Tagung über

TRANSFORMATIONSGRUPPEN UND INVARIANTENTHEORIE

22. - 28. September 1978

Universität Bonn

Sonderforschungsbereich 40

Theoretische Mathematik

Wegelerstraße 10

D-5300 B o n n

September 1978

INHALT

T E I L N E H M E R

K. Angermüller (Erlangen)

D. Bartels (Wuppertal)

W. Borho (Wuppertal)

J. F. Boutot (Strasbourg)

M. Brodmann (Münster)

C. De Concini (Pisa)

A. Del Fra (Rom)

J. Dixmier (Paris)

V. Dlab (Carleton/Poitier)

D. Eisenbud (Brandeis)

R. Elkik (Paris)

N. Goldstein (Cornell)

R. Großer (Bonn)

W. Haboush (Albany, N. Y.)

D. Happel (Bielefeld)

G. Harder (Bonn/Wuppertal)

M. Hazewinkel (Rotterdam)

H. Helling (Bielefeld)

J. Helmstetter (Grenoble)

J. Herzog (Essen)

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J. C. Jantzen (Bonn)

A. Joseph (Tel Aviv)

V. Kac (Cambridge)

S. Koizumi (Bonn)

M. Krämer (Bayreuth)

V. Lakshmibai (z.Zt. Rom)

A. Lascoux (Paris VII)

D. Luna (Grenoble)

G. Mazzola (Zürich)

J. Mennicke (Bielefeld)

H.-M. Meyer (z.Zt. Bonn)

W. Meyer (Bonn)

H. Morikawa (Nagoya)

U. Oberst (Innsbruck)

F. Pauer (Innsbruck)

D. Peterson (Cambridge)

K. Pommerening (Mainz)

H. Popp (Mannheim)

A. S. Procesi (Rom)

C. Procesi (Rom)

R. Rentschler (Paris)

R. W. Richardson (Canberra)

C.-M. Ringel (Bielefeld)

H. Schiek (Bonn)

K. Shinoda (z.Zt. Paris)

G. Schwarz (Brandeis)

J. Schwermer (Bonn)

P. Slodowy (Bonn)

N. Spaltenstein (Lausanne)

M. Steurich (Essen)

E. Strickland (Rom)

Y. Teranishi (Nagoya)

P. Tomter (Tromsö)

Th. Vust (Genf)

B. Wajuryb (Bonn)

A. Wiedemann (Stuttgart)

D. Ziplies (Düsseldorf)

Mathematisches Institut
der Universität Bonn

Programm der Tagung über "Transformationsgruppen und Invariantentheorie"

Freitag, den 22.9.:

16.00 - 17.00 Uhr: Tee (Wegelerstr. 10)

17.00 - 17.15 Uhr: Eröffnung (Großer Hörsaal)

17.15 - 18.15 Uhr: R. W. Richardson: Invariant Theory, some History, some Techniques, some New Results.

Samstag, den 23.9.:

10.00 - 11.00 Uhr: P. Slodowy: Deformation of simple singularities in simple groups

12.00 - 13.00 Uhr: C. Procesi: Normality of closures of conjugacy classes

17.00 - 18.00 Uhr: D. Luna: SL_2-embeddings

Sonntag, den 24.9.:

10.00 - 11.00 Uhr: N. Spaltenstein: Duality for some unipotent classes

12.00 - 13.00 Uhr: D. Peterson: Affine cross sections (?)

17.00 - 18.00 Uhr: G. Schwarz: Lifting vector fields from orbit spaces I

Die Vorträge finden alle im "Großen Hörsaal" (Wegelerstraße 10) statt.

Erfrischungspausen mit Tee: Samstag und Sonntag vormittags von 11.15-12.00 Uhr vor dem Großen Hörsaal, nachmittags ab 16.00 Uhr im Diskussionsraum Beringstr. 1.

Die Post liegt während der Vormittags-Teepausen aus.

Tischtennis im Keller des Hauses Beringstraße 4.

Den Tagungsbeitrag bitte an Frau Barrón (Beringstr. 1, Zi. 14) bezahlen.

Alle Teilnehmer mögen sich bitte in die Teilnehmerliste eintragen. Teilnehmerlisten und andere Informationen liegen vor dem Diskussionsraum Beringstraße 1 aus.

Die Vortragenden werden gebeten, die Kurzfassungen (summeries) möglichst bald an einen der Tagungsleiter abzugeben (in der Beringstr. 4 ist ein Schreibzimmer eingerichtet).

Alle Tagungsteilnehmer mit ihren Damen oder Herren sind herzlich zur Party in der Beringstr. 1, Montagabend ab 20.00 Uhr eingeladen.

Mathematisches Institut

der Universität Bonn

Programm der Tagung über "Transformationsgruppen und Invariantentheorie" (II)

Montag, den 25.9.:

10.00 - 11.00 Uhr:	J. F. Boutot: Rational singularities and quotients by reductive groups.
12.00 - 13.00 Uhr:	V. Kac: Infinite dimensional Lie algebras, representations of graphs and Invariant Theory.
17.00 - 18.00 Uhr:	V. Lakshmibai: Standard Monomial Theory.

Dienstag, den 26.9.:

10.00 - 11.00 Uhr:	G. Schwarz: Lifting vector fields from orbit spaces II.
12.00 - 13.00 Uhr:	A. Lascoux: Finite linear groups and "charge" of Young tableaux.
14.00 Uhr ff.	Spaziergang nach Villip (Abmarsch von der Beringstr. 1)

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Den Tagungsbeitrag bitte an Frau Barrón (Beringstr. 1, Zi. 14) bezahlen.

Alle Teilnehmer mögen sich bitte in die Teilnehmerliste eintragen. Teilnehmerlisten und andere Informationen liegen vor dem Diskussionsraum Beringstraße 1 aus.

Die Vortragenden werden gebeten, die Kurzfassungen (summeries) möglichst bald an einen der Tagungsleiter abzugeben (in der Beringstr. 4 ist ein Schreibzimmer eingerichtet).

Alle Tagungsteilnehmer mit ihren Damen oder Herren sind herzlich zur Party in der Beringstr. 1, Montagabend ab 20.00 Uhr eingeladen.

Mathematisches Institut
der Universität Bonn

Programm der Tagung über "Transformationsgruppen und Invariantentheorie" (III

Mittwoch, den 27.9.:

10.00 - 11.00 Uhr: H. Popp: Some applications of moduli theory of algebraic varieties.

12.00 - 13.00 Uhr: M. Hazewinkel: Dynamical systems and invariant theory.

17.00 - 18.00 Uhr: W. Haboush: Central differential operators in positive characteristic.

Donnerstag, den 28.9.:

10.00 - 11.00 Uhr: H. Morikawa: On some applications of differential invariants of linear differential operators.

12.00 - 13.00 Uhr: J. Dixmier: On enveloping algebras and invariant theory.

17.00 - 18.00 Uhr: C. De Concini: Some remarks on Sp_n.

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Den Tagungsbeitrag bitte an Frau Barrón (Beringstr. 1, Zi. 14) bezahlen.

Alle Teilnehmer mögen sich bitte in die Teilnehmerliste eintragen. Teilnehmerlisten und andere Informationen liegen vor dem Diskussionsraum Beringstraße 1 aus.

Important:

!!! Die Vortragenden werden gebeten, die Kurzfassungen (summeries) möglichst bald an einen der Tagungsleiter abzugeben (in der Beringstr. 4 ist ein Schreibzimmer eingerichtet).

Tagung über

TRANSFORMATIONSGRUPPEN UND INVARIANTENTHEORIE

Bonn, 22. - 28. September 1978

Titel: Invariant Theory, Some History, Some Techniques and Some New Results

Autor: R. W. Richardson

Adresse: Australian National University, Canberra

Notation: If X is an affine algebraic variety, $A(X)$ denotes the algebra of regular functions on X. If G is a group of automorphisms of X, then $A(X)^G$ denotes the ring of G-invariants, that is the ring of regular functions constant on G-orbits.

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An example $M = M_n(\mathbb{C})$ = all $n \times n$ complex matrices.
$A(M)^{GL_n}$ = all poly. fctns constant on conjugacy classes:
$f(gXg^{-1}) = f(X)$. $X \in M$, $g \in GL_n(\mathbb{C}) = G$

Let $\chi_X(t)$ = characteristic polynom. of X

$\chi_X(t) = \det(X - tI) = (-1)^n t^n + (-1)^{n-1} P_1(X) t^{n-1} + \ldots + P_n(X)$. Note $P_1(X) = \mathrm{tr}(X)$, $P_n(X) = \det X$.

THM. $A(M)^G$ is a graded polynomial algebra with generators $P_1, \ldots, P_n$ (and no relations).

Thus $P_1, \ldots, P_n$ give "all" of the polyn. invariants in the sense that every invariant is (uniquely) a polynom. in $P_1, \ldots, P_n$.

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Remark Let the reductive complex algebraic group G act on the smooth complex affine algebraic variety X. Let Q_1,

..., Q_r generate $A(X)^G$. Then it was shown by Schwarz (for G compact) and Luna (general case) that every C^∞ (resp holomorphic) invariant fctn $f(x)$ on X can be written as a C^∞ (resp holom.) fctn of $Q_1(x), \ldots, Q_r(x)$. Thus polynomial invariants give as much information as C^∞ or holomorphic invariants.

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Two basic questions. Let G be a group of automorphisms of the complex affine variety X. 1. Are there only a "finite number of invariants"? (Precisely this means: "Is $A(X)^G$ a finitely generated $\mathbb{C}$ algebra?"). 2. If $x, y \in G$ are on different orbits, is there an $f \in A(X)^G$ s.t. $f(x) \neq f(y)$.

Even for such a nice case as $GL_n(\mathbb{C})$ acting on $M_n(\mathbb{C})$ the answer to question 2 is no. For example if $X \neq 0$ is a nilpotent matrix, then $f(X) = f(0) = 0$ for every $f \in A(M)^G$. However one can separate non conjugate diagonalizable matrices by invariant polynomials.

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Some history: Let $G = SL_2(\mathbb{C})$ act on the space H_n of homogeneous polynomial functions on $\mathbb{C}^2$ of degree n in the obvious way. If $\varphi \in H_n$, then $\varphi(x, y) = a_0 x^n + a_1 x^{n-1} y + \cdots + a_n$. Thus an element of $A(H_n)$ is a polynomial $P(a_0, \ldots, a_n)$ in the coefficients $a_0, \ldots, a_n$. Roughly there were two basic questions: (1) Is $A(H_n)^G$ finitely generated? (2) If so, find explicit generators and relations.

Question 1 was answered in the affirmative by

Gordan in 1869. There was a great deal of work done ~~since~~ in the late 19th century on Question 2. Complete answers were obtained through $n=6$. Recently Shioda has given complete information for the case $n = 8$.

There was also a great deal of work done trying to prove finite generation of the ring of invariants for SL_3 (or SL_n) acting on homogeneous polynomials of degree n. This problem was solved by Hilbert in the 1890's.

<u>Hilbert's proof of finite generation of ring of invariants</u>

Using integration, it is possible to give an elementary proof of Hilbert's finiteness proof for compact groups.

<u>Thm</u> Let K be a compact sbgp of $GL_n(\mathbb{C})$. Then $A(\mathbb{C}^n)^K$ is a finitely generated $\mathbb{C}$-algebra.

<u>Sketch of proof</u> If $f \in A(\mathbb{C}^n)$, define $f^\natural$ by
$$f^\natural(x) = \int_K f(kx)\,dk$$
(dk - normalized Haar measure on K) $f \to f^\natural$ is projection of $A(\mathbb{C}^n)$ onto $A(\mathbb{C}^n)^K = J$. Let $J = \bigoplus_{m \geq 0} J_m$ (J_m homog invariants of deg m)
$J_+ = \bigoplus_{m>0} J_m$. The ideal $J_+ A(\mathbb{C}^n)$ has a finite set of generators (by the Hilbert basis theorem; this application was the motivation for Hilbert's proof.) We may choose homogeneous generators $\varphi_1, \dots \varphi_s$ from J_+. <u>Lemma</u> $J = \mathbb{C}[\varphi_1, \dots, \varphi_s]$. Proof. Suffices to show $J_m \subset \mathbb{C}[\varphi_1, \dots, \varphi_s]$ $\forall m$. Clear for $m=0$. Assume for $t < m$. Let $f \in J_m$; then $f = a_1\varphi_1 + \dots + a_s\varphi_s$ (*) a_j homog elts of $A(\mathbb{C}^n)$ s.t. $\deg a_i + \deg \varphi_i = m$. We use basic property of $h \mapsto h^\natural$: if $\varphi \in J$ and $h \in A(\mathbb{C}^n)$, then $(\varphi h)^\natural = \varphi h^\natural$. Apply the projection to * above

$f = f^{\natural} = a_1^{\natural}\varphi_1 + \cdots + a_s^{\natural}\varphi_s$. If $\deg a_j > 0$, then by induction $a_j^{\natural} \in \mathbb{C}[\varphi_1, \ldots, \varphi_s]$; follows immediately that $f \in \mathbb{C}[\varphi_1, \ldots, \varphi_s]$. q.e.d.

<u>Note</u> With minor technical modifications same proof works for $G \subseteq GL_n(k)$ G reductive, k alg closed of char 0.

<u>Consequences of Hilbert's result</u> k = base field of char 0. Let reductive group G act on affine variety X. Then $A(X)^G$ is finitely generated. Let $X/G = \mathrm{Spec\,max}\, A(X)^G$ and $\pi : X \to X/G$ correspond to $A(X)^G \hookrightarrow A(X)$. Then π is surjective and and fibre $\pi^{-1}(\xi)$, $\xi \in X/G$ contains a unique <u>closed</u> orbit. In particular X/G is a "good" orbit space iff all G-orbits in X are closed.

<u>Note</u> It follows from Haboush's recent proof of the Mumford Conjecture and earlier work of Nagata that <u>all</u> of above consequences <u>hold</u> for base field k of char. $p > 0$.

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Another influential work on invariant theory was H. Weyl's book, "The Classical Groups." Among other things Weyl gives a complete treatment (more or less) of the invariant theory of $G = SL_n(\mathbb{C}), SO_n(\mathbb{C}), O_n(\mathbb{C})$ and $Sp_n(\mathbb{C})$ acting on $\mathbb{C}^n \oplus \ldots \oplus \mathbb{C}^n$ (any number of copies). Recently this "classical invariant theory" has been redone in a characteristic free manner by Procesi and de Concini; apparently their treatment is an improvement on Weyl's even in characteristic 0.

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We now skip about 30 years to the early 1960's. <u>Mumford - Geometric Invariant Theory</u>. <u>Problem</u> (roughly) Let a family of algebraic structures of some type (e.g.

algebraic curves, vector bundles on curves) be parametrized by an algebraic variety X; a reductive group G acts on X, two points of X give equivalent structures iff they lie on same G-orbit. Want, in so far as possible, to give structure of algebraic variety to set of G-orbits (i.e. equivalence classes of structures). This is possible only for stable points of X.

A simple version Let G be reductive act linearly on a vector space V, hence G acts on $X = \mathbb{P}(V)$. Let $m = \max_{v \in V} \dim G\cdot v$; assume exists a closed orbit of dim m.

Defn $v \in V$ is stable if $G\cdot v$ is closed of dim. m.
$v \in V$ is semi-stable if $0 \notin$ closure of $G\cdot v$
$v \in V$ is unstable if it is not stable.

V_s = stable pts, V_{ss} = semi stable pts. Let ~~V~~ $p: V - \{0\} \to X = \mathbb{P}(V)$ be canonical map. $X_s = p(V_s)$, $X_{ss} = p(V_{ss})$. Let $Y = \mathrm{Proj}\,(A(V)^G)$; Y is the "projective" variety corresponding to the graded ring $A(V)^G$. Then is a surjective ~~map~~ morphism $\tau: X_{ss} \to Y$. Y is complete (roughly compact), $\tau(X_s)$ is an open subset of Y and τ induces a bijective correspondence between G-orbits in X_s (all closed) and pts of $\tau(X_s)$. Thus $\tau(X_s)$ is a "good" orbit space for the set X_s of stable points of $\mathbb{P}(V)$.

EXAMPLE $G = SL_2(\mathbb{C})$ acting on H_n. One can identify $\mathbb{P}(H_n)$ with $(\mathbb{P}^1)^n$/Symmetric group = set of ~~and~~ unordered n-tuples of points of $\mathbb{P}^1$ - (Idea: if $f \in H_n$, zero set of f is a set of n lines (~~and~~ allowing multiplicities) in $\mathbb{C}^2$, thus n points in $\mathbb{P}^1$) An unordered n-tuple $(P_1, \ldots, P_n)$ is stable (resp semi stable) if no. point occurs with multiplicity $\geq \frac{n}{2}$ (resp.

$> \frac{n}{2}$.

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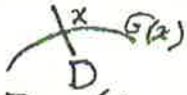
Luna's étale slice Motivation. Let the compact Lie group G act differentiably on the smooth manifold M. By an averaging argument, one can assume M Riemannian and G acts by isometries. Let $x \in M$ and let D be a small geodesic disk transversal to the orbit $G(x)$



Let E denote the homogeneous disk bundle $G *_{G_x} D$ (base $G/G_x = G(x)$ fibre D, structure group G_x) Then the map $G \times D \to M$ $(g, y) \mapsto g \cdot y$ ~~is~~ determines a G-isomorphism of E onto an open neighbourhood of the orbit $G(x)$.

Luna has proved a precise analogue for a reductive group G (in characteristic zero) acting on an affine variety. He proves that there exists an "étale slice" ~~is~~ in a neighbourhood of each closed orbit. Precisely he proves the following result: Let (G, X) be an affine G-space (G reductive). $\pi_X : X \to X/G$, $\xi \in X/G$, $x \in$ unique closed orbit in $\pi^{-1}(\xi)$. Then there exist 1) affine open sets $U' \subset X/G$, $\xi \in U'$ and $U = \pi_X^{-1}(U')$; 2) an affine variety V and a surjective étale morphism $\varphi : V \to U'$; and 3) an affine G_x-variety D — such that the fibre product $V \times_{U'} U$ is isomorphic as a G-variety to the homogeneous fibre bundle $G *_{G_x} D$."

For the complex case this implies the existence of "true" slices in the framework of complex analytic geometry.

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~~Cofree representations~~

<u>Coregular representations</u> Let $\rho : G \to GL(V)$ be a rep. of a red. grp. G (in char 0). We say that the rep. ρ is <u>coregular</u> if the ring of ~~rep.~~ invariants $A(V)^G$ is a regular ring (i.e V/G is a smooth variety). In this case it is easy to show that $A(V)^G$ is, in fact, a graded polynomial algebra (Hence $V/G \cong k^r$).

<u>Examples</u> 1. Example on page 1.

2. $G = S_n$ (symmetric group), ρ usual rep of G on $\mathbb{C}^n$. $A(\mathbb{C}^n)^{S_n}$ is a (graded) polynomial algebra.

3. $\underline{g}$ a semi simple Lie algebra, G = adjoint group, $\underline{h}$ a Cartan subalgebra, W = Weyl group. $A(\underline{h})^W$ is a graded poly. alg. The restriction map $A(\underline{g})^G \to A(\underline{h})^W$ is an isomorphism. Thus the adjoint rep of G is coregular. These results are due to Chevalley.

In the general case, a school of Russian mathematicians (Vinberg, Kac, Popov, assistance from others) have given a complete list of all ~~(G,V)~~ coregular (G,V) with G simple and V irred.

