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TUBULAR NEIGHBORHOODS OF LOCAL MODELS

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Abstract. We show that the v-sheaf local models of [SW20] are unibranch. In particular, this proves that the scheme-theoretic local models defined in [AGLR22] are always normal with reduced special fiber, thereby establishing the remaining cases of the geometric part of the Scholze–Weinstein conjecture when $p \leq 3$. Our methods are general, topological and simplify those of [Zhu14] for tamely ramified groups in positive characteristic. As a technical input, we generalize a comparison theorem of nearby cycles of [Hub96] to the v-sheaf setup.

Contents

1. Introduction

Local models were introduced in the nineties to study the singularities of Shimura varieties, namely in the works of Chai–Norman [CN90], de Jong [dJ93] and Deligne–Pappas [DP94], and have found various applications. The theory was systematized in the book of Rapoport–Zink [RZ96], via linear algebraic moduli problems. Later, it underwent a significant transformation when Görtz [Gör01, Gör03] embedded their special fibers in certain infinite-dimensional flag varieties. This was subsequently exploited by Faltings, Pappas, Rapoport and Zhu [Fal03, PR08, Zhu14, PZ13] to great effect. More recently, Scholze–Weinstein [SW20] proposed a fully functorial avenue to study local models in mixed characteristic via perfectoid geometry. This program was pursued in [AGLR22]. We use the following notation in the paper.

Definition 1.1. Let F be a local field, O its ring of integers, and k its residue field. Let G be a reductive connected F-group, $\mathcal{G}$ a parahoric O-model of G, μ a geometric conjugacy class of cocharacters, and E its reflex field. We denote by $\mathcal{M}_{\mathcal{G},\mu}$ the v-sheaf given as the closure of $\mathrm{Gr}_{G,\mu} \subset \mathrm{Gr}_{G,E}$ inside the Beilinson–Drinfeld Grassmannian $\mathrm{Gr}_{\mathcal{G},O_E}$ of [SW20, FS21]. If F is of positive characteristic or μ is minuscule, we denote by $\mathcal{M}^{\mathrm{sch}}_{\mathcal{G},\mu}$ the canonical weakly normal[1] proper O_E-scheme representing $\mathcal{M}_{\mathcal{G},\mu}$.

If F is p-adic and μ is minuscule, then there is a unique weakly normal scheme $\mathcal{M}^{\mathrm{sch}}_{\mathcal{G},\mu}$ representing $\mathcal{M}_{\mathcal{G},\mu}$ by [AGLR22, Theorem 1.1]. If F has positive characteristic, then $\mathcal{M}^{\mathrm{sch}}_{\mathcal{G},\mu}$ is defined

[1] Recall that a scheme is weakly normal if every finite, birational, universally homeomorphic morphism with reduced source is an isomorphism.

as the weakly normal scheme representing $\mathcal{M}_{\mathcal{G},\mu}$ whose generic fiber is a Schubert variety in positive characteristic.

Historically, an important goal has been to show that the special fiber of $\mathcal{M}^{\mathrm{sch}}_{\mathcal{G},\mu}$ is reduced, see [PRS13] for various ad hoc definitions of $\mathcal{M}^{\mathrm{sch}}_{\mathcal{G},\mu}$. Görtz [Gör01, Gör03] proved this for split classical groups of PEL type via so-called straightening laws, but the proof does not extend to the general case. An important development is due to Pappas–Rapoport [PR08], who formulated the coherence conjecture to address reducedness via coherent cohomology of ample line bundles. Finally, Zhu [Zhu14] proved the conjecture for tame groups, by translating the problem to equicharacteristic, and constructing a global Frobenius splitting of the local model compatibly with the special fiber. Recall that a scheme X in characteristic p is Frobenius split if the $\mathcal{O}_X$-module homomorphism $\mathcal{O}_X \to \varphi_*\mathcal{O}_X$ given by Frobenius splits. Note that Frobenius split schemes are necessarily reduced. We refer to [BK05, BS13] for proper introductions to the subject.

Most modern results in the literature concerning the reduced structure of the local models rely on [Zhu14] through reductions and comparisons. Interestingly, the heart of Zhu's proof lies in characteristic p. This becomes problematic in the perfectoid perspective of [SW20, AGLR22], because it is not clear how to work with Frobenius splitting techniques in this context. Fortunately, we have the following crucial fact that allow us to bypass them:

Lemma 1.2. *If $\mathcal{M}^{\mathrm{sch}}_{\mathcal{G},\mu}$ is unibranch, then its special fiber is reduced.*

Proof. This is [AGLR22, Lemma 7.26], but we repeat the crux of the argument for future reference and the reader's convenience. We know already that the perfection of the special fiber of $\mathcal{M}^{\mathrm{sch}}_{\mathcal{G},\mu}$ equals the so-called μ-admissible locus $\mathcal{A}_{\mathcal{G},\mu}$, see [AGLR22, Theorem 6.16]. In particular, the union of maximal orbits defines a smooth open of $\mathcal{M}^{\mathrm{sch}}_{\mathcal{G},\mu}$ with dense geometric fibers, see [Ric16, Corollary 2.14] and [AGLR22, Equation (7.44)], so the special fiber satisfies Serre's condition R_0. As $\mathcal{M}^{\mathrm{sch}}_{\mathcal{G},\mu}$ is weakly normal per definition, the unibranch assumption in the statement implies normality already. In particular, the special fiber is S_1 by the Serre criterion for normality plus flatness. The Serre criterion for reducedness yields our claim. □

The unibranch property is already amenable to a formulation in terms of perfectoids, because it is topological in nature. Indeed, it suffices to know that the tubular neighborhoods of [Gle22, Definition 4.38] at all the closed points of the special fiber are connected. In turn, being connected is a cohomological invariant and it is natural to expect that a deeper study of the nearby cycles initiated in [AGLR22] would yield the result. Over a p-adic field and for non-minuscule coweights μ, the v-sheaf $\mathcal{M}_{\mathcal{G},\mu}$ does not come from a scheme. Nevertheless, $\mathcal{M}_{\mathcal{G},\mu}$ is a kimberlite, the v-sheaf analogue of a formal scheme, and being unibranch (or topologically normal) still makes sense in this context, see [Gle22, Definition 4.52]. Our main result is:

Theorem 1.3. *The kimberlite $\mathcal{M}_{\mathcal{G},\mu}$ is unibranch for all pairs $(\mathcal{G},\mu)$ from Definition 1.1.*

As an immediate corollary, we get the geometric part of the Scholze–Weinstein conjecture in full generality, removing certain exceptions found in [AGLR22, Theorem 7.23] for $p \leq 3$.

