MATHEMATISCHE ARBEITSTAGUNG 1978

UNIVERSITÄT BONN

Sonderforschungsbereich 40

Theoretische Mathematik

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Programm der Mathematischen Arbeitstagung 1978 (I)

<u>Freitag, den 16.6.:</u>

17.15 - 18.15 Uhr: M.F. Atiyah: Yang-Mills instantons and algebraic geometry

<u>Samstag, den 17.6.:</u>

10.00 - 11.00 Uhr: E. Calabi: SU(n)- and Sp(n)-manifolds

12.00 - 13.00 Uhr: H. Jacquet: From GL(2) to GL(3)

17.00 - 18.00 Uhr: J.-P. Bourguignon: Differential-geometry of the Yang-Mills equation

<u>Sonntag, den 18.6.:</u>

10.00 - 10.15 Uhr: Festlegung der nächsten Vorträge

10.15 - 11.15 Uhr: J. Eells: Holomorphic and harmonic maps of surfaces

12.00 - 13.00 Uhr: J. Steenbrink: Non-rationality of the quartic threefold

17.00 - 18.00 Uhr: B. Gross: The Chowla-Selberg formula

Die Vorträge finden alle im "Großen Hörsaal" (Wegelerstraße 10) statt.

Erfrischungspausen mit Tee: Samstag und Sonntag vormittags von 11.15-12.00 Uhr vor dem Großen Hörsaal, nachmittags ab 15.30 Uhr im Diskussionsraum Beringstr. 1.

Die Post liegt während der Vormittags-Teepausen aus.

Tischtennis im Keller des Hauses Beringstraße 4.

Den Tagungsbeitrag bitte an Frau Gerber (SFB-Büro Beringstr. 4) bezahlen.

Alle Teilnehmer mögen sich bitte in die Teilnehmerliste eintragen. Teilnehmerlisten und andere Informationen liegen vor dem Diskussionsraum Beringstraße 1 aus.

Alle Tagungsteilnehmer mit ihren Damen oder Herren sind herzlich zum Empfang des Rektors eingeladen. Zeit: Montag, den 19.6., 20.00 Uhr. Ort: Festsaal der Universität (Hauptgebäude), Eingang von der Straße "Am Hof" durch das Tor gegenüber Buchhandlung Röhrscheid.

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Programm der Mathematischen Arbeitstagung 1978 (II)

Montag, den 19.6.:

10.00 - 11.00 Uhr: B. Mazur: Rational points on elliptic curves and congruences of L-series

12.00 - 13.00 Uhr: D. Burghelea: Computation of homotopy groups of diffeomorphism groups of compact manifolds

17.00 - 18.00 Uhr: P. Schweitzer: Residues of real foliation singularities

Dienstag, den 20.6.:

8.15 - ca. 9.15 Uhr: (Sondervortrag) T. Banchoff: The fourth dimension and computer animated geometry

9.45 - 10.00 Uhr: Festlegung der restlichen Vorträge

10.00 - 11.00 Uhr: J. Milnor: Volume of hyperbolic manifolds

12.30 - ca. 20.00 Uhr: Dampferfahrt auf dem Rhein nach Leutesdorf; Abfahrt am "Alten Zoll" mit Motorschiff "Carmen Silva"

Mittwoch, den 21.6.:

10.00 - 11.00 Uhr: K. Ueno: Birational geometry of fibre spaces

Die Vorträge finden alle im "Großen Hörsaal" (Wegelerstraße 10) statt. Erfrischungspausen mit Tee: Montag vormittag von 11.15-12.00 Uhr vor dem Großen Hörsaal, nachmittags ab 15.30 Uhr im Diskussionsraum Beringstraße 1; Mittwoch vormittag von 11.15-12.00 Uhr vor dem Großen Hörsaal. Die Post liegt während der Vormittags-Teepausen aus. Tischtennis im Keller des Hauses Beringstraße 4. Den Tagungsbeitrag bitte an Frau Gerber (SFB-Büro Beringstraße 4) bezahlen. Alle Teilnehmer mögen sich bitte in die Teilnehmerliste eintragen. Alle Informationen und die Teilnehmerliste liegen während der Teepausen aus.

! Am Montag um 20.00 Uhr ist der Empfang des Rektors im Festsaal der Universität
! (Hauptgebäude). Alle Tagungsteilnehmer mit ihren Damen oder Herren sind
! herzlich dazu eingeladen.

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Programm der Mathematischen Arbeitstagung 1978 (III)

Mittwoch, den 21.6.:

12.00 - 13.00 Uhr: F. Adams: Finite H-spaces and algebras over the Steenrod Algebra

17.00 - 18.00 Uhr: N. Hitchin: Twistor spaces

Donnerstag, den 22.6.:

10.00 - 11.00 Uhr: A. Todorov: Surfaces with $Pg = 1$ and $K^2 = 1$

12.00 - 13.00 Uhr: P. Baum: K-homology and Riemann-Roch

17.00 - 18.00 Uhr: J. Brüning: Representations of compact Lie groups and elliptic operators

Freitag, den 23.6.:

10.00 - 11.00 Uhr: I. Piatetski-Shapiro: Automorphic forms on the metaplectic group

12.00 - 13.00 Uhr: F. Waldhausen: Algebraic K-theory of topological spaces

Die Vorträge finden alle im "Großen Hörsaal" (Wegelerstraße 10) statt.

Erfrischungspausen mit Tee: Mittwoch, Donnerstag und Freitag vormittags von 11.15 Uhr - 12.00 Uhr vor dem großen Hörsaal, Mittwoch und Donnerstag nachmittags ab 15.30 Uhr im Diskussionsraum Beringstraße 1.

Die Post liegt während der Vormittags-Teepausen aus.

! Die Referenten werden nochmals gebeten, ihre Kurzfassungen möglichst bald
! bei Herrn Kraft abzugeben, da wir den Tagungsbericht allen Teilnehmern noch
! vor ihrer Abreise aushändigen möchten.

Title: YANG-MILLS FIELDS AND ALGEBRAIC GEOMETRY

Name of author: M. F. ATIYAH

Address: Mathematical Institute, Oxford, England

Bibliography:

M. F. Atiyah, N. J. Hitchin, V. G. Drinfeld and Yu. I. Manin, Construction of instantons, Phys. Lett. 65A (1978), 185–187

The Yang-Mills equations arise from theoretical physics as a non-linear generalization of Maxwell's equations. In the Euclidean rather than Minkowski domain they may be extended naturally to the 4-sphere S^4 by conformal invariance. The geometrical data consists of a compact Lie group G (eg $G = SU(2)$) a principal fibre bundle Q over S^4 with fibre G, a connection A and its curvature F. The Yang-Mills functional is the natural L^2-norm $\|F\|^2$ of F. The Euler equations are the Y-M equations. There is a topological invariant k of the bundle Q (for G simple and non-abelian) and it is related to F by the formula

$$-8\pi^2 k = \|F^+\|^2 - \|F^-\|^2$$

$$\|F\|^2 = \|F^+\|^2 + \|F^-\|^2$$

where F^+ and F^- are the components of F relative to the $*$ operator on 2-forms (satisfying $*^2 = 1$). From these equations we see that $\|F\|^2 \geq 8\pi^2 |k|$

with equality if and only if $*F = \pm F$, the sign depending on sign k. A k-instanton is a connection A with $*F = \pm F$ thus giving an absolute minimum for $\|F\|^2$.

Ideas of Penrose and Ward explained at the last Arbeitstagung enable one to reinterpret instantons in terms of algebraic geometry on $P_3(C)$. The basic geometry involves a natural fibration $P_3(C) \to S^4$ with fibres $P_1(C)$. Here one has a "conjugation" σ on P_3 with no "real points" but with the fibres above as "real lines". This is a real form of the Klein picture of lines in P_3. The result of this reinterpretation is that instantons correspond to vector bundles on $P_3(C)$ with certain extra conditions. For $G = SU(2)$ we get rank 2 bundles (algebraic) E such that

(i) $c_1(E) = 0$, $c_2(E) = k$
(ii) E is algebraically trivial on all real lines
(iii) σ lifts to a σ on E with $\sigma^2 = -1$.

As a consequence of (ii) one can prove that the sheaf cohomology group $H^1(P_3, E(-2))$ vanishes: this is because one can identify it with the space of solutions of a linear differential operator on S^4 which is positive.

Now Horrocks has given a simple

construction for a certain family of bundles on P_3 and Barth has shown that this family is characterized by stability (weaker than (ii) above) and the vanishing of the above $H^1(P_3, E(-2))$. Thus every instanton turns out to be given by the Horrocks construction supplemented by appropriate reality conditions. This gives a complete elementary description of instantons (not only for $SU(2)$ but for all classical groups) in terms of rational functions of the coordinates of S^4 and certain parameters. The parameters are constrained by algebraic relations and the exact nature of this parameter (or moduli) space has yet to be determined. However it is already known that it is highly non-trivial topologically. For example for $G = SU(2)$ the parameter space for large k contains all the homology of $\Omega^3(SU(2))$, the 3-fold loop space.

Title: SU(n)- and Sp(n)- manifolds

Name of author: E. Calabi

Address: Math. Dept.- University of Pennsylvania - Philadelphia PA 19174 USA

Bibliography:

[1] E. Calabi, Métriques kähleriennes et fibrés holomorphes, Ann. Sci. de l'Ec. Norm. Sup., (in preparation)

[2] Tohru Eguchi & Andrew J. Hanson — Asymptotically Flat, self-dual Solutions to Euclidean Gravity, Manuscript dated January 1978, submitted to Physics Letters B.

[3] G. W. Gibbons & C. N. Pope, CP^2 as a Gravitational Instanton, Preprint, 1978 (DAMTP)

[4] V. A. Belinskii, G. W. Gibbons, D. N. Page & C. N. Pope, Asymptotically Euclidean Bianchi IX Metrics in Quantum Gravity, Preprint, 1978.

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A fully detailed exposition of the subject will appear in [1].

Let M be a complex manifold with a Kähler metric $ds^2 = 2g_{\alpha\bar\beta}dz^\alpha d\bar z^\beta$, where locally $g_{\alpha\bar\beta} = \frac{\partial^2\Phi}{\partial z^\alpha \partial \bar z^\beta}$ and the real valued function Φ is called a Kähler potential; let $E \xrightarrow{\pi} M$ be a holomorphic vector bundle of rank m with local, holomorphic, fibre-linear coordinates $\underline{\zeta} = (\zeta^1, \dots, \zeta^m)$, equipped with an hermitian norm $t = a_{\lambda\bar\mu}(z,\bar z)\zeta^\lambda \bar\zeta^\mu$, the latter defining a reduction of the structure group of E from $GL(m, \mathbb{C})$ to $U(m)$. We consider the class of Kähler metrics related to the above data, that are defined in E or in a subdomain of E, defined by restricting the values of the hermitian norm t to an interval, such as

$\{t \mid 0 \le t < t_0\}$. Such metrics are uniquely determined by a suitable choice of $U(m)$-invariant Kähler metric in the typical fibre (respectively, a spherical domain or spherical shell). They can be derived from a local Kähler potential Ψ in E of the form

(1) $\Psi(z,\bar{z};\varsigma,\bar{\varsigma}) = \Phi(z,\bar{z}) + u(t)$,

where Φ is a Kähler potential for the given metric in M and $u(t)$ a function of one real variable t, where t is the value of the given hermitian form in E and $u(t)$ is defined in an interval, and satisfies certain differential inequalities that ensure that the resulting Hermitian form in E (or in the corresponding subdomain) is positive definite. We call such a metric adapted to the data of the bundle $\{E \xrightarrow{\pi} M;\ \partial\bar\partial\Phi,\ t\}$.

Theorem 1. Given any holomorphic vector bundle $E \xrightarrow{\pi} M$ with the additional structural data $\partial\bar\partial\Phi$ on M and $t : E \to \mathbb{R}$ as above, then there exists a Kähler metric on E adapted to the data and such that E is complete, if and only if: i) M is complete with respect to the given Kähler structure $\partial\bar\partial\Phi$, and ii) the curvature of the Cartan connection derived from the hermitian structure $\{t\ M \to \mathbb{R}\}$ has a uniform lower bound $-b > -\infty$.

Key to the proof. If the curvature is non-negative everywhere, it is sufficient to take the fibre-flat metric derived from local potentials Ψ as in (1) with $u(t) = t$. If the lower bound $-b$ of the curvature is strictly negative, one may take the potential Ψ to be determined by the following choice for $u(t)$

$$u(t) = \frac{2}{3b}\log(e^b + t) - \frac{1}{3}\log\log(e^b + t).$$

The title of this communication is justified by the following applications of Kähler metrics adapted to holomorphic bundles

Ⓐ SU(n)-manifolds

Let M be a complete Einstein-Kähler manifold with $\dim_{\mathbb{C}} M = n-1$ and constant Ricci curvature k_0 (i.e., without loss of generality, satisfying $\det(g_{\alpha\bar\beta}) = e^{-k_0\Phi}$) and let $E \xrightarrow{\pi} M$ be a holomorphic line bundle of an equivalence class that admits an hermitian form $t = a|\varsigma|^2$ with constant curvature $\ell \neq 0$ (the locally trivial case $\ell = 0$ is less interesting for our purposes); in particular, if M is compact and $k_0\ell \neq 0$, this means that $c_1(M, E) = \frac{\ell}{k_0} c_1(M)$, and consider various Kählerian forms adapted to the bundle structure of E.

Theorem 2 (a) A Kähler metric in E, adapted to the above data, is an Einstein-Kähler metric in a subdomain of E, whose constant Ricci curvature has any preassigned, real value k, if and only if the function $u(t)$ in (1) satisfies the differential equation

(2) $(1+\ell t u'(t))^{n-1}(t u''(t) + u'(t)) = c\, t^{\ell^{-1}(k_0-k-\ell)} e^{-ku(t)}$ $(c = \text{const} > 0)$

which can be integrated to

(2') $\frac{k_0}{n}\left[(1+\ell t u')^n - 1\right] - \frac{k}{n+1}\left[(1+\ell t u')^{n+1} - 1\right] = c\ell^2 t^{\ell^{-1}(k_0-k)} e^{-ku(t)} + c'$,

This last equation can be solved by quadratures of implicit functions, yielding a function $u(t)$ that generates a Kähler metric in a subdomain of E defined, usually by $0 \le t_1 < t < t_2$, including any desired value $t = t_0 > 0$.

(b) A solution $u(t)$ of (2') defines a Kähler metric extendable to a subdomain of E defined by $0 \le t < t_2$, if and only if

$$k = k_0 - \ell \quad \text{and} \quad c' = 0,$$

reducing (2') to the form

(2'') $\frac{k_0}{n}\left[(1+\ell t u')^n - 1\right] - \frac{k_0-\ell}{n+1}\left[(1+\ell t u')^{n+1} - 1\right] = c\ell^2 t e^{-(k_0-\ell)u}$

(c) If $k = k_0 - \ell \leq 0$ and $\ell > 0$, then the maximal intervals for nonsingular solutions of (2") ~~are~~ include intervals $0 \leq t < t_2$ ($t_2 = \infty$ for $k = 0$, $t_2 < \infty$ for $k < 0$); in these cases the corresponding subdomains of E are complete with respect to the resulting Einstein-Kähler metric. In particular, we have —

(d) If $k_0 = \ell < 0$, then the functions

$$u(t) = \int (k_0 t)^{-1}\left[(1+nck_0 t)^{1/n} - 1\right] dt =$$

$$= k_0^{-1}\left[n\sqrt[n]{1+nck_0 t} - \sum_{\nu=1}^{n-1}(1-\omega^\nu)\log\left(\sqrt[n]{1+nck_0 t} - \omega^\nu\right)\right],$$

$$\omega = e^{2\pi i/n}$$

define a complete, Ricci flat Kähler metrics, each of which is the canonical line bundle over a (necessarily compact) complex manifold M with a Kähler metric with constant, positive Ricci curvature. The resulting Ricci flat Kähler manifolds can be characterized equivalently as $2n$-(real) dimensional Riemannian manifolds, whose holonomy group is contained in (and usually equals) $SU(n) \subset O(2n)$. The simplest case of such $SU(n)$-manifolds is the unique one possible for $n = 2$, the canonical line bundle over $P_1(\mathbb{C})$ (the latter with the usual metric with constant curvature). This particular example has been constructed recently, by entirely different methods, by several other authors who interpreted Ricci flat Kähler surfaces as "instantons", [2], [3], [4].

(B) $Sp(n)$-Manifolds (Hyperkähler manifolds)

One can construct special examples of Riemannian metrics in $4n$ real dimensions with holonomy group $Sp(n)$; the spaces carrying such metrics may even be complete in at least one case for each value of n; these manifolds, for reasons made clear below, can rightly be called hyperkähler manifolds.

