SO(3) MONOPOLES, LEVEL-ONE SEIBERG-WITTEN MODULI SPACES, AND WITTEN'S CONJECTURE IN LOW DEGREES

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ABSTRACT. We prove Witten's formula relating the Donaldson and Seiberg-Witten series modulo powers of degree $c+2$, with $c = -\frac{1}{4}(7\chi + 11\sigma)$, for four-manifolds obeying some mild conditions, where χ and σ are their Euler characteristic and signature. We use the moduli space of SO(3) monopoles as a cobordism between a link of the Donaldson moduli space of anti-self-dual SO(3) connections and links of the moduli spaces of Seiberg-Witten monopoles. Gluing techniques allow us to compute contributions from Seiberg-Witten moduli spaces lying in the first (or 'one-bubble') level of the Uhlenbeck compactification of the moduli space of SO(3) monopoles.

1. INTRODUCTION

1.1. **Main results.** In the present article we extend our results in [17, 18], showing that Witten's conjecture [53] relating the Donaldson and Seiberg-Witten series holds in 'low degrees' for a broad class of four-manifolds. We apply our work on gluing SO(3) monopoles [19, 21] — restricting to the case of one 'instanton bubble' in this article — to prove that these two series are equivalent through a higher range of degrees than was possible in [17, 18]. We assume throughout that X is a closed, connected, smooth four-manifold with an orientation for which $b_2^+(X) > 0$. The Seiberg-Witten (SW) invariants (see §2.3.1) comprise a function, $SW_X : \mathrm{Spin}^c(X) \to \mathbb{Z}$, where $\mathrm{Spin}^c(X)$ is the set of isomorphism classes of spin^c structures on X. For $w \in H^2(X;\mathbb{Z})$, define the *Seiberg-Witten series* by

$$(1.1) \qquad \mathbf{SW}_X^w(h) = \sum_{\mathfrak{s}\in\mathrm{Spin}^c(X)} (-1)^{\frac{1}{2}(w^2+c_1(\mathfrak{s})\cdot w)} SW_X(\mathfrak{s}) e^{\langle c_1(\mathfrak{s}),h\rangle}, \quad h \in H_2(X;\mathbb{R}),$$

and let $\mathbf{D}_X^w(h)$ denote the Donaldson series (see [31, Theorem 1.7] or §2.5 here). There is a map $c_1 : \mathrm{Spin}^c(X) \to H^2(X;\mathbb{Z})$ and the image of the support of SW_X is the set B of SW-basic classes [53]. A four-manifold X has SW-simple type, when $b_1(X) = 0$, if $c_1(\mathfrak{s})^2 = 2\chi + 3\sigma$ for all $c_1(\mathfrak{s}) \in B$, where χ and σ are the Euler characteristic and signature of X. Let $B^\perp \subset H^2(X;\mathbb{Z})$ denote the orthogonal complement of B with respect to the intersection form Q_X on $H^2(X;\mathbb{Z})$. Denote $c(X) = -\frac{1}{4}(7\chi + 11\sigma)$. Our main result is

Theorem 1.1. *Let X be four-manifold with $b_1(X) = 0$ and odd $b_2^+(X) \geq 3$. Assume X is abundant, SW-simple type, and effective. Then there exist $\Lambda \in B^\perp$ and $w \in H^2(X;\mathbb{Z})$ for which $\Lambda^2 = 4 - (\chi + \sigma)$ and $w - \Lambda \equiv w_2(X) \pmod 2$. For any such Λ and w, and any $h \in H_2(X;\mathbb{R})$, one has*

$$(1.2) \qquad \begin{aligned} \mathbf{D}_X^w(h) &\equiv 0 \equiv \mathbf{SW}_X^w(h) \pmod{h^{c(X)-2}}, \\ \mathbf{D}_X^w(h) &\equiv 2^{2-c(X)} e^{\frac{1}{2}h\cdot h} \mathbf{SW}_X^w(h) \pmod{h^{c(X)+2}}. \end{aligned}$$

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Noting that $c(X) = \chi_h(X) - c_1^2(X)$ (see §1.2), Theorem 1.1 can be used to compute Donaldson invariants for four-manifolds in the region $c_1^2 \leq \chi_h$ of the (χ_h, c_1^2) plane; all simply connected, compact, complex algebraic surfaces with odd $b_2^+ \geq 3$ are abundant (see below) and SW-simple type.

We shall explain below the terminology and notation in the statement of Theorem 1.1. Witten's conjecture [53] asserts that a four-manifold X with $b_1(X) = 0$ and odd $b_2^+(X) \geq 3$ has SW-simple type if and only if it has KM-simple type, that is, simple type in the sense of Kronheimer and Mrowka (see Definition 1.4 in [31]), and that the SW-basic and KM-basic classes (see Theorem 1.7 in [31]) coincide; if X has simple type, then

$$\mathbf{D}_X^w(h) = 2^{2-c(X)} e^{\frac{1}{2}Q_X(h,h)} \mathbf{SW}_X^w(h), \quad h \in H_2(X;\mathbb{R}). \tag{1.3}$$

The quantum field theory argument giving equation (1.3) when $b_2^+(X) \geq 3$ has been extended by Moore and Witten [38] to allow $b_2^+(X) \geq 1$, $b_1(X) \geq 0$, and four-manifolds X of non-simple type. Recall that $b_2^+(X)$ is the dimension of a maximal positive-definite linear subspace $H^{2,+}(X;\mathbb{R})$ for the intersection pairing Q_X on $H^2(X;\mathbb{R})$.

In [17, 18] we proved that the second equation (1.2) holds modulo $h^{c(X)}$, with $\Lambda^2 = 2-(\chi+\sigma)$ but otherwise identical hypotheses. If one desires a mod h^δ relation such as (1.2) for larger values of δ relative to $c(X)$, one must allow more bubbles and the difficulty of the calculations rapidly increases: see [18, §1] for a more detailed discussion. In particular, to prove Witten's conjecture (1.3) in full, we would need to prove that equation (1.2) holds modulo h^δ for all $\delta \in \mathbb{Z}_{\geq 0}$: this is the goal of the remaining papers [19, 20, 21] in our series.

If the hypotheses concerning a class Λ are omitted, one still obtains a formula for Donaldson invariants in terms of Seiberg-Witten invariants (see Theorem 1.5), but it has a more complicated structure and one cannot immediately see if it has the shape (1.2) consistent with Witten's conjecture.

Definition 1.2. [14, p. 169] We say that a closed, oriented four-manifold X is *abundant* if the restriction of the intersection form to $B^\perp$ contains a hyperbolic sublattice.

The abundance condition is just a convenient way of formulating the weaker, but more technical condition that one can find (for example) classes $\Lambda_j \in B^\perp$ such that $\Lambda_j^2 = 2j - (\chi+\sigma)$, for $j = 1, 2$: this is the only property of $Q_X|B^\perp$ which we use to prove Theorem 1.1. All compact, complex algebraic, simply-connected surfaces with $b_2^+ \geq 3$ are abundant [17, Theorem A.1].

We expect the second equation (1.2) to hold modulo h^δ, for any δ, without reference to whether a class Λ exists with the properties stated and without the constraint on w. For an explanation of why the limitations of the present article prevent us from relaxing the constraints on w in Theorem 1.1, see Remark 6.8.

In the present article and its companion [18] we prove Theorem 1.1 using the moduli space $\mathcal{M}_\mathfrak{t}$ of SO(3) monopoles [45] to provide a cobordism between the link of the moduli space M_κ^w of anti-self-dual connections and links of moduli spaces of Seiberg-Witten moduli spaces, $M_\mathfrak{s}$, these moduli spaces being (topologically) embedded in $\mathcal{M}_\mathfrak{t}$. Pairing certain cohomology classes with the links of these moduli spaces gives multiples of the Donaldson and Seiberg-Witten invariants, respectively, and thus provides a relation between the two types of invariants. Let $\bar{\mathcal{M}}_\mathfrak{t}$ denote the Uhlenbeck compactification (see Theorem 2.2) of $\mathcal{M}_\mathfrak{t}$ in the space of ideal SO(3) monopoles, $\cup_{\ell=0}^\infty(\mathcal{M}_{\mathfrak{t}_\ell} \times \mathrm{Sym}^\ell(X))$.