Corollary 1.4. *If $F = k(\!(t)\!)$ or μ is minuscule, the underlying scheme $\mathcal{M}^{\mathrm{sch}}_{\mathcal{G},\mu}$ is normal with reduced special fiber. Under* [AGLR22, Assumption 1.9]*, $\mathcal{M}^{\mathrm{sch}}_{\mathcal{G},\mu}$ is Cohen–Macaulay with Frobenius split special fiber.*

In the last sentence, we must exclude wild odd unitary groups when $p = 2$, and it follows from the normality and Frobenius splitness results for Schubert varieties in positive characteristic, see [FHLR22, Theorem 4.1, Theorem 5.3] for the least restrictive hypothesis, and the minuscule comparison of [AGLR22, Theorem 3.16] in the p-adic case, compare with the discussion surrounding [AGLR22, Conjecture 7.25].

Remark 1.5. Another direct corollary of Theorem 1.3 is that moduli of p-adic shtukas in the sense of [Gle21, Definition 2.27] or the more restrictive [PR21, Definition 3.2.1] are also unibranch.

Let us explain the strategy behind Theorem 1.3. The main idea is that the stalks of the first non-trivial cohomology sheaf of the nearby cycles $R\Psi(\mathrm{IC}_\mu)$ detects the number of connected components of the tubular neighborhoods. In this way, a disconnection of a tubular neighborhood produces a disconnection étale locally. This however involves a comparison between the analytic nearby cycles defined in [Sch17] and the formal nearby cycles defined in [Gle22]. In general, such a result only holds if the analytic nearby cycles are already algebraic, see Definition 4.6. Fortunately, algebraicity of $R\Psi(\mathrm{IC}_\mu)$ was proved in [AGLR22]. We expect Theorem 4.7 to find broader applications in formal geometry.

If $\mathcal{G}$ is Iwahori (enough for our purposes by Lemma 5.2 and [Gle22, Lemma 5.26]), one uses the theory of Wakimoto filtrations in mixed characteristic developed in [ALWY22] to bound the dimension of stalks of $R\Psi(\mathrm{IC}_\mu)$ along orbits in codimension at most 1 and to prove directly that $\mathcal{M}_{\mathcal{G},\mu}$ is unibranch away from a subset of codimension 2. To prove unibranchness in deeper strata, one uses the perversity of $R\Psi(\mathrm{IC}_\mu)$ due to [ALWY22] and a combinatorial argument. To deal with codimension 2 subsets in the special fiber, we observe that $\mathcal{A}_{\mathcal{G},\mu}$ is perfectly S_2 (i.e. the perfection of a scheme that satisfies Serre's condition S_2).[2] The S_2 property can be neatly expressed combinatorially due to [HH94], so it reduces to positive characteristic. Morally, the S_2 property implies that the normalization map has to be an isomorphism. Unfortunately, that map is not available for kimberlites, so we argue more carefully. We exploit the S_2 property and unibranchness on codimension 1 strata to prove that an étale-formal local disconnection of the generic fiber forces a large open subset to specialize to a small stratum contradicting the perversity of $R\Psi(\mathrm{IC}_\mu)$.

In positive characteristic, we reprove normality for tame groups using $\mathbb{G}_{m,k}$-actions, following techniques of Le–Levin–Le Hung–Morra [LLHLM20], who study the unibranch property for the more singular crystalline local models. The strategy for proving that the local model is unibranch in this case was therefore known to the authors of [LLHLM20], but they did not seem to know of Lemma 1.2.

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2. New proof of Zhu's theorem

In this section, we establish a particular case of Theorem 1.3, originally due to [Zhu14], via global methods specific to positive characteristic. Let k be algebraically closed of characteristic

[2] One should even expect $\mathcal{A}_{\mathcal{G},\mu}$ to be the perfection of a Cohen–Macaulay scheme by [HR19].

p and G a connected reductive $\mathbb{G}_{m,k}$-group that splits over the finite étale cover $\mathbb{G}_{m,k} \to \mathbb{G}_{m,k}$ given by raising to the e-th power, with $(e,p)=1$. Let $\mathcal{G}$ be the $\mathbb{A}^1_k$-model of G built out of a parahoric $k[\![t]\!]$-model of $G_{k(\!(t)\!)}$ and G via Beauville–Laszlo descent. We regard the Beilinson–Drinfeld Grassmannian $\mathrm{Gr}_{\mathcal{G}}$ as being an ind-(perfect scheme) defined over the absolute integral closure $\bar{\mathbb{A}}^1_k$, the local model $\mathcal{M}_{\mathcal{G},\mu}$ given as the closure of $\mathrm{Gr}_{G,\mu}$ inside $\mathrm{Gr}_{\mathcal{G}}$, and finally $\mathcal{M}^{\mathrm{sch}}_{\mathcal{G},\mu}$ as the canonical weakly normal deperfection of $\mathcal{M}_{\mathcal{G},\mu}$ finitely presented over $\bar{\mathbb{A}}^1_k$. Apologies to the reader are in order for deviating from the notation in Definition 1.1, which referred to the associated v-sheaves over a complete local ring, but it is not hard to reconcile both perspectives.

We aim to reprove [Zhu14, Theorem 3]. Thanks to Lemma 1.2, this can proceed along the lines of [LLHLM20, Section 3].

Theorem 2.1 ([Zhu14])**.** *The flat projective scheme $\mathcal{M}^{\mathrm{sch}}_{\mathcal{G},\mu}$ is normal with Frobenius split, reduced fiber over* 0.

A special feature of tame groups in equicharacteristic is that $\mathrm{Gr}_{\mathcal{G}}$ carries the so-called rotation $\mathbb{G}^{\mathrm{pf}}_{m,k}$-action which lifts the e-th power of the natural one on the base $\bar{\mathbb{A}}^1_k$, see [Zhu14, Section 5]. Also, we can regard a maximal F-split torus $S \subset G$ as defined over k up to unique isomorphism, and we get a natural action of S^{pf} on $\mathrm{Gr}_{\mathcal{G}}$ linear over $\bar{\mathbb{A}}^1_k$. For any coweight χ of $S \times \mathbb{G}_{m,k}$, the induced $\mathbb{G}^{\mathrm{pf}}_{m,k}$-action on $\mathrm{Gr}_{\mathcal{G}}$ is Zariski locally linearizable in the sense of [Ric19]. This is seen by reduction to GL_n, where we reason via lattices as in [HR21, Lemma 3.3]. Now, the attractor $\mathrm{Gr}^+_{\mathcal{G}}$ exists by [Ric19, Theorem 1.8] and [HR19, Theorem 2.1], and is representable by a disjoint union of locally closed sub-ind-schemes. By compactifying $\mathrm{Gr}_{\mathcal{G}}$ to $\bar{\mathbb{P}}^1_k$ (simply extend $\mathcal{G}$ further to a parahoric $\mathbb{P}^1_k$-group scheme, see [Lou19, Définition 4.2.8]), we see by [Ric19, Lemma 1.11] that the attractor $\mathrm{Gr}^+_{\mathcal{G}}$ maps surjectively to $\mathrm{Gr}_{\mathcal{G}}$ if χ is contracting on $\bar{\mathbb{A}}^1_k$. We denote by $\mathcal{F}\ell_{\mathcal{G}}$ the fiber of $\mathrm{Gr}_{\mathcal{G}}$ over 0.