A hyperkähler manifold can be regarded, in more than one way, as a Kähler manifold, in the sense that it has a fixed metric, and not just one, but at least three almost complex structures

$\underline{I}, \underline{J}, \underline{K}$ that are endomorphisms of the tangent bundle
satisfying $\underline{I}^2 = \underline{J}^2 = \underline{K}^2 = -1$, $\underline{I}\underline{J} = -\underline{J}\underline{I} = \underline{K}$, $\underline{J}\underline{K} = -\underline{K}\underline{J} = \underline{I}$, $\underline{K}\underline{I} = -\underline{I}\underline{K} = \underline{J}$,
and in addition all of them covariant constant with respect
to the metric. Hence for any point $(a,b,c) \in S^2$, $(a^2+b^2+c^2 = 1)$
the endomorphism $a\underline{I} + b\underline{J} + c\underline{K}$ defines an almost complex
structure in the manifold, integrable to a complex analytic one
such that the given metric is Kählerian with respect to it.
Fixing our attention to the complex structure defined, say
by $\underline{I}$, the ~~fo~~ bilinear form $\varphi(U,V) = \langle U, \tfrac{1}{2}(\underline{J}V - \sqrt{-1}\,\underline{K}V)\rangle$
~~is of type (2,0) and co~~ an exterior form of type (2,0), of
maximal rank everywhere, and holomorphic, since its covariant
derivative is identically zero. Hence every hyperkähler manifold
V, referred to any of its complex analytic structures, is also
a complex symplectic manifold. For this reason, a good
candidate for a hyper~~complex~~kähler manifold is the cotangent
vector bundle of a suitable Kähler manifold M, $\dim_{\mathbb{C}} M = n$. For
other intuitive reasons M is chosen to be a Fubini space
with constant holomorphic sectional curvature $4K \neq 0$. In terms
of local coordinates $(z^1, \dots, z^n)$ for M, the Kähler-Fubini-Study
metric is derived from $\Phi = +K^{-1}\log(1+K|z|^2)$ $\left(|z|^2 = \sum_{\alpha=1}^n |z^\alpha|^2\right)$.
Let $V \xrightarrow{\pi} M$ denote the cotangent bundle, ~~co~~ with corresponding
local coordinates $(z, \zeta) = (z^1, \dots, z^n; \zeta_1, \dots, \zeta_n)$ at the point (cotangent
vector) $f = \sum_{\alpha=1}^n \zeta_\alpha dz^\alpha$. We take as the norm in V the function

$$t = g^{\alpha\bar\beta}\zeta_\alpha\bar\zeta_\beta = (1+K|z|^2)\left(\sum_{\alpha=1}^n |\zeta_\alpha|^2 - K\left|\sum_{\alpha=1}^n z^\alpha \zeta_\alpha\right|^2\right)$$

<u>Theorem 3</u> The canonical 2-form $\sum_{\alpha=1}^n dz^\alpha \wedge d\zeta_\alpha$ ~~in~~ V is a covariant constant with respect to a Kähler metric adapted to the bundle ~~struct~~ structure and determined by a function $u(t)$, if and only if $Kt(u'(t))^2 + u'(t) = c > 0$ (c = constant), or equivalently

$$u(t) = \frac{2c}{K}\left(\sqrt{1+4cKt} - \log\left(1+\sqrt{1+4cKt}\right)\right) + c_0$$

In this case, the resulting Kähler metric defines a hyperkähler structure in its domain of regularity. If $K > 0$ this hyperkähler metric ~~de~~ is regular and complete on the whole cotangent bundle of $P_n(\mathbb{C})$ for each n. If $K < 0$, the metric becomes singular on the boundary of the domain where $t < (-4cK)^{-1}$ and the resulting manifold is not complete. Other examples are being sought.

1

Title: From GL(2) to GL(3).

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[1] I. M. Gelfand and D. A. Kazdan, Representations of the group GL(n, K) where K is a local field, in "Lie groups and their representations", John Wiley & Sons, (1975) 95-118.

[2] R. Godement and H. Jacquet, Zeta functions of simple algebras, Springer-Verlag, Lecture Notes 260 (1972).

[3] H. Jacquet and R. Langlands, Automorphic forms on GL(2), Springer-Verlag, L.N. 114 (1970).

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1 - Let F be a number field or a function field (e.g. $F = \mathbb{Q}$) and $\mathbb{A}$ the ring of adèles of F. Let G be the group $GL(n)$; then $G(F) = GL(n, F)$ is a discrete subgroup of $G(\mathbb{A}) = GL(n, \mathbb{A})$.

Roughly speaking an automorphic form φ on G is a function on $G(\mathbb{A})$, with complex values, which is invariant on the left under $G(F)$. An irreducible representation π of $G(\mathbb{A})$ is said to be automorphic if there is a space V of automorphic forms which is invariant under right translations and such that the representation of $G(\mathbb{A})$ on V is equivalent to π.

An irreducible representation π of $G(\mathbb{A})$ can always be written as an "infinite tensor product"

$$(1.1) \qquad \pi = \bigotimes_v \pi_v$$

where, for each place v, the representation π_v is an irreducible representation of the local group $G_v = GL(n, F_v)$; in addition, for almost all finite places v, the representation π_v contains, with multiplicity one, the trivial representation of the maximal compact subgroup

$$K_v = GL(n, R_v),$$

where R_v is the ring of integers of F_v. Representations of this type are

said to be "unramified"; they are parametrized by the semi-simple conjugacy classes in $GL(n, \mathbb{C})$. Hence if π is an irreducible representation of $G(\mathbb{A})$, we get, for almost all v, an element $A_v \in GL(n, \mathbb{C})$, defined up to conjugation.

In particular, if π is automorphic, the infinite product

$$(1.2) \qquad \prod_{a.a.\, v} \det(1 - A_v q_v^{-s})^{-1}$$

converges absolutely for $\operatorname{Re} s$ large enough; it can be shown ([2] and [7]) that it extends to the whole complex plane as a meromorphic function of s. Moreover it satisfies a functional equation. Indeed let $\tilde{\pi}$ be the representation of $G(\mathbb{A})$ defined by:

$$(1.3) \qquad \tilde{\pi}(g) = \pi({}^t g^{-1}).$$

It is automorphic; the corresponding elements in $GL(n, \mathbb{C})$ are just the A_v^{-1} and, apart from elementary factors, the analytic continuation of (1.2) is equal to the analytic continuation of

$$(1.4) \qquad \prod_{a.a.\, v} \det(1 - A_v^{-1} q_v^{-1+s})^{-1}.$$

2 - One way to establish this is to introduce the notion of "Mellin transform" of an automorphic form φ. We first review the case $n = 2$. We let π be an automorphic representation, V the corresponding space of automorphic forms. For simplicity we assume that each $\varphi \in V$ is cuspidal, i.e. that

$$(2.1)\quad \int_{\mathbb{A}/F} \varphi\left[\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} g\right] dx = 0, \quad \text{all } g \in G(\mathbb{A}).$$

Then the Mellin transform of φ is just the integral

$$(2.2)\quad \int_{F^\times_{\mathbb{A}}/F^\times} \varphi\left[\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}\right] |a|^{s-1/2} d^\times a \; ;$$

here $F^\times_{\mathbb{A}}$ is the idèle-group, $d^\times a$ its Haar measure, $|a|$ the module of a, and s some complex number. The integral converges for all s and defines therefore an holomorphic function of s.

Now let ψ be a non trivial (additive) character of $\mathbb{A}/F$. Set:

$$(2.3)\quad W(g) = \int_{\mathbb{A}/F} \varphi\left[\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} g\right] \overline{\psi}(x)\, dx .$$

Then the other non-trivial characters of $\mathbb{A}/F$ have the form $x \mapsto \psi(\alpha x)$ with $\alpha \in F^\times$. Moreover:

$$(2.4)\quad W\left[\begin{pmatrix} \alpha & 0 \\ 0 & 1 \end{pmatrix} g\right] = \int \varphi\left[\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} g\right] \overline{\psi}(\alpha x)\, dx .$$

Thus expanding φ into its Fourier series we get:

$$(2.5)\quad \varphi(g) = \sum_{\alpha \in F^\times} W\left[\begin{pmatrix} \alpha & 0 \\ 0 & 1 \end{pmatrix} g\right].$$

Replacing in (2.2) φ by this series we get

$$(2.6)\quad \int_{F^\times_{\mathbb{A}}/F^\times} \varphi\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} |a|^{s-1/2} d^\times a$$

$$= \int \sum_{\alpha \in F^\times} W\left[\begin{pmatrix} \alpha a & 0 \\ 0 & 1 \end{pmatrix}\right] |a|^{s-1/2} d^\times a$$

$$= \int \sum_{\alpha \in F^\times} W\left[\begin{pmatrix} \alpha a & 0 \\ 0 & 1 \end{pmatrix}\right] |\alpha a|^{s-1/2} d^\times a$$

$$= \int_{F^\times_{\mathbb{A}}} W\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} |a|^{s-1/2} d^\times a .$$

More precisely the last integral converges for Re s large enough and is then equal to the first one. This shows that the last integral, which is a priori defined only for Re $s >> 0$, extends to an holomorphic function of s. Now the last integral is, up to elementary factors, equal to the product (1.2). So we have obtained the analytic continuation of (1.2).

To obtain the functional equation, we replace φ by the function

(2.7) $\tilde{\varphi}(g) = \varphi({}^t g^{-1})$.

The space $\tilde{V}$ spanned by the functions $\tilde{\varphi}$ with $\varphi \in V$ is invariant under $G(A)$ and the representation of $G(A)$ on $\tilde{V}$ equivalent to $\tilde{\pi}$. Moreover the function W defined by (2.3) with $\tilde{\varphi}$ instead of φ is related to W by

(2.8) $\tilde{W}(g) = W(w\,{}^t g^{-1})$, $w = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.

Finally:

(2.9) $$\int_{F_A^\times / F^\times} \varphi\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} |a|^{s-1/2} d^\times a = \int \tilde{\varphi}\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} |a|^{1/2-s} d^\times a$$

This implies:

(2.10) $$\int_{F_A^\times} W\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} |a|^{s-1/2} d^\times a = \int \tilde{W}\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} |a|^{1/2-s} d^\times a;$$

this equality, which holds only in the sense of analytic continuation, gives indeed the required functional equation.

3 – Now we discuss the case $n = 3$. Here V and π have the same meaning as before. Again we assume the elements of φ to be cuspidal: this means now that

(3.1) $\int \varphi(ug)\, du = 0$, all g

where $u \in U(A)/U(F)$ and

$$U = \left\{ \begin{pmatrix} 1 & 0 & * \\ 0 & 1 & * \\ 0 & 0 & 1 \end{pmatrix} \right\} \quad \text{or} \quad = \left\{ \begin{pmatrix} 1 & x & x \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right\}.$$

For $\varphi \in V$ we set

(3.2) $$V_\varphi(g) = \iint_{(A/F)^2} \varphi\left[\begin{pmatrix} 1 & 0 & x \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} g \right] \bar{\psi}(y)\, dx\, dy.$$

Then

$$V_\varphi\left[\begin{pmatrix} \alpha & \beta & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} g \right] = V_\varphi(g), \quad \alpha \in F^\times, \beta \in F.$$

The Mellin transform of φ is the integral

(3.3) $$\int_{F_A^\times / F^\times} V_\varphi\left[\begin{pmatrix} a & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right] |a|^{s-1} d^\times a;$$

it is always convergent. On the other hand we have also a Fourier series for V_φ:

(3.4) $$V_\varphi(g) = \sum_{\alpha \in F^\times} W\left[\begin{pmatrix} \alpha & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} g \right]$$

where

(3.5) $$W(g) = \int_{A/F} V_\varphi\left[\begin{pmatrix} 1 & u & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} g \right] \bar{\psi}(u)\, du.$$

Then, just as before,

(3.6) $$\int_{F_A^\times / F^\times} V_\varphi\left[\begin{pmatrix} a & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right] |a|^{s-1} d^\times a = \int_{F_A^\times} W\left[\begin{pmatrix} a & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right] |a|^{s-1} d^\times a$$

Again this formula gives the analytic continuation of (1.2).

To obtain the functional equation we replace φ by $\tilde{\varphi}$ as in (2.7). Then $\tilde{V}$ is replaced by $\hat{V}$, π by $\tilde{\pi}$, and W by $\tilde{W}$: relation (2.8) still holds with now

$$w = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

What's new is the relation between V_φ and $V_{\tilde{\varphi}}$:

(3.7) $$\int_{\mathbb{A}} V_\varphi\left[\begin{pmatrix} 1 & 0 & 0 \\ x & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} g\right] dx = V_{\tilde{\varphi}}\left[w' \, {}^t g^{-1}\right]$$

where

(3.8) $$w' = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}.$$

Then (2.9) is replaced by

(3.9) $$\iint V_\varphi\left[\begin{pmatrix} a & 0 & 0 \\ x & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\right] |a|^{s-1} d^\times a \, dx = \int V_{\tilde{\varphi}}\begin{pmatrix} a & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} |a|^{-s} d^\times a$$

$(a \in F^\times_{\mathbb{A}}/F^\times, \ x \in \mathbb{A})$ and (2.10) by

(3.10) $$\iint W\left[\begin{pmatrix} a & 0 & 0 \\ x & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\right] |a|^{s-1} d^\times a \, dx = \int \tilde{W}\left[\begin{pmatrix} a & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} w'\right] |a|^{-s} d^\times a$$

$(a \in F^\times_{\mathbb{A}}, \ x \in \mathbb{A})$. In (3.10) the left-hand side is still equal to the product (1.2) times an elementary factor, ~~so~~ this gives the required functional equation.

4 - The above theory can be made much more precise: to every irreducible representation π_v of G_v we attach a factor $L(s, \pi_v)$ equal to $(1 - A_v q_v^{-s})^{-1}$ if

π_v is unramified; we also attach an exponential function of s noted $\varepsilon(s, \pi_v, \psi_v)$. If π_v is unramified and ψ_v has order 0 then this factor is one. Now we set

$$L(s, \pi) = \prod_v L(s, \pi_v),$$

$$\varepsilon(s, \pi) = \prod_v \varepsilon(s, \pi_v, \psi_v).$$

The first product converges for $\mathrm{Re}\, s >> 0$ and the second product has almost all its factors equal to one. If π is as before, then $L(s, \pi)$ extends to an holomorphic function of s and:

$$L(s, \pi) = \varepsilon(s, \pi) L(1-s, \tilde{\pi}).$$

There is also a converse theorem. Assume that for a given irreducible representation π all the L-functions attached to the representations

$$\pi \otimes \chi : g \mapsto \pi(g)\chi(\det g),$$

where χ is a character of $F^\times_{\mathbb{A}}/F^\times$, satisfy the above conditions. Then π is automorphic.

Finally let us say that the direct part of the theory extends to all $n > 3$ but not the converse part.

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Title: DIFFERENTIAL GEOMETRY OF THE YANG-MILLS EQUATION

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Let (M,g) be a compact oriented Riemannian manifold. Let G be a compact Lie group and $\mathfrak{g}$ its Lie algebra.

We will be interested in finding adapted connections to a vector bundle $\pi : E \to M$ with structure group G. We denote by R^∇ the curvature of the connection ∇ (R^∇ is a $\mathfrak{g}_E$-valued two-form on M where $\mathfrak{g}_E$ is the bundle of Lie algebras of infinitesimal automorphisms of E). It is classical that with respect to the exterior differential d^∇ of vector-valued forms defined by ∇, $d^\nabla R^\nabla = 0$ (this is the Bianchi identity).

We denote the affine space of G-connections over E by $\mathcal{C}$. This space is acted upon by the gauge group (group of sections of the bundle of G-automorphisms of E).

The Yang-Mills functional $\mathcal{YM}$ on $\mathcal{C}$ is defined as : $\mathcal{YM}(\nabla) = \int_M \|R^\nabla\|^2$. It is invariant under the gauge group, (cf [2]).

A critical point for $\mathcal{YM}$, called a Yang-Mills connection, satisfies $\delta^\nabla R^\nabla = 0$, where δ^∇ is the adjoint of d^∇ in other words R^∇ is <u>harmonic</u>.

This can also be written as $d^\nabla(R^\nabla \circ *) = 0$, making the case $\dim M = 4$ special, since there R^∇ splits into $R^{\nabla +}$ and $R^{\nabla -}$ which turn out to be both harmonic. In this case appears also a constraint : the Pontrjagin number $p(E)$. The absolute minimum of $\mathcal{YM}$ is then achieved by $\pm$ self dual connections depending on whether $p(E)$ is positive or negative.

By using a Weitzenböck formula (which compares the Laplacian $d^\nabla\delta^\nabla + \delta^\nabla d^\nabla$ to the Laplacian $\nabla^*\nabla$), one can prove the following :

THEOREM.- (cf [1]). On S^n ($n \geqslant 3$) a Yang-Mills field which is uniformly small is zero (more precisely if at each point $\|R^{\nabla}\|^2 < 1/2 \binom{n}{2}$, in fact $R^{\nabla} = 0$).

which can be refined in dimension 4 to

THEOREM.- (cf [1]). On S^4 , if $\|R^{\nabla\pm}\|^2 < 3$, then $R^{\nabla\pm} = 0$.

As a corollary of this, we prove that an explicit C^0-neighboorhood of $\pm$ self dual Yang-Mills connections does not contain other Yang-Mills connections.

One can derive more information from the operator of the second variational formula which is, as expected, an elliptic self-adjoint second order differential operator.

A critical point ∇ is called stable (resp. weakly stable) if at ∇ the second variation is positive (resp. non-negative).

Then we have the following :

THEOREM (J. Simons).- On S^n ($n \geqslant 5$) , there is no weakly stable Yang-Mills field.

THEOREM.- (cf [1]). If the structure group of the bundle is SU_2 or SU_3 , any weakly stable Yang-Mills field on S^4 is $\pm$ self-dual.

For other groups the situation is more subtle : for SO_4 for example, the Euler number appears as a new constraint and the usual notion od self-duality must be refined. With this new notion analogous results can be obtained. This seems to be especially interesting since a lot of seven-dimensional exotic spheres can be obtained as total spaces of sphere bundles of SO_4-bundles over S^4 .

Title: Holomorphic and harmonic maps of surfaces.

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1. The rendering problems.

(A) Let M and N be compact smooth surfaces, and $\mathcal{H}$ a homotopy class of smooth maps $M \to N$. Firstly, suppose we work in the oriented context (i.e., M, N are oriented, and $\mathcal{H}$ consists of orientation-preserving maps).