Definition 1.3. [17, Definition 1.3] We say that a closed, oriented, smooth four-manifold X with $b_1(X) = 0$ and $b_2^+(X) \geq 1$ is *effective* if X satisfies Conjecture 3.1 in [14]. This conjecture asserts that for a moduli space of ideal Seiberg-Witten monopoles, $M_{\mathfrak{s}}\times\mathrm{Sym}^\ell(X)$, appearing in level $\ell\geq 0$ of $\bar{\mathcal{M}}_{\mathfrak{t}}$, the intersection number

$$\#\left(\bar{\mathcal{V}}(z)\cap\bar{\mathcal{W}}^\eta\cap\bar{\mathbf{L}}_{\mathfrak{t},\mathfrak{s}}\right) \tag{1.4}$$

is a multiple of the Seiberg-Witten invariant $SW_X(\mathfrak{s})$. Here, $\bar{\mathcal{V}}(z)$ and $\bar{\mathcal{W}}$ are the closures in $\bar{\mathcal{M}}_{\mathfrak{t}}/S^1$ of geometric representatives of Donaldson-type cohomology classes on the top stratum of $\mathcal{M}_{\mathfrak{t}}/S^1$ and $\bar{\mathbf{L}}_{\mathfrak{t},\mathfrak{s}}$ is the link of $(M_{\mathfrak{s}}\times\mathrm{Sym}^\ell(X))\cap\bar{\mathcal{M}}_{\mathfrak{t}}/S^1$ in $\bar{\mathcal{M}}_{\mathfrak{t}}/S^1$, and $\eta\geq 0$ is an integer for which $\deg(z)+2(\eta+1)=\dim\mathcal{M}_{\mathfrak{t}}$. In particular, the intersection number (1.4) is zero when the Seiberg-Witten invariant $SW_X(\mathfrak{s})$ is zero.

When $b_1(X)>0$, one replaces the Seiberg-Witten invariants, $SW_X(\mathfrak{s})$, mentioned in Definition 1.3 with values of the Seiberg-Witten function $SW_{X,\mathfrak{s}}$ — see [18, Equation (1.7) and §4.1] for definitions. The motivation for Conjecture 3.1 of [14] and a more detailed explanation appears in [14, §3.1] together with an explanation of its role in the proofs of the main results of that paper; see also [15]. It is almost certainly true that this conjecture holds for all four-manifolds, based on our work in [19, 20, 21], and it is a simple consequence of our Conjecture 1.6 for the general form of the pairings of Donaldson-type cohomology classes with links of ideal Seiberg-Witten moduli spaces in $\bar{\mathcal{M}}_{\mathfrak{t}}$.

We verify Conjecture 3.1 in [14] by direct calculation in the present article (see Theorem 6.1 and Proposition 6.5) for Donaldson invariants defined by $M^w_\kappa\hookrightarrow\mathcal{M}_{\mathfrak{t}}$ and Seiberg-Witten moduli spaces $M_{\mathfrak{s}}$ embedded in the first level, $\mathcal{M}_{\mathfrak{t}_1}\times X$, while in [17, 18] we verified the conjecture for Seiberg-Witten moduli spaces $M_{\mathfrak{s}}$ contained in the top level, $\mathcal{M}_{\mathfrak{t}}$, of the Uhlenbeck compactification $\bar{\mathcal{M}}_{\mathfrak{t}}$ (see Theorem 4.13 and Proposition 4.22 in [18]). However, we strongly expect the conjecture to hold for Seiberg-Witten moduli spaces $M_{\mathfrak{s}}$ contained in any level of the compactification $\bar{\mathcal{M}}_{\mathfrak{t}}$.

Equation (1.2) is a special case of a more general formula for Donaldson invariants which we now describe; the hypotheses still include an important restriction which guarantees that Seiberg-Witten moduli spaces with non-trivial invariants do not lie in the second or lower levels of the compactified SO(3)-monopole moduli space. For $\Lambda\in H^2(X;\mathbb{Z})$, define

$$i(\Lambda)=\Lambda^2+c(X)+\chi+\sigma. \tag{1.5}$$

If $S(X)\subset\mathrm{Spin}^c(X)$ denotes the support of the function $SW_X:\mathrm{Spin}^c(X)\to\mathbb{Z}$, let

$$r(\Lambda,c_1(\mathfrak{s}))=-(c_1(\mathfrak{s})-\Lambda)^2-\tfrac{3}{4}(\chi+\sigma)\quad\text{and}\quad r(\Lambda)=\min_{\mathfrak{s}\in S(X)}r(\Lambda,c_1(\mathfrak{s})). \tag{1.6}$$

See Remark 3.36 in [18] for a discussion of the significance of $r(\Lambda,c_1(\mathfrak{s}))$ and $r(\Lambda)$, while the significance of $i(\Lambda)$ is explained in [18, §1.2 & §4.6]; see also [14]. For the statement of Theorem 1.4 below, we refer the reader to §2.5 for a definition of the Donaldson invariants, $D^w_X(h^{\delta-2m}x^m)$. Recall that $P^{a,b}_d(\zeta)$, the *Jacobi polynomial* [27, §8.960], is defined by

$$P^{a,b}_d(\zeta)=\frac{1}{2^d}\sum_{v=0}^d\binom{d+a}{v}\binom{d+b}{d-v}(\zeta-1)^{d-v}(\zeta+1)^v,\quad\zeta\in\mathbb{C}, \tag{1.7}$$

just as in [18, Equation (4.27)]; if $d=0$, then $P^{a,b}_d(0)=1$.

Theorem 1.4. *Let X be a four-manifold with $b_1(X)=0$ and odd $b_2^+(X)\geq 1$. Assume X is effective. Suppose $\Lambda,w\in H^2(X;\mathbb{Z})$ are classes such that $w-\Lambda\equiv w_2(X)\pmod 2$ and, if $b_2^+(X)=1$, the class $w \pmod 2$ admits no torsion integral lifts. Suppose $h\in H_2(X;\mathbb{R})$, $x\in H_0(X;\mathbb{Z})$ is the positive generator, and δ is a non-negative integer. If $\delta<i(\Lambda)$ and*

$\delta = r(\Lambda)+4$, then

$$
\begin{aligned}
D^w_X(h^{\delta-2m}x^m) &= 2^{1-\frac{1}{4}i(\Lambda)-\frac{3}{4}\delta}(-1)^{m+\frac{1}{2}(\sigma-w^2)} \sum_{\substack{\mathfrak{s}\in\mathrm{Spin}^c(X)\\ r(\Lambda,c_1(\mathfrak{s}))=\delta}} (-1)^{\frac{1}{2}(w^2+c_1(\mathfrak{s})\cdot(w-\Lambda))} \\
&\quad \times (-2)^{d_s(\mathfrak{s})/2} P^{a-1,b}_{d_s(\mathfrak{s})/2}(0) SW_X(\mathfrak{s})\langle c_1(\mathfrak{s})-\Lambda,h\rangle^{\delta-2m} \\
&\quad + 2^{1-\frac{1}{4}i(\Lambda)-\frac{3}{4}\delta}(-1)^{m+\frac{1}{2}(\sigma-w^2)} \sum_{\substack{\mathfrak{s}\in\mathrm{Spin}^c(X)\\ r(\Lambda,c_1(\mathfrak{s}))=\delta-4}} (-1)^{\frac{1}{2}(w^2+c_1(\mathfrak{s})\cdot(w-\Lambda))} \\
&\quad \times (-2)^{d_s(\mathfrak{s})/2} P^{a,b}_{d_s(\mathfrak{s})/2}(0) SW_X(\mathfrak{s}) \\
&\quad \times \Big(a_0\langle c_1(\mathfrak{s})-\Lambda,h\rangle^{\delta-2m} + b_0\langle c_1(\mathfrak{s})-\Lambda,h\rangle^{\delta-2m-1}\langle\Lambda,h\rangle \\
&\qquad + a_1\langle c_1(\mathfrak{s})-\Lambda,h\rangle^{\delta-2m-2}Q_X(h,h)\Big),
\end{aligned}
\tag{1.8}
$$

where all terms on the right which would have a negative exponent are omitted and the coefficients $a_0(c_1(\mathfrak{s}),\Lambda,\delta,m)$, $a_1(\delta,m)$, and $b_0(\delta,m)$ are given by

$$
\begin{aligned}
a_0 &= 3(c_1(\mathfrak{s})-\Lambda)^2 + c_1^2(X) + 2(c_1(\mathfrak{s})-\Lambda)\cdot\Lambda + 4\delta - 12m, \\
b_0 &= 2(\delta-2m)\frac{P^{a-1,b+1}_{d_s(\mathfrak{s})/2}(0)}{P^{a,b}_{d_s(\mathfrak{s})/2}(0)}, \\
a_1 &= 4\binom{\delta-2m}{2},
\end{aligned}
\tag{1.9}
$$

with

$$
a = -\tfrac{1}{2}d_s(\mathfrak{s}) + \tfrac{1}{4}(i(\Lambda)-\delta) \quad \text{and} \quad b = -\tfrac{1}{2}d_s(\mathfrak{s}) - \tfrac{1}{4}(\chi+\sigma),
$$

and $P^{a,b}_{d_s(\mathfrak{s})/2}(0)$ given by definition (1.7). If $d_s(\mathfrak{s})=0$, then $P^{a,b}_{d_s(\mathfrak{s})/2}(0)=1$. If $b_2^+(X)=1$, then all invariants in equation (1.8) are evaluated with respect to the chambers determined by the same period point in the positive cone of $H^2(X;\mathbb{R})$.

