehat{W}_{mm} \iff \begin{cases} w \text{ is dominant}, \\ v(r) \in D_{mm} \cap Q^\vee . \end{cases}$
2. *The mapping* $\widehat{W}_{mm} \to D_{mm} \cap Q^\vee$, $w = v{\cdot}t_r \mapsto v(r)$, *is a bijection.*

It can be shown that there is an element of $\widehat{W}$ that takes D_{mm} to

$$\{x \in V \mid 0 \le (x, \alpha) \le 2 \ \ \forall \alpha \in \Pi \ \ \& \ \ h - 1 \le (x, \theta) \le h + 1\} .$$

But, I do not see how it could help us to count the integral points in D_{mm}.
Recall that, for any strictly positive ideal $I \subset \Delta^+$ and any $\gamma \in I$, we have defined the numbers

$$l(\gamma, I) = \max\{m \mid \gamma = \sum_{i=1}^{m} \varkappa_i, \ \ \varkappa_i \in I\} , \quad k(\gamma, I) = \min\{n \mid \gamma = \sum_{i=1}^{n} \nu_i, \ \ \nu_i \in \Delta^+ \setminus I\} .$$

Using results of Section 2 we immediately obtain various reformulations of the minimax property.

4.3 Theorem. *Suppose I is a strictly positive ideal and w is either $w_{min}(I)$ or $w_{max}(I)$. Then the following conditions are equivalent:*

(i) $w \in \widehat{W}_{mm}$ *;*
(ii) $k(\gamma, I) - 1 = l(\gamma, I)$ *for all* $\gamma \in I$*;*
(iii) $R_I = w^{-1} {*} \mathcal{A}$*;*
(iv) R_I *consists of a single alcove.*

Proof. Use Eq. (2.13) and Theorem 2.19. □

Thus, the minimax elements (ideals) are in a bijection with the dominant regions of the Catalan arrangement consisting of a single alcove.
Now, we give a description of minimax Abelian ideals. Let $I \subset \Delta^+$ be a nontrivial Abelian ideal and $w = w_{min}(I)$ the corresponding minimal element. It was shown in [10, Prop. 2.5] that $w(\alpha_0) = -\delta + \nu$, where $\nu \in \Delta_l^+$. In particular, $\mathrm{rt}(w) = \mathrm{rt}(I)$ is a positive root.

4.4 Theorem. *An Abelian ideal I is minimax if and only if $\mathrm{rt}(I)$ is not a simple root.*

Proof. 1. Assume $w(\alpha_0) + \delta = \alpha_i \in \Pi$. Then $w^{-1}(\alpha_i) = 2\delta - \theta$, i.e., w is not maximal.

2. Since I is Abelian, $l(\gamma, I) = 1$ for all $\gamma \in I$. Therefore, in view of Theorem 4.3(ii), the minimax property is equivalent to that any $\gamma \in I$ is a sum of two elements of $\Delta^+ \setminus I$. Assume $w(\alpha_0) + \delta = \mu \notin \Pi$. Then $\mu = \mu_1 + \mu_2$, where $\mu_i \in \Delta^+$. Hence $w^{-1}(\mu_i) \in \widehat{\Delta}^+$ and $2\delta - \theta = w^{-1}(\mu_1) + w^{-1}(\mu_2)$ is a sum of two positive roots. The only possibility for this is $\theta = \gamma_1 + \gamma_2$ and $w^{-1}(\mu_i) = \delta - \gamma_i$. Since $w(\delta - \gamma_i) = \mu_i \succ 0$, we have $\gamma_i \notin I$. Thus, the required property is satisfied for $\theta \in I$. To prove this for any $\mu \in I$, we use a descending induction. Indeed, if $\theta - \alpha \in I$ for some $\alpha \in \Pi$, then the condition $\gamma_1 + \gamma_2 - \alpha \in I \subset \Delta^+$ implies that $\alpha \neq \gamma_i$ and at least one of $\gamma_i - \alpha$ is in Δ^+ (and hence in $\Delta^+ \setminus I$), see [11, Lemma 2.3]. Hence $\theta - \alpha$ is a sum of two elements from $\Delta^+ \setminus I$. This can be continued further. □

It should already be clear that a minimax ideal is not necessarily Abelian. The simplest example is the ideal in $\mathfrak{sl}_5$ generated by $\{\alpha_1+\alpha_2, \ \alpha_3+\alpha_4\}$. More precisely, applying results of Section 3 to minimax elements, we obtain

4.5 Proposition. *A Heisenberg type element $w = vs_0$ is minimax if and only if either $v = w_\nu$, where $\nu \in \Delta_l^+ \setminus \Pi$, or $v = s_\nu w_\nu$, where $\nu \in \Delta_l^+ \setminus \Pi$ and $(\theta, \nu) = 0$. The minimax ideals of the second kind are not Abelian.*

It is natural to look at Δ_{mm}, the set of rootlets of all minimax elements (ideals). Clearly, we have the inclusion $\{\nu \in \Delta_l \mid (\nu, \theta) \ge 0\} \setminus \{\pm\Pi\} \subseteq \Delta_{mm}$. But, in general, it is a strict containment. This can be expressed in the following way. Consider the mapping $\zeta : \mathfrak{A}\mathfrak{d}_{mm} \to \Delta_{mm}$ that associates the rootlet to a minimax ideal. Then, in general, there exist fibres of ζ that do not contain a Heisenberg type ideal. Here is an example. Let $\mathfrak{g} = \mathfrak{sl}_6$. Take the ideal generated by the roots $\alpha_1+\alpha_2,\ \alpha_3+\alpha_4$. A straightforward computation shows that it is minimax and the rootlet is $\nu = -(\alpha_1+\alpha_2+\alpha_3)$. Thus, ν is negative and $(\nu, \theta) \ne 0$.