Lemma 2.2. *For every S-fixed point $w \in \mathcal{F}\ell_{\mathcal{G}}$, there exists $\chi_w \colon \mathbb{G}_{m,k} \to S \times \mathbb{G}_{m,k}$ such that $w \in \mathcal{F}\ell^0_{\mathcal{G}}$ is isolated and the connected component of $\mathcal{F}\ell^+_{\mathcal{G}}$ containing w is open in $\mathcal{F}\ell_{\mathcal{G}}$.*

Proof. Recall that $S \times \mathbb{G}_{m,k}$ supports a Kac–Moody root system, see [Lou19, Définition 4.2.1, Lemme 4.2.2]. Let $\chi \colon \mathbb{G}_{m,k} \to S \times \mathbb{G}_{m,k}$ be a coweight lying in the w-conjugate of the anti-dominant facet of type corresponding to $\mathcal{G}$. Then, the connected component of $\mathcal{F}\ell^+_{\mathcal{G}}$ containing w equals the open left translate $w \cdot L^{--}\mathcal{G} \cdot e \subset \mathcal{F}\ell_{\mathcal{G}}$ of the big cell, see [Lou19, Corollaire 4.2.11]. □

Notice that χ_w is anti-dominant for the Kac–Moody root system, meaning it acts on the variable t_1 defining the flag variety of $\mathcal{F}\ell_{\mathcal{G}}$ by negative powers, whereas we expect χ_w to contract the affine line $\bar{\mathbb{A}}^1_k$ that serves as base of the local model. The change of sign occuring here is explained by the fact that the definition of $\mathrm{Gr}_{\mathcal{G}}$ involves an auxiliary formal variable t_2, so that $r^{-1}t_1 - t_2$ and $t_1 - rt_2$ define the same Cartier divisor.

The next task is to globalize the open set of the previous lemma. In the case of the Iwahori model of GL_n, the desired open neighborhood is constructed explicitly in [LLHLM20, Lemma 3.2.7]. Instead, we provide an abstract argument.

Lemma 2.3. *Let χ_w be as in Lemma 2.2. Then, $w \in \mathrm{Gr}^0_{\mathcal{G}}$ is isolated and the connected component of $\mathrm{Gr}^+_{\mathcal{G}}$ containing w is open in $\mathrm{Gr}_{\mathcal{G}}$.*

Proof. Choose a presentation of a connected component of $\mathrm{Gr}_{\mathcal{G}}$ containing w by an increasing union of $\mathbb{G}^{\mathrm{pf}}_{m,k}$-stable perfect varieties X. A simple finiteness argument with reduced words, see also [HLR18, Theorem 2.5], shows that every sufficiently large element of the Iwahori–Weyl group is bigger than w. Thus, we may and do assume that every irreducible component of the fiber X_0 over 0 already contains the point w. Let U be the connected component of X^+ containing the generic point of X. Since $U \subset X$ is locally closed and X is irreducible, it must be an open subset. Also, its fiber U_0 over 0 is non-empty, as it contains the fixed points U^0. As every generic point of X_0 specializes to w, we see that U_0 intersects $w \cdot L^{--}\mathcal{G} \cdot e$ non-trivially. In particular, U is contracting to w. □

We also need the following helpful criterion to detect the unibranch property.

Lemma 2.4 ([LLHLM20]). *Let X be a perfect k-variety with monoid $\mathbb{A}_k^{1,\mathrm{pf}}$-action, such that $X^0 = \{x\}$. Then X is unibranch at x.*

Proof. We explain the idea of [LLHLM20, Lemma 3.4.8] for the reader's convenience. Note that the $\mathbb{A}_k^{1,\mathrm{pf}}$-action extends to the normalization Y. The closed subspace Y^0 is the fiber over x. So now the Białynicki-Birula map $Y \to Y^0$ shows that the right side is connected, as Y is irreducible. □

Proof of Theorem 2.1. Thanks to Lemma 2.3, we can produce for any given $S^{\mathrm{pf}} \times \mathbb{G}_{m,k}^{\mathrm{pf}}$-stable point $w \in \mathcal{M}_{\mathcal{G},\mu}(k)$, a χ_w-stable open neighborhood $\mathcal{N}_w$ of w in $\mathcal{M}_{\mathcal{G},\mu}$ such that $\mathcal{N}_w = \mathcal{N}_w^+$ and $\mathcal{N}_w^0 = w$. Now, $\mathcal{N}_w$ is irreducible, so Lemma 2.4 shows that $\mathcal{M}_{\mathcal{G},\mu}$ is unibranch at w. By $L^+\mathcal{G}$-equivariance, this holds at any closed point of the special fiber of $\mathcal{M}_{\mathcal{G},\mu}$, hence it must be unibranch. By Lemma 1.2, we conclude that the special fiber of $\mathcal{M}_{\mathcal{G},\mu}^{\mathrm{sch}}$ is reduced.

Passing to a z-extension, we may and do assume that G_{der} is simply connected. Let $\mathcal{M}_{\mathcal{G},\mu}^{\mathrm{fl}}$ be the flat closure inside $\mathrm{Gr}_{\mathcal{G}}^{\mathrm{sch}}$, the scheme-theoretic Beilinson–Drinfeld Grassmannian. Then, the natural morphism $\mathcal{M}_{\mathcal{G},\mu}^{\mathrm{sch}} \to \mathcal{M}_{\mathcal{G},\mu}^{\mathrm{fl}}$ is an isomorphism. By Nakayama's lemma, we may pass to special fibers and check that it becomes a closed immersion. But we know by [PR08, Theorem 8.4] that the right side contains the deperfected admissible locus $\mathcal{A}_{\mathcal{G},\mu}^{\mathrm{sch}}$, which is weakly normal and Frobenius split, so the map on special fibers must be a closed immersion, as the left side is reduced. □

Remark 2.5. There are two ways in which the results in [Zhu14] seem to differ from Theorem 2.1:

(1) The original [Zhu14, Theorem 3] refers to a local model defined as flat closure inside the ind-scheme $\mathrm{Gr}_{\mathcal{G}}^{\mathrm{sch}}$ when $G^{\mathrm{sc}} \to G$ has an étale kernel, which does not involve any weak normalization. During our proof of Theorem 2.1, we showed this after passing to a z-extensions, which is both necessary and sufficient to avoid certain pathological non-normal Schubert varieties discovered in [HLR18, Theorem 2.5].
(2) The finer [Zhu14, Theorem 6.10] asserts that the local model is compatibly Frobenius split with its special fiber. We stress that our methods do not yield this stronger claim, as we work over the perfection.