In a remarkable computational tour-de-force, G. Schwarz has recently extended the Russian list to a ~~list~~ list of all coregular (G,V) with G simple.

Conjugacy classes in semi-simple L.A.: Let $\mathfrak{g}$, G be as in example 3 preceding and ~~$\pi : \mathfrak{g} \to G$~~ $\pi : \mathfrak{g} \to \mathfrak{g}/G$ the standard map. Kostant ("Lie group reps. on polynomial rings") has given detailed information on G-orbits on $\mathfrak{g}$ (i.e. conjugacy classes), the fibres $\pi^{-1}(\xi)$, and the G-module structure of the graded ring $A(\mathfrak{g}) = \bigoplus_{n \geq 0} A(\mathfrak{g})_n$. In particular all fibres are normal and reduced, ~~fib~~ fibres of π are equidimensional, and each fibre has only a finite number of orbits (which can be rather precisely described). Recently there has been a great deal of detailed investigation of conjugacy classes and the geometry of the orbits. For $\mathfrak{g} = \mathfrak{sl}_n(\mathbb{C})$ it has been recently shown that the closure of any conjugacy class is normal; for $\mathfrak{g}$ of type B_n, this is apparently not so (Kraft - Procesi).

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Geometry of generalized flag manifolds (G/B and G/P), Schubert varieties (Bruhat cells) etc. Special examples of these varieties (e.g. Grassmann varieties) have long been of interest. The general case has recently been studied (in arbitrary characteristic or over $\mathbb{Z}$) by, among others, Kempf, Demazure, Seshadri, Musili, Lakshmibai, ... A number of good results have been obtained but many problems remain (particularly over $\mathbb{Z}$).

Tag ng über

TRANSFORMATIONSGRUPPEN UND INVARIANTENTHEORIE

Bonn, 22. - 28. September 1978

Titel: Deformations of simple singularities in simple Lie groups

Autor: Peter Slodowy

Adresse: Math. Institut, Universität Bonn, Wegeler Str. 10

Consider the quotient $\mathbb{C}^2/F$ of $\mathbb{C}^2$ by the natural action of a finite subgroup F of $SL_2(\mathbb{C})$. Such groups F are of the following kind: $\mathcal{C}_n$ = cyclic group of order n, $\mathcal{D}_n$ = binary diedral group of order $4n$, $\mathcal{T}$ = binary tetrahedral group, $\mathcal{O}$ = binary octahedral group or $\mathcal{J}$ = binary icosahedral group. The quotients $\mathbb{C}^2/F$ are isolated surface singularities and can be resolved minimally. The dual graph of the exceptional divisor turns out to be one of the homogeneous Dynkin-diagrams according to the following correspondence:

$$\mathcal{C}_n \quad — \quad A_{n-1}$$

$$\mathcal{D}_n \quad — \quad D_{n+2}$$

$$\mathcal{T} \quad — \quad E_6$$

$$\mathcal{O} \quad — \quad E_7$$

$$\mathcal{J} \quad — \quad E_8$$

Accordingly the quotients $\mathbb{C}^2/_F$ are called simple singularities, rational double points or Kleinian singularities of the respective types A, D, E.

The following theorem was conjectured by Grothendieck and proved by Brieskorn.

Theorem 1 (cf. [1]): Let $\chi : \mathfrak{g} \to \mathfrak{h}/W$ be the quotient of a simple Liealgebra $\mathfrak{g}$ of type A, D, E by the action of the adjoint group G ($\mathfrak{h}$ Cartansubalgebra, W Weylgroup) and $S \subset \mathfrak{g}$ a transversal slice to the subregular nilpotent Orbit in $\mathfrak{g}$. Then the intersection $S \cap N(\mathfrak{g})$ of S with the nilpotent variety $N(\mathfrak{g}) = \chi^{-1}(\chi(0))$ is a simple singularity of the corresponding type and the restriction $\chi|_S : S \to \mathfrak{h}/W$ of χ to the slice S realises a semiuniversal deformation of this singularity.

Performing the same construction as in the theorem above for simple Lie algebras of type B_n, C_n, F_4, G_2 one could identify $S \cap N(\mathfrak{g})$ as a simple singularity of type A_{2n-1}, D_{n+1}, E_6, D_4 respectively (Steinberg, Tits).

Define now simple singularities of type B, C, F, G as couples of singularities X_0 and groups Γ

acting on them:

type		(X_0, Γ)
B_n	—	$(A_{2n-1}, \Gamma = \mathcal{D}_n/\mathcal{C}_{2n})$
C_n	—	$(D_{n+1}, \Gamma = \mathcal{D}_{2n-2}/\mathcal{D}_{n-1})$
F_4	—	$(E_6, \Gamma = \mathcal{O}/\mathcal{T})$
G_2	—	$(D_4, \Gamma = \mathcal{O}/\mathcal{D}_2)$

Here the groups $\Gamma = F'/F$ act naturally on the singularities $X_0 = \mathbb{C}^2/F$.

A deformation of a couple (X_0, Γ) is a deformation $\mathcal{X} : X \longrightarrow (S, s)$ of $X_0 \cong X_s$ with a Γ-action on X s.t. $\mathcal{X}$ is an invariant morphism and the induced Γ-action on $X_s \cong X_0$ is the given one. There is an obvious notion of semiuniversal deformation for the couple (X_0, Γ).

<u>Theorem 2</u> (cf. [2]) With the above definitions theorem 1 is also true in the cases B_n, C_n, F_4, G_2.

As a corollary of theorems 1 and 2 one obtains a description of the singularities in the neighbouring fibres of the semiuniversal deformations. For details cf.:

[1] Brieskorn, E.: Singular elements of algebraic groups, in Proceed. of the I.M.C at Nice 1970

[2] Slodowy, P.: Einfache Singularitäten und einfache algebraische Gruppen, Regensburger Math. Schriften 2, 1978

Tagung über

TRANSFORMATIONSGRUPPEN UND INVARIANTENTHEORIE

Bonn, 22. - 28. September 1978

Titel: NORMALITY OF CONJUGACY CLASSES

Autor: PROCESI CLAUDIO

Adresse: ISTITUTO MATEMATICO - UNIVERSITA DI ROMA

(Joint work with H.Kraft)

Let $\mathfrak{g}$ be a semisimple Lie algebra, G the adjoint group, $x \in \mathfrak{g}$ and $\overline{Gx}$ the orbit closure. We analize the problem: Is $\overline{Gx}$ a normal variety?

We will restrict our attention to nilpotent orbits.The answer is yes if $\mathfrak{g} = sl_n$ and no in general,althout for the classical group one has infinitely many nilpotent orbit closures yhich are and infinitely many wich are not normal.

<u>The method for sl_n</u>

Let $X \in M$ be a nilpotent $n \times n$ matrix,thought as a map $X:U \to U$.

$$X \sim \begin{pmatrix} J_{p_1} & 0 & \cdots & 0 \\ 0 & J_{p_2} & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & J_{p_r} \end{pmatrix}, \quad J_t = \begin{pmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & & \vdots \\ 0 & 0 & 0 & 1 & \ddots & 0 \\ \vdots & & & 0 & \ddots & 1 \\ 0 & \cdots & & & \cdots & 0 \end{pmatrix}$$

a Jordan block, $p_1 \geq p_2 \geq \cdots \geq p_r$. We draw the partition $p_1 p_2 \ldots p_r$ as a diagram,e.g.  is 3,2,2 and let $\check{p}_1 \check{p}_2 \ldots \check{p}_t$ be the dual partition of columns.

The rank of X^i is $n_{t-i} = \check{p}_{i+1} + \check{p}_{i+2} + \ldots$ .We form vector spaces of dimension $n_1, n_2, \ldots$: $U_1, U_2, \ldots$ and consider the affine space M of maps $U_1 \underset{B_1}{\overset{A_1}{\rightleftarrows}} U_2 \underset{B_2}{\overset{A_2}{\rightleftarrows}} U_3 \rightleftarrows \cdots\cdots \underset{B_{t-1}}{\overset{A_{t-1}}{\rightleftarrows}} U_t$.

In M we look at the variety Z given by the equations:

$$\begin{aligned}
B_1A_1 &= 0\\
B_2A_2 &= A_1B_1\\
B_3A_3 &= A_2B_2\\
&\vdots\\
B_{t-1}A_{t-1} &= A_{t-2}B_{t-2}
\end{aligned}$$

Theorem 1) Z under the previous equations is a complete intersection ,non singular in codimension 1.

2) The mappimg $(A_1, B_1, \ldots\ldots, A_{t-1}, B_{t-1}) \longrightarrow A_{t-1}B_{t-1}$ restricted to Z is a quotient under the group $Gl(U_1) \times Gl(U_2) \times \ldots\ldots \times Gl(U_{t-1})$ with image the closure of the orbit of X .

sketch of the proof

2) The previous equations and the classical theorems of invariant theory,that the multiplication of matrices is a quotient, imply easily that the map is in fact a quotient . The fact that it maps onto the closure V of the orbit of X comes from two remarks:

a) $V = \{ Y \mid \operatorname{rk} Y^i \leq \operatorname{rk} X^i \text{ for all } i \}$

b) We can choose isomorphisms $U_i \simeq X^{t-i}(U)$ and thus a point in Z associated to the diagram

$0 \underset{X}{\overset{i}{\rightleftarrows}} X^{t-1}(U) \underset{X}{\overset{i}{\rightleftarrows}} X^{t-2}(U) \underset{X}{\overset{i}{\rightleftarrows}} \ldots\ldots$ (i means inclusion).

1) Looking at the equations defining Z as fiber of a map from M to $\operatorname{End}(U_1) \times \operatorname{End}(U_2) \times \ldots\ldots \times \operatorname{End}(U_{t-1})$ one checks

that the differential of such a map is onto at all points for which, for each j, either A_j or B_j is of maximal rank. This gives a hold of a big set Z^o of smooth points of Z, then one has to show that $\dim Z - Z^o \leqslant \dim Z - 2$. This is obtained stratifying Z by strata which are itereted fiber products and of which we can compute the dimension. The strata are those for which each pair A_i, B_i lies in a fixed orbit of pairs of maps $U_{i-1} \underset{B_i}{\overset{A_i}{\rightleftarrows}} U_i$ under $Gl(U_{i-1}) \times Gl(U_i)$. One uses the classification of such orbits to compute their dimension and so that of the stratum.

A similar approach can be used for the other classical groups. In this case if U is a space with a form, symmetric or antisymmetric, $X: U \rightarrow U$ a skew map, X induces a non degenerate form of opposite type on $X(U)$ and so we have a picture: a) Spaces and maps $U_1 \xrightarrow{X_1} U_2 \xrightarrow{X_2} U_3 \xrightarrow{X_3} \cdots \xrightarrow{X_{t-1}} U_t$ alternatively orthogonal and symplectic,

b) Equations $X_1^* X_1 = 0,\ X_2^* X_2 = X_1 X_1^*, \ldots$

The variety Z obtained is a complete intersection but may be singular in codimension 1. The mapping $(X_1, \ldots\ldots, X_{t-1}) \longrightarrow X_{t-1} X_{t-1}^*$ is a quotient of Z onto the orbit closure.

Theorem Z is non singular in codimension 1 if the partition (of X) has not 4 consecutive rows $k\ k\ h\ h$, $h < k$, with k, h even in the orthogonal case and odd in the symplectic.

Corollary If the complement of the orbit GX of X in its closure has codimension $\geqslant 4$ then $\overline{GX}$ is normal.

Finally for example, for the partition 3 3 1 1 the orbit closure in sp_8 is not normal.

Tagung über

TRANSFORMATIONSGRUPPEN UND INVARIANTENTHEORIE

Bonn, 22. - 28. September 1978

Titel: SL(2)-Embeddings

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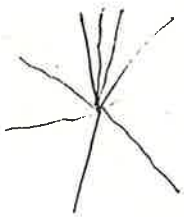
Let G be a connected linear algebraic group. An <u>embedding</u> of G is an irreducible reduced algebraic G-variety with a given point $x \in X$ such that the orbit map $s \in G \mapsto s.x \in X$ is an open embedding. The main purpose of the talk was to give the classification of all (complete) normal embeddings of $SL(2,\mathbb{C})$. (joint work with Th. Vust)

Consider $\mathbb{P}_1 = \mathbb{P}_1(\mathbb{C})$ with discrete topology and let us designe by $\left({}^{SL(2)}/_{U}\right)_{\mathbb{R}}$ the topological space $\mathbb{P}_1 \times [0,1] / \mathbb{P}_1 \times \{0\}$



(as many intervals $[0,1]$ as points in $\mathbb{P}_1$, glued together in one point). If $\sigma \in \left({}^{SL(2)}/_{U}\right)_{\mathbb{R}}$, let us designe by $|\sigma|$ its natural projection on $[0,1]$.

A (complete) <u>fan</u> in $(\mathcal{X}(\mathcal{C})/\Gamma)_{\mathbb{R}}$ is given by a choice of a finite set F of rational points on $(\mathcal{X}(\mathcal{C})/\Gamma)_{\mathbb{R}}$ with two restrictions:

1) If all the points are chosen in one interval, there must be a point with $| \; | < \frac{1}{2}$.

2) It is not allowed to chose just two points with $| \; |$ equal to 1.

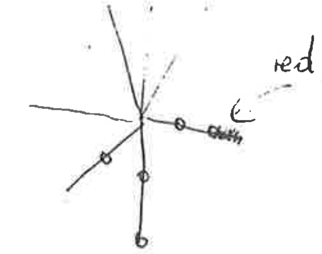
Let us call <u>segments</u> the connected components of $(\mathcal{X}(\mathcal{C})/\Gamma)_{\mathbb{R}} - F$. A segment of type "all the points on an interval satisfying $| \; | > \alpha$, with $\alpha > \frac{1}{2}$" is called a special segment.

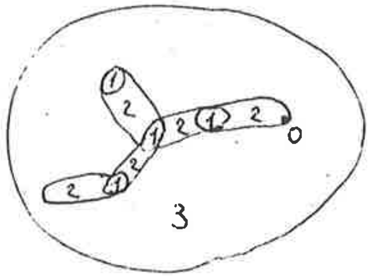
A <u>painted fan</u> is a fan with some of the special segments painted in red.

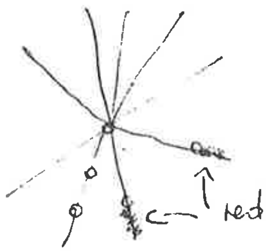
<u>Theorem</u> The complete normal imbeddings of $SL(2)$ are classified (up to isomorphisme) by painted fans.

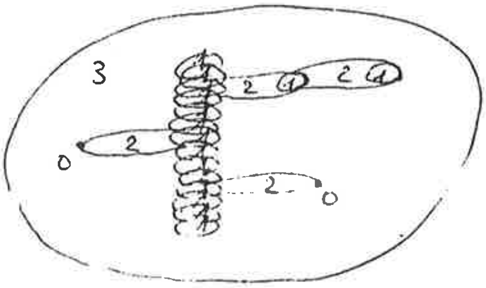
There is a "dictionary" permitting to interpret painted fans: plain segments correspond to $\mathbb{P}_1$'s (closed orbits of dimension 1), red segments to fixed points, the points of $F - \{\text{middle point}\}$ to orbits of dimension 2, ..., with obvious rules for specialization, ...

Examples:









<u>Proposition</u> A complete embedding of $SL(2)$ is projective if and only if there is at most one fixed point.

The main tool in the proofs is the notion of G-valuation, the discrete valuations of the field of rational functions on G, which are invariant under left translations.

Tagung über

TRANSFORMATIONSGRUPPEN UND INVARIANTENTHEORIE

Bonn, 22. - 28. September 1978

Titel: Duality for some unipotent classes. . .

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G a connected reductive algebraic group $/k$, algebraically closed. $X = X^G = \{$unipotent classes of $G\}$, ordered by inclusion of Zariski closures. If $G = GL_n$; X has a natural decreasing involution d. We try to get something similar in the general case.

Let $P \subset G$ be a parabolic subgroup, L a Levi factor, U_P the unipotent radical. Define the Richardson class C_P by: $C_P \cap U_P$ is dense in U_P. Let $u \in L$ be a regular unipotent element of L. Let $\hat{C}_P$ be the class of u in G.

For GL_n, we have $d(\hat{C}_P) = C_P$ (and $d(C_P) = \hat{C}_P$). In general, it is known that $\hat{C}_P = \hat{C}_{P'} \Leftrightarrow P$ and P' have conjugate Levi factors $\Rightarrow C_P = C_{P'}$.

We look now for a decreasing map $d : X \to X$ such that:

I) $d(\hat{C}_P) = C_P$

II) $d^2(x) \geq x$ for all $x \in X$.

Let $e = d^2$ and $\tilde{X} = d(X)$. Then $d^3 = d$, $e^2 = e$, $e(X) = \tilde{X}$, and the restriction of d to $\tilde{X}$ is a decreasing involution.

Thm 1. Such a map d always exists.

Thm 2. If G is of type A_n, B_n, C_n or D_n, then d is unique, $d(x) = \sup\{\hat{C}_P \mid C_P \geq x\}$ and $e(x) = \inf\{C_P \mid C_P \geq x\}$.

Dependence with respect to p:

Take G' a connected reductive group $/\mathbb{C}$, same type as G.

Choose $B\subset G$, $B'\subset G'$ (Borel subgroups) and $T\subset B$, $T'\subset B'$ (max. tori). Identify the root systems.
Let $\Pi=\{P\subset G \mid P\supset B\}$, $\Lambda=\{L\supset T \mid L \text{ Levi factor of some } P\in\Pi\}$, and define Π', Λ' in a similar way. We have natural bijections $\Lambda'\leftrightarrow\Pi'\leftrightarrow\Pi\leftrightarrow\Lambda$.

If $L\in\Lambda$, unipotent elements of L belong also to G, and we get a natural map $i_{L,G}: X^L\to X^G$.

<u>Proposition</u>. There is a unique increasing map $\pi_G: X^{G'}\to X^G$ such that:

1) If $L\in\Lambda$ and $L'\in\Lambda'$ correspond to each other, then we have $i_{L,G}\circ\pi_L \le \pi_G\circ i_{L',G'}$ in the diagram:

$$\begin{array}{ccc} X^{L'} & \xrightarrow{\pi_L} & X^L \\ {\scriptstyle i_{L',G'}}\downarrow & & \downarrow{\scriptstyle i_{L,G}} \\ X^{G'} & \xrightarrow{\pi_G} & X^G \end{array}$$

2) If $C'\in X^{G'}$ and $C=\pi_G(C')$, then $\operatorname{codim}_{\mathcal{U}(G)} C = \operatorname{codim}_{\mathcal{U}(G')} C'$, where $\mathcal{U}(G)$, $\mathcal{U}(G')$ are the unipotent varieties of G, G' respectively.

If $P\in\Pi$ and $P'\in\Pi'$ correspond to each other, we have $C_P\in X^G$, $\hat{C}_P\in X^G$, $C_{P'}\in X^{G'}$, $\hat{C}_{P'}\in X^{G'}$, and we define $C_P^\circ=\pi_G(C_{P'})$, $\hat{C}_P^\circ=\pi_G(\hat{C}_{P'})$. Let also $X^\circ=\pi_G(X^{G'})$, $X=X^G$, and identify $X^{G'}$ with X°.

<u>Proposition</u>. 1) $C_P^\circ = C_P$

2) the map $f: X\to X^\circ$, $x\mapsto \sup_{X^\circ}(x)$ is well defined.

