ON THE DIMENSION OF SOME LINEAR SPACES OVER $\mathbb{Q}$

Part 3.

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Deeds of the very distant days...
A.S.Pushkin
(Tanslated from Russian by L.A.G.)

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§1 Foreword

In sections 1 I introduce and study my initial auxiliary functions. I follow there [23].

In [22] was proved the followimg Theorem . The $\mathbb{Z}-$ module with generators

$$f_1 = \begin{pmatrix} \log 4 \\ \zeta(2) \end{pmatrix}, f_2 = \begin{pmatrix} \zeta(2) \\ 3\zeta(3) \end{pmatrix}, f_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, f_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

has these four generators as free generators, it is not a module Liouville and its mesure of nondiskretness is not bigger than

$$\gamma = \frac{\log(\rho_2)}{\log(\rho_1)},$$

where

$$\rho_1^{\sim} = \left(5 + 4\sqrt{2} + 2\sqrt{2\sqrt{2} - 2} + 4\sqrt{2\sqrt{2} + 2}\right)\exp(-3), \rho_2^{\sim} = \rho_1 \exp(6).$$

The proof published in [22] was not the initial proof. In the section 2 I am happy to present to the mathematical public the initial proof of this result . I do this because in [34] - [38] I prove a similar result, which proof is development of the mentioned initial proof. The situation in [22] is much simpler than in [34]-[38].

Therefore, I hope, the reading of the initial proof of the result considered in [22] will be helpful for more easier understanding of the papers [34]-[38].

In the section 3 I list the some key moments of the proof presented in the section 2.

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§ 1. My initial auxiliary functions.

In this section, I use the general functions of C.S.Mejer, which are defined as follows ([20], ch. 5). In what follows, we denote by $\mathfrak{F}$ the Riemann surface of the function $Log(z)$ and identify it with the direct product of the multiplicative group $\mathbb{R}_+^* = \{r \in \mathbb{R} |\ r > 0\}$ of positive real numbers with the operation $\times$, not to be written down explicitly as usual, and the additive group $\mathbb{R}$ of all the real numbers, so that

$$z_1 z_2 = (r_1 r_2, \phi_1 + \phi_2)$$

for any two points $z_1 = (r_1, \phi_1)$ and $z_2 = (r_2, \phi_2)$ on this Riemann surface. Let $\theta_0(z) = re^{i\phi}$ for every $z = (r, \phi) \in \mathbb{R}_+^* \times \mathbb{R}$. It is clear that θ_0 is a surjective homomorphism of $\mathbb{R}_+^* \times \mathbb{R}$ on the multiplicative group of the field $\mathbb{C}$. For each $z = (r, \phi) \in \mathfrak{F}$, we denote r by $|z|$, ϕ by $\mathrm{Arg}\,(z)$ and the complex number number $r + i\phi$ by $\mathrm{Log}\,(z)$. Clearly, $\mathrm{Log}\,(z_1 z_2 = \mathrm{Log}\,(z_1) + \mathrm{Log}\,(z_2)$ for any $z_1 \in \mathfrak{F}$ and $z_2 \in \mathfrak{F}$. Clearly, $\mathfrak{F}$ is a metric space relatively to the distance $\rho(z_1, z_2) = |Log(z_1) - Log(z_2)|$. Clearly, $\theta_0(z) = \exp(\mathrm{Log}\,(z))$ for any $z \in F$. Clearly,

$$|\theta_0(z_1) - \theta_0(z_2)| = |\exp(\mathrm{Log}\,(z_1)) - \exp(\mathrm{Log}\,(z_2))| =$$

$$|\exp(\mathrm{Log}\,(z_2)||\exp(\mathrm{Log}\,(z_1) - \mathrm{Log}\,(z_2)) - 1| \leq$$

$$|z_2||\exp(|\mathrm{Log}\,(z_1) - \mathrm{Log}\,(z_2)|) - 1| =$$

$$|z_2|(\exp(\rho(z_1, z_2)) - 1).$$

Consequently,

$$|\theta_0(z_1) - \theta_0(z_2)| \leq (\exp(\rho(z_1, z_2)) - 1) \min(|z_1|, |z_2|). \tag{3.1.1}$$

Terefore the map θ_0 is continuous. For each $z=(r,\phi)\in\mathfrak{F}$, let $\theta_1(z)=(r,\phi-\pi)\in\mathfrak{F}$. So, the map $z\to\theta_1(z)$ is the bijection of $\mathfrak{F}$ onto $\mathfrak{F}$. Clearly,

$$\theta_0((\theta_1)^m(z))=(-1)^m\theta_0(z)$$

for each $z=(r,\phi)\in\mathfrak{F}$ and $m\in\mathbb{Z}$. Let us fix a domain D in $\mathfrak{F}$. For a complex-valued function $f(z)$ on D, let $f(z)=f^\wedge(r,\phi)$. It is well known that $f(z)$ is holomorphic in D if the complex-valued function $f^\wedge(r,\phi)$ of two real variables r and ϕ has continuous partial derivatives in D and, for every point $z=(r,\phi)\in D$, the Cauchy-Riemann conditions

$$r\left(\left(\frac{\partial}{\partial r}f^\wedge\right)(r,\phi)\right)=-i\left(\left(\frac{\partial}{\partial\phi}f^\wedge\right)(r,\phi)\right):= \tag{3.1.2}$$

$$(\delta f)(z):=\theta_0(z)\left(\left(\frac{\partial}{\partial z}f\right)(z)\right)$$

are satisfied. The equalities (3.1.1) deternine a differentiations δ and $\frac{\partial}{\partial z}$ on the ring of all the holomorphic in D functions. In particular, the function $\operatorname{Log}(z)$ is holomorphic on $\mathfrak{F}$ and

$$\left(\frac{\partial}{\partial z}\operatorname{Log}\right)(z)=\theta_0(z^{-1}),(\delta\operatorname{Log})(z)=1. \tag{3.1.3}$$

Let

$$G_{p,q}^{(m,n)}\left(z\middle|\begin{matrix}a_1, & \dots & a_p\\ a_1, & \dots & a_p\end{matrix}\right)=\frac{1}{2\pi i}\int\limits_L g(s)ds, \tag{3.1.4}$$

where $z\in\mathfrak{F}$,

$$g=g(s)=g(z,s)=g_{p,q}^{(m,n)}\left(z\middle|\begin{matrix}a_1, & \dots & a_p\\ a_1, & \dots & a_p\end{matrix}\right)(s)=$$

$$\frac{\exp(s\operatorname{Log}(z))\prod\limits_{j=1}^{m}\Gamma(b_j-s)\prod\limits_{j=1}^{n}\Gamma(1-a_j+s)}{\prod\limits_{j=1+m}^{q}\Gamma(1-b_j+s)\prod\limits_{j=n+1}^{p}\Gamma(a_j-s)}, \tag{3.1.5}$$

an empty product is, by definition, equal to 1, $0\le m\le q$, $0\le n\le p$; the parameters $a_j\in\mathbb{C}$, $j=1,\dots,p$, and $b_k\in\mathbb{C}$, $k=1,\dots,q$, are chosen in such a way that, if $P_r(g)$ denotes the set of all poles of the functions $\Gamma(b_k - s),\ k = 1, \dots, m$, and $P_l(g)$ denotes the set of all poles of the functions $\Gamma(1 - a_j + s),\ j = 1, \dots, n$, then $P_l(g) \cap P_r(g) = \oslash$. There are 3 possibilities to choose the curve L.

(A) First, the curve $L = L_0$ may be chosen to pass from $-i\infty$ to $+i\infty$ in such a way that the set $P_r(g)$ lies to the right of it and the set $P_l(g)$ lies to the left of it. The integral (3.1.4) is convergent in either of the following two cases:

$(A1)$ $|\arg(z)| < (m + n - p/2 - q/2)\pi$;

$(A2)$ $|\arg(z)| \leq (m + n - p/2 - q/2)\pi$ and $(p - q)/2 + \mathrm{Re}\,(\Delta^*(g)) < -1$, where

$$\Delta^*(g) = \sum_{k=1}^{q} b_k - \sum_{j=1}^{p} a_j. \tag{3.1.6}$$

(B) Second, the curve $L = L_1$ may be chosen to pass from $+i\infty$ to $+i\infty$, in such a way that the set $P_r(g)$ lies to the right of it and the set $P_l(g)$ lies to the left of it. The integral (3.1.4) is convergent in each of the following three cases:

$(B1)$ $p < q$;

$(B2)$ $1 \leq p \leq q$ and $|z| < 1$;

$(B3)$ $1 \leq p \leq q, |z| \leq 1$ and $\mathrm{Re}\,(\Delta^*(g)) < -1$.

(C) Third, the curve $L = L_2$ may be chosen to pass from $-\infty$ to $-\infty$, in such a way that the set $P_r(g)$ lies to the right of it and the set $P_l(g)$ lies to the left of it. The integral (3.1.4) is convergent in each of the following 3 cases:

$(C1)$ $q < p$;

$(C2)$ $1 \leq q \leq p$ and $|z| > 1$;

$(C3)$ $1 \leq q \leq p, |z| \geq 1$ and $\mathrm{Re}\,(\Delta^*(g)) < -1$.

If both conditions (A) and (B) are satisfied, then each of these conditions leads to the same result. If both conditions (A) and (C) are satisfied, then each of these conditions leads to the same result.

Let G denote the integral (3.1.4) with $L = L_k$, where $k = 1, 2$, and let S_k be the set of all the unremovable singularities of g encircled by L_k. If for $k = 1$ one of conditions (B) is satisfied or for $k = 2$ one of conditions (C) is satisfied, then

$$G = (-1)^k \sum_{s \in S_k} \mathrm{Res}\,(g; s), \tag{3.1.7}$$

Remark 3.1.1. In each of the three cases (A), (B) and (C) the curve of integration L is chosen in such a way that the set $P_r(g)$ lies to the right of it and the set $P_l(g)$ lies to the left of it. The proof of the all these assertions may be foundd in [34].

I shall work with the set

$$\Omega^{\wedge} = \{z \in \mathfrak{F}\colon |z| \geq 1$$

and its subset

$$\Omega^{\vee} = \{z \in \mathfrak{F}\colon |z| > 1\}.$$

I shall work with also with the auxiliary set $\Omega^{(h\wedge)} = \{z \in \mathfrak{F}\colon |z| \le 1\}$ and its subset

$$\Omega^{(h\vee)} = \{z \in \mathfrak{F}\colon |z| < 1\}.$$

Clearly,

$$\log(\theta_1(z)) = \log(-z) = \log z - (-1)^k i\pi, \tag{3.1.8}$$

if $z \in \mathfrak{F}$ and $k \in \mathbb{Z}$. Let $m \in \mathbb{N}, n \in \mathbb{N}, \nu \in \mathbb{N}, d^\vee \in \mathbb{N}, d^\wedge \in \mathbb{N}, \beta^\vee \in \mathbb{N}-1, \beta^\wedge \in \mathbb{N}-1$,

$$c_1(\nu) = d^\vee\nu + \beta^\vee, c_2(\nu) = d^\wedge\nu + \beta^\wedge, 2 \le (\beta^\wedge + 1)m - n\beta^\vee, nd^\vee \le md^\wedge \tag{3.1.9}$$

where $\nu \in \mathbb{N} - 1$. In view of (3.1.9),

$$2 \le (c_2\nu + 1)m - c_1(\nu)n \tag{3.1.10}$$

where $\nu \in \mathbb{N} - 1$.

To introduce the first of my auxiliary function $f_1(z,\nu)$, I use the auxiliary set $\Omega^{(h\wedge)}$. I shall prove that, for each $\nu \in \mathbb{N}$, the function $f_1(z,\nu)$ belongs to $\mathbb{Q}[\theta_0(z)]$; therefore using the principle of analytic continuation we may regard it as being defined in $\mathfrak{F}$, and consequently, in $\Omega^\wedge$. Let

$$f_1(z,\nu) = (-1)^{nc_1(\nu)}\frac{((c_2(\nu))!)^m}{(c_1(\nu)!)^n}\times \tag{3.1.11}$$

$$G^{(1,n)}_{m+n,m+n}\left((\theta_1)^{m+1}(z)\middle| \begin{matrix} \overbrace{-c_1(\nu) \quad \dots \quad -c_1(\nu)}^{n \text{ times}} & \overbrace{c_2(\nu)+1 \quad \dots \quad c_2(\nu)+1}^{m \text{ times}} \\ 0 \quad \dots \quad 0 & 0 \quad \dots \quad 0 \end{matrix}\right) =$$

$$\frac{((c_2(\nu))!)^m}{(c_1(\nu)!)^n}\frac{(-1)^{nc_1(\nu)}}{2\pi i}\int\limits_{L_1} g^{(1,n)}_{m+n,m+n}(s)ds,$$

where $z \in \Omega^{(h\wedge)}$, $\nu \in \mathbb{N}$,

$$g^{(1,n)}_{m+n,m+n} = g^{(1,n)}_{m+n,m+n}(s) =$$

$$g^{(1,n)}_{m+n,m+n}\left((\theta_1)^{m+1}(z)\middle| \begin{matrix} \overbrace{-c_1(\nu) \quad \dots \quad -c_1(\nu)}^{n \text{ times}} & \overbrace{c_2(\nu)+1 \quad \dots \quad c_2(\nu)+1}^{m \text{ times}} \\ 0 \quad \dots \quad 0 & 0 \quad \dots \quad 0 \end{matrix}\middle| s\right) =$$

$$(\exp(s \operatorname{Log}((\theta_1^) m + 1(z))\Gamma(-s)(\Gamma(1+s))^{-m-n+1} \times$$
$$(\Gamma(1+c_1(\nu)+s))^n(\Gamma(c_2(\nu)+1-s))^{-m},$$

and the curve L_1 passes from $+\infty$ to $+\infty$ in the negative direction in such a way, that the set $\mathbb{N}-1$ lies to the right from it but the set $-\mathbb{N}$ lies to the left from it. Here $m+n$ plays the role of $p=q$ in (3.1.4), the number 1 plays the role of m, the number n in (3.1.4) plays the role of n,

$$a_k = -c_1(\nu)$$

where $k=1, \dots, n$,

$$a_k = c_2(\nu)$$

where $k = n+1, \dots, n+m$,

$$b_k = 0,$$

where $k=1, \dots, n+m$. In view of (3.1.10) and (3.1.6),

$$\Delta^* = \sum_{k=1}^{q} b_k - \sum_{j=1}^{p} a_j = nc_1(\nu) - mc_2(\nu) - n \leq -2 < -1$$

and the condition (B3) of convergence of the integral (3.1.9) is fulfilled; I use (3.1.7) to compute this integral. The set of all unremovable singular points of the function $g^{(1,n)}_{m+n,m+n}(s)$ encircled by the curve L_1, consists of the points $s=0, \dots, c_2(\nu)$, each of these points is a pole of the first order. Therefore

$$\operatorname{Res}(g^{(1,n)}_{m+n,m+n}; k) = \lim_{s\to k}((s+k)g^{(1,n)}_{m+n,m+n}(s)),$$

where $k=0, \dots, d^\wedge\nu + \beta^\wedge$. Let

$$s = k+u, H_1(u,k,\nu) = g^{(1,n)}_{m+n,m+n}(k+u) =$$

$$\exp((k+u)\operatorname{Log}(\theta^{m+1}(z))\Gamma(-k-u)(\Gamma(1+k+u))^{-m-n+1} \times$$
$$(\Gamma(1+c_1(\nu)+k+u))^n(\Gamma(c_2(\nu)+1-k-u))^{-m} =$$
$$\prod_{\kappa=0}^{k}(-k+\kappa-u)^{-1} \times$$
$$\exp((k+u)\operatorname{Log}(\theta_1^m(z))\Gamma(1-u)(\Gamma(1+k+u))^{-m-n+1} \times$$
$$(\Gamma(1+c_1(\nu)+k+u))^n(\Gamma(1+c_2(\nu)-k-u))^{-m},$$

where $k = 0, \dots, c_2(\nu)$ and $\nu \in \mathbb{N} - 1$. Therefore

$$\operatorname{Res}(g^{(1,n)}_{m+n,m+n}; k) =$$

$$\lim_{u \to 0}(uH_1(u,k,\nu)) =$$

$$-(-1)^{mk}(k!)(\theta_0(z))^k(k!)^{-m-n}((c_1(\nu)+k)!)^n((c_2(\nu)-k)!)^{-m} =$$

$$-\left(\frac{((c_2(\nu))!)^m}{(c_1(\nu)!)^n}\right)^{-1}(-1)^{mk}(\theta_0(z))^k\binom{c_1(\nu)+k}{k}^n\binom{c_2(\nu)}{k}^m.$$

Consequently, the function $f_1(z,\nu)$ is equal to a finite sum

$$f_1(z,\nu) = \sum_{k=0}^{c_2(\nu)}(-1)^{nc_1(\nu)+mk}\binom{c_1(\nu)+k}{k}^n\binom{c_2(\nu)}{k}^m(\theta_0(z))^k,$$

where $z \in \Omega^{(h\wedge)}$ and $\nu \in \mathbb{N}$.