3. Combinatorics of admissible sets

In contrast with the introduction, we no longer assume that F is a local field, but rather a complete discretely valued field with algebraically closed residue field k of positive characteristic. Let G be a connected reductive F-group and $\mathcal{I}$ a Iwahori O-model of G.

Fix a quadruple (G, S, T, B) where S is a maximal F-split torus whose apartment contains the alcove fixed by $\mathcal{I}$, T is the maximal torus given as $Z_G(S)$, and B is some F-Borel containing S. We denote $N = N_G(S)$ and let $\tilde{W} = N(F)/\mathcal{T}(O)$ be the Iwahori–Weyl group of G. It is an extension of $W_0 = N(F)/T(F)$ by $T(F)/\mathcal{T}(O)$. We consider the Kottwitz map $T(F)/\mathcal{T}(O) \to X_*(T)_I$ to the group of inertia coinvariants of T-coweights, inducing a bijection with inverse $\bar{\nu} \mapsto t_{\bar{\nu}}$. Note that, according to the sign conventions of [BT72, BT84], the action of $t_{\bar{\nu}}$ on the apartment of S is given by $-\bar{\nu}$.

Recall that the Witt flag variety $\mathcal{F}\ell_{\mathcal{I}}$ of [Zhu17] is an ind-(perfect scheme) by [BS17] stratified in $L^+\mathcal{I}$-orbits indexed by $\tilde{W}$. The reduction of the local model $\mathcal{M}_{\mathcal{I},\mu}$ embeds in $\mathcal{F}\ell_{\mathcal{I}}$ and it was shown in [AGLR22, Theorem 6.16] that it coincides with the μ-admissible locus $\mathcal{A}_{\mathcal{I},\mu}$. This perfect $L^+\mathcal{I}$-stable subscheme is defined via the μ-admissible set $\mathrm{Adm}(\mu)$ of Kottwitz–Rapoport [KR00]: the lower poset generated by $t_{\bar{\nu}}$ with $\bar{\nu}$ running over the $\hat{T}^I$-weights $\Omega(\mu)$ of the highest weight $\hat{G}$-representation V_μ. Its maximal elements form a W_0-orbit [Hai18, Theorem C] and we denote by $\Lambda(\mu) \subset X_*(T)_I$ its image under the Kottwitz map.

Lemma 3.1 ([Hai04])**.** *Suppose $x \in \mathrm{Adm}(\mu)$ has codimension 1. Then, there are at most two distinct $\bar{\nu}_i \in \Lambda(\mu)$ such that $x \leq t_{\bar{\nu}_i}$ for $i = 1, 2$.*

Proof. We proceed as in the last paragraph of [Hai04, Proposition 8.7]. There is an affine reflection s_i, such that $x = s_i t_{\bar{\nu}_i}$ due to the codimension hypothesis. Mapping x to the group W_0 of euclidean transformations, we see that the fixed hyperplanes of the s_i must all be parallel, so the $\bar{\nu}_i$ lie in a $\mathbb{R}$-line. But they all have the same length by maximality, and a $\mathbb{R}$-line cannot cross a $\mathbb{R}$-sphere more than twice. □

The next remark was explained to us by Haines. Although it will not be needed, we leave it for the interested reader.

Remark 3.2 (Haines)**.** [Hai04, Proposition 8.7] can be adapted to describe the set $\mathrm{Irr}(x) \subset \Lambda(\mu)$ whose translations lie above x with codimension 1. Write $x = t_{\bar{\nu}} s_\beta$ for some $\bar{\nu} \in \Omega(\mu)$ and some positive root β of the échelonnage root system Σ, see [BT72, 1.4.1].

(a) If $x < s_\beta x$, then $\mathrm{Irr}(x) = \{\bar{\nu}, s_\beta(\bar{\nu})\}$.
(b) If $s_\beta x < x$, then $\mathrm{Irr}(x) = \{\bar{\nu} + \beta^\vee, s_\beta(\bar{\nu} + \beta^\vee)\}$. If μ is minuscule with respect to Σ, this does not occur.

Another important tool for us will be the S_2 property of Serre for μ-admissible loci. We say that a perfect k-scheme X perfectly of finite presentation is perfectly S_2 if it is the perfection of an S_2 finite type scheme.

Lemma 3.3. *If X is equidimensional, there is up to isomorphism a unique perfectly finite birational morphism $X^{S_2} \to X$ such that X^{S_2} is perfectly S_2 and identifies with the right side away from codimension 2. In particular, X is perfectly S_2 if and only if some (equiv. every) weakly normal finite type deperfection X_0 is S_2.*

Proof. Take a finite type reduced deperfection X_0 of X, an S_2 open subset $U_0 \subset X_0$ with complement of codimension 2 and consider the S_2-ification $X_0^{S_2} \to X_0$ given as the normalization of $U_0 \to X_0$, see [Čes21, Lemma 2.11, Corollary 2.14]. Passing to the perfection, we get the desired morphism with the stated property. Indeed, given a finite universal homeomorphism $X_1 \to X_0$ of reduced schemes such that the preimage $U_1 \to U_0$ is also S_2, the local sections of $\mathcal{O}_{U_1}$ are iterated p-th roots of those of $\mathcal{O}_{U_0}$, so the same holds for the integral closures. For the last claim, observe that X being perfectly S_2 implies $X_0^{S_2} \to X_0$ is a finite universal homeomorphism and birational, hence an isomorphism by weak normality of X_0. □

Proposition 3.4. *The following are equivalent:*

(1) *The μ-admissible locus $\mathcal{A}_{\mathcal{I},\mu}$ is perfectly S_2.*
(2) *For any $x \in \mathrm{Adm}(\mu)$, the (undirected) Bruhat graph of*

$$\mathrm{Codim}_{\leq 1}(x) = \{y \in \mathrm{Adm}(\mu) \colon x \leq y \text{ has codimension at most } 1\} \tag{3.1}$$

is connected.

Proof. Let $\mathcal{A}_{\mathcal{I},\mu}^{\mathrm{sch}}$ be the canonical deperfection in the sense of [AGLR22, Definition 3.14], which is weakly normal by construction. We know by Lemma 3.3 that this scheme is S_2 if and only if $\mathcal{A}_{\mathcal{I},\mu}$ is perfectly S_2. Consider the Hochster–Huneke graph $\mathrm{HH}(x)$ of the local ring of $\mathcal{A}_{\mathcal{I},\mu}^{\mathrm{sch}}$

at some closed point in the x-stratum, see [HH94, Definition 3.4]. Recall that the vertices of $\mathrm{HH}(x)$ are enumerated by the irreducible components of the local ring, and the edges connecting two of those by prime divisors contained in their intersection. Hence, $\mathrm{HH}(x)$ and $\mathrm{Codim}_{\leq 1}(x)$ have the same number of connected components. Now, as the irreducible components of $\mathcal{A}^{\mathrm{sch}}_{\mathcal{I},\mu}$ are unibranch by [AGLR22, Proposition 3.7], the Hochster–Huneke graph does not change under completion in our situation, so we apply [HH94, Theorem 3.6], which states that the fibers of closed points in the x-stratum of the S_2-ification of $\mathcal{A}^{\mathrm{sch}}_{\mathcal{I},\mu}$ are singletons exactly when $\mathrm{HH}(x)$ is connected. Here, we are also using the fact that S_2-ifications commute with completion, see [HH94, Proposition 3.8]. □

It would be interesting to find a purely combinatorial proof of the S_2 property of admissible sets. Instead, we apply the previous criterion twice to reduce to positive characteristic geometry.