3) $f(\hat{C}_P)=\hat{C}_P^\circ$.

<u>Corollary</u>. Let $d_0: X^\circ\to X^\circ$ be a decreasing map with properties (I) and (II) (for G'). Then $d=d_0\circ f: X\to X$ is a decreasing map with properties I and (II) (for G).

<u>Proposition</u>. All decreasing maps $d: X\to X$ satisfying (I) and (II) are of this type.

<u>Thm 3</u>. As an ordered set with a decreasing involution, $(\tilde{X}, d)$ depends only on W (the Weyl group of G).

(for exceptional groups we get actually a family of such pairs, and this family depends only on W).

We know already that $(\tilde{X}, d)$ is independent of the characteristic p of k. To check that we get the same result for B_n and C_n it is then sufficient to take the case $p = 2$, where this is obvious.

<u>Demonstrations.</u>

For E_6, E_7, E_8, F_4, G_2 : look at the tables.

For classical groups : take first $p \neq 2$, $G = Sp_{2n}$, SO_{2n+1} or O_{2n} (it is then quite easy to go from O_{2n} to SO_{2n}), $G \subset GL_N$ ($N = 2n$ or $2n+1$).

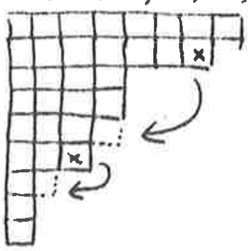
Let $\mathcal{P} = \{\text{partitions of } N\}$. If $\lambda \in \mathcal{P}$ let λ^* be the dual of λ. Taking the dimension of Jordan blocks, we get a bijection $\mathcal{P} \leftrightarrow X^{GL_N}$, and X corresponds to a subset $\mathcal{X}$ of $\mathcal{P}$. (If $G = Sp_{2n}$ (resp. SO_{2n+1} or O_{2n}), then $\lambda \in \mathcal{X}$ iff λ has an even number of parts equal to $2i-1$ (resp. $2i$), $i \geq 1$). Let ρ_p (resp. λ_p) be the partition corresponding to C_p (resp. $\hat{C}_p$).

<u>Lemma</u>. If $\lambda, \mu \in \mathcal{P}$, then $\inf_{\mathcal{P}}(\lambda, \mu)$ exists.

<u>Lemma</u>. If $\lambda \in \mathcal{P}$, then $\inf_{\mathcal{X}}(\lambda)$ exists.

These are very easy to prove.

The map $\lambda \mapsto \inf_{\mathcal{X}}(\lambda)$ can be described as follows. Represent λ as a diagram (the lines corresponding to the parts of λ), and "make λ into an element of $\mathcal{X}$" starting from the top. For example, if $G = Sp_{2n}$ and $\lambda = (8, 7, 4, 4, 3, 3, 1, 1, 1, 0, \ldots)$, we have:



<u>Corollary</u>: If $A \subset \mathcal{X}$, then $\inf_{\mathcal{X}}(A)$ and $\sup_{\mathcal{X}}(A)$ exist.

We now define $d : \mathcal{X} \to \mathcal{X}$, $\lambda \mapsto \inf_{\mathcal{X}}(\lambda^*)$

Let also $e = d^2$, $\tilde{\mathcal{X}} = e(\mathcal{X})$.

<u>Lemma</u>. $\lambda \le d^2(\lambda)$ for all $\lambda \in \mathcal{X}$.

Since $d(\lambda) = \inf_{\mathcal{X}}(\lambda^*) \le \lambda^*$ we have also $\lambda \le d(\lambda)^*$. Therefore $\lambda = \inf_{\mathcal{X}}(\lambda) \le \inf_{\mathcal{X}}(d(\lambda)^*) = d^2(\lambda)$.

<u>Lemma</u>. $d(\lambda_P) = \rho_P$.

This is a reformulation of known results.

So d satisfies the conditions (I) and (II). Define now d', e' by: $d'(\lambda) = \sup_{\mathcal{X}}\{\lambda_P \mid \rho_P \ge \lambda\}$, $e'(\lambda) = \inf_{\mathcal{X}}\{\rho_P \mid \rho_P \ge \lambda\}$ (this makes sense). To check that $d' = d$ and $e' = e$, it can be seen that it is enough to prove that if $\lambda \in \tilde{\mathcal{X}}$, then $\lambda = \sup_{\mathcal{X}}\{\lambda_P \mid \lambda_P \le \lambda\}$. This is done by exhibiting a suitable family of P's such that $\lambda_P \le \lambda$.

Once we know that $d(x) = \sup_X\{\hat{C}_P \mid C_P \ge x\}$ and $e(x) = \inf_X\{C_P \mid C_P \ge x\}$, the proof of unicity can be reduced to the following condition: if $x \in X$ is such that $\hat{C}_P \le x \Rightarrow e(\hat{C}_P) \le x$, then $x \in \tilde{X}$. This again is done by exhibiting a suitable family of parabolic subgroups.

In the case where $p = 2$, we start by checking that π_G exists and has the required properties. Using the fact that thm 2 is true for $p = 0$, one shows that it is sufficient to check the following conditions:

1) if $x \in \tilde{X}^0$ $(= \pi_G(\tilde{X}^{G'}))$, then $x = \sup_X\{\hat{C}_P \mid \hat{C}_P \le x\}$

2) if $x \in X$ is such that $\hat{C}_P \le x \Rightarrow e(\hat{C}_P) \le x$ $(e = e_0 \circ f)$, then $x \in X^0$.

For this we use a suitable family of P's.

Tagung über

TRANSFORMATIONSGRUPPEN UND INVARIANTENTHEORIE

Bonn, 22. - 28. September 1978

Titel: Affine Cross-Sections and Dixmier Sheets of $sl(n)$

Autor: Dale Peterson

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For an algebraic group action $G \times X \to X$ and a subvariety Y of X, set $Y^{(n)} = \{y \in Y : \dim Gy = n\}$, $Y^{reg} = \{y \in Y : \forall y' \in Y, \dim Gy \geq \dim Gy'\}$; these are locally-closed sets. In case $\mathfrak{g}$ is a complex semisimple Lie algebra and G is the adjoint group of $\mathfrak{g}$ with the adjoint action, an irreducible component of any $\mathfrak{g}^{(n)}$ is a <u>sheet</u>, and a sheet containing a semisimple element is a <u>Dixmier sheet</u>.

Dixmier sheets were studied by Dixmier [3] for $\mathfrak{g} = sl(n)$, and by Kraft [5] and Borho-Kraft [2].

From now on, $\mathfrak{g}$ denotes a complex semisimple Lie algebra. A <u>polarization</u> of $x \in \mathfrak{g}$ is a (necessarily parabolic) subalgebra $\mathfrak{p}$ of $\mathfrak{g}$ which is a maximal isotropic subspace of $\mathfrak{g}$ for the alternating form $B_x(y,z) = B(x,[y,z])$, where B is the Killing form of $\mathfrak{g}$. If $\mathfrak{p}$ is a parabolic subalgebra of $\mathfrak{g}$, and $\underline{r}$ and $\underline{n}$ are the solvable and nilpotent radicals of $\mathfrak{p}$, then $\underline{r}^{reg} = \{x \in \mathfrak{g} : x \text{ is polarized by } \mathfrak{p}\}$, and $\underline{n}^{reg} \subset \underline{r}^{reg}$. One has:

Theorem 1 ([2]). For any parabolic subalgebra $\mathfrak{p}$ of $\mathfrak{g}$, $D(\mathfrak{p}) = G \cdot \underline{r}^{reg}$ is a Dixmier sheet of $\mathfrak{g}$, and all Dixmier sheets are obtained in this way.

From the point of view of representation theory, Dixmier sheets are interesting because of Theorem 1 and the "orbit picture". In this picture, primitive ideals should "look like" the ideals of the corresponding orbits. Good results have been obtained here for the regular sheet $\mathfrak{g}^{reg}$ and for $\mathfrak{g} = \underline{sl}(n)$ [1].

The Dixmier sheets of $\underline{sl}(n)$ are described in [3]. One knows that they form a disjoint cover of $\underline{sl}(n)$. The following two results describe the orbits of $\underline{sl}(n)$ and the orbit-spaces $D(\mathfrak{p})/SL(n)$.

Theorem 2 [6]. The closure of any orbit of $SL(n)$ in $\underline{sl(n)}$ is a normal variety.

Theorem 3 [7,8]. Let D be a Dixmier sheet of $\mathfrak{g} = \underline{sl(n)}$. Then D is smooth, and there exists an affine subspace A of $\underline{sl(n)}$ such that:

1) $A \subset D$, and for any $x \in D$, Gx intersects A transversely in a single point.
2) Restriction gives an isomorphism $\mathbb{C}[D]^G \cong \mathbb{C}[A]$.

In the case of arbitrary $\mathfrak{g}$, one cannot expect such nice properties for a "cross-section" of a Dixmier level, even a smooth one [8, ch.4].

The following result gives some information on invariant subvarieties of a complex semisimple $\mathfrak{g}$. It is correct only on the level of locally-bounded functions, i.e. regular functions on the normalization.

Theorem 4 [8]. Let $\mathfrak{g}$ be a complex semisimple Lie algebra, $\mathfrak{p}$ a parabolic subalgebra of $\mathfrak{g}$, $x \in \mathfrak{g}$ a nilpotent polarized by $\mathfrak{p}$. Assume:

1) $\overline{Gx}$ is a normal variety.
2) $G^x = P^x$.

Set $H = \sum_{\substack{y \in Gx \\ k \in \mathbb{N}}} \mathbb{C}\, x^k \subset S(\mathfrak{g}) \cong \mathbb{C}[\mathfrak{g}]$.

Let V be any G-invariant subvariety of $\mathfrak{g}$ such that $V = \overline{V \cap D(\mathfrak{p})}$. Then we have G-module isomorphisms

$$R^b(V) = R^b(V \cap D(\mathfrak{p})) \cong R^b(V \cap D(\mathfrak{p}))^G \otimes H$$
$$= R^b(V)^G \otimes H ,$$

where the isomorphism maps tensor product to multiplication, and R^b denotes the ring of locally-bounded rational functions.

In particular, Theorem 4 always holds for $\mathfrak{g} = \underline{sl(n)}$. We note that Theorem 3 generalizes a result of Kostant [4] for $\mathfrak{g} = \underline{sl(n)}$, and Theorem 4 generalizes another result of Kostant.

Bibliography:

1 W. Borho, Definition einer Dixmier-Abbildung für $sl(n, \mathbb{C})$. Inv. Math. 40(1977), 143-169.

2 W. Borho and H. Kraft, Über Bahnen und deren Deformationen bei linearen Aktionen reduktiver Gruppen. Preprint, Bonn, 1977.

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3 J. Dixmier, Polarisations dans les algebres de Lie semi-simples complexes. Bull. Sci. Math. 99 (1975), 45-63.

4 B. Kostant, Lie group representations on polynomial rings. Amer. J. Math. 85 (1963), 327-404

5 H. Kraft, Parametrisierung der Konjugationsklassen in sl_n. Preprint, Bonn, 1977.

6 H. Kraft and C. Procesi, Closures of conjugacy classes in gl_n are normal. Preprint, 1978.

7 D. Peterson, Affine cross-sections for gl/GL. Inv. Math. (to appear)

8 D. Peterson, <u>Geometry of the Adjoint Representation of a Complex Semisimple Lie Algebra</u>. Thesis, Harvard U., 1978.

Tagung über

TRANSFORMATIONSGRUPPEN UND INVARIANTENTHEORIE

Bonn, 22. - 28. September 1978

Titel: Lifting Vector Fields From Orbit Spaces I+II

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I. Introduction.

G will always denote a reductive complex algebraic group, V will always denote a rational representation ~~of~~ space of G. V/G denotes the variety corresponding to the invariants $\mathbb{C}[V]^G$, and $\pi_G : V \longrightarrow V/G$ denotes the canonical map.

Let Gv be closed. The *slice representation* at v is the representation of G_v on $V/T_v(Gv)$. If H is a reductive subgroup of G, then $(V/G)_{(H)}$ denotes the closed orbits Gv with $G_v \in (H)$, and (H) is called an isotropy class of V if $(V/G)_{(H)} \neq \emptyset$. We say that $(H) \leq (L)$ if H is conjugate to a subgroup of L.

<u>Prop. (Luna)</u>: ① $(V/G)_{(H)}$ is constructible and

$$\overline{(V/G)_{(H)}} = \bigcup_{(L)\geq(H)} (V/G)_{(L)} = \pi_G(V^H).$$

② the stratification of V/G is ~~finite~~ finite.

③ there is a unique minimal isotropy class (H_0), ~~and~~ and $V/G = \pi_G(V^{H_0})$. (H_0) is called the principal isotropy class.

<u>Defn.</u> $I_{(H)} = \{f \in C[V/G] : f = 0 \text{ on } (V/G)_{(H)}\}$.

Let $\mathcal{X}(V/G)$ denote the derivations of $C[V/G]$ which preserve the ideals $I_{(H)}$, and let $\mathcal{X}(V)$ denote the derivations of $C[V]$. Then the natural map $(\pi_G)_*$ sends $\mathcal{X}(V)^G$ into $\mathcal{X}(V/G)$.

<u>Main Theorem (abbreviated MT)</u>: Suppose that V is an orthogonal representation. Then $(\pi_G)_*$ maps $\mathcal{X}(V)^G$ onto $\mathcal{X}(V/G)$.

<u>Example:</u> Suppose $\dim V/G = 1$ and suppose G acts non-trivially on V. Then $\mathbb{C}[V]^G$ is generated by a form f of degree $m > 1$, and $V/G \simeq f(V) = \mathbb{C}$ with strata $\{0\}$ and $\mathbb{C} - \{0\}$. Thus $\mathcal{X}(V/G)$ has a generator A corresponding to $z\frac{\partial}{\partial z}$, i.e. $A(f) = f$. Let $B = \sum z_i \frac{\partial}{\partial z_i}$ where the z_i are co-ordinates on V. Then $B(f) = mf$, so $(\pi_G)_* \frac{1}{m} B = A$.

<u>Remark.</u> We did not ~~assume~~ use the orthogonality assumption above. A similar but more complicated argument can be used in the case V orthogonal, $\dim V/G = 2$. ~~can be used~~

II. Motivation

Let M be a K-space (e.g. a smooth K-manifold) where K is a compact Lie group. One approach to classifying such M is the following:

① Construct a universal K-space (or manifold) M_0 such that every M we are interested in is a pull-back:

$$\begin{array}{ccc} f^* M_0 = M & \longrightarrow & M_0 \\ \downarrow f^*\pi & & \downarrow \pi \\ M/K & \xrightarrow{f} & M_0/K \end{array}$$

② Show that if $f, f' : M/K \to M_0/K$ are homotopic in the appropriate category, then $f^* M_0$ is K-isomorphic to $(f')^* M_0$.

<u>Case 1.</u> Free K-actions. Here M_0 is the universal principal K-bundle and ② is the usual homotopy lifting theorem.

<u>Case 2.</u> K actions on topological spaces. Here problems ① and ② have been solved by Palais (1960). The maps between orbit spaces must preserve the isotropy type strata.

Case 3. Smooth actions. There has been much work on part ① by Jänich, the Hsiangs, Bredon, Davis, Erle, Hirzebruch, and others. They have also proved special cases of ② – as has also Bierstone.

One can reduce the appropriate version of ② to a problem of lifting strata preserving ~~and~~ C^∞ vector fields on orbit spaces W/L, where W is a representation space of the compact Lie group L. One can use ideas of Malgrange to reduce to the case of polynomial vector fields, and by complexifying (i.e. passing from (W, L) to $(W \otimes_{\mathbb{R}} \mathbb{C}, L_{\mathbb{C}} = \text{cxif. of } L)$) we reduce to proving the main theorem.

III. The General Case

From now on we always assume that V is an orthogonal representation space of G.

Using Serre's results in GAGA, it is equivalent to prove the MT or its complex analytic analogue (where we consider holomorphic vector fields). Let $X \in \mathcal{X}(V/G)$. By induction we may assume the MT holds for representations of proper reductive subgroups of G, and then using the holomorphic version of Luna's slice theorem we can assume there are local lifts A_α of X on G-invariant open sets $\mathcal{U}_\alpha$ covering $V - \pi_G^{-1}(0)$. The obstruction to patching the A_α is an element $\beta_X \in H^1(V/G - 0, \underset{\sim}{M}^h)$ where $\underset{\sim}{M}^h$ is the complex analytic sheaf corresponding to the $\mathbb{C}[V/G]$-module $M = \mathrm{Ker}(\pi_G)_*$. Since we have solved the case $\dim V/G = 1$ (even $\dim V/G = 2$), we may assume $\dim V/G \geq 2$, in which case one can show $\mathrm{codim}\, \pi_G^{-1}(0) \geq 2$. Then if $\beta_X = 0$, X lifts to $V - \pi_G^{-1}(0)$, hence to all of V by Hartog's extension theorem.

Now assume:

(I) (V,G) has finite principal isotropy groups (abbreviated FPIG).

Now $\mathfrak{g}$ acts on V as vector fields which annihilate $\mathbb{C}[V]^G$, and there is an injection (assuming (I)) of $(\mathbb{C}[V]\otimes\mathfrak{g})^G \underset{\text{defn.}}{=} \mathcal{X}_{\mathrm{Ad}\,G}(V)^G$ into M.

Also assume:

(II) $\qquad M = \mathcal{X}_{\mathrm{Ad}G}(V)^G$.

Note that if (I) and (II) hold for (V,G), then they hold for any slice representation. Using commutative algebra and the theory of local cohomology one can show that our H^1 problem with coefficients in $\mathcal{X}_{\mathrm{Ad}G}(V)^G$ vanishes if $\operatorname{codim} \pi_G^{-1}(0) \geq 3$. Using Hilbert-Mumford theory one can ~~show~~ prove the estimate:

$$\operatorname{codim} \pi_G^{-1}(0) \geq \tfrac{1}{2}(\dim V - \dim G + \operatorname{rank} G + \nu_0)$$

where ν_0 is the multiplicity of the zero weight space. Since we are assuming (I), $\dim V - \dim G = \dim V/G$. We may assume $\operatorname{rank} G \geq 1$ (if G is finite then $M = 0$!) and that $\dim V/G \geq 3$ (we have the cases $\dim V/G \leq 2$). Thus we get the required estimate unless $\operatorname{rank} G = 1$, $\dim V/G = 3$, $\nu_0 = 0$. One can give a separate argument in this case (it turns out that $G^0 = \mathbb{C}^*$). Thus we have:

<u>Theorem MT'</u>: Suppose (I) and (II) hold. Then $(\pi_G)_* \mathcal{X}(V)^G = \mathcal{X}(V/G)$.

We can make condition (II) less mysterious: We say that (V, G) has S^3 strata if there is a slice representation (V', G') where $(V', (G')^0) = (2\mathbb{C}^2 + \text{trivial}, SL_2)$. Then one can prove:

Prop. Assume (I). Then (II) holds $\Longleftrightarrow$ (V,G) has no S^3 strata.

IV. Outline of the Proof

One can show that if the MT holds for representations of G, then it holds for any extension of G by tori $(\mathbb{C}^*)^m$ or by finite groups. Thus we may reduce to the case that G is a product of connected simple groups. We now state two more reduction results and we show below how to apply them to prove the MT.

Proposition A: Suppose (V,G) has principal isotropy groups (H) and that

$$(*) \quad \operatorname{res}_H \mathcal{X}(V)^G = \mathcal{X}(V^H)^{N_G(H)/H}$$

where res_H denotes restriction to V^H. If $(V^H, N_G(H)/H)$ has the lifting property, (i.e. the MT holds) then so does (V,G).