Problem 1. When do complex structures exist on M, N relative to which $\mathcal{H}$ contains a holomorphic representative?

The solution follows easily from recent work of Edmonds [1]: iff $d_{\mathcal{H}} \geq j_{\mathcal{H}}$, with equality when and only when $\mathcal{H}_*$ is injective.

Here $d_{\mathcal{H}} =$ degree φ for all $\varphi \in \mathcal{H}$.

$j_{\mathcal{H}} = \text{index}\,\{\pi_1(N) : \varphi_* \pi_1(M)\}$. $\mathcal{H}_* = \{\varphi_* : \varphi \in \mathcal{H}\}$.

Problem 2. When do Riemannian metrics exist on M, N relative to which $\mathcal{H}$ contains a harmonic representative?

The solution consists in combining several different results in the theory (see [2] for background): Always, except for the case genus $M = 1$, genus $N = 0$, $d_{\mathcal{H}} = 1$.

(B) If we drop all orientation assumptions, then Problem 1 has no full generalisation — for branched covers are oriented maps. However, Problem 2 remains. Hopefully, it has the following solution (in preparation by Eells–Lemaire):

Always, with the following five exceptions:

$$T \xrightarrow{\varphi} S \xrightarrow{\theta} P, \quad T \xrightarrow{\tilde{\varphi}} P \qquad\qquad P \xrightarrow{\psi} S \xrightarrow{\theta} P, \quad P \xrightarrow{\tilde{\psi}} P$$

degree $\varphi = \pm 1$ degree$_2$ $\psi = 1$. Here θ is the orientable covering map of the sphere onto the projective plane. "Hopefully" means that there is one case whose details have not been written down as yet; and in another case we haven't verified fully a certain boundary regularity matter (p.5). Nevertheless, the above assertion is fully established for oriented maps.

2. Methods for the solution of Problem 2.

(A) In case $N \neq S$ or P, the solution follows from the existence theorem in [3].

(B) In the oriented case, with $N = S$, sufficiently symmetric Riemannian metrics exist on M relative to which the existence of a harmonic map in $\mathcal{H}$ can be established by solving a Dirichlet problem and then applying careful reflections (Lemaire Thesis); we must exclude the case $M = T$.

(C) In the oriented case, the exception was established in [4]. A simplification in the proof can be given using a construction in [5]. In fact, $T^{*\prime}(M) \otimes \varphi^{-1}T'(N)$ and $T^{*\prime}(M) \otimes \varphi^{-1}T''(N)$ are holomorphic line bundles $/M$, relative to a Riemannian metric h on N; and the partial differentials

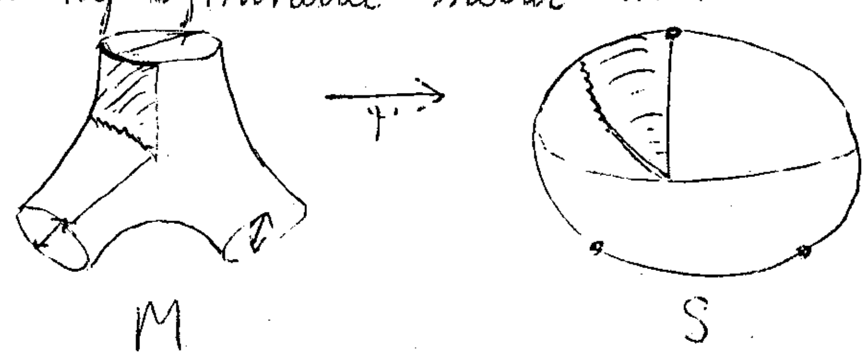
$\partial'\varphi$ and $\partial''\varphi$ are holomorphic sections, provided that φ is harmonic.

(D) Case M nonorientable, $N=S$

If $M=P$, then every harmonic map $\varphi: P \to S$ is constant. Indeed, the covering map $\tilde{\varphi}: S \to S$ is harmonic; by (C) above it is $\pm$holomorphic; but $d_{\tilde{\varphi}}=0$. These assertions are valid for arbitrary Riemannian metrics on P, S. They provide the other two exceptions in §2B.

If $M=K$ (=Klein bottle), the Gauss map of the Delaunay nodoid (a surface of constant mean curvature in $\mathbb{R}^3$ obtained by rotating about a line L the trace of a focus of a hyperbola rolling along L) factors through K to give a harmonic map $\varphi: K \to S$ of $\text{degree}_2 \varphi = 1$. This is Calabi's interpretation of a harmonic map $T \to S$ constructed by Smith [6].

If M is arbitrary, then we can put a sufficiently symmetric metric on it:



We map each of the circles of M onto the corresponding symmetrically placed points on S — and then solve the problem of constructing a harmonic map between the shaded regions. Successive reflections define a map $\varphi: M \to S$, harmonic on interior regions. A reflection principle insures that φ is harmonic across reflections. Intersection points of reflection lines are removable singularities, since the energy of φ is finite. At points where reflection lines meet the circles, hopefully Hildebrandt's boundary

regularity theorem insures that φ is smooth there as well.

1

Title: Non-rationality of some quartic threefold.

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References: A. Collino : private communications.

We use degeneration methods to prove:

<u>Theorem</u> : <u>Not every quartic threefold in $\mathbb{P}^4(\mathbb{C})$ is rational</u>.

Starting point is the intermediate Jacobian of a threefold X, smooth and projective, given by

$$J^3(X) = (H^{3,0} \oplus H^{2,1}) \backslash H^3(X,\mathbb{C}) / H^3(X,\mathbb{Z})$$

which is an abelian variety if $H^{3,0} = 0$ and carries a principal polarization induced by the intersection form on $H^3(X,\mathbb{C})$. Clemens and Griffiths showed, that for any rational threefold X, $J^3(X)$ is isomorphic, as a principally polarized abelian variety, to a product of Jacobians of curves. Moreover they showed that the intermediate Jacobian of a cubic threefold is not isomorphic to a product of Jacobians.

If one has a family of smooth threefolds X_s over a curve S where X_0 gets singular, one also has a family $J^3(X_s)$ of intermediate Jacobians which, by work of S. Zucker and JS, tends to a generalized intermediate Jacobian $J^3(X_0)$.

<u>Theorem</u>: (A. Collino) If a family $\mathcal{A}$ of products of Jacobians over a curve S has stable reduction at s, then $\mathcal{A}_s$ is isomorphic to a product of generalized Jacobians of curves.

So suppose that every quartic threefold is rational. Then for every degeneration $X \to S$ of quartic threefolds, the generalized intermediate Jacobian $J^3(X_0)$ would be isomorphic to a product of generalized Jacobians. In particular the abelian part of $J^3(X_0)$ is isomorphic to a product of Jacobians.

Consider the degeneration of a quartic threefold which acquires an ordinary triple point for $s=0$. E.g.

$$X = \{ (z,t) \in \mathbb{P}^4 \times S \mid z_0 f(z_1,..,z_4) + g(z_1,..,z_4) + t\, h(z_0,--,z_4) = 0 \}$$

with f,g,h homogeneous of degrees $3,4,4$, sufficiently general. X_0 has triple point $(1,0,..,0)$. One shows: $J^3(X_0)$ has abelian part which contains as a factor the intermediate Jacobian of the cubic threefold with equation $z_0^3 = f(z_1,..,z_4)$ so $J^3(X_0)$ cannot be a product of generalized Jacobians. As Namikawa remarks, one can also use a degeneration into the union of a cubic threefold and a hyperplane to get this result.

With more refined methods one can even handle the (generic) cubic threefold: take a degeneration into a cubic threefold with one ordinary double point. The abelian part of $J^3(X_0)$ is isomorphic to the Jacobian of a curve C, which is the intersection of a cubic and a quadric in $\mathbb{P}^3$. If The extensions of $J(C)$ by $\mathbb{G}_m$ are parametrised by $J(C)$, and for a curve C' with double point x and C as its normalization, the generalized Jacobian of C' corresponds to the element $cl((x_1)-(x_2))$ in $J(C)$, where $x_1, x_2 \in C$ map to x. One shows that the element in $J(C)$ representing $J^3(X_0)$ is obtained as follows: take lines ℓ and m on the quadric containing C and not belonging to the same pencil; then take $cl(\ell - m)\cdot C \in J(C)$.
One shows that this is not of the form $cl((x_1)-(x_2))$ for $x_1, x_2 \in C$. Hence $J^3(X_0)$ is not a generalized Jacobian of any C'.

Title: The Chowla-Selberg Formula

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Let k be an imaginary quadratic field of discriminant $-d$ and class number h. Let w be the number of roots of unity in k, and let ε be the Dirichlet character modulo d which corresponds to this quadratic extension. Define the number

$$b_k = \sqrt{\pi} \prod_{0<a<d} \Gamma(a/d)^{w\varepsilon(a)/4h}$$

Chowla and Selberg [1] prove that, up to algebraic numbers, b_k gives the periods of any elliptic function field with algebraic coefficients and a non-trivial multiplier in k.

In [4] we prove the following generalization:

<u>Theorem 1.</u> Let A be an abelian variety of dimension n which is defined over $\overline{Q}$. Assume that A has complex multiplication by k and that the representation of k on $\mathrm{Lie}(A)$ decomposes as p copies of one embedding and $q = n-p$ copies of the conjugate embedding.

Then there is a sub-Hodge structure M of $H^n(A)$ of rank 2, type $(p,q)+(q,p)$, with complex multiplication by k. One has $M_Q = \tilde{\Lambda}_k H^1(A,Q)$; if $\omega \in M^{pq}$ and $\nu \in M^{qp}$ are forms of pure type defined over $\overline{Q}$, then

$$(\text{Periods of } \omega) \sim (b_k)^p \, (2\pi i/b_k)^q$$
$$(\text{Periods of } \nu) \sim (b_k)^q \, (2\pi i/b_k)^p$$

where $a \sim b$ means that a/b is an algebraic number.

When $n=1$, A is an elliptic curve with complex multiplication by k, and this is just a restatement of the result of Chowla and Selberg. When $p=q$ it is useful in the analysis of certain exceptional Hodge classes. The theorem is proved by first making a specific calculation on the Jacobian of the Fermat curve (using results of Rohrlich [4] and Weil [5]) and then transferring this result to other abelian varieties in a Shimura family, using the existence of two global sections of the middle cohomology which are horizontal for the Gauss-Manin connection.

By a powerful extension of these methods, Deligne [2] has recently obtained:

<u>Theorem 2.</u> Let A be an abelian variety which is defined over $\overline{Q}$, and M a sub-Hodge structure of rank 1 of $H^{2p}(A)$ (a Hodge class). If $\omega \in M^{pp}$ is defined over $\overline{Q}$, then

$$(\text{Periods of } \omega) \sim (2\pi i)^p.$$

As a corollary, he shows that the period conjecture

of [4] is true for abelian varieties. For the relation between this conjecture and the special values of Hecke L-series, see the papers of Deligne [3] and Weil [5].

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Title: Rational points of elliptic curves and congruences of L series.

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B. Mazur: Modular curves and the Eisenstein ideal.
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In considering arithmetic questions concerning an elliptic curve E over $\mathbb{Q}$ it is natural to suppose that E is expressed as the quotient of one of the modular curves $X_0(N)/\mathbb{Q}$ ($N = 1, 2, \cdots$). If the conjecture of Taniyama-Weil is true, one loses no generality by this hypothesis while one gains much structure.

For example, if N is a prime number and K a quadratic imaginary number field in which (N) splits into a product of distinct primes, then, given a parametrization of E by $X_0(N)$, one can produce a point e_K of E rational over K (the Birch-Heegner point).

In general, if $L(E, \chi, s)$ is the Hasse-Weil L series of E, "twisted by a finite Dirichlet character χ", it converges (as an infinite product) only in the half-plane $\mathrm{Re}(s) > 3/2$.

When E is parametrizable by $X_0(N)$, then $L(E, \chi, s)$ has an analytic extension to the entire plane. Moreover the "special values" $L(E, \chi, 1)$ are "essentially cyclotomic integers" in a sense to be made

precise in an example below.

Example of $E = X_0(11)$:

For the first ten values of N, $X_0(N)$ is a curve of genus zero and hence admits no elliptic curves as quotients. The curve $X_0(11)$ is of genus 1. Being the "first" elliptic curve gotten from the series $X_0(N)$ it has been the subject of extensive study and indeed provides an excellent proving-ground for general theories. For this curve,

$$L(E, \chi, s) = \left(\frac{1}{1 - \chi(11) \cdot 11^{-s}}\right) \prod_{\substack{p \neq 11 \\ p \text{ prime}}} \left(\frac{1}{1 - a_p \chi(p) \cdot p^{-s} + p^{1-2s}}\right)$$

where the a_p are the integers occurring in $\sum_{n=1}^{\infty} a_n x^n = x \cdot \prod_{m=1}^{\infty} (1 - x^m)^2 (1 - x^{11m})^2$.

Then if χ is a primitive character of conductor m, one has that

$$c^{\operatorname{sign}\chi}(E) \cdot \tau(\bar{\chi}) \cdot L(E, \chi, 1) \overset{\text{defn.}}{=} \Lambda(E, \chi)$$

is a cyclotomic integer (i.e. is in $\mathbb{Z}[\chi]$

where $\tau(\bar{\chi})$ is the Gauss sum and $c^+(E) = 5/\text{real period of } E$; $c^-(E) = 2/\text{imaginary period of } E$

Questions:

Although we have questions of a quite general nature in mind and some results* for $X_0(N)$ where N is any prime, not much might is lost if we specialize them to the traditional example $X_0(11)$; we do so, then, in what follows.

(I) Formula mod p? For p a prime number, is there a formula for $\Lambda(E,\chi)$ mod p which depends only on the representation $\rho_p: \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \longrightarrow GL_2(\mathbb{F}_p)$ defined by the natural action of Galois on p-division points in E.

(II) Analogue of Kummer's mod p regularity condition? If K is a quadratic imaginary field with character χ of conductor prime to $p \neq 11$ are these statements equivalent?

ⓐ $\Lambda(E,\chi) \not\equiv 0 \mod p$

ⓑ The Mordell-Weil group E(K) is finite and there are no points of order p in the Shafarevitch-Tate group of E over K?

---

* relative to the Eisenstein primes of odd residual characteristic.

For our example $E = X_0(11)$ and $p = 5$, the representation decomposes into the direct sum of two 1-dimensional representations and thus one might expect a particularly simple formula in answer to question (I).

Indeed we obtain the following formula. There is an element $c \in \mathbb{F}_5^*$ such that for all primitive characters $\chi: (\mathbb{Z}/m\mathbb{Z})^* \longrightarrow A^*$ where A is any $\mathbb{F}_5$-algebra, and all m prime to 5,

$$c \cdot \Lambda(E,\chi) = \left(1 - \chi^{-1}(11)\right) B_{1,\chi} \cdot B_{1,\chi^{-1}}$$

where $B_{1,\chi}$ is the "first Bernoulli number":

$$B_{1,\chi} = \sum_{a=1}^{m-1} a\cdot\chi(a)$$

By means of this, plus a 5-descent, one can show that the answer to question (II) is yes, if $p = 5$.

More precise information concerning the Mordell-Weil group E(K) can be proved for quadratic imaginary fields if one supposes that $(m, 5) = 1$ and the class number of K is prime to 5.

Then:

If 11 doesn't split into the product of two distinct primes of K, the Mordell-Weil group is finite.

If 11 does split, the Birch-Heegner point $e_K \in E(K)$ is (defined, and is) of infinite order. The subgroup generated by e_K is of finite index in $E(K)$.

Title: *Computation of homotopy groups of diffeomorphism groups of compact manifolds*

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For a differentiable compact manifold (with or without boundary) M^n, we denote by $\mathrm{Diff}(M^n)$ the group of C^∞-diffeomorphisms endowed with the C^∞-topology. At this time a good deal of information is known about the homotopy type of $\mathrm{Diff}(M^n)$ for instance:

1) The existence of a stability range $\omega(n)$ (which is certainly $\geq n - 25/6$ but probably $\geq n-3$) so that the homotopy type of the $(\omega(n)-1)$- Postnikov term of $Diff(M^n)$ can be described away of the prime "2" as a twisted product of two pieces $T_1(M)$ and $T_2(M)$, with $T_1(M)$ an homotopy invariant and $T_2(M)$ a geometrical type invariant. $T_2(M)$ can be computed by the means of surgery and $T_1(M)$ can be ~~compu~~ rationally reduced to the computation of a Quillen type algebraic K-theory of the space M defined by Waldhausen.

2) Good upper bound estimates on the homotopy groups of $T_1(M)$ can be obtained for 1- connected case and some exact computations can be also produced in few particular cases (They have been obtained independentely by Hsiang-Staffeldt and Burghelea)

Beside the known exemples of the exact values of $\pi_i(Diff\, T^n)\otimes Q$, $\pi_i(Diff(D^n))\otimes Q$, $\pi_i(Diff\, \Sigma^n/G)\otimes Q$ .... (see the papers of Hsiang and .... in Proc. Symp. in pure Math. vol 32 as also [3]) we prove the following theorems

<u>Theorem 1</u> : a) $\pi_i(Diff\, CP^n)\otimes Q = \pi_i(PGL_{n+1}(C))\otimes Q + \oplus G_i$ if $i \leq \omega(2n)-1$ with $G_i = \begin{cases} 0 & \text{if } i=2k \\ Q^{[\frac{n}{2}]+1} & i=4k-1 \\ Q^{[\frac{n-1}{2}]+1} & i=4k-3 \end{cases}$

b) $\pi_i(Diff\, HP^n)\otimes Q = \pi_i(PGL_{n+1}(H))\otimes Q \oplus \overline{G}_i$ if $i \leq \omega(4n)-1$ with $\overline{G}_i = \begin{cases} 0 & \text{if } i \not\equiv -1\ (4) \\ Q^n & \text{if } i = 4k-1 \end{cases}$; here PGL denotes the projective group.