Therefore, as it has been already remarked, using the principle of analytic continuation we may regard it as being defined in $\mathfrak{F}$ and consequently defined for $z \in \Omega^\wedge$.

Now, let me introduce the other my auxiliary functions

$$f_2(z,\nu), \dots, f_{2+n-1}(z,\nu),$$

defined in $\Omega^\wedge$. Let

$$c_1^\vee(\nu) = -c_1(\nu), c_2^\vee(\nu) = 1 + c_2(\nu), \mathfrak{m}(\mu) = m + 1 + \mu,$$

where $\nu \in \mathbb{N} - 1, \mu = 0, \dots, n-1$,

$$f_{2+\mu}(z,\nu) = (-1)^{nc_1(\nu)}\frac{((c_2(\nu))!)^m}{(c_1(\nu)!)^n}\times \tag{3.1.12}$$

$$G^{(\mathfrak{m}(\mu),n)}_{m+n,m+n}\left((\theta)_1^{}\mu+1(z)\left|\begin{matrix}\overbrace{c_1^\vee(nu) & \dots & c_1^\vee(nu)}^{n\text{ times}} & \overbrace{c_2^\vee(nu) & \dots & c_2^\vee(nu)}^{m\text{ times}}\\ 0, & \dots & 0 & 0 & \dots & 0\end{matrix}\right.\right) =$$

$$\frac{(-1)^{nc_1(\nu)}}{2\pi i}\frac{((c_2(\nu))!)^m}{(c_1(\nu)!)^n}\times$$

$$\int_{L_2} g^{(m+1+\mu,n)}_{m+n,m+n}(s)ds,$$

where $z \in \Omega^\wedge$, $\mu = 0, \ldots, n-1$, and $\nu \in \mathbb{N}$,

$$g_{m+n,m+n}^{(m+1+\mu,n)} = g_{m+n,m+n}^{(m+1+\mu,n)}(s) =$$

$$g_{m+n,m+n}^{(\mathfrak{m}(\mu),n)}\left((\theta_1)^{\mu+1}(z)\left|\begin{matrix} \overbrace{c_1^\vee(nu) & \ldots & c_1^\vee(nu)}^{n \text{ times}} & \overbrace{c_2^\vee(nu) & \ldots & c_2^\vee(nu)}^{m\text{times}} \\ 0 & \ldots & 0 & 0 & \ldots & 0 \end{matrix}\right|s\right) =$$

$$(\exp(s \operatorname{Log}((\theta_1)^{\mu+1}(z))))\times$$

$$(\Gamma(-s))^{m+1+\mu}(\Gamma(1+s))^{-n+1+\mu}(\Gamma(1+c_1(nu)+s))^n(\Gamma(1+c_2(nu)-s))^{-m},$$

and the curve L_2 passes from $-\infty$ to $-\infty$ in the positive direction in such a way, that the set $\mathbb{N}-1$ lies to the right from it but the set $-\mathbb{N}$ lies to the left from it. Here $m+n$ plays the role of $p=q$ in (3.1.4), the number $m+1+\mu$ plays the role of m, the number n in (3.1.4), plays the role of n.

$$a_k = -c_1(\nu)$$

where $k = 1, \ldots, n$,

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and the condition (C3) of convergence of the integral (3.1.4) is fulfilled; I use (3.1.7) to compute this integral. The set of all unremovable singular points of the function $g_{m+n,m+n}^{(m+1+\mu,n)}(s)$ encircled by the curve L_2, consists of the points $s = -1 - c_1(\nu) - k$ with $k \in \mathbb{N}-1$; each of these points is a pole, which, as it will be shown below, has the order equal to $1+\mu$. Let

$$s = -c_1(\nu) - 1 - k + u, H^*(u,k,\nu) =$$

$$(\Gamma(c_1(\nu)+1+k-u))^{m+n}(\Gamma(1+k-u))^{-n}(\Gamma(2+c_1(\nu)+c_2(\nu)+k-u))^{-m},$$

and

$$R^{\vee}(t,\nu) = \frac{\left(\prod_{\kappa=1}^{c_1(\nu)}(t-\kappa)\right)^n}{\left(\prod_{\kappa=0}^{c_2(\nu)}(t+\kappa)\right)^m}, \tag{3.1.13}$$

where $\nu \in \mathbb{N}$ and sufficiently big. Let further $T = -s = c_1(\nu)+1+k-u$. Then

$$H^*(u,k,\nu) = \frac{\left(\prod_{\kappa=1}^{c_1(\nu)}(c_1(\nu)+1+k-u-\kappa)\right)^n}{\left(\prod_{\kappa=0}^{c_2(\nu)}(c_1(\nu)+1+k-u+\kappa)\right)^m} = R^{\vee}(T,\nu)$$

where $\nu \in \mathbb{N}-1$. Let

$$H_{2+\mu}(u,k,\nu) =$$
$$g_{m+n,m+n}^{(m+1+\mu,n)}(-c_1(\nu)-1-k+u) =$$
$$(\exp((-c_1(\nu)-1-k+u)\,\mathrm{Log}\,((\theta_1)^{\mu+1}(z))))\times$$
$$(\Gamma(c_1(\nu)+1+k-u))^{m+1+\mu}\times$$
$$(\Gamma(-c_1(\nu)-k+u))^{-n+1+\mu}(\Gamma(-k+u))^n(\Gamma(2+c_2(\nu)+c_1(\nu)+k-u))^{-m} =$$
$$(\exp((-c_1(\nu)-1-k+u)\,\mathrm{Log}\,((\theta_1)^{\mu+1}(z))))\times$$
$$(-1)^{(c_1(\nu)+k+1)(n-1-\mu)+(k+1)n}\left(\frac{\pi}{\sin(u\pi)}\right)^{1+\mu}\times$$
$$(\Gamma(c_1(\nu)+1+k-u))^{m+n}(\Gamma(1+k-u))^{-n}(\Gamma(2+c_2(\nu)+c_1(\nu)+k-u))^{-m} =$$
$$(\exp((-c_1(\nu)-1-k+u)\,\mathrm{Log}\,((\theta_1)^{\mu+1}(z))))\times$$
$$(-1)^{(c_1(\nu)+k+1)(1+\mu)+nc_1(\nu)}\left(\frac{\pi}{\sin(u\pi)}\right)^{1+\mu}H^*(u,k,\nu) =$$
$$(\exp((-c_1(\nu)-1-k)\,\mathrm{Log}\,((\theta_1)^{\mu+1}(z))))(\exp(u\,\mathrm{Log}\,((\theta_1)^{\mu+1}(z))))\times$$
$$\left(\frac{\pi}{\sin(u\pi)}\right)^{1+\mu}(-1)^{(c_1(\nu)+k+1)(1+\mu)+nc_1(\nu)}\times$$
$$H^*(u,k,\nu) =$$

$$(\theta_0((\theta_1)^{\mu+1}(z)))^{-c_1(\nu)-1-k})(\exp(u\,\mathrm{Log}\,((\theta_1)^{\mu+1}(z))))\times$$

$$\left(\frac{\pi}{\sin(u\pi)}\right)^{1+\mu}(-1)^{(c_1(\nu)+k+1)(1+\mu)+nc_1(\nu)}\times$$

$$H^*(u,k,\nu)=$$

$$((-1)^{\mu+1}\theta_0(z))^{-c_1(\nu)-1-k})(\exp(u\,\mathrm{Log}\,((\theta_1)^{\mu+1}(z))))\times$$

$$\left(\frac{\pi}{\sin(u\pi)}\right)^{1+\mu}(-1)^{(c_1(\nu)+k+1)(1+\mu)+nc_1(\nu)}\times$$

$$H^*(u,k,\nu)=$$

$$(\theta_0(z))^{-c_1(\nu)-1-k})(\exp(u\,\mathrm{Log}\,((\theta_1)^{\mu+1}(z))))\times$$

$$\left(\frac{\pi}{\sin(u\pi)}\right)^{1+\mu}(-1)^{nc_1(\nu)}\times$$

$$H^*(u,k,\nu)=$$

$$(-1)^{nc_1(\nu)}(\theta_0(z))^{-c_1(\nu)-1-k})(\exp(u\,\mathrm{Log}\,((\theta_1)^{\mu+1}(z))))\times$$

$$\left(\frac{\pi}{\sin(u\pi)}\right)^{1+\mu}\times$$

$$H^*(u,k,\nu)=$$

$$(-1)^{nc_1(\nu)}(\theta_0(z))^{-c_1(\nu)-1-k})(\exp(u\,\mathrm{Log}\,((\theta_1)^{\mu+1}(z))))\times$$

$$\left(\frac{\pi}{\sin(u\pi)}\right)^{1+\mu}\times$$

$$R^\vee(T,\nu)$$

$$\left(\frac{\pi}{\sin(u\pi)}\right)^{1+\mu}(-1)^{c_1(\nu)}(-(-1)^{\mu}z)^u(z)^{-d^\vee\nu-1-k}R^\vee(T,\nu).$$

So, in fact, the set of all unremovable singular points of the function $g_{m+n,m+n}^{(m+1+\mu,m)}(s)$ encircled by the curve L_2, consists of the points $s=-c_1(\nu)-1-k$ with $k\in\mathbb{N}-1$; each of these points is a pole,which has the order equal to $1+\mu$. Let

$$H_\mu^\vee(u,z)=$$

$$\left(\frac{u\pi}{\sin(u\pi)}\right)^{1+\mu}\exp(u\,\mathrm{Log}\,((\theta_1)^{\mu+1}(z))),$$

where we may count $\frac{u\pi}{sin(u\pi)}$ regular at the point $u = 0$ and equal to 1 in this point. Clearly also that for each $k \in \mathbb{N} - 1$ the function

$$\left(\frac{\partial}{\partial u}\right)^{\kappa} \exp(u \operatorname{Log}((\theta_1)^{\mu+1}(z))) = (\operatorname{Log}((\theta_1)^{\mu+1}(z)))^{\kappa} \exp(u \operatorname{Log}((\theta_1)^{\mu+1}(z)))$$

iz regular in the domain $|u| < 1$, if $z \in \Omega^{\wedge}$; moreover it is continuous on the set $|u| < 1, |z| \geq 1$. holomorphic in the domain $|u| < 1, |z| > 1$. Therefore the function $H_{\mu}^{\vee}(u, z)$ is regular in the domain $|u| < 1$, if $z \in \Omega^{\wedge}$; moreover it is continous on the set $|u| < 1, |z| \geq 1$ and holomorphic in the domain $|u| < 1, |z| > 1$; we remark also that this function is independent from $k \in \mathbb{N} - 1$, that

$$H_{\mu}^{\vee}(0, z) = 1 \tag{3.1.14}$$

and that for each $\kappa \in \mathbb{N} - 1$ the function

$$H_{\mu,\kappa}^{\wedge}(z) = \frac{1}{\kappa!}\left(\left(\frac{\partial}{\partial u}\right)^{\kappa} (H_{\mu}^{\vee}(u, z)\right)\Bigg|_{u=0}$$

is continuous on the set $\Omega^{\wedge}$ and holomorphic in the domain $\Omega^{\vee}$. Therefore

$$\operatorname{Res}\left(g_{m+n,m+n}^{(m+1+\mu,n)}; -1 - c_1(\nu) - k\right) = \tag{3.1.15}$$

$$\frac{1}{\mu!}(-1)^{nc_1(\nu)}(\theta_0(z))^{-c_1(\nu)-1-k}(\frac{\partial}{\partial u})^{\mu}(H_{\mu}^{\vee}(u, z)H^*(u, k, \nu))\Bigg|_{u=0} =$$

$$\frac{1}{\mu!}(-1)^{nc_1(\nu)}(\theta_0(z))^{-c_1(\nu)-1-k}\times$$

$$\sum_{\kappa=0}^{\mu}\binom{\mu}{\kappa}\left(\left(\frac{\partial}{\partial u}\right)^{\kappa} H_{\mu}^{\vee}(u, z)\right)\Bigg|_{u=0}\left(\left(\frac{\partial}{\partial u}\right)^{\mu-\kappa} H^*(u, k, \nu)\right)\Bigg|_{u=0} =$$

$$(-1)^{nc_1(\nu)}(\theta_0(z)) - c_1(\nu) - 1 - k\times$$

$$\sum_{\kappa=0}^{\mu}(H_{\mu,\kappa}^{\wedge}(z)\frac{(-1)^{\mu-\kappa}}{(\mu-\kappa)!}\left(\left(\frac{\partial}{\partial T}\right)^{\mu-\kappa} R^{\vee}(T, \nu)\right)\Bigg|_{T=c_1(\nu)+1+k}.$$

Let, in correspondance to (3.1.12) and (3.1.13),

$$R(t, \nu) = \frac{((c_2(\nu))!)^m}{((c_1(\nu))!)^n}R^{\vee}(t, \nu) = \frac{((c_2(\nu))!)^m}{((c_1(\nu))!)^n}\frac{\left(\prod_{\kappa=1}^{c_1(\nu)}(t-\kappa)\right)^n}{\left(\prod_{\kappa=0}^{c_2(\nu)}(t+\kappa)\right)^m}.$$

It may be remarked that the function $R(t,\nu)$ coincides with the function $R(t;\nu)$, defined in paragraph 6 of the paper [25]. Let in correspondence to (33) of the work [25]

$$\psi_\mu(\theta_0(z);\nu)=\sum_{t=1}^{\infty}\frac{(-1)^\mu}{\mu!}\left(\left(\frac{\partial}{\partial t}\right)^\mu R\right)(t,\nu)(\theta_0(z))^{-t}, \tag{3.1.16}$$

where $z\in\Omega^\wedge$. Since

$$\left(\left(\frac{\partial}{\partial t}\right)^\mu R\right)(t,\nu)=0$$

for $t=1,\dots,d^\vee\nu+k$ and $\mu=0,\dots,n-1$, it follows that

$$\psi_\mu(\theta_0(z);\nu)=\frac{(-1)^\mu}{\mu!}\times \tag{3.1.17}$$

$$\sum_{t=1+c_1(\nu)+k}^{\infty}\left(\left(\frac{\partial}{\partial t}\right)^\mu R\right)(t,\nu)(\theta_0(z))^{-t},$$

Now, after the summation of the equality (3.1.14) over $k\in\mathbb{N}-1$, according to (3.1.12), (3.1.14), (3.1.15), (3.1.17),we come to equalities

$$f_{2+\mu}(z,\nu)= \tag{3.1.18}$$

$$\sum_{\kappa=0}^{\mu}H^\wedge_{\mu,\kappa}(z)\sum_{k=0}^{+\infty}\frac{(-1)^{\mu-\kappa}}{(\mu-\kappa)!}\left(\left(\frac{\partial}{\partial T}\right)^{\mu-\kappa}R^\vee(T,\nu)\right)\Bigg|_{T=c_1(\nu)+1+k}(\theta_0(z))^{-c_1(\nu)-1-k}=$$

$$\sum_{\kappa=0}^{\mu}(H^\wedge_{\mu,\kappa}(z)\psi_{\mu-\kappa}(\theta_0(z);\nu)=$$

$$\psi_\mu(\theta_0(z);\nu)+\sum_{\kappa=1}^{\mu}H^\wedge_{\mu,\kappa}(z)\psi_{\mu-\kappa}(\theta_0(z);\nu),$$

where $z\in\Omega^\wedge$. We return now to the general properties of Mejer's functions. Let the operator $\nabla_{b;k}$, with $k=1,\dots,q$ (respectively $\nabla_{a;j}$, with $j=1,\dots,p$) when acting on the function g replaces parameter b_k by b_k+1 (respectively a_j by a_j+1), and let δ denotes the operator $z\frac{\partial}{\partial z}$. It is clear that

$$P_r(\nabla_{b;k}g)\subset P_r(g) \tag{3.1.19}$$

for $k=1,\dots,q$, and

$$P_l((\nabla_{a;j})^{-1}g)\subset P_l(g) \tag{3.1.20}$$

for $j = 1, \dots, p$. In view (3.1.19) and (3.1.20),

$$\nabla_{b;k} G = \frac{1}{2\pi i} \int\limits_L (\nabla_{b;k} g)(s) ds, \tag{3.1.21}$$

where $k = 1, \dots, q$, and

$$(\nabla_{a;j})^{-1} G = \frac{1}{2\pi i} \int\limits_L ((\nabla_{a;j})^{-1} g)(s) ds, \tag{3.1.22}$$

where $j = 1, \dots, p$, assuming, of course, that one of the conditions (A), (B), (C) of the absolute convergence of the integral (3.1.21) (respectively (3.1.22)) is satisfied; here the contour of integration L is taken to be the same as in (3.1.4).