Remark. It turns out that (*) holds in almost all cases we had to consider. Lifting from $\mathcal{X}(V^H)^{N_G(H)}$ to $\mathcal{X}(V)^G$ fails $\Longleftrightarrow$ it fails near the codimension one strata.

Proposition B. Let V be a representation of $G \times G_1$, where $G \neq \{id\} \neq G_1$. Suppose that lifting holds for all proper reductive subgroups of $G \times G_1$, and that $\mathbb{C}[V]^{G_1}$ is a regular ring. Then lifting holds for $(V, G \times G_1)$.

Our last ingredient is the notion of index (the notion goes back to Dynkin, but our application follows from ideas of Andreev, Elashvili, and Vinberg). Let G be simple. Define $\operatorname{ind}_G(V) = \dfrac{\operatorname{trace}_V(X^2)}{\operatorname{trace}_{\mathfrak{g}}(X^2)}$

where $0 \neq X \in \mathfrak{g}$ is semisimple. ~~Note that if~~ $\operatorname{ind}_G(V)$ is independent of X. Note that if

H is the isotropy group of a closed orbit with slice representation S, then $V \simeq \mathfrak{g}/\mathfrak{h} \oplus S$ as a representation space of H. Hence there is a connection between the indices of ~~the~~ slice representations and ~~those~~ that of the ~~whole group~~ original representation. One can show:

<u>Thm.</u> Let $G = G_1 \times \dots \times G_r$ be a product of connected simple groups. Suppose that $\mathrm{ind}_{G_i}(V) = 1$, $1 \le i < j$, and $\mathrm{ind}_{G_i}(V) > 1$, $j \le i \le r$. Then H^0 is a torus, $H^0 \subset G_1 \times \dots \times G_j$ where H is any principal isotropy group. If $H^0 = \{id\}$, then (V, G) has no S^3 strata.

<u>Outline of the proof of MT</u>: We may assume $G = G_1 \times \dots \times G_r$ is a product of connected simple groups. Whenever $\mathrm{ind}_{G_i}(V) < 1$ we show that Prop. A or Prop. B applies. (We only end up applying Prop. B in the case of simple

groups with S^3 strata — we show how to establish the hypotheses of Prop. B in the next section.). Thus one is left with the cases ~~where~~ that $\mathrm{ind}_{G_i}(V) \geq 1$ for all i. By applying (A) again one is able to reduce to the case of FPIG. Then MT follows from theorem MT' and the theorem above.

IV. Simple Groups with S^3 Strata

Let V be a representation space of a connected simple group G with S^3 strata. We must show:

(a) $C[V]^G$ is a regular ring

(b) The MT holds for (V, G).

Defn. Let V have principal isotropy class (H). An isotropy class (L) is subprincipal if $(H) \leq (K) \leq (L)$, (K) an isotropy class, implies that $(K) = (H)$ or (L). In all the S^3 strata cases we have to consider it turns out that:

(A) the isotropy class $(L = SL_2)$ of the S^3 stratum is the unique subprincipal isotropy class. The ideal $I_{(L)}$ is a principal prime ideal (f_L).

(B) (V, G) has trivial principal isotropy groups.

<u>Defn.</u> Let N denote $N_G(L)/L$ where (L) is the isotropy class of a codimension one stratum. Then $\mathcal{X}^+(V^L)^N$ denotes the elements of $\mathcal{X}(V^L)^N$ preserving $\mathrm{res}_L\, C[V]^G$.

<u>Thm.</u> Assume (A) above. Suppose further that

(1) $C[V]^G$ is a regular ring.

(2) $\mathcal{X}^+(V^L)^N \subset \mathrm{res}_L\, \mathcal{X}(V)^G + \mathrm{Ker}(\pi_N)_*$.

(3) (V^L, N) has the lifting property.

Then (V, G) has the lifting property.

An instructive example where the above theorem applies is the case

$(V, G) = (2n\,\mathbb{C}^{2n}, Sp_{2n})$. ~~Choose~~ Here $L = Sp_2$, $N = Sp_{2n-2}$.

To finish our proof of the MT it suffices to establish (1) and (2) of the theorem in each ~~S^3 stratum~~ ~~case~~ remaining S^3 stratum case. We only comment here on establishing (1):

<u>Lemma 1.</u> Assume (A). Let $d = \dim \mathbb{C}[V]^G$. Suppose there are forms $p_1, \dots, p_d$ in $\mathbb{C}[V]^G$ such that the $p_i' = \mathrm{res}_L\, p_i$ are a minimal generating set for $\mathrm{res}_L\, \mathbb{C}[V]^G$. Further assume that:

(1) The relations of the p_i' are generated by a ~~form~~ polynomial $f(p_1', \dots, p_d')$.
(2) $\deg f \le \deg f_L$.

Then $\mathbb{C}[V]^G = \mathbb{C}[p_1, \dots, p_d]$ is a regular ring.

One can often estimate $\deg f_L$

G. Schwarz

using the following:

Lemma 2. Suppose $V = V_1 \oplus V_2$ is a direct sum of orthogonal representation spaces and V satisfies (A). Let (H_1) be the principal isotropy class of (V_1, G), and suppose (V_2, H_1) has codimension one strata whose ideal is generated by a form f. Then $\deg f_L > (0, \deg f)$. [Note: f_L must be bihomogeneous.]

Example: Let $(V, G) = (3\varphi_1, G_2) =$ 3 copies of the seven dimensional representation of G_2. Using lemma 2 one estimates $\deg f_L \geq (2, 2, 2)$. L equals SL_2 where $\varphi_1|_L = 2C^2 + \text{trivial}$. $N = SO_3$ where $(V^L, N) = (3C^3, SO_3)$. By classical invariant theory, $C[V^L]^N = C[\alpha'_{ij}, \beta']$ $1 \leq i, j \leq 3$ where the α'_{ij} are inner products, β' the determinant. Thus f is the relation: $(\beta')^2 = \det(\alpha'_{ij})$ and

$\deg f = (2,2,2)$. The α'_{ij} are the restrictions of inner products α_{ij} of $(3C^7, G_2)$, and β' is the restriction of an invariant $\beta \in \Lambda^3 \varphi_1 \subset S^3(3\varphi_1)$. By ~~the theorem~~ Lemma 1, $C[V]^G = C[\alpha_{ij}, \beta]$ is a regular ring.

Tagung über

TRANSFORMATIONSGRUPPEN UND INVARIANTENTHEORIE

Bonn, 22. - 28. September 1978

Titel: Rational singularities and quotients by reductive groups

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We consider schemes of finite type over a field of characteristic zero and call such an object a scheme.

Recall the following definitions.

① A ring homomorphism $A \to A'$ is pure if, for every A-module M, the map $M \to A' \otimes_A M$ is injective. A morphism of affine schemes $X' \to X$ is pure if the induced homomorphism $\Gamma(X, \underline{O}_X) \to \Gamma(X', \underline{O}_{X'})$ is pure.

[for instance, $A \to A'$ is pure if A is a direct summand of A' as A-module].

② A scheme X has rational singularities if it is normal and if $R^i f_* \underline{O}_Z = 0$, for $i > 0$, where $f : Z \to X$ is a resolution of singularities of X.

[it follows from Hironaka's work, that this definition is independent of the chosen resolution].

One can prove that a scheme with rational singularities is Cohen-Macaulay.

<u>Theorem</u>. Let $X' \to X$ be a pure morphism of affine schemes. Then, if X' has rational singularities, X also has rational singularities.

<u>Corollary</u>. Let k be a field of characteristic zero, G a reductive algebraic group over k acting k-rationally on an affine k-scheme $X = \mathrm{Spec}(R)$ with rational singularities. Then $X/G = \mathrm{Spec}(R^G)$ has rational singularities.

This follows from the theorem, since R^G is a direct summand of R. It is an extension of a result of Hochster and Roberts who proved that X/G is Cohen-Macaulay, when X is regular. Their proof proceeds by reduction to characteristic p while our proof uses Grauert-Riemenschneider's vanishing theorem in char. 0.

Some special cases of the corollary have been proved by R. Kempf.

Tagung über

TRANSFORMATIONSGRUPPEN UND INVARIANTENTHEORIE

Bonn, 22. - 28. September 1978

Titel: Infinite-dimensional Lie algebras, representation of graphs and Invariant theory.

Autor: . . . .

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Let S be an oriented graph, let $S_0 = \{p_1, \dots, p_n\}$ be the set of vertices of S and S_1 the set of edges of S; we denote by $i(\ell)$ and $f(\ell)$ the initial and final vertices of an (oriented) edge $\ell \in S_1$. ~~The graph S~~ We assign with a (non-oriented) graph S a symmetric matrix $A = (a_{ij})$ as follows: $a_{ii} = 2$, $a_{ij} = -$(number of edges, connecting p_i and p_j) for $i \neq j$.

A Representation of an oriented graph S is a collection U of finite-dimensional vector spaces $U_{p_1}, \dots, U_{p_n}$ and linear maps for each oriented edge $\{p_i, p_j\}$ from U_{p_i} to U_{p_j}. This notion was introduced by Gabriel [1] in connection with the representation theory of finite-dimensional associative algebras.

A representation U is called <u>indecomposable</u> if it can not be represented as a direct sum of two non-zero representations.

Let Γ be a free abelian group of rank n with generators $\alpha_1, \dots, \alpha_n$, ~~The~~ and Γ^+ be a subsemigroup of Γ, generated by $\alpha_1, \dots, \alpha_n$. The <u>dimension</u> of a representation U is the element $\alpha = \sum_i (\dim U_{p_i}) \alpha_i$. In [1] Gabriel discovered a remarkable correspondance between indecomposable representations of Dynkin graphs A_n, D_n, E_6, E_7, E_8 and ~~dime~~ root systems of the corresponding Lie algebras. Our purpose is to establish a correspondance between the dimensions of indecomposable representations of any oriented graph S and the infinite root system, associated with the corresponding matrix A.

Let $A = (a_{ij})$ be a matrix such that: $a_{ii} = 2$, $-a_{ij} \in \mathbb{Z}_+$, $a_{ij} = 0 \Rightarrow a_{ji} = 0$. A <u>root system</u> assosiated with A is a subset Δ^+ in Γ^+ which is uniquely defined by the following properties [2]:

1) $\alpha_1, \dots, \alpha_n \in \Delta^+$; $2\alpha_i \notin \Delta^+$;

2) the set of elements of the form $\alpha + s\alpha_i$, $s \in \mathbb{Z}$, is a progression $\alpha - p\alpha_i, \dots, \alpha + q\alpha_i$, where $p - q = \sum_j a_{ij} k_j$, where $\alpha = \sum_i k_i \alpha_i$.

<u>Theorem 1</u>. Let W be the group generated by reflections r_i, given by: $r_i(\alpha) = \alpha - (\sum_j a_{ij} k_j)\alpha_i$ for $\alpha = \sum_j k_j \alpha_j$. We set $\Delta^+_{re} = (\bigcup_i W(\alpha_i)) \cap \Gamma^+$ and $\Delta^+_{im} = W(C)$, where $C \subset \Gamma^+$ is defined by the following conditions:

1) $\alpha = \sum_i k_i \alpha_i \in C \Rightarrow \sum_j a_{ij} k_j \leq 0$, $i = 1, \dots, n$;

2) $\alpha = \sum_i k_i \alpha_i \in C \Rightarrow \{p_i \mid k_i \neq 0\}$ is a connected subdiagram of the corresponding graph (Dynkin diagram of the matrix A).

Using the results from [4] we prove

<u>Lemma 1</u>. If $\alpha \in C$, then a generic representation of the graph S of dimension α is indecomposable (except for the case when $\sum_j a_{ij} k_j = 0$, $i = 1, \dots, n$).

The proof is based on the remarks that

1) there is a correspondance between indecomposable representations of S of dimension $\alpha = \sum k_i \alpha_i$ and the orbits of the group $GL_{k_1} \times \ldots \times GL_{k_n}$ in the space $\bigoplus_{\ell \in S_1} \mathrm{Hom}(U_{i(\ell)}, U_{f(\ell)})$;

2) the stabiliser of an indecomposable representation is a unipotent group.

Definition. Let G be a linear algebraic group, acting in a vector space V. An orbit of G in V is called quasifree if its stabiliser is a unipotent group.

Conjecture (*). The sets of quasifree orbits of G in V and V^* have the same cardinalities and depend on the same number of parameters.

By making use of reflection functors [3] we prove modulo conjecture (*) the following result.

Theorem 2. Let S be an oriented graph and $\alpha = \sum k_i \alpha_i \in \Gamma_+$.

a) α is a dimension of an indecomposable representation of $S \Leftrightarrow \alpha \in \Delta^+$.

b) $\alpha \in \Delta^+_{re} \Leftrightarrow \exists!$ indecomposable representation.

c) $\alpha \in \Delta^+_{im} \Leftrightarrow \exists$ an infinite number of indecomposable representations of dimension α, which depends on $1 - \frac{1}{2}\sum_{i,j} a_{ij} k_i k_j > 0$ parameters ($\alpha = \sum_i k_i \alpha_i$).

In the case of semidefinite matrix A this theorem follows from [5] and in the case of (2×2)-matrix it follows from [6] (in the latter case our proof does not use the conjecture).

As a prize to the solution of conjecture (*) I offer a drink of choice.

References.

[1]. Gabriel P., Man. Math. 6 (1972), 71-103.

[2]. Kac V. G., Math. USSR, Izvestija 2 (1968), 1271-1311.

[3]. Bernstein I.N., Gelfand I. M., Ponomarev V. A., Russian Math. Surveys 28 (1973), 17-32.

[4]. Andreev E.M., Vinberg E.B., Elashvili A.G., Functional Anal. Appl., 1 (1967), 257-261.

[5]. Nazarova L. A., Math. USSR, Izvestija, 7 (1973), 752-791.

[6]. Ringel C. M., Math. Z. (1978).

Tagung über

TRANSFORMATIONSGRUPPEN UND INVARIANTENTHEORIE

Bonn, 22. - 28. September 1978

Titel: Standard Monomial Theory

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Let G be a semi-simple simply-connected chevalley group defined over a field k of arbitrary characteristic. Let T be a maximal torus of G and B, a Borel sub-group, $B \supset T$. Relative to T and B, let

Δ = system of roots

S = set of simple roots.

Let $M = X(T) \otimes \mathbb{Q}$, where $X(T) = \mathrm{Hom}(T, G_m)$. Let $(\, , \,)$ be a +ve definite W-invariant scalar product on M (here W denotes the Weyl group of G). Let us denote $\alpha^* = 2\alpha/(\alpha,\alpha)$, $\alpha \in \Delta$. Let $\{\omega_i\}_{1 \le i \le n}$, n being the rank of G, be the fundamental weights of G. Recall that ω_i's are elements of M defined by

$$\langle \omega_i, \alpha_j^* \rangle = \delta_{ij}, \quad \alpha_j \in S.$$

Defn 1: A fundamental weight ω (or the associated maximal parabolic sub-group P, $P \supset B$) is said to be of classical type if $|\langle \omega, \alpha^* \rangle| \le 2$, $\forall \alpha \in \Delta$. This definition has a geometrical interpretation as follows. For $w \in W$, let $X(w)$ denote the Schubert variety in G/P, associated to w, i.e., $X(w)$ is the Zariski closure with the canonical reduced structure of the Bruhat cell Bwe_P where e_P is the point of G/P determined by the coset P. It is easily seen that the mapping $w \longmapsto X(w)$ induces a bijection of W/W_P onto the set of Schubert sub-varieties of G/P. This bijection enables one to define a partial order, in a natural way, on W/W_P as follows. given $w_1, w_2 \in W/W_P$, let us define $w_1 \geq w_2$ if $X(w_1) \supseteq X(w_2)$. For $w \in W/W_P$, let $[X(w)]$ denote the class of $X(w)$ in the chow ring, $Ch(G/P)$, of G/P. Then, if H denotes the unique Codim-1 Schubert Subvariety in G/P, then it can be shown that

$$[X(w)]\cdot[H] = \sum_i d_i [X(w_i)], \quad d_i \in \mathbb{Z}^+$$

where $X(w_i)$ runs over all Schubert subvarieties of $X(w)$ of codim 1 (here $\cdot$ denotes multiplication in $Ch(G/P)$). We call d_i the intersection multiplicity of $X(w_i)$ in $[X(w)]\cdot[H]$. We see that P is of classical type if and only if $d_i \leq 2$, for every (w, w_i) related as above.

Remark: If G is a classical group, then every maximal parabolic subgroup of G is of classical type.

Defn 2: A pair of elements (τ, φ) in W/W_P is called an admissible pair if either

(i) $\tau = \varphi$, in which case, it is called a trivial admissible pair, or,

(ii) there exists a sequence $\tau\ \tau_0 > \tau_1 > \tau_2 > \cdots > \tau_m = \varphi$ such that $X(\tau_i)$ is a codim 1 Schubert subvariety of $X(\tau_{i-1})$, occurring with multiplicity 2.

Theorem I: Let L be the ample generator of $Pic(G/P)$. There exists a G-sub-module $V \subset H^0(G/P; L)$ (it will turn out that V is in fact $= H^0(G/P, L)$) and a basis $\{p_{\lambda,\mu}\}$ of V indexed by all admissible pairs in W/W_P such that

① $p_{\lambda\mu}$ is a weight vector of weight $-\frac{1}{2}(\lambda(\omega)+\mu(\omega))$

②. $p_{\lambda,\mu}\big|_{X(\varphi)} \neq 0 \iff \varphi \geq \lambda$.

③. The linear system V on G/P defines a projective embedding of G/P in $\mathbb{P}(V^*) = \operatorname{Proj} S(V^*)$, where $S(V^*)$ is the symmetric algebra of V^*

④. An element of the form

$$p_{\tau_1,\varphi_1} \cdot p_{\tau_2,\varphi_2} \cdots p_{\tau_m,\varphi_m}$$

is called a standard monomial of degree m on $X(\varphi)$, $\varphi \in W/W_P$ if $\varphi \geq (\tau_1,\varphi_1) \geq (\tau_2,\varphi_2) \geq \cdots \geq (\tau_m,\varphi_m)$. (An admissible pair (τ_1,φ_1) is $\geq (\tau_2,\varphi_2)$ if $\varphi_1 \geq \tau_2$).

Then distinct standard monomials of degree m on $X(\varphi)$ form a basis of $H^0(X(\varphi), L^m)$.

More generally, let Q be a parabolic subgroup of G, $Q \supset B$ such that $Q = \bigcap_{i=1}^{r} P_i$, where all P_i's are of classical type. We call such a parabolic subgroup, a parabolic subgroup of classical type. Note that if G is a classical group then every parabolic subgroup of G is of classical type.

Defn 3: Let Q be a parabolic subgroup (of G) of classical type. A Young diagram of type $(m_1,\ldots,m_r)$, $m_i \in \mathbb{Z}^+$, on G/Q is a pair (λ,μ) where $\lambda = (\lambda_{ij})$, $\mu = (\mu_{ij})$, $1 \leq i \leq r$, $1 \leq j \leq m_i$ and (λ_{ij}, μ_{ij}) is an admissible pair in W/W_{P_i}

Defn 4: A Young diagram (λ,μ) of type $(m_1,\ldots m_r)$ is said to be standard on the Schubert variety $X(\varphi) \subset G/Q$, if there exists a pair (θ, σ), $\theta = (\theta_{ij})$, $\sigma = (\sigma_{ij})$, $1 \le i \le r$, $1 \le j \le m_i$ such that

(i) $X(\varphi) \supset X(\theta_{11}) \supset X(\sigma_{11}) \supset X(\theta_{12}) \cdots \supset X(\sigma_{1m_1}) \supset X(\theta_{21}) \cdots$

(ii) $\pi_i(\theta_{ij}) = \lambda_{ij}$; $\pi_i(\sigma_{ij}) = \mu_{ij}$, $1 \le i \le r$, $1 \le j \le m_i$ where π_i is the canonical projection of G/Q onto G/P_i.