Theorem 1.2 in [18] provides a formula analogous to (1.8) under similar hypotheses for $D^w_X(h^{\delta-2m}x^m)$, but for $\delta=r(\Lambda)$, and a vanishing result for $\delta<r(\Lambda)$.

Specializing Theorem 1.4 to case where X has SW-simple type and Λ is in $B^\perp$ and recalling that in this situation one has [18, §4.6]

$$
r(\Lambda) = -\Lambda^2 + c(X) - (\chi+\sigma),
\tag{1.10}
$$

yields

Theorem 1.5. *Continue the hypotheses of Theorem 1.4. We further assume that X has $b_2^+(X)>1$, Seiberg-Witten simple type and that $\Lambda\in B^\perp$. Then*

$$
\begin{aligned}
D^w_X(h^{\delta-2m}x^m) &= 2^{-\frac{1}{2}(c(X)+\delta)}(-1)^{m+\frac{1}{2}(\sigma-w^2)} \\
&\quad \times \sum_{\mathfrak{s}\in\mathrm{Spin}^c(X)} (-1)^{\frac{1}{2}(w^2+w\cdot c_1(\mathfrak{s}))} SW_X(\mathfrak{s}) \\
&\quad \times \Big(a_0\langle c_1(\mathfrak{s})-\Lambda,h\rangle^{\delta-2m} + b_0\langle c_1(\mathfrak{s})-\Lambda,h\rangle^{\delta-2m-1}\langle\Lambda,h\rangle \\
&\qquad + a_1\langle c_1(\mathfrak{s})-\Lambda,h\rangle^{\delta-2m-2}Q_X(h,h)\Big),
\end{aligned}
\tag{1.11}
$$

where the coefficients $a_0(\Lambda,\delta,m)$, $b_0(\delta,m)$, *and* $a_1(\delta,m)$ *are given by*

$$\begin{aligned} a_0 &= 4c_1^2(X)+\Lambda^2+4\delta-12m,\\ b_0 &= 2(\delta-2m),\\ a_1 &= 4\binom{\delta-2m}{2}. \end{aligned} \tag{1.12}$$

Theorem 1.4 in [18] provides a formula analogous to (1.11) under similar hypotheses for $D_X^w(h^{\delta-2m}x^m)$, but for $\delta=r(\Lambda)$, and a vanishing result for $\delta<r(\Lambda)$.

Our main result, Theorem 1.1, then follows from Theorem 1.5 and [18, Theorem 1.1].

1.2. **Some applications to minimal surfaces of general type.** If X is any closed, oriented, smooth four-manifold we may define (by analogy with their values when X is a complex surface)

$$c_1^2(X)=2\chi+3\sigma, \tag{1.13}$$

and

$$\chi_h(X)=\tfrac{1}{4}(\chi+\sigma). \tag{1.14}$$

Then, the topological invariant $c(X)$ acquires a more familiar interpretation:

$$c(X)=-\tfrac{1}{4}(7\chi+11\sigma)=\chi_h(X)-c_1^2(X). \tag{1.15}$$

Theorem 1.1 has some immediate applications to surfaces of general type: it can be used to compute previously unknown Donaldson invariants. If X is a simply-connected, minimal surface of general type, then, according to [3, Theorem VII.1.1(iv')], these surfaces obey the Noether inequality:

$$\chi_h(X)\le\begin{cases}\frac{1}{2}c_1^2(X)+3, & \text{if } c_1^2(X) \text{ is even},\\ \frac{1}{2}c_1^2(X)+\frac{5}{2}, & \text{if } c_1^2(X) \text{ is odd}.\end{cases}$$

Hence, Theorem 1.1 computes previously unknown Donaldson invariants $D_X^w(x^mh^{\delta-2m})$ for minimal surfaces of general type for δ obeying $\delta\le c(X)$ and

$$\delta\le\begin{cases}3-\frac{1}{2}c_1^2(X), & \text{if } c_1^2(X) \text{ is even},\\ \frac{5}{2}-\frac{1}{2}c_1^2(X), & \text{if } c_1^2(X) \text{ is odd}.\end{cases}$$

This gives non-trivial results for Donaldson invariants when $c_1^2(X)\le 6$.

1.3. **Discussion of the hypotheses of Theorems 1.1, 1.4, and 1.5.** To prove Theorem 1.4 (and thus Theorems 1.5 and 1.1), we employ the compactified moduli space of SO(3) monopoles, $\bar{\mathcal{M}}_{\mathfrak{t}}/S^1$, as a cobordism between a link of the moduli space of anti-self-dual connections, M_κ^w, and the links of moduli spaces of ideal Seiberg-Witten monopoles, $M_{\mathfrak{s}}\times\mathrm{Sym}^\ell(X)$. Our application of the cobordism method in this article requires that

1. The codimension of M_κ^w in $\mathcal{M}_{\mathfrak{t}}$, given by twice the complex index of a Dirac operator, is positive (used in Proposition 3.29 in [18]), and
2. Only the top or first levels of the Uhlenbeck compactification $\bar{\mathcal{M}}_{\mathfrak{t}}$ can contain Seiberg-Witten moduli spaces $M_{\mathfrak{s}}\times\mathrm{Sym}^\ell(X)$ (where $\ell=0,1$) with non-trivial invariants (used in §2.6 to eliminate the more intractable terms in the sum (2.52)).

In the proof of Theorem 1.4, one has $2\delta=\deg(z)=\dim M_\kappa^w$ and the hypotheses $\delta<i(\Lambda)$ and $\delta=r(\Lambda)+4$ ensure that Conditions (1) and (2) hold, respectively.

The hypotheses imply that the cobordism $\bar{\mathcal{M}}_{\mathfrak{t}}/S^1$ yields an equality between pairings with the link of M_κ^w and a sum of pairings with the links of $M_{\mathfrak{s}} \times \operatorname{Sym}^\ell(X)$ where $\ell = 0, 1$. The same remarks apply to the hypotheses in Theorem 1.5.

The assumption that $b_1(X) = 0$ could be relaxed by generalizing our calculation of the Segre classes for the virtual normal bundle $N_{\mathfrak{t},\mathfrak{s}} \to M_{\mathfrak{s}}$ to the case $b_1(X) > 0$. We computed those Segre classes in [17, Corollary 3.32], [18, Lemma 4.7] when $b_1(X) > 0$, under a technical assumption on $H^1(X;\mathbb{R})$, but we have not used this generalization here as it would greatly complicate our main formulae.

When $b_2^+(X) = 1$, we assume that $w \pmod 2$ does not admit a torsion integral lift in order to avoid complications in defining the chamber in the positive cone of $H^2(X;\mathbb{R})$ with respect to which the Donaldson and Seiberg-Witten invariants are computed. See the comments at the end of §3.4.2 in [18] and before Lemma 4.1 in [18] for further discussion.

The proof of Theorem 1.1 requires one to choose classes $\Lambda \in B^\perp$ with optimally prescribed even square in order to obtain the indicated vanishing results for the Donaldson and Seiberg-Witten series, as well as compute the first non-vanishing terms. The hypothesis that X is abundant guarantees that one can find such classes, though such choices are also possible for some non-abundant four-manifolds [14]. The constraints on the pair w, Λ were discussed in §1.1.

1.4. **Role of the present article in the SO(3)-monopole program.** Since our series of articles on the SO(3)-monopole approach to the proof of Witten's conjecture is quite long, it is perhaps worth mentioning how the present article fits into this program. We gave a broad (but now slightly dated) review of the program in [15], while we provided a more modern outline in the introductions to [17] and [18].

We proved the basic transversality and compactness properties of the moduli space of SO(3) monopoles in [16, 13]. The local gluing results for SO(3) monopoles — focusing on analytical steps involved in construction of the links of lower-level Seiberg-Witten moduli spaces — are the subject of [19, 20], while the global gluing results — focusing on topological steps involved in construction of these links — are considered in [21]. At the time of writing, work on [20] and [21] is still in progress, though we believe we have surmounted most of the difficult technicalities.

Turning to computational aspects of our work on Witten's conjecture, we show in [17, 18] that, accepting some mild hypotheses, the Donaldson series is equal to $2^{2-c(X)}e^{\frac{1}{2}Q_X}$ times the Seiberg-Witten series, at least through terms h^δ of degree $\delta \leq c(X)$ (compare Theorem 1.1 here). In our articles [17, 18] we did not consider contributions from Seiberg-Witten moduli spaces other than those in the top level, so we could not compare higher-degree terms in those articles. In order to compare terms of arbitrarily high degree we need to compute contributions of the form (1.4) from Seiberg-Witten moduli spaces in arbitrarily low levels: partial computations in this vein are given in [21]. Though we only compute contributions from the first level in the present article, thus extending the calculations of [17, 18], whenever possible we present the calculations in sufficient generality that they apply to arbitrary levels: see §5 and §6 for details. Finally, the article [14] is an application of the main conclusions of [18] and [21].

