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POINCARÉ DUALIZATION AND FORMAL DOMINATION

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Abstract. We consider the question of whether formality of the domain of a non-zero degree map of closed manifolds implies formality of the target. Though there are various situations where this is indeed the case, we show the answer is negative in general, with a counterexample given by a non-zero degree map from a formal manifold to one that carries a non-vanishing quadruple Massey product. This violates a general heuristic that the domain of a non-zero degree map should be more complicated than its target. For the construction of the counterexample we introduce a method to turn a cdga into one that satisfies Poincaré duality, which is natural in certain situations.

Contents

1. Introduction: Formality and non-zero degree maps

The existence of non-zero degree maps between closed manifolds, giving a relation going by the name of *domination* or *Gromov's partial order*, has been of substantial interest going back at least to work of Gromov, Milnor, and Thurston in the 1970's, see [CT89, p.173]. The general empirical observation is that the domain of a non-zero degree map should be "more complicated" than its target, see e.g. loc. cit. In view of this heuristic, one might expect that rational homotopy theoretic formality is preserved under such maps.

Indeed, formal spaces are those having the "simplest" rational homotopy type with a given cohomology algebra, where simplicity refers to the vanishing of certain higher structures on the cohomology, the most prominent incarnation of which are Massey products.

This embeds into the more general question of under which circumstances a map $Y \to X$ of topological spaces with formal domain Y lets us deduce something about the formality of X, or at least about the vanishing of certain Massey products on X. A naive first guess could be that positive conclusions can be drawn as soon as the map on cohomology $H(X) \to H(Y)$ is injective, since then the vanishing Massey products on $H(Y)$ together with naturality might imply the vanishing of Massey products on $H(X)$. However, while this certainly works for the binary product, i.e. the cup product, it is not hard to see that the argument fails in general for triple and higher products, due to the presence of indeterminacy (see Remark 3.3).

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If we restrict ourselves to the more geometric setting of closed orientable manifolds (or more generally rational Poincaré duality spaces) and non-zero degree maps between them, then, as predicted by the aforementioned heuristic of domination, there are indeed a number of positive results in the above direction:

- Taylor [Ta10] proved that a non-vanishing Massey triple product on X pulls back to a non-vanishing Massey triple product on Y.
- By the argument in [DGMS75, Theorem 5.22], if $Y \to X$ is a holomorphic map of compact complex manifolds of non-zero degree, and Y satisfies the $\partial\overline{\partial}$-lemma, then X also satisfies the $\partial\overline{\partial}$-lemma; recall from loc. cit. that a manifold satisfying the $\partial\overline{\partial}$-lemma is formal.
- Suppose we have a non-zero degree map $Y \xrightarrow{f} X$ of rational Poincaré duality spaces of dimension $\leq 5n+2$, where X is rationally n–connected. Then Y being formal implies that X is formal. Indeed, by [CN20], formality of X is equivalent to the Bianchi–Massey tensor vanishing; a non-trivial Bianchi–Massey tensor pulls back non-trivially, obstructing formality of Y.

We add to this by giving an alternative proof of the aforementioned result from [Ta10], as well as by showing that formality is preserved under non-zero degree maps of closed manifolds of dimension ≤ 4 when the domain has trivial rational cup product $H^1 \otimes H^1 \to H^2$.

Despite the above evidence and the general heuristic, we show that formality of the domain does not imply formality of the target in full generality. We prove the following, our main result:

Theorem A. There exists a non-zero degree map $Y \to X$ between smooth, simply-connected, closed manifolds, such that Y is formal and X admits a non-trivial quadruple Massey product.

The strategy is to start on the algebraic level and construct a suitable map between commutative differential graded algebras (cdga's) whose cohomology satisfies Poincaré duality. This can then be realized geometrically by using results from [Su77]. To construct our algebraic example, we develop a procedure to modify any cdga (satisfying suitable finiteness conditions) into one whose cohomology satisfies Poincaré duality and study naturality properties of the construction. We refer to this procedure as *Poincaré dualization*; the construction depends on a fixed integer n. We believe this might prove to be a useful technical tool to construct manifold examples from general cdga phenomena, akin to taking the boundary or double of a thickening of a cell complex in Euclidean space.

Theorem B. There is an association of cdga's $A \mapsto P_nA$ such that

1. (1) A is naturally a sub-cdga of P_nA and the inclusion induces an injection in cohomology.
2. (2) If A is cohomologically connected and has finite dimensional cohomology concentrated in degrees between 0 and n, then P_nA is a Poincaré duality cdga.
3. (3) P_n is functorial for pairs (f, r) where $f : A \to B$ is a map of cdga's and $r : B \to A$ is a map of dg-A-modules.
4. (4) If A admits a non-vanishing Massey product, the induced Massey product in P_nA is non-trivial.
5. (5) If A is a Sullivan cdga that is cohomologically connected, has finite dimensional cohomology concentrated in degrees between 0 and $n/2$ and is formal, then P_nA is also formal.

In view of the above, the following questions are natural to ask and remain to be systematically addressed:

Question. Let $Y \to X$ be a non-zero degree map of rational Poincaré duality spaces, and suppose Y is formal. Under what conditions does it follow that X is formal? If m is some non-vanishing obstruction to formality on X that can be pulled back to Y, under what conditions is the pullback f^*m non-vanishing?

The article is organized as follows: in Section 2 we review the necessary concepts from (rational) homotopy theory we discuss. In Section 3 we give the aforementioned alternative argument to Taylor's theorem on triple Massey products pulling back non-trivially, and prove the mentioned results on manifolds of dimension ≤ 4. In Section 4 we introduce the Poincaré dualization construction and prove some basic properties. We then use this machinery in Section 5 to construct a non-zero degree map of Poincaré duality cdga's with a quadruple Massey product on the domain, and formal target. This then translates, using results of Sullivan, into a geometric map of closed manifolds where a non-trivial quadruple Massey product pulls back trivially.

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2. Preliminaries

We recall the concepts used in the introduction and throughout, and set notation. Let K be a field. We will consider graded commutative algebras (A, d) over K, where we allow entries in negative degrees, i.e. $A = \bigoplus_{k \in \mathbb{Z}} A^k$, but most of the time we restrict to *cohomologically connected* cdga's, i.e. those where $H^k(A) = 0$ for $k < 0$ and $H^0(A) = K$. If $A^k = 0$ for $k < 0$ and $A^0 = K$, we say A is connected. We call A a *K–Poincaré duality cdga* if its cohomology satisfies Poincaré duality. That is, there is an index n such that $H^n(A, d) \cong K$ and the pairing

$$H^k(A, d) \otimes H^{n-k}(A, d) \to H^n(A, d) \cong K$$

given by

$$\alpha \otimes \beta \mapsto \alpha\beta$$

is non-degenerate.