Lemma 3.1.1 *Let, as before, $g = g(s) = g(z, s)$ be the integrand of the integral (3.1.4). Let, further, $\varepsilon_1(k) = -1$ if $k = 1, \dots, m$, and $\varepsilon_1(k) = 1$ if $k = m+1, \dots, q$. Let finally $\varepsilon_2(j) = 1$ if $j = 1, \dots, n$, and $\varepsilon_2(j) = -1$, if $j = n+1, \dots, p$. If one of the conditions of the absolute convergence (A), (B), (C) is satisfied for the integral (3.1.21) (respectively (3.1.22),) then the corresponding condition of the absolute convergence (A), (B), (C) is satisfied also for the integral (3.1.4). Moreover, if one of the conditions (B1), (B2), (C1), (C2) is satisfied for the integral (3.1.4), then the corresponding condition of the absolute convergence is satisfied for each of the integrals (3.1.21), (3.1.22) and*

$$\varepsilon_1(k) \nabla_{b;k} G = (\delta - b_k) G, \tag{3.1.23}$$

where $k = 1, \dots, q$, and

$$\varepsilon_2(j)(\nabla_{a;j})^{-1} G = (\delta + 1 - a_j) G, \tag{3.1.24}$$

where $j = 1, \dots, p$.

Proof *may be found in [35], section 1, Lemma 2.1.1.* ■

Corollary. *If one of conditions, (B1), (B2), (C1), (C2) is satisfied, then the function G given by the integral (3.1.4) is holomorphic in the corresponding domain and, for each $d \in \mathbb{N}$, the equalities*

$$(\nabla_{b;k})^d G = (\varepsilon_1(k))^d \left(\prod_{\kappa=0}^{d-1} (\delta - b_k - \kappa) \right) G, \tag{3.1.25}$$

where $k = 1, \dots, q$, and

$$(\nabla_{a;j})^{-d} G = (\varepsilon_2(j))^d \left(\prod_{\kappa=0}^{d-1} (\delta - a_j + 1 + \kappa) \right) G, \tag{3.1.26}$$

where $j = 1, \dots, p$, hold true.

Proof *may be found in [35], section 1, Corollary to the Lemma 2.1.1.* ■

Let $\bigtriangledown$ denote the operator replacing each of the parameters b_k, with $k = 1, \dots, q$, by $b_k + 1$ and each of the parameters a_j, with $j = 1, \dots, p$, by $a_j + 1$. As in [8], for each $A \subset \mathbb{C}$ and each $c \in \mathbb{C}$, the set $\{c + a \colon a \in A\}$ is denoted by $A + c$. It is clear that

$$P_r(g) \neq P_r(\bigtriangledown g) = P_r(g) + 1 \subset P_r(g), \tag{3.1.27}$$

as in (3.1.19). But

$$P_l(g) \subset P_l(g) + 1 = P_l(\bigtriangledown g) \neq P_l(g), \tag{3.1.28}$$

as in (3.1.20).The first inequality in (3.1.27) holds because the set $P_r(g)$ includes some (possibly not unique) number β with the smallest real part; then $\beta \notin P_r(g)+1$. The last inequality in (3.1.28) holds because the set $P_l(g)+1$ includes some (possibly not unique) number α with the biggest real part; then $\alpha \notin P_l(g)$. Nevertheless

$$(P_l(g) + 1) \bigcap (P_r(g) + 1) = P_l(g) \bigcap P_r(g) = \emptyset. \tag{3.1.29}$$

Therefore for the same contour of integration L, which is used in the integral (2.1.1), we let

$$(\bigtriangledown G)(z) = \frac{1}{2\pi i} \int\limits_{L+1} (\bigtriangledown g)(z, s) ds, \tag{3.1.30}$$

assuming, of course, that one of the conditions (A), (B), (C) of the absolute convergence of the integral (3.1.30) is satisfied.

Lemma 3.1.2 *If, for the integral (3.1.4), one of the conditions (B1), (B2), (C1), (C2) is satisfied, then the corresponding condition of the absolute convergence of the integral (3.1.30) is also satisfied, and*

$$(\bigtriangledown G)(z) = \theta_0(z) G(z). \tag{3.1.31}$$

Proof *may be found in [35], section 1, Lemma 2.1.2.* ■

Remark 3.1.2. *In view of (3.1.28), if not only $P_l(g)$ but also $P_l(g) + 1$ lies to the left of the contour of integration L and one of the conditions (B1), (B2), (C1), (C2) is satisfied, then*

$$(\bigtriangledown G)(z) = \frac{1}{2\pi i} \int\limits_{L} (\bigtriangledown g)(z, s) ds.$$

This remark is important since the location of the curve L must be taken into account. We shall make use of this remark when we apply both Lemma 3.1.1 and Lemma 3.1.2 simultaneously.

Lemma 3.1.3. *If the conditions mentioned in Remark 3.1.2 are satisfied, then*

$$(-1)^{m+p-n}\theta_0(z)\left(\left(\prod_{j=1}^{p}(\delta+1-a_j)\right)G\right)(z)= \left(\left(\prod_{k=1}^{q}(\delta-b_k)\right)G\right)(z). \tag{3.1.33}$$

Proof *may be found in [35], section 1, Lemma 2.1.3.* ■

Lemma 3.1.4. *Let H stand for either $\mathbb{C}$ or $\mathfrak{F}$, let D be a domain in H, and let $f(z)$ be a function holomorphic in D. Given any $a \in H$, satisfying the condition $a \neq 0$ if $H = \mathbb{C}$, let $\eta_a(z) = az$ for $z \in H$. Then*

$$(\delta(f \circ \eta_a))(z) = ((\delta f) \circ \eta_a)(z)$$

for $z \in a^{-1}D$ and the function $\delta(f \circ \eta_a)$ (and therefore $(\delta f) \circ \eta_a$) is holomorphic in the domain $a^{-1}D$.

Proof *may be found in [35], section 1, Lemma 2.1.4.* ■

Corollary 1. *Let Ω be a domain in H, and let $a \in \mathbb{C}\setminus 0$, if $H = C$. If the function $f(z)$ is holomorphic in the domain $\eta_a\Omega = \{az : z \in \Omega\}$, then*

$$\delta(f \circ \eta_a))(z) = ((\delta f) \circ \eta_a)(z)$$

and the function $\delta(f \circ \eta_a)$ (and therefore $(\delta f) \circ \eta_a$) is holomorphic in Ω.

Proof. *The assertion follows from Lemma 3.1.4 with $D = \eta_a\Omega$.* ■

Corollary 2. *Let Ω be a domain in $\mathfrak{F}$. If the function $f(z)$ is holomorphic in the domain $\theta_1(\Omega) = \{\theta_1(z) : z \in \Omega\}$, then*

$$(\delta(f \circ \theta_1))(z) = ((\delta f) \circ \theta_1)(z)$$

and the function $\delta(f \circ \theta_1)$ (and therefore $(\delta f) \circ \theta_1$) is holomorphic in Ω.

Proof. *The assertion follows from Corollary 1 on taking $a = (1, -\pi)$.* ■

Let as before G denote the integral (3.1.4). It has been proved in [34], that the function $G(z)$ is holomorphic in the domain $\Omega = \mathfrak{F}$, if one of the conditions (B1) or (C1) is satisfied, that it is holomorphic in the domains $\Omega = \Omega^{(h\vee)}$ if condition (B2) is satisfied, and that it is holomorphic in the domains $\Omega = \Omega^{\vee}$ if condition (C2) is satisfied. In each of these cases,$\eta_a(\Omega) = \Omega$ for any $a \in SO_2 = \{a \in \mathfrak{F}\colon |a| = 1.\}$

Lemma 3.1.5. *Let $d \in \mathbb{N}$ and suppose that one of the conditions (B1), (C1), (B2), (C2) is satisfied. Then*

$$(\nabla_{b;k})^d (G \circ \theta_1) = (\varepsilon_1(k))^d \left(\prod_{\kappa=0}^{d-1} (\delta - b_k - \kappa)\right) (G \circ \theta_1), \tag{3.1.34}$$

where $k = 1, \dots, q$,

$$(\nabla_{a;j})^{-d} (G \circ \theta_1) = (\varepsilon_2(j))^d \left(\prod_{\kappa=0}^{d-1} (\delta - a_j + 1 + \kappa)\right) (G \circ \theta_1), \tag{3.1.35}$$

where $j = 1, \dots, p$, and

$$(\nabla (G \circ \theta_1))(z) = \theta_1(z)(G \circ \theta_1)(z). \tag{3.1.36}$$

If the conditions of Remark 3.1.3 are satisfied, then

$$(-1)^{m+p-n+1}\theta_0(z) \left(\left(\prod_{j=1}^{p} (\delta + 1 - a_j)\right) (G \circ \theta_1)\right)(z) = \tag{3.1.37}$$

$$\left(\left(\prod_{k=1}^{q} (\delta - b_k)\right) (G \circ \theta_1)\right)(z).$$

Proof. *may be found in [35], section 1, Lemma 2.1.5 and its corollary.* ■

Corollary. *Let $d \in \mathbb{N}$ and suppose that one of the conditions (B1), (C1), (B2), and (C2) is satisfied. Then*

$$(\nabla_{b;k})^d (G \circ \theta_1) = (\varepsilon_1(k))^d \left(\prod_{\kappa=0}^{d-1} (\delta - b_k - \kappa)\right) (G \circ \theta_1), \tag{3.1.38}$$

where $k = 1, \dots, q$,

$$(\nabla_{a;j})^{-d} (G \circ \theta_1) = (\varepsilon_2(j))^d \left(\prod_{\kappa=0}^{d-1} (\delta - a_j + 1 + \kappa)\right) (G \circ \theta_1), \tag{3.1.39}$$

where $j = 1, \dots, p$, and

$$(\nabla (G \circ \theta_1))(z) = -\theta_0(z)(G \circ \theta_1)(z). \tag{3.1.40}$$

Moreover, if the conditions of Remark 3.1.2 are satisfied, then

$$(3.1.41)\qquad (-1)^{m+p-n+\mu}\theta_0(z)\left(\left(\prod_{j=1}^{p}(\delta+1-a_j)\right)(G\circ(\theta_1)^{\mu})\right)(z)=$$

$$\left(\left(\prod_{k=1}^{q}(\delta-b_k)\right)(G\circ(\theta_1)^{\mu})\right)(z),$$

where $\mu\in\mathbb{N}-1$.

Proof. *To establish the assertions of the corollary it suffices to let* $a=(1,-\pi)$. ■

I wish to apply the Lemmas 3.1.1 - 3.1.5 and their corollaries to my initial auxilliary functions. For them we have the equality $P_r(g)=\mathbb{N}-1, P_l(g)=-\mathbb{N}-c_1(\nu)$. *Therefore the conditions of the Remark 3.1.2 hold for* $\nu\in\mathbb{N}$. *According to (3.1.41) and (3.1.11)*

$$(3.1.42)\qquad (-1)^{m+n+1+n+m+1}\theta_0(z)(\delta+1+c_1(\nu))^n(\delta-c_2(\nu))^m f_1=\delta^{m+n}f(\nu)$$

where $\nu\in\mathbb{N}$. *According to (3.1.41) and (3.1.12)*

$$(3.1.43)\quad (-1)^{m+n+(m+\mu+1)+n+\mu+1}\theta_0(z)(\delta+1+c_1(\nu))^n(\delta-c_2(\nu))^m f_{2+\mu}(z,\nu)=$$

$$\delta^{m+n}f_{2+\mu}(z,\nu)$$

where $\mu=0,\ldots,n-1$ *and sufficiently big* $\nu\in\mathbb{N}$. *According to (3.1.42) and (3.1.43),*

$$(3.1.44)\qquad \theta_0(z)(\delta+1+c_1(\nu))^n(\delta-c_2(\nu))^m f_\mu(z,\nu)=\delta^{m+n}f_\mu(z,\nu)$$

where $\mu=1,\ldots,n+1$ *and* $\nu\in\mathbb{N}$. *In correspondence with (3.1.11), let*

$$(3.1.45)\qquad G_1^{\vee}=$$

$$G^{(1,n)}_{m+n,m+n}\left((\theta_1)^{m+1}(z)\middle|\begin{matrix}\overbrace{-c_1(\nu)\ \ \ldots\ \ -c_1(\nu)}^{n\text{ times}} & \overbrace{c_2(\nu)+1\ \ \ldots\ \ c_2(\nu)+1}^{m\text{ times}}\\ 0\ \ \ldots\ \ 0 & 0\ \ \ldots\ \ 0\end{matrix}\right)$$

and,in correspondence with (3.1.12), let

$$(3.1.46)\qquad G_{2+\mu}^{\vee}=$$

$$G_{m+n,m+n}^{(\mathfrak{m}(\mu),n)}\left((\theta_1)^{\mu+1}(z)\left|\begin{array}{c}\overbrace{c_1^{\vee}(nu)\quad\ldots\quad c_1^{\vee}(nu)}^{n\text{ times}}\,\overbrace{c_2^{\vee}(nu)\quad\ldots\quad c_2^{\vee}(nu)}^{m\text{ times}}\\ 0,\quad\ldots\quad 0\quad 0\quad\ldots\quad 0\end{array}\right.\right).$$