1.5. **Comparison with** [34]**.** We use our gluing theorems [19, 20] to describe the topology of an open neighborhood in $\bar{\mathcal{M}}_{\mathfrak{t}}$ of the Seiberg-Witten 'stratum' $(M_{\mathfrak{s}} \times \operatorname{Sym}^\ell(X)) \cap \bar{\mathcal{M}}_{\mathfrak{t}}$, restricting our attention in the present article to the case $\ell = 1$. We then compute pairings with (see Theorem 6.1) the circle-quotient of the boundary of this open neighborhood, the link $\bar{\mathbf{L}}_{\mathfrak{t},\mathfrak{s}}$, by generalizing techniques developed in [54] and [34] to compute wall-crossing contributions to Donaldson invariants of four-manifolds with $b_2^+ = 1$ (see Lemma 5.24) to compute contributions to Donaldson invariants Seiberg-Witten strata in $\bar{\mathcal{M}}_{\mathfrak{t}}$. There are two novel features here:

1. The moduli space of 'reducibles', $M_{\mathfrak{s}}$, can be positive-dimensional.
2. There is an obstruction to gluing arising from the cokernel of a Dirac operator on a complex-rank eight Clifford module over S^4.

In [34, 54] the strata of 'reducibles' have the form $([A] \times \mathrm{Sym}^\ell(X)) \cap \bar{M}_\kappa^w(g_I)$, where $[A]$ is a single point represented by a reducible anti-self-dual connection and $M_\kappa^w(g_I)$ is the parametrized (by a path of metrics g_I with $I = (-1,1)$) moduli space of anti-self-dual connections on an SO(3) bundle over X with second Stiefel-Whitney class w (mod 2) and Pontrjagin number $-\kappa/4$. However, as we have already computed [17, 18] the Segre classes of a 'virtual' or 'stabilized' normal bundle of the stratum $M_{\mathfrak{s}} \hookrightarrow \mathcal{M}_{\mathfrak{t}_\ell}$, the problem that $M_{\mathfrak{s}}$ is positive-dimensional can be addressed via an elementary formula, Proposition 5.20, relating the Thom class of an equivariant normal bundle with the Segre classes. (The authors are not aware if this formula is already known — the derivation is straightforward, but it clarifies the relation between the equivariant localization computations of Seiberg-Witten wall-crossing formula in [9] and the Segre class computations of the same formula in [36] and [41].)

The second novel feature is the presence of an obstruction to gluing, described in §3.6.2, arising from the cokernel of the twisted Dirac operator on S^4: this produces a term in the formula (6.2) with coefficient b_0, which has no counterpart in the corresponding formulae of [34] and [54].

1.6. **Outline of the computation of Seiberg-Witten link pairings.** The central problem in the application of the cobordism method to a proof of Witten's conjecture is to compute the intersection number (1.4). It seems worthwhile to outline the basic steps involved in this calculation for the case $\ell = 1$. These steps comprise the proof of Theorem 6.1, the computational core of the current article. In our outline we try to draw a distinction between the technical points which are at the heart of the difficulty underlying the cobordism approach to a proof of Witten's conjecture and those which are more tractable.

1.6.1. *Defining the virtual link.* The domain of the gluing map (see Theorem 3.8) is the space

$$\bar{\mathcal{M}}^{\mathrm{vir}}_{\mathfrak{t},\mathfrak{s}}/S^1 = N_{\mathfrak{t}_1,\mathfrak{s}}(\varepsilon) \times_{S^1} \bar{\mathrm{Gl}}_{\mathfrak{t}_1}(\delta), \tag{1.16}$$

where $N_{\mathfrak{t}_1,\mathfrak{s}}(\varepsilon) \to M_{\mathfrak{s}}$ is a complex disk bundle (of radius ε) homeomorphic to a neighborhood of $M_{\mathfrak{s}}$ in a virtual moduli space containing a neighborhood of $M_{\mathfrak{s}}$ in the moduli space $\mathcal{M}_{\mathfrak{t}_1}$ and $\bar{\mathrm{Gl}}_{\mathfrak{t}_1}(\delta)$ is a space of gluing parameters, containing the moduli space of instantons on S^4 and the bundle frames necessary to splice instantons onto SO(3) monopoles represented by points in $N_{\mathfrak{t}_1,\mathfrak{s}}(\varepsilon)$. We define a *virtual link*, $\bar{\mathbf{L}}^{\mathrm{vir}}_{\mathfrak{t},\mathfrak{s}}$, as the boundary of the domain (1.16).

1.6.2. *Passage from an intersection product to a pairing of dual cohomology classes with the fundamental class of the virtual link.* In §5.1, we prove that the intersection number (1.4) is equal to a pairing of a product of cohomology classes with a homology class $[\bar{\mathbf{L}}^{\mathrm{vir}}_{\mathfrak{t},\mathfrak{s}}]$ which is, effectively, the fundamental class of the virtual link. We first show that the intersection number is equal to the pairing of a relative cohomology class defined by the geometric representatives and the relative Euler class of the obstruction bundle and obstruction section with the relative fundamental class of a codimension-zero subspace of the top stratum of the virtual link (see (5.23)). The proofs given in §5.1 are specific to the topology of the case $\ell = 1$, but the results should hold for all $\ell > 0$.

1.6.3. *Integration over the fiber.* We then consider a subspace, $M_{\mathfrak{s}} \times \partial\bar{\mathrm{G}}\mathrm{l}_{\mathfrak{t}_1}(\delta)/S^1$, of the virtual link, $\bar{\mathbf{L}}^{\mathrm{vir}}_{\mathfrak{t},\mathfrak{s}}$. We use division by the Poincaré dual of the subspace $M_{\mathfrak{s}} \times \partial\bar{\mathrm{G}}\mathrm{l}_{\mathfrak{t}_1}(\delta)/S^1$ to relate pairings with $[\bar{\mathbf{L}}^{\mathrm{vir}}_{\mathfrak{t},\mathfrak{s}}]$ to pairings with the fundamental class of this subspace (see equation (5.71)). An analogous subspace appears in the virtual link for when $\ell > 1$ and the Poincaré dual division argument will translate to the general case.

1.6.4. *Kunneth-type formula and Seiberg-Witten pairings.* Next we write pairings with $[M_{\mathfrak{s}} \times \partial\bar{\mathrm{G}}\mathrm{l}_{\mathfrak{t}_1}(\delta)/S^1]$ as a sum of products of pairings with $[M_{\mathfrak{s}}]$, yielding multiples of the Seiberg-Witten invariant, $SW_X(\mathfrak{s})$, and pairings with $[\partial\bar{\mathrm{G}}\mathrm{l}_{\mathfrak{t}_1}(\delta)/S^1]$; see equations (6.12), (6.13), and the identities that follow.

1.6.5. *Instanton link pairings.* The pairings with $[\partial\bar{\mathrm{G}}\mathrm{l}_{\mathfrak{t}_1}(\delta)/S^1]$ are almost identical with those found in [30]. It is the calculation of these pairings (and indeed the definition of the space $\bar{\mathrm{G}}\mathrm{l}_{\mathfrak{t}_\ell}(\delta)$ for arbitrary levels $\ell > 0$) which causes by far the most difficulty. In this article we use the computations of [34], recorded here in equation (5.76), to compute these pairings; see equation (6.18).

1.7. **Extension of results from level-one case to higher levels.** Although Theorem 1.1 falls well short of a complete proof of Witten's conjecture, it nonetheless provides further confirmation of the promise of the SO(3)-monopole approach. Our proof of Theorem 1.1 addresses many of the technical issues needed for our later work on a proof of Conjecture 1.6 and hence Witten's conjecture. As explained in more detail in §1.7.2, our proof of Theorem 1.1 (in particular, Theorem 6.1) illustrates that a crucial step is to evaluate a certain 'instanton link' pairing (see (1.18)), of the kind that arise in the Göttsche-Kotschick-Morgan wall-crossing formula for Donaldson invariants [25, 30].