Restricting to the case of $K = \mathbb{Q}$, a commutative differential graded algebra (cdga) is said to be *formal* if there is a chain of quasi-isomorphisms of cdga's

$$\begin{tikzcd}[column sep=small] & (B_1, d) \arrow[dl] \arrow[dr] & & \cdots \arrow[dl] \arrow[dr] & & (H, 0) \arrow[dl] \\ (A, d) & & (B_2, d) & & (B_r, d) & \end{tikzcd}$$

connecting (A, d) to a cdga with trivial differential. For A cohomologically connected, one may pick a Sullivan model $(\Lambda V, d) \to A$, i.e. a connected cdga that is free as an algebra, satisfying a nilpotence condition (c.f. [FHT12]), with a quasi-isomorphism to A. One may even pick $(\Lambda V, d)$ to be minimal, i.e. $d(\Lambda V) \subseteq \Lambda^{\geq 2} V$; [FHT12, p.191]. In terms of such a model, formality of A is equivalent to the existence of a quasi-isomorphism $(\Lambda V, d) \to (H(A), 0)$. (That is, we may replace the chain of quasi-isomorphisms by a single 'roof'.) This reformulation of formality can be further rephrased as saying that one may pick a complement N to the space of closed generators $C \subseteq V$ such that the ideal generated by N in ΛV is acyclic. We will also consider the following weakening of the notion of formality (c.f. [FM05]): A minimal Sullivan cdga $(\Lambda V, d)$ is called s–formal, where $s \in \mathbb{Z}_{\geq 1}$ if for each $i \leq s$ the closed generators $C^i \subseteq V^i$ have a complement N^i such that any closed element in the ideal $I \subseteq \Lambda^{\leq s} V$ generated by $\bigoplus_{i \leq s} N^i$ is exact in ΛV. Then an arbitrary (cohomologically connected) cdga is called s-formal if its minimal Sullivan model is s–formal.

Computable obstructions to formality are given by (ad hoc) Massey products [M58, Section 2]. Given three pure-degree classes $[x], [y], [z] \in H(A)$ such that $xy = da$, $yz = db$, the element $az - (-1)^{|x|} xb$ is closed and therefore gives rise to a cohomology class. Modulo the ideal generated by $[x]$ and $[y]$, this class is well-defined and independent of the choices of representatives and primitives. It is called the triple Massey product and denoted $\langle [x], [y], [z] \rangle := [az - (-1)^{|x|} xb] \in H(A)/([x], [y])$.

Quadruple Massey products are defined similarly: Given four classes $[w], [x], [y], [z] \in H(A)$, which for simplicity we assume to have pure even degree (which will be the case for us below), a defining system for the quadruple product $\langle [w], [x], [y], [z] \rangle \subseteq H(A)$ is a collection of pure-degree elements a, b, c, f, g such that $da = wx$, $db = xy$, $dc = yz$ and $df = ay - wb$ and $dg = bz - xc$. For any such defining system, one obtains a cohomology class $[wg + ac + zf] \in H(A)$. The quadruple Massey product $\langle [w], [x], [y], [z] \rangle \subseteq H(A)$ is then defined to be the collection of classes obtained from all such defining systems. Again, this collection is independent of the chosen representatives for the classes. The quadruple product is said to be trivial (or vanish) if $0 \in \langle [w], [x], [y], [z] \rangle$. The definitions for quintuple and higher products are similar; we refer the reader to [K66].

Massey products are invariants of the quasi-isomorphism type of a cdga, and on formal cdga's all Massey products vanish.

To a topological space X we can associate its connected cdga $A_{PL}(X)$ of rational piecewise-linear forms [Su77], [DGMS75]; this cdga computes the rational cohomology of X (see e.g. [DGMS75, Theorem 2.1], [H07, Theorem 1.21]). We say the space X is formal if $A_{PL}(X)$ is formal as a cdga [DGMS75, p.260], [H07, Definition 2.1]. A space is a *rational Poincaré duality space* if its rational cohomology satisfies Poincaré duality.

3. Some positive results

In this section, $K = \mathbb{Q}$ and $H(X)$ denotes rational cohomology. First let us give an alternative proof of Taylor's theorem [Ta10] mentioned above.

Proposition 3.1. Let $Y \to X$ be a non-zero degree map between rational Poincaré duality spaces and let $a, b, c \in H(X)$ with $ab = bc = 0$. If

$$m := \langle a, b, c \rangle \neq 0 \in \frac{H(X)}{a \cup H(X) + H(X) \cup c},$$

then also

$$f^*(m) \neq 0 \in \frac{H(Y)}{f^* a \cup H(Y) + H(Y) \cup f^* c}.$$

Proof. The map $f^* : H(X) \to H(Y)$ has a one-sided inverse given by $f'_* := \frac{1}{\deg f} f_*$, i.e. $f'_* f^* = Id_{H(X)}$. Here f_* denotes the pushforward, determined by f^* and Poincaré duality.

This yields a splitting

$$H(Y) \stackrel{(f'_*,\mathrm{pr})}{\longrightarrow} H(X) \oplus H(Y)/f^*H(X).$$

By the projection formula

$$f_*(f^*x \cup y) = x \cup f_*y \quad \text{for } x \in H(X), y \in H(Y),$$

this (additive) splitting is compatible with the natural $H(X)$-module structures on both sides (given by f^* on the left and (Id, f^*) on the right). Therefore, writing $K := H(Y)/f^*H(X)$, the domain of definition of the Massey product decomposes as

$$\frac{H(Y)}{f^*a \cup H(Y) + H(Y) \cup f^*c} \xrightarrow{\sim} \frac{H(X)}{a \cup H(X) + H(X) \cup c} \oplus \frac{K}{f^*a \cup K + K \cup f^*c},$$

and under this splitting, we have $f^*(m) = (m, 0)$. □

Remark 3.2. The above proof works for maps of Poincaré duality cdga's over any field, as long as we interpret $\deg f \neq 0$ to mean that $\deg f$ is invertible.