Defn 5. Let (λ, μ) be a standard Young diagram on $X(\varphi)$ of type $(m_1, \ldots m_r)$. To (λ, μ) is associated the element

$$p_{\lambda,\mu} = \prod_{i=1}^{r} \prod_{j=1}^{m_i} p_{\lambda_{ij}, \mu_{ij}}$$

Note that $p_{\lambda,\mu} \in H^0(X(\varphi), L)$ where $L = L_1^{m_1} \otimes \cdots \otimes L_r^{m_r}$, L_i being the ample generator of $\mathrm{Pic}(G/P_i)$. The element $p_{\lambda,\mu}$ is called a standard monomial on $X(\varphi)$ of deg $(m_1, \ldots m_r)$.

Theorem II. Distinct standard monomials of deg $(m_1, \ldots m_r)$ on $X(\varphi)$ form a basis of $H^0(X(\varphi), L)$

Sketch of proof of Theorem I. Let $G_{\mathbb{Z}}$ be a Chevalley group scheme$/\mathbb{Z}$ so that, the group G over any field k is given by $G = G_{\mathbb{Z}} \otimes_{\mathbb{Z}} k$. Let $P_{\mathbb{Z}}$ be the parabolic subgroup scheme of $G_{\mathbb{Z}}$, $X_{\mathbb{Z}}(\varphi)$, the Schubert sub-scheme of $G_{\mathbb{Z}}/P_{\mathbb{Z}}$; $L_{\mathbb{Z}}$ the ample generator of $\mathrm{Pic}(G_{\mathbb{Z}}/P_{\mathbb{Z}})$ The first step would be the construction of $p_{\lambda,\mu}$. For this, we proceed as follows. Let $k = \mathbb{C}$. Then one knows that $H^0(G/P, L)$ is an irreducible G-module with highest weight $i(\omega)$, where i is the Weyl involution. Let V_ω = dual of $H^0(G/P, L)$ so that V_ω is an irreducible G-module with highest weight ω. Let e be a highest weight vector in V_ω. Let $V_{\mathbb{Z}} = U_{\mathbb{Z}} \cdot e$, where $V_{\mathbb{Z}}$ is the $\mathbb{Z}$-form of $U = U(\mathrm{Lie}\, G)$, the universal enveloping algebra of $\mathrm{Lie}\, G$. Then one knows that $V_{\mathbb{Z}}$ is a $G_{\mathbb{Z}}$-$\mathbb{Z}$ module. For $\varphi \in W/W_P$, let $e_\varphi = \varphi \cdot e$ and $V_{\mathbb{Z}}(\varphi) = U_{\mathbb{Z}}^{+} \cdot e_\varphi$. We have the

Proposition 1. Let $\bar{e}$ be the $\mathbb{Z}$-valued point of $\mathbb{P}(V_{\mathbb{Z}}^{*})$ corresponding to $\mathbb{Z}\cdot e$. We have (a) the isotropy subgroup scheme of $G_{\mathbb{Z}}$ at e is $P_{\mathbb{Z}}$, so that we have a canonical closed immersion

$$j: G_{\mathbb{Z}}/P_{\mathbb{Z}} \hookrightarrow \mathbb{P}(V_{\mathbb{Z}}^{*}).$$

The pull back of the tautological line bundle on $\mathbb{P}(V_{\mathbb{Z}}^{*})$ is $L_{\mathbb{Z}}$.

(b). Let $h_{\mathbb{Z}}$ be the canonical homomorphism $V_{\mathbb{Z}}^{*} \to H^0(G_{\mathbb{Z}}/P_{\mathbb{Z}}, L_{\mathbb{Z}})$ induced by j. Then the kernel of the canonical restriction homomorphism $V_{\mathbb{Z}}^{*} \longrightarrow H^0(X_{\mathbb{Z}}(\varphi), L_{\mathbb{Z}})$ coincides with the canonical surjective homomorphism $V_{\mathbb{Z}}^{*} \to V_{\mathbb{Z}}^{*}(\varphi)$. In particular, we have a canonical injective homomorphism

$$h_{\mathbb{Z}}(\varphi): V_{\mathbb{Z}}^{*}(\varphi) \hookrightarrow H^0(X_{\mathbb{Z}}(\varphi), L_{\mathbb{Z}})$$

(c). For every field k, the canonical k-linear map $V_{\mathbb{Z}}^{*}(\varphi) \otimes k \longrightarrow H^0(X(\varphi), L)$ induced by $h_{\mathbb{Z}}(\varphi)$ is injective, equivalently $V_{\mathbb{Z}}^{*}(\varphi)$ is a direct summand in $H^0(X_{\mathbb{Z}}(\varphi), L_{\mathbb{Z}})$. (Using some results of Demazure in characteristic zero we see that $V_{\mathbb{Z}}^{*}(\varphi) \otimes \mathbb{Q} \approx H^0(X_{\mathbb{Z}}(\varphi)\otimes\mathbb{Q}, L_{\mathbb{Z}}(\varphi)\otimes\mathbb{Q})$) so that (c) is in fact equivalent to the fact that $h_{\mathbb{Z}}(\varphi)$ is an isomorphism.

Proposition 2. Let $\varphi \in W/W_P$. There exists a basis $\{Q_{\lambda,\mu}\}$ of $V_{\mathbb{Z}}(\varphi)$ indexed by admissible pairs on $X(\varphi)$ such that

(1) $Q_{\lambda,\mu}$ is a weight vector of weight $\frac{1}{2}(\lambda(\omega)+\mu(\omega))$

(2). Let $\theta \leq \varphi$ and $W(\theta)$ be the $\mathbb{Z}$ submodule of $V_{\mathbb{Z}}(\varphi)$ generated by $\{Q_{\lambda,\mu}\}$ such that $\theta \geq \lambda$. Then $W(\theta) = V_{\mathbb{Z}}(\theta)$. In particular, $V_{\mathbb{Z}}(\theta)$ is a direct summand in $V_{\mathbb{Z}}(\varphi)$.

As a particular case of Proposition 2, we obtain a basis $\{Q_{\lambda,\mu}\}$ for $V_{\mathbb{Z}}$ ($= V_{\mathbb{Z}}(w_0)$, w_0 being the unique element of W of maximal length) indexed by all admissible pairs in W/W_P. Now, the parts (1), (2), (3) of Theorem I follow if we define $p_{\lambda,\mu} = P_{\lambda,\mu} \otimes 1$, for every field k, where $\{P_{\lambda,\mu}\}$ is the basis of $V_{\mathbb{Z}}^{*}$ dual to $\{Q_{\lambda,\mu}\}$

For proving (4) of Theorem I we first prove the following

Proposition 3. Distinct standard monomials of degree m on $X(\varphi)$ form a linearly independent subset of $H^0(X(\varphi), L^m)$ (in any characteristic p).

The proof of this proposition is obtained by using induction on m, induction on $\dim X(\varphi)$ and using two special quadratic relations on $X(\varphi)$, viz,

on $X(\varphi)$, $p^2_{\tau,\varphi} = \pm p_\tau \cdot p_\varphi$

$$p_{\tau,\varphi_1} \cdot p_{\tau,\varphi_2} = p_\tau \cdot F$$

where $F^2 = p_{\varphi_1} \cdot p_{\varphi_2}$. As a consequence of Proposition 3 we obtain the following: consider $j : G_{\mathbb{Z}}/P_{\mathbb{Z}} \hookrightarrow \mathbb{P}(V^*_{\mathbb{Z}})$. Let $R_{\mathbb{Z}}(\varphi)$ be the homogeneous coordinate ring of $X_{\mathbb{Z}}(\varphi)$ for the projective embedding given by j. One knows that $R_{\mathbb{Z}}(\varphi)_m \otimes k \hookrightarrow H^0(X(\varphi), L^m)$ for every field k, which is an isomorphism if $k = \mathbb{Q}$. Let $S_{m,\mathbb{Z}}(\varphi)$ be the $\mathbb{Z}$-submodule of $R_{\mathbb{Z}}(\varphi)_m$ generated by standard monomials of degree m on $X(\varphi)$. Then as a consequence of Proposition 3, we obtain that $S_{m,\mathbb{Z}}(\varphi)$ is a direct summand in $R_{\mathbb{Z}}(\varphi)_m$. We prove that $S_{m,\mathbb{Z}}(\varphi)$ is in fact $= R_{\mathbb{Z}}(\varphi)_m$, by proving that $S_{m,\mathbb{Z}}(\varphi) \otimes \mathbb{Q} = R_{\mathbb{Z}}(\varphi)_m \otimes \mathbb{Q}$. Again, using the form of some quadratic relations, the proof of this for a general m is reduced to the case $m=2$ and the case $m=2$ is proved by showing that

$\dim H^0(X(\varphi), L^2)$ in characteristic zero $= \#\{$standard monomials of degree 2 on $X(\varphi)\}$. In proving this one uses the generalization (in characteristic 0) to a Schubert variety of the Weyl character formula, due to Demazure. Thus, we obtain that the ring $R(\varphi)$ (in any characteristic p) has a basis given by standard monomials on $X(\varphi)$. Now the proof of (4) of Theorem I is completed by proving that the cone $\hat{X}(\varphi)$ for the projective embedding $G/P \hookrightarrow \mathbb{P}(V_{\mathbb{Z}}^{*} \otimes k)$ (for any field k) is normal. And the proof of this is obtained by (i) exhibiting a 2-regular sequence, viz (p_φ, p_{Id}) in $R(\varphi)$ (ii) using the fact that Schubert varieties are non-singular in codim. 1 (due to Chevalley).

The spirit of the proof of Theorem II is more or less the same as the spirit of proof of Theorem I except the fact that the proofs here become more complicated. As before the essential steps are the following

① Proof of linear independence of standard monomials on $X(\varphi)$

② Consider the canonical immersion $G/Q \hookrightarrow G/P_1 \times \cdots \times G/P_r$. Now, each G/P_i has a canonical projective embedding and hence we obtain a canonical embedding of G/Q into a multi-projective space. Let $R(\varphi)$ denote the multigraded homogeneous coordinate ring of $X(\varphi)$ ($\subset G/Q$) ~~under~~ for this embedding. Then one proves

ⓐ $R(\varphi)_{(m_1, \ldots, m_r)}$ has a basis given by standard monomials of deg $(m_1, \ldots, m_r)$ on $X(\varphi)$

ⓑ. The multi-cone $\hat{X}(\varphi)$ is normal.

This completes the ~~proo~~ sketch of proof of Theorems I and II.

Comments ① As a consequence of Theorems I & II, we obtain a proof of Demazure's conjecture. Let us recall Demazure's conjecture. Let G be a semisimple algebraic group of any type and λ a positive character. Let V_λ be the irreducible G-module with highest weight λ. Let e be a highest weight vector in V_λ and let $V_{\mathbb{Z}} = U_{\mathbb{Z}} \cdot e$. For $\varphi \in W$, define $V_{\mathbb{Z}}(\varphi) = U_{\mathbb{Z}}^{+} e_\varphi$ where $e_\varphi = \varphi \cdot e$. Then Demazure's conjecture is that $V_{\mathbb{Z}}(\varphi)$ is a direct summand in $V_{\mathbb{Z}}$. To see the proof of this conjecture as a consequence of the standard monomial theory, let us suppose that $\lambda = \sum a_i \omega_i$, $a_i \in \mathbb{Z}^{+}$, where the ω_i's appearing with non-zero coeffs are of classical type. Note

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that λ could be arbitrary if G is a classical group. Now, we had identified $V_{\lambda}^{*} \otimes k$ with $H^0(G/P, L)$ and $V_{\lambda}^{*}(\varphi) \otimes k$ with $H^0(X(\varphi), L)$ where $L = L_1^{a_1} \otimes \cdots \otimes L_n^{a_n}$ (k being any field). As a consequence of our main theorems, we obtain that the canonical restriction map $H^0(G/P, L) \longrightarrow H^0(X(\varphi), L)$ is surjective (for every field k) and this implies that the canonical map ~~$V_{\lambda}^{*}(\varphi) \otimes k$~~ $V_{\lambda}^{*} \otimes k \longrightarrow V_{\lambda}^{*}(\varphi) \otimes k$ is surjective for every k and this proves the conjecture.

(2). When $G = GL(n)$ or $SL(n)$, our definition of standard Young diagrams coincides with the classical notion of Young diagrams. The classical notion leads to the notion of what we call "weakly standard monomials"; but this is not enough if one wants to have a notion valid for any Schubert variety in G/B. In fact this is not enough even for the big cell, if G is of type D_n, though it turns out to be O.K for the big cell if G is of type A_n, B_n or C_n.

(3). As a consequence of the standard monomial theory, we obtain vanishing theorems for positive bundles.

(4). The problem of standard monomial theory for non-classical fundamental representation is open.

(5). It looks hopeful for computing the dimension of the irreducible G-module V_λ with highest weight λ in positive characteristic as an application of the standard monomial theory. To be more precise, let $\lambda = \sum a_i \omega_i$ be such that the ω_i's occuring with non zero coefts are of classical type. Fix a highest weight vector e in the irreducible $\mathfrak{g}$-module V_λ ~~with highest wt~~ Then $V_{\mathbb{Z}} = U_{\mathbb{Z}} \cdot e$ is a minimal admissible $\mathbb{Z}$-form ~~(see [illegible])~~

for $V_{\mathbb{Z}}$. (An admissible $\mathbb{Z}$-form for V_λ is a lattice in V_λ which is $U_{\mathbb{Z}}$-stable). And $V_{\mathbb{Z}}^*$, the $\mathbb{Z}$-dual of $V_{\mathbb{Z}}$, is a maximal admissible $\mathbb{Z}$-form in $V_{i(\lambda)}$, where i is the Weyl involution. For simplicity, let us suppose that $i(\lambda) = \lambda$ so that $V_{\mathbb{Z}} \hookrightarrow V_{\mathbb{Z}}^*$. Now $V_{\mathbb{Z}}^*$ has a basis in terms of standard monomials. We will be through, if we could ~~get~~ guess a nice basis for $V_{\mathbb{Z}}$ and get the matrix expressing this basis in terms of the standard monomials. Very little is known regarding this problem, i.e., results of Jantzen which give the dimension for the case of $SL(4)$ and results of Braden (the case of $SL(3)$) which give the dimension for ~~the~~ a sub-class of the Steinberg class.

Tagung über

TRANSFORMATIONSGRUPPEN UND INVARIANTENTHEORIE

Bonn, 22. - 28. September 1978

Titel: Finite linear groups and charge of Young Tableaux

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For many geometrical or algebraïc problem, one needs a better numerical knowledge of the representations of the symmetric and linear groups than that is the case for the moment. The appropriate tool in this field is the so-called Schur functions; this point of view is mainly due to D.E Littlewood (Group characters, Oxford, 2nd ed 1950). In modern language, we should say that he studies the λ-ring structure of $Z[a,b,..]$, a,b... belonging to a finite alphabet A.

Recent progress have been made by lifting properties of Schur functions to a non-commutative ring, the ring of the monoïd plaxique (it is the quotient of the free monoid by the relations $xzy \equiv zxy$ and $zty \equiv tzy$ for all letters such that $x \leq y < z \leq t$, the alphabet having been totally ordered)

Let us consider the problem of describing the representations (on $\mathbb{C}$) of a finite linear group $Gl(n,F_q)$. There exist also symmetric functions, which are called Hall-Littlewood functions, allowing to compute the characters. We give a precise definition at the end of this paper . These functions P_I are indexed by partitions I , symmetrical in the elements of A , but this time with coefficients belonging to $Z[q]$. In fact, they also form, as Schur functions do, a basis of the $Z[q]$ module of symmetric polynomials with coefficients in $Z[q]$. So, for example, we can express the Schur functions S_J in this basis :

$$S_J = \sum_I F(I,J) \quad P_I$$

The polynomial coefficients are called <u>FOULKES polynomials</u> ; Foulkes conjectured the following theorem we prove:

THEOREM . $F(I,J)$ is a polynomial with positive integral coefficients ; moreover, it is monic (the coefficient of the maximal power of q is 1), and its degree is

$i_n - j_n + 2(i_{n-1} - j_{n-1}) + 3(i_{n-2} - j_{n-2}) + \ldots$

As the value of $F(I,J)$ for $q=1$ is known to be the cardinal of a certain set of combinatorial objects, Young tableaux belonging to a certain family $T(I,J)$, it is clear that the theorem is implied by

THEOREM' . There exists a function (the charge)

$$\nu \ : \ T(I,J) \longrightarrow \mathbb{N}$$

such that

$$F(I,J) = \sum_{t \in T(I,J)} q^{\nu(t)}$$

More explicitely : in the free monoid A , a row is a word xyz... such that $x \leq y \leq z \leq \ldots$; a row r is bigger than another $r' = x'y'\ldots$ if $\mathrm{degree}(r) \leq \deg(r')$ and $x > x'$, $y > y'$,.... A word t is a Young tableau if for its decomposition in maximal subrows ($t = r_1 r_2 \ldots$) r_i is bigger than r_{i+1} for all i ; the sequence of degrees of the successive rows is called the diagram of the Young Tableau.

On the other hand, a standard word is a word whose commutative evaluation is a segment in the alphabet , e.g. dcb ,and not dab . For a standard word w , we define its charge $\nu(w)$ by the following law : we index the letters by positive integers, in the lexigraphical order ; c_i implies d_i if d is left of c , d_{i+1} if d is right of c ; begin by a_0 .

DEFINITION. $\nu(w)$ is the sum of the indices .

Example. Let $w = e_1\ f_2\ b_0\ d_1\ a_0\ c_1$, then the indexation is the one written and its charge is $1+2+1+1 = 5$.

Now given a word w with $d^\circ_a(w) \geq d^\circ_b(w) \geq \ldots$, we extract from it successive standard subwords (its filières): Starting from the right, take the first a , and suppose you have choosen b,...,c . Now you pick out the first d on the left of c, if none, the first d starting again from the right of the word . When this process ends, you have obtained the first filière of the word, you erase it and build the second filière with the remaining word .

DEFINITION. The charge of such a word is the sum of the charges of its filières .

Example.

b	d	c	a	a	b	c	
b_o	d_1	.	.	a_o	.	c_o	first filière
.	.	c_1	a_o	.	b_1	.	2nd

charge = 2 + 2 = 4

We now need only to define the family $T(I,J)$ to complete the formulation of the theorem : $T(I,J)$ is the set of Young tableaux of diagram J and of evaluation $a^{i_n} b^{i_{n-1}} \ldots$

Example. $T(1122,24)$ is the following set

	bcaabd	bdaabc	bbaacd	cdaabb
charge of first filière	o1.o.2	o1.o.1	.o.o12	12.o.1
charge of 2nd filière	..o.1.	..o.1.	o.o...	..o.1.
charge	4	3	3	5

So, according to the theorem, $F(1122,24) = q^5 + q^4 + 2q^3$.

The demonstration of the theorem uses exact sequences in the ring of the plaxique monoid, which extend exact sequences of Schur representations ; it will be found in Schützenberger exposé in Séminaire Pisot (fev.1978) .