<u>Theorem 2</u> : If M^n is a 1- connected compact differentiable manifold, then $\dim \pi_i(Diff(M^n))\otimes Q < \infty$ if $i \leq \omega(n)-1$.

These two theorems are applications of Theorems A and D bellow.

<u>Theorem A</u>: ([3]) There exists a homotopy functor $\mathcal{S}$: $: CW \rightsquigarrow \tilde{\Omega}^{\infty h}$ (CW denotes the category of finite CW-complexes and $\tilde{\Omega}^{\infty h}$ the homotopy category of ∞-loop spaces) so that:

a) $\mathcal{S}(M^n) \overset{\omega(n)}{\sim} \mathbb{B}\, C(M^n)$ where $C(M^n)$ is the topological group of the diffeomorphisms of $M^n \times I$ which restrict to identity on $M^n \times \{0\} \cup \partial M^n \times I$ and $\mathbb{B}$, the "classifying space"- functor.

b) There exists a natural map $\theta(X) : \underline{H}(X) \to \mathcal{S}(X)$ where $\underline{H}(X)$ denotes the associative H-space of simple homotopy equivalences of X (the meaning of naturality needs explanations)

c) For any stable spherical fibration ξ on X, there exists an involution $\tau(\xi) : \mathcal{S}(X) \circlearrowleft$ which decomposes $\mathcal{S}(X)_{odd}$ as a product of $\mathcal{S}^+(X) \times \mathcal{S}^-(X)$; all these involutions agree homotopically after localisation to "0", i.e on the rational homotopy type of $\mathcal{S}(X)$. (*

Let $\widetilde{Diff}(M^n)$ be the topological group of block diffeomorphisms (described simplicially by its k-simplexes, diffeomorphisms $h : \Delta[k] \times M \to \Delta[k] \times M$ with $h(d_i(\Delta[k]) \times M) \subset d_i(\Delta[k]) \times M$ and $h / \Delta[k] \times \partial M = id$ and $\boxed{I : \widetilde{Diff}(M^n) \hookrightarrow \underline{H}(M^n)}$; the map I can be

*) There are many other interesting properties of this homotopy functor ~~but~~ (see for instance [3])

completely understood in terms of surgery theory. Let $\boxed{\mathrm{II}: \mathcal{E}^{\varepsilon(n)}(M^n) \longrightarrow \underline{\mathcal{H}}(M^n)}$ $\varepsilon(n) = \begin{cases} + \text{ if } n \text{ even} \\ - \text{ if } n \text{ odd} \end{cases}$ be the homotopy theoretic fibre of the composition $\underline{\mathcal{H}}(M^n) \xrightarrow{\Theta(M^n)} \mathcal{I}(M^n) \to \mathcal{I}^{\varepsilon(n)}(M^n)$ where + or − are taken with respect to the involution produced by the tangent bundle of M^n. Clearly II is an invariant of the homotopy type of the "compact manifold with boundary", $(M^n, \partial M^n)$.

<u>Theorem B</u> ([3]) $\mathcal{D}iff(M)_{odd}$ is $(\omega(n)-1)$- homotopy equivalent with the "odd"- localisation of the pull back $T(M)$ of (I) and (II); so $\widetilde{\mathcal{D}iff}(M^n) = T_2(M)$, $\Omega \mathcal{I}^{\varepsilon(n)}(M^n) = T_1(M)$ and the twisting is produced by $\Theta(M)$ (fig 1)

$$
\begin{tikzcd}
\Omega \mathcal{I}^{\varepsilon(n)}_{(M)} \arrow[r, dotted, no head] \arrow[d, equal] & \mathcal{E}^{\varepsilon(n)}(M) \arrow[r, "\text{homotopy theory}" above, "(\mathrm{II})" below] & \underline{\mathcal{H}}(M) \arrow[r, "\Theta^{\varepsilon(n)}(M)"] & \mathcal{I}^{\varepsilon(n)}_{(M)} \\
\Omega \mathcal{I}^{\varepsilon(n)}_{(M)} \arrow[r, dotted, no head] & T(M) \arrow[u] \arrow[r] & \widetilde{\mathcal{D}iff}(M) \arrow[u, "\text{surgery}" right, "(\mathrm{I})" right] &
\end{tikzcd}
$$

fig 1.

<u>Exemple:</u> If $M = N \times I$ and $\tau : \mathcal{D}iff(M) \circlearrowleft$ is the involution induced by conjugating with $id_N \times \nu$, where ν is the reflection through midle point of I, $T_1(M) = \mathcal{D}iff(M)^+_{odd}$ $T_2(M) = \mathcal{D}iff(M)^-_{odd} = \widetilde{\mathcal{D}iff}(M)_{odd}$.

Applying the surgery theory one proves that $\dim(\pi_i(\widetilde{\mathcal{D}iff}(M^n) \otimes Q) < \infty$ if $\pi_1(M) = 0$, and that for $M = CP^n$ or HP^n $\pi_i(\widetilde{\mathcal{D}iff}(\dots P^n)) \otimes Q = \pi_i(PGL_{n+1}(\dots)) \otimes Q \oplus \oplus G_i(\bar{G}_i)$ as indicated in the statement of Theorem 1. One checks that in this last case the twisting is rationally trivial.

Waldhausen has extended the Quillen higher algebraic K-theory for unitary associative rings to unitary associative topological (semisimplicial) rings and defined the space $\underline{\underline{K}}(X) = B\widetilde{GL}^{\bullet}_{+}(Z\Omega X) \times \mathbb{Z}$ for any based pointed

semisimplicial complex $(X, *)$, ΩX being the loop space considered as a semisimplicial group and $Z(\Omega X)$ the semisimplicial group ring associated with [redacted] ΩX.

If one denotes by ${}^{Diff}Wh_{alg}(X)$ the fibre of $\underline{\underline{K}}(X) \to \underline{\underline{K}}^S(X) = \varinjlim \Omega^n K(\Sigma^n(X \cup p))$ one has

<u>Theorem C</u> ([4]) $\Omega\, {}^{Diff}Wh_{alg}(X)$ and $\mathcal{I}(X)$ are rationally homotopy equivalent.

We have computed $\pi_i(\underline{\underline{K}}(X)) \otimes Q$ for $X = K(G, 2r)$ and have also found upper bounds for $\dim \pi_i(\underline{\underline{K}}(X)) \otimes Q$ in terms of $\dim \pi_i(X) \otimes Q$ for all X, 1-connected, proving they are finite if $\dim \pi_i(X) \otimes Q$ are finite. (Similar results have been independently obtained by Hsiang & Staffeldt). This proves Theorem 2

In particular we have proved

<u>Theorem D</u> ([5]) 1). $\underline{\underline{K}}(X)$ has the rational homotopy type of $\underline{\underline{K}}(pt) \times \prod_{i=1}^{\infty} K(Z; 2ri)$ if $X = K(Z; 2r)$

2). ${}^{Diff}Wh_{alg}(X)$ and ${}^{Diff}Wh_{alg}(*)$ have the same rational homotopy type.

Combined with the Hsiang's computations of ${}^{Diff}Wh^{\pm}_{alg}(*)$ one obtains Theorem 1.

Title: Residues of real foliation singularities

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By analogy with Hopf's Theorem that for a vector field X on a compact manifold M with finite singularities $x_1, \dots, x_r \in M$ ($X(x_i)=0$), the Euler characteristic $\chi(M) = \sum_{i=1}^{r} \mathrm{Index}(X, x_i)$, with the corollary that X can be non-singular $\Leftrightarrow \chi(M) = 0$, the Bott Vanishing Theorem for certain polynomials in the Pontryagin (resp. Chern) classes of the normal vector bundle to a non-singular C^∞ (resp. holomorphic) foliation suggests the existence of invariants which measure the deviation from Bott vanishing in the neighborhood of singular sets of foliations. In the holomorphic case, Baum and Bott [1] have given such a Residue Theorem, and we give an analogous result for real C^∞ foliations with singularities (see [3] for the case that the singularities are isolated points.)

Notation. Let M be a manifold (complex analytic or oriented C^∞) of dimension n; and $\mathcal{F}$ a foliation (holomorphic or C^∞) with singularities on M, of codimension q. S denotes the singularity set, and Z a compact connected component of S. We define a C^∞ foliation $\mathcal{F}$ of M with singularity set S to be a C^∞ foliation of $M-S$. Baum and Bott define a holomorphic foliation $\mathcal{F}$ of M with singular set S to be an integrable, full, coherent subsheaf ξ of the sheaf of germs of holomorphic vector fields on M, such that $\xi|M-S$ is exactly the sheaf of germs of holomorphic sections of the tangent bundle of a holomorphic foliation of $M-S$. (In the holomorphic case, dimension and codimension are complex.)

In the holomorphic case, S is an analytic subvariety. In the C^∞ case, assume that S is a union of a finite number of smooth submanifolds meeting transversely. Set $s = \dim(S)$.

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1. The Holomorphic Residue Existence Theorem [1] Let $\mathcal{F}$ be a holomorphic foliation of codimension q of M with singularity set S, as above, and let Z be a compact connected component of S. Let $\varphi = \varphi(C_1,\dots,C_n)$ be a weighted homogeneous polynomial of degree $k>q$, where $\deg(C_i)=i$. Then there is defined a residue

$$\mathrm{Res}_\varphi(\mathcal{F},Z)\in H_{2n-2k}(Z;\mathbb{C})$$

such that

(1) $\mathrm{Res}_\varphi(\mathcal{F},Z)$ depends only on φ and the local structure of $\mathcal{F}$ near Z.

(2) If M is compact, then

$$\varphi(c_1(Q),\dots,c_n(Q)) = \sum_{Z \text{ components of } S} \mu_* \mathrm{Res}_\varphi(\mathcal{F},Z),$$

where Q is the (virtual) normal complex vector bundle to $\mathcal{F}$ over M, and μ_* is the composition $H_{2n-2k}(Z;\mathbb{C})\xrightarrow{i_*} H_{2n-2k}(M;\mathbb{C})\xrightarrow{\gamma} H^{2k}(M;\mathbb{C})$ (i_* induced by the inclusion $i: Z\hookrightarrow M$, γ = Poincaré duality), while $c_i(Q)$ is the ith Chern class of Q.

Remarks. $Q|M-S$ is a complex vector bundle. As a virtual bundle it is extended over M via a finite resolution of ξ. (See [1].)

Baum and Bott also give a formula for $\mathrm{Res}_\varphi(\mathcal{F},Z)$ in certain cases, in terms of the Grothendieck residue symbol [1, Theorem 3].

2. The Real Residue Existence Theorem [3] Let $\mathcal{F}$ be a C^∞ foliation of codimension q on $M-S$, as above, and let Z be a compact connected component of S. Let $\varphi = \varphi(P_1,\dots,P_\ell)$ be a weighted homogeneous polynomial of degree $4k>2q$, where $\deg(P_i)=4i$. Suppose also that

(*) $\Delta+1+4\ell<n$.

Then there is defined a residue $\mathrm{Res}_\varphi(\mathcal{F},Z)\in H_{n-4k}(Z;\mathbb{R})$ such that

(1) $\mathrm{Res}_\varphi(\mathcal{F},Z)$ depends only on the foliation $\mathcal{F}$ near Z.

(2) If M is compact, then

$$\varphi((j^*)^{-1}p_1(Q),\dots,(j^*)^{-1}p_\ell(Q)) = \sum_{Z \text{ components of } S}\mu_*\mathrm{Res}_\varphi(\mathcal{F},Z)$$

where $Q = T(M-S)/T\mathcal{F}$ is the normal vector bundle to $\mathcal{F}$ over $M-S$, μ_* is the composition $H_{n-4k}(Z,\mathbb{R})\xrightarrow{i_*}H_{n-4k}(M;\mathbb{R})\xrightarrow{\gamma}H^{4k}(M;\mathbb{R})$,

$p_i(Q)$ is the ith Pontryagin class of Q, $p_i(Q)\in H^{4i}(M-S;\mathbb{R})$ and $j^*: H^{4i}(M;\mathbb{R})\to H^{4i}(M-S;\mathbb{R})$ is the isomorphism (for $i\le\ell$, by (*)) induced by the inclusion $j: M-S\hookrightarrow M$.

Remarks. 1. The holomorphic structure over S assumed in [1] has been replaced by the hypothesis (*), which assures that no linking is possible between cycles of dimensions $4k$ and Δ, respectively.

2. The analogous non-linking hypothesis $\Delta+\ell<n$ can replace the holomorphic structure of ξ on S (assuming only that $\mathcal{F}$ is a holomorphic foliation of $M-S$), provided that only the variables $C_1,\dots,C_\ell$ appear in φ.

3. For Riemannian foliations, the hypothesis $4k>2q$ can be relaxed to $4k>q$.

4. $\mathrm{Res}_\varphi(\mathcal{F},Z)$ is invariant under concordance of $\mathcal{F}$.

5. If H is a Haefliger Γ_q-structure on $M-S$, and the other hypotheses are maintained, then there is an analogous R.E.T. involving $\mathrm{Res}_\varphi(H,Z)$, with Q the underlying vector bundle of H. Thus the type of singularity which appears in a Γ_q-structure is not detected by the residue.

6. Clearly $\mathrm{Res}_\varphi(\mathcal{F},Z)=0$ if $\mathcal{F}$ extends to a foliation (or a Γ_q-structure) on $(M-S)\cup Z$. The residue can be interpreted as an obstruction.

7. As in the case of [1, Theorem 4], Heitsch rigidity holds for the residue except perhaps in the lowest dimension where the residue can be defined:

Rigidity Theorem. Let M and $Z\subset M$ be as before, and let U be an open set containing the compact set Z. Let $\mathcal{F}_t$ ($t\in[0,1]$) be a C^∞ family of codimension q foliations of $U-Z$. If $\deg(\varphi)=4k>2q+2$, then $\mathrm{Res}_\varphi(\mathcal{F}_t,Z)\in H_{n-4k}(Z;\mathbb{R})$ is independent of t.

(This follows by an argument similar to the proof of the Real R.E.T. since the foliations $\mathcal{F}_t$ define a codimension $(q+1)$ foliation of $(U-Z)\times[0,1]$.)

8. The hypothesis C^∞ can be relaxed to C^2 without difficulty.

3. Outline of Proof of the Real R.E.T. Assume that $\mathcal{F}$ is a smooth foliation of $M-S$, and let Z be a compact connected component of S. Assume the hypotheses of the Real R.E.T.. Let U be an open neighborhood of Z with smooth boundary, such that Z is a deformation retract of Z.

Choose a connection ∇ on Q that is basic with respect to $\mathcal{F}$ (see [3] or Bott, Springer Lect. Notes in Math. No 279 for details), and let $K \in \Lambda^2(M-S, \mathfrak{gl}_q)$ be the curvature 2-form of ∇. Define closed forms $\alpha_i \in \Lambda^{4i}(M-S)$, $i=1,\dots,[q/2]$, by

$$\det\left(I + \frac{\sqrt{-1}}{2\pi} K\right) = 1 + \alpha_1 + \alpha_2 + \dots + \alpha_{[q/2]}.$$

Then $p_i(Q) = [\alpha_i] \in H^{4i}(M-S;\mathbb{R})$. For $i=1,\dots,\ell$, extend $\alpha_i | M-S-U$ to a closed form $\alpha_i' \in \Lambda^{4i}(M-S \cup Z)$. This is possible and unique, up to addition of an exact form on U, since $H^{4i+1}(U;\mathbb{R}) = H^{4i}(U;\mathbb{R}) = 0$ by hypothesis (*).

Now since $\deg(\varphi) = 4k > 2q$, $\varphi(\alpha_1,\dots,\alpha_\ell) = 0 \in \Lambda^{4k}(M-S)$ according to the Bott vanishing property. ~~Define~~ Then $[\varphi(\alpha_1',\dots,\alpha_\ell')] \in H^{4k}(M, M-U)$.

Define $\operatorname{Res}_\varphi(\mathcal{F}, Z) = (\bar{\mu}_*)^{-1}[\varphi(\alpha_1',\dots,\alpha_\ell')] \in H_{n-4k}(Z;\mathbb{R})$ where $\bar{\mu}_*$ is the composition $H_{n-4k}(Z;\mathbb{R}) \xrightarrow{i_*} H_{n-4k}(U;\mathbb{R}) \xrightarrow{\gamma} H^{4k}(M, M-U;\mathbb{R})$. The definition is independent of the choices, proving (1).

To prove (2), observe that the construction of the forms α_i' can be done simultaneously extending over all components of S, so that one has $\alpha_i' \in \Lambda^{4i}(M)$ with $(j^*)^{-1} p_i(Q) = [\alpha_i'] \in H^{4i}(M;\mathbb{R})$. Then (2) follows.

4. Some examples. (See [3] for details)

1. For $M^{8k} = S^{4k} \times S^{4k}$, $x_0 \in M$, $\exists$ codimension $4k-1$ foliation $\mathcal{F}$ of $M-\{x_0\}$ such that $\operatorname{Res}_{p_k^2}(\mathcal{F}, x_0) \neq 0$. (For $k=1,2$, one can use the quaternionic and Cayley projective planes for M in this example.)
2. For $m > r > 0$ $(m \geq 4)$, $M = P_m(\mathbb{C})$, $q = 2r$, $S = P_{m-r-2}(\mathbb{C})$, $\ell = [\frac{r+1}{2}]$, and for any $\varphi(P_1,\dots,P_\ell)$ of degree $\geq 4(r+1)$, $\exists\, \mathcal{F}$ on $M-S$ such that $\operatorname{Res}_\varphi(\mathcal{F}, S) \neq 0$.