Often in examples one has a subalgebra of invariant forms, satisfying Poincaré duality, and containing an invariant volume form, in the algebra of all forms A_X on a closed manifold X. By duality, this algebraic map is injective on cohomology. Then the same argument as above applies to show that a non-vanishing triple Massey product calculated on the invariant subalgebra continues to be non-vanishing on A_X; that is, if there is some defining system for the triple product on A_X making it trivial, then there was a defining system on the subalgebra making it trivial to begin with.

Remark 3.3. Without the Poincaré duality assumption, it is generally easy to find examples of cohomologically injective maps of cdga's such that a non-vanishing triple Massey product in the domain vanishes in the target. For example, consider the inclusion of cdga's

$$A := (\Lambda(x, y, z), dz = xy) \hookrightarrow B := (\Lambda(x, y, z, u, v), dz = xy, dv = xz - yu)\,,$$

where all generators are in degree 1. This induces an inclusion $A' := A/A^{\geq 3} \hookrightarrow B' := B/B^{\geq 3}$ which is injective on cohomology. Now, $\langle x, x, y\rangle$ is a non-vanishing triple product in A', while in B' it is represented by $[xz] = [yu]$, which lies in the indeterminacy.

We also record the following result in which formality is preserved under a non-zero degree map.

Proposition 3.4. Let $Y \xrightarrow{f} X$ be a cohomologically injective map of topological spaces, where the cup product $H^1(Y) \otimes H^1(Y) \to H^2(Y)$ is trivial. If Y is 1–formal, then X is 1–formal.

By [FM05, Theorem 3.1], a closed manifold in dimensions ≤ 4 is formal if and only if it is 1–formal. This directly implies the following:

Corollary 3.5. Let $Y \xrightarrow{f} X$ be a non-zero degree map of orientable closed n-manifolds (or more generally rational Poincaré duality spaces) where $n \leq 4$ and the cup product $H^1(Y) \otimes H^1(Y) \to H^2(Y)$ is trivial. If Y is formal then so is X.

Notice that our assumptions imply that $H^1(X) \otimes H^1(X) \to H^2(X)$ is trivial as well. Indeed, suppose $ab \neq 0$ for some $a, b \in H^1(X)$. Then we would have $(f^*a)(f^*b) = f^*(ab) \neq 0$ by the injectivity of f^*.

Remark 3.6. (The condition of having trivial $H^1(Y) \otimes H^1(Y) \xrightarrow{\cup} H^2(Y)$ on 3–manifolds)

There exist both formal and non-formal closed 3–manifolds satisfying this condition, e.g. connected sums of $S^1 \times S^2$, resp. Heisenberg manifolds.

By a theorem of Sullivan [Su75], for any finite–dimensional rational vector space H and any skew–symmetric trilinear form $H^{\otimes 3} \to \mathbb{Q}$, there is an oriented 3–manifold M realizing this data as the trilinear form $H^1(M)^{\otimes 3} \to \mathbb{Q}$. In the case of 3–manifolds, our result applies to the case of the zero form.

If there were $a, b \in H^1(M)$ such that $ab \neq 0$, then by Poincaré duality we could choose $c \in H^1(M)$, linearly independent of a and b, such that $abc \neq 0$. Choosing integral representatives of non-zero multiples of a, b, c hence gives us a non-zero degree map $M \to T^3$.

Proof of Proposition 3.4. Take minimal models $M_X = (\Lambda V^X, d)$ and $M_Y = (\Lambda V^Y, d)$ for X and Y. We will argue that M_X is 1–formal. Namely, we will show there is a splitting of the space of degree 1 generators $V_1^X = C_1^X \oplus N_1^X$, where $C_1^X = \ker d$, such that any closed element in $\Lambda(V_1^X)$ in the ideal generated by N_1^X is exact in M_X.

Before doing that, let us show that V_1^X injects into V_1^Y under the induced map on models, which we also denote by f^*. Certainly C_1^X injects (into C_1^Y) since f^* is injective on cohomology. Consider the increasing filtration of V_1^X given by $F^0 = C_1^X$ and $F^i = d^{-1}(\Lambda^2 F^{i-1})$ for $i \geq 1$. This filtration is exhaustive and preserves the differential by nilpotency. Now take a non-closed element in F^1 (if it is closed, it pulls back non-trivially); then the image of its differential under f^* is non-zero by freeness, and hence the image of the element is non-zero. Inductively we obtain our claim. In fact, we have shown that the cdga $\left(\Lambda V_1^X, d\right)$ injects into M_Y.

Now we observe that, in general, on a 1–formal minimal cdga $(\Lambda V, d)$ with trivial cup product $H^1 \otimes H^1 \to H^2$, we can choose *any* complement to the closed elements in degree 1 in the definition of 1–formality. Indeed, by 1–formality we have some splitting $V_1 = C_1 \oplus N_1$ with the desired properties. Choose another complement N_1' to C_1, and take a closed element in ΛV_1 in the ideal generated by N_1'. We can write this element as a sum of an element in the ideal of N_1 and a product of elements in C_1. Since the latter is closed, the former is closed as well, and hence it is exact in $(\Lambda V, d)$ by assumption. The product of elements in C_1 is also exact in $(\Lambda V, d)$ by the assumption on the cup product.

Now choose any complement N_1^X to C_1^X in V_1^X. Mapping over (injectively) to M_Y via f^*, we can complete a basis of C_1^X to a basis of C_1^Y, and a basis of N_1^X to a basis for a complement to C_1^Y in V_1^Y. Suppose now that we have a closed element in $\Lambda(V_1^X)$ in the ideal generated by N_1^X. Pulling back to M_Y, this is now a closed element in $\Lambda(V_1^Y)$ in the ideal generated by N_1^Y. By 1–formality of M_Y, it is exact. Hence, by the injectivity of f^* on cohomology, the element must have been exact in M_X to begin with. □

4. Poincaré dualization

Motivated by the example in Remark 3.3, we detail a construction that "completes" any cohomologically connected cdga to one satisfying Poincaré duality on its rational cohomology, which in certain cases is functorial.

Fix a natural number n and a field K. Let (A, d) a complex of K-vector spaces. We define the (n-th) **dual complex** $D_n A$ by $(D_n A)^k := (A^{n-k})^\vee$ with differential $(D_n A)^k \to (D_n A)^{k+1}$ given on pure-degree elements $\varphi \in D_n A$ by $d(\varphi)(a) := (-1)^{|\varphi|-1}\varphi(da)$ for any $a \in A$. Clearly, D_n is a contravariant functor and

$$H^k(D_n A) = (H^{n-k}(A))^\vee.$$

Now let us assume A carries in addition the structure of a graded-commutative algebra, such that d is a derivation (i.e. A is a cdga).