Let further

$$G_\mu^{\wedge}=\left(\left(\prod_{j=1}^{n}(\nabla_{a;j})^{-d^{\vee}}\right)\left(\prod_{j=n+1}^{j=n+m}(\nabla_{a;j})^{d^{\wedge}}\right)\right)G_\mu^{\vee},$$

$$G_\mu^{*}=\left(\prod_{j=1}^{n}(\nabla_{a;j})^{-d^{\vee}}\right)G_\mu^{\vee}$$

where $\mu=1,\,\ldots\,,\,n+1$. *Clearly,*

$$\left(\prod_{j=n+1}^{n+m}(\nabla_{a;j})^{-d^{\wedge}}\right)G_\mu^{\wedge}=\left(\prod_{j=1}^{n}(\nabla_{a;j})^{-d^{\vee}}\right)G_\mu^{\vee}=G_\mu^{*},\tag{3.1.47}$$

where $\mu=1,\,\ldots\,,\,n+1$. *According to the equality (3.1.47) and to the Lemma 3.1.5,*

$$(-1)^{md^{\wedge}}\left(\prod_{\kappa=0}^{d^{\wedge}-1}(\delta-c_2(\nu)+\kappa)\right)^{m}G_\mu^{\wedge}=\tag{3.1.48}$$

$$\left(\prod_{\kappa=0}^{d^{\vee}-1}(\delta+c_1(\nu)+1+\kappa)\right)^{n}G_\mu^{\vee},$$

where $\mu=1,\,\ldots\,,\,n+1$. *In view of (3.1.11) and (3..12),*

$$f_\mu(z,\nu)=(-1)^{nc_1(\nu)}\frac{(((c_2(\nu))!)^m}{((c_1(\nu))!)^n}G_\mu^{\vee},\tag{3.1.49}$$

$$f_\mu(z,\nu+1)=(-1)^{nc_1(\nu+1)}\frac{(((c_2(\nu+1))!)^m}{((c_1(\nu+1))!)^n}G_\mu^{\wedge},\tag{3.1.50}$$

where $\mu=1,\,\ldots\,,\,n+1$, *and according to (3.1.48),*

$$\left(\left(\prod_{\kappa=0}^{d^{\wedge}-1}\frac{-1}{c_2(\nu)+1+\kappa}(\delta-c_2(\nu)+\kappa)\right)^{m}f_\mu\right)(z,\nu+1)=\tag{3.1.51}$$

$$\left(\left(\prod_{\kappa=0}^{d^\vee-1}\frac{-1}{c_1(\nu)+1+\kappa}(\delta+c_1(\nu)+1+\kappa)\right)^n f_\mu\right)(z,\nu),$$

where $\mu=1,\ldots,n+1$. *Let*

$$D^\wedge(z,\nu,w)=\theta_0(z)(w+1+c_1(\nu))^n(\delta-c_2(\nu))^m-w^{m+n}, \tag{3.1.52}$$

$$P_0^\wedge(\nu,w)=\left(\prod_{\kappa=0}^{d^\vee-1}\frac{-1}{c_1(\nu)+1+\kappa}(w+c_1(\nu)+1+\kappa)\right)^n, \tag{3.1.53}$$

$$P_1^\wedge(\nu,w)=\left(\prod_{\kappa=0}^{d^\wedge-1}\frac{-1}{c_2(\nu)+1+\kappa}(w-c_2(\nu)+\kappa)\right)^m \tag{3.1.54}$$

According to (3.1.44) and (3.1.52)

$$D^\wedge(z,\nu,\delta)f_\mu=0, \tag{3.1.55}$$

where $\mu=1,\ldots,n+1$, *According to (3.1.51), (3.2.53) and (3.1.54),*

$$(P_1^\wedge(\nu,\delta)f_\mu)(z,\nu+1)=(P_0^\wedge(\nu,\delta)f_\mu)(z,\nu). \tag{3.1.56}$$

As in [35], let ν^{-1} *be an independent variable taking its values in* $\mathbb{C}$ *including* 0. *We denote by* $\mathfrak{O}$ *the ring consisting all the the elements in* $\mathbb{Q}(\nu^{-1})$, *which are well defined at the point* $\nu^{-1}=0$. *Let, for* $i=0,1$,

$$D_i^*(z,\nu^{-1},w)=\nu^{-m-n}D^\wedge(z,\nu+i,\nu w)= \tag{3.1.57}$$

$$=(\theta_0(z)-1)w^{m+n}+\sum_{k=0}^{m+n-1}b_{i,k}^\vee(z,\nu^{-1})w^k,$$

$$P^*(\nu^{-1},w)=P_0^*(\nu^{-1},w)=P_0^\wedge(\nu,\nu w)= \tag{3.1.58}$$

$$\left(\prod_{\kappa=0}^{d^\vee-1}\frac{-1}{d^\vee+\nu^{-1}(\beta^\vee+1+\kappa)}(w+d^\vee\nu^{-1}(\beta^\vee+1+\kappa))\right)^n,$$

$$Q^*(\nu^{-1},w)=P_1^*(\nu^{-1},w)=P_1^\wedge(\nu,\nu w)= \tag{3.1.59}$$

$$\left(\prod_{\kappa=0}^{d^\wedge-1}\frac{-1}{d^\wedge+\nu^{-1}(\beta^\wedge+1+\kappa)}(w-d^\wedge+\nu^{-1}(-\beta^\wedge+\kappa))\right)^m$$

Then

$$D_0^*(z,\nu^{-1},w)\in\mathbb{Q}[\theta_0(z),\nu^{-1},w], D_1^*(z,\nu^{-1},w)\in\mathbb{Q}[\theta_0(z),\nu^{-1},w],$$

$$D_0^*(z,0,w)=D_1^*(z,0,w)=D^\sim(z,w)=\theta_0(z)(w+d^\vee)^n(w-d^\wedge)^m-w^{m+n}, \tag{3.1.60}$$

$$P^*(\nu^{-1},w)=P_0^*(\nu^{-1},w)\in\mathfrak{O}[w], Q^*(\nu^{-1},w)=P_1^*(\nu^{-1},w)\in\mathfrak{O}[w], \tag{3.1.61}$$

$$P^*(0,w)=P_0^*(0,w)=P^\sim(w)=\left(\frac{-1}{d^\vee}(w+d^\vee)\right)^{nd^\vee}, \tag{3.1.62}$$

$$Q^*(0,w)=P_1^*(0,w)=Q^\sim(w)=\left(\frac{-1}{d^\wedge}(w-d^\wedge)\right)^{md^\wedge}, \tag{3.1.63}$$

According to (3.1.55), and (3.1.57),

$$(D_i^*(z,\nu^{-1},\nu^{-1}\delta)f_\mu)(z,\nu+i)=0, \tag{3.1.64}$$

where $i=0,\,1$ *and* $\mu=1,\,\dots\,,\,n+1$. *According to (3.1.56),(3.1.58) and (3.1.59),*

$$(P_1^*(\nu^{-1},\nu^{-1}\delta)f_\mu)(z,\nu+1)=(P_0^*(\nu^{-1},\nu^{-1}\delta)f_\mu)(z,\nu). \tag{3.1.65}$$

For $\mu=1,\,\dots\,,\,n+1$ *let* $X_\mu(z,\nu)$ *denote the column consisting of the* $m+n$ *elements*

$$((\nu^{-1}\delta)^{j-1}f_\mu^*)(z,\nu)$$

where $j=1,\,\dots\,,\,m+n$. *For* $i=0,1$, *let* $B_i(z,\nu^{-1})=(a_{kj})$ *is a* $(m+n)\times(m+n)$*-matrix defined as follows: on intersection of its first* $m+n-1$ *rows and last* $m+n-1$ *stays the* $(m+n-1)\times(m+n-1)$ *unit matrix, the first* $m+n-1$ *elements of its first column equal to zero and for* $j=1,\,\dots\,,\,m+n$, *the* $j-$*th element of its last row is equal to* $b_{i,j-1}^\vee(z,\nu^{-1})/(1-\theta_0(z))$. *Following the well known procedure of representation of a differential equation in matrix form, we deduce from the relation (3.1.64) that*

$$(\nu^{-1}\delta)X_\mu(z,\nu+i)=B_i(z,\nu^{-1})X_\mu(z,\nu+i) \tag{3.1.66}$$

for $i = 0, 1$, $\mu = 1, \ldots, n+1$, $\nu \in \mathbb{N}$, *and* $z \in \Omega_1^*$. *According to (3.1.58) and (3.1.59), the operator* $\nu^{-1}\delta$ *commutes with the operator* $P_i^*(\nu^{-1}, \nu^{-1}\delta)$ *for* $i = 0, 1$. *Therefore it follows from the relation (3.1.65),(3.1.58),(3.1.59) that*

$$(3.1.67) \qquad Q^*(\nu^{-1}, \nu^{-1}\delta)X_\mu(z, \nu+1) = P_1^*(\nu^{-1}, \nu^{-1}\delta)X_\mu(z, \nu+1) =$$

$$P_0^*(\nu^{-1}, \nu^{-1}\delta)X_\mu(z, \nu) = P^*(\nu^{-1}, \nu^{-1}\delta)X_\mu(z, \nu)$$

for $\mu = 1, \ldots, m+1$, $\nu \in \mathbb{N}$, *and* $z \in \Omega^\vee$. *Let* $\mathrm{Mat}_n(K)$, *for* $n \in \mathbb{N}$, *denote the set of all the* $n \times n$*-matrices with entries in the subset* K *of a given ring. In view of (3.2.57),*$B_0(z, 0) = B_1(z, 0)$; *we denote this matrix by* $B^\sim(z)$. *So,*

$$(3.1.68) \qquad B^\sim(z) = B_0(z, 0) = B_1(z, 0).$$

Regarded as a polinomial of w, *the polynomial*

$$(3.1.69) \qquad D_1^\sim(z, w) = (1/(\theta_0(z) - 1))D^\sim(z, w) = w^{m+n} - \sum_{k=1}^{m+n} b_k^\sim(z)w^{k-1},$$

coincides with the characteristic polynomial of the matrix $B^\sim(z)$, *so that,in view of (3.1.60),*

$$(3.1.70) \qquad \det(d^\wedge E - B^\sim(z)) = D_1^\sim(z, d^\wedge) = \frac{-1}{\theta_0(z) - 1}(d^\wedge)^{m+n}.$$

Therefore

$$(3.1.71) \qquad (B^\sim(z) - d_2E)^{-1} \in \mathrm{Mat}_{m+n}(\mathbb{Q}[\theta_0(z)]);$$

moreover, the last column of this matrix consists of the elements of the ideal $(\theta_0(z) - 1)\mathbb{Q}[\theta_0(z)]$ *in the ring* $\mathbb{Q}[\theta_0(z)]$. *The functions* $\theta_0(z)$ *and* $(\theta_0(z) - 1)^{-1}$ *are holomorphic on* $\Omega^\vee$, *and, clearly, are transcendental elements over* $\mathbb{C}$ *and, particularly, over* $\mathbb{Q}$. *Therefore we may consider the rings* $\mathbb{Q}[\theta_0(z)]$ *and* $\mathbb{Q}[(\theta_0(z) - 1)^{-1}]$. *Let*

$$K_r = (\theta_0(z) - 1)^{-r}\mathbb{Q}[\theta_0(z)] \cap \mathbb{Q}[(\theta_0(z) - 1)^{-1}],$$

$$K_r^* = (\theta_0(z) - 1)^{-r}\mathfrak{O}[\theta_0(z)] \cap \mathfrak{O}[(\theta_0(z) - 1)^{-1}],$$

where $r \in \mathbb{N} - 1$. *Clearly,* $K_0 = \mathbb{Q}$, $K_0^* = \mathfrak{O}$, *and*

$$B_i(z, \nu^{-1}) - B^\sim(z) \in \nu^{-1}\,\mathrm{Mat}_{m+n}(K_1^*)$$

for $i = 0, 1$.

Lemma 3.1.6. *For $i = 0, 1$ and $r \in \mathbb{N}$, suppose that*

$$H^{\sim}_{i,m-1} \in \operatorname{Mat}_{m+n}(K_{r-1}), \quad H^{*}_{i,m-1} \in \operatorname{Mat}_{m+n}(K^{*}_{r-1}),$$

$$H_{i,r-1} = H^{\sim}_{i,r-1} + \nu^{-1} H^{*}_{i,r-1}, \quad b_{i,r}(\nu^{-1}) \in \mathfrak{O},\ c_{i,r}(\nu^{-1} \in \mathfrak{O}Q.$$

Then, for $i = 0, 1$, there exists $H^{*}_{i,r}(z, \nu^{-1}) \in \operatorname{Mat}_{m+n}(K^{*}_{m})$ such that

$$((b_{i,r}(\nu^{-1})\nu^{-1}\delta + c_{i,r}(\nu^{-1}))H_{i,r-1}X_{\mu})(z, \nu + i) = \tag{3.1.72}$$

$$(H^{\sim}_{i,r-1}(b_{i,r}(0)B^{\sim}(z) + c_{i,m}(0)E) + \nu^{-1}H^{*}_{i,r}(z, \nu^{-1}))X_{\mu}(z, \nu + i),$$

where E is the unit $(m + n) \times (m + n)$-matrix, $\mu = 1, \ldots, n + 1$, $\nu \in \mathbb{N}$, $\nu \in \mathbb{N}$, and $z \in \Omega^{*}_{1}$.

Proof. *Since*

$$B_i(z, \nu^{-1}) - B^{\sim}(z) \in \nu^{-1} \operatorname{Mat}_{m+n}(K^{*}_{1}),$$

$$b_{i,r}(\nu^{-1}) - b_{i,r}(0) \in \nu^{-1}K^{*}_{1},$$

$$c_{i,r}(\nu^{-1}) - c_{i,r}(0) \in \nu^{-1}K^{*}_{1},$$

we can let

$$H^{*}_{i,m}(z, \nu^{-1}) = (b_{i,r}(\nu^{-1})\delta + \nu(c_{i,r}(\nu^{-1}) - c_{i,r}(0))H_{i,r-1}+$$

$$H^{*}_{i,r-1}(b_{i,r}(\nu^{-1})B_i(z, \nu^{-1}) + c_{i,r}(\nu^{-1})E) +$$

$$H_{i,r}\nu(b_{i,r}(\nu^{-1})B_i(z, \nu^{-1}) - b_{i,r}(0)B_i(z, 0)). \blacksquare$$

Corollary. *Let $b_{i,s}(\nu^{-1}) \in \mathfrak{O}$ and $c_{i,s}(\nu^{-1}) \in \mathfrak{O}$ be given, for $i = 0, 1$ and $s = 1, \ldots, r$ with $r \in \mathbb{N}$. Then, for each $i = 0, 1$, there exists a matrix $H^{*}_{i,r} \in$* $\operatorname{Mat}_{(} m + n)(K^{*}_{m})$ *depending on $b_{i,1}(\nu^{-1}), \ldots, b_{i,r}(\nu^{-1})$, $c_{i,1}(\nu^{-1}), \ldots, c_{i,r}(\nu^{-1})$ and such that*

$$((\prod_{s=1}^{r}(b_{i,r+1-s}(\nu^{-1})\nu^{-1}\delta + c_{i,r+1-s}(\nu^{-1}))X_{\mu})(z, \nu + i) =$$

$$(\nu^{-1}H^{*}_{i,r} + \prod_{s=1}^{r}(c_{i,r+1-s}(0)B^{\sim}(z) + b_{i,r+1-s}(0)E))X_{\mu}(z, \nu + i),$$

where E is the unit $(m + n) \times (m + n)$-matrix, $\mu = 1, \ldots, n + 1$, $\nu \in \mathbb{N}$, $\nu \in \mathbb{N}$, and $z \in \Omega^{*}_{1}$.

Proof. *The proof is by induction on r. For $r = 1$, the assertion of the corollary follows from Lemma 3.1.6 with $r = 1$ on letting $H^{\sim}_{i,0} = E$ and $H^{*}_{i,0}$ equal to the zero $(m+n) \times (m+n)$-matrix. Let $r \geq 2$ and suppose the assertion of the lemma holds for $r - 1$. Then the assertion of the lemma for r follows from Lemma 3.1.6 on letting*

$$\prod_{s=1}^{r-1}(b_{i,r-s}(0)B^{\sim}(z) + c_{i,r-s}(0)E) = H^{\sim}_{i,r-1}. \blacksquare$$

Lemma 3.1.7. *Let*

$$\mathfrak{W} = \mathrm{Mat}_{\,m+n}(K^{*}_{nd^{\vee}+md^{\wedge}})$$

for $i = 0, 1$. Then, for $i = 0, 1$, there is a matrix $U_i(z, \nu^{-1})$ satisfying the following conditions:

$$U_i(z, \nu^{-1}) \in \mathfrak{W},$$

$$(U_0(z, \nu^{-1}) - P^{\sim}(B^{\sim}(z)) \in \nu^{-1}\mathfrak{W}, \tag{3.1.73}$$

$$U_1(z, \nu^{-1}) - Q^{\sim}(B^{\sim}(z)) \in \nu^{-1}\mathfrak{W}, \tag{3.1.74}$$

$$P^{*}(\kappa^{-1}, \kappa^{-1}\delta)X_{\mu}(z, \kappa) = U_0(z, \kappa^{-1})X_{\mu}(z, \kappa), \tag{3.1.75}$$

and

$$Q^{*}(\kappa^{-1}, \kappa^{-1}\delta)X_{\mu}(z, \kappa + 1) = U_1(z, \kappa^{-1})X_{\mu}(z, \kappa + 1), \tag{3.1.76}$$

where $\mu = 1, \ldots, n+1$, $\kappa \in \mathbb{N}$, $\nu \in \mathbb{N}$, and $z \in \Omega^{*}_{1}$.