Corollary 3.5. *The μ-admissible locus $\mathcal{A}_{\mathcal{I},\mu}$ is perfectly S_2.*

Proof. We may and do assume that G is adjoint. By Proposition 3.4, it suffices to prove an analogous combinatorial property for $\mathrm{Adm}(\mu)$. Observe that $\tilde{W}$ embeds into the group of affine transformations of the échelonnage root system Σ, its quotient W_0 identifies with vector transformations of Σ, and $\mathrm{Adm}(\mu)$ is the lower poset generated by the W_0-orbit of some translation element, thanks to [Hai18, Theorem C]. But this can be realized as $\mathrm{Adm}(\mu')$ for a split $k(\!(t)\!)$-group G' by the classification of affine Coxeter groups, see [BT72, 1.3.17, 1.4.6]. Applying again Proposition 3.4, we need to show that $\mathcal{A}_{\mathcal{I}',\mu'}$ is perfectly S_2 under the previous restrictions. In those cases, it follows from Theorem 2.1 and [HMS14, Theorem A.3] that $\mathcal{A}^{\mathrm{sch}}_{\mathcal{I}',\mu'}$ is already Cohen–Macaulay[3], hence its perfection is perfectly S_2. □

4. Comparison of nearby cycles

In this section, we compare the analytic nearby cycles of [Sch17] to the formal ones of [Gle22, Remark 4.29] under an algebraicity assumption. In the classical setting of formal schemes, our results are known by the work of Huber [Hub96], so the scheme-theoretically inclined reader can safely skip this section.

Fix a rank 1 valuation ring O and a prekimberlite X over $\mathrm{Spd}\, O$ in the sense of [Gle22, Definition 4.15]. Let $j : Y = X^{\mathrm{an}} \to X$, $i : Z^{\diamond} \to X$ be the natural inclusions, where $Z = X^{\mathrm{red}}$. Given an ℓ-torsion coefficient ring Λ with $\ell \neq p$, the first author defines in [Gle22] a naive nearby cycles functor

$$R\Psi' : D_{\text{ét}}(Y, \Lambda) \to D_{\text{ét}}(Z, \Lambda). \tag{4.1}$$

This arises as the (left-completion of the) derived pushforward of a morphism of sites $\Psi' \colon Y_{\text{ét}} \to Z_{\text{ét}}$ induced by canonical liftings of étale neighborhoods [Gle22, Theorem 4.27]. On the other hand, we have the nearby cycles functor of Scholze's theory $R\Psi : D_{\text{ét}}(Y, \Lambda) \to D_{\text{ét}}(Z^{\diamond}, \Lambda)$ given by $R\Psi = i^* Rj_*$.

In order to compare them, we can consider the fully faithful functor (Proposition 4.2):

$$D_{\text{ét}}(Z, \Lambda) \xrightarrow{c_Z^*} D_{\text{ét}}(Z^{\Diamond}, \Lambda) \xrightarrow{t_Z^*} D_{\text{ét}}(Z^{\diamond}, \Lambda) \tag{4.2}$$

where c_Z^* is the functor from [Sch17, §27] and $t_Z : Z^{\diamond} \to Z^{\Diamond}$ is the natural inclusion. By construction, $R\Psi'$ lands in the full subcategory of $D_{\text{ét}}(Z^{\diamond}, \Lambda)$ just described, while in general $R\Psi$ might not. In this section, we prove that under some conditions if $R\Psi(A) \in D_{\text{ét}}(Z, \Lambda)$ then $R\Psi(A) = t_Z^* c_Z^* R\Psi'(A)$. We do this in a series of lemmas.

[3] In this generality, this result is original due to Haines–Richarz [HR19] for $p > 2$, relying explicitly on the global Frobenius splitting of [Zhu14]. This simpler proof was found in [FHLR22].

Lemma 4.1. *Let U be a perfect scheme separated over $\mathbb{F}_p$. When U is affine denote by $t_{U^\dagger} U^\dagger \to U^\diamondsuit$ the inclusion of the closure of $U^\diamond$ in $U^\diamondsuit$. For every $B \in D_{\text{ét}}(U, \Lambda)$ we have canonical identifications $R\Gamma(U^\diamond, t_U^* c_U^* B) = R\Gamma(U^\diamondsuit, c_U^* B) = R\Gamma(U, B)$. Moreover, if U is affine $R\Gamma(U^\dagger, t_{U^\dagger}^* c_U^* B) = R\Gamma(U^\diamond, t_U^* c_U^* B)$.*

Proof. The second equality is [Sch17, Proposition 27.2]. The first equality is done following the same reduction steps in the proof loc. cit. reducing to the case where $U = \mathrm{Spec}(\mathbb{A}^I)$ for a set I. This case is settled during the proof of [Sch17, Proposition 27.2] by appealing to the invariance under change of base field [Sch17, Theorem 19.5] and comparing cohomology on affine space $\mathbb{A}^I_C$ and the unit ball $\mathbb{B}^I_C$.

If $i : U^\diamond \to U^\dagger$ is the natural inclusion, then a Postnikov tower argument, quasicompact basechange, [Sch17, Proposition 17.6], and that $c_U^* B$ is overconvergent imply that $Ri_* t_{U^\diamond}^* c_U^* B = t_{U^\dagger}^* c_U^* B$. This proves the second claim. □

Proposition 4.2. *Let U be a perfect scheme separated over $\mathbb{F}_p$, the functor $t_U^* c_U^*$ is fully faithful.*