Definition 4.1. Let $(A, \wedge, d)$ be as above. The n-th **Poincaré dualization** of A is given, as a complex, by

$$P_n A := A \oplus D_n A,$$

with multiplication (extending that on A) defined on pure-degree elements $a \in A$, $\varphi \in D_nA$ by the dual complex element given by

$$\begin{aligned}(a \wedge \varphi)(b) &:= (-1)^{|a||\varphi|}\varphi(a \wedge b),\\ (\varphi \wedge a)(b) &:= \varphi(a \wedge b),\end{aligned}$$

and setting $\varphi \wedge \psi = 0$ for $\varphi, \psi \in D_nA$.

Lemma 4.2. The Poincaré dualization P_nA is indeed a cdga.

Proof. Graded commutativity of the multiplication holds by definition. For associativity, we only need to check the case $\varphi \in D_nA$ and $a, b \in A$ as all other combinations of products of three elements are either zero or entirely in A, where associativity holds since A is a cdga. We compute, for $c \in A$:

$$\begin{aligned}((\varphi \wedge a) \wedge b)(c) &= (\varphi \wedge a)(b \wedge c)\\ &= \varphi(a \wedge b \wedge c)\\ &= (\varphi \wedge (a \wedge b))(c).\end{aligned}$$

That d is a derivation again only has to be checked on products of the form $\varphi \wedge a$ with $\varphi \in D_nA$ and $a \in A$. In this case, we compute:

$$\begin{aligned}(d\varphi \wedge a)(b) &= d\varphi(a \wedge b)\\ &= (-1)^{|\varphi|-1}\varphi(d(a \wedge b))\\ &= (-1)^{|\varphi|-1}\varphi(da \wedge b)) + (-1)^{|\varphi|-1+|a|}\varphi(a \wedge db)\\ &= (-1)^{|\varphi|-1}(\varphi \wedge da)(b) + (-1)^{|\varphi\wedge a|-1}(\varphi \wedge a)(db)\\ &= (-1)^{|\varphi|-1}(\varphi \wedge da)(b) + d(\varphi \wedge a)(b).\end{aligned}$$

□

For simplicity, let us now further assume that A is cohomologically connected, $H(A)$ is finite-dimensional, and $H^k(A) \neq 0$ at most for $0 \leq k < n/2$.

Lemma 4.3. The cohomology $H(P_nA)$ is finite dimensional and concentrated in degrees $0, ..., n$. Further, P_nA is a Poincaré duality cdga, i.e. for any integer k, the pairing

$$H^k(P_nA) \times H^{n-k}(P_nA) \xrightarrow{\wedge} H^n(P_nA) \cong K$$

is non-degenerate.

Proof. Without loss of generality, let $k < n/2$. By construction, $H^k(P_nA) = H^k(A)$ and $H^{n-k}(P_nA) = H^{n-k}(D_nA) = (H^k(A))^\vee$, and the pairing is given (up to a non-zero scalar) by evaluation. □

Remark 4.4. In fact, $P_n(A)$ is an oriented differential Poincaré duality algebra in the sense of [LS08] (though without the assumption of no elements in negative degree as therein), i.e. it satisfies Poincaré duality already on the chain level and there is a canonical orientation (i.e. an isomorphism $(P_nA)^n \cong K$) induced by the dual of the unit map $K \to A$ sending $1 \mapsto 1$. However, we will not make use of this additional structure here.

Example 4.5.

- Let $A = \big(\Lambda(x)/x^2, d = 0\big)$ with $|x| \geq 1$ and let $n > |x|$. Then

$$P_n(A) \cong \big(\Lambda(x, y)/(x^2, y^2), d = 0\big)$$

with $|y| = n - |x|$, i.e. we obtain the cohomology algebra of the product of spheres $S^{|x|} \times S^{|y|}$.

- Take now, for example, A to be a minimal model for S^2, i.e. $A = \big(\Lambda(x,y), dy = x^2\big)$. Then $P_n(A)$ is generated as an algebra by x, y, and a dual basis $\{\widehat{x^k}, \widehat{x^k y}\}_{k\geq 0}$ for $\{x^k, x^k y\}$, in degrees $n-2k$ and $n-2k-3$ respectively. satisfying
$$\widehat{x^k y} \wedge x^l y = \widehat{x^{k-l}},\quad \widehat{x^k y} \wedge x^l = \widehat{x^{k-l} y},\quad \widehat{x^k} \wedge x^l y = 0,\quad \widehat{x^k} \wedge x^l = \widehat{x^{k-l}}$$
for $k \geq l$. All other products of dual elements with basis elements from A are zero. The differential is determined by $d(\widehat{x}) = 0$ and $d(\widehat{x^k}) = -\widehat{x^{k-2}y}$ for $k \geq 2$. A quasi-isomorphism from $P_n(A)$ to $\big(H(S^2 \times S^{n-2}), d = 0\big)$ is given by sending x and $\widehat{x}$ to the volume classes of S^2 and S^{n-2} respectively (and 1 to 1), and all other generators to zero.
- For a non-manifold example, consider the formal space $S^2 \vee S^3$, with model
$$\big(\Lambda(x,y)/(x^2, xy), d = 0\big)$$
with $|x| = 2$, $|y| = 3$. Then for $n \geq 4$, its n-th Poincaré dualization is the cohomology ring of the manifold $\big(S^2 \times S^{n-2}\big) \# \big(S^3 \times S^{n-3}\big)$ equipped with trivial differential. Note that for large enough n, the boundary of a thickening of $S^2 \vee S^3$ in $\mathbb{R}^n$ is $\big(S^2 \times S^{n-2}\big) \# \big(S^3 \times S^{n-3}\big)$.

The above examples suggest that there is a certain quasi-isomorphism invariance in the Poincaré dualization construction, and a geometric construction mirrored by it. It would be interesting, but not necessary for the present purposes, to pursue these points further.