For example, let $M = P_{10}(\mathbb{C})$, $q=6$, $S = P_5(\mathbb{C})$, $\varphi = p_1^4, p_1^2 p_2, p_2^2, p_1^5, p_1^3 p_2$, or $p_1 p_2^2$.

Title: Volume of Hyperbolic Manifolds

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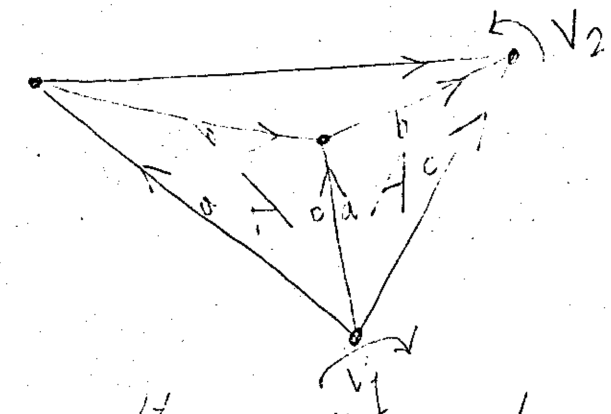
By a *hyperbolic manifold* I mean a complete Riemannian manifold of dimension $n \geq 2$ with finite volume and with (sectional curvature) $\equiv -1$.

One class of examples, due essentially to Bianchi, is the following. Let $\mathcal{O}$ be the ring of integers in an imaginary quadratic field $\mathbb{Q}(\sqrt{-d})$. Then $PSL_2\mathcal{O}$ is a discrete subgroup of finite covolume in $PSL_2\mathbb{C}$, which can be identified with the group of orientation preserving isometries of the hyperbolic space H^3. If $\Gamma \subset PSL_2\mathcal{O}$ is a torsionfree subgroup of finite index, it follows that H^3/Γ is a hyperbolic 3-manifold.

Here is a more geometric example, due to Gieseking. Identify the faces of a regular tetrahedron, in pairs, as follows. Identify the front two faces by rotating $120°$ about the vertex v_1 as indicated, and the back two faces by rotating about the vertex v_2 in the same way.



The resulting cell complex has 1 vertex, 1 edge, 2 faces and 1 three-cell. It is a manifold except at the vertex, whose neighborhood is homeomorphic to a cone over a Klein bottle. Removing this bad point, we obtain a (non-orientable) hyperbolic 3-manifold, as one sees by choosing the original tetrahedron to be the <u>ideal</u> regular 3-simplex in H^3, that is the 3-simplex with all vertices on the sphere of "points at infinity" in the unit disk model for H^3. The dihedral angles are then all 60°, hence six of them fit smoothly together to yield a non-singular manifold, even along the edge.

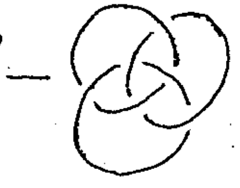
Thurston has pointed out that the orientable 2-fold covering of this Gieseking manifold is homeomorphic to $S^3 -$ . Proof: direct triangulation of this knot complement. Riley has shown that

$S^3 -$  $\cong H^3/\Gamma$

where $\Gamma \subset PSL_2\mathbb{Z}[\omega]$ is the subgroup of index 12 generated by the matrices

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 & 0 \\ -\omega & 1 \end{bmatrix},$$

and where $\omega = (-1+\sqrt{-3})/2 = e^{2\pi i/3}$. Similarly, N. Wielenberg, in a letter to Thurston, showed that both

$S^3 -$  and $S^3 -$ 

are diffeomorphic to H^3/Γ for appropriately chosen subgroups $\Gamma \subset PSL_2\mathbb{Z}[i]$ of finite index.

Following Humbert, the hyperbolic volume of such a manifold can be computed by the formula

$$\text{volume}\,(H^3/PSL_2\mathcal{O}) = \frac{D\sqrt{D}}{24}L(2)$$

where $-D$ is the discriminant of $F = \mathbb{Q}(\sqrt{-d})$ and

$$L(s) = \frac{\zeta_F(s)}{\zeta_{\mathbb{Q}}(s)} = \prod_{\text{primes}} \frac{1}{\left(1-\left(\frac{-D}{p}\right)\frac{1}{p^s}\right)} = \sum_{n>0}\left(\frac{-D}{n}\right)\frac{1}{n^s}.$$

Here the quadratic residue symbol $\left(\frac{-D}{n}\right)$ is defined to be multiplicative in n, with $\left(\frac{-D}{p}\right)=0$ if $p|D$, and $\left(\frac{-D}{2}\right)=+1$ if $-D\equiv 1 \pmod 8$ or -1 if $-D\equiv 5 \pmod 8$. Recall that $D=d$ if $d\equiv 3 \pmod 4$ and $D=4d$ otherwise if $d\geq 1$ is squarefree.

<u>Example.</u> If $D=d=3$, then

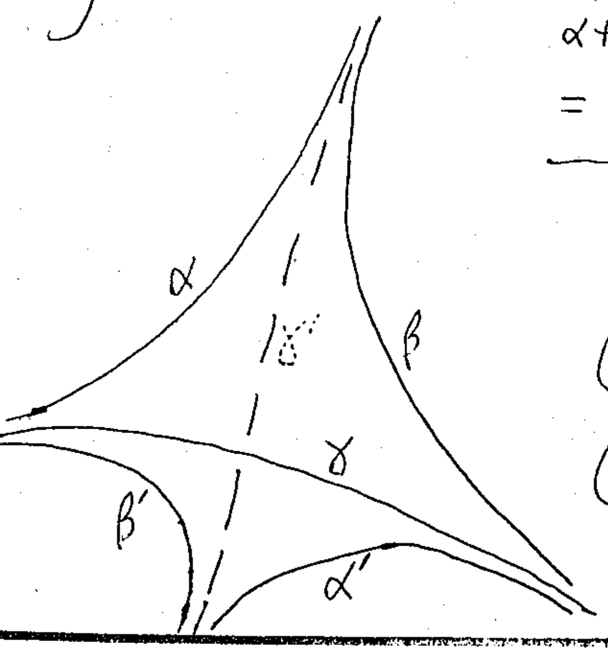
$$\text{volume}\left(H^3/PSL_2\mathbb{Z}[\omega]\right)=\frac{\sqrt{3}}{8}\left(1-\frac{1}{2^2}+\frac{1}{4^2}-\frac{1}{5^2}+-\cdots\right)$$

hence

$$\text{volume}\left(S^3-8\right)=\frac{3\sqrt{3}}{2}\left(1-\frac{1}{2^2}+\cdots\right)=2.029\ldots$$

Using methods due to Lobachevsky this same volume can be computed as follows. Consider an ideal 3-simplex with dihedral angles as follows. Then

$\alpha+\beta+\gamma'=\alpha'+\beta+\gamma=\alpha'+\beta'+\gamma'$
$=\alpha+\beta'+\gamma=\pi$, hence:



(1) $\alpha=\alpha'$, $\beta=\beta'$, $\gamma=\gamma'$

(2) $\alpha+\beta+\gamma=\pi$

(3) Theorem
volume $=\Lambda(\alpha)+\Lambda(\beta)+\Lambda(\gamma)$

Here $\Lambda(\theta)$ denotes the "Lobachevsky function"

$$\Lambda(\theta)=-\int_0 \log|2\sin\theta|\,d\theta,$$

periodic of period π.

This volume is maximized when $\alpha=\beta=\gamma=\pi/3$, hence

$$\text{volume}\left(\overset{\text{any}}{3}\text{-simplex}\right)\leq V_{Max}=3\Lambda\left(\frac{\pi}{3}\right)=1.01494\ldots$$

In particular, the volume of S^3-8 is just twice this.

To reconcile these computations, note that

$$\Lambda(\theta)=\sum_{n>0}\frac{\sin(2n\theta)}{2n^2},$$

while the function $n\mapsto\left(\frac{-D}{n}\right)$ on $\mathbb{Z}/D$ has Fourier transform

$$\sum_{k \bmod D}\left(\frac{-D}{k}\right)e^{2\pi i kn/D}=\left(\frac{-D}{n}\right)\sqrt{-D}.$$

Now multiply by $1/n^2$, sum over $n>0$, take the imaginary part of both sides, and multiply by $D/24$ $\Longrightarrow$

$$\text{volume}\left(H^3/PSL_2\mathcal{O}\right)=\frac{D}{12}\sum_{k \bmod D}\left(\frac{-D}{k}\right)\Lambda\left(\frac{\pi k}{D}\right)$$

For example if $D=3$, volume $=\frac{1}{4}\left(\Lambda\left(\frac{\pi}{3}\right)-\Lambda\left(\frac{2\pi}{3}\right)\right)=\frac{1}{2}\Lambda\left(\frac{\pi}{3}\right)$

Here are three properties:

① $Л(\theta+\pi) = Л(\theta)$

② $Л(n\theta) = n \sum_{j \bmod n} Л(\theta + \frac{j}{n}\pi)$

③ $Л(-\theta) = -Л(\theta)$.

(Properties ①, ② essentially describe a "distribution" in the sense of Lang and Kubert, Math. Z. 1975-6.)

<u>Conjecture</u>. All $\mathbb{Q}$-linear relations between the $Л(\theta)$, with θ/π rational, are consequences of ①, ②, ③.

Equivalently, for each n the real numbers $Л(j\pi/n)$ with $0 < j < \frac{n}{2}$ and j prime to n are independent over $\mathbb{Q}$.

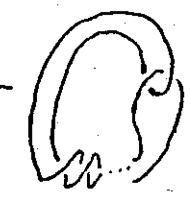
If a hyperbolic manifold has an end homeomorphic to $S^1 \times S^1 \times [0,\infty)$, then:

<u>Thurston Theorem</u>. For all but a finite number of the ways of replacing this end by a solid torus, we obtain another

manifold $M_i = M - (S^1 \times S^1 \times (0,\infty)) \cup_{f_i} S^1 \times D^2$ which is again hyperbolic. Furthermore:

$$\mathrm{vol}(M_i) < \mathrm{vol}(M), \quad \lim_{i\to\infty} \mathrm{vol}(M_i) = \mathrm{vol}(M)$$

Example:

$M = S^3 -$  $= S^3 -$  and $M_i = S^3 -$ 

<u>Gromov Theorem</u>. For a singular chain $c = \sum r_i f_i$ where $r_i \in \mathbb{R}$, $f_i : \Delta^n \to X$ let $\|c\| = \sum |r_i|$.

For $\gamma \in H_n(X;\mathbb{R})$ let $\|\gamma\| \geq 0$ be the infimum of $\|z\|$ as z ranges over all cycles representing γ.

Then $\|[M^n]\| = \mathrm{vol}(M^n)/\sigma_n$ for any closed hyperbolic manifold M^n with fundamental class $[M^n]$.

Here $\sigma_2 = \pi$, $\sigma_3 = 1.01494\ldots$,

σ_n = maximum volume of n-simplex in H^n.

Title: Birational geometry of fibre spaces

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In the following, by an algebraic variety V we mean that V is irreducible, complete and defined over $\mathbb{C}$. A non-singular algebraic variety is called an algebraic manifold. A complex variety is,by definition, a reduced irreducible compact complex space.

By definition, two complex varieties M_1, M_2 are birationally (bimeromorphically) equivalent, if there exists a rational (meromorphic) mapping $f : M_1 \longrightarrow M_2$ such that there exists the inverse $f^{-1}: M_2 \longrightarrow M_1$. Such a mapping is called a birational (bimeromorphic) mapping. By virtue of the Hironaka theorem, each birationally equivalent class of complex varieties (resp. algebraic varieties) contains a compact complex manifold(resp. projective manifold). For each birationally equivalent class, we can construct three different fibre spaces.

The meromorphic function field $\mathbb{C}(M)$ of a complex variety M is an algebraic function field, that is, there is a projective manifold W such that $\mathbb{C}(M) \simeq \mathbb{C}(W)$. Hence there exists a rational mapping $\varphi : M \longrightarrow W$ which induces the isomorphis of the function fields. Taking a suitable birationally equivalent model, we have a morphis $\varphi^* : M^* \longrightarrow W$ which is called the algebraic reduction.

$a(M) = \text{tr. deg}\,\mathbb{C}(M)$ is called the <u>algebraic dimension</u> of M. This is a birational invariant.

Let K_M be the canonical bundle of a compact complex manifold M. The <u>m-genus</u> $P_m(M) = h^0(M, K_M^{\otimes m})$ is a birational invariant for any positive integer m. Put $\mathbb{N}(M) = \{ m > 0 \mid P_m(M) > 0 \}$. If $\mathbb{N}(M) \neq \emptyset$, for $m \in \mathbb{N}(M)$, we let $\{\varphi_0, \ldots\ldots, \varphi_N\}$ be a basis of $H^0(M, K_M^{\otimes m})$. We have a rational mapping (m-th canonical mapping)

$$\Phi_{mK} : \begin{array}{ccc} M & \longrightarrow & W_m = \Phi_{mK}(M) \subset \mathbb{P}^N \\ \cup & & \cup \\ z & \longmapsto & (\varphi_0 : \ldots\ldots : \varphi_N). \end{array}$$

The <u>Kodaira dimension</u> $k(M)$ of M is defined by

$$k(M) = \begin{cases} \max\limits_{m \in \mathbb{N}(M)} \dim W_m, & \text{if } \mathbb{N}(M) \neq \emptyset, \\ -\infty, & \text{if } \mathbb{N}(M) = \emptyset. \end{cases}$$

This is also a birational invariant. For sufficient large $m \in \mathbb{N}(M)$, taking suitable birationally equivalent models of M and W_m, we obtain a morphism $\varphi : M^* \longrightarrow W$ which is called the <u>pluricanonical fibration</u>.

The third fibration is constructed by means of the Albanese mapping. For a compact complex manifold M, we have the Albanese mapping $\alpha : M \longrightarrow A(M)$ which is characterized by certain universal properties. A(M) is a complex torus and $t(M) = \dim A(M)$ ia called the <u>Albanese dimension</u>. This is a birational invariant. As ,in general, $\alpha : M \longrightarrow \alpha(M)$ has no connected fibres, we take the Stein factorization

$$\begin{tikzcd} M \arrow[rr, "\alpha"] \arrow[dr, "\beta"'] & & \alpha(M) \\ & W \arrow[ur, "\gamma"'] & \end{tikzcd}$$

$\beta : M \longrightarrow W$ is called the fibre space associated with the Albanese mapping.

The following table gives the important infomations on these fibre spaces. The detailed discussions can be found in the above references.

	algebraic reduction	pluricanonical fibration	Albanese mapping: general	Albanese mapping: $a(M) = 0$	Albanese mapping: $k(M) = 0$, M algebraic
birational invariant	$a(M) \leq \dim M$	$k(M) \leq \dim M$	$t(M) \leq g_1(M)$ $(g_1(M) = h^0(M, \Omega_M^1))$		$t(M) = g_1(M)$
general fibre F	$k(F) \leq 0$	$k(F) = 0$		α has connected fibres. $k(F) \leq 0$? (OK, if $t(M) \leq \dim M - 2$	<u>conjecture</u> $k(F) = 0$ (OK, if $\dim M \leq 2$ or $\dim M = 3, g_1 \geq 2$)
base space W	$\dim W = a(M)$	$\dim W = k(M)$	$k(\alpha(M)) \geq 0$ The equality holds iff α is surjective.	α is surjective $a(W) = 0$	<u>conjecture</u> α is surjevt (OK, if $\dim M \leq 3$)
Inequalities for a fibre space $\psi : M \longrightarrow N$ F ,a general fibre.	$a(N) \leq a(M)$ $a(M) \leq a(N) + a(F)$	$k(M) \leq k(F) + \dim N$ <u>conjecture C.</u> $k(M) \geq k(F) + k(W)$ OK, if $\dim F = 1$ or $\dim N = 1$ $k(N) = 1$ or $F \sim$ abelian var. (M algebraic)			

Title: Finite H-spaces and algebras over the Steenrod Algebra.

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Steenrod has posed the following problem, which has applications to the study of finite H-spaces. Let p be a fixed prime, and let $F_p[x_1, x_2, \ldots, x_l]$ be a (graded) polynomial algebra on generators $x_1, x_2, \ldots, x_l$ of degrees $2n_1, 2n_2, \ldots, 2n_l$; then is there or is there not a space X with $H^*(X; F_p) \cong F_p[x_1, x_2, \ldots, x_l]$?

Collaboration between C. Wilkerson and myself (1) has completed the solution of this problem when p is sufficiently large, in the sense that p does not divide $n_1 n_2 \cdots n_l$. The solution is given by the following results.

Theorem 1. Let H^* be an algebra over the mod p Steenrod algebra. In order that H^* should admit an embedding $H^* \subset H^*(BT^n; F_p)$ (for some suitable value of n) the following four conditions are necessary and sufficient.

(i) H^* is zero in odd degrees.

(ii) H^* is an integral domain.

(iii) H^* satisfies the "unstable condition".

(iv) There is an upper bound to the number of elements in H^* which can be algebraically independent over F_p.

Theorem 2. In order that there should exist an isomorphism $H^* \xrightarrow{\cong} H^*(BT^l; F_p)^W$ (for some suitable l and some group W of automorphisms of $H^*(BT^l; F_p)$) the following two conditions, in addition to (i) – (iv) above, are sufficient.