This joint work has been announced in the

C.R. Acad. Paris, 286, (1978) 323-324 .

N.B. The easiest way to define Hall-Littlewood polynomials is to take a vector space V of dimension m on C, and to consider the total flag manifold D of V. On it exist "tautological" line bundles $L_1, \ldots, L_m$ which are the successive quotients in the fibration of D into projective spaces. Let I be a partition : $0 < i_1 \leqslant i_2 \leqslant \ldots i_m$ and L^I be the tensor product $(L_1)^{i_1} \otimes \ldots \otimes (L_m)^{i_m}$. Moreover, let Ω^j be the sheaf of j-differential forms on D. The Euler charateristic of a vector bundle M on D, denoted $\chi(M)$, is the (formal) vector space $\sum_{i,j} (-q)^j \; (-1)^i \; H^i(D, M \otimes \Omega^j)$. On $(1-q)^m \chi(L^I)$ acts $Gl(m,C)$, and so let $G(I,J)$, which belongs to $Z[q]$, be the multiplicity of the component of index J (the irreducible representations of the linear group on $\mathbb{C}$ are indexed by partitions).

DEFINITION. Let S_J be the Schur symmetric function of index J; then the Littlewood-polynomial of index I is

$$Q_I = \sum_J G(I,J) \; S_J \quad .$$

The Hall-polynomial P_I is equal to Q_I / a_I, with $a_I = \prod_j (1-q) \ldots (1-q^{m_j})$, m_j being the number of parts of I equal to j. So, the polynomials $F(I,J)$ are the elements of the inverse matrix of $G(I,J) / a_I$.

One has also

DEFINITION. The Green function of index I is equal to

$$\sum_J F(I,J)_{1/q} \; \chi_J \quad ,$$

χ_J being the character of the irreducible representation of index J of the symmetric group (on $\mathbb{C}$).

One has also another basis of the symmetric polynomials in (a,b,...) over $Z[[q]]$, the modified Schur functions S'_J . They are defined through the intermediary of an accessory sufficiently numerous alphabet X . By definition,

$$\prod_{x\in X} (1-qax)(1-qbx)\dots/(1-ax)(1-bx)\dots = \sum_J S_J(X)\ S'_J .$$

Then one can show :

PROPOSITION. $Q_I = \sum_J F(I,J)\ S'_J$

The fact that we find the same coefficients than for the expression of S_J in terms of P_I is equivalent to the "orthogonality" of Green characters, but in fact, it is a purely formal property of symmetric polynomials .

For the classical litterature on the subject, we send to A.O.MORRIS in Combinatoire du Groupe Symétrique, Strasbourg 1976, Lect.Notes n 589 .

We shall end by a table of Foulkes polynomials , writing 123 for the modified Schur function S'_{123} .

TABLE DES POLYNOMES DE FOULKES

Poids 5

$$Q_5 = \underline{5}$$
$$Q_{14} = q\underline{5} + \underline{14}$$
$$Q_{23} = q^2\ \underline{5} + q\ \underline{14} + \underline{23}$$
$$Q_{113} = q^3\ \underline{5} + (q+q^2)\ \underline{14} + q\ \underline{23} + \underline{113}$$
$$Q_{122} = q^4\ \underline{5} + (q^2+q^3)\ \underline{14} + (q+q^2)\ \underline{23} + q\ \underline{113} + \underline{122}$$
$$Q_{1112} = q^6\ \underline{5} + (q^3+q^4+q^5)\ \underline{14} + (q^2+q^3+q^4)\ \underline{23} + (q+q^2+q^3)\ \underline{113} + (q+q^2)\ \underline{122} + 1112$$
$$Q_{11111} = q^{10}\ \underline{5} + (q^6+q^7+q^8+q^9)\ \underline{14} + (q^4+q^5+q^6+q^7+q^8)\ \underline{23} + (q^3+q^4+2q^5+q^6+q^7)\ \underline{113}$$
$$+ (q^2+q^3+q^4+q^5+q^6)\ \underline{122} + (q+q^2+q^3+q^4)\ \underline{1112} + \underline{11111}$$

Poids 6

$$Q_6 = \underline{6}$$
$$Q_{15} = q\ \underline{6} + \underline{15}$$
$$Q_{24} = q^2\ \underline{6} + q\ \underline{15} + \underline{24}$$
$$Q_{33} = q^3\ \underline{6} + q^2\ \underline{15} + q\ \underline{24} + \underline{33}$$
$$Q_{114} = q^3\ \underline{6} + (q+q^2)\ \underline{15} + q\ \underline{24} + \underline{114}$$
$$Q_{123} = q^4\ \underline{6} + (q^2+q^3)\ \underline{15} + (q+q^2)\ \underline{24} + q\ \underline{33} + q\ \underline{114} + \underline{123}$$
$$Q_{222} = q^6\ \underline{6} + (q^4+q^5)\ \underline{15} + (q^2+q^3+q^4)\ \underline{24} + q^3\ \underline{33} + q^3\ \underline{114} + (q+q^2)\ \underline{123} + \underline{222}$$
$$Q_{1113} = q^6\ \underline{6} + (q^3+q^4+q^5)\ \underline{15} + (q^2+q^3+q^4)\ \underline{24} + q^3\ \underline{33} + (q+q^2+q^3)\ \underline{114} + (q+q^2)\ \underline{123}$$
$$+ \underline{1113}$$
$$Q_{1122} = q^7\ \underline{6} + (q^4+q^5+q^6)\ \underline{15} + (2q^3+q^4+q^5)\ \underline{24} + (q^2+q^4)\ \underline{33} + (q^2+q^3+q^4)\ \underline{114}$$
$$+ (q+2q^2+q^3)\ \underline{123} + q\ \underline{222} + q\ \underline{1113} + \underline{1122}$$
$$Q_{11112} = q^{10}\underline{6} + (q^6+q^7+q^8+q^9)\ \underline{15} + (q^4+q^5+2q^6+q^7+q^8)\ \underline{24} + (q^4+q^5+q^7)\ \underline{33}$$
$$+ (q^3+q^4+2q^5+q^6+q^7)\ \underline{114} + (q^2+2q^3+2q^4+2q^5+q^6)\ \underline{123} + (q^2+q^4)\ \underline{222}$$
$$+ (q+q^2+q^3+q^4)\ \underline{1113} + (q+q^2+q^3)\ \underline{1122} + \underline{11112}$$
$$Q_{111111} = q^{15}\underline{6} + (q^{10}+q^{11}+q^{12}+q^{13}+q^{14})\ \underline{15} + (q^7+q^8+2q^9+q^{10}+2q^{11}+q^{12}+q^{13})\ \underline{24}$$
$$+ (q^6+q^8+q^9+q^{10}+q^{12})\ \underline{33} + (q^6+q^7+2q^8+2q^9+2q^{10}+q^{11}+q^{12})\ \underline{114}$$
$$+ (q^4+2q^5+2q^6+3q^7+3q^8+2q^9+2q^{10}+q^{11})\ \underline{123} + (q^3+q^5+q^6+q^7+q^9)\ \underline{222}$$
$$+ (q^3+q^4+2q^5+2q^6+2q^7+q^8+q^9)\ \underline{1113} + (q^2+q^3+2q^4+q^5+2q^6+q^7+q^8)\ \underline{1122}$$
$$+ (q+q^2+q^3+q^4+q^5)\ \underline{11112} + \underline{111111}$$

On a écrit $\underline{123}$ pour la fonction de Schur (modifiée) indexée par la partition (1,2,3) .

Tagung über

TRANSFORMATIONSGRUPPEN UND INVARIANTENTHEORIE

Bonn, 22. - 28. September 1978

Titel: Some applications of moduli theory.

Autor: Herbert Popp

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Let $f: M \to S$ be a fibre space, i.e M and S are smooth projective varieties of dimension n and m respectively and f is a surjective morphism with connected fibres. Let M_s be the general fibre of f.

Then a proof of the following conjecture on the Kodaira dimension $\varkappa(\ )$ of the varieties M, S, M_s is an important open problem.

<u>Conjecture $C_{n,m}$:</u> $\varkappa(M) \geq \varkappa(S) + \varkappa(M_s)$

For fibre spaces of curves the conjecture is known and was proved by Viehweg using the moduli theory of stable curves. Also for fibre spaces where the general fibre is an abelian variety a proof is known by Ueno. Otherwise very little is known.

Viehwegs proof is as follows.

The first observation is that in order to prove conjecture $C_{n,m}$ it suffices to show the following statement $C'_{n,m}$

<u>Statement $C'_{n,m}$:</u> Let $\pi_1 : V_1 \to W_1$ be a fibre space, $n = \dim V_1$, $m = \dim W_1$. There exists a birationally equivalent fibre space $\pi : V \to W$ of smooth projective varieties such that $\varkappa(\omega_V \otimes (\pi^* \omega_W)^{-1}, V) \geq \varkappa(V_w)$, where ω_V and ω_W are the canonical sheafs on V and W respectively and $\varkappa(\omega_V \otimes (\pi^* \omega_V)^{-1}, V)$ is the dimension of V respectively the sheaf $\omega_V \otimes (\pi^* \omega_W)^{-1}$.

More precisely the following ~~[illegible]~~ holds

Assume that $C'_{r+l,l}$ holds for all $l \leq m$ and $r = n - m$. Then $C_{n,m}$ holds.

Next moduli theory of stable curves allow to reduce the proof of $C'_{n,n-1}$ to stable curves.

~~Roughly speaking the situation is as follows~~

Roughly speaking every fibre space of curves can be obtained as pullback from the family of stable curves with level n-structure up to taking the quotient by a finite group.

There exists a diagram as follows

$$
(*)\qquad
\begin{tikzcd}
V \arrow[d, dash] & V' \arrow[l, "h"'] \arrow[d] & V'_s \arrow[l, "\psi"'] \arrow[r] \arrow[d] & \overline{\Pi}^{(n)} \arrow[d] \\
W & W' \arrow[l, "g"'] & W'_s \arrow[l, "\varphi"'] \arrow[r, "\varrho"] & \overline{M}_g^{(n)}
\end{tikzcd}
$$

such that

1) h and g are birational maps
2) φ and ψ are finite galois coverings
3) $V'_s|W'_s$ is a family of stable curves, pullback of the "universal" family $\overline{\Pi}^{(n)} \to \overline{M}_g^{(n)}$ of stable curves with level n-structure.

A close inspection of the dualizing sheaf of the diagram (*) yields the inequality

$$\chi(\omega_{V'|W'}, V') \geq \chi(\omega_{V'_s|W'_s}, V'_s)$$

and allows to restrict to families of stable curves for proving $C'_{n,n-1}$.

Let $V \xrightarrow{\pi} W$ be a family of stable curves (with level n-structure) then the following explicite description of the relative canonical sheaf is known

$$\omega_{V|W}^{\frac{1}{2}g(g+1)} \sim \overset{g}{\Lambda}\pi_* \omega_{V|W} + WP_{V|W} + E$$

where $WP_{V|W}$ is the divisor of Weierstraßpoints of V/W and E a positive divisor ~~contentrated~~ in singular fibres of $V \to W$

For the proof of statement $C'_{n,n-1}$ it suffices to produce at least two sections of $\overset{g}{\Lambda}\pi_* \omega_{V|W}$.

This is done by looking ~~to~~ the family $\overline{\Gamma}^{(n)} \to \overline{M}_g^{(n)}$ of stable curves and the sheaf $\overset{g}{\Lambda}\pi_* \omega_{\overline{\Gamma}^{(n)}/\overline{M}_g^{(n)}}$.

Then Mumford's proof of the projectivity of $\overline{M}_g^{(n)}$ gives these sections.

There is a second way ~~[illegible]~~ using Hodge theory to obtain sections of $\overset{g}{\Lambda}\pi_* \omega_{\overline{\Gamma}^{(n)}/\overline{M}_g^{(n)}}$

Consider the Torelli map $\Phi: M_g \to H_g/\Gamma$ from the moduli space of smooth curves to the moduli space of principally polarized abelian varieties. Then the automorphic forms of H_g of weight m lead to sections of the sheaf

$$\left(\bigwedge^{g} \pi_* \,\omega_{\overline{\Gamma}(n)/\overline{H}_g(n)}\right)^{\otimes m}.$$

Having these sections the statement $C'_{n,n-1}$ is immediate.

The proof of Ueno of Conjecture $C_{n,m}$ for fibre spaces of abelian varieties goes along the same lines. Instead of the family of stable curves the compact family of polarized quasiabelian varieties with level n-structure constructed by Namikawa must be used.

Tagung über

TRANFORMATIONSGRUPPEN UND INVARIANTENTHEORIE

Bonn, 22. - 28. September 1978

Titel: Dynamical systems and invariant theory

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1. A linear, time invariant, dynamical system is a set of differential equations or difference equations

(1) $\dot{x} = Fx + Gu$, $y = Hx$ $\qquad x_{t+1} = Fx_t + Gu_t$, $y_t = Hx_t$

where $x \in \mathbb{R}^n$ = state space, $u \in \mathbb{R}^m$ = input space, $y \in \mathbb{R}^p$ = output space and where F, G, H are matrices with time independent coefficients. In the difference equations case (= discrete time case) one can take any field k instead of $\mathbb{R}$ or $\mathbb{C}$.

The group Gl_n acts on $L_{m,n,p}$ = space of all triples of matrices (F, G, H) of sizes $n \times n$, $n \times m$, $p \times n$ respectively, as follows

(2) $(F, G, H)^S = (SFS^{-1}, SG, HS^{-1})$

This action corresponds to base change in state space. Associated to (1) is an input/output map $u(t) \mapsto y(t)$, which in the continuous time case is given by

(3) $u(t) \mapsto y(t) = \int_0^t He^{F(t-\tau)} G u(\tau) d\tau$

(start at state zero at time zero and feed in $u(\tau)$, $0 \le \tau \le t$). This map is what is observable of the system. It is also clearly invariant under the action of Gl_n defined by (2). The question arises whether this input/output map embodies all invariants.

More generally one is, for various reasons, interested in the "orbit space" $L_{m,n,p}/Gl_n$ and the map $L_{m,n,p} \xrightarrow{\pi} L_{m,n,p}/Gl_n$ (Identification of systems, construction of control loops, "recovering" (F,G,H) from the input/output map (realization theory)).

Mathematically what we are doing is studying representations of the quiver

$$\begin{tikzcd} \bullet \arrow[r] & \bullet \arrow[r] \arrow[loop above] & \bullet \end{tikzcd}$$

under the equivalence relation defined by allowing arbitrary isomorphisms of the middle space and identity only for the two end spaces. This is a finer notion of equivalence than the usual one in this context.

2. $(F,G,H) \in L_{m,n,p}$ is said to be <u>completely reachable</u> (cr) if the $n \times (n+1)m$ matrix

$$(4) \qquad R(F,G) = \left(G \mid FG \mid \cdots \mid F^n G \right)$$

has rank n. Dually (F,G,H) is said to be <u>completely observable</u> (co) if the $(n+1)p \times n$ matrix

$$(5) \qquad Q(F,H) = \begin{pmatrix} H \\ HF \\ \vdots \\ HF^n \end{pmatrix}$$

Let $L^{co,cr}_{m,n,p}$ be the open subvariety of co and cr triples.

<u>Theorem</u> 1. The quotients $L^{co,cr}_{m,n,p}/Gl_n = M^{co,cr}_{m,n,p}$ and $L^{cr}_{m,n,p}/Gl_n = M^{cr}_{m,n,p}$ exist and are smooth quasi projective varieties. The morphisms $\pi: L^{cr}_{m,n,p} \to M^{cr}_{m,n,p}$ is a principal Gl_n fibre bundle which admits a section (i.e. is trivial) if and only if $m = 1$. Its restriction to $M^{cr,co}_{m,n,p}$ is trivial if and only if $m = 1$ or $p = 1$.

3. An explicit description of the moduli space $M^{co,cr}_{m,n,p}$ is obtained as follows. Let

(6) $$\mathcal{H}(F,G,H) = Q(F,H)\,R(F,G) = \begin{pmatrix} A_0 & A_1 & \cdots & A_n \\ A_1 & & \iddots & \vdots \\ \vdots & \iddots & & \vdots \\ A_{2n} & \cdots & & A_{2n} \end{pmatrix}$$

where $A_i = HF^iG$.

Theorem 2. $\mathcal{H}$ defines an embedding $M^{cr,co}_{m,n,p}$ into the affine space of all $(n+1)p \times (n+1)m$ matrices. Its image consists of those matrices $\mathcal{H}$ such that (i) they have the block Hankel structure indicated in (6), (ii) $\operatorname{rank}(\mathcal{H}) = n$, (iii) the rank of the submatrix obtained by omitting the last row and column of blocks is also n.

4. Fine moduli property. A family of linear dynamical systems over a scheme (variety, space) V consists of a vector bundle $\mathcal{E}$ of dimension n over V, together with a vector bundle endomorphism F of $\mathcal{E}$ and morphisms of vector bundles $G: \mathcal{A}_m \to \mathcal{E}$, $H: \mathcal{E} \to \mathcal{A}_p$, where $\mathcal{A}_m$, $\mathcal{A}_p$ are the trivial vector bundles over V of dimension m and p respectively. There is an obvious notion of isomorphism of which the one defined by (2) corresponds to the case $V =$ one point. A family of systems is co and cr if it is so over every point.

Theorem 3. $M^{co,cr}_{m,n,p}$ is a fine moduli space for the functor defined just above. I.e. there exists over $M^{co,cr}_{m,n,p}$ a (universal) co and cr family of systems $\Sigma^u = (\mathcal{E}^u, F^u, G^u, H^u)$ such that every co and cr family Σ over V is uniquely obtained (up to isomorphism) by pulling back Σ^u along a morphism $V \to M^{co,cr}_{m,n,p}$ (which is uniquely determined by Σ).

5. The time varying case. Now consider systems (1) with the coefficients of F, G, H in, say, the field of meromorphic functions in t over $\mathbb{R}$ or $\mathbb{C}$, and allow time varying base change. So instead of (2) one obtains the differential action

$$(7) \qquad (F, G, H)^S = (SFS^{-1} + \dot{S}S^{-1}, SG, HS^{-1})$$

In this setting one defines

$$(8) \qquad R(F,G) = (G_0 \vdots G_1 \vdots \quad \vdots G_n), \quad Q(F,H) = \begin{pmatrix} H_0 \\ H_1 \\ \vdots \\ H_n \end{pmatrix}$$

with $G_0 = G$, $G_i = FG_{i-1} - \dot{G}_{i-1}$, $H_0 = H$, $H_i = H_{i-1}F + \dot{H}_{i-1}$. Theorem 1 now goes through in this setting with the difference that π is now a map of differential algebraic varieties. There is also an analogue of theorem 2 (with $\mathcal{H}(F,G,H) = Q(F,H)R(F,G)$ as before); the rank conditions are the same, but instead of the Block Hankel condition one requires that the matrices are of the form

$$\begin{pmatrix} A_{00} & \cdots & A_{0n} \\ \vdots & & \vdots \\ A_{n0} & \cdots & A_{nn} \end{pmatrix}$$

with $A_{i+1,j} - A_{i,j+1} = \dot{A}_{i,j}$.

In particular all differential invariants of the action (7) are functions of the coordinates of $Q(F,H)R(F,G)$.