1.7.1. *Level two.* The most immediate extension would involve level-two Seiberg-Witten moduli spaces. The first main change would be to replace Lemma 5.24, which records formulae from [34] for pairings with the boundary of the level-one gluing-data bundle $\partial\bar{\mathrm{G}}\mathrm{l}_{\mathfrak{t}_1}(\delta)/S^1$, with the analogous results from [34] for the boundary of the level-two gluing-data $\partial\bar{\mathrm{G}}\mathrm{l}_{\mathfrak{t}_2}(\delta)/S^1$. The second main change would be to compute the Euler class of the instanton obstruction bundle for the level-two case, generalizing the current Lemma 4.12 which addresses the level-one case. For the Euler class calculation, the description of the cokernel of the twisted Dirac operator [12, Lemma 3.3.28] might provide a useful starting point. (Note that the description of this cokernel bundle in [2] does not suffice as we must work with an equivariant extension over the gluing-data bundle.) We would then need to replace in the proof of Theorem 6.1 the resulting formulae for pairings with $[\partial\bar{\mathrm{G}}\mathrm{l}_{\mathfrak{t}_1}(\delta)/S^1]$ by formulae for level-two pairings and replace the Euler class of the instanton obstruction bundle with its level-two counterpart. The end result would be versions of Theorems 1.4 and 1.5 valid for $\delta \leq r(\Lambda) + 8$ and an extension of Theorem 1.1 to a mod $h^{c(X)+4}$ equivalence in equation (1.2) (using $\delta = c(X) + 2$ when $\Lambda \in B^\perp$ obeys $\Lambda^2 = 6 - (\chi + \sigma)$, noting that δ must obey the inequality $\delta < i(\Lambda)$, with $i(\Lambda)$ as given in definition (1.5), and obey $\delta \equiv c(X) + \Lambda^2 \pmod 4$ to give non-zero Donaldson invariants.) From [14, Theorem 1.3] we know that the Donaldson invariants $D^w_X(z)$ vanish, for suitable w (for example, w characteristic), when $\deg(z) < c(X) - 2$ (at least for abundant four-manifolds of Seiberg-Witten simple type), so this calculation should give a direct verification that Kronheimer-Mrowka simple type implies Seiberg-Witten simple type for a non-trivial case with $\deg(z) \leq c(X) + 2$:

$$D^w_X(h^{c(X)-2}x^2) = 4D^w_X(h^{c(X)-2}).$$

Hence, this computation could be very useful for future attempts to understand the relationship between Kronheimer-Mrowka and Seiberg-Witten simple type.

1.7.2. *Higher levels and comparison with the Kotschick-Morgan conjecture.* When $b_2^+(X) = 1$, the work of Kotschick and Morgan (see [30, Theorem 3.0.1] or [25, Theorem 2.3]) expresses the difference in Donaldson invariants of X, defined by two different chambers $\mathcal{C}_\pm$ in the same connected component of the positive cone of $H^2(X;\mathbb{R})$, as an alternating sum

$$\begin{aligned} &D^w_{X,\mathcal{C}_+}(x^m h^{\delta-2m}) - D^w_{X,\mathcal{C}_-}(x^m h^{\delta-2m}) \\ &\quad = \sum_{\xi\in S(\mathcal{C}_+,\mathcal{C}_-,w,\delta)} (-1)^{\varepsilon(w,\xi,\delta)}\delta^w_{X,\xi}(x^m h^{\delta-2m}), \end{aligned}$$

where the precise definition of the index set $S(\mathcal{C}_+,\mathcal{C}_-,w,\delta)\subset H^2(X;\mathbb{Z})$ and sign $\varepsilon(w,\xi,\delta)$ are not relevant to this discussion. The classes $\xi\in S(\mathcal{C}_+,\mathcal{C}_-,w,\delta)$ correspond to points $[A_\xi]\in M^w_{\kappa-\ell}(g_0)$ represented by reducible connections. The terms $\delta^w_{X,\xi}(x^m h^{\delta-2m})$ in the preceding sum are given by [30, Definition 5.1.1]

$$\delta^w_{X,\xi}(x^m h^{\delta-2m}) = \left\langle \bar\mu(x^m h^{\delta-2m}), [\partial\left(B^n_{\mathbb{C}}(\varepsilon)\times_{S^1}\bar{\mathrm{Gl}}_{\xi,\ell}(\delta)\right)]\right\rangle, \tag{1.17}$$

where $B^n_{\mathbb{C}}(\varepsilon)\subset\mathbb{C}^n$ is the ball of radius ε centered at the origin. The space $B^n_{\mathbb{C}}(\varepsilon)\times_{S^1}\bar{\mathrm{Gl}}_{\xi,\ell}(\delta)$ is homeomorphic to a neighborhood of the reducible ideal anti-self-dual connections $[A_\xi]\times\mathrm{Sym}^\ell(X)$ in a compactified moduli space $\bar M^w_\kappa(g_I)$ of anti-self-dual ideal connections parametrized by a path of metrics g_I. A neighborhood of the reducible connection $[A_\xi]$ in the parametrized moduli space $M^w_{\kappa-\ell}(g_I)$ is homeomorphic to $B^n_{\mathbb{C}}(\varepsilon)/S^1$. The space of global gluing data, $\bar{\mathrm{Gl}}_{\xi,\ell}(\delta)$, is (roughly) the union of the spaces of stratum-wise gluing data, $\bar{\mathrm{Gl}}_{\xi,\ell}(\Sigma,\delta)$, as Σ ranges over the smooth strata of $\mathrm{Sym}^\ell(X)$, patched together via (non-canonical) transition maps obeying a cocycle condition. Naturally, the case $\ell=1$ is simplest, as no transition map is needed, while if $\ell=2$ the transition map does not need to satisfy a cocycle condition: the real problem of constructing $\bar{\mathrm{Gl}}_{\xi,\ell}(\delta)$ arises when $\ell\geq 3$. An approach to constructing the space of global gluing data $\bar{\mathrm{Gl}}_{\xi,\ell}(\delta)$ is indicated in [30, Theorem 4.4.2], although the construction of the space of global gluing data and the global gluing map — in a form suitable for our purposes — is a difficult analytical and topological problem.

The terms $\delta^w_{X,\xi}(x^m h^{\delta-2m})$ are computed directly in [34] for the cases $\ell=1$ and $\ell=2$; we exploit the computation when $\ell=1$ in the present article, where the result is recorded as Lemma 5.24. It is natural then to ask if one can use the computations of $\delta^w_{X,\xi}(x^m h^{\delta-2m})$ in [25] to compute the intersection numbers (1.4) with links in the moduli space of SO(3) monopoles.

There are some differences worth noting between the link of $[A_\xi]\times\mathrm{Sym}^\ell(X)$ in $\bar M^w_\kappa(g_I)$,

$$\bar{\mathbf{L}}^w_{\kappa,\xi} := \partial(B^n_{\mathbb{C}}(\varepsilon)\times_{S^1}\bar{\mathrm{Gl}}_{\xi,\ell}(\delta)),$$

and the links $\bar{\mathbf{L}}^{\mathrm{vir}}_{\mathfrak{t},\mathfrak{s}}$ of $M_\mathfrak{s}\times\mathrm{Sym}^\ell(X)$ in $\bar{\mathcal{M}}^{\mathrm{vir}}_{\mathfrak{t},\mathfrak{s}}/S^1$ or the links $\bar{\mathbf{L}}_{\mathfrak{t},\mathfrak{s}}$ of $M_\mathfrak{s}\times\mathrm{Sym}^\ell(X)$ in $\bar{\mathcal{M}}_\mathfrak{t}/S^1$. A neighborhood of the point $[A_\xi]$ in the background moduli space $M^w_{\kappa-\ell}(g_0)$ is homeomorphic to $B^n_{\mathbb{C}}(\varepsilon)/S^1$, while a neighborhood of the Seiberg-Witten moduli space (which can have positive dimension) in the background virtual moduli space, $\mathcal{M}^{\mathrm{vir}}_{\mathfrak{t}_1,\mathfrak{s}}/S^1$, is homeomorphic to the S^1-quotient of the complex disk bundle, $N_{\mathfrak{t}_1,\mathfrak{s}}(\varepsilon)/S^1\to M_\mathfrak{s}$. However, the Poincaré-dual division argument mentioned in §1.6 reduces both the pairing (1.17) and the intersection number (1.4) to a sum of terms of the form

$$\left\langle \nu^n_{\mathfrak{t}_\ell}\smile\pi_X^*\alpha, [\partial\bar{\mathrm{Gl}}_{\mathfrak{t}_\ell}/S^1]\right\rangle, \tag{1.18}$$

where $\nu_{\mathfrak{t}_\ell}$ is the first Chern class of the circle bundle $\partial\bar{\mathrm{Gl}}_{\mathfrak{t}_\ell} \to \partial\bar{\mathrm{Gl}}_{\mathfrak{t}_\ell}/S^1$, the map π_X is a projection $\partial\bar{\mathrm{Gl}}_{\mathfrak{t}_\ell}/S^1 \to \mathrm{Sym}^\ell(X)$, and α is a cohomology class on $\mathrm{Sym}^\ell(X)$. (One has $\xi = \Lambda - c_1(\mathfrak{s})$ in the proof of Theorem 6.1; see equation (6.12).) Thus, when comparing the Kotschick-Morgan conjecture and conjectures about the intersection number (1.4), we will discuss pairings of the form (1.18).