Proof. *The asertion of the Lemma is direct corollary of the equalities (3.1.58), (3.1.59), (3.1.62), (3.1.63), of the Lemma 3.1.6 and its Corollary.* $\blacksquare$

Remark 3.1.3. *The assertions of Lemmata 3.1.6 and 3.1.7 become obvious if one observes that the operator $\nu^{-1}\delta$ maps the ring $\mathfrak{O}[1/(\theta_0(z) - 1)]$ (respectively $\mathfrak{O}[\theta_0(z)]$-module $(\theta_0(z) - 1)^{-r}\mathfrak{O}[z]$) into its ideal $\nu^{-1}\mathfrak{O}[1/(\theta_0(z) - 1)]$ (respectively into $\mathfrak{O}[z]$-module $\nu^{-1}(\theta_0(z) - 1)^{-r-1}\mathfrak{O}[z]$ and that*

$$\nu^{-1}(\theta_0(z) - 1)^{-r-1}\mathfrak{O}[z] \supset (\theta_0(z) - 1)^{-r}\mathfrak{O}[z] \times \nu^{-1}(\theta_0(z) - 1)^{-1}\mathfrak{O}[z].$$

In view of the relations (3.1.70),(3.1.71) and (3.1.63),

$$\det(B^{\sim}(z) - d^{\wedge}E) = \frac{-1}{\theta_0(z) - 1}(-d^{\wedge})^{m+n},$$

$$\det\left(\frac{-1}{d^{\wedge}}(B^{\sim}(z)-d^{\wedge}E)\right)=\frac{-1}{\theta_0(z)-1},$$

$$\det(Q^{\sim}(B^{\sim}(z))=\left(\det\left(\frac{-1}{d^{\wedge}}(B^{\sim}(z)-d^{\wedge}E)\right)\right)^{md^{\wedge}}=(1-\theta_0(z))^{-md^{\wedge}}, \tag{3.1.77}$$

$$Q^{\sim}((B^{\sim}(z))^{-1}\in \mathrm{Mat}_{\,m+n}(\mathbb{Q}[\theta_0(z)]),$$

and

$$A^{\sim}(z)=(Q^{\sim}(B^{\sim}(z)))^{-1}P^{\sim}(B^{\sim}(z))\in \mathrm{Mat}_{\,m+n}(\mathbb{Q}[\theta_0(z),1/(\theta_0(z)-1)]). \tag{3.1.78}$$

Let

$$A(z,\nu^{-1})=(U_1(z,\nu^{-1}))^{-1}U_0(z,\nu^{-1}). \tag{3.1.79}$$

Clearly,

$$A(z,\nu^{-1})\in \mathrm{Mat}_{\,m+n}(\mathbb{Q}(\theta_0(z),\nu^{-1})).$$

A rational function $R(x)\in k(x), x:=(x_1,\dots,x_n)$, is said to be well defined in a point $a\in\bar{k}^n$ if $R(x)=P(x)Q(x)^{-1}$ with $P(x)\in k[x]$, $Q(x)\in k[x]$, $Q(a)\neq 0$, where $\bar{k}$ stands for the algebraic closure of the field k. For $\varepsilon>0$, let $\mathfrak{N}_0(\varepsilon)=\{\nu^{-1}\in\mathbb{C}\colon\ |\nu^{-1}|\leq\varepsilon\}$. We denote by Z the above function $\theta_0(z)$, which is holomorphic on $\Omega^{\vee}$ and is transcendental over $\mathbb{C}$; we name an element $T\in\mathbb{C}(Z),\nu^{-1}$ as well defined on the point $(z,\nu^{-1})\in\mathfrak{F}\times\mathfrak{N}_0(\varepsilon)$, if it is well defined on the point $(\theta_0(z),\nu^{-1})\in\mathbb{C}\times\mathfrak{N}_0(\varepsilon)$ and we name an element $T\in\mathbb{C}(Z)$ as well defined on the point $z\in\mathfrak{F}$, if it is well defined on the point $\theta_0(z)\in\mathbb{C}$. Given a subset M of $\mathfrak{F}$, let $\mathfrak{H}(M,\varepsilon)$ stand for the subset of the field $\mathbb{Q}(\theta_0(z),\nu^{-1})$ consisting of those elements of that field, which are well defined in every point of the set $M\times\mathfrak{N}_0(\varepsilon)$, and let $\mathfrak{H}_0(M)$ stand for the subset of the field $\mathbb{Q}(\theta_0(z)$ consisting of those elements of that field, which are well defined in every point of the set M.

Lemma 3.1.8. *Let F be a closed bounded subset of the set $\Omega^{\wedge}$ and $1\notin\theta_0(F)$. There exists an $\varepsilon_0=\varepsilon_0(F)\in(0,1)$ such that*

$$A(z,\nu^{-1})\in \mathrm{Mat}_{\,m+n}(\mathfrak{H}(F,\varepsilon_0));$$

moreover, $A(z,0)=A^{\sim}(z)$ for each $z\in F$.

Proof. *Since $1\notin\theta_0(F)$, it follows that*

$$U_i(z,\nu^{-1})\in \mathrm{Mat}_{\,m+n}(\mathfrak{H}(F,1)),\ i=0,1.$$

Therefore $\det(U_i(z,\nu^{-1}))\in\mathfrak{H}(F,1)$. *By Lemma 3.1.7,*

$$U_0(z,0)=P^{\sim}(B^{\sim}(z))\in\mathfrak{H}_0(F),U_1(z,0)=Q^{\sim}(B^{\sim}(z))\in\mathfrak{H}_0(F).$$

Let $\mu = \min\{|\det(Q^{\sim}(B^{\sim}(z)))| : z \in F\}$. *In view of (3.1.77),* $\mu > 0$. *Since* F *is compact, there is an* $\varepsilon_0 = \varepsilon_0(F) \in (0,1)$ *such that*

$$|\det(U_2(z,\nu^{-1})) - \det(U_2(z,0))| < \frac{\mu}{2},$$

where $z \in F$ *and* $|\nu^{-1}| \leq \varepsilon_0$. ■

Lemma 3.1.9. *As above, let* F *be a closed bounded subset of* $\mathfrak{F}$ *(in particular,* F *may be an one-point set) and let* $\varepsilon_0 \in (0,1)$. *For* $n \in \mathbb{N}$, $z \in F$, $\nu \in \mathbb{N} + [1/\varepsilon_0]$, *let*

$$A(z,\nu^{-1}) \in \mathrm{Mat}_n(\mathfrak{H}(F,\varepsilon_0)) \tag{3.1.80}$$

and let the sequence $X_\nu \in (\mathbb{C}^F)^n$ *satisfies the relation*

$$X_{\nu+1}(z) = A(z,\nu^{-1})X_\nu(z) \tag{3.1.81}$$

for any $z \in F$. *Moreover, suppose that, for each* $z \in F$, *the characteristic polynomial*

$$\det(\lambda E - A^{\sim}(z)) \tag{3.1.82}$$

for $A^{\sim}(z) = A(z,0)$ *has only simple roots, that none of the elements of the first row of the matrix*

$$C = C(z) \in \mathrm{Mat}_n(\mathbb{C}^F) \tag{3.1.83}$$

vanishes, that $\det(C(z)) \neq 0$, *and that the matrix* $(C(z))^{-1}A(z,0)C(z)$ *is diagonal. Then there is* $\varepsilon_3 \in (0,\varepsilon_0)$ *such that, for any* $z \in F, \nu \in \mathbb{N}+[1/\varepsilon_3]$, *the first element* $x_\nu(z)$ *of the column* $X_\nu(z)$ *satisfies the equation*

$$x_{\nu+n}(z) + \sum_{j=0}^{n-1} q_j^*(z,\nu^{-1})x_{\nu+j}(z) = 0, \tag{3.1.84}$$

moreover,

$$q_j^*(z,\nu^{-1}) \in \mathfrak{H}(F,\varepsilon_3), \tag{3.1.85}$$

for $j = 0,\dots,n-1$, *and the polynomial*

$$\lambda^n + \sum_{j=0}^{n-1} q_j^*(z,0)\lambda^{\nu+j}(z) \tag{3.1.86}$$

coincides with the polinomial (3.1.82).

Proof *may be found in [35], section 4, Lemma 2.4.4.* ■

Lemma 3.1.10 *Let D be a bounded domain in $\mathfrak{F}$. and let F be closure of D. Let, as in previous Lemma, $\varepsilon_0 \in (0,1)$ and for $n \in \mathbb{N}$, $z \in F$, $\nu \in \mathbb{N} + [1/\varepsilon_0]$, we have (3.1.80). Let is given the sequence $X_\nu = X_\nu(z), \nu \in \mathbb{N}$, of continuous maps of D in $\mathbb{C}^n$ satisfying to the relation (3.1.81) for $z \in D$, (but not necessary for $z \in F$) Let , as in previous Lemma, for each $z \in F$, the characteristic polynomial (3.1.82) has only simple roots. Let further $C = C(z)$ is continuous map of gothF in $\mathrm{Mat}_n(\mathbb{C})$ such that, for each $z \in F$ the following conditions are fulfilled:*

(a) none of the elements of the first row of the matrix $C = C(z)$ vanishes;

(b) $\det(C(z)) \neq 0$;

(c) the matrix $(C(z))^{-1}A(z,0)C(z)$ is diagonal.

Let, finally,the first element $x_\nu(z)$ of the column $X_\nu(z)$, is cotinuous on F

Then there exists a $\varepsilon_3 \in (0, \varepsilon_0)$ such that, for any $z \in F, \nu \in \mathbb{N} + [1/\varepsilon_3]$, the relations (3.1.84) and (3.1.85) hold, and the polynomial (3.1.86) coincides with the polynomial (3.1.82).

Proof *Repeating the arguments of the proof of the Lemma 2.4.4 in [35], section 4, we see that there exist $\varepsilon_2 \in (0, \varepsilon_0)$ such that the equality (3.1.84) holds for any $z \in D$ and $\nu \in \mathbb{N} + [\varepsilon_2]$ with the continuous on $\mathfrak{F} \times \mathfrak{N}_0(\varepsilon_2) coefficients$; since $x_\nu(z), \nu \in \mathbb{N}$, also is continuous on F the equality 3.1.84 holds for any $z \in F$. The proofs of the other aqsserions of the lemma may be found in [35], section 4, Lemma 2.4.4.* ■

Lemma 3.1.11. *Let M be nonempty subset of $\mathfrak{F}$ (in particular, M may consist of a single point) and let $n \in \mathbb{N}$. Let*

$$A^\sim(z) \in Mat_n(\mathfrak{H}_0(M)), \tag{3.1.87}$$

let $b^\sim_j(z) \in \mathfrak{H}_0(M)$ for $0 \leq j \leq n$, and let $b^\sim_n(z) = 1$. Suppose that, for each $z \in M$, all the roots of the polynomial

$$D^\sim(z, \lambda) = \sum_{j=0}^{n} b^\sim_j(z)\lambda^j \tag{3.1.88}$$

are simple. Let $B^\sim(z)$ be an $n \times n$ - matrix defined as follows: for $l = 1, \ldots, n$ the element the $l-$th column and the last row is equal to $-b^\sim_{l-1}(z)$; if $n > 1$, then the first $n-1$ elements of the first column are equal to 0; finally, the first $n-1$ rows and the last $n-1$ columns are occupied by the unit $(n-1)\times(n-1)$ matrix. Let $P^\sim(z, \lambda) \in \mathfrak{H}_0(M)[\lambda]$ and $Q^\sim(z, \lambda) \in \mathfrak{H}_0(M)[\lambda])$. Suppose that, for any $z \in M$, the polynomial $Q^\sim(z, \lambda)$ does not vanish when λ runs through set of the roots of the polynomial $D^\sim(z, \lambda)$, the map

$$\lambda \to P^\sim(z, \lambda)/Q^\sim(z, \lambda) \tag{3.1.89}$$

is injective on that set, and $A^{\sim}(z)$ *be an* $n \times n$ *- matrix such that*

$$Q^{\sim}(z, B^{\sim}(z))A^{\sim}(z) = P^{(}z, B^{\sim}(z)) \tag{3.1.90}$$

for $z \in M$*. Then, for every* $z \in M$*, all the roots of*

$$\det(\lambda E - A^{\sim}(z))$$

are simple and there exists a matrix $C = C(z) \in Mat_n(\mathbb{C}^M)$ *with the following properties: a) det*$(C(z)) \neq 0$ *every* $z \in M$*; b)* $(C(z))^{-1}A^{\sim}(z)C(z)$ *is a diagonal matrix for every* $z \in M$*; c) for every* $z \in M$*, no element in the first row of the matrix* $C(z)$ *is equal to zero.*

Proof may be found in [35], Section 4, Lemma 2.4.5, or in [31], pp. 19 - 21, Lemma 5. There we have supposed that M is closed and bounded, but have not used this condition. ■

Lemma 3.1.12. *Let* M *be simply connected domain in* $\mathfrak{F}$*. Let further are fulfilled all the conditions of the previous Lemma. Then all the elaments of the matrix* $C = C(z)$ *in the asserion of the previous lemma may be chosen as holomorphic functins of* $z \in M$*.*

Proof. Since M be simply connected domain, it follows that the roots

$$\lambda_i = \lambda_i(z), i = 1, \dots, n,$$

of the polynomial (3.1.88) may be represented as holomorphic functions of $z \in M$. Therefore the constructed in the proof of the Lemma 2.4.5 in [35], Section 4,all the elements of the matrix $C = C(z)$ are holomorphic functions of $z \in M$. ■

The substitution

$$w = \frac{d^{\wedge}\eta + d^{\vee}}{\eta - 1} \tag{3.1.91}$$

transforms the polynomials (3.1.60), (3.1.62), (3.1.63) and the rational function $\dfrac{P^{\sim}(w)}{Q^{\sim}(w)}$, respectively, in

$$D^{\sim}(z, w) = -(\eta - 1)^{-m-n}(\delta^{\wedge})m + n((\eta + \delta_1)^{m+n} - (1 + \delta)^{m+n}\theta_0(z)z\eta^n, \tag{3.1.92}$$

$$P^*(0, w) = P_0^*(0, w) = P^{\sim}(w) = \left(\frac{-1}{d^{\vee}}(w + d^{\vee})\right)^{nd^{\vee}} =$$

$$\left(\frac{\eta(1 + d^{\vee}/d^{\wedge})}{(1 - \eta)(d^{\vee}/d^{\wedge})}\right)^{nd^{\vee}}, \tag{3.1.93}$$

$$Q^*(0,w) = P_1^*(0,w) = Q^{\sim}(w) = \left(\frac{-1}{d^{\wedge}}(w-d^{\wedge})\right)^{md^{\wedge}} =$$

$$\left(\frac{(1+d^{\vee}/d^{\wedge})}{(1-\eta)}\right)^{md^{\wedge}}, \tag{3.1.94}$$

$$P^{\sim}(w)/Q^{\sim}(w) \ = \ h^{\sim}(\eta) = \tag{3.1.95}$$

$$\left(\frac{\eta}{d^{\vee}/d^{\wedge}}\right)^{nd^{\vee}} \left(\frac{1-\eta}{1+d^{\vee}/d^{\wedge}}\right)^{md^{\wedge}-nd^{\vee}}.$$

Let

$$D^{\wedge}(z,\eta) = (\eta + d^{\vee}/d^{\wedge})^{m+n} - (1+d^{\vee}/d^{\wedge})^{m+n}\theta_0(z)\eta^n. \tag{3.1.96}$$

The substitution (3.1.91) and inverse substitution

$$\eta = (w+d^{\vee})/(w-d^{\wedge}) \tag{3.1.97}$$

relate the roots w of the polynomial $D^{\sim}(z,w)$ with the different from 1 roots η of the polynomial $D^{\wedge}(z,\eta)$.