It is intuitively clear that if a (dominant) region of $\mathrm{Cat}(\Delta)$ consists of a single alcove, then this region must be sufficiently close to the origin. Equivalently, the corresponding ideal must be not too large. We now discuss a precise meaning that can be given to this intuitive feeling. Given $I \in \mathfrak{A}\mathfrak{d}$, consider the quantity $\#\Gamma(I) + \#\Xi(I)$. It is not hard to realize that its maximal value is $2p - 1$, which is attained for $I = \Delta^+ \setminus \Pi$.

4.6 Proposition. *Suppose $I \in \mathfrak{A}\mathfrak{d}_{mm}$; then*

(i) $\#\Gamma(I) + \#\Xi(I) \le p + 1$;
(ii) *If $\#\Gamma(I) + \#\Xi(I) = p + 1$, then h is odd.*

Proof. (i) Set $w = w(I)$ and write $w^{-1}(\alpha) = k_\alpha\alpha+\nu_\alpha$. Since w is simultaneously minimal and maximal, $k_\alpha \in \{-1, 0, 1\}$ for any $\alpha \in \widehat{\Pi}$. (Notice that $\nu_\alpha \in \Delta^+$, if $k_\alpha = -1$; $\nu_\alpha \in -\Delta^+$, if $k_\alpha = 1$.) Making use of Propositions 2.7 and 2.14 we obtain

$$\gamma \in \Gamma(I) \iff \text{there is an } \alpha \in \widehat{\Pi} \text{ such that } w^{-1}(\alpha) = -\delta + \gamma,$$
$$\mu \in \Xi(I) \iff \text{there is an } \alpha \in \widehat{\Pi} \text{ such that } w^{-1}(\alpha) = \delta - \mu.$$

This means that the elements of $\Gamma(I)$ (resp. $\Xi(I)$) are in a bijection with the affine simple roots such that $k_\alpha = -1$ (resp. $k_\alpha = 1$). Since $\#\widehat{\Pi} = p + 1$, we are done.

(ii) It follows from the previous part of proof that if $\#\Gamma(I) + \#\Xi(I) = p+1$, then $k_\alpha \ne 0$ for all α. Therefore we obtain the partition $\widehat{\Pi} = \widehat{\Pi}_+ \sqcup \widehat{\Pi}_-$, according to whether $k_\alpha = +1$ or -1. Recall that the coefficients c_α ($\alpha \in \Pi$) are defined by the equality $\theta = \sum_{\alpha\in\Pi} c_\alpha\alpha$. We also set $c_\alpha = 1$ for $\alpha = \alpha_0$. Then

$$\delta = \sum_{\alpha\in\widehat{\Pi}} c_\alpha\alpha \quad \text{and} \quad \sum_{\alpha\in\widehat{\Pi}} c_\alpha = h\ .$$

Because also $\delta = \sum c_\alpha w^{-1}(\alpha)$, we obtain

$$1 = \sum_{\alpha\in\widehat{\Pi}_+} c_\alpha - \sum_{\alpha\in\widehat{\Pi}_-} c_\alpha = h - 2 \sum_{\alpha\in\widehat{\Pi}_-} c_\alpha\ .$$

That is, h is odd. $\square$

As is well-known, h is odd if and only if $\mathfrak{g} = \mathfrak{sl}_{2n+1}$. The following example shows that the above upper bound is exact. Consider the ideal in $\mathfrak{sl}_{2n+1}$ such that

$$\Gamma(I) = \{\alpha_1+\alpha_2, \alpha_3+\alpha_4, \dots, \alpha_{2n-1}+\alpha_{2n}\} .$$

Then

$$\Xi(I) = \{\alpha_1, \alpha_2+\alpha_3, \dots, \alpha_{2n-2}+\alpha_{2n-1}, \alpha_{2n}\} .$$

It follows from the description of minimax ideals in type $\mathbf{A}_p$, see Corollary 6.5, that this ideal is minimax. For all other simple Lie algebras, it is possible to exhibit a minimax ideal such that $\#\Gamma(I) + \#\Xi(I) = p$.

5. Counting minimax elements/ideals in the simple Lie algebras

By Proposition 4.2, the number of minimax elements of $\widehat{W}$ is equal to the cardinality of $D_{mm} \cap Q^\vee$. Unfortunately, D_{mm} is not a simplex, so that it is not easy to find a uniform expression for $\#(D_{mm} \cap Q^\vee)$. However, for each simple algebra one has a certain system of inequalities and one may try to solve these systems individually. It turns out that, for practical computations, it is easier to deal with the coweight lattice in V, denoted $P^\vee$. The number $[P^\vee : Q^\vee]$ is called the *index of connection* of Δ.