(v) H^* is integrally closed in its field of fractions.

(vii) H^* is generated as an F_p-algebra by a finite number of elements h_i of degrees $2n_i$ with $n_i \not\equiv 0$ mod p for each i.

Corollary 3. If H^* is an unstable algebra over the mod p Steenrod algebra which as an F_p-algebra is a polynomial algebra $F_p[x_1, x_2, \dots, x_l]$ on l generators x_i with x_i of degree $2n_i$ and $n_i \not\equiv 0$ mod p, then

$$H^* \cong H^*(BT^l; F_p)^W$$

where W is a p-adic generalized reflection group.

The point of this is twofold. First, the possible groups W are known, and listed for example in (2). Secondly, given such a W, there is a way to construct a space X with

$$H^*(X; F_p) \cong H^*(BT^l; F_p)^W \cong H^*.$$

So all the algebras H^* considered in Corollary 3 do arise as the cohomology rings

of spaces, and we have a list of them.

Working in a suitable category of algebras over the mod p Steenrod algebra which satisfy (i) (ii) (iii), we can set up a theory of "algebraic closure" analogous to the usual theory of "algebraic closure" for fields.

Proposition 5. Each object H^* in this category has an algebraic closure $H^* \subset K^*$. If H^* satisfies (iv), then so does K^*.

Theorem 6. The algebraically closed objects in this category which satisfy (iv) are precisely the algebras $H^*(BT^l; F_p)$ for $l = 0, 1, 2, \dots$

Corollary 7. We can prove Theorem 1 by exhibiting the following canonical embedding: compose the embedding $H^* \subset K^*$ of Proposition 5 with the isomorphism $K^* \cong H^*(BT^l; F_p)$ of Theorem 6.

In order to prove Theorem 2, we need a Galois group to use as W.

Proposition 4. If H^* satisfies (vii), then the extension $H^* \subset K^*$ of Proposition 5 is separable.

Title: TWISTOR SPACES

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The aim of the Penrose programme is to translate the geometry of physical fields into complex analysis, and the starting point is to encode the <u>conformal</u> and <u>metric</u> geometry of four-dimensional real space-time in the complex geometry of a 3-dimensional <u>twistor</u> space Z. The situation becomes simpler if we begin with a positive-definite space-time which is a necessary assumption anyway to make functional integrals converge.

§1. Conformal structure

Let M^4 be a Riemannian manifold, then its twistor space is defined to be the projective spinor bundle $\mathbb{P}(V_-)$ or alternatively $P/U(2)$ where P is the principal bundle of orthonormal frames. Using the Riemannian connection, $Z = P/U(2)$ has a canonical almost complex structure J since Z is the universal bundle of all complex structures on the tangent space compatible with the metric. J depends only on the conformal structure of M^4.

J is integrable iff $W_- = 0$ where $W = W_+ + W_-$ is the Weyl tensor.

If this holds, we have:

1) a complex manifold $Z \xrightarrow{p} M$
2) $p^{-1}(m)$ is a holomorphic curve $\mathbb{P}_1$ with normal bundle $H \otimes (V_+)_m$ $\quad (H = \mathcal{O}(1))$.
3) a fixed point free antiholomorphic involution $\sigma : Z \to Z$ induced by the quaternionic structure of the bundle V_-.

PROP: This is the conformal structure: $p^{-1}(m)$ belongs to a complete 4-dimensional family of curves parametrized by a complex 4-manifold M^c. The curves left stable under σ are parametrized by the real subvariety $M \subset M^c$. The tangent space $T_m^c \cong H^0(p^{-1}(m), \mathcal{O}(H \otimes (V_+)_m))$ and the null-cone in T_m^c is the set of sections of the normal bundle which vanish somewhere on the line $p^{-1}(m)$.

EXAMPLES:

1) $M = S^4$, $Z = \mathbb{P}_3$
2) $M = S^1 \times S^3$, $Z = (\mathbb{P}_3 - 2\mathbb{P}_1)/G$, where G is the group generated by $\begin{pmatrix} \lambda & & & \\ & \lambda & & \\ & & \mu & \\ & & & \mu \end{pmatrix}$ $\lambda \neq \mu$.

These are both conformally flat.

3) $M = \mathbb{P}_2$, $Z = \mathbb{F}_3$, the flag manifold $SU(3)/T$.

A ~~[illegible]~~ line in $\mathbb{F}_3$ ~~is~~ is obtained by taking a point p and a skew line $l \in \mathbb{P}_2^*$ and considering all lines through p and their intersection with l.

For all twistor spaces there are no non-trivial alternating or symmetric holomorphic forms and no sections of the pluricanonical bundles: they are of Kodaira dimension $-\infty$.

§2. Einstein's equations.

If in addition to $W_- = 0$, the Ricci tensor of M vanishes, then V_- is a flat bundle and the horizontal sections of $Z \to M$ define:

1) a holomorphic projection $Z \xrightarrow{\pi} \mathbb{P}_1$
2) a non-vanishing holomorphic section s of $K \otimes \pi^*(H^4)$.

The horizontal sections in fact give M an $SU(2)$-metric, and the covariant constant holomorphic 2-form in the fibres of π defines s.

PROP: This is the metric.
We already have the conformal structure, and s defines a volume form on each fibre, hence a metric.

EXAMPLES:

1) $M = \mathbb{R}^4$, Z = bundle $H \oplus H$ over $\mathbb{P}_1$.

2) M = K3 surface, Yau's proof of the Calabi conjecture gives an $SU(2)$ metric. Z = ? some interesting non-Kähler manifold.

3) Metric of Calabi (see lecture) and Eguchi & Hanson (ref. 3).
M = resolved quadric cone $z^2 - xy = 0$
Let $f(x,y,z) = z^2 - xy$, then f defines a map

$$\begin{tikzcd} H^2 \otimes \mathbb{C}^3 \arrow[r, "f"] \arrow[d] & H^4 \arrow[d, shift left] \\ \mathbb{P}_1 & \mathbb{P}_1 \arrow[u, shift left, "s"'] \end{tikzcd}$$

If $s = z_0^2 z_1^2$, then $f^{-1}(s)$ is a 3-fold. ~~manifold~~ fibering over $\mathbb{P}_1$. Resolve the singularities over 0 and ∞ within the ~~family~~ and you get Z.
$\pi^{-1}(u)$ for $u \neq 0$ or ∞ is an affine quadric $z^2 - xy = u^2$.
In dimension $4n$, the appropriate generalization of the above would be to $Sp(1) \times Sp(n)$ and $Sp(n)$ manifolds.

□.

Title: Surfaces with $p_g = 1$ $(K^2) = 1$

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Introduction. (see (1)). It is well known fact that the classifying space for Hodge Structures on a algebraic surfaces (with $p_g \geq 1$) with fixed polorization is $SO(2p_g, h^{1,1}-1)/SU(p_g) \times SO(h^{1,1}-1)$, where $p_g = \dim H^0(\Omega^2)$ and $h^{1,1} = \dim_{\mathbb{C}} H^1(\Omega^1)$. Notice that $SU(p_g) \times SO(h^{1,1}-1)$ is a maximal compact subgroup of $SO(2p_g, h^{1,1}-1) \iff p_g = 1$. So we may hope that only surfaces with $p_g = 1$ will give us that the period domain is more firmly connected with the moduli space (for example take K-3 surfaces). The aim of this talk is to give example of surfaces with $p_g = 1$ $(K^2) = 1$, which have the following properties: 1) The local moduli space is non singular and has the same dimension as the period domain
2) The period map is not a local isomorphism
3) These surfaces are simply connected.

Theorem 1. There exists a surface with $p_g = 1$, $(K^2) = 1$ and this surface is simply connected

Outline of the proof: Let $G = \mathbb{Z}/6\mathbb{Z}$ acts on $\mathbb{P}^3$ in the following manner:

$$(x_0 : x_1 : x_2 : x_3) \to (x_0 : \varepsilon x_1 : \varepsilon^2 x_2 : -x_3), \text{ where } \varepsilon^3 = 1, \varepsilon \neq 1.$$

Let f_6 is a polynomial of degree 6 with the following properties

a) f_6 is invariant under the action of G

b) f_6 defines a non-singular surface $X \subset \mathbb{P}^3$

c) f_6 intersects the line $l_1 = \{x_0 = 0 \ \& \ x_3 = 0\}$ in 6 distinct pts (Example $f = x_0^6 + x_1^6 + x_2^6 + x_3^6$)

Let $\bar{Y} = X/G$ and Y be the desingularization of $\bar{Y}$

Claim I: $p_g(Y) = 1$ $(K_Y^2) = 1$.

First notice that $K_X = \mathcal{O}_{\mathbb{P}^3}(2)|_X$ (see (2)) and by easy computation we have:

$$\dim_{\mathbb{C}} H^0(X, \Omega_X^2)^G = 1$$

Now let ω be the only invariant two holomorphic form on X. The divisor of $(\omega) = H_0 + H_3$, where H_0 is given by $x_0 = 0$ on X and H_3 is $x_3 = 0$. We can prove that 1) $\bar{Y}$ has 3 rational double pts of type A_3 2) $K_{\bar{Y}}$ is the image of the curve H_0 on X under the map $X \to X/G = \bar{Y}$.

From the adjunctional formula it follows: ~~that~~

$$p_g(K_Y) = \frac{K_Y(K_Y+K_Y)}{2} + 1 \Rightarrow (K_Y)^2 = p_g - 1$$

So it is enough to compute the genus of K_Y, which is verry easy. Use Hurwitz formula (2) So $p_g = 1$ $(K^2) = 1$.

Claim II: Y is a simply connected manifold (I suppose that X is given by f_6 in which x_0 is always with even degree Example X: $x_0^6 + x_1^6 + x_2^6 + x_3^6 = 0$)

Bombieri proved that $|2K|$ always gives a regular map $Y \to \mathbb{P}^2$ (see (3)). It is not verry difficult to prove that the degree of $|2K|$ is 4 and that in our special case Y is a Galois covering of $\mathbb{P}^2$ with Galois group $\mathbb{Z}_2 \times \mathbb{Z}_2$. The two involutions are given by $\sigma_1: x_0 \to -x_0$ and σ_2 ($\{x_1 \to x_2 \ \& \ x_2 \to x_1\}$). So from here it follows that $\sigma_1|_K$ is the identity map, so $Y/\sigma_1 = Z$ and Z is a K-3 surface. Notice that σ_1 has 9 fixed points. So from here it follows that $\pi_1(Y) = 0$.

Remarks 1) We can look at the surfaces we just constructed as a hipersurfaces in a wheighted projective space $\mathbb{P}^3(1,3,3,2)$ defined by $f_6 = 0$.

2) One can easily compute the ramification divisor of $Z \to \mathbb{P}^2$. It is two cubics intesecting in 9 pts.

Theorem 2. If Y is a Galoise covering of $\mathbb{P}^2 \Rightarrow$ Gal $(\mathbb{C}(Y)/\mathbb{C}(\mathbb{P}^2)) = \mathbb{Z}_2 \times \mathbb{Z}_2$ and $H^2(Y, \Theta) = 0$.

Cor. 1. The local modulli space of surfaces with $p_g = 1$ $(K^2) = 1$ with $G(\mathbb{C}(Y)/\mathbb{C}(\mathbb{P}^2)) \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ is non singular

Cor. 2. $\dim H^1(Y, \Theta_Y) = 18$.

Theorem 3. If Y is a surface with $p_g = 1$, $(K^2) = 1$ with a) non-singular canonical class and b) Y is a Galoise covering of $\mathbb{P}^2$ $\Rightarrow$ local Torelli theorem fails.

Outline of the proof: Notice that the tangent space to the local moduli at Y is $H^1(Y, \Theta_Y)$. On the other hand we can see that the tangent space to the period domain is $\mathrm{Hom}(H^{1,1}, H^{2,0})$; this is because we can look at the period domain as $Gr(p_g, \dim H^2(X,\mathbb{C}))$. So the condition that the period map is a local isomorphism is that we must have an injection : $H^1(X,\Theta) \hookrightarrow \mathrm{Hom}(H^{1,1}, H^{2,0})$. Griffith gave an explicit description of that map in [(2)]. If we dualize this condition we will get that the period map is a local isomorphism iff the natural pairing

$$(*) \qquad H^1(\Omega^1) \otimes H^0(\Omega^2) \to H^1(\Omega^1 \otimes \Omega^2)$$

is surjective. Now look at the exact sequence:

$$(**) \qquad 0 \to \Omega^1_Y \xrightarrow{\otimes\omega} \Omega^1_Y(K_Y) \to \Omega^1_Y(K_Y)\big|_{K_Y} \to 0$$

and

$$(***) \qquad \to H^1(\Omega^1_Y) \xrightarrow{\otimes\omega} H^1(\Omega^1_Y(K_Y)) \to H^1(\Omega^1_Y(K_Y)\big|_{K_Y}) \to 0$$

$= H^2(\Omega^1_Y) = 0$ (Y is simply connected)

So if $H^1(\Omega^1(K_Y)\big|_{K_Y}) \neq 0$ then local Torelli fails.

Notice that σ_1 is acting on $\Omega^1_Y\big|_{K_Y}$, so $\Omega^1_Y\big|_K = \Omega^1_Y\big|^+_K \oplus \Omega^1\big|^-_K$

$\Omega^1_Y\big|^+_K = \Omega^1_K \quad \Omega^1_Y\big|^-_K = N^*_{Y/K}$, so $\Omega^1_Y\big|_{K_Y} \otimes \mathcal{O}_Y(K) = \Omega^1_K \otimes N_{Y/K} \oplus$

$N_{Y/K} \otimes N^*_{Y/K} = \mathcal{O}_K$ and $\dim H^1(\mathcal{O}_K) = 2$, so we get what we need.

Problem 1. How many components have the moduli space of surfaces with $p_g = 1$ $(K^2) = 1$? (May be one!)

Problem 2. Prove the global Torelli theorem for $p_g = 1 (K^2) = 1$

Title: K homology and Riemann-Roch

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Let X be a projective algebraic variety over the complex numbers $\mathbb{C}$. Grothendieck defined two K groups for X: $K^0_{alg}(X)$ and $K_0^{alg}(X)$. These are, respectively, the Grothendieck of algebraic vector-bun on X and the Grothendieck group of coherent algebraic sheaves on X.

In topology, Atiyah and Hirzebruch defined $K^{0}_{top}(X)$: the Grothendieck group of topological vector-bundles on X. By forgetting some structure an algebraic vector-bundle on X gives a topological vector-bundle on X. So there is the evident map

$$K^{0}_{alg}(X) \longrightarrow K^{0}_{top}(X)$$

This is a natural transformation of contravariant functors.

The purpose of this lecture is to define a natural transformation of covariant functors

$$K_0^{alg}(X) \longrightarrow K_0^{top}(X)$$

Here K_*^{top} denotes the homology theory that goes with the Atiyah-Hirzebruch cohomology theory. Thus K_*^{top} is the homology theory associated to the BU spectrum. In this lecture K_*^{top} will

be defined in terms of objects and an equivalence relation on the objects.

Once the map $\alpha : K_0^{alg}(X) \to K_0^{top}(X)$ has been defined, Riemann-Roch is an immediate corollary. To see this, recall that there is the homology Chern character $ch_\bullet : K_*^{top} \to H_*(\ ;\mathbb{Q})$. Let $\mathcal{O}_X$ denote the structure sheaf of X. Set $td(X) = ch_\bullet(\alpha(\mathcal{O}_X))$. $td(X) \in H_*(X;\mathbb{Q})$. $td(X)$ is the Todd class of X. If X is non-singular, then $td(X)$ is the Poincaré dual of the Todd class of the tangent bundle of X. Let $\epsilon : X \to \cdot$ be the map of X to a point. $H_*(\cdot\,;\mathbb{Q}) = \mathbb{Q}$, so we have

$$\epsilon_* : H_*(X;\mathbb{Q}) \longrightarrow \mathbb{Q}$$

Given any algebraic vector-bundle

E on X, $H^i(X,E)$ denotes the i-th cohomology group of X with respect to the sheaf of germs of algebraic sections of E. $n = \dim_{\mathbb{C}} X$. Set $\chi(X,E) = \sum_{i=0}^{n}(-1)^i \dim_{\mathbb{C}} H^i(X,E)$.

Then:

$$\chi(X,E) = \epsilon_*(ch(E) \cap td(X))$$

If X is non-singular this formula becomes the R. R. formula of Hirzebruch.

<u>Definition of K_*^{top}</u>. Let A be a topological space. Let $\Gamma(A)$ be the collection of all triples (M,E,f) such that:

(i) M is a compact stably almost complex manifold without boundary

(ii) E is a vector-bundle on M. (E is a topological complex vector-bundle.)

(iii) f is a continuous map from M to A.

In $\Gamma(A)$, let $\sim$ be the equivalence relation generated by 4 elementary steps:

Isomorphism
Bordism
Direct Sum - Disjoint Union
Vector-bundle modification

Then $\Gamma(A)/\sim \; = K_*^{top}(A) = K_0^{top}(A) \oplus K_1^{top}(A)$. $K_0^{top}(A)$ and $K_1^{top}(A)$ are given, respectively, by those (M,E,f) such that each connected component of M is even-dimensional, respectively odd dimensional. The group operation is given by disjoint union.