Remark 3.1.4. *It may be remarked that for the real $z > 1$ and $0 < \eta < 1$ the equation*

$$D^{\wedge}(z,\eta) = 0 \tag{3.1.97}$$

is equivalent to the equation

$$f_0(\alpha,\delta,z,\eta) = 0, \tag{3.1.98}$$

where

$$f_0(\alpha,\delta,z,\eta) = (\eta+\delta)^{\alpha+1} - (1+\delta)^{\alpha+1}z\eta,$$

$\alpha = m/n, \delta = d^{\vee}/d^{w}edge$, was considered in (14) of paper [35]. Moreover, if $0 < \eta < 1$, then $P^{\sim}(w)/Q^{\sim}(w)$ in (3.1.94) coincides with $(\tilde{\rho}(\alpha,\delta\eta))^{nd^{\wedge}}$, where $(\tilde{\rho}(\alpha,\delta,\eta)$ was considered in [35], section 3.

Lemma 3.1.13. *If for the integral (3.1.4) the condition (C3) is satsified then this integral is a continuous function of $z \in \Omega^{\wedge}$.*

Proof. The assertion of the lemma will be established, if we prove that the integral

$$\frac{1}{2\pi i}\int\limits_{L_2} g(z,s)ds$$

is uniformly convergent in each boundet subset of the set $\Omega^{\wedge}$. If the condition (C3) is satsified then in correspondence to (1.22) in [34], $g(s) = O(1)|\sigma|^{\mathrm{Re}\,(\Delta^*)}$ on the rays $|\,\mathrm{Im}\,(s)| = \tau_0 > 0$, $\mathrm{Re}\,(s) \leq -\sigma_0 < 0$ and $O(1)$ uniformly bounded in in each bounded subset of the set $\Omega^{\wedge}$. ■

§ 2. Short proof an old result.

Here I consider the most simple case, when $m = n = 2, d^\vee = d^\wedge = 1, \beta^\vee = \beta^\wedge = 0$. Therefore inequalities (3.1.9) and (3.1.10) hold. The rational function (3.1.95) take the form

$$h^\sim(\eta) = \eta^2 \tag{3.2.1}$$

The polinomial (3.1.96) take the form

$$D^\wedge(z,\eta) = (\eta+1)^4 - 2^4\theta_0(z)\eta^2, \tag{3.2.2}$$

where $z \in \Omega^\wedge$. Therefore we can represent z in the form $z = (r^2, \phi)$, where $1 \le r, \phi \in \mathbb{R}$ Let further $\psi = \phi/2 - k\pi$, where $k = 0, 1$. Then

$$D^\wedge(z,\eta) = \prod_{k=0}^{1} \left((\eta+1)^2 + 4r\exp(i\phi/2 - ik\pi)\eta\right) \tag{3.2.3}$$

So, we have to study the zeros of the polynomial

$$D_1^\vee(r,\psi,\eta) = (\eta+1)^2 + 4r\exp(i\psi)\eta, \tag{3.2.4}$$

where $1 \le r, \psi \in \mathbb{R}$ The equality (3.2.3) may be rewritten in the form

$$D^\wedge(z,\eta) = \prod_{k=0}^{1} D_1^\vee(r, \phi/2 - k\pi, \eta), \tag{3.2.5}$$

where $z = (r^2, \phi), 1 \le r, \phi \in \mathbb{R}$. Since $D^\wedge((r^2, phi + 2\pi, \eta) = D^\wedge((r^2, phi, \eta)$, i follows that we can consider only $phi \in [0, 2\pi]$; therfore $\psi = \phi/2 - k\pi \in [-\pi, \pi]$, where $k = 0, 1$, and this corresponds to the equality $D_1^\vee(r, psi + 2\pi, \eta) = D_1^\vee(r, psi, \eta)$; therefore we may consider only $\psi \in [-\pi, \pi]$. Clearly,

$$D_1^\vee(r,\pi,\eta) = D_1^\vee(r,-\pi,\eta). \tag{3.2.6}$$

Clearly, the polynomial $D^\vee(1,\psi,\eta)$ may be rewritten in the form

$$\begin{gathered} D_1^\vee(1,\psi,\eta) = (\eta + 1 + 2\exp(i\psi))^2 - 4\exp(2i\psi) - 4\exp(i\psi) = \\ (\eta + 1 + 2\exp(i\psi))^2 - 8\exp(3i\psi/2)\cos(\psi/2). \end{gathered} \tag{3.2.7}$$

and has exactly two roots:

$$\eta_k^*(\psi) = -1 - 2\exp(i\psi) - (-1)^k\sqrt{8\cos(\psi/2)}\exp(3i\psi/4), \tag{3.2.8}$$

where $k = 0, 1$ and $\psi \in [-\pi, \pi]$. Clearly,

$$\eta_k^*(-\psi) = \overline{\eta_k^*(\psi)} \tag{3.2.9}$$

where $\psi \in [-\pi, \pi]$. Further we have

$$|\eta_k^*(\psi)|^2 = \tag{3.2.10}$$

$$5 + 8cos(\psi/2) + 4\cos(\psi) + 2(-1)^k\sqrt{8\cos(\psi/2)}\cos(3\psi/4)+$$
$$4(-1)^k\sqrt{8\cos(\psi/2)}\cos(\psi/4) =$$
$$5 + 8cos(\psi/2) + 4\cos(\psi) + 4(-1)^k\sqrt{8\cos(\psi/2)}\cos(\psi/2)\cos(\psi/4)+$$
$$2(-1)^k\sqrt{8\cos(\psi/2)}\cos(\psi/4),$$

where $\psi \in [-\pi, \pi]$. Since

$$4(-1)^k\sqrt{8\cos(\psi/2)}\cos(\psi/4) =$$

$$0 \le 4\sqrt{8\cos(\psi/2)}\cos(\psi/2)\cos(\psi/4)+$$
$$2\sqrt{8\cos(\psi/2)}\cos(\psi/4)$$

for $\psi \in [-\pi, \pi]$, it follows that

$$|\eta_1^*(\psi)| \le |\eta_0(\psi)^*|, \tag{3.2.11}$$

where $\psi \in [-\pi, \pi]$ and equality reaches only for $\psi = \pm\pi$. It follows from (3.2.9) that $|\eta_0^*(\psi)|$ decreases together with increasing $\psi \in [0, \pi]$. According to (3.2.4),

$$\eta_1^*(\psi)\eta_0^*(\psi) = 1, \tag{3.2.12}$$

where $\psi \in [-\pi, \pi]$. Therefore, $|\eta_1(\psi)|$ increases together with increasing $\psi \in [0, \pi]$. It follows from (3.2.5) that the four numbers

$$\eta_l^*(\phi/2 - k\pi), \tag{3.2.13}$$

where $k = 0, 1$ and $l = 0, 1$ compose all the roots of the polynomial $D^\wedge((1, \phi), \eta)$. In particular, the four numbers

$$\eta_0^*(\pi/2) = -1 - 2i - 2^{5/4}\exp(3i\pi/8) = \tag{3.2.14}$$

$$-1 - 2i - 2\left(\sqrt{\frac{\sqrt{2}-1}{2}} + i\sqrt{\frac{\sqrt{2}+1}{2}}\right) =$$

$$-1-2i-\left(\sqrt{\sqrt{8}-2}+i\sqrt{\sqrt{8}+2}\right),$$

$$\eta_1^*(\pi/2)=1/\eta_0^*(\pi/2)\eta_0, \tag{3.2.15}$$

$$\eta_0^*(-\pi/2)\eta_0=\overline{\eta}_0^*(\pi/2) \tag{3.2.16}$$

and

$$\eta_1^*(-\pi/2)=\overline{\eta}_1^*(\pi/2) \tag{3.2.16}$$

compose all the roots of the polynomial $D^\wedge((1,\pi),\eta)$. Moreover, since

$$\mathrm{Re}\,(\eta_0^*(\pi/2))\,\mathrm{Im}\,(\eta_0^*(\pi/2))=\left(1+\sqrt{\sqrt{8}-2}\right)\left(2+\sqrt{\sqrt{8}+2}\right)>0,$$

it follows that

$$h^\sim(\eta_l^*(\pi/2-k\pi))=(\eta_l^*(\pi/2-k\pi))^2\notin\mathbb{R} \tag{3.2.17}$$

where $k=0,1$ and $l=0,1$ and therefore, in view of (3.2.11), all the four numbers

$$h^\sim(\eta_l^*(\pi/2-k\pi))$$

where $k=0,1$ and $l=0,1$ are pairwise distinct. Let

$$D_2^\vee(r,\psi,\eta)=\eta^2-(2+4r\exp(i\psi))^2-2)\eta+1. \tag{3.2.18}$$

Then the roots of the polynomial (3.2.18) are squares of the roots of the polynomial $D_1^\vee(r,\psi,\eta)$ and the roots of the polynomial

$$D_2^\wedge(z,\eta)=\prod_{k=0}^{1}D_2^\vee(r,\phi/2-k\pi,\eta), \tag{3.2.19}$$

where $z=(r^2,\phi), 1\le r,\phi\in\mathbb{R}$, are squares of the roots of the polynomial $D^\wedge(z,\eta)$, where $z=(r^2,\phi)$, $1\le r$, $\phi\in\mathbb{R}$. Let $\mathfrak{D}_0^\wedge(z)$ stand for the discriminant of the polynomial $D_2^\wedge(z,\eta)$. Since all the numbers (3.2.17) are distinct, it follows that

$$\mathfrak{D}_0^\wedge((1,\pi))\neq 0. \tag{3.2.20}$$

It follows from (3.2.10), and (3.2.12) that

$$|\eta_0^*(\pm\pi/2)|^2 > 5, |\eta_1^*(\pm\pi/2)|^2 < 1/5 \tag{3.2.21}$$

Let $\varepsilon \in (0, 1)$,

$$U_\varepsilon^\vee = \{z \in \mathfrak{F}\colon 1 < |z| < 1 + \varepsilon/2, pi - \varepsilon/2 \leq \operatorname{Arg}(z) < \pi + \varepsilon/2\},$$

$$F_\varepsilon = \{z \in \mathfrak{F}\colon 1 \leq |z| \leq 1 + \varepsilon/2, pi - \varepsilon/2 \leq \operatorname{Arg}(z) \leq \pi + \varepsilon/2\},$$

$$U_\varepsilon = \{z \in \mathfrak{F}\colon 1 - \varepsilon < |z| < 1 + \varepsilon, pi - \varepsilon < \operatorname{Arg}(z) \leq \pi + \varepsilon\}.$$

Clearly F_ε is the closure of the domain $U_\varepsilon^\vee$ and $F_\varepsilon \subset U_\varepsilon$ for any $\varepsilon \in (0, 1)$. In view of (3.2.20) there exist $\varepsilon_1 \in (0, 1)$, such that

$$\mathfrak{D}_0^\wedge(z) \neq 0 \tag{3.2.22}$$

for all the $z \in U_{\varepsilon_1}$. Since the domain U_{ε_1} is simply connected there exist four holomorphic in U_{ε_1} functions

$$\eta_k^{**}(z), k = 0,\ 1,\ 2,\ 3,$$

such that $D^\wedge(z, \eta_k^{**}(z)) = 0$ for $k = 0,\ 1,\ 2,\ 3$ and

$$\eta_{k+2l}^{**}((1, \pi)) = \eta_l^*((-1)^k\pi/2) \tag{3.2.23}$$

for $k = 0, 1$, $l = 0, 1$. In view of (3.2.21) and (3.2.23), thre exist $\varepsilon_2 \in (0, \varepsilon_1)$ such that

$$|\eta_k^{**}(z)|^2 > 4, |\eta_{2+k}^{**}(z)|^2 < 1/4 \tag{3.2.24}$$

for all the $z \in U_{\varepsilon_2}$. In view of (3.2.22), all the four numbers

$$h^\sim(\eta_k^{**}(z)) = (\eta_k^{**}(z))^2,\ k = 0, 1, 2, 3, \tag{3.2.25}$$

are pairwise distinct Since

$$\theta_0(z) \neq 1, \tag{3.2.26}$$

for all the $z \in U_{\varepsilon_2}$, it follows that the matrix (3.1.68) is well defined for all the $z \in U_{\varepsilon_2}$.

I wish to apply the Lemma 3.1.11 and the Lemma 3.1.12 now taking on the role of the set M the domain U_{ε_2}, taking on the role of the variable λ in (3.1.88) the variable w in (3.1.57) - (3.1.63), taking on the role of the number n in (3.1.88) the number 4, taking on the role of the polynomial $D^{\sim}(z,\lambda)$ in (3.1.88) the polynomial $D^{\sim}(z,w)$ in (3.1.60) with $m=n=2$, taking on the role of the polynomials $P^{\sim}(z,\lambda)\in\mathfrak{H}_0(M)[\lambda]$ and $Q^{\sim}(z,\lambda)\in\mathfrak{H}_0(M)[\lambda]$) of the Lemma (3.1.11) respectively the polynomial $P^{\sim}(w)$ in (3.1.62) with $n=2$ and $d^{\vee}=1$ and the polynomial $Q^{\sim}(w)$ in (3.1.63 *with* $m=2$ *and* $d^{\wedge}=1$, *taking on the role of the matrix* $B^{\sim}(z)$ *in (3.1.90) the matrix* $B^{\sim}(z)$ *in (3.1.68) In view of (3.1.94), (3.1.92) and (3.2.24) the polynomial* $Q^{\sim}(w)$ *does not vanish on the set of the roots of the polynomial* $D^{\sim}(z,w)$. *Theefore the matrix* $Q^{\sim}((B^{\sim}(z))^{-1}$ *in (3.1.77) and the matrix* $A^{\sim}(z)$ *in (3.1.78) are defined for all the* $z\in U_{\varepsilon_2}$ *and this matrix plays now the role of te matrix* $A^{\sim}(z)$ *in (3.1.90). Since all the four numbers (3.2.25) are distinct the map*

$$w\to P^{\sim}(w)/Q^{\sim}(w)=h^{\sim}(\eta)=\eta^2,$$

where,in view of (3.1.97),$\eta=\frac{w+d^{\vee}}{w-d^{\wedge}}$ *is injective on the set of all the roots of the polynomial* $D^{\sim}(z,w)$. *We see that all the condiions of the lemma 3.1.12 are fullfiled. Therefore, for every* $z\in U_{\varepsilon_2}$, *all the roots of*

$$\det(\lambda E-A^{\sim}(z))$$

are simple and there exists the square $n\times n-$*matrix* $C=C(z)$, *whose elements are holomorphic functions of* $z\in U_{\varepsilon_2}$, *with the following properties:*

a) $det(C(z))\neq 0$ *every* $z\in U_{\varepsilon_2}$;

b) $(C(z))^{-1}A^{\sim}(z)C(z)$ *is a diagonal matrix for every* $z\in U_{\varepsilon_2}$;

c) for every $z\in U_{\varepsilon_2}$, *no element in the first row of the matrix* $C(z)$ *is equal to zero.*

I wish to apply the Lemma 3.1.10 now, taking on the role of the matrix $A(z,\nu^{-1})$ *in (3.1.81) the matrix* $A(z,\nu^{-1})$ *in (3.1.79). The equality (3.1.81) we consider in the domain* $U^{\vee}_{\varepsilon_2}$, *but because, in view of (3.1.79), (3.1.73), (3.1.74), (3.1.78), the matrix* $A(z,0)=A^{\sim}(z)$ *is well defined in* $U_{\varepsilon_2}\supset F_{\varepsilon_2}$, *according to the Lemma 3.1.8, there exists an* $\varepsilon_0=\varepsilon_0(F_{\varepsilon_2})\in(0,1)$ *such that*

$$A(z,\nu^{-1})\in \mathrm{Mat}_{\,m+n}(\mathfrak{H}(F_{\varepsilon_2},\varepsilon_0)).$$