Recall from Section 2 that $\{\varpi_i^\vee\}_{i=1}^p$ is the dual basis of V for $\{\alpha_i\}_{i=1}^p$. Then the lattice generated by the ϖ_i's is $P^\vee$. If $y = \sum_i y_i \varpi_i \in P^\vee$, then $y \in Q^\vee$ if and only if a certain congruence condition is satisfied for $(y_1, \dots, y_p) \in \mathbb{Z}^p$. Recall that $\theta = \sum_{i=1}^p c_i\alpha_i$. The equations of D_{mm} in Proposition 4.2 show that $\#(D_{mm} \cap P^\vee)$ equals the number of solutions of the following system:

$$\text{(5.1)} \qquad \left\{ \begin{array}{l} y_i \in \{-1, 0, 1\} \ \ (i = 1, 2, \dots, p) \\ \ \ 0 \le c_1y_1 + \ldots + c_py_p \le 2 \ . \end{array} \right.$$

It is convenient to introduce one more variable, y_0, which also ranges over $\{-1, 0, 1\}$, and add it to the expression in the second inequality, so that the total sum to be equal to 1. (We also set $c_0 = 1$.) Then the above system becomes

$$\text{(5.2)} \qquad \left\{ \begin{array}{l} y_i \in \{-1, 0, 1\} \ \ (i = 0, 1, \dots, p) \\ \ c_0y_0 + c_1y_1 + \ldots + c_py_p = 1 \ . \end{array} \right.$$

In a sense, this procedure corresponds to taking the extended Dynkin diagram of $\mathfrak{g}$. For this reason, system (5.2) will be referred to as the *extended* system.

It is easily seen that the number of solutions of system (5.2) is nothing but the coefficient of x in the expansion of the Laurent polynomial

$$\text{(5.3)} \qquad \prod_{i=0}^{p}(x^{-c_i} + 1 + x^{c_i}) .$$

An important particular case is that where all $c_i = 1$. The coefficients in the expansion of $(x^{-1} + 1 + x)^n$ are called *trinomials*, and we write $X_k(n)$ for the coefficient of x^k. The coefficient of x^0 (resp. x) is called the *central* (resp. *next-to-central*) trinomial. It is easily seen that

$$X_k(n) := \sum_{l\ge 0} \frac{n!}{l!(k+l)!(n-2l-k)!} \ .$$

The trinomial coefficients satisfy the following relation

$$X_k(n{+}1) = X_{k-1}(n) + X_k(n) + x_{k+1}(n) \ . \tag{5.4}$$

To get the number $\#(D_{mm} \cap Q^\vee)$, one should add to both systems the above-mentioned congruence condition. Recall that $[P^\vee : Q^\vee]$ is equal to the number of 1's among the coefficients c_i, $i = 0, 1, \dots, p$. Our first goal is to show that the congruence condition can be omitted.

5.5 Theorem. $\#(D_{mm} \cap P^\vee) = [P^\vee : Q^\vee] \cdot \#(D_{mm} \cap Q^\vee)$.

Proof. The only proof I know is case-by case (see also Question 4 in Section 7). In considering particular cases, our numbering of α_i's and hence of c_i's correspond to the numbering adopted in [17, Tables]. The explicit form of the congruence condition can be extracted from Table 3 in loc. cit.

1) $\mathfrak{g} = \mathfrak{sl}_{n+1}$. Here all $c_i = 1$ and $[P^\vee : Q^\vee] = n + 1$. The extended system is

$$\begin{cases} y_0 + y_1 + \ldots + y_n = 1 \\ \quad y_i \in \{-1, 0, 1\} \ , \end{cases} \tag{5.6}$$

and the congruence is $ny_1 + (n-1)y_2 + \ldots + y_n \in (n+1)\mathbb{Z}$. Obviously, the symmetric group $\mathbb{S}_{n+1}$ acts on the set of solutions of system (5.6). Let $\tilde c \in \mathbb{S}_{n+1}$ be the transformation $y_i \mapsto y_{i-1}$ $(i > 0)$, $y_0 \mapsto y_n$. (It is a Coxeter element of $\mathbb{S}_{n+1}$.) It is easily seen that all orbits of $\tilde c$ on the set of solutions of Eq. (5.6) have cardinality $n + 1$. Furthermore, making use of the relation $y_0 + y_1 + \ldots + y_n = 1$, one easily verifies that

$$\tilde c(ny_1 + (n-1)y_2 + \ldots + y_n) - (ny_1 + (n-1)y_2 + \ldots + y_n) \equiv 1 \pmod{n+1} \ .$$

This yields the desired relation.

2) $\mathfrak{g} = \mathfrak{sp}_{2n}$. Here $[P^\vee : Q^\vee] = 2$, the extended system is

$$\begin{cases} y_0 + 2(y_1 + \ldots + y_{n-1}) + y_n = 1 \\ \quad y_i \in \{-1, 0, 1\} \ , \end{cases} \tag{5.7}$$

and the congruence is $y_n \in 2\mathbb{Z}$. It follows that $y_0 + y_n$ is always odd for a solution of (5.7) and that the permutation of y_0 and y_n takes the solutions with y_n even to those with y_n odd. Hence the result.

3) $\mathfrak{g} = \mathfrak{so}_{2n+1}$. Here $[P^\vee : Q^\vee] = 2$, the extended system is

$$\begin{cases} y_0 + y_1 + 2(y_2 + \ldots + y_n) = 1 \\ \quad y_i \in \{-1, 0, 1\} \ , \end{cases} \tag{5.8}$$

and the congruence is $y_1 + y_3 + y_5 + \ldots \in 2\mathbb{Z}$. Here one may use the permutation of y_0 and y_1, as in part 2).