We note that the terms $\delta^w_{X,\xi}(x^m h^{\delta-2m})$ in [25] are computed under the assumption that the four-manifold X has $b_2^+(X) = 1$, so the computational methods of [25] do not immediately apply to the pairings (1.18), where we allow $b_2^+(X) \geq 1$.

While pairings with links of reducible SO(3) monopoles involve some cohomology classes differing from those appearing in (1.17) (specifically, a cohomology class (2.44) associated with the circle action on $\mathcal{M}^{*,0}_{\mathfrak{t}}$), these pairings can still be expressed in terms of pairings of the form (1.18). Thus, no new difficulties are posed by those additional cohomology classes.

A final difference between computation of the pairings in [30] and the computation of pairings with Seiberg-Witten links is due to the presence of obstructions to gluing SO(3) monopoles. The link of a family of $M_{\mathfrak{s}} \times \mathrm{Sym}^\ell(X)$ in $\bar{\mathcal{M}}_{\mathfrak{t}}/S^1$ is given not by the space $\bar{\mathbf{L}}^{\mathrm{vir}}_{\mathfrak{t},\mathfrak{s}}$, the SO(3)-monopole analogue of $\bar{\mathbf{L}}^w_{\kappa,\xi}$, but rather by the zero-locus of a section of an obstruction bundle over $\bar{\mathbf{L}}^{\mathrm{vir}}_{\mathfrak{t},\mathfrak{s}}$. However, in Lemmas 4.11 and 4.12 we prove that for $\ell = 1$, the Euler class of this obstruction bundle can be expressed in terms of $\nu_{\mathfrak{t}_\ell}$ and $\pi_X^*\Lambda$. We expect a similar result will hold for $\ell > 1$. In addition, we expect the arguments of §5.1 representing the zero-locus of the obstruction section as dual to extensions of the Euler class of the obstruction bundle to hold for $\ell > 1$. Thus, in spite of the obstruction to gluing, the pairings with the link of reducible SO(3) monopoles can still be understood in terms of the pairings (1.18).

The crux of the proof of Witten's conjecture, via SO(3) monopoles, is then to define the link components $\partial\bar{\mathrm{Gl}}_{\mathfrak{t}_\ell}(\delta)/S^1$ and compute the pairings (1.18): one can see how they arise in the present article in the course of the proof of Theorem 6.1. Short of proving Witten's conjecture outright, one can aim instead to prove that (by analogy with the conjecture of Kotschick and Morgan, as phrased by Göttsche [25, Remark 4.3]):

Conjecture 1.6. Let X be a closed, oriented, smooth four-manifold with $b_1(X) = 0$ and odd $b_2^+(X) > 1$. Suppose $w, \Lambda \in H^2(X;\mathbb{Z})$ obey $w - \Lambda \equiv w_2(X) \pmod 2$. Let δ, m be non-negative integers for which $m \leq [\delta/2]$ and $\delta \equiv -w^2 - \frac{3}{4}(\chi+\sigma) \pmod 4$. Then let $\mathfrak{t}$ be the spin^u structure over (X,g) with $c_1(\mathfrak{t}) = \Lambda$, $p_1(\mathfrak{t})$ determined by $\delta = -p_1(\mathfrak{t}) - \frac{3}{4}(\chi+\sigma)$, and $w_2(\mathfrak{t}) \equiv w \pmod 2$. (See §2.1 here or §2.1 and §2.2 in [17].) Suppose $\mathfrak{s}$ is a spin^c structure over (X,g) with $\dim M_{\mathfrak{s}} = 0$ and $\ell \equiv \frac{1}{4}(\delta - r(\Lambda, c_1(\mathfrak{s}))) \geq 0$. Let $\bar{\mathbf{L}}_{\mathfrak{t},\mathfrak{s}}$ denote the link of $M_{\mathfrak{s}} \times \mathrm{Sym}^\ell(X)$ in $\bar{\mathcal{M}}_{\mathfrak{t}}/S^1$ and denote $\eta = \frac{1}{4}(p_1(\mathfrak{t}) + \Lambda^2 - \sigma) - 1$. Let $x \in H_0(X;\mathbb{Z})$ be the positive generator. Then for any $h \in H_2(X;\mathbb{R})$, and $z = h^{\delta-2m}x^m$, one has

$$
\begin{aligned}
&\#\left(\bar{\mathcal{V}}(z) \cap \bar{\mathcal{W}}^\eta \cap \bar{\mathbf{L}}_{\mathfrak{t},\mathfrak{s}}\right) \\
&\quad = \pm 2^{a(\chi,\sigma,\delta,m,\ell)} SW_X(\mathfrak{s}) \sum_{j=0}^{q} p_{\delta,m,,\ell,j}\left(\langle c_1(\mathfrak{s}), h\rangle, \langle \Lambda, h\rangle\right) Q_X^j(h,h),
\end{aligned}
\tag{1.19}
$$

where $q = \min(\ell, [\delta/2] - m)$, $a(\cdot)$ is a linear polynomial, and $p_{\delta,m,\ell,j}$ is a degree-$(\delta - 2m - 2j)$ homogeneous polynomial in two variables with coefficients which are degree-$(\ell - j)$ polynomials in $2\chi \pm 3\sigma$, $(c_1(\mathfrak{s}) - \Lambda)^2$, Λ^2, $(c_1(\mathfrak{s}) - \Lambda)\cdot c_1(\mathfrak{s})$, and also depend on δ, m, ℓ.

The assumption that $\dim M_{\mathfrak{s}} = 0$ can be relaxed; the coefficients on the right-hand side of equation (1.19) will then additionally depend on $d_s(\mathfrak{s}) = \dim M_{\mathfrak{s}}$. Conjecture 1.6 is motivated by our explicit calculations of the pairings (1.19) when $\ell = 0$ in [18] and $\ell = 1$

here, as well as partial calculations when $\ell \geq 2$. Our development of a proof of Conjecture 1.6 is work in progress [21]. Conjecture 1.6 also implies the 'multiplicity conjecture', namely Conjecture 2.5, on which our proof of the Mariño-Moore-Peradze conjecture [14] relies. Conjecture 1.6 is of interest to us because of

Conjecture 1.7. Witten's formula (1.3) is implied by Conjecture 1.6.

Göttsche's calculation of the wall-crossing formula for Donaldson invariants [25], assuming the Kotschick-Morgan conjecture [30] and Fintushel and Stern's proof of the general blow-up formula for Donaldson invariants lends weight to our expectation that Conjecture 1.7 holds. Even though there are many unknown universal coefficients in our conjectured formula (1.19) for the pairings for arbitrary $\ell(\mathfrak{t},\mathfrak{s}) \geq 0$, that formula still yields a qualitative version of Witten's conjecture — that Donaldson invariants are determined by Seiberg-Witten invariants — and implies that a formula (1.3) of the type predicted by Witten should exist.

1.8. **Guide to the article.** We now outline the contents of the remainder of this article and indicate the principal steps in the proofs of our principal results.

In §2 we recall the definition of the SO(3) monopole moduli space and its basic properties, describe the strata given by the moduli spaces of anti-self-dual SO(3) connections and Seiberg-Witten monopoles, and review the definitions of the Donaldson and Seiberg-Witten invariants. We briefly describe the links of these strata (deferring a detailed account of the Seiberg-Witten links until §3) and the resulting SO(3)-monopole cobordism formula relating the invariants.

In §3 we define the domain of the gluing map [19, 20], the topological model space which parametrizes a neighborhood of the stratum $(M_{\mathfrak{s}} \times \mathrm{Sym}^{\ell}(X)) \cap \bar{\mathcal{M}}_{\mathfrak{t}}$ when $\ell = 1$; the detailed definition occupies §3.1 through §3.5. We identify the obstruction bundle in §3.6 and discuss the gluing theorem in §3.7. We define the virtual link in §3.8 as the boundary of the domain of the gluing map; the actual link is given by the image of the zero-locus of a section of the obstruction bundle under the gluing map. In §3.9, we define an orientation for the virtual link and compare this orientation to orientations previously defined in [18].

In §4 we identify and extend the universal cohomology classes. Specifically, in §4.1 we compute the pullback of the cohomology classes on $\mathcal{M}_{\mathfrak{t}}$ (both Donaldson-type and those specific to the SO(3)-monopole moduli space) to the domain of the gluing map. In §4.2 we compute the Euler classes of the instanton and Seiberg-Witten components of the obstruction bundle.