Now consider a cdga B, satisfying the same finiteness and connectedness conditions as A, and a map of cdga's $f : A \to B$. In general, it is not true that this can be extended to a map $P_n A \to P_n B$. However, one has:

Lemma 4.6. Given a map $r : B \to A$ of dg-A-modules, i.e. a map of complexes satisfying $r(f(a) \wedge b) = a \wedge r(b)$, the map
$$f \oplus D_n r : P_n A \to P_n B$$
is a map of cdga's. When $r(1) = 1$ (or equivalently $r \circ f = \mathrm{id}$), the map $f \oplus D_n r$ has non-zero degree.

Proof. Because both $f : A \to B$ and $D_n r : D_n A \to D_n B$ are maps of complexes (the latter follows from a direct calculation, using that r is a map of complexes), so is $f \oplus D_n r$, where f and $D_n r$ are extended trivially to all of $P_n A$. It thus remains to show that the map is compatible with the product. If both factors are in $A \subseteq P_n A$, this is true since f is an algebra map. If both entries are in $D_n A$, their product is zero, and so is the product of their images under $D_n r$. The remaining case, $a \in A, \varphi \in D_n A$, follows from:
$$\begin{aligned}
[(f \oplus D_n r)(\varphi \wedge a)](b) &= D_n r(\varphi \wedge a)(b) \\
&= (\varphi \wedge a)(r(b)) \\
&= \varphi(a \wedge r(b)) \\
&= \varphi(r(f(a) \wedge b)) \\
&= [D_n r(\varphi) \wedge f(a)](b) \\
&= [(f \oplus D_n r)(\varphi) \wedge (f \oplus D_n r)(a)](b).
\end{aligned}$$
The statement about the degree follows since the top-degree cohomology in $P_n A$ and $P_n B$ is generated by any class that evaluates non-trivially on 1. ∎

Remark 4.7. Consider the category whose objects are cdga's satisfying the conditions of A above and morphisms $A \to B$ given by pairs (f, r) as above, with composition $(g, s) \circ (f, r) = (g \circ f, r \circ s)$. Then Poincaré dualization P_n defines a functor from this category to that of Poincaré duality cdga's. Restricted to degree-wise finite dimensional cdga's, it is fully faithful.

From now on, we will take our ground field K to be the rationals $\mathbb{Q}$.

Proposition 4.8. Let $(\Lambda V, d)$ be a formal Sullivan cdga. Assume that $H(\Lambda V)$ is finite dimensional and concentrated in degree $\leq k$ with $2k < n$. Then $P_n(\Lambda V, d)$ is formal.

Proof. We assume first that $(\Lambda V, d)$ is minimal. Consider the canonical map $\varphi\colon (\Lambda V, d) \to P_n(\Lambda V, d)$ and note that φ induces an isomorphism in cohomology up until degree $s := \lfloor n/2 \rfloor$. Following the usual algorithm for the construction of a minimal model $(\Lambda W, d) \xrightarrow{\simeq} P_n(\Lambda V, d)$ (see [Su77, Section 5]), we may arrange that $W^{\leq s} = V^{\leq s}$ and that φ factors as

$$(\Lambda V, d) \xrightarrow{\psi} (\Lambda W, d) \to P_n(\Lambda V, d)$$

with ψ being the identity in degrees $\leq s$. Note that though $P_n(\Lambda V, d)$ may contain elements in negative degree, due to it being cohomologically connected, the construction in [Su77, Section 5] carries through verbatim.

Now since $(\Lambda V, d)$ is formal we may write $V = C \oplus N$ such that $d(C) = 0$, $d|_N$ is injective, and every closed element in the ideal $N \cdot \Lambda V$ is exact [DGMS75, Theorem 4.1]. We claim that the decomposition of $W^{\leq s} = V^{\leq s}$ into $C^{\leq s} \oplus N^{\leq s}$ satisfies the condition of s-formality for $(\Lambda W, d)$. To check this we only need to verify that any closed element in $N^{\leq s} \cdot \Lambda V^{\leq s}$ becomes exact in $(\Lambda W, d)$. But this holds since such an element lies in the image of ψ and is already exact in $(\Lambda V, d)$. Now by [FM05, Theorem 3.1], s-formality of $(\Lambda W, d)$ already implies formality (note that the theorem therein is stated for closed manifolds, but the proof holds for rational Poincaré duality cdga's).

Now, in general, if $(\Lambda V, d)$ is only a Sullivan cdga, it decomposes (uniquely) as a tensor product $(\Lambda V, d) \cong (\Lambda U, d) \otimes (\Lambda(C \oplus dC), d)$ of a minimal cdga with a contractible one [Su77, Theorem 2.2]. Note that the inclusion $\iota\colon (\Lambda U, d) \to (\Lambda U, d) \otimes (\Lambda(C \oplus dC), d)$ admits a retraction r which is the identity on U and sends $C \oplus dC$ to 0. The pair ι, r, satisfies the requirements of Lemma 4.6 and thus induces a quasi–isomorphism $P_n(\Lambda U, d) \to P_n(\Lambda V, d)$. □

Proposition 4.9. Let (A, d) be a cdga which admits a non-trivial Massey product. Then also $P_n(A, d)$ has a non-trivial Massey product.

Proof. By construction, $P_n A = A \oplus D_n A$ with A a subalgebra and $D_n A$ an ideal. Thus, the inclusion of cdga's $i : A \to P_n A$ admits a one-sided inverse map of cdga's $r : P_n A \to A$ with $r \circ i = \mathrm{id}$. Now for any non-trivial Massey product $m \in H(A)$, $i(m)$ is non-trivial as $(r \circ i)(m) \subseteq m$. Recall, we treat a Massey product as the set of cohomology classes obtained via any possible defining system (see [K66]), with the Massey product being trivial if the zero class is contained in this set. □

5. A counterexample

Our goal is to construct a non-zero degree map $P_1 \to P_2$ between cohomologically connected Poincaré duality cdga's such that P_2 is formal while P_1 is not, as a counterexample to the heuristic that domination should preserve formality. In view of Propositions 4.8, 4.9 and Lemma 4.6 it suffices to construct Sullivan cdga's with finite-dimensional cohomology A, B such that B is formal and A admits a non-trivial Massey product, together with a cdga morphism $f\colon A \to B$ and a differential graded A-module homomorphism $r\colon B \to A$ sending

$1 \mapsto 1$. Then the induced map $P_n(A) \to P_n(B)$ will have the desired properties for large enough n.