4) $\mathfrak{g} = \mathfrak{so}_{2n}$, $n \geq 4$. Now $[P^\vee : Q^\vee] = 4$ and the extended system is

$$(5.9) \qquad \begin{cases} y_0 + y_1 + 2(y_2 + \ldots + y_{n-2}) + y_{n-1} + y_n = 1 \\ y_i \in \{-1, 0, 1\} . \end{cases}$$

But the congruence condition depends on the parity of n.

(A) $n = 2l$. Here we actually have two conditions:

$$(5.10) \qquad \begin{cases} y_{n-1} + y_n \in 2\mathbb{Z} \\ y_1 + y_3 + \ldots + y_{n-1} \in 2\mathbb{Z} \end{cases} .$$

Here we consider the cyclic permutation $\tilde{c} : \ y_0 \to y_1 \to y_{n-1} \to y_n \to y_0$. Since the sum $y_0 + y_1 + y_{n-1} + y_n$ is always odd, all orbits of this permutation on the set of solutions of system (5.9) have cardinality 4. It is not hard to verify that each orbit contains a unique representative satisfying (5.10). Let us give some details. It follows from system (5.9) that $-1 \leq y_2 + \ldots + y_{n-2} \leq 2$. That is, there are four cases:

(a_1) $y_2 + \ldots + y_{n-2} = -1$ and hence $y_0 + y_1 + y_{n-1} + y_n = 3$;

(a_2) $y_2 + \ldots + y_{n-2} = 0$ and hence $y_0 + y_1 + y_{n-1} + y_n = 1$;

(a_3) $y_2 + \ldots + y_{n-2} = 1$ and hence $y_0 + y_1 + y_{n-1} + y_n = -1$;

(a_4) $y_2 + \ldots + y_{n-2} = 2$ and hence $y_0 + y_1 + y_{n-1} + y_n = -3$;

The number of possibilities for (y_0, y_1, y_2, y_3) in these four cases is equal to $4, 16, 16$, and 4, respectively. (For instance, in case (a_2), this number is $X_1(4) = 16$.) Hence, the total number is 40, and one have to test only 10 orbits of $\tilde{c}$ (actually, only five orbits in view of the symmetry). Each orbit contains two representatives satisfying the first condition in (5.10); for these two representatives, the parity of y_1 is different, so that only one of them satisfies the second condition. Notice also that we have proved that the total number of the solutions of system (5.9) is equal to

$$4X_{-1}(n-3) + 16X_0(n-3) + 16X_1(n-3) + 4X_2(n-3) .$$

(B) $n = 2l + 1$. Here we have one congruence condition:

$$2(y_1 + y_3 + \ldots + y_{n-2}) + y_{n-1} - y_n \in 4\mathbb{Z} .$$

However, we can split it in two conditions: $y_{n-1} - y_n \in 2\mathbb{Z}$ and $y_1 + y_3 + \ldots + y_{n-2} + \frac{1}{2}(y_{n-1} - y_n) \in 2\mathbb{Z}$, so that the previous argument goes through verbatim.

5) $\mathfrak{g} = \mathbf{E}_6$. Here $[P^\vee : Q^\vee] = 3$, the extended system is

$$(5.11) \qquad \begin{cases} y_0 + y_1 + 2y_2 + 3y_3 + 2y_4 + y_5 + 2y_6 = 1 \\ y_i \in \{-1, 0, 1\} , \end{cases}$$

and the congruence condition is $y_1 - y_2 + y_4 - y_5 \in 3\mathbb{Z}$. Then one may use the permutation

$$\tilde{c} : \begin{cases} y_1 \to y_5 \to y_0 \to y_1 \\ y_2 \to y_4 \to y_6 \to y_2 \end{cases} ,$$

since $y_1 - y_2 + y_4 - y_5 - \tilde{c}(y_1 - y_2 + y_4 - y_5) \equiv 1 \pmod 3$

6) $\mathfrak{g} = \mathbf{E}_7$. Here $[P^\vee : Q^\vee] = 2$, the extended system is

$$\text{(5.12)} \qquad \begin{cases} y_0 + y_1 + 2y_2 + 3y_3 + 4y_4 + 3y_5 + 2y_6 + 2y_7 = 1 \\ y_i \in \{-1, 0, 1\} \,, \end{cases}$$

and the congruence condition is $y_1 + y_3 + y_7 \in 2\mathbb{Z}$. Then one may use the involution

$$\tilde{c} : \begin{cases} y_1 \to y_0, \ y_2 \to y_6 \\ y_3 \to y_5, \ y_4 \to y_4, \ y_7 \to y_7 \,. \end{cases}$$

since $y_1 + y_3 + y_7 - \tilde{c}(y_1 + y_3 + y_7)$ is odd.

7) Finally, we have $P^\vee = Q^\vee$ for $\mathbf{G}_2, \mathbf{F}_4, \mathbf{E}_8$, and the theorem is proved. □

Now, we are prepared to compute the number of minimax elements in all simple Lie algebras. For the classical Lie algebras, we obtain some well-known combinatorial quantities.

5.13 Theorem. *The number of minimax elements in* $\widehat{W}(\mathfrak{sl}_{n+1})$ *is equal to the* n-th Motzkin number, M_n.

5.14 Theorem. *The number of minimax elements in* $\widehat{W}(\mathfrak{sp}_{2n})$ *or* $\widehat{W}(\mathfrak{so}_{2n+1})$ *is equal to the number of* directed animals of size n, *denoted* dir_n.