The proof that the intersection number in (1.4) can be expressed cohomologically appears in §5.1. The topological computations described in §1.6 for the cohomology of the virtual link appear in §5.2.

We give the computations leading to Theorem 1.1 in §6. The main technical result of the article is Theorem 6.1 in which we compute the intersection number appearing on the left-hand-side of (1.19) for a level-one link. Theorem 6.1 follows from the results of §5 and some algebraic computations performed in §6.1. Because the Donaldson and Seiberg-Witten invariants are defined with the aid of the blow-up formula (to avoid technical difficulties associated with flat (reducible) SO(3) or U(1) connections), in §6.2 we prove a blow-up formula for level-one links. The proofs of Theorems 1.4, 1.5, and 1.1 are then completed in §6.3.

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2. Preliminaries

We begin in §2.1 by recalling the definition of the moduli space of SO(3) monopoles and its basic properties [16, 13]. In §2.2 we describe the stratum of zero-section monopoles, or anti-self-dual connections. In §2.3 we discuss the strata of reducible, or Seiberg-Witten monopoles, together with their 'virtual' neighborhoods and normal bundles. In §2.4 we define the cohomology classes which will be paired with the links of the anti-self-dual and Seiberg-Witten moduli spaces. In §2.5 we review the definition of the Donaldson series. Lastly, in §2.6 we describe the basic relation between the pairings with links of the anti-self-dual and Seiberg-Witten moduli spaces provided by the SO(3)-monopole cobordism.

2.1. **The moduli space of SO(3) monopoles.**

Throughout this article, (X,g) will denote a closed, connected, oriented, smooth, Riemannian four-manifold.

2.1.1. *Clifford modules.* Let V be a Hermitian vector bundle over (X,g) and let $\rho : T^*X \to \operatorname{End}_{\mathbb{C}}(V)$ be a real-linear map satisfying

$$\rho(\alpha)^2 = -g(\alpha,\alpha)\mathrm{id}_V \quad \text{and} \quad \rho(\alpha)^\dagger = -\rho(\alpha), \quad \alpha \in C^\infty(T^*X). \tag{2.1}$$

The map ρ uniquely extends to an algebra homomorphism, $\rho : \Lambda^\bullet(T^*X)\otimes_{\mathbb{R}}\mathbb{C} \to \operatorname{End}_{\mathbb{C}}(V)$, and gives V the structure of a Hermitian Clifford module for the complex Clifford algebra $\mathbb{C}\ell(T^*X)$. There is a splitting $V = V^+\oplus V^-$, where $V^\pm$ are the ∓ 1 eigenspaces of $\rho(\mathrm{vol})$. A unitary connection A on V is *spin* if

$$[\nabla_A, \rho(\alpha)] = \rho(\nabla\alpha) \quad \text{on } C^\infty(V), \tag{2.2}$$

for any $\alpha \in C^\infty(T^*X)$, where ∇ is the Levi-Civita connection.

A Hermitian Clifford module $\mathfrak{s} = (\rho, W)$ is a spin^c structure when W has complex rank four; it defines a class

$$c_1(\mathfrak{s}) = c_1(W^+), \tag{2.3}$$

and every class in $H^2(X;\mathbb{Z})$ lifting the second Stiefel-Whitney class, $w_2(X)\in H^2(X;\mathbb{Z}/2\mathbb{Z})$, arises this way.

We call a Hermitian Clifford module $\mathfrak{t} = (\rho, V)$ a spin^u structure when V has complex rank eight. Recall that $\mathfrak{g}_V \subset \mathfrak{su}(V)$ is the SO(3) subbundle given by the span of the sections of the bundle $\mathfrak{su}(V)$ which commute with the action of $\mathbb{C}\ell(T^*X)$ on V. We obtain a splitting

$$\mathfrak{su}(V^+) \cong \rho(\Lambda^+)\oplus i\rho(\Lambda^+)\otimes_{\mathbb{R}}\mathfrak{g}_V\oplus\mathfrak{g}_V, \tag{2.4}$$

and similarly for $\mathfrak{su}(V^-)$. The fibers V_x^+ define complex lines whose tensor-product square is $\det(V_x^+)$ and thus a complex line bundle over X,

$$\det{}^{\frac{1}{2}}(V^+). \tag{2.5}$$

A spin^u structure $\mathfrak{t}$ thus defines classes,

$$c_1(\mathfrak{t}) = \tfrac{1}{2}c_1(V^+), \quad p_1(\mathfrak{t}) = p_1(\mathfrak{g}_V), \quad \text{and} \quad w_2(\mathfrak{t}) = w_2(\mathfrak{g}_V). \tag{2.6}$$

Given W, one has an isomorphism $V\cong W\otimes_{\mathbb{C}} E$ of Hermitian Clifford modules, where E is a rank-two Hermitian vector bundle [17, Lemma 2.3]; then

$$\mathfrak{g}_V = \mathfrak{su}(E) \quad \text{and} \quad \det{}^{\frac{1}{2}}(V^+) = \det(W^+)\otimes_{\mathbb{C}}\det(E).$$

2.1.2. SO(3) *monopoles.* We fix a smooth unitary connection A_Λ on the line bundle $\det^{\frac{1}{2}}(V^+)$, let $k \geq 2$ be an integer, and let $\mathcal{A}_{\mathfrak{t}}$ be the affine space of L^2_k spin connections on V which induce the connection $2A_\Lambda$ on $\det(V^+)$. If A is a spin connection on V then it defines an SO(3) connection $\hat{A}$ on the subbundle $\mathfrak{g}_V \subset \mathfrak{su}(V)$ [17, Lemma 2.5]; conversely, every SO(3) connection on $\mathfrak{g}_V$ lifts to a unique spin connection on V inducing the connection $2A_\Lambda$ on $\det(V^+)$ [17, Lemma 2.11].

Let $\mathcal{G}_{\mathfrak{t}}$ denote the group of L^2_{k+1} unitary automorphisms of V which commute with $\mathbb{C}\ell(T^*X)$ and which have Clifford-determinant one (see [17, Definition 2.6]). Define

$$\tilde{\mathcal{C}}_{\mathfrak{t}}(X) = \mathcal{A}_{\mathfrak{t}}(X) \times L^2_k(X, V^+) \quad \text{and} \quad \mathcal{C}_{\mathfrak{t}} = \tilde{\mathcal{C}}_{\mathfrak{t}}/\mathcal{G}_{\mathfrak{t}}. \tag{2.7}$$

The action of $\mathcal{G}_{\mathfrak{t}}$ on V induces an adjoint action on $\mathrm{End}_{\mathbb{C}}(V)$, acting as the identity on $\rho(\Lambda^\bullet_{\mathbb{C}}) \subset \mathrm{End}_{\mathbb{C}}(V)$ and inducing an adjoint action on $\mathfrak{g}_V \subset \mathrm{End}_{\mathbb{C}}(V)$ (see [17, Lemma 2.7]). The space $\tilde{\mathcal{C}}_{\mathfrak{t}}$ and hence $\mathcal{C}_{\mathfrak{t}}$ carry circle actions induced by scalar multiplication on V:

$$S^1 \times V \to V, \quad (e^{i\theta}, \Phi) \mapsto e^{i\theta}\Phi. \tag{2.8}$$

Because this action commutes with that of $\mathcal{G}_{\mathfrak{t}}$, the action (2.8) also defines an action on $\mathcal{C}_{\mathfrak{t}}$. Note that $-1 \in S^1$ acts trivially on $\mathcal{C}_{\mathfrak{t}}$. Let $\mathcal{C}^0_{\mathfrak{t}} \subset \mathcal{C}_{\mathfrak{t}}$ be the subspace represented by pairs whose spinor components are not identically zero, let $\mathcal{C}^*_{\mathfrak{t}} \subset \mathcal{C}_{\mathfrak{t}}$ be the subspace represented by pairs where the induced SO(3) connections on $\mathfrak{g}_V$ are irreducible, and let $\mathcal{C}^{*,0}_{\mathfrak{t}} = \mathcal{C}^*_{\mathfrak{t}} \cap \mathcal{C}^0_{\mathfrak{t}}$.