Since now $n=2$, *it follows the first componente of the column* $X_\nu(z)$ *are the functions* $f_k(z;\nu), k=1,2,3$ *and,according to the lemma 3.1.13, they are continuous functions in* $F_{\varepsilon_2}\subset\Omega^{\wedge}$. *We see that all the conditions of the Lemma 3.1.10 are fulfilled. Therefore there exists a* $\varepsilon_3\in(0,\varepsilon_0)$ *such that, for any* $z\in F,\nu\in\mathbb{N}+[1/\varepsilon_3]$, *the relations (3.1.84) and (3.1.85) hold, and the polynomial (3.1.86) coincides with the polynomial (3.1.82). Therefore there is* $\varepsilon_3\in(0,\varepsilon_0)$ *such that, for any* $z\in F,\nu\in\mathbb{N}+[1/\varepsilon_3]$, *the first element* $x_\nu(z)=f_k(z;\nu), k=1,2,3$ *of the column* $X_\nu(z)$ *satisfies the equation*

$$x_{\nu+4}(z)+\sum_{j=0}^{3}q^*_j(z,\nu^{-1})x_{\nu+j}(z)=0, \tag{3.2.27}$$

moreover,

$$q_j^*(z,\nu^{-1}) \in \mathfrak{H}(F,\varepsilon_3), \tag{3.2.28}$$

for $j=0,\dots,3$, *and the polynomial*

$$\lambda^4 + \sum_{j=0}^{3} q_j^*(z,0)\lambda^{\nu+j}(z) \tag{3.2.29}$$

coincides with the polinomial (3.1.82). In view of (3.1.18), $x_\nu = \psi_0(\theta_0(z);\nu) = f_2(z,\nu)$ *and* $x_\nu = \psi_1(\theta_0(z);\nu)$ *also are solutions of the equation (3.2.27). The function* $R(t,\nu)$ *in (3.1.16) takes the following form*

$$R(t,\nu) = \frac{\left(\prod\limits_{\kappa=1}^{\nu}(t-\kappa)\right)^2}{\left(\prod\limits_{\kappa=0}^{\nu}(t+\kappa)\right)^2}. \tag{3.2.30}$$

In view of (3.1.17),

$$\psi_\mu(\theta_0((1,\pi);\nu) = \frac{(-1)^\mu}{\mu!}\times \tag{3.2.31}$$

$$\sum_{t=1+\nu}^{\infty}\left(\left(\frac{\partial}{\partial t}\right)^\mu R\right)(t,\nu)(-1)^{-t},$$

where $\mu=0,1$ *and* $\nu\in\mathbb{N}$. *According to (3.2.30), if* $t\ge\nu+1, \nu\in\mathbb{N}$, *then* $|R(t,\nu)|\le 1/t^2, |\left(\frac{\partial}{\partial t}R\right)(t,\nu)/R(t,\nu)|\le 4\nu+2$. *Therefore, in view of (3.2.31)*

$$|\psi_0(\theta_0((1,\pi);\nu)|\le 1/\nu, |\psi_1(\theta_0((1,\pi);\nu)|\le 6, \tag{3.2.32}$$

where $\nu\in\mathbb{N}$. *Clearly,*

$$R(t,\nu) = \sum_{k=0}^{\nu}\sum_{\lambda=1}^{2}\frac{\alpha_{k,\lambda}^{\sim}(\nu)}{(t+k)^\lambda}, \tag{3.2.33}$$

where

$$\alpha_{k,2}^{\sim}(\nu) = \binom{\nu+k}{k}^2\binom{\nu}{k}^2, \tag{3.2.34}$$

$$\alpha^{\sim}_{k,1}(\nu) = 2\binom{\nu+k}{k}^2\binom{\nu}{k}^2\times \tag{3.2.35}$$

$$\left(\left(\sum_{\kappa=1}^{k}\frac{1}{\kappa}\right)-\left(\sum_{\kappa=1}^{\nu-k}\frac{1}{\kappa}\right)-\left(\sum_{\kappa=k+1}^{k+\nu}\frac{1}{\kappa}\right)\right).$$

Since

$$\sum_{t=1}^{\infty}(\theta_0(z))^{-t}/(t+k)^{\lambda+\mu} = (\theta_0(z))^k L_{\lambda+\mu}((\theta_0(z))^{-1}) - \sum_{l=1}^{k}(\theta_0(z))^{k-l}/l^{\lambda+\mu}$$

with $\lambda = 1,\, 2,\, \mu \in \mathbb{N}-1$, *it follows from (3.1.16) that*

$$\psi_\mu(z;\nu) = -P_{3+\mu}(\theta_0(z);\nu)+ \tag{3.2.36}$$

$$\sum_{\lambda=1}^{2}P_\lambda(\theta_0(z);\nu)\binom{\mu+\lambda-1}{\mu}L_{\lambda+\mu}((\theta_0(z))^{-1}),$$

where

$$P_\lambda(\theta_0(z);\nu) = \sum_{k=0}^{\nu}\alpha^{\sim}_{k,\lambda}(\nu)(\theta_0(z))^k, \tag{3.2.37}$$

$$P_{3+\mu}(z;\nu) = \sum_{l=0}^{\nu-1}(\theta_0(z))^l\sum_{k=l+1}^{\nu}\sum_{\lambda=1}^{2}\binom{\mu+\lambda-1}{\mu}\frac{\alpha^{\sim}_{k,\lambda}(\nu)}{(k-l)^{\lambda+\mu}} \tag{3.2.38}$$

for $\lambda = 1,\, 2,\, \mu \in \mathbb{N}-1$. *Let* p *is prime number,* $p > 3$. *According to the Lemma 9 in [25], in view of (3.2.34),*

$$\alpha^{\sim}_{0,2}(p) = 1, \alpha^{\sim}_{p,2}(p) \in 4 + p\mathbb{Z}, \tag{3.2.39}$$

$$\alpha^{\sim}_{k,2}(p) \in p^2\mathbb{Z},\, k = 1,\, \dots,\, p-1. \tag{3.2.40}$$

Therefore, in view of (3.2.37), (3.2.39), (3.2.40),

$$P_2(-1;p) \in -3 + p\mathbb{Z}. \tag{3.2.41}$$

In view of (3.2.35)

$$\operatorname{ord}_p(\alpha^{\sim}_{0,1}(p) + 2/p) \geq 0,\, \operatorname{ord}_p(\alpha^{\sim}_{p,1}(p) - 2/p) \geq 0, \tag{3.2.42}$$

$$\operatorname{ord}_p(\alpha^{\sim}_{k,1}(p)) \geq 1,\ k = 1, \ldots, p-1. \tag{3.2.43}$$

Therefore in view of (3.2.37), (3.2.42), (3.2.43),

$$\operatorname{ord}_p(P_1(-1;p) + 4/p) \geq 0 \tag{3.2.44}$$

In view of (3.2.38), (3.2.39), (3.2.40), (3.2.42), (3.2.43),

$$\operatorname{ord}_p\left(P_{3+\mu}(-1,;p) - \sum_{l=0}^{p-1}(-1)^l \sum_{\lambda=1}^{2}\binom{\mu+\lambda-1}{\mu}\frac{\alpha^{\sim}_{p,\lambda}(p)}{(p-l)^{\lambda+\mu}}\right) \geq 1, \tag{3.2.45}$$

$$\operatorname{ord}_p\left(\sum_{l=0}^{p-1}\sum_{\lambda=1}^{2}\binom{\mu+\lambda-1}{\mu}\frac{(-1)^l\alpha^{\sim}_{p,\lambda}(p)}{(p-l)^{\lambda+\mu}} - \sum_{\lambda=1}^{2}\binom{\mu+\lambda-1}{\mu}\frac{\alpha^{\sim}_{p,\lambda}(p)}{p^{\lambda+\mu}}\right) \geq -1, \tag{3.2.46}$$

$$\operatorname{ord}_p\left(\sum_{\lambda=1}^{2}\binom{\mu+\lambda-1}{\mu}\frac{\alpha^{\sim}_{p,\lambda}(p)}{(p)^{\lambda+\mu}} - \binom{\mu+1}{\mu}\frac{\alpha^{\sim}_{p,\lambda}(p)}{(p)^{2+\mu}}\right) \geq -1-\mu, \tag{3.2.47}$$

where $\mu = 0,\ 1$. It follows from (3.2.41), (3.2.44) - (3.2.47) that

$$\operatorname{ord}_p(P_{1+\mu}(-1;p)) = 1-\mu, \operatorname{ord}_p(P_{3+\mu}(-1;p)) = 2+\mu, \tag{3.2.48}$$

where $\mu = 0, 1$. The following Lemma is direct corollary of (3.2.48).

Lemma 3.2.1. *For every $\kappa \in \mathbb{N}$ the four sequences*

$$\{P_k(-1, \kappa+\omega) : \omega \in \mathbb{N}\}, \tag{3.2.49}$$

with $k = 1, \ldots, 4$, are linearly independent over $\mathbb{C}$.

Proof. The assertion of the Lemma is evident (see also [25], Lemma 14). ■

It follows from (3.2.36) that

$$\psi_\mu((1,\pi);\nu) = -P_{3+\mu}(-1;\nu) + \sum_{\lambda=1}^{2} P_\lambda(-1;\nu)\binom{\mu+\lambda-1}{\mu}L_{\lambda+\mu}(-1), \tag{3.2.50}$$

and, more detailed,

$$\psi_0((1,\pi);\nu) = (P_1(-1;\nu)/2)\log(4) + (P_2(-1;\nu)/2)\zeta(2) - P_3(-1;\nu) \tag{3.2.51}$$

$$\psi_1((1,\pi);\nu) = (P_1(-1;\nu)/2)\zeta(2) + (P_2(-1;\nu)/2)3\zeta(3) - P_4(-1;\nu) \tag{3.2.52}$$

It follows from (3.2.50), (3.2.51) and Lemma 3.2.1 that

$$x_\nu = \psi_0((1,\pi);\nu), x_\nu = \psi_1((1,\pi);\nu) \tag{3.2.53}$$

are nonzero solutions of the equation (3.2.27). In view of (3.2.32) and according to the Perron's Theorem, all the solutions (3.2.53) correspond to the following set of two roots of the characteristic polynomial (3.2.29)

$$(\eta_1^*(\pi/2-k\pi))^2 = \overline{(\eta_1^*(\pi/2-(1-k)\pi))^2} = \tag{3.2.54}$$

$$1/(\eta_0^*(\pi/2-k\pi))^2 = \overline{1/(\eta_0^*(\pi/2-(1-k)\pi))^2},$$

where $k = 0, 1$, and therefore

$$\limsup_{\nu\to\infty} |\psi_\lambda((1,\pi);\nu)|^{1/\nu} = |\eta_1^*(\pi/2-k\pi)|^2 = 1/|\eta_0^*(\pi/2-k\pi)|^2. \tag{3.2.55}$$

Let, as usual, $\pi(x)$ stand for the number of rational primes p with $p \le x$, $x \in \mathbb{R}$, so that $\pi(x) = 0$ for $x < 2$. Let $\{\alpha, \beta, \gamma\} \subset \mathbb{R}$. If $\pi(\alpha) < \pi(\beta)$ and $1 \le \gamma$, let $d_{\alpha,\beta,\gamma}$ denote the smallest positive integer d satisfying the condition

$$\operatorname{ord}_p\left(\frac{d}{\kappa}\right) \ge 0 \tag{3.2.56}$$

for all primes p in the interval $\alpha < p \le \beta$ and all the positive integers κ with $\kappa \le \gamma$. We let, by definition, $d_{\alpha,\beta,\gamma} = 1$ if ether $\gamma < 1$, or $\pi(\beta) \le \pi(\alpha)$. Clearly,

$$d_{\alpha,\beta,\gamma} = \prod_{\alpha<p\le\beta} p^{[ln(\gamma)/ln(p)]} \le \gamma^{max(\pi(\beta)-\pi(\alpha),0)}. \tag{3.2.57}$$

Lemma 3.2.2. *If*

$$2 \le \alpha, \alpha \le \gamma \le \gamma_0\alpha \tag{3.2.58}$$

for some $\gamma_0 \ge 1$ then

$$d_{\alpha,\beta,\gamma} = \exp\left(\alpha\left(1 + O(1)\frac{1}{\log(\alpha)}\right)\right), \tag{3.2.59}$$

where $O(1)$ depends only from γ_0

Proof. Let $\alpha \ge \gamma_0$. Then $\sqrt(\gamma) \le \sqrt(\gamma_0\alpha) \le \alpha$ and, since $[\log(\gamma)/\log(p)] = 1$ for $p \in (\sqrt(\gamma), \gamma]$ it follows that

$$\log(d_{1,\alpha,\gamma}) = \sum_{p\le\alpha} \log(p)[\log(\gamma)/\log(p)] =$$

$$\log(d_{1,\alpha,\gamma}) = \sum_{p\le\sqrt{\gamma}} \log(p)[\log(\gamma)/\log(p)]+$$

$$\sum_{\sqrt{\gamma}<p\le\alpha} \log(p) =$$

$$\alpha + O(\sqrt{\gamma_0\alpha}\log(\gamma_0\alpha)/log(2)) + O(\alpha/\log(\alpha)) = \alpha + O(1)\alpha/\log(\alpha),$$

when $\alpha \to \infty$. ∎

According to (3.2.59),

$$d_{1,\nu,2\nu} = \exp\left(\nu\left(1 + O(1)\frac{1}{\log(\nu)}\right)\right), \tag{3.2.60}$$

and

$$\lim_{\nu\to\infty} (d_{1,\nu,2\nu})^{1/\nu} = e. \tag{3.2.61}$$

Lemma 3.2.3. *If* $p > \nu \in \mathbb{N} + 1$, *then*

$$\mathrm{ord}_p(\alpha^{\sim}_{k,\lambda}(\nu)) \ge 0 \tag{3.2.62}$$

for $k = 0, \ldots, \nu$, $\lambda = 1, 2$, $\nu \in \mathbb{N}$..

Proof. In view of (3.2.33) for $\lambda = 2$ the assertion of the lemma is trivial. Let $\lambda = 1$. If $p > \nu + k$, then the asserted inequality (3.2.62) is an immediate consequence of the equalities (3.2.35). Suppose that

$$\nu < p \le \nu + k. \tag{3.2.63}$$

Then

$$2\nu < 2p < p^2, \tag{3.2.64}$$

and, since $k \le \nu$ in (3.2.35), it follows that

$$\mathrm{ord}_p\left(\binom{\nu) + k}{k}^2\right) = 2. \tag{3.2.65}$$

Since the variable κ in the equality (3.2.35) satisfies the inequality

$$\kappa \le \nu + k \le 2\nu \tag{3.2.66}$$

it folows that

$$\operatorname{ord}_p\left(\binom{\nu+k}{k}^2\binom{\nu+k}{k}^2/\kappa\right)\geq 2. \tag{3.2.67}$$

■

Lemma 3.2.4. *Each of the numbers*

$$\alpha^{\sim}_{k,\lambda}(\nu)(d_{1,\nu,2\nu}))^{2-\lambda}$$

for $k=0,\ldots,\nu,\lambda=1,\ ,\ 2,\nu\in\mathbb{N}$*, is contained in* $\mathbb{Z}$*.*

Proof. For $\lambda=1$ the assertion of the Lemma follows directly from (3.2.35) and Lemma 3.2.3. For $\lambda=2$ the assertion of the Lemma follows directly from (3.2.34).
■

Lemma 3.2.5. *Let*

$$P^*_s(\theta_0(z);\nu)=d^3_{1,\nu,2\nu}P_s(\theta_0(z);\nu), \tag{3.2.68}$$

where $s=1,\ 2,\ 3,\ 4;\ \nu\in\mathbb{N}$*. Then*

$$P^*_s(\theta_0(z);\nu)\in\mathbb{Z}[\theta_0(z)]$$

where $s=1,\ 2,\ 3,\ 4;\ \nu\in\mathbb{N}$*.*

Proof. If $s=\lambda=1,\ 2$ then, in view of (3.2.37) and Lemma 3.2.4,

$$P^*_\lambda(\theta_0(z);\nu)=\sum_{k=0}^{\nu}(d^3_{1,\nu,2\nu}\alpha^{\sim}_{k,\lambda}(\nu))(\theta_0(z))^k\in\mathbb{Z}[\theta_0(z)]$$

where $\lambda=1,\ 2;\ \nu\in\mathbb{N}$. If $s=3+\mu$, where $\mu=0,\ 1$, then, in view of (3.2.38) and Lemma 3.2.4,

$$P^*_{3+\mu}(z;\nu)=(d_{1,\nu,2\nu})^{1-\mu}\times$$

$$\sum_{l=0}^{\nu-1}(\theta_0(z))^l\sum_{k=l+1}^{\nu}\sum_{\lambda=1}^{2}\binom{\mu+\lambda-1}{\mu}(d_{1,\nu,2\nu})^{2-\lambda}\alpha^{\sim}_{k,\lambda}(\nu)\frac{(d_{1,\nu,2\nu})^{\lambda+\mu}}{(k-l)^{\lambda+\mu}}\in\mathbb{Z}[\theta_0(z)].$$

■

In view of (3.2.14),

$$|\eta^*_0(\pi/2)|^2=\left(1+\sqrt{2(\sqrt{2}-1)}\right)^2+\left(2+2\sqrt{(\sqrt{2}+1)/2}\right)^2. \tag{3.2.69}$$

To calculate the value (3.2.69), I use calculator of the firm Casio. So,

$$1+\sqrt{\sqrt{8}-2}=1.910179721,\ \left(1+\sqrt{\sqrt{8}-2}\right)^2=3.648786567,$$

$$2+\sqrt{\sqrt{8}+2}=4.197368227,\ \left(2+\sqrt{\sqrt{8}+2}\right)^2=17.61790003,$$

$$|\eta_0^*(\pi/2)|^2=21.2666866>\exp(3)=20.08553692\,. \tag{3.2.70}$$

Lemma 3.2.6. *If $r\in\mathbb{N}$, then the functions*

$$1, L_1(w)\,,\ldots\,, L_r(w), \tag{3.2.71}$$

where $w\in\mathbb{C}, |w|>1$ are linear independent over $\mathbb{C}(w)$.