5.15 Theorem. *The number of minimax elements in* $\widehat{W}(\mathfrak{so}_{2n})$ *is equal to* $2dir_{n-2} + dir_{n-1}$.

We refer to [1], [6], and [16, Ex. 6.37] for basic facts on Motzkin numbers and to [7], [16, Ex. 6.46] for "directed animal" results; see also explicit formulae below. The first few terms of these sequences are

n	1	2	3	4	5	6	7	8
M_n	1	2	4	9	21	51	127	323
dir_n	1	2	5	13	35	96	267	750

Proof of Theorem 5.13.
Consider the extended system (5.6). Here the number of solutions is just

$$X_1(n{+}1) = \sum_{k\ge 0} \frac{(n+1)!}{k!(k+1)!(n-2k)!} \,.$$

It then follows from Theorem 5.5 that the number of minimax elements is

$$\frac{X_1(n{+}1)}{n+1} = \sum_{k\ge 0} \binom{n}{2k} C_k \,,$$

where $C_k = \frac{1}{k+1}\binom{2k}{k}$ is the k-th Catalan number. But the right-hand-side is a well-known formula for M_n, see [6] and [7, Eq. (21)]. □

Proof of Theorem 5.14.
In both cases, the extended systems (5.7), (5.8) are the same, up to renumbering y_i. Let us work with (5.7). Here we have the constraint $0 \le y_1 + y_2 + \ldots + y_{n-1} \le 1$. That is, there are two cases:

(a_1) $y_1 + y_2 + \ldots + y_{n-1} = 0$ and hence $y_0 + y_n = 1$;

(a_2) $y_1 + y_2 + \ldots + y_{n-1} = 1$ and hence $y_0 + y_n = -1$.

This means that system (5.7) has $2X_0(n-1) + 2X_1(n-1)$ solutions. Thus, the number of minimax elements is the sum of central and next-to-central trinomials

$$X_0(n-1) + X_1(n-1) = \sum_{k\ge 0} \frac{(n-1)!}{k!k!(n-2k-1)!} + \sum_{k\ge 0} \frac{(n-1)!}{k!(k+1)!(n-2k-2)!} =$$
$$= \sum_{k\ge 0} \binom{2k}{k}\binom{n-1}{2k} + \sum_{k\ge 0} \binom{2k+1}{k}\binom{n-1}{2k+1} = \sum_{q\ge 0} \binom{q}{[q/2]}\binom{n-1}{q},$$

which is a well-known expression for dir_n, see [7, Eq. (27)]. □

Notice that we have also derived the relation

$$X_0(n-1) + X_1(n-1) = \mathrm{dir}_n . \tag{5.16}$$

Proof of Theorem 5.15.
All essential work is already done in the proof of Theorem 5.5, part 4. Namely, the number of minimax elements is one fourth of the total number of solutions of system (5.9), i.e.,

$$\frac{1}{4}\Big(4X_{-1}(n-3) + 16X_0(n-3) + 16X_1(n-3) + 4X_2(n-3)\Big) .$$

Making use of Eq. (5.4) and (5.16), one transforms this sum to $2\mathrm{dir}_{n-1} + \mathrm{dir}_{n-2}$. □

Remarks. 1. Although $\#\mathfrak{Ad}_{mm}(\mathfrak{sp}_{2n}) = \#\mathfrak{Ad}_{mm}(\mathfrak{so}_{2n+1})$, the ideals occurring in these two sets have different algebraic properties. For instance, $\mathfrak{Ad}_{mm}(\mathfrak{sp}_8)$ contains 8 Abelian ideals (of 13), whereas $\mathfrak{Ad}_{mm}(\mathfrak{so}_9)$ contains 11.

2. Using the relation $\mathrm{dir}_n = 3\,\mathrm{dir}_{n-1} - M_{n-2}$, we may also write $\#\mathfrak{Ad}_{mm}(\mathfrak{so}_{2n}) = 5\,\mathrm{dir}_{n-2} - M_{n-3}$.

5.17 Theorem. *1. For $\mathfrak g = \mathbf{E}_p$, $p = 6, 7, 8$, the number of minimax elements in $\widehat{W}$ equals 67, 217, and 834, respectively.*

2. For $\mathfrak g = \mathbf{F}_4$ and $\mathbf{G}_2$, there are 17 and 3 minimax elements, respectively.

Proof. In each case, one can directly compute the coefficient of x in the respective Laurent polynomial (5.3), which gives the number $\#(D_{mm} \cap P^\vee)$, and then use Theorem 5.5. □

5.18 Example. For $\mathfrak g = \mathbf{F}_4$, the coweight and coroot lattices are equal and system (5.1) is

$$\begin{cases} \ y_i \in \{-1, 0, 1\} \ \ (i = 1, \ldots, 4) , \\ 0 \le 2y_1 + 4y_2 + 3y_3 + 2y_4 \le 2 \ . \end{cases}$$

It has 17 solutions, and a description of the corresponding non-trivial Abelian minimax ideals can be extracted from [10, Table 1], using Theorem 4.4 (there are 12 such ideals). The non-abelian minimax ideals are described in the next table.