<u>Riemann-Roch</u>. The R.R. map $K_0^{alg}(X) \longrightarrow K_0^{top}(X)$ can be described as follows. Let M be a non-singular projective algebraic variety. $\underline{E}$ denotes the sheaf of germs of algebraic sections of E. Let $f: M \longrightarrow X$ be a morphism of algebraic varieties. As in

Grothendieck, set $f_!(\underline{E}) = \sum_q (-1)^q R^q f_*(\underline{E})$. Thus $f_!(\underline{E}) \in K_0^{alg}(X)$. Such $f_!(\underline{E})$ generate $K_0^{alg}(X)$. By forgetting some structure, regard (M, E, f) as giving an element in $K_0^{top}(X)$. The R.R. map $K_0^{alg}(X) \longrightarrow K_0^{top}(X)$ is given by $f_!(\underline{E}) \longrightarrow (M, E, f)$.

Title: REPRESENTATIONS OF COMPACT LIE GROUPS AND ELLIPTIC OPERATORS

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References: J. Brüning + E. Heintze, Representations of compact Lie groups and elliptic operators
To appear

The following is joint work with E. Heintze. Let G be a compact Lie group, M a compact Riemannian manifold and E a Hermitian G-vector bundle over M, such that G acts on M by isometries and unitarily on the fibers. Then $(gf)(p) := g(f(g^{-1}(p)))$ defines a unitary representation of G in $L^2(E)$. Moreover let R be a selfadjoint operator in $L^2(E)$ commuting with G. Then if

$$R = \int_{-\infty}^{+\infty} t \, dR_t$$

is its spectral resolution, each R_t is a G-invariant subspace of $L^2(E)$. This leads to the following

<u>Problem</u>: For a given irreducible unitary representation $\rho : G \longrightarrow \mathrm{Aut}(V)$, determine the multiplicity of ρ in R_t — at least asymptotically.

We put $N(t) :=$ multiplicity of ρ in R_t and $N(\infty) :=$ multiplicity of ρ in $L^2(E)$. Then $N(\infty) = \lim_{t \to \infty} N(t)$. If $L^2(E)_\rho$ denotes the ρ-part of $L^2(E)$ we have the following natural isomorphisms

$$L^2(E)_\rho \simeq \mathrm{Hom}_G(L^2(E) \otimes V^*) \otimes V \simeq (L^2(E) \otimes V^*)^G \otimes V$$
$$\simeq L^2(E \otimes V^* \otimes V)^G$$

where $E \otimes V^*$ and $E \otimes V^* \otimes V$ are G-bundles with G-action ρ^* on V^* and trivial action on V. This allows reduction to the trivial representation. Now let M_0 be the union of principal G-orbits in M which is an open and dense subset of M with $\mathrm{vol}\, M = \mathrm{vol}\, M_0$ and make M_0/G a Riemannian manifold by requiring the orbit map $\pi_G : M_0 \to M_0/G$ to be a Riemannian submersion. Then $h(q) := \mathrm{vol}\, \pi_G^{-1}(q)$, $q \in M_0/G$, defines a positive C^∞-function on M_0/G.

<u>Theorem 1</u>: There is a Hermitian vector bundle F over M_0/G with $F_q \simeq E_p^{G_p}$ for $q \in M_0/G$, $p \in \pi_G^{-1}(q)$ and an isomorphism of Hilbert spaces

$$\Phi : L^2(E)^G \longrightarrow L^2(F,h).$$

This result may be viewed as a generalization of the Frobenius-Weil-reciprocity theorem, to which it reduces if M is homogeneous.

<u>Corollary</u>: $N(\infty) > 0$ iff $\dim \mathrm{Hom}_{G_p}(\gamma, E_p) > 0$ and in this case

$$N(\infty) = \begin{cases} \dim \mathrm{Hom}_{G_p}(\gamma, E_p) & \text{if } M \text{ is homogeneous,} \\ \infty & \text{otherwise.} \end{cases}$$

Thus γ occurs in $L^2(E)$ iff it occurs in some R_t. Now we assume that M is not homogeneous i.e. $\dim M_0/G > 0$. We want to investigate the behavior of N as $t \to \infty$. We have the sequence of maps

$$L^2(E) \xrightarrow{Q} L^2(E)^G \xrightarrow{\Phi} L^2(F,h),$$

where Q is given by integration of l_g over G. Thus $S := R \mid L^2(E)^G$ is a selfadjoint operator in $L^2(E)^G$ and therefore $T := \Phi S \Phi^{-1}$ is selfadjoint in $L^2(F,h)$. Denoting the spectral family of T by $(T_t)_{t \in \mathbb{R}}$, we see that

$$N(t) = \dim T_t.$$

We now restrict attention to the case that R is generated by a differential operator P on the sections of E, i.e. $C_0^\infty(E) \subset \mathrm{dom}\, R$ and $R \mid C_0^\infty(E) = P$.

<u>Theorem 2</u>: If R is generated by a differential P, then T is generated by a differential operator P' of the same order. If P is elliptic so is P'.

<u>Theorem 3</u>: If R is generated by an elliptic differential operator P of order $2k$, then

$$N(t) \sim \mathrm{const} \cdot t^{\frac{m}{2k}}, \quad t \to \infty,$$

where $m = \dim M_0/G$. If the principal symbol of P is given by

$$\sigma(P)(p,\xi) = |\xi|^2 \, \mathrm{id}_{E_p}, \quad p \in M, \ \xi \in T_p^* M,$$

then

$$N(t) \sim \frac{\omega_m}{(2\pi)^m} \dim F \ \mathrm{vol}\, M_0/G \ t^{\frac{m}{2}},$$

where ω_m = volume of the unit ball in $\mathbb{R}^m$.

We remark that the theorem holds under more general conditions, e.g. if M is a compact Riemannian manifold with boundary and R is generated by an elliptic boundary value problem. It has been proved by Huber for M a compact Riemann surface of genus ≥ 2 and $P = -\Delta$ and recently by Donelly for general compact M and $P = -\Delta$. The main ingredients of the proof are Hörmander's estimate for the spectral function of T and the following lemma:

<u>Lemma</u>: For $\alpha > 0$ and $s > 0$ we have uniformly in $p \in M$

$$\frac{\mathrm{vol}\, G_p}{\mathrm{vol}\, G} \int_G e^{-\frac{d_M^\alpha(g(p),p)}{s}} \, dG(g) \leq \mathrm{const} \ s^{\frac{\alpha}{\dim G_p}}.$$

Here d_M denotes the Riemannian metric on M.

It is natural to ask for remainder estimates. In this direction we have the following partial result.

<u>Theorem 4</u>: Let G be a finite group and let R be generated by a suitable elliptic operator (e.g. Δ on functions or forms). Then

$$N(t) = \mathrm{const} \cdot t^{\frac{m}{2k}} + O(t^{\frac{m-1}{2k}} \log t)$$

as $t \to \infty$.

However we can use this result to improve the asymptotic estimates for the spectra of compact quotients of semisimple Lie groups which have been derived by Gelfand/Graev/Piatetski-Shapiro, Gangolli and Wallach.

<u>Theorem 5</u>: Let G be a connected semisimple Lie group with finite center, Γ a discrete cocompact subgroup, K a maximal compact subgroup and $N := \{g \in G \mid g \text{ acts trivially on } G/K\}$. Provide G and $\Gamma \backslash G$ with the metric coming from the Killing form and for an irreducible unitary representation ω of G denote by $n_\Gamma(\omega)$ the multiplicity of ω in $L^2(\Gamma \backslash G)$ and by λ_ω the value of the

Casimir element on ω. Then for $t \to \infty$

$$\sum_{\substack{\omega \in \hat{G} \\ \lambda_\omega \leq t}} n_\Gamma(\omega)\,[\omega|K:\tau] = \frac{\omega_m}{(2\pi)^m}\, V^{\Gamma\cap H}\, \frac{\mathrm{vol}\,\Gamma\backslash G\ \ |\Gamma\cap H|}{\mathrm{vol}\,K}\, t^{\frac{m}{2}}$$

$$+\ O\left(t^{\frac{m-1}{2}} \log t\right)$$

for any irreducible unitary representation $\tau : K \to \mathrm{Aut}(V)$ of K. Here $m = \dim \Gamma\backslash G/K$.

Title: Automorphic forms on the metaplectic group

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Let k be a local field and $\mathbb{A}$ be the adeles of a global field. Denote by G_k the group $GL(2,k)$. It is known that there exists a covering $\bar{G}_k$ of degree two, which is usually called the metaplectic covering. If $G = GL(2,\mathbb{A})$ it is possible to define the similar metaplectic covering $\bar{G}_{\mathbb{A}}$ of degree 2, which splits over the principal adeles (see for instance [2])

It is known that automorphic forms of half-integral weight can be considered as automorphic forms on the metaplectic group [2].

The aim of this talk is to explain how to attach to every irreducible automorphic cuspidal representation $\bar{\pi}$ of $\bar{G}_{/A}$ an L-function. It is proved that these L functions are L-functions of automorphic representations of $GL(2, A)$. This gives the map from the set of irreducible automorphic cuspidal representations of $\bar{G}_{/A}$ onto the set of irreducible automorphic representations of $G_{/A}$. In the classical language of holomorphic automorphic forms the existence of the corresponding map was proved by Shimura [1]. The result of Shimura suggests the conjecture: there exists a map from the set of irreducible genuine representations of $\bar{G}_k$ onto the set of irreducible representations of G_k. (genuine means that the kernel of the map from $\bar{G}_k$ to G_k acts nontrivially) In order to obtain this result it suffices to get a functional equation for the L-function twisted with an arbitrary character. Shimura himself obtained a functional equation only for characters with conductor coprime

with the level. In order to obtain a functional equation for all characters it seems we have to consider the theory of "Jacquet-Langlands type" i.e. to introduce the appropriate Whittaker functions, to prove a uniqueness result for Whittaker functions, to introduce local functional equations and so on. But each step for the metaplectic group much harder than in the classical Jacquet-Langlands theory. It is also proved that our map is injective. All of the results described were obtained jointly with S Gelbart [see 3].

Title: Algebraic K-theory of topological spaces

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In order to understand the homotopy types of diffeomorphism spaces and homeomorphism spaces one needs, besides surgery, an understanding of concordance spaces. The algebraic K-theory of spaces is a functor which is helpful for this understanding. I denote this functor $X \mapsto A(X)$ ('A' for 'algebraic') in order to avoid confusion with topological K-theory with which it has little to do. The functor is part of an elaborate machine and admits many variations, some of which are useful. Also useful are its connections with other functors, particularly the algebraic K-theory of Quillen.

In the sequel I shall describe those properties of A(X) which appear most prominent at the present time, and list some computations. I start by reviewing background material from geometric topology, and more specifically concordance theory; references for this material may be found in the articles cited. I have divided the material - somewhat arbitrarily - into four parts:

1. Inputs from geometric topology
2. The space $A(*)$
3. The functor $A(X)$, and general properties
4. Some computations

1. Inputs from geometric topology. The smooth concordance space of a compact smooth manifold M of dimension n is defined to be

$$C'(M) = \mathrm{Diff}(M \times I\ ,\ M \times 0 \cup \partial M \times I)$$

the space of those diffeomorphisms (with the C^∞ topology) of $M \times I$ to itself which are the identity on $M \times 0$ and $\partial M \times I$; here I denotes the unit interval. There is a map, canonical up to homotopy.

$$C'(M) \longrightarrow C'(M \times I)$$

and there is a stability theorem (Hatcher-Burghelea-Lashof) which asserts that this map is an isomorphism on π_i provided that i is in a stability range $i \leq i_o(n)$ (it is known that $i_o(n) = n/6 - 6$ will do, but it is likely that this is far from best possible). Hence information about the <u>stable concordance space</u>

$$C(M) = \varinjlim_k C'(M \times I^k)$$

will translate into information about actual concordance spaces in the stability range.

By standard arguments (exhausting a 'reasonable' space by finite subspaces, and embedding these in manifolds) one extends to a functor $X \mapsto C(X)$ on the homotopy category, and by arguments which are (slightly) less standard one produces a canonical double de-loop of $C(X)$ (this is to avoid a dimension shift with respect to algebraic K-theory). The resulting space $Wh^{Diff}(X)$, the <u>differential Whitehead space</u>, is connected, its fundamental group is the Whitehead group $Wh(\pi_1 X)$, and its double loop space is $C(X)$.

One similarly constructs the <u>piecewise linear</u> and <u>topological Whitehead spaces</u> $Wh^{PL}(X)$ and $Wh^{Top}(X)$. As the canonical map $Wh^{PL}(X) \rightarrow Wh^{Top}(X)$ is a homotopy equivalence, one of the two is redundant (which one is a matter of taste - or convenience). By contrast the map $Wh^{Diff}(X) \rightarrow Wh^{PL}(X)$ is not a homotopy equivalence, but one knows that its homotopy theoretic fibre as a functor of X is a <u>homology theory</u> (the graded abelian group of homotopy groups satisfies the Eilenberg-Steenrod axioms except for the dimension axiom). These facts were made explicit by Burghelea-Lashof, they are implicit in earlier work by Kirby-Siebenmann.

At this point one can have a certain amount of fun by turning ones attention to some homotopy properties that functors may or may not have. These also allow for neater formulation of later theorems.

Let F be a functor from (reasonable) spaces to spaces. F will be called <u>connective</u> if it takes n-connected maps to n-connected maps for every n (or, what is more appropriate in the present framework, if this is so for every n not smaller than 2). F will be called <u>strongly connective</u> if in addition it has the following property. For every pushout diagram of cofibrations

$$\begin{array}{ccc} X & \longrightarrow & Y \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} \\ X' & \longrightarrow & Y' \end{array}$$

in which the horizontal and vertical maps are m-connected and n-connected, respectively, the induced map

$$\text{fibre}(f) \longrightarrow \text{fibre}(g)$$

must be $(m + n - c)$-connected, where c is a constant (roughly 3 , in practice).

For example the identity functor is strongly connective (by the homotopy excision theorem). Results of Hatcher say that the Whitehead space functors Wh^{Diff} and Wh^{PL} are strongly connective.

For any functor F from spaces to spaces one defines its <u>stabilization</u> F^S to be

$$F^S(X) = \varinjlim_n \Omega^n \,\text{fibre}(\, F(S^n \wedge (X \cup *)) \longrightarrow F(*)\,)$$

where the maps are the maps of the homotopy fibres in the diagrams

$$\begin{array}{ccc} F(S^{n-1} \wedge (X \cup *)) & \longrightarrow & F(D^n_+ \wedge (X \cup *)) \\ \downarrow & & \downarrow \\ F(D^n_- \wedge (X \cup *)) & \longrightarrow & F(S^n \wedge (X \cup *)) \end{array}$$

obtained by decomposition of the n-sphere into its lower and upper hemisphere. In view of its definition, F^S comes equipped with a natural transformation $F \rightarrow F^S$. If F is strongly connective then F^S is a homology theory (unreduced version) as one easily checks.

In particular, as Wh^{Diff} and Wh^{PL} are strongly connective their stabilizations are homology theories. But

<u>the stabilization</u> $(\mathrm{Wh}^{\mathrm{Diff}})^S$ actually <u>is the trivial theory</u>. This was shown by Burghelea-Lashof-Rothenberg. Subsequently Hatcher pointed out how it can be derived as an elementary consequence of Morlet's disjunction lemma.

In order to focus attention on a phenomenon that is of some importance in our context, let us introduce a further notion. If G is a functor from spaces to spaces, let an <u>approximation of</u> G consist of a functor F together with a natural transformation $F \longrightarrow G$ whose homotopy fibre is a homology theory.

For example it was mentioned before that $\mathrm{Wh}^{\mathrm{Diff}} \to \mathrm{Wh}^{\mathrm{PL}}$ is an approximation in this sense. The following says that the functor $\mathrm{Wh}^{\mathrm{PL}}$ is computable, more or less, from any of its approximations.

<u>Lemma</u> 1.1. If $f\colon F \longrightarrow \mathrm{Wh}^{\mathrm{PL}}$ is an approximation then the 'coefficients' of the homology theory fibre(f) is the space $F(*)$.

Proof. $\mathrm{Wh}^{\mathrm{PL}}(*)$ is contractible by the s-cobordism theorem and the Alexander trick, and the 'coefficients' are given by the homotopy fibre of $F(*) \longrightarrow \mathrm{Wh}^{\mathrm{PL}}(*)$, q.e.d.

Approximations to $\mathrm{Wh}^{\mathrm{PL}}$ are also relevant for $\mathrm{Wh}^{\mathrm{Diff}}$ as the following argument shows which was pointed out by Jahren. The case of the identity approximation leads to an amusing observation due to Hatcher.

<u>Lemma</u> 1.2. If $F \to \mathrm{Wh}^{\mathrm{PL}}$ is an approximation then there is a homotopy fibration of functors

$$\mathrm{Wh}^{\mathrm{Diff}} \longrightarrow F \longrightarrow F^S$$

Proof. Let the diagram

$$\begin{tikzcd}
F(X) \arrow[r] \arrow[d] & \mathrm{Wh}^{\mathrm{PL}}(X) \arrow[d] & \mathrm{Wh}^{\mathrm{Diff}}(X) \arrow[l] \arrow[d] \\
F^S(X) \arrow[r] & (\mathrm{Wh}^{\mathrm{PL}})^S(X) & (\mathrm{Wh}^{\mathrm{Diff}})^S(X) \arrow[l]
\end{tikzcd}$$

be obtained by stabilization from the upper row. As the homotopy fibre of the left map in the upper row is a homology theory already it is unchanged by stabilization. Hence the left hand square is homotopy cartesian. Hence the homotopy

fibre of the left vertical map is mapped by homotopy equivalence to the homotopy fibre of the middle vertical map. Arguing similarly with the right hand square, and noting that the homotopy fibre of the right vertical map is $\mathrm{Wh}^{\mathrm{Diff}}(X)$ since $(\mathrm{Wh}^{\mathrm{Diff}})^S(X)$ is contractible, the assertion results.