References:
1. M. Hazewinkel, Moduli and invariants for linear dynamical systems II. J. Math. System Theory 11 (1977)
2. — Moduli and invariants for linear dynamical systems III. In Proc 1976 AMES research centre (NASA) conference on geometric control theory. Math. Sci. Press, 1977
3. — Moduli and invariants for time varying systems. Preprint ~~[illegible]~~ Erasmus Univ., Rotterdam, 1977.

Tagung über

TRANSFORMATIONSGRUPPEN UND INVARIANTENTHEORIE

Bonn, 22. - 28. September 1978

Titel: CENTRAL DIFFERENTIAL OPERATORS FOR SEMI-SIMPLE GROUPS OF POSITIVE CHARACTERISTIC

Autor: WILLIAM J. HABOUSH

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THIS WORK IS ADDRESSED TO THE PROBLEM OF FINDING THE COMPOSITION FACTORS OF WEYL MODULES OVER FIELDS OF POSITIVE CHARACTERISTIC. WE BEGIN BY PROVING THAT THE INVARIANT DIFFERENTIAL OPERATORS OF THE SEMI-SIMPLE GROUP G OVER ANY BASE, k, ARE ISOMORPHIC TO THE tensor product, $\mathcal{U}_{\mathbb{Z}} \otimes_{\mathbb{Z}} k$ where $\mathcal{U}_{\mathbb{Z}}$ denotes the Kostant $\mathbb{Z}$-form of the type corresponding to G, provided that G is split over k. ~~Moreover~~ Let $D_{G/k}$ denote the invariant differential operators and let B denote a Borel subgroup. Then $D_{B/k} \subset D_{G/k}$ and one shows that $D_{B/k}$ is the subalgebra generated by the $X_\alpha^{[n]}$ for α positive and all n positive. (Here this denotes $\frac{X_\alpha^n}{n!} \otimes 1$ where X_α is an elt of a Chevalley basis of the Lie algebra of G over $\mathbb{Z}$). For each algebra morphism $\lambda : D_{B/k} \longrightarrow k$ one can construct a Verma module $V_\lambda = D_{G/k} \otimes_{D_{B/k}} k_\lambda$ where k_λ is the field k with the $D_{B/k}$ action defined by λ. One proves that ~~B~~ the set of algebra morphisms $\lambda : D_{B/k} \to k$, coincides with the algebra morphisms $D_{T/k} \to k$ where T is a maximal torus in B. By Cartier Duality for formal groups, this set may be identified with $\hat{\mathbb{Z}}_{(p)} \otimes_{\mathbb{Z}} P$, where $p = \mathrm{char}.(k) > 0$ and P is the module of weights of T. Let U^- denote the unipotent radical of the Borel subgroup opposite to B. $D_{U^-/k}$ is the subalgebra of $D_{G/k}$ generated by the $X_\alpha^{(n)}$ for α negative. Then one can prove:

1. Theorem: Let $V_\lambda = D_{G/k} \otimes_{D_{B/k}} k_\lambda$. Then

1) V_λ is free rank one over $D_{U^-/k}$. Hence it has a basis consisting of $D_{T/k}$ weight vectors, ~~of~~ each of which is of the form $\lambda - \gamma$, γ a positive integral linear combination of positive roots. It contains a unique vector $1 \otimes 1$ of weight λ.

2) V_λ has a unique irreducible quotient I_λ. I_λ is generated by a vector of weight λ.

3) Let $\rho = \frac{1}{2}\sum$ (positive roots). Let α be a simple positive root. Let $m = \alpha^\vee(\lambda + \rho)$

Write $n = \sum_{j=0}^{\infty} \nu_j p^j$ and let $(n)_g = \sum_{j=0}^{g-1} \nu_j p^j$. Then There is a vector in V_λ of weight $\lambda - (n)_g \alpha$ which is a $D_{B/k}$ weight for each g and each simple root α.

This, ~~is~~ relation, $\lambda \sim \lambda - (n)_g \alpha$ & induces a topologically closed equivalance relation on $\hat{P}$ and one major problem is to describe it. What we can say is the following.

Let the closed equivalence relation be denoted $\sim$.

2. Proposition: Let E be the sub-algebra of $\mathrm{Hom}_{\hat{\mathbb{Z}}_p}(\hat{P}, \hat{P})$ generated by all elements $\alpha^\vee \otimes \beta$ (i.e. $(\alpha^\vee \otimes \beta)(\gamma) = \alpha^\vee(\gamma)\,\beta$) Let ~~$U_0 \subset \mathrm{Hom}_{\hat{\mathbb{Z}}_p}(\hat{P}, \hat{P}$~~ $U \subset E$ be the multiplicative subgroup generated by all elements ~~of the form~~ $\sigma \in U$ such that $\sigma \equiv id \mod (p \cdot \hat{R}_\Phi)$ ($\hat{R}_\Phi$ = closure of the root lattice) Let $\tilde{W}_0 = U \cdot W \subset \mathrm{Hom}_{\hat{\mathbb{Z}}_p}(\hat{P}, \hat{P})$ where W is the Weyl group. Then

$$\lambda \sim \lambda' \iff \sigma(\lambda + \rho) = \lambda' + \rho \quad (\sigma \in \tilde{W}_0)$$

Thus one has a tentative kind of extended Weyl group.

The next problem is to describe the center of $D_{G/k}$. Let it be denoted $\mathfrak{Z}_{G/k}$. Then if $z \in \mathfrak{Z}_{G/k}$, z acts on I_λ as a constant. This defines a map $\eta: \mathfrak{Z}_{G/k} \longrightarrow \mathcal{F}(\hat{P}, k)$ where the latter is the ring of locally constant functions on $\hat{P}$ with values in k. Write $z(\lambda)$ for $\eta(z)(\lambda)$.

3. <u>CONJECTURE</u> $\eta(\mathfrak{Z}_{G/k})$ is the ring of functions on $\hat{P}$ constant on $\sim$ equivalence classes. That is it is the set of locally constant functions on the orbits of $\tilde{W}_0$ under the shifted action

One would like to prove 3, But one can not quite make it. What one can do is the following. Let $D^{(\nu)}_{G/k}$ denote the sub algebra of $D_{G/k}$ generated by all elements $X_\alpha^{[r]}$, $\binom{H_i}{r}$ with $r < p^\nu - 1$

4. THEOREM Let $u \in Z_{G/k} \cap D^{(\nu)}_{G/k}$. Let $\lambda = \sum m_i \omega_i$, $0 \le m_i \le p^{\nu}-1$. Let ξ be any weight in $(p\hat{P}) \cap \hat{R}$. Then there is a differential operator $\tilde{u}$ in $Z_{G/k} \cap D^{(\nu+1)}_{G/k}$ such that the following holds for $\lambda_0 = \sum_i r_i \omega_i$ $0 \le r_i \le p-1$.

$$\tilde{u}\left(\lambda_0 + p\lambda + p^{\nu+1}\xi\right) = \begin{cases} u(\lambda) & \text{if } \lambda_0 = (p-1)\rho \\ 0 & \text{if not.} \end{cases}$$

This seems odd but by the theorem Kac-Weisfeiler, there are sufficiently many elements of $Z_{G/k} \cap D^{(\nu)}_{G/k}$ for char $p \neq 2$ to conclude that the following is true.

5. COROLLARY: Let U be the multiplicative subgroup of E consisting of elements ~~which are~~ $\sigma \ni \sigma \equiv id \mod ((p\hat{P}) \cap \hat{R}_{\Phi})$. Let $\tilde{W}_1 = U \cdot W \subset \mathrm{Hom}_{\hat{Z}_p}(\hat{P}, \hat{P})$. Then $\eta(Z_{G/k})$ contains the locally constant functions on $\hat{P}$ constant on the shifted $\tilde{W}_1$ orbits of W.

Thus we have too large a group of relations for the conjecture (5). It would suffice by 4, however to construct a finite set of differential operators to make up the difference. Moreover I very strongly suspect the following.

6. CONJECTURE: There exists a set of automorphisms of $D_{G/k}$, purely as an algebra (not as a co-algebra) which leave $D_{B/k}$ stable and which duplicate the action of $\overline{W}_0$ on the weights of $D_{T/k}$.

Tagung über

TRANSFORMATIONSGRUPPEN UND INVARIANTENTHEORIE

Bonn, 22. - 28. September 1978

Titel: On differential invariants of holomorphic projective curves.

Autor: Hisasi Morikawa

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Our final purpose is to do something on the following difficult problem:

"Characterize projective automorphic varieties by mean of analytic terms."

I should like to say my talk has some relation with this problem.

1. Classical invariant theory on several binary forms can be easily generaled to formal power series

$$f_j(\zeta_j | z) = \sum_{\ell=0}^{\infty} \binom{w_j}{\ell} \zeta_j^{(\ell)} z^\ell \qquad (1 \leq j \leq N)$$

with Aronhold operators (a realization of $sl(2)$)

$$\mathcal{D} = \sum_{j=1}^{N} \sum_{\ell=0}^{\infty} \ell \zeta_j^{(\ell-1)} \frac{\partial}{\partial \zeta_j^{(\ell)}}, \quad \Delta = \sum_j \sum_\ell (w_j - \ell) \zeta_j^{(\ell+1)} \frac{\partial}{\partial \zeta_j^{(\ell)}}$$

$$H = \sum_j \sum_\ell (w_j - 2\ell) \zeta_j^{(\ell)} \frac{\partial}{\partial \zeta_j^{(\ell)}},$$

where $w_1, \cdots, w_N$ are complex numbers $\neq 0, 1, 2, \cdots$.

Only germ of $SL(2, \mathbb{C})$ acts on $f_j(\zeta_j | z)$ as follows;

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$$f_j\left(\begin{pmatrix}\delta & \beta\\ \gamma & \alpha\end{pmatrix}\xi_j \,\Big|\, z\right) = (\gamma z+\delta)^{w_j} f_j\left(\xi_j \,\Big|\, \frac{\alpha z+\beta}{\gamma z+\delta}\right) \quad (1\leq j\leq N).$$

A formal power series with coefficients in $K[\xi]$ is called a covariant of index u, if germ of $SL(2,\mathbb{C})$ acts as follows

$$F\left(\begin{pmatrix}\delta & \beta\\ \gamma & \alpha\end{pmatrix}\xi\,;\, z\right) = (\gamma z+\delta)^{u} F\left(\xi\,;\,\frac{\alpha z+\beta}{\gamma z+\delta}\right).$$

An element $\varphi(\xi)$ in $K[\xi]$ is called a semi-invariant of index u if

$$\mathcal{D}\varphi = 0, \quad \mathcal{N}\varphi = u\varphi.$$

Robert's theorem and Gram's theorem are also true for our case.

2. A holomorphic variable transformation

$$(u,y) \longmapsto (u(z), \lambda(z)y) \quad \left(\lambda(0)\neq 0,\ \frac{du(0)}{dz}\neq 0\right)$$

induces a map of vectors of holomorphic functions around origin;

$$(\varphi_1(u),\cdots,\varphi_n(u)) \longmapsto (\psi_1(z),\cdots,\psi_n(z))$$
$$= (\lambda(z)\varphi_1(u(z)),\cdots,\lambda(z)\varphi_n(u(z))).$$

Denote

$$L_n(p\,|\,u,y) = \sum_{\ell=0}^{n}\binom{n}{\ell} p_\ell(u)\left(\frac{d}{du}\right)^{n-\ell} y$$
$$= (-1)^{n-1}\begin{vmatrix}\varphi_1, & \cdots, & \varphi_n\\ \vdots & & \vdots\\ \left(\frac{d}{du}\right)^{n-1}\varphi_1, & \cdots, & \left(\frac{d}{du}\right)^{n-1}\varphi_n\end{vmatrix}^{-1}\begin{vmatrix}y, & \varphi_1, & \cdots, & \varphi_n\\ \vdots & & & \vdots\\ \left(\frac{d}{du}\right)^{n}y, & \left(\frac{d}{du}\right)^{n}\varphi_1, & \cdots, & \left(\frac{d}{du}\right)^{n}\varphi_n\end{vmatrix}$$

$$L_n(q\,|\,z,y) = \sum_{\ell=0}^{n}\binom{n}{\ell} q_\ell(z)\left(\frac{d}{dz}\right)^{n-\ell} y$$
$$= (-1)^{n-1}\begin{vmatrix}\psi_1 & \cdots, & \psi_n\\ \vdots & & \vdots\\ \left(\frac{d}{dz}\right)^{n-1}\psi_1, & & \left(\frac{d}{dz}\right)^{n-1}\psi_n\end{vmatrix}^{-1}\begin{vmatrix}y & \cdots & \psi_n\\ \vdots & & \vdots\\ \left(\frac{d}{dz}\right)^{n}y & \cdots & \left(\frac{d}{dz}\right)^{n}\psi_n\end{vmatrix}.$$

Then the variable transformation $(u, y) \longmapsto (u(z), \lambda(z)y)$ induces a transformation

$$L_n(p|u,y) \longmapsto L_n(Q|z,y)$$

$$= \lambda(z)^{-1}\left(\frac{du}{dz}\right)^{-n}\sum \binom{n}{l} p_l(u(z))\left(\frac{d}{du(z)}\right)^{n-l}(\lambda(z)y).$$

A linear differential operator $L_n(Q|z,y)$ is called to be canonical, if $Q_1 \equiv Q_2 \equiv 0$.

Theorem 1 (Forsyth) $\exists\, (u,y) \longmapsto (u(z), \lambda(z)y)$ such that $L_n(p|u,y) \longmapsto L_n(Q|z,y)$ (canonical). Moreover if $L_n(Q|u,y) \longmapsto L_n(Q'|z,y)$

(a canonical $\longmapsto$ a canonical),

then $(u(z), \lambda(z)y) = \left(\frac{\alpha z+\beta}{\gamma z+\delta}, \frac{cy}{(\gamma z+\delta)^{n-1}}\right)$.

We call the above z a canonical variable.

Roughly speaking, a differential invariant of weight p of $L_n(p|u,y)$ is a polynomial

$$\Phi(\cdots, \left(\tfrac{d}{du}\right)^l p_j(u), \cdots)$$

such that

$$\Phi(\cdots, \left(\tfrac{d}{du}\right)^l p_j(u), \cdots)(du)^p$$

is invariants for any $(u,y) \longmapsto (u(z), \lambda(z)y)$.

Forsyth gave the fundamental differential invariants of a canonical form $L_n(Q|z,y)$:

$$\theta_p(z) = \frac{1}{2}\sum_s \frac{(-1)^s (p-2)!\, p!\, (2p-s-2)!}{(p-s-1)!\,(p-s)!\,(2p-3)!\,s!}\left(\frac{d}{dz}\right)^s Q_{p-s}(z)$$

$(3 \leq p \leq n)$.

<u>Theorem 2</u> Let $L_n(Q|z,y)$ be a <u>canonical</u> form. Then

$$\{\text{differential invariants}\} = \{\text{covariants of } \theta_3(z), \cdots, \theta_n(z)\},$$

where

$$\theta_p(z) = \sum_{\ell=0}^{\infty} \binom{-2p}{\ell} \mathfrak{z}_p^{(\ell)} z^\ell \quad (\text{i.e. } w_p = -2p).$$

3. <u>Equivalence classes of curves.</u>

$C_\varphi : u \longmapsto (\varphi_1(u), \cdots, \varphi_n(u)) = \varphi(u)$

$C_\psi : z \longmapsto (\psi_1(z), \cdots, \psi_n(z)) = \psi(z)$

$$C_\varphi \sim C_\psi \underset{\text{def}}{\Longleftrightarrow} \begin{array}{l} \exists\, (u,y) \mapsto (u(z), \lambda(z) y) \\ \exists\, A \in PGL(n,\mathbb{C}) \end{array} \quad \text{s.t. } \psi(z) = \lambda(z)\varphi(u(z))A.$$

[a classes of curves]

$\updownarrow$

[canonical forms $L_n(Q|z,y)$]/$PSL(2,\mathbb{C})$

$\updownarrow$

$$\left[\left(\theta_3(z)(dz)^3, \cdots, \theta_n(z)(dz)^n\right) + \text{canonical variable } z\right]$$

$\updownarrow$ <u>Theorem 3</u>

[systems of holomorphic differential forms

$$\left(\theta_3(u)(du)^3, \cdots, \theta_n(u)(du)^n, \chi_3(u)(du)^8, \cdots, \chi_n(u)(du)^{2n+2}\right)$$

satisfying

$$2(2j+1)\chi_j\,\theta_k^2 - 2(2k+1)\chi_k\,\theta_j^2$$

$$= \frac{1}{j}\theta_j\,\theta_k^2\left(\frac{d}{du}\right)^2\theta_j - \frac{1}{k}\theta_k\,\theta_j^2\left(\frac{d}{du}\right)^2\theta_k$$

$$- \frac{2j+1}{2j^2}\theta_k^2\left(\frac{d\theta_j}{du}\right)^2 + \frac{2k+1}{2k^2}\theta_j^2\left(\frac{d\theta_k}{du}\right)^2 \qquad (3 \le j < k \le n).$$

<u>Theorem 4</u>. Let $\theta_3, \cdots, \theta_n$ be the fundamental differential invariants of a canonical form $L_n(Q|z, y)$, and denote

$$\chi_j = \begin{vmatrix} \theta_j & \frac{-1}{2j}\frac{d}{dz}\theta_j \\ \frac{-1}{2j}\frac{d}{dz}\theta_j & \frac{1}{2j(2j+1)}\left(\frac{d}{dz}\right)^2\theta_j \end{vmatrix}$$

$$\theta_{jk} = \begin{vmatrix} \theta_j & \theta_k \\ \frac{-1}{2j}\frac{d}{dz}\theta_j & \frac{-1}{2k}\frac{d}{dz}\theta_k \end{vmatrix}, \quad \chi_{jk} = \begin{vmatrix} \chi_j & \theta_k \\ \frac{-1}{4j+4}\frac{d}{dz}\chi_j & \frac{-1}{2k}\frac{d}{dz}\theta_k \end{vmatrix}$$

$$\tilde{C}_\varphi : z \longmapsto \left(\theta_3(z)(dz)^3, \cdots, \chi_3(z)(dz)^8, \cdots, \theta_{jk}(z)(dz)^{j+k+1}, \cdots, \chi_{jk}(z)(dz)^{2j+k+3} \cdots\right)$$

in weighted projective space.

<u>stable case</u>

$$\exists\,\theta_{jk} \text{ or } \chi_{jk} \not\equiv 0 \iff \tilde{C}_\varphi \ne \text{a point} \iff \left([C_\varphi] \underset{\text{bijective}}{\longleftrightarrow} \tilde{C}_\varphi\right)$$

<u>semi-stable case</u>

$$\forall\,\theta_{jk} \equiv \chi_{jk} \equiv 0 \iff \tilde{C}_\varphi = \text{a point} \iff$$

$$C_\varphi \sim \left(z^{\lambda_1}, z^{\lambda_1}\log z, \cdots, z^{\lambda_1}(\log z)^{m_1}, \cdots, z^{\lambda_r}, \cdots, z^{\lambda_r}(\log z)^{m_r}\right),$$

where C_φ is a holomorphic projective curve corresponding to $L_n(Q|z, y)$.