Proof may be found in [36], Theorem 3.2.1 or in [30], pp. 25 - 30. ■

Since $x_\nu=\psi_0((1,\pi);\nu)$ and $x_\nu=\psi_1((1,\pi);\nu)$ are solutions of the equation (3.2.27) it follows from (3.2.36) and the Lemma 3.2.6 that

$$x_\nu=P_s(\theta_0((1,\pi));\nu) \tag{3.2.72}$$

with $s=1,\,2,\,3,\,4$ also are solutions of the equation (3.2.27); and moreover it follows from the Lemma 3.2.1 that solutions (3.2.72) are nonzero solutions of the equation (3.2.27).

Lemma 3.2.7. *All the solutions (3.2.72) correspond to the followig set of two roots of the characteristic polynomial (3.2.29)*

$$(\eta_0^*(\pi/2-k\pi))^2=\overline{(\eta_0^*(\pi/2-(1-k)\pi))^2}=$$

$$1/(\eta_1^*(\pi/2-k\pi))^2=\overline{1/(\eta_1^*(\pi/2-(1-k)\pi))^2},$$

where $k=0,1,$ and therefore

$$\limsup_{\nu\to\infty}|P_s(\theta_0((1,\pi);\nu)|^{1/\nu}= \tag{3.2.73}$$

$$|\eta_0^*(\pi/2-k\pi)|^2=1/|\eta_1^*(\pi/2-k\pi)|^2.$$

where $k=0,\ 1;\ s=1,\ 2,\ 3,\ 4$

Proof. The contrary means that for some $s\in\{1,\,2,\,3,\,4\}$ the following equality holds:

$$\limsup_{\nu\to\infty}|P_s(\theta_0((1,\pi);\nu)|^{1/\nu}=|\eta_1^*(\pi/2-k\pi)|^2=1/|\eta_0^*(\pi/2-k\pi)|^2.$$

where $k = 0$, 1. Then, in view of (3.2.70) and (3.2.61),

$$\limsup_{\nu\to\infty} |d_{1,\nu,2\nu} P_s(\theta_0((1,\pi);\nu))|^{1/\nu} =$$

$$\lim_{\nu\to\infty} (d_{1,\nu,2\nu})^{1/\nu} \times$$

$$\limsup_{\nu\to\infty} |P_s(\theta_0((1,\pi);\nu))|^{1/\nu} =$$

$$\exp(3)/|\eta_0^*(\pi/2)|^2 < 1.$$

On the other hand, according to the Lemma 3.2.5, $d^3_{1,\nu,2\nu} P_s(\theta_0(z);\nu) \in \mathbb{Z}$ and, according to the Lemma 3.2.1 does not vanish on the infinite set of numbers $\nu \in \mathbb{N}$. Therefore

$$\limsup_{\nu\to\infty} |d_{1,\nu,2\nu} P_s(\theta_0((1,\pi);\nu))|^{1/\nu} \geq 1.$$

■

Lemma 3.2.8. *Let $s \in \mathbb{N}-1, n \in \mathbb{N}, a_i^{\sim} \in \mathbb{C}$, $a_i(\nu) \in \mathbb{C}$, $a_n(\nu) = 1$, $a_i(\nu) - a_i^{\sim} = O(1/\nu)$ for $\nu \in \mathbb{N}$ and $0 \leq i \leq n$. For $m \in \mathbb{N}$ let V_m denotes linear over $\mathbb{C}$ space of solutions $y = y(\nu)$ of the equation $a_0(\nu)y(\nu+0) + \ldots + a_n(\nu)y(\nu+n) = 0$, where $\nu \in m+\mathbb{N}$. Let the absolute values of all the roots of the polynomial $a_0^{\sim} z^0 + \ldots + a_n^{\sim} z^n$ are among the numbers $\{\rho_i : 1 \leq i \leq 1+s\}$, such that $\rho_{s+1} = 0$ and $\rho_j < \rho_i$ for $1 \leq i < j \leq s+1$. Let e_i and k_i denote respectively the sum of and the maximum of the multiplicities of those roots, whose absolute value is equal to ρ_i, $1 \leq i \leq 1+s$. For given $y = y(\nu)$ in V_m, let $\omega_{n,y}(\nu) = \max(|y(\nu)|, \ldots, |y(\nu+n-1)|)$. Then there exist $m \in \mathbb{N}$, the subspaces $V^{\vee}_{m,1}, \ldots, V^{\vee}_{m,s+1}$ of the space V_m and positive number A, which depends only from the equation, such that $\dim_{\mathbb{C}}(V^{\vee}_{m,i}) = e_i$, $1 \leq i \leq s+1, V_m = V^{\vee}_{m,1} \oplus \ldots, \oplus V^{\vee}_{m,s+1}$ and, moreover, the spaces $V^{\wedge}_{m,j} = V^{\vee}_{m,j} \oplus \ldots \oplus V^{\vee}_{m,s+1}$, $1 \leq j \leq s+1$ have the folloving property:*
if $y \in V^{\wedge}_{m,\theta} \backslash V^{\wedge}_{m,\theta+1}$ for some $\theta \in \{1, ..., s\}$, then

$$\exp(-A(ln(\nu) + \nu^{1-1/k_\theta}))\beta \leq \omega_{n,y}(\nu)(\rho_\theta)^{-\nu} \leq \exp(A(ln(\nu) + \nu^{1-1/k_\theta}))\omega_{n,y}(m),$$

where $\beta = \beta(y) > 0$ depends only from equation and from y (but does not depend on ν); if $\mathfrak{k} := k_{s+1} > 0$ and $y \in V^{\vee}_{m,s+1} (= V^{\wedge}_{m,s+1})$, then

$$|y(\nu)| \leq (A/\nu)^{\nu/\mathfrak{k}} \omega_{n,y}(m).$$

Proof may be found in [32].

I wish to apply the Lemma 3.2.8 to the case of the equation (3.2.27) . In this case $n = 4$, $s = 2, \rho_3 = 0, \mathfrak{k} = k_{s+1} = e_{s+1} = 0$,

$$\rho_1 = |\eta_0^*(\pi/2 - k\pi)|^2, k_1 = 1, e_1 = 2,$$

where $k = 1, 2$,

$$\rho_2 = |\eta_1^*(\pi/2 - k\pi)|^2, k_2 = 1, e_2 = 2,$$

where $k = 1, 2$. According to the Lemma 3.2.27, the solutions (3.2.72) of the equation (3.2.27) are contained in the set $V^{\wedge}_{m,1} \backslash V^{\wedge}_{m,2}$ of the Lemma 3.2.8. In view of the (3.2.55), the solutions (3.2.53) of the equation (3.2.27) are contained in the set $V^{\wedge}_{m,2} \backslash V^{\wedge}_{m,3}$ of the Lemma 3.2.8. In view of (3.2.61) and according to the Lemma 3.2.8, for each $\varepsilon \in (0,1)$ there exist positive constants $C_1(\varepsilon)$, $C_2(\varepsilon)$, $C_3(\varepsilon)$ and $C_4(\varepsilon) \geq 1$ such that

$$C_1(\varepsilon)(\exp(3)\rho_2(1-\varepsilon))^{\nu} \leq \tag{3.2.74}$$

$$|\omega(4, d_{1,\nu,2\nu}\psi_{\mu}((1,\pi);\nu))|^{1/\nu} \leq C_2(\varepsilon)(\exp(3)\rho_2(1+\varepsilon))^{\nu},$$

where $\mu = 0, 1$

$$C_3(\varepsilon)(\exp(3)\rho_1(1-\varepsilon))^{\nu} \leq |d_{1,\nu,2\nu}\omega(4, P_s(\theta_0((1,\pi);\nu))|^{1/\nu} \leq \tag{3.2.75}$$

$$C_4(\varepsilon)(\exp(3)\rho_1 \exp(3)(1+\varepsilon))^{\nu},$$

where $s = 1, 2, 3, 4$; $\nu \in m + \mathbb{N}$. Since, in view of (3.2.61)

$$\exp(3)\rho_2 = \exp(3)/|\eta_0^*(\pi/2)|^2 < 1 \tag{3.2.76}$$

we chose below $\varepsilon \in \left(0, min\left(1, \frac{1}{\exp(3)rho_2} - 1\right)\right)$. Then

$$1 < R_2 = \frac{1}{\exp(3)\rho_2(1+\varepsilon)} < R_1 = \frac{1}{\exp(3)\rho_2(1-\varepsilon)}. \tag{3.2.77}$$

Lemma 3.2.9. *Let are given four natural numbers l, m, q, and r. Let $a_{j,k} \in \mathbb{R}$ for $1 \leq j \leq q$, $1 \leq k \leq r$ and $\alpha^{\wedge}_j(\nu) \in \mathbb{Z}$ for $1 \leq j \leq r+q$, $\nu \in m + \mathbb{N}$. Let*

$$y^{\wedge}_k(\nu) = \alpha^{\wedge}_1(\nu)a_{1,k} + \ldots + \alpha^{\wedge}_q(\nu)a_{q,k} - \alpha^{\wedge}_{q+k}(\nu)$$

for $\nu \in \mathbb{N} + m$ and $1 \leq k \leq r$. Suppose there are $\gamma_0 > 0$, $r^{\wedge}_1 \geq 1, \ldots, r^{\wedge}_q \geq 1$ such that

$$|\alpha^{\wedge}_j(\nu)| < \gamma_0(r^{\wedge}_j)^{\nu}$$

for $\nu \in \mathbb{N} + m$, $1 \leq j \leq q$. For $X = (x_1, \ldots, x_r)$, $X \in \mathbb{C}^r$, let

$$\phi_j(X) = a_{j,1}x_1 + \ldots + a_{j,r}x_r,\ 1 \leq j \leq q.$$

Suppose there are $R_1 \geq R_2 > 1$, $\gamma_1 > 0$, $\gamma_2 > 0$ such that

$$\gamma_1 R_1^{-\nu} h(X) \leq \sum_{\kappa=0}^{l-1} |y^{\wedge}(X; \nu + \kappa)| \leq \gamma_2 R_2^{-\nu} h(X)$$

Let $\alpha_j = (\log(r_j^\wedge R_1/R_2))/\log(R_2)$, $1 \le j \le q$. *Then there exists* $\gamma_4 > 0$ *such that*

$$\|\phi_1(X)\|(h(X))^{\alpha_1} + \ldots + \|\phi_q(X)\|(h(X))^{\alpha_q} \ge \gamma_4$$

for $X \in \mathbb{Z}^r \setminus \{0\}$, *where, as usual,* $\|x\|$ *stands for the distance from* $x \in \mathbb{R}$ *to* $\mathbb{Z}$.

Proof may be found in [37], proof of Theorem 2.3.1, or in [31], pp. 64 - 67.

Remark. We use this opportunity to correct two misprints in [37].

Equality (2.3.16) should read:

$$\alpha_j = (log(r_j^\wedge R_1/R_2))/log(R_2) \tag{2.3.16}$$

(instead of $\alpha_j = (log(r_j^\wedge R_1))/log(R_2))$*).*

Inequality (2.3.21) should read:

$$q_\infty^{(\mathfrak{K})}(y^\wedge(X,\nu)) \le \gamma_2(R_2)^{-\nu} \tag{2.3.21}$$

(instead of $q_\infty^{(\mathfrak{K})}(y^\wedge(X,\nu)) \le \gamma_1(R_2)^{-\nu}$*).*

Proof the Theorem mentioned in the foreword. I apply now the Lemma 3.2.9. I take $l = 4$. The number m will be detrmined by the Lemma 3.2.8. Let $r = q = 2$. In correspondence with (3.2.51) and (3.2.52) let

$$a_{1,1} = log(4),\ a_{1,2} = a_{2,1} = \zeta(2),\ a_{2,2} = 3\zeta(3).$$

We take further

$$a_s(\nu) = 2d_{1,\nu,2\nu}^3 P_s(-1;\nu),$$

where $s = 1,\ 2,\ 3,\ 4;\ \nu \in \mathbb{N}$. According to the Lemma 3.2.5 and in view of (3.2.51), (3.2.52), $a_s(\nu) \in \mathbb{Z}$ for $s = 1,\ 2,\ 3,\ 4;\ \nu \in \mathbb{N}$. The role $y_k^\wedge(\nu)$ in the Lemma 3.2.9 play now $2d_{1,\nu,2\nu}^3\psi_{k-1}((1,\pi);\nu)$ with $\psi_{k-1}((1,\pi);\nu)$ from (3.2.51) and (3.2.52), here $k = 1,\ 2$. The role of the numbers $r_1^\wedge \ge 1, \ldots, r_q^\wedge \ge 1$ in the Lemma 3.2.9 play now the number $\rho_1 exp(3)(1+\varepsilon)$ in (3.2.75). The role of the number γ_0 in the Lemma 3.2.9 play now the number $2C_4(\varepsilon)$ in (3.2.75). The role of the number γ_k, where $k = 1,\ 2$ in the Lemma 3.2.9 play now the number $2C_k(\varepsilon)$ in (3.2.74). The role of the number R_k for $k = 1, 2$ play now the number in the Lemma 3.2.9 play now the number R_k in (3.2.77). We see that all the conditions of the Lemma 3.2.9 are fulfilled. Therefore the conclusion of the Lemma 3.2.9 also is fulfilled with

$$\alpha_j = (\log(\rho_1 \exp(3)(1+\varepsilon)^2/(1-\varepsilon))/\log(1/(\rho_2 \exp(3)(1+\varepsilon)) =$$

$$\alpha_j = (\log(\rho_1^\sim \exp(6)(1+\varepsilon)^2/(1-\varepsilon))/\log(\rho_1^s im(1+\varepsilon)),$$

where $rho_1^\sim = \rho_1/exp(3)$. Our theorem follows now from the equality

$$\lim(\log(\rho_1^\sim \exp(6)(1+\varepsilon)^2/(1-\varepsilon))/\log(\rho_1^\sim(1+\varepsilon)) =$$

$$\frac{\log(\rho_2^\sim)}{\log(\rho_1^\sim)}.$$

■

§ 3. The some key moments of the previous proof (the discussion).

Here I shall list the some key moments of the previous proof. First it would be remarked the inequality (3.2.26) for $z = (1, \pi)$. This inequality allows us to consider the matrix (3.1.68) at the point $z = (1, \pi)$.

The equalites (3.2.1), (3.2.12) and the inequality (3.2.20) allow us to apply the Lemmata 3.1.12 and 3.1.10 and to get the equality (3.2.27).

Making use of the Lemma 3.2.1 we show that solutions (3.2.53) and (3.2.72) are nonsero solutions of the equation (3.2.27).

And making use of the Lemma 3.2.8 we obtained the lower estimates, which have been required by the condition of the Lemma 3.2.9. .

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