$\Gamma(I)$	$\#(I)$	$\#(I^2)$	$w(\alpha_0)$	y
$[1,2,1,1]$	9	1	$-\delta-[0,2,1,0]$	$(1,-1,0,1)$
$[1,1,1,1]$	10	1	$-\delta-[2,2,1,0]$	$(-1,0,0,1)$
$[0,2,2,1],\ [2,2,1,0]$	10	2	$-2\delta+[2,4,2,1]$	$(1,1,-1,-1)$
$[0,2,1,1],\ [2,2,1,0]$	12	3	$-2\delta+[2,2,1,0]$	$(0,1,0,-1)$

For all cases we have $I^3=\varnothing$, so that $\ell(w(I))=\#(I)+\#(I^2)$. We write $[n_1,n_2,n_3,n_4]$ for the root $\sum_i n_i\alpha_i$. The numbering of simple roots follows [17, Tables]. The last column presents the corresponding solution of the above system. The ideals given in the first two rows are of Heisenberg type, and the explicit description of $w(I)$ for them can be extracted from Proposition 3.8(ii) and Theorem 3.12(ii).

6. A DESCRIPTION OF THE MINIMAX IDEALS FOR $\mathfrak{sl}_{n+1}$ AND $\mathfrak{sp}_{2n}$

In this section, we give an explicit matrix description of the minimax ideals in $\mathfrak{sl}_{n+1}$ and $\mathfrak{sp}_{2n}$. The idea is as follows. In both cases, we describe a certain set of ideals containing the minimax ones. Then we count the number of ideals in this, a priori, larger set. It turns out that we again obtain the Motzkin numbers and the number of directed animals of size n, respectively, which solves the problem.

$\underline{\mathfrak{g}=\mathfrak{sl}_{n+1}}$.
We choose $\mathfrak{b}$ (resp. $\mathfrak{t}$) to be the set of upper-triangular (resp. diagonal) matrices of size $n+1$. With the usual numbering of rows and columns of $n+1$ by $n+1$ matrices, we identify the positive roots with the pairs (i,j), where $1\le i<j\le n+1$. Here $\alpha_i=(i,i+1)$ and therefore $(a,b)=\alpha_a+\ldots+\alpha_{b-1}$. Then an antichain in $\Delta^+(\mathfrak{sl}_{n+1})$ is an arbitrary sequence $\Gamma=\{(a_1,b_1),\ldots,(a_k,b_k)\}$, where $a_i<b_i$ and the two sequences $\{a_i\}$ and $\{b_i\}$ are strictly increasing.

6.1 Definition. Let us say that the generators of an ideal $I\in\mathfrak{Ad}(\mathfrak{sl}_{n+1})$ *do not meet* or I has *non-meeting generators*, if the antichain $\Gamma(I)$ has the property that $b_j\ne a_i+1$ for all i,j.

The equality $b_j=a_i+1$ means that the last root in the expression for (a_j,b_j) is equal to the first root in the expression for (a_i,b_i). This explains the term.

6.2 Proposition. *If I is a minimax ideal, then its generators do not meet.*

Proof. Suppose I has meeting generators; i.e., $\Gamma(I)\supset\gamma_1,\gamma_2$, where $\gamma_1=\alpha_i+\alpha_{i+1}+\ldots+\alpha_j$ and $\gamma_2=\alpha_j+\alpha_{j+1}+\ldots+\alpha_k$ $(i<j<k)$. Consider $\gamma:=\gamma_1+\gamma_2-\alpha_j=\alpha_i+\ldots+\alpha_k$. It is easily seen that, for any presentation of γ as a sum of <u>two</u> roots, the summand containing α_j lies in I, while the other summand is in $\Delta^+\setminus I$. This means that $\gamma\notin I^2$, i.e., $l(\gamma,I)=1$, and $k(\gamma,I)\ge 3$. Thus, condition (ii) in Theorem 4.3 is not satisfied, and therefore I is not minimax. $\square$

6.3 Theorem. *The number of ideals in $\Delta^+(\mathfrak{sl}_{n+1})$ with k non-meeting generators is equal to $\binom{n}{2k}C_k$. In particular, the total number of ideals with non-meeting generators is M_n.*

Proof. Let I be an ideal with non-meeting generators and $\Gamma(I)=\{(a_1,b_1),\dots,(a_k,b_k)\}$. In order to compute the number of such ideals, it is convenient to replace b_j with $\tilde b_j = b_j-1$. In view of Definition 6.1, the numbers $\{a_i,\tilde b_j \mid 1\le i,j\le n\}$, which belong to $\{1,2,\dots,,n\}$, are pairwise different. They also satisfy the conditions

$$a_i<\tilde b_i,\quad a_1<\dots<a_k,\quad \tilde b_1<\dots<\tilde b_k\ . \tag{6.4}$$

To obtain such a collection of numbers, one should first choose arbitrarily $2k$ numbers from $\{1,2,\dots,n\}$. Next, given $2k$ numbers, one should choose k numbers among them and call them $a_1,\dots,a_k$ (in the increasing order!). Then the remaining k numbers form the sequence of $\tilde b_j$'s. But the choice of the a_i's cannot be arbitrary, for Eq. (6.4) must be satisfied. It is easily seen that the number of admissible choices is C_k. Indeed, given an ordered sequence of $2k$ elements $v_1\,v_2\,\dots v_{2k}$ from $\{1,2,\dots,n\}$, we assign to v_l the value $+1$, if $v_l=a_i$; and -1, if $v_l=\tilde b_j$. Then Eq. (6.4) is satisfied if and only if all partial sums $\sum_{i\le m} v_i$ are nonnegative. But the number of such sequences $v_1\,v_2\,\dots v_{2k}$ is the k-th Catalan number, see [16, Ex. 6.19(r)].
Hence, there are $\sum_{k\ge 0}\binom{n}{2k}C_k$ ideals with non-meeting generators, and it is the same number that occurs in the proof of Theorem 5.13. □