Any approximation to $\mathrm{Wh}^{\mathrm{Diff}}$ is also one to $\mathrm{Wh}^{\mathrm{PL}}$ (just compose with the map from the former to the latter), but the converse need not necessarily be true. So one should expect that approximations to $\mathrm{Wh}^{\mathrm{Diff}}$ are harder to produce. In fact, the following shows that whenever one has produced one in a non-trivial way, one is in for a surprise.

<u>Lemma</u> 1.3. If $F \longrightarrow \mathrm{Wh}^{\mathrm{Diff}}$ is an approximation then there is a natural splitting

$$F(X) \simeq F^S(X) \times \mathrm{Wh}^{\mathrm{Diff}}(X)$$

Proof. Stabilization gives a diagram

$$\begin{tikzcd}
\text{fibre} \arrow[r] \arrow[d, "\simeq"] & F(X) \arrow[r] \arrow[d] & \mathrm{Wh}^{\mathrm{Diff}}(X) \arrow[d] \\
\text{fibre} \arrow[r] & F^S(X) \arrow[r] & (\mathrm{Wh}^{\mathrm{Diff}})^S(X)
\end{tikzcd}$$

in which the left hand vertical map is a homotopy equivalence and $(\mathrm{Wh}^{\mathrm{Diff}})^S(X)$ is contractible, q.e.d.

An important step to relate the Whitehead spaces to the outside world of non-manifolds was taken by Hatcher. Hatcher's theorem applies to the PL Whitehead space. It says that the loop space $\Omega \mathrm{Wh}^{\mathrm{PL}}(X)$ is naturally homotopy equivalent to the geometric realization of the category $\mathcal{J}(X)$ defined as follows. Let us assume X is a polyhedron. Then the objects of $\mathcal{J}(X)$ are the polyhedra Y containing X as a deformation retract and satisfying that $\mathrm{Cl}(Y-X)$ is compact, and the morphisms in $\mathcal{J}(X)$ are the maps $Y \to Y'$ which restrict to the identity map on X and satisfy the condition that every point in Y' has contractible pre-image in Y .

This theorem is used in showing that the functor $A(X)$ is an approximation to $\mathrm{Wh}^{\mathrm{PL}}(X)$, and this in turn is used in the corresponding result for $\mathrm{Wh}^{\mathrm{Diff}}(X)$. It is an interesting question if there is an alternative route between $A(X)$

and the Whitehead spaces, proceeding directly to $Wh^{Diff}(X)$ and not using this theorem.

2. <u>The space</u> $A(*)$. This may be considered as a representing space for a Grothendieck construction in the homotopy theory of finite pointed spaces. As it would be rather technical to give precise meaning to this statement I refrain here from doing so. But let me point to the resulting interpretation of the space $A(*)$ as providing a higher order theory of the Euler characteristic. Specifically, any finite pointed space represents canonically an element of $\pi_0 A(*) \approx \mathbb{Z}$, and the integer represented is just the (reduced) Euler characteristic of the space. The higher homotopy of $A(*)$ is distilled from auto-homotopy-equivalences of finite pointed spaces in the same sense that the Quillen K-theory of the integers is distilled from automorphisms of finitely generated abelian groups (all of them, not just the free abelian ones). The notion of (not necessarily split) 'exact sequence' must play a crucial role in this setup.

There are several ways to actually construct the space $A(*)$. One of them (the one hinted at) involves an extension of the machinery of Quillen's Q-construction to make it apply in non-additive situations. This is one of the main ingredients of the theory, it is discussed at some length in both the references cited. The machine will produce a homotopy type whenever it is fed the following data: a category together with a pair of subcategories the morphisms of which are called 'cofibrations' and 'weak equivalences', respectively, and which satisfy certain familiar properties suggested by their names. The homotopy type of $A(*)$ drops out if the machine is fed the category of finite pointed spaces where 'cofibration' has its usual meaning and 'weak equivalence' means homotopy equivalence.

As with the algebraic K-theory of rings, the different constructions of $A(*)$ are on different levels of sophistication, and the mutual equivalence of any two of them may be a non-trivial result. The simplest definition of $A(*)$ is given by

$$A(*) \simeq \mathbb{Z} \times (\varinjlim_{n,k} B\mathcal{H}(\textstyle\bigvee^k S^n))^+$$

Here,

$\bigvee^k S^n$ = wedge of k spheres of dimension n

$\mathcal{H}(..)$ = the topological monoid of homotopy equivalences

$B\mathcal{H}$ = its classifying space, and

$(..)^+$ = the + construction of Quillen which abelianizes the fundamental group without altering the homology;

the factor $\mathbb{Z}$ has to be artificially resurrected with this definition (this is analogous to the situation in algebraic K-theory where the corresponding construction will fail to produce the projective class group).

As the algebraic K-theory of the integers may be defined as

$$K(\mathbb{Z}) \simeq \mathbb{Z} \times (\varinjlim_{k} B\, GL_k(\mathbb{Z}))^+$$

the map

$$\mathcal{H}(\textstyle\bigvee^k S^n) \longrightarrow \mathrm{Aut}(H_n(\bigvee^k S^n))$$

induces a map $A(*) \to K(\mathbb{Z})$. (From a more categorical point of view, and with suitable definitions of $A(*)$ and $K(\mathbb{Z})$, this map may be interpreted as being induced from the functor which in homotopy theory induces the Hurewicz map).

<u>Lemma</u> 2.1. The map $A(*) \to K(\mathbb{Z})$ is a rational homotopy equivalence.

The lemma exploits that the stable homotopy groups of spheres are finite above degree 0.

The Barratt-Priddy theorem says that

$$\Omega^\infty S^\infty \simeq \mathbb{Z} \times (\varinjlim_{k} B\mathcal{H}(\textstyle\bigvee^k S^0))^+$$

since $\mathcal{H}(\bigvee^k S^0)$ is isomorphic to Σ_k, the symmetric group on k letters. Consequently one has a map $\Omega^\infty S^\infty \to A(*)$. Its composite with $A(*) \to K(\mathbb{Z})$ is the usual map from the stable homotopy of spheres to the K-theory of the integers.

<u>Theorem</u> 2.2. $\Omega^\infty S^\infty \longrightarrow A(*)$ is a coretraction, up to homotopy.

If one considers how the map is defined, this is a surprising result. The composite map $\Omega^\infty S^\infty \longrightarrow K(\mathbb{Z})$ does certainly not split off as one sees from the induced map $\pi_3^S \longrightarrow K_3(\mathbb{Z})$ whose source is $\mathbb{Z}/24$ and whose target is $\mathbb{Z}/48$ (Lee-Sczcarba). The distinction suggests the possibility that the structure of $A(*)$ might be easier to understand than that of $K(\mathbb{Z})$.

3. <u>The functor</u> $A(\ )$. For any given space X , the space $A(X)$ may also be considered as a representing space for a Grothendieck construction in a homotopy theory, this time the <u>equivariant homotopy theory parametrized by</u> X . Apart from technical distinctions (such as working with simplicial sets on the one hand, or topological spaces on the other) there are two ways for setting up this theory which I will describe now very briefly.

The first setting is essentially that of the Ex-spaces of James. One works in the category of <u>retractive spaces over</u> X whose objects are the triples (Y,r,s) where $r\colon Y \to X$ is a retraction, and s is a section of r . There are two ways to imposing a finiteness condition on these objects. We may ask that the homotopy fibre of r be finite (up to homotopy). But this is not the correct condition for the construction of $A(X)$. Rather we must ask that s have finite cofibre, or more precisely, that up to homotopy Y can be obtained from X by attaching finitely many cells.

In the other setting one replaces a connected X by any loop group G (a topological group such that there exists a principal G-bundle over X with contractible total space) and works with G-spaces which are pointed (as G-spaces). The wrong finiteness condition above corresponds here to considering homotopy finite spaces on which G acts. The correct finiteness condition presupposes a freeness condition (the action must be as free as compatible with the pointing). While this works beautifully in a simplicial context it looks awkward in our topological context. A convenient way to overcome the difficulty is to introduce the notion of G-CW complex.

Very little imagination is required for this notion: just interpret 'attaching of a G-cell' as 'attaching of $G \times$ cell in an equivariant way', and the elementary theory of CW complexes goes through. In particular it is now easy to arraign the finiteness condition.

There are good functors back and forth between the two settings. If P is the principal bundle in question then one of the functors is given by $(Y,r,s) \longmapsto r^*(P)/P$ while the other one takes a pointed G-space to the associated sectioned bundle.

In the framework of G-spaces, the quick definition of $A(*)$ admits an analogue. It is

$$A(X) \simeq \mathbb{Z} \times \big(\varinjlim_{n,k} B\,\mathcal{H}_G(\textstyle\bigvee^k S^n \wedge (G \cup *)) \big)^+$$

where $\mathcal{H}_G$ denotes the topological monoid of G-homotopy equivalences. The factor $\mathbb{Z}$ looks artificial here. It comes from the insistence on a finiteness condition. Had we replaced this by the condition 'dominated by finite', the factor would be a projective class group.

As in the special case $X = *$ it is hard to derive many properties from this particular definition. By working with suitable alternative definitions, most of the following properties are comparatively easy to derive:

1. $X \mapsto A(X)$ can be a covariant functor; its values are infinite loop spaces;

2. $X \mapsto A(X)$ is strongly connective in the sense of section 1, in particular the stabilization A^S is a homology theory;

3. there is a map $A(X) \longrightarrow K(\mathbb{Z}[\pi_1 X])$, this map is a rational homotopy equivalence if (and only if) $X \to B\pi_1 X$ is;

4. there is an external pairing $A(X) \wedge A(X') \longrightarrow A(X \times X')$

5. if $X'' \longrightarrow X$ has homotopy finite fibre then there is a transfer map $A(X) \longrightarrow A(X'')$; the pairing and the transfer have the properties one can reasonably expect, and are related in the usual way;

6. there are λ-operations $\lambda^k\colon A(X) \longrightarrow A(X^k \times_{\Sigma_k} E\Sigma_k)$

where Σ_k denotes a symmetric group acting on X^k by permutation of the factors, and $E\Sigma_k$ a universal Σ_k-bundle; the operations satisfy a Cartan formula; their definition is an adaption of Segal's definition of the corresponding operations in stable homotopy (which they extend);

7. there is a canonical involution on $A(X)$; in the case $X = *$ it is induced by Spanier Whitehead duality; in the general case one manufactures a version of this duality in the equivariant homotopy theory.

The followin^ ties up $A(X)$ with the material in section 1.

Theorem 3.1. There are natural transformations from $A(X)$ to $Wh^{PL}(X)$ and $Wh^{Diff}(X)$ whose fibres are homology theories.

According to lemma 1.3 the latter signifies that

$$A(X) \simeq Wh^{Diff}(X) \times A^S(X)$$

Theorem 3.2. There is a map $A^S(X) \longrightarrow \Omega^\infty S^\infty(X \cup *)$ so that

$$\begin{tikzcd}
 & \Omega^\infty S^\infty(X\cup *) \arrow[dl] & \\
A(X) \arrow[rr] & & A^S(X) \arrow[ul]
\end{tikzcd}$$

commutes up to homotopy.

In particular, $A^S(X) \simeq \Omega^\infty S^\infty(X \cup *) \times ?$. Putting the two splittings together one obtains a threefold splitting

$$A(X) \simeq Wh^{Diff}(X) \times \Omega^\infty S^\infty(X \cup *) \times ?$$

But a threefold natural splitting, with no a priori reason for it, is a rather unheard of thing. This suggests contemplating the possibility if in fact $A(X)$ might be homotopy equivalent to $Wh^{Diff}(X) \times \Omega^\infty S^\infty(X \cup *)$.

4. Some computations. In view of lemma 2.1 one has, thanks to Borel,

$$\pi_i A(*) \otimes \mathbb{Q} \approx K_i(\mathbb{Z}) \otimes \mathbb{Q} \approx \begin{cases} \mathbb{Q}, & i = 0 \\ \mathbb{Q}, & i = 5, 9, \ldots \\ 0, & \text{otherwise} \end{cases}$$

The stabilization of $A(X)$ can be mimicked with a stabilization of algebraic K-theory (this requires a framework of simplicial rings), and the rational homotopy equivalence $A(*) \rightarrow K(\mathbb{Z})$ has a companion $A^S(*) \rightarrow K^S(\mathbb{Z})$.

Among other things, the algebraic K-theory of Quillen provides an organizing center for the study of the homology of general linear groups, with untwisted coefficients. The kind of algebraic K-theory involved with the present theory is also related to the homology of general linear groups, but the coefficients are not necessarily untwisted. In particular one has the following.

Lemma 4.1 There is a spectral sequence

$$H_p(GL(\mathbb{Z}), \pi_q K^S(\mathbb{Z})) \Longrightarrow H_{p+q}(GL(\mathbb{Z}), M(\mathbb{Z}))$$

where $M(\mathbb{Z})$ denotes the essentially finite matrices over $\mathbb{Z}$ upon which $GL(\mathbb{Z})$ acts by conjugation, and the coefficients on the left are untwisted.

It has been shown by Farrell-Hsiang (and probably also Borel) that the trace map $M(\mathbb{Q}) \rightarrow \mathbb{Q}$ induces an isomorphism

$$H_*(GL(\mathbb{Z}), M(\mathbb{Q})) \xrightarrow{\approx} H_*(GL(\mathbb{Z}), \mathbb{Q})$$

In view of the spectral sequence comparison theorem, therefore

$$\pi_i A^S(*) \otimes \mathbb{Q} \approx \pi_i K^S(\mathbb{Z}) \otimes \mathbb{Q} \approx \begin{cases} \mathbb{Q}, & i = 0 \\ 0, & \text{otherwise} \end{cases}$$

Putting this together with the rational computation of $A(*)$, and the homotopy equivalence $A(*) \simeq A^S(*) \times Wh^{Diff}(*)$, one obtains

Theorem 4.2.

$$\pi_i \; Wh^{Diff}(*) \otimes \mathbb{Q} \approx \begin{cases} \mathbb{Q}, & i = 5, 9, 13, \ldots \\ 0, & \text{otherwise} \end{cases}$$

As explained in section 1 this implies the rational knowledge of certain concordance spaces in a stable range. From this in turn one obtains, in view of the fibration

$$\mathrm{Diff}(D^{n+1}, \partial) \longrightarrow \mathrm{Diff}(D^{n+1}, D^n_-) \longrightarrow \mathrm{Diff}(D^n, \partial)$$

in which the middle term is a concordance space,

Corollary 4.3. For every j and every sufficiently large n (depending on j) the group

$$\pi_{4j+3}\,\mathrm{Diff}(D^{n+1},\partial) \quad \oplus \quad \pi_{4j+3}\,\mathrm{Diff}(D^{n},\partial)$$

is infinite.

In low degrees one can state explicit results without prior rationalization. The known computations in concordance theory (Hatcher-Wagoner for π_0 and K. Igusa for π_1) can be derived from these.

The first deviation in the map $A(X) \longrightarrow K(\mathbb{Z}[\pi_1 X])$ occurs as $\pi_2$fibre . This can be entirely computed, the computation is given in the first reference.

Let us concentrate now on the particular case $X = *$. The next computation depends on a theorem of Dennis to the effect that H_1 of the Steinberg group $St(\mathbb{Z})$ acting on integral matrices by conjugation, is 0. Using this theorem one can show that $\pi_3$fibre is either $\mathbb{Z}/2$ or 0. Thus a part of the long exact sequence of homotopy groups is:

$$\begin{array}{ccccccccccc} K_4(\mathbb{Z}) & \longrightarrow & \pi_3\text{fibre} & \longrightarrow & \pi_3A(*) & \longrightarrow & K_3(\mathbb{Z}) & \longrightarrow & \pi_2\text{fibre} & \longrightarrow & \pi_2A(*) \\ ? & & (\mathbb{Z}/2) & & & & \mathbb{Z}/48 & & \mathbb{Z}/2 & & \end{array}$$

The non-trivial element of $\pi_2$fibre is represented by the non-trivial element of

$$\pi_2\, B\,\mathcal{H}(S^n) \quad (n \text{ large}) \quad \approx \pi_1^S$$

The map $K_3(\mathbb{Z}) \longrightarrow \pi_2$fibre must be surjective since π_3^S splits off $\pi_3A(*)$ (theorem 2.2) but not $K_3(\mathbb{Z})$ (Lee-Sczcarba). Incidentally, this gives a kind of interpretation to the exotic element (or rather coset) of $K_3(\mathbb{Z})$.

The prospective non-trivial element of $\pi_3$fibre is represented by the non-trivial element of

$$\pi_3\, B\,\mathcal{H}(S^n) \quad (n \text{ large}) \quad \approx \pi_2^S$$

It could be zero in $\pi_3$fibre because it might have been hit by a differential originating from an H_2 (with twisted coefficients) of what is (essentially) the Steinberg group.

Hence

$$\pi_3A(*) \;\approx\; \pi_3^S \oplus ((\mathbb{Z}/2))$$

where the double bracket means that even if the element survives to $\pi_3$fibre , it still might be hit by $K_4(\mathbb{Z})$. As

$$\pi_3A(*) \;\approx\; \pi_3^S \oplus \pi_3 Wh^{Diff}(*) \oplus \;?$$

(theorems 3.1 and 3.2) we have thus recovered the theorem which K. Igusa proved by entirely different means, namely that $\pi_3 Wh^{Diff}(*)$, or what is the same, π_1 of the smooth concordance space of the n-disk (n large), is at most of order 2. It would be very interesting to know if it is zero or not.