Proof. We may assume $A \in D^+_{\text{ét}}(U, \Lambda)$, since full faithfulness extends formally to left-completions. By descent, we may also assume U is defined over $\mathrm{Spec}(\overline{\mathbb{F}}_p)$. For $A \in D_{\text{ét}}(U, \Lambda)$ we verify $A \to Rc_{U,*} Rt_{U,*} t_U^* c_U^* A$ is an isomorphism by checking this on sections. Let $B = c_U^* A$, we also denote by B the evident pullback to different loci. By Lemma 4.1 one reduces to proving $R\Gamma(Q, B) = R\Gamma(V^\diamond, B)$ for sufficiently small étale neighborhoods $V \to U$, for $Q = U^\diamond \times_{U^\diamondsuit} V^\diamondsuit$. Let $W \subseteq U$ be an open subset and let $V_W = V \times_U W$, observe that $W^\diamond \times_{U^\diamond} Q = W^\diamond \times_{W^\diamondsuit} V_W^\diamondsuit$. Applying descent to an open cover $\coprod W \to U$ and shrinking V we may assume that U and V are affine. Let $\overline{V}$ be the closure of $V^\diamond$ in Q, arguing as in Lemma 4.1 we have $R\Gamma(V^\diamond, B) = R\Gamma(\overline{V}, B)$ so it suffices to see that $R\Gamma(Q, j_! B) = 0$ for the open immersion $j : Q \setminus \overline{V} \to Q$. By Noetherian approximation $V \to U$ arises as the base change of an étale map $S \to T$ with S and T the spectrum of finite type $\overline{\mathbb{F}}_p$-algebras. All spaces above come from basechange by the map $\pi : U^\diamond \to T^\diamondsuit$ which is qcqs. Indeed, Q corresponds to $S^\diamondsuit$, $V^\diamond$ corresponds to $S^\diamond$ and $\overline{V}$ corresponds to $S^\dagger$. To see the last identification recall that $U^\dagger \times_{T^\dagger} S^\dagger = V^\dagger$ as it is evident from the moduli description [Gle22, Proposition 2.22], and that $V^\diamond$ is always dense in $V^\dagger$ [Gle22, Proposition 2.24]. By [Sch17, Proposition 17.6], it suffices to prove $R\Gamma(S^\diamondsuit, Rs_! s^* R\pi_* B) = 0$ where $s : S^\diamondsuit \setminus S^\dagger \to S^\diamondsuit$ is the natural inclusion. We claim $R\Gamma(S^\diamondsuit, Rs_! K) = 0$ for any K. Finding a closed immersion $S \to \mathbb{A}^N_{\overline{\mathbb{F}}_p}$, it suffices to prove this if $S = \mathbb{A}^N_{\overline{\mathbb{F}}_p}$. This follows from [FS21, Theorem IV.5.3], since $S^\diamondsuit \setminus S^\dagger$ is a spatial diamond partially proper over $\mathrm{Spd}(\overline{\mathbb{F}}_p)$. Indeed, it is the pointed formal completion of the divisor at infinity in $\mathbb{P}^N_{\overline{\mathbb{F}}_p}$. □

Lemma 4.3. *Let $A \in D_{\text{ét}}(Y, \Lambda)$. Then $R\Gamma(X, Rj_! A) = 0$, or equivalently $R\Gamma(Y, A) = R\Gamma(Z, R\Psi(A))$.*

Proof. Since $j_!$ commutes with canonical truncations a Postnikov limit argument allows us to assume $A \in D^+_{\text{ét}}(Y, \Lambda)$. Since X is a specializing v-sheaf there is a hypercover of $X_\bullet \to X$ where each X_i is of the form $\coprod_{j \in I_i} \mathrm{Spd}(R_j^+)$ for I_j a set and the $\mathrm{Spa}(R_j, R_j^+)$ are strictly totally disconnected perfectoid spaces. By v-hyperdescent [Sch17, Proposition 17.3] and proper basechange we may assume $X = \mathrm{Spd}(R^+)$. At this point we may cite [FS21, Remark V.4.3]. Let us explain a detail. A choice of pseudouniformizer defines a qcqs map $\mathrm{Spd}(R^+) \to \mathrm{Spd}(W(k)[\![t]\!])$ and [Sch17, Proposition 17.6] reduces the computation to the case $X = \mathrm{Spd}(W(k)[\![t]\!])$ which follows from [FS21, Theorem IV.5.3]. □

We also have a map of sites $f : X_v \to Z_{\text{ét}}$ induced again by canonical liftings of étale neighborhoods. Restricted to $Z_v^\diamond$, $f \circ i_v$ sends an étale map $U \to Z$ to $U^\diamond \to Z^\diamond$. In the case of Y_v, $f \circ j_v$ factors as the composition of $\nu_Y : Y_v \to Y_{\text{ét}}$ with $\Psi' : Y_{\text{ét}} \to Z_{\text{ét}}$. We get a functor $Rf_{v,*} : D(X_v, \Lambda) \to D_{\text{ét}}(Z, \Lambda)$ that factors through $D_{\text{ét}}(X, \Lambda)$, since its right adjoint factors through the inclusion $D_{\text{ét}}(X,\Lambda) \subseteq D(X_v,\Lambda)$. We denote by Rf_* the induced map on this latter category.

Lemma 4.4. *Let $A \in D_{\text{ét}}(Y,\Lambda)$. Then $R(f\circ i)_* R\Psi(A) = R\Psi' A$.*

Proof. We have an exact triangle $Rj_!A \to Rj_*A \to i_*R\Psi(A)$ in $D_{\text{ét}}(X,\Lambda)$, to which we apply Rf_*. Now, by [Sch17, Proposition 14.10, Proposition 14.11] $Rf_*Rj_* = R(f_v\circ j_v)_*\nu_Y^* = R\Psi'\nu_{Y,*}\nu_Y^* = R\Psi'$. It suffices to prove $Rf_*j_!A = 0$. It suffices to prove $R\Gamma(U, Rf_*j_!A) = 0$ for all $U \in Z_{\text{ét}}$. We compute directly $R\Gamma(U, Rf_*j_!A) = R\Gamma(\widehat{X}_U, j_!A) = 0$. The last equality follows from Lemma 4.3, and the fact that passing to étale formal neighborhoods preserves being a prekimberlite. □

Lemma 4.5. *We have canonical identifications $R(f\circ i)_* t_Z^* c_Z^* A = Rc_{Z,*}c_Z^*A = A$.*

Proof. The second one is [Sch17, Proposition 27.2]. Let $h\colon U \to Z$ any étale neighborhood and let $t_U : U^\diamond \to U^{\diamondsuit}$ denote the natural map. We compute as follows:

$$\begin{aligned} R\Gamma(U, h^*Rc_{Z,*}c_Z^*A) &= R\Gamma(U^{\diamondsuit}, h^{\diamondsuit,*}c_Z^*A) \\ &= R\Gamma(U^{\diamondsuit}, c_U^*h^*A) \\ &= R\Gamma(U^\diamond, t_U^*h^{\diamondsuit,*}c_Z^*A) \\ &= R\Gamma(U^\diamond, h^{\diamond,*}t_Z^*c_Z^*A) \\ &= R\Gamma(U, h^*R(f\circ i)_*t^*c_Z^*A), \end{aligned} \tag{4.3}$$

where we have applied [Sch17, Proposition 27.1] twice to commute c^* and h^*, and also Lemma 4.1 in the middle equality. □

From now on assume Y is a spatial diamond that has finite cohomological dimension as in [Sch17, Proposition 20.10] so that $D(Y_{\text{ét}},\Lambda) = D_{\text{ét}}(Y,\Lambda)$. We also assume that Z is perfectly of finite type over the residue field of O so that $D(Z_{\text{ét}},\Lambda) = D_{\text{ét}}(Z,\Lambda)$. With these hypothesis $R\Psi' : D_{\text{ét}}(Y,\Lambda) \to D_{\text{ét}}(Z,\Lambda)$ is defined site theoretically (without having to left-complete) and taking stalks is well-behaved.