4. Application to higher dimensional case

$z=(z_1,\cdots,z_r)$, $\mathbb{C}\{z\}$ = ring of convergent power series

$d^l z_j$ $(1\le j\le r;\ l=\phi,2,3,\cdots)$; independent variables over $\mathbb{C}\{z\}$. d: a derivation of $\mathbb{C}\{z\}[\cdots,d^l z_j,\cdots]$ given by

$$d(d^l z_j)=d^{l+1}z_j,\quad d\varphi=\sum_{j=1}^{r}\frac{\partial\varphi}{\partial z_j}dz_j.$$

For each holomorphic curve: $u\longmapsto(z_1(u),\cdots,z_r(u))$ the map $d^l\varphi\longmapsto\left(\frac{d}{du}\right)^l\varphi(z(u))$ is a differential algebra homomorphism of $\mathbb{C}\{z\}[\cdots,d^l z_j,\cdots]$ into $\mathbb{C}\{u\}$.

$$\tilde{W}_\varphi=\begin{vmatrix}\varphi_1 & \cdots & \varphi_n\\ \vdots & & \\ d^{n-1}\varphi_1 & \cdots & d^{n-1}\varphi_n\end{vmatrix},$$

$$\tilde{L}(\tilde{p}\,|\,y)=\sum\binom{n}{l}\tilde{p}_l\,d^{n-l}y,$$

$$=(-1)^{n-1}\tilde{W}_\varphi^{-1}\begin{vmatrix}y & \varphi_1 & \cdots & \varphi_n\\ \vdots & & & \vdots\\ d^n y & d^n\varphi_1 & \cdots & d^n\varphi_n\end{vmatrix}$$

where $\varphi_1,\cdots,\varphi_n$ are holomorphic functions in $z_1,\cdots,z_r$.

Differential invariants of $\varphi=(\varphi_1,\cdots,\varphi_n)$ or $\tilde{L}(\tilde{p}\,|\,y)$ is given as follows

$$\left\{\begin{array}{l}\Phi\left(\cdots,\left(\frac{d}{du}\right)^l p_j(u),\cdots\right)\quad\text{differential invariants in one variable } u\\ \qquad\downarrow \text{ replacing}\\ \tilde{\Phi}=\Phi(\cdots,d^l\tilde{p}_j,\cdots)\end{array}\right.$$

Differential invariants, for examples
$\tilde{\theta}_3, \cdots, \tilde{\chi}_3, \cdots, \tilde{\theta}_{jk}, \cdots, \tilde{\chi}_{jk}, \cdots, \tilde{\chi}_{nn}$,
must be important something associated with
$V_\varphi : z \longmapsto (\varphi_1(z), \cdots, \varphi_n(z)) \subset \mathbb{P}^{n-1}$.

5. Some nice examples for more easy problem than the first difficult one: Let G be a simply connected complex Lie group, and let $\omega_{\alpha\beta\lambda\mu}$ $(0 \leq \alpha < \beta \leq n,\ 0 \leq \lambda, \mu \leq n)$ be holomorphic left invariant 1-forms on G. Consider a system of differential equations

$$(*) \quad y_\beta dy_\alpha - y_\alpha dy_\beta - \sum_{\lambda\mu} \omega_{\alpha\beta\lambda\mu} y_\lambda y_\mu = 0 \quad (0 \leq \alpha < \beta \leq n).$$

Let W be the set of initial points in $\mathbb{P}^n$ of non-trivial holomorphic solutions (local). Then W is a projective algebraic variety such that

$$W = \bigcup W_j,$$

where W_j are homogeneous spaces of G.

Tagung über

TRANSFORMATIONSGRUPPEN UND INVARIANTENTHEORIE

Bonn, 22. - 28. September 1978

Titel: On enveloping algebras and invariant theory.

Autor: J. DIXMIER

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Let k be an algebraically closed field of characteristic 0, $\mathfrak{g}$ a Lie algebra over k, $U(\mathfrak{g})$ its enveloping algebra, $K(\mathfrak{g})$ the field of fractions of $U(\mathfrak{g})$.

I. Structure of the center $Z(\mathfrak{g})$ of $U(\mathfrak{g})$.

1) The Duflo isomorphism. Let $S(\mathfrak{g})$ be the symmetric algebra of $\mathfrak{g}$, $Y(\mathfrak{g})$ the subalgebra of invariant elements for the adjoint representation. There is a canonical algebra isomorphism $Y(\mathfrak{g}) \to Z(\mathfrak{g})$. (It is difficult to describe, and the proof is highly analytical).

2) $\mathfrak{g}$ nilpotent. If $\dim \mathfrak{g} \leq 4$, $Z(\mathfrak{g})$ is a polynomial algebra. In general, it can even happen that $Z(\mathfrak{g})$ is not of finite type. If $\mathfrak{g}$ is a maximal ad-nilpotent subalgebra of a semisimple Lie algebra, $Z(\mathfrak{g})$ is a polynomial algebra.

3) $\mathfrak{g}$ semisimple. Then $Z(\mathfrak{g})$ is a polynomial algebra. Let $\mathfrak{h}$ be a Cartan subalgebra, W the Weyl group. We have the canonical Harish-Chandra isomorphism $Z(\mathfrak{g}) \to S(\mathfrak{h})^W$ and Chevalley isomorphism $Y(\mathfrak{g}) \to S(\mathfrak{h})^W$ related to the Duflo isomorphism by a commutative triangle.

4) If $\mathfrak{g}$ is one of the evident semidirect products $sl(n) \ltimes k^n$, $so(n) \ltimes k^n$, $sp(n) \ltimes k^n$, $Z(\mathfrak{g})$ is a polynomial algebra.

5) Suppose that the radical of $\mathfrak{g}$ is nilpotent. Then $Z(\mathfrak{g}) \neq k$ (if $\mathfrak{g} \neq 0$), deg. trans. $Z(\mathfrak{g}) \equiv \dim \mathfrak{g}$ (mod. 2), $Z(\mathfrak{g})$ is factorial, the center

$C(\mathfrak{g})$ of $K(\mathfrak{g})$ is the field of fractions of $Z(\mathfrak{g})$. If moreover $\mathfrak{g}$ is solvable (and maybe in general), $C(\mathfrak{g})$ is a pure extension of k.

II. A generalization.

A few results of I extend to $U(\mathfrak{g})/\mathfrak{J}$ ($\mathfrak{J}$ a 2-sided ideal of $U(\mathfrak{g})$) and the center $Z(\mathfrak{g};\mathfrak{J})$ of $U(\mathfrak{g})/\mathfrak{J}$. Maybe these algebras $Z(\mathfrak{g};\mathfrak{J})$ can be used to define structures of quasi-affine varieties on the different pieces of Prim $U(\mathfrak{g})$.

III. Another generalization.

Instead of considering only <u>the invariants</u> of $S(\mathfrak{g})$, one can consider the whole decomposition of the adjoint action in $S(\mathfrak{g})$, and in some quotients of $S(\mathfrak{g})$. If Ω is an orbit in $\mathfrak{g}$ for the adjoint action, what is the decomposition of the adjoint action in $k[\Omega]$ and $k[\overline{\Omega}]$? This can be done for $\mathfrak{g} = \mathfrak{sl}(n)$, due to several theorems, in particular the normality of $\overline{\Omega}$. The result can be used to construct the map $(\mathfrak{sl}(n))^* \to$ Prim $U(\mathfrak{sl}(n))$ of the orbit method.

IV. Another generalization.

<u>Invariant theory</u> introduces itself naturally in noncommutative contexts. Examples. 1) Commutant of $\mathfrak{h}$ in $U(\mathfrak{g})$ (notation of I.3).
2) Commutant of $\mathfrak{k}$ in $U(\mathfrak{g})$ if $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ is a Cartan decomposition.
3) Commutant of $\mathfrak{n}$ in $\mathfrak{b}$ ($\mathfrak{b}$ a Borel subalgebra, $\mathfrak{n} = [\mathfrak{b},\mathfrak{b}]$); this is a commutative algebra, even a polynomial algebra.

Tagung über

TRANSFORMATIONSGRUPPEN UND INVARIANTENTHEORIE

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Titel: Some remarks on $Sp(n)$

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Let R be any field or the integers. Let X be the variety whose points over any R-algebra C are the $2n \times m$ matrices with entries in C such that if $M \in X$, ${}^tMJM = 0$, $J = \begin{pmatrix} 0 & \\ 1 \cdots -1 & 0 \end{pmatrix}$. Then the group $G = Sp(n) \times Gl(m)$, naturally acts on X hence on the coordinate ring A of X.

We study A as a representation of G.

Def. Let $P = (i_s, \ldots, i_1 | k_1, \ldots, k_s)$, $s \leq \min(n, m)$, denote the function on X defined to be the determinant of the minor formed by the rows $1 \leq i_1 < i_2 < \cdots < i_s \leq 2n$, and columns $1 \leq k_1 < k_2 < \cdots < k_s \leq m$. Then P is called admissible if the following condition holds:

Let $1 \leq i_1 < \cdots < i_t \leq n$ and $n < i_{t+1} < \cdots < i_s \leq 2n$, define $I = \{i_1, \ldots, i_t\}$, $J = \{2n - i_{t+1} + 1, \ldots, 2n - i_s + 1\}$; then if $M = I \cap J = \{\gamma_1 < \cdots < \gamma_e\}$, P ~~is called as~~ there exists $\Lambda \subset \{1, \ldots, n\} - I \cup J$, $\Lambda = \{\lambda_1, \ldots, \lambda_e\}$ with $\lambda_1 > \gamma_1, \ldots, \lambda_e > \gamma_e$.

Suppose such a Λ exists and let Λ_0 be minimal

among such Λ's, $I' = (I - \Gamma) \cup \Lambda_0 = \{ i'_1 < \cdots < i'_t \}$
$J' = (J - \Gamma) \cup \Lambda_0 = \{ 2n - i'_{t+1} + 1 > \cdots > 2n - i'_s + 1 \}$, then P will be denoted by

$$\left(\begin{matrix} i'_s & \cdots & i'_{t+1} & i_t & \cdots & i_1 \\ i_s & & i_{t+1} & i'_t & \cdots & i'_1 \end{matrix}\,\middle|\, k_1, \ldots k_s \right)$$

Def. A product $T = P_1 \cdots P_z$ of admissible minors,

$$P_1 = \left(\begin{matrix} i_{s_1 1} & \cdots & i_{11} \\ j_{s_1 1} & & j_{11} \end{matrix}\,\middle|\, k_{11} \cdots k_{1 s_1}\right), \ldots, P_z = \left(\begin{matrix} i_{s_z z} & \cdots & i_{1z} \\ j_{s_z z} & & j_{1z} \end{matrix}\,\middle|\, k_{z1} \cdots k_{z s_z}\right)$$

will be called a symplectic (double) tableau. The partition $\sigma = (s_1, \ldots, s_z)$ is called the shape of T. Such a tableau will be called standard if both tableaux

$$\left(\begin{array}{ccc} i_{s_1 1} & & i_{11} \\ j_{s_1 1} & & j_{11} \\ \vdots & & \vdots \\ i_{s_z z} & & i_{1z} \\ j_{s_z z} & & j_{1z} \end{array}\right) \quad \text{and} \quad \left(\begin{array}{ccc} k_{11} & \cdots & k_{1 s_1} \\ \vdots & & \vdots \\ k_{z1} & \cdots & k_{z s_z} \end{array}\right)$$

are standard Young tableaux (with repetitions).

Theorem 1 The standard symplectic tableaux form a basis of A over R.

We call a symplectic tableau right semicanonical if it is of the form

$$\left(\text{anything}\,\middle|\, \begin{matrix} 1 & 2 & \cdots & s_1 \\ \vdots & & & \\ 1 & 2 & \cdots & s_z \end{matrix}\right)$$

Theorem 2. The right semicanonical symplectic standard tableaux of shape $\sigma = (s_1, \ldots, s_z)$ form a basis for $H^0(Sp(n)/B, L)$, $L = L_1^{m_1} \cdots L_n^{m_n}$, where L_i is the i-th tautological line bundle on $Sp(n)/B$ and m_1 is the number of s_i which are equal to $1, \ldots, m_n$ the number of s_i which are equal to n.

(it can be shown that this basis coincides with given by Lakshmibai, Musili, Seshadri)

<u>Theorem</u> 3 There is a canonically defined G-invariant filtration $\mathcal{F}$ of A such that

$$Gr_{\mathcal{F}} A = \bigoplus_{m_1 \cdots m_k} H^0(Sp(n)/B, L_1^{m_1} \cdots L_k^{m_k}) \otimes H^0(Gl(m)/\bar{B}, \tilde{L}_1^{m_1} \cdots \tilde{L}_k^{m_k})$$

where $\tilde{L}_i$ is the i-th tautological line bundle on ~~Gl (m)~~ $Gl(m)/\bar{B}$, $k = \min(n, m)$.

In particular if $k = n = m$ and R is a field of characteristic 0, this sets a canonical one one correspondence between $Gl(n)$ polynomial irreducible representations and $Sp(n)$ polynomial irreducible representations.

Consider now the following situation. Let V be an even dimensional vector space over R (resp. if $R = \mathbb{Z}$ a free module of even rank) together with a non degenerate bilinear form $\langle\ ,\ \rangle$. Let $Sp(V)$ be the symplectic group of $\langle\ ,\ \rangle$.

Consider the homomorphism

$$\varphi_{ij} : V^{\otimes m} \longrightarrow V^{\otimes m-2} \qquad i < j$$

defined by

$$\varphi_{ij}(v_1 \otimes \cdots \otimes v_m) = \langle v_i, v_j \rangle v_1 \otimes \cdots \otimes \hat{v}_i \otimes \cdots \otimes \hat{v}_j \cdots \otimes v_m.$$

Then it is clear that if we consider $Sp(V)$ acting on $V^{\otimes m}$ by $g(v_1 \otimes \cdots \otimes v_m) = gv_1 \otimes \cdots \otimes gv_m$, $g \in Sp(V)$, the subspace $W_m = \bigcap \ker \varphi_{ij}$ is $Sp(V)$-invariant.

One can use the above basis to prove

Theorem 4 (with E. Strickland) Let R have more than m elements then:

$$\mathrm{End}_{Sp(V)}(W_n) \cong R[S_m]/I, \text{ where}$$

S_m is the symmetric group on m letters, $I = 0$ if $m \leq n$, I is the ideal generated by the element

$$\sum_{\sigma \in S_{\frac{\dim V}{2}+1}} \varepsilon_\sigma \sigma \quad \text{if } m > n.$$

In the case in which R is a field of characteristic 0 this is proved in H. Weyl's book "The Classical groups.

Autor: VUST Seite Nr.: 1

Foncteurs polynomiaux et théorie des invariants.

On note $\mathcal{V}$ la catégorie des espaces vectoriels de dimension finie (corps de base k de caractéristique nulle) et $\mathcal{Aff}$ la catégorie des k-variétés algébriques affines (non nécessairement de type fini).

Un foncteur polynomial est un foncteur covariant $F : \mathcal{V} \to \mathcal{V}$ tel que, pour tout P, Q, l'application

$$\begin{array}{ccc} \mathrm{Hom}(P,Q) & \longrightarrow & \mathrm{Hom}(F(P),F(Q)) \\ u & \longmapsto & F(u) \end{array}$$

est polynomiale ; on dira que F est de hauteur finie si F est naturellement isomorphe à un sous-foncteur de $\otimes^{a(1)} \oplus \dots \oplus \otimes^{a(r)}$ pour un $a : \{1,\dots,r\} \to \mathbb{N}$.

On se donne un objet N de $\mathcal{V}$ et G un sous-groupe de $GL(N)$; on considère les foncteurs

$$\begin{array}{ccc} \mathcal{V} & \longrightarrow & \mathcal{Aff} \\ P & \longmapsto & \mathrm{Spec}\left(k[\mathrm{Hom}(P,N)]^G\right) \overset{\text{not.}}{=} \mathrm{Hom}(P,N)/G \end{array}$$

et

$$P \longmapsto \text{dual de } F(P) \overset{\text{not.}}{=} F'(P)$$

pour F polynomial.

<u>Théorème</u>.- On suppose que, pour un espace vectoriel I avec $\dim I \geq \dim N$, la variété $\mathrm{Ham}(I,N)/_{G}$ est de type fini. Il existe alors une résolution (dans Aff)

$$\mathrm{Ham}(\cdot,N)/_{G} \longrightarrow F_1' \longrightarrow F_2'$$

où F_1 et F_2 sont polynomiaux de hauteur finie.

<u>Réf.</u> Sur la théorie classique des invariants; Comm. math. Helv. (1977).

Autor: William J. Haboush.

Problems:

1) (Due to A. Faunlkercy): Let R be commutative Noetherian normal. Let M be a finite torsion free R-module. Let $\mathbb{A}(M)$ be the direct sum

$$\mathbb{A}(M) = \bigoplus_{r \geq 0} S^r(M)^{**} \qquad \text{(double duals)}.$$

Is $\mathbb{A}(M)$ finitely generated?
If so this $\Rightarrow$ that if $\mathbb{G}_a$ acts on the scheme X, and one considers only the points with finite stabilizers, $X^0 \subset X$, then $X^0/\mathbb{G}_a$, the orbit space exists in any characteristic.

2) Let $\mathcal{U}_{\mathbb{L},\mathbb{Z}}$ denote the Kostant $\mathbb{Z}$ form associated to the split semi-simple Lie algebra $\mathbb{L}$ over $\mathbb{Z}$. Let k be a field of positive characteristic. Let $D_{\mathbb{L}/k} = \mathcal{U}_{\mathbb{L}\mathbb{Z}} \otimes_{\mathbb{Z}} k$. Let D_H denote the sub-algebra generated by all elements of the form $\binom{H_i}{r}$ H_i in the Cartan sub algebra of $\mathbb{L}$. Describe all the automorphisms of $D_{\mathbb{L}/k}$, $\sigma \ni \sigma(D_H) \subset D_H$.

Autor: V. G. Kac

PROBLEMS:

1. $G : V$ is a reductive linear group, admitting an open orbit. Is it true that ~~Then~~ all the orbits are conical (i.e. any two non-zero proportional vectors are G-equivalent).

2. Let G be an affine algebraic group and H be a closed subgroup. It is easy to show that if G/H is an affine variety, then any 1-parameter unipotent subgroup in H, which can be included in a 3-dimensional simple subgroup (TDS) of G, can be included in a TDS in H.

Is the inverse statement true? I can prove it for $\dim H = 1$. It is known for G reductive.

3. Is it true that ~~Any~~ any orbit of an algebraic group G is a rational variety? (It is true for G solvable and $G = GL_3$).

4. Find a necessary and sufficient conditions for G/H to be quasiaffine (a group-theoretic one).

Autor: R. W. Richardson

Problem Let (x, h, y) be an $\underline{sl}_2(\mathbb{C})$ triple in a semi simple L.A. $\underline{g}$ and let $\underline{s}$ be the subalgebra spanned by ~~x, y~~ (x, h, y). Assume that $\underline{z}_{\underline{g}}(\underline{s})$, the centralizer of $\underline{s}$ in $\underline{g}$, is equal to $\{0\}$. Show that all eigenvalues of ad h on $\underline{g}$ are even. (This has been proved by R. Carter using a very lengthy classification argument. It would be convenient to have an elementary proof)

More generally $\underline{g}$ a s.s. L.A over $\mathbb{R}$ and (x, h, y) an $sl_2(\mathbb{R})$ triple in $\underline{g}$; $\underline{s}$ = L.A. spanned by (x, h, y). Let G = adjoint group of $\underline{g}$. Let $Z_G(\underline{s}) = \{ g \in G \mid \mathrm{Ad}\, g = 1 \text{ on } \underline{s} \}$; $Z_G(\underline{s})$ is a reductive real algebraic group. Assume that $Z_G(\underline{s})$ is compact. Show that all eigenvalues of ad h on $\underline{g}$ are even. (This is some slight evidence for this, but I have not checked many cases)