Remark 5.1. In view of Proposition 3.1 the above datum $f, r\colon A \leftrightarrows B$ with B formal should not exist in case A admits a non-trivial triple Massey product. Indeed, consider a triple Massey product $\langle [x],[y],[z]\rangle$ in A. Then, as the Massey product $\langle [f(x)],[f(y)],[f(z)]\rangle$ vanishes in B, we can find a defining system $a, b \in B$ with $da = f(x)f(y)$, $db = f(y)f(z)$ such that $af(z) - (-1)^{|x|}f(x)b$ is exact. But then using that r is a dg-A-module morphism we find that $r(a), r(b)$ is a defining system for $\langle [x],[y],[z]\rangle$ and the representing cocycle

$$r(a)z - (-1)^{|x|}xr(b) = r(af(z) - (-1)^{|x|}f(x)b)$$

is exact. This shows triviality of the Massey product $\langle [x],[y],[z]\rangle$.

Hence in our counterexample we will need to construct at least a non-trivial quadruple Massey product. In order to motivate what is happening in the counterexample it is rather instructive to check where the above argument fails for quadruple Massey products. To this end consider a quadruple Massey product $\langle [w],[x],[y],[z]\rangle$ in A. As before, choose a defining system a, b, c, g, h for $\langle [f(w)],[f(x)],[f(y)],[f(z)]\rangle$ such that $da = f(w)f(x)$, $db = f(x)f(y)$, $dc = f(y)f(z)$, $dg = f(a)y - f(w)b$ and $dh = bf(z) - f(x)c$. While it still holds that $r(a), r(b), r(c), r(g), r(h)$ is a defining system for $\langle [w],[x],[y],[z]\rangle$ it is in general no longer true that the cocycles representing the Massey products get mapped to one another, i.e. we might have

$$wr(h) + r(a)r(c) + zr(g) \neq r(f(w)h + ac + f(z)g)$$

in case $r(ac) \neq r(a)r(c)$, which can happen since r is not fully multiplicative. In particular the right hand side being exact does not force the left hand side to be so. In other words: while a non-trivial triple Massey product would obstruct the construction of the module retract r in the counterexample below, the freedom of choosing $r(ac)$ will allow us to construct r even in the presence of a non-trivial quadruple Massey product. ☐

We begin with the construction of A. Set $(A, d) = (\Lambda(X, Y, a, b, c, e, f, h, i) \otimes \Lambda(V), d)$ where

degree	generators	differential
2	X, Y	$X, Y \mapsto 0$
3	a, b, c	$a \mapsto X^2 \quad b \mapsto XY \quad c \mapsto Y^2$
4	e, f	$e \mapsto Ya - Xb \quad f \mapsto Yb - Xc$
5	h, i	$h \mapsto Xe + ab \quad i \mapsto Yf + bc$

and $V = V^{\geq 6}$ is a vector space which we construct inductively in order to eliminate all cohomology in degrees ≥ 7. To be precise, we first choose cycles representing a basis for degree 7 cohomology. Then for each these element introduce a generator in V^6 and map it to the chosen cycle under the differential. The resulting algebra will have trivial degree 7 cohomology while cohomology in degrees ≤ 6 remains unchanged. Now repeat this process inductively for all higher degrees.

Lemma 5.2. The cohomology of (A, d) is generated by the linearly independent cohomology classes of the cocycles $1, X, Y, m$, where $m = Ye + ac + Xf$. Furthermore the Massey product $\langle [X],[X],[Y],[Y]\rangle$ is non-trivial and represented by $[m]$.

Proof. Clearly $1, X, Y$ generate cohomology in degrees ≤ 2. Furthermore $A^3 = \langle a, b, c\rangle$ maps isomorphically onto $\Lambda^2(X, Y)$ so there is no cohomology in degree $3, 4$ in $\Lambda(X, Y, a, b, c)$. This changes in degree 5, where the $\ker d$ is generated by $Ya - Xb$, $Yb - Xc$. Note that the corresponding cohomology classes do indeed form a basis of $H^5(\Lambda(X, Y, a, b, c), d)$, since d vanishes on the degree 4 span of the above generators. Thus after introducing e, f we obtain

$H^5(\Lambda(X,Y,a,b,c,e,f),d) = 0 = H^4(\Lambda(X,Y,a,b,c,e,f),d)$. At this point we compute that the degree 6 part of $\ker d$ is $\langle Xe+ab, m, Yf+bc\rangle \oplus \Lambda^3(X,Y)$. The differential maps the degree 5 span of the above generators onto $\Lambda^3(X,Y)$ so the cocycles in the left hand factor yield a basis for the cohomology at this stage, after introducing h,i,V only the class of m remains, generating $H^6(A)$. This proves the first part of the Lemma. The reader can also verify this with the "Commutative Differential Graded Algebras" module in [Sage]:

```
A.<X,Y,a,b,c,e,f,h,i>= GradedCommutativeAlgebra(QQ, degrees = (2,2,3,3,3,4,4,5,5))
B = A.cdg_algebra(a : X*X, b : X*Y, c : Y*Y, e : Y*a - X*b, f : Y*b - X*c, h : X*e + a*b, i : Y*f + b*c)
[B.cohomology(i) for i in [1..7]]
```

When writing down a defining system for the Massey product $\langle [X],[X],[Y],[Y]\rangle$, there is no choice for the primitives of X^2, XY, Y^2, except for a,b,c. When choosing primitives p_1,p_2 for the cocycles $Ya-Xb$ and $Yb-Xc$, we get $p_1 = e+\alpha_1$, $p_2 = f+\alpha_2$ for some $\alpha_i \in (\ker d)^4 = \Lambda^2(X,Y)$. Then the resulting representative of $\langle [X],[X],[Y],[Y]\rangle$ is $m+Y\alpha_1+X\alpha_2$. Independent of the choice of α_i, this is cohomologous to m. Hence we get a unique non-trivial cohomology class representing $\langle [X],[X],[Y],[Y]\rangle$. □

Now we come to the construction of B. Consider first the algebras $H_1 = \Lambda(x,y)/\Lambda^{\geq 2}(x,y)$ with $|x| = |y| = 2$ and $H_2 = \Lambda(\tilde{a},\tilde{c})$ with $|\tilde{a}| = |\tilde{c}| = 3$. Our algebra B will be of the formal rational type of the product[1] of the augmented cdga's H_1 and H_2, i.e. $(H,d) = (H_1 \oplus_{\mathbb{Q}} H_2, 0)$, where $(H_1 \oplus_{\mathbb{Q}} H_2)^k = H_1^k \oplus H_2^k$ for $k \geq 1$ and $(H_1 \oplus_{\mathbb{Q}} H_2)^0 = \langle (1,1)\rangle$, with multiplication $(n_1,n_2)\wedge(m_1,m_2) = (n_1m_1, n_2m_2)$. However since we want to construct a A-module retraction $B \to A$, we will need a better representative of the quasi–isomorphism type of H.