Definition 4.6. Identify $D_{\text{ét}}(Z,\Lambda)$ with its essential image in $D_{\text{ét}}(Z^\diamond,\Lambda)$ under $t_Z^*c_Z^*$. We say $K \in D_{\text{ét}}(Z^\diamond,\Lambda)$ is algebraic if $K \in D_{\text{ét}}(Z,\Lambda)$.

Recall that for a closed point $\overline{x} \in Z$, we denote by $\widehat{X}_{/\overline{x}}$ the formal neighborhood at $\overline{x}$ in the sense of [Gle22, Definition 4.18] and by $X^{\circledcirc}_{/\overline{x}}$ the tubular neighborhood in the sense of [Gle22, Definition 4.38].

Theorem 4.7. *Let the notation be as above and $A \in D_{\text{ét}}(Y)$. The following hold:*

1. *If $\overline{x} \in Z$ is a closed point, then $(R\Psi A)_{\overline{x}} = R\Gamma(X^{\circledcirc}_{/\overline{x}}, A)$.*
2. *$(R\Psi' A)_{\overline{x}} = \varinjlim_{\overline{x}\in U} R\Gamma(X^{\circledcirc}_{/U}, A)$.*
3. *If $R\Psi A$ is algebraic then $R\Psi A = R\Psi' A$ and $R\Gamma(X^{\circledcirc}_{/\overline{x}}, A) = \varinjlim_{\overline{x}\in U} R\Gamma(X^{\circledcirc}_{/U}, A)$.*

Proof. The third claim follows directly from Lemma 4.5 and Lemma 4.4, and from the second and first claim. Let $\iota_{\overline{x}}$ denote the inclusion of $\overline{x}$. Notice that $\iota_{\overline{x}}$ factors through the open immersion $\widehat{X}_{/\overline{x}} \to X$. By smooth base change and Lemma 4.3 applied to $\widehat{X}_{/\overline{x}} \to X$ we get $(R\Psi A)_{\overline{x}} = R\Gamma(X^{\circledcirc}_{/\overline{x}}, A)$, proving the first claim. Finally, by [Sch17, Proposition 27.1.(ii)] $\iota_{\overline{x}}^{\diamondsuit,*}c_Z^*(R\Psi' A) = c_{\overline{x}}^*(R\Psi' A_{\overline{x}})$ and the latter is by definition computed by $\varinjlim_{\overline{x}\in U} R\Gamma(X^{\circledcirc}_{/U}, A)$, since it is site theoretic. □

5. Proof of unibranchness

In this section, we prove Theorem 1.3. During this section, we set $\Lambda = \mathbb{F}_\ell$ and consider $\mathcal{M}_{\mathcal{G},\mu}$ already after base change to $\operatorname{Spd} O_C$, where C is a complete algebraic closure of F with ring of integers O_C. The reader who is only interested in the scheme-theoretic local models can imagine this takes place in the realm of formal and rigid-analytic geometry.

The object $\mathcal{Z}_\mu = R\Psi(\mathrm{IC}_\mu)$ allows us to read off the set of connected components of tubes.

Lemma 5.1. *There is an equality* $\#\pi_0(\mathcal{M}^{\circledcirc}_{\mathcal{G},\mu/x}) = \dim_{\mathbb{F}_\ell} \mathcal{H}_x^{-\langle 2\rho,\mu\rangle} \mathcal{Z}_\mu$.

Proof. Let $j_\mu \colon \mathrm{Gr}^\circ_{G,\mu} \to \mathrm{Gr}_G$ denote the orbit inclusion in the generic fiber. It follows from perverse left t-exactness of $Rj_{\mu*}$ and Schubert varieties being unibranch that

$$\mathcal{H}^{-\langle 2\rho,\mu\rangle}(\mathrm{IC}_\mu) = \mathcal{H}^0(Rj_{\mu*}\mathbb{F}_\ell) = \mathbb{F}_\ell \tag{5.1}$$

is a constant sheaf on the generic fiber. (Strictly speaking, the B^+_{dR}-affine Grassmannian is not an ind-scheme, but facts such as these can be reduced via [FS21, Corollary VI.6.7] to the Witt affine Grassmannian of the split form of G, which is an ind-perfect scheme.) This implies that we may replace the right side of the claimed equality by $\dim_{\mathbb{F}_\ell} i_x^* R^0 j_* \mathbb{F}_\ell$, with $j \colon \mathrm{Gr}_G \to \mathrm{Gr}_{\mathcal{G}}$ the generic fiber inclusion and $i_x \colon \operatorname{Spd} k \to \mathrm{Gr}_{\mathcal{G}}$ the inclusion of the point x. But the latter equals $\#\pi_0(\mathcal{M}^{\circledcirc}_{\mathcal{G},\mu/x})$ by Theorem 4.7. □

Thanks to [Gle22, Lemma 5.26], we can reduce the proof of Theorem 1.3 to the case when $\mathcal{G}$ is Iwahori, provided we verify the following.

Lemma 5.2. *If $\mathcal{I} \to \mathcal{G}$ is a Iwahori dilation, then the geometric fibers of $\pi \colon \mathcal{A}_{\mathcal{I},\mu} \to \mathcal{A}_{\mathcal{G},\mu}$ are connected.*

Proof. By $\mathcal{I}$-equivariance, we are reduced to considering the fiber over the image $w_{\mathcal{G}}$ of a $L^+\mathcal{T}$-fixed point $w \in \mathcal{A}_{\mathcal{I},\mu}$. We may and do assume that w is minimal in its right $W_{\mathcal{G}}$-coset. Using Demazure resolutions, one sees that the intersection of $\pi^{-1}(w_{\mathcal{G}})$ with any Schubert variety is connected. As all of those subschemes must contain w by minimality, the fiber itself must be connected. □

From now on, we work with a Iwahori model $\mathcal{I}$. To calculate in codimension 1, we are going to apply the Wakimoto filtration of the Gaitsgory central functor studied in [Gai01, AB09] in equicharacteristic and in [AGLR22, ALWY22] in mixed characteristic.