6.5 Corollary. *For $\mathfrak{sl}_{n+1}$, the minimax ideals are precisely the ideals with non-meeting generators.*

$\underline{\mathfrak{g}=\mathfrak{sp}_{2n}}$.
We use a standard matrix model of $\mathfrak{sp}_{2n}$ corresponding to a Witt basis for alternating bilinear form. For this basis of $\mathbb{C}^{2n}$, the algebra $\mathfrak{sp}_{2n}$ has the following block form:

$$\mathfrak{sp}_{2n}=\{\begin{pmatrix} A & B\\ C & D\end{pmatrix} \mid B=\widehat{B},\ C=\widehat{C},\ D=-\widehat{A}\}\ ,$$

where A,B,C,D are $n\times n$ matrices and $A\mapsto\widehat{A}$ is the transpose relative to the antidiagonal. If $\bar{\mathfrak{b}}$ is the standard Borel subalgebra of $\mathfrak{sl}_{2n}$, then $\mathfrak{b}:=\bar{\mathfrak{b}}\cap\mathfrak{sp}_{2n}$ is a Borel subalgebra of $\mathfrak{sp}_{2n}$. (See also [11, 5.1].) We identify the positive roots of $\mathfrak{sp}_{2n}$ with the set $\{(i,j)\mid i<j,\ i+j\le 2n+1\}$. Here the simple roots are $\alpha_i=(i,i+1)$, $1\le i\le n$, and therefore:

$$(i,j)=\begin{cases}\alpha_i+\dots+\alpha_{j-1}, & \text{if } j\le n+1\ ,\\ \alpha_i+\dots+\alpha_{2n-j}+2(\alpha_{2n-j+1}+\dots+\alpha_{n-1})+\alpha_n, & \text{if } j>n+1\ .\end{cases}$$

The ideals for $\mathfrak{sp}_{2n}$ can be identified with the ideals for $\mathfrak{sl}_{2n}$ that are symmetric with respect to the antidiagonal (= *self-conjugate*). In other words, there is a natural bijection between the ideals in $\Delta^+(\mathfrak{sp}_{2n})$ and the self-conjugate ideals in $\Delta^+(\mathfrak{sl}_{2n})$. More precisely, suppose $\bar I\in\mathfrak{Ad}(\mathfrak{sl}_{2n})$ and $\Gamma(\bar I)=\{(i_1,j_1),\dots,(i_k,j_k)\}$ with $i_1<\dots<i_k$, where we use our convention on the roots of $\mathfrak{sl}_{2n}$. Then $\bar I$ is self-conjugate if and only if $i_m+j_{k+1-m}=2n+1$ for all m. The corresponding ideal $I\in\mathfrak{Ad}(\mathfrak{sp}_{2n})$ has the generators

$\Gamma(I)=\{(i_m,j_m)\mid m\le[(k+1)/2]\}$. We shall say that $\bar I\in\mathfrak{Ad}(\mathfrak{sl}_{2n})$ is the *symmetrization* of $I\in\mathfrak{Ad}(\mathfrak{sp}_{2n})$.

6.6 Proposition. *If $I\in\mathfrak{Ad}(\mathfrak{sp}_{2n})$ is a minimax ideal, then the symmetrization $\bar I$ has non-meeting generators.*

Proof. Suppose the symmetrization $\bar I$ has some meeting generators. Then arguing as in Proposition 6.2, we find a root $\bar\gamma\in\bar I$ such that $l(\bar\gamma,\bar I)=1$ and $k(\bar\gamma,\bar I)\ge 3$. To any root $\bar\gamma\in\Delta^+(\mathfrak{sl}_{2n})$, one naturally associate the root $\gamma\in\Delta^+(\mathfrak{sp}_{2n})$. With our identification for both root systems, this can be formalized as follows. If $\bar\gamma=(i,j)$ and $i+j\le 2n+1$, then set $\gamma=\bar\gamma$. If $i+j\ge 2n+2$, then set $\gamma=(2n{+}1{-}j,2n{+}1{-}i)$. This yields a surjective mapping $\Delta^+(\mathfrak{sl}_{2n})\to\Delta^+(\mathfrak{sp}_{2n})$ and, in particular, $\bar I\to I$ for any $I\in\mathfrak{Ad}(\mathfrak{sp}_{2n})$. The last and easy observation is that $(\bar I)^l=\overline{I^l}$ for any $I\in\mathfrak{Ad}(\mathfrak{sp}_{2n})$. So that $l(\gamma,I)=1$ and $k(\gamma,I)\ge 3$ as well. Thus, I is not minimax. □

6.7 Theorem. *The number of ideals in $\Delta^+(\mathfrak{sp}_{2n})$ with q generators, whose symmetrization has non-meeting generators, is equal to $\binom{2q-1}{q-1}\binom{n-1}{2q-1}+\binom{2q}{q}\binom{n-1}{2q}$. In particular, the total number of such ideals is $\sum_{q\ge 0}\binom{q}{[q/2]}\binom{n-1}{q}=dir_n$.*