Consider $f_1\colon A \to H_1$ which sends $X \mapsto x$, $Y \mapsto y$ and all other generators to 0. In order to check that this morphism is compatible with the differential we only need to check that for every generator v, we have $f_1(dv) = df_1(v) = 0$, which is clearly satisfied (note that while we have not explicitly specified the differential on V, $d(V)$ lies in degrees ≥ 7 and thus maps to 0). In the same fashion we check that the algebra morphism $f_2\colon A \to H_2$ which sends $a \mapsto \tilde{a}$, $c \mapsto \tilde{c}$, and all other generators to 0 is a morphism of cdga's. These two morphisms piece together to form the components of a cdga morphism $f = (f_1,f_2)\colon A \to H$, where e.g. $f(X) = (x,0)$ and $f(a) = (0,\tilde{a})$. The cdga B is now defined as the relative minimal model of f. Thus we have a commutative diagram

$$\begin{tikzcd} (A,d) \arrow[r, "f"] \arrow[dr, "\iota"'] & (H,0) \\ & (A\otimes\Lambda W, d) \arrow[u, "\varphi"'] \end{tikzcd}$$

with φ a quasi isomorphism, ι being the standard inclusion and the bottom algebra being defined as B. Since $A^1 = H^1 = 0$ there is an explicit degreewise inductive procedure (see [Su77, §5]) to compute ΛW, φ, and the differential in terms of A and f. We will need to do so up until degree 5. The result is shown in the following table.

degree	generators	differential	image under φ
3	α,γ	$\alpha,\gamma \mapsto 0$	$\alpha \mapsto \tilde{a} \quad \gamma \mapsto \tilde{c}$
4	$s_{X\alpha}, s_{Y\alpha}, s_{X\gamma}, s_{Y\gamma}$	$s_* \mapsto *$	$s_* \mapsto 0$
5	$t_1,\ldots,t_9$	$t_i \mapsto v_i$ (as defined below)	$t_i \mapsto 0$

[1] i.e. the pullback in the category of augmented cdga's of the diagram $(H_1,0) \to (\mathbb{Q},0) \leftarrow (H_2,0)$ of augmentation maps

Let us briefly reason that this is indeed the beginning of a relative minimal model of f. We add new generators by induction over degree turning φ into a quasi isomorphism. The original map f is cohomologically injective as can be seen by applying it to the generators of $H(A)$ described in Lemma 5.2 (note that the element m maps to $\tilde{a}\tilde{c}$). The only degree in which f^* is not an isomorphism is in degree 3 where $\operatorname{coker}(f^*) = \langle \tilde{a}, \tilde{c} \rangle$ (as the differential on A is injective on $\langle a, c\rangle$) . Thus we introduce the generators α, γ to add to cohomology and map them onto $\tilde{a}, \tilde{c}$. At this point φ is cohomologically surjective so it remains to eliminate $\ker \varphi^*$. The introduction of α, γ has created new unwanted cohomology in degree 5 in form of the classes of $X\alpha$, $Y\alpha$, $X\gamma$, and $Y\gamma$. Thus we introduce the generators $s_{X\alpha}$, $s_{Y\alpha}$, $s_{X\gamma}$, $s_{Y\gamma}$ to make these exact. Finally, we check that at this stage a basis of $H^6(A \otimes \Lambda(\alpha, \gamma, s_{X\alpha}, s_{Y\alpha}, s_{X\gamma}, s_{Y\gamma}), d)$ is generated by m as well as the cocycles

$$\begin{array}{lll} v_1 = m - \alpha\gamma, & v_2 = Y s_{X\alpha} - X s_{Y\alpha}, & v_3 = Y s_{X\gamma} - X s_{Y\gamma}, \\ v_4 = a\alpha - X s_{X\alpha}, & v_5 = b\alpha - X s_{Y\alpha}, & v_6 = c\alpha - Y s_{Y\alpha}, \\ v_7 = a\gamma - X s_{X\gamma}, & v_8 = b\gamma - X s_{Y\gamma}, & v_9 = c\gamma - Y s_{Y\gamma}, \end{array}$$

where the v_i generate a basis of $(\ker \varphi^*)^6$. Hence we introduce new generators $t_1, \ldots, t_9$ in degree 5 with $dt_i = v_i$. The procedure of course carries on indefinitely but we will not need to describe $W^{\geq 6}$ explicitly.

It remains to construct a dg-A-module retract of the map $\iota\colon A \to A \otimes \Lambda W$. In order to do this we recall the following

Definition 5.3. Let A be a dga and (M, d) be a dg-A-module. Then a semi-free extension of (M, d) is a dg-A-module of the form $(M \oplus (A \otimes V), d)$, where V is a graded vector space and $d(1 \otimes V) \subset M$.

For us this concept is helpful due to the following standard Lemma. Part (1) is an immediate observation, while part (2) is a more explicit form of [FHT12, Lemma 14.1] which will prove useful when dealing with the explicit example.

Lemma 5.4.
1. Let $f\colon M \to N$ a morphism of dg-A-modules and $(M \oplus (A \otimes V), d)$ a semi-free extension of M. Let $(v_i)_{i \in I}$ be a basis of V, and let $(\alpha_i)_{i\in I}$ be a collection of elements in N with $d\alpha_i = f(dv_i))$. Then f extends to a morphism of dg-A-modules $M \oplus (A \otimes V) \to N$ by setting $f(v_i) = \alpha_i$.
2. Let $A \to A \otimes \Lambda W$ be a relative minimal cdga with $A^1 = W^1 = 0$. For $0 \leq j \leq i$, set $V_{(i,j)} = (\Lambda^{i-j} W)^{2i-j}$. Then $\Lambda W = \bigoplus_{0 \leq j \leq i} V_{(i,j)}$ and for any (i, j) as above the inclusion
$$A \otimes \left(\bigoplus_{(k,l)<(i,j)} V_{(k,l)} \right) \to A \otimes \left(\bigoplus_{(k,l) \leq (i,j)} V_{(k,l)} \right)$$
is a semi-free extension, where we use the lexicographical order on tuples.