Theorem 5.3 ([ALWY22])**.** *The functor $R\Psi$ is perverse t-exact. Moreover, the perverse sheaf $R\Psi(\mathrm{Sat}(V))$ admits a filtration with subquotients isomorphic to $\mathcal{J}_{\bar\nu} \otimes V(w_0\bar\nu)$.*

The Wakimoto sheaves $\mathcal{J}_{\bar\nu}$ depend crucially on the choice of a Borel subgroup $B \subset G$. For the images $\bar\nu$ of B-dominant coweights, $\mathcal{J}_{\bar\nu}$ are defined as the costandard object $\nabla_{\bar\nu} = Rj_{\bar\nu,*}\mathbb{F}_\ell[\ell(t_{\bar\nu})]$, which admits the standard object $\Delta_{\bar\nu} = j_{\bar\nu,!}\mathbb{F}_\ell[\ell(t_{\bar\nu})]$ as a left and right inverse for convolution. Here $j_{\bar\nu} : \mathcal{F}\ell^\circ_{\mathcal{I},t_{\bar\nu}} \to \mathcal{F}\ell_{\mathcal{I}}$ is the natural orbit inclusion, which is affine, so the derived functors are perverse t-exact by [BBDG18, Corollaire 5.1.3]. The definition extends by convolution and linearity to the other $\bar\nu$. Just as in [AB09, Theorem 5], it turns out that $\mathcal{J}_{\bar\nu}$ is a perverse sheaf concentrated in $\mathcal{F}\ell_{\mathcal{I},t_{\bar\nu}}$ and restricts to $\mathbb{F}_\ell[\ell(t_{\bar\nu})]$ on $\mathcal{F}\ell^\circ_{\mathcal{I},t_{\bar\nu}}$. The proof of Theorem 5.3 in [ALWY22] relies not only on the $\mathcal{J}_{\bar\nu}$, but also on $R\Psi(\mathrm{Sat}(V))$ being central, see [AGLR22, Proposition 6.17], and the computation of its constant terms, see [AGLR22, Equation (6.32)].

Lemma 5.4. *For any $x \in \mathcal{F}\ell_{\mathcal{I}}(k)$, the $\mathbb{F}_\ell$-vector space $\mathcal{H}_x^{-\ell(t_{\bar\nu})}\mathcal{J}_{\bar\nu}$ is either zero or one-dimensional.*

Proof. Consider the adjunction map $\mathcal{J}_{\bar{\nu}} \to \nabla_{\bar{\nu}}$. It follows that the kernel and cokernel have support contained in $\mathcal{F}\ell_{\mathcal{I},t_{\bar{\nu}}} \setminus \mathcal{F}\ell^{\circ}_{\mathcal{I},t_{\bar{\nu}}}$. In particular, they are concentrated on degrees strictly larger than $-\ell(t_{\bar{\nu}})$ by perversity, so that

$$\mathcal{H}^{-\ell(t_{\bar{\nu}})}_x \ker(\mathcal{J}_{\bar{\nu}} \to \nabla_{\bar{\nu}}) = \mathcal{H}^{-\ell(t_{\bar{\nu}})}_x \operatorname{coker}(\mathcal{J}_{\bar{\nu}} \to \nabla_{\bar{\nu}}) = 0. \tag{5.2}$$

Taking the long exact sequence of cohomology, this yields an inclusion $\mathcal{H}^{-\ell(t_{\bar{\nu}})}_x \mathcal{J}_{\bar{\nu}} \subset \mathcal{H}^{-\ell(t_{\bar{\nu}})}_x \nabla_{\bar{\nu}}$. The latter sheaf is constant equal to $\mathbb{F}_\ell$, because the Schubert perfect variety $\mathcal{F}\ell_{\mathcal{I},t_{\bar{\nu}}}$ is normal. □

Proposition 5.5. *Given $x \in \mathcal{A}_{\mathcal{I},\mu}(k)$ whose $L^+\mathcal{I}$-orbit has codimension at most 1, we have an equality $\mathcal{H}^{-\langle 2\rho,\mu\rangle}_x \mathcal{Z}_\mu = \mathbb{F}_\ell$.*

Proof. This is evident in codimension 0. By abuse of notation, we denote by x the corresponding element in $\operatorname{Adm}(\mu)$ and let $x < t_{\bar{\nu}_i}$ with $\bar{\nu}_i \in \Lambda(\mu)$ with $i = 1, 2$ be the only (possibly equal) maximal elements above x by Lemma 3.1. Assume without loss of generality that $\bar{\nu}_1 \in X_*(T)^-_I$. Consider the Wakimoto filtration of the perverse sheaf $\mathcal{Z}_\mu$ given in Theorem 5.3. Only $\mathcal{J}_{\bar{\nu}_i} \otimes V(w_0\bar{\nu}_i)$ contribute to the stalk at x. Since we chose $\mathcal{J}_{\bar{\nu}_1}$ to be standard it does not contribute to the stalk at x. Consequently, $\mathcal{H}^{-\langle 2\rho,\mu\rangle}_x \mathcal{Z}_\mu = V(w_0\bar{\nu}_2) \otimes \mathcal{H}^{-\langle 2\rho,\mu\rangle}_x (\mathcal{J}_{\bar{\nu}_2})$. As $V_\mu(w_0\bar{\nu}_2) = \mathbb{F}_\ell$ for extremal weights, Lemma 5.4 bounds the dimension above by 1. On the other hand, Lemma 5.1 and density of the generic fiber bound the dimension below by 1. (This also implies $\bar{\nu}_1 \neq \bar{\nu}_2$.) □

We have now all the necessary tools at our disposal to finish the proof of the main theorem.

Proof of Theorem 1.3. Assume there is $x \in \mathcal{A}_{\mathcal{I},\mu}(k)$ such that $\mathcal{M}^{\circledcirc}_{\mathcal{I},\mu/x}$ is disconnected. By Theorem 4.7 there is a connected étale neighborhood $x \in U \to \mathcal{A}_{\mathcal{I},\mu}$ such that

$$\widehat{\mathcal{M}}^{\mathrm{an}}_{\mathcal{I},\mu/U} = V_1 \sqcup V_2 \tag{5.3}$$

is a union of two non-empty clopen subsets. Let $W \subset U$ be the open subset obtained by pulling back the union of $L^+\mathcal{I}$-orbits of $\mathcal{A}_{\mathcal{I},\mu}$ of codimension at most 1. This is connected by Corollary 3.5, as the S_2 property is stable under étale maps. Apply Lemma 5.1, Proposition 5.5 and [Gle22, Lemma 4.55] to conclude that $\widehat{\mathcal{M}}^{\mathrm{an}}_{\mathcal{I},\mu/W}$ is connected as well, so $\operatorname{sp}^{-1}(W)$ is entirely contained in V_1, say.

On the other hand, the disjoint union induces a direct sum decomposition $\mathcal{Z}_\mu|_U = A_1 \oplus A_2$ into non-zero perverse sheaves, where the A_i are the nearby cycles of IC_μ after pulling it back to V_i. We know by construction that $\mathcal{H}^{-\langle 2\rho,\mu\rangle} A_2$ does not vanish, and also that the support of A_2 has codimension at least 2 in U by the previous paragraph. These two facts contradict A_2 being perverse, so our initial assumption was wrong. □

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