Proof. Let $\bar I\in\mathfrak{Ad}(\mathfrak{sl}_{2n})$ be a self-conjugate ideal and $\Gamma(\bar I)=\{((i_1,j_1),\dots,(i_k,j_k)\}$ the sequence of its generators. Then

$$i_m<j_m,\quad 1\le i_1<i_2<\ldots<i_k,\quad j_1<j_2<\ldots<j_k\le 2n\ ,$$

and the symmetry condition $i_m{+}j_{k+1-m}=2n{+}1$ is satisfied for any m. If the generators do not meet, then all the numbers $\{i_l,\tilde j_m=j_m{-}1\mid 1\le l,m\le k\}$ are different. They form a set $E\subset\{1,2,\dots,2n{-}1\}$ consisting of $2k$ elements. Because of the symmetry, E is completely determined by the first k elements, which lie in $\{1,2,\dots,n\}$. Moreover, the symmetry and "non-meeting" condition readily imply that $n\not\in E$. Thus, $\frac12E:=E\cap\{1,\dots,n\}$ actually belongs to $\{1,2,\dots,n-1\}$ and $\#(\frac12E)=k$. Notice that E (and hence $\frac12E$) arises as a disjoint union of its i-part and j-part. So, the problem is to count the admissible partitions in two parts of all k-element subsets of $\{1,\dots,n{-}1\}$. To this end, one should first choose arbitrarily k numbers from $\{1,2,\dots,n{-}1\}$, and then to choose a partition of this set into i- and j-parts. In order to compute the number of admissible partitions, we restate the problem, as in the proof of Theorem 6.3, in terms of sequences of $+1$ and -1: Given a k-element subset $v_1<v_2<\ldots<v_k$ of $\{1,\dots,n{-}1\}$, we assign the value $+1$ to all elements lying in the i-part, and -1 to all elements lying in the j-part. It is easily seen that such a sequence corresponds to $\frac12E$ for a suitable subset E if and only if all partial sums $\sum_{s\le m}v_s$ are non-negative. It is not hard to prove (e.g. using a lattice path interpretation and the reflection principle) that the number of such sequences is equal to $\binom{k}{[k/2]}$. It remains to remember that the ideal $I\in\mathfrak{Ad}(\mathfrak{sp}_{2n})$, corresponding to $\bar I$, has $[(k{+}1)/2]$ generators, so that the ideals with q generators arise if $k=2q{-}1,\ 2q$. □

6.8 Corollary. *For $I\in\mathfrak{Ad}(\mathfrak{sp}_{2n})$, the following conditions are equivalent:*

(i) $I \in \mathfrak{Ad}_{mm}(\mathfrak{sp}_{2n})$;
(ii) $\bar{I} \in \mathfrak{Ad}_{mm}(\mathfrak{sl}_{2n})$;
(iii) $\bar{I}$ *has non-meeting generators.*

6.9 Example. By Theorem 6.7 and Corollary 6.8, the number of minimax ideals with one generator is equal to $(n-1)^2$. It is easy to verify that the set of positive roots occurring in this way is $\Delta^+(\mathfrak{sp}_{2n}) \setminus (\Pi \cup \{\alpha_i + \ldots + \alpha_n \mid i = 1, 2, \ldots, n-1\})$.

In [11], we considered the statistic on $\mathfrak{Ad}$ which assigns to an ideal the number of its generators. The corresponding generating functions (polynomials) turn out to be always palindromic. It is also makes sense to compute the respective generating functions for various classes of ideals. Theorems 6.3 and 6.7 give us essentially these generating functions for $\mathfrak{Ad}_{mm}(\mathfrak{sl}_{n+1})$ and $\mathfrak{Ad}_{mm}(\mathfrak{sp}_{2n})$:

$$\mathcal{F}_{mm}(\mathfrak{sl}_{n+1}; t) = \sum_{k \ge 0} \binom{n}{2k} C_k t^k \ ,$$

$$\mathcal{F}_{mm}(\mathfrak{sp}_{2n}; t) = \sum_{k \ge 0} (\binom{2k-1}{k-1} \binom{n-1}{2k-1} + \binom{2k}{k} \binom{n-1}{2k}) t^k \ .$$

It would be interesting to compute the polynomials $\mathcal{F}_{mm}(\mathfrak{g}; t)$ for all simple Lie algebras. It is likely that these polynomials coincide for $\mathfrak{sp}_{2n}$ and $\mathfrak{so}_{2n+1}$, but I have no suggestion for $\mathfrak{so}_{2n}$.

7. Concluding remarks

Here we state several questions/problems related to minimax elements.

1. Is there a uniform expression for $\#(\widehat{W}_{mm})$ for all simple Lie algebras?

2. Consider the set $\bigcup_{w \in \widehat{W}_{mm}} w^{-1} {*} \overline{\mathcal{A}} \subset V$. It is just the union of the closures of all dominant regions consisting of a single alcove. Is there a reasonable description of this set? Note that it is not convex in general.

3. Describe combinatorial properties of the polytope D_{mm} defined in Proposition 4.2. Compute the Ehrhart quasi-polynomial for D_{mm}.

4. Find a uniform proof of Theorem 5.5. It is worth noting that the similar statement can be proved a priori for the simplices D_{min} and D_{max}. Unfortunately, this does not immediately imply the validity of Theorem 5.5 for $D_{mm} = D_{min} \cap D_{max}$.

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