Proof. Part (1) is straightforward verification. For the proof of part (2) we observe that due to $W = W^{\geq 2}$ we indeed have

$$\Lambda W = \bigoplus_{0 \leq 2k \leq l} (\Lambda^k W)^l = \bigoplus_{0 \leq j \leq i} (\Lambda^{i-j} W)^{2i-j}.$$

It remains to check that $d(V_{(i,j)}) \subset A \otimes \left(\bigoplus_{(k,l)<(i,j)} V_{(k,l)} \right)$. To see this, we investigate the differential with respect to its bidegree $A \otimes ((\Lambda^p W)^q)$, where p is the wordlength degree in W and q is the cohomological degree in ΛW. If p does not increase then q decreases by at least 1 due to minimality and $A^1 = 0$. Furthermore p can decrease by at most 1 in which case q decreases by 2 since $d(W) \cap A$ lies in degrees ≥ 3. Consequently

$$d\left((\Lambda^{i-j}W)^{2i-j}\right) \subset A \otimes \left((\Lambda^{\geq i-j+1}W)^{\leq 2i-j+1} \oplus (\Lambda^{i-j}W)^{\leq 2i-j-1} \oplus (\Lambda^{i-j-1}W)^{\leq 2i-j-2}\right)$$

which proves the claim. □

Thus by this lemma, in order to define the retraction $r\colon A \otimes \Lambda W \to A$ it suffices to inductively specify images of a suitable basis of ΛW and extend A-linearly. In fact, even less is sufficient by the following:

Lemma 5.5. Any morphism

$$r\colon A \otimes \left(\bigoplus_{(i,j)\leq(4,3)} V_{(i,j)} \right) \to A$$

of dg-A-modules extends to $A \otimes \Lambda W$.

Proof. Recall that by part (1) of Lemma 5.4 the only obstruction to extend r over a new generator v is that the class $[r(dv)] \in H^*(A)$ has to vanish. By definition, for $(i,j) > (4,3)$ the space $V_{(i,j)}$ is concentrated in cohomological degrees ≥ 6 (since $V_{(i,i)} = 0$ for $i \neq 0$) while $H(A)$ is concentrated in degrees ≤ 6. □

Furthermore note that for $(i,j) \leq (4,3)$, we have $V_{(i,j)} \subset \Lambda(W^{\leq 5})$ which means we have already computed all the required algebra generators. We define r according to the following table, where we list all non-trivial $V_{(i,j)}$ with $(i,j) \leq (4,3)$ in their order of occurrence.

extension	generators	image under r
$V_{(0,0)}$	1	$1 \mapsto 1$
$V_{(2,1)}$	α, γ	$\alpha, \gamma \mapsto 0$
$V_{(3,1)}$	$s_{X\alpha}, s_{Y\alpha}, s_{X\gamma}, s_{Y\gamma}$	$s_* \mapsto 0$
$V_{(4,2)}$	$\alpha\gamma$	$\alpha\gamma \mapsto m$
$V_{(4,3)}$	$t_1, \ldots, t_9$	$t_i \mapsto 0$

One checks that indeed for any of the generators v above we have $r(dv) = dr(v)$. Then by Lemmas 5.4, 5.5 we obtain the desired retraction $r\colon A \otimes \Lambda W \to A$.

In conclusion, applying the Poincaré dualization construction, we have the desired result of this section:

Theorem 5.6. There is a non-zero degree map of cohomologically connected rational Poincaré duality cdga's $P_1 \to P_2$ such that P_1 carries a non-trivial quadruple Massey product, and P_2 is formal.

The above map of cdga's can be realized, up to a lift of a non-zero grading automorphism on the cohomology of P_2, by a non-zero degree smooth map of closed manifolds. First of all, there is a topological map of simply connected rational spaces $M_{\mathbb{Q}} \to N_{\mathbb{Q}}$ corresponding to this map of cdga's. For this, one checks that the algorithms for building a minimal model and the induced map between the models [Su77, Section 5], [FHT12, Proposition 12.9], usually stated for connected cdga's, work essentially verbatim for cohomologically connected cdga's (we already made use of the former in Proposition 4.8). Thus, we may replace the map between P_1 and P_2 by one between connected cdga's and hence by one between rational spaces. We choose n not divisible by 4, and apply the realization theorem [Su77, Theorem 13.2] to obtain simply connected closed smooth manifolds M and N with rationalization maps $M \to M_{\mathbb{Q}}$ and $N \to N_{\mathbb{Q}}$. We hence consider the following lifting problem:

$$\begin{tikzcd} M \arrow[r, dashed] \arrow[d] & N \arrow[d] \\ M_{\mathbb{Q}} \arrow[r] & N_{\mathbb{Q}} \end{tikzcd}$$

The obstructions to lifting (up to homotopy) the composite $M \to N_{\mathbb{Q}}$ lie in $H(M; \pi_{*-1}(F))$, where F is the homotopy fiber of the map $N \to N_{\mathbb{Q}}$. Since this map is a rationalization, the homotopy groups of F are torsion. Now, since M has the homotopy type of a simply connected finite complex, and is formal, there are sufficiently many endomorphisms of M that allow us to surpass the obstructions to lifting, upon precomposing with such an endomorphism [Su77, Theorem 12.2] (we remark that there is an implicit nilpotency assumption on the space therein). We refer to [A15, Corollary 4.3] for full details of the argument. To see that $M_{\mathbb{Q}}$ satisfies the hypothesis of [A15, Corollary 4.3], i.e. that it admits a self-map with k^{th} multiples for all large enough k, see the proof of [A15, Corollary 4.4]. Namely, the map in question is a lift of any grading automorphism on the cohomology; such a lift exists by formality [Su77, Theorem 12.7].

Ultimately we obtain a non-zero degree endomorphism f of M and a homotopy commutative diagram

$$\begin{tikzcd} M \arrow[r, "f"'] \arrow[d] \arrow[rr, bend left] & M \arrow[d] & N \arrow[d] \\ M_{\mathbb{Q}} \arrow[r] & M_{\mathbb{Q}} \arrow[r] & N_{\mathbb{Q}} \end{tikzcd}$$

giving us the following:

Corollary 5.7. There is a non-zero degree map from a formal closed smooth simply connected manifold to a closed smooth simply connected manifold that has a non-trivial quadruple Massey product.

Note that the smallest dimension in which our construction provides such an example is thirteen.

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