We call a pair (A, Φ) in $\tilde{\mathcal{C}}_{\mathfrak{t}}$ a SO(3) *monopole* if

$$\begin{aligned} \mathrm{ad}^{-1}(F^+_{\hat{A}}) - \tau\rho^{-1}(\Phi \otimes \Phi^*)_{00} &= 0, \\ D_A\Phi + \rho(\vartheta) &= 0. \end{aligned} \tag{2.9}$$

Here, $D_A = \rho \circ \nabla_A : C^\infty(X, V^+) \to C^\infty(X, V^-)$ is the Dirac operator; the isomorphism $\mathrm{ad} : \mathfrak{g}_V \to \mathfrak{so}(\mathfrak{g}_V)$ identifies the self-dual curvature $F^+_{\hat{A}}$, a section of $\Lambda^+ \otimes \mathfrak{so}(\mathfrak{g}_V)$, with $\mathrm{ad}^{-1}(F^+_{\hat{A}})$, a section of $\Lambda^+ \otimes \mathfrak{g}_V$; the section τ of $\mathrm{GL}(\Lambda^+)$ is a perturbation close to the identity; the perturbation ϑ is a complex one-form close to zero; $\Phi^* \in \mathrm{Hom}(V^+, \mathbb{C})$ is the pointwise Hermitian dual $\langle \cdot, \Phi \rangle$ of Φ; and $(\Phi \otimes \Phi^*)_{00}$ is the component of the section $\Phi \otimes \Phi^*$ of $i\mathfrak{u}(V^+)$ lying in $\rho(\Lambda^+) \otimes \mathfrak{g}_V$ with respect to the splitting $\mathfrak{u}(V^+) = i\underline{\mathbb{R}} \oplus \mathfrak{su}(V^+)$ and decomposition (2.4) of $\mathfrak{su}(V^+)$.

Equation (2.9) is invariant under the action of $\mathcal{G}_{\mathfrak{t}}$. We let $\mathcal{M}_{\mathfrak{t}} \subset \mathcal{C}_{\mathfrak{t}}$ be the subspace represented by pairs satisfying equation (2.9) and write

$$\mathcal{M}^*_{\mathfrak{t}} = \mathcal{M}_{\mathfrak{t}} \cap \mathcal{C}^*_{\mathfrak{t}}, \quad \mathcal{M}^0_{\mathfrak{t}} = \mathcal{M}_{\mathfrak{t}} \cap \mathcal{C}^0_{\mathfrak{t}}, \quad \text{and} \quad \mathcal{M}^{*,0}_{\mathfrak{t}} = \mathcal{M}_{\mathfrak{t}} \cap \mathcal{C}^{*,0}_{\mathfrak{t}}. \tag{2.10}$$

Since equation (2.9) is invariant under the circle action induced by scalar multiplication on V, the subspaces (2.10) of $\mathcal{C}_{\mathfrak{t}}$ are also invariant under this action.

Theorem 2.1. [13, Theorem 1.1], [52] *Let X be a closed, oriented, smooth four-manifold and let V be a complex rank-eight, Hermitian vector bundle over X. Then for parameters $(\rho, g, \tau, \vartheta)$, which are generic in the sense of* [13], *and* $\mathfrak{t} = (\rho, V)$, *the space $\mathcal{M}^{*,0}_{\mathfrak{t}}$ is a smooth manifold of the expected dimension,*

$$\begin{aligned} \dim \mathcal{M}^{*,0}_{\mathfrak{t}} = d_a(\mathfrak{t}) + 2n_a(\mathfrak{t}), \quad \textit{where } d_a(\mathfrak{t}) &= -2p_1(\mathfrak{t}) - \tfrac{3}{2}(\chi + \sigma), \\ n_a(\mathfrak{t}) &= \tfrac{1}{4}(p_1(\mathfrak{t}) + c_1(\mathfrak{t})^2 - \sigma). \end{aligned} \tag{2.11}$$

For the remainder of the article, we assume that the perturbation parameters in (2.9) are chosen as indicated in Theorem 2.1. For each non-negative integer ℓ, let $\mathfrak{t}_\ell = (\rho, V_\ell)$, where

$$c_1(V_\ell) = c_1(V), \quad p_1(\mathfrak{g}_{V_\ell}) = p_1(\mathfrak{g}_V) + 4\ell, \quad \text{and} \quad w_2(\mathfrak{g}_{V_\ell}) = w_2(\mathfrak{g}_V).$$

We let $\bar{\mathcal{M}}_{\mathfrak{t}}$ denote the closure of $\mathcal{M}_{\mathfrak{t}}$ in the space of ideal monopoles,

$$I\mathcal{M}_{\mathfrak{t}} = \bigsqcup_{\ell=0}^{\infty}(\mathcal{M}_{\mathfrak{t}_\ell} \times \mathrm{Sym}^\ell(X)), \tag{2.12}$$

with respect to an Uhlenbeck topology [16, Definition 4.19] and call the intersection of $\bar{\mathcal{M}}_{\mathfrak{t}}$ with $\mathcal{M}_{\mathfrak{t}_\ell} \times \mathrm{Sym}^\ell(X)$ its *ℓ-th level*.

Theorem 2.2. [16, Theorem 1.1] *Let X be a Riemannian four-manifold with spin^u structure $\mathfrak{t}$. Then there is a positive integer N, depending at most on the curvature of the chosen unitary connection on $\det(V^+)$ together with $p_1(\mathfrak{t})$, such that the Uhlenbeck closure $\bar{\mathcal{M}}_{\mathfrak{t}}$ of $\mathcal{M}_{\mathfrak{t}}$ in $\sqcup_{\ell=0}^{N}(\mathcal{M}_{\mathfrak{t}_\ell} \times \mathrm{Sym}^\ell(X))$ is a second countable, compact, Hausdorff space. The space $\bar{\mathcal{M}}_{\mathfrak{t}}$ carries a continuous circle action, which restricts to the circle action defined on $\mathcal{M}_{\mathfrak{t}_\ell}$ on each level.*

2.2. **Stratum of anti-self-dual or zero-section solutions.** From equation (2.9), we see that the stratum of $\mathcal{M}_{\mathfrak{t}}$ represented by pairs with zero spinor is identified with

$$\{A \in \mathcal{A}_{\mathfrak{t}} : F_{\hat{A}}^+ = 0\}/\mathcal{G}_{\mathfrak{t}} \cong M_\kappa^w(X,g),$$

the moduli space of g-anti-self-dual connections on the SO(3) bundle $\mathfrak{g}_V$, where $\kappa = -\frac{1}{4}p_1(\mathfrak{t})$ and $w \equiv w_2(\mathfrak{t}) \pmod 2$. For a generic Riemannian metric g, the space $M_\kappa^w(X,g)$ is a smooth manifold of the expected dimension $-2p_1(\mathfrak{t}) - \frac{3}{2}(\chi+\sigma) = d_a(\mathfrak{t})$.

As explained in [17, §3.4.1], it is desirable to choose $w \pmod 2$ so as to exclude points in $\bar{\mathcal{M}}_{\mathfrak{t}}$ with associated flat SO(3) connections, so we have a *disjoint* union,

$$\bar{\mathcal{M}}_{\mathfrak{t}} \cong \bar{\mathcal{M}}_{\mathfrak{t}}^{*,0} \sqcup \bar{M}_\kappa^w \sqcup \bar{\mathcal{M}}_{\mathfrak{t}}^{\mathrm{red}}, \tag{2.13}$$

where $\bar{\mathcal{M}}_{\mathfrak{t}}^* \subset \bar{\mathcal{M}}_{\mathfrak{t}}$ is the subspace represented by triples whose associated SO(3) connections are irreducible, $\bar{\mathcal{M}}_{\mathfrak{t}}^0 \subset \bar{\mathcal{M}}_{\mathfrak{t}}$ is the subspace represented by triples whose spinors are not identically zero, $\bar{\mathcal{M}}_{\mathfrak{t}}^{*,0} = \bar{\mathcal{M}}_{\mathfrak{t}}^* \cap \bar{\mathcal{M}}_{\mathfrak{t}}^0$, while $\bar{\mathcal{M}}_{\mathfrak{t}}^{\mathrm{red}} \subset \bar{\mathcal{M}}_{\mathfrak{t}}$ is the subspace $\bar{\mathcal{M}}_{\mathfrak{t}} - \bar{\mathcal{M}}_{\mathfrak{t}}^*$ represented by triples whose associated SO(3) connections are reducible. We recall the

Definition 2.3. [18, Definition 3.20] A class $v \in H^2(X;\mathbb{Z}/2)$ is *good* if no integral lift of v is torsion.

If $w \pmod 2$ is good, then the union (2.13) is disjoint, as desired. In practice, rather than constraining $w \pmod 2$ itself, we replace X by the blow-up $X\#\overline{\mathbb{CP}}^2$ and w by $w + \mathrm{PD}[e]$ (where $e \in H_2(X;\mathbb{Z})$ is the exceptional class), noting that $w + \mathrm{PD}[e] \pmod 2$ is always good, and define gauge-theoretic invariants of X in terms of moduli spaces on $X\#\overline{\mathbb{CP}}^2$. When $w \pmod 2$ is good, we define [17, Definition 3.7] the link of $\bar{M}_\kappa^w$ in $\bar{\mathcal{M}}_{\mathfrak{t}}/S^1$ by

$$\mathbf{L}_{\mathfrak{t},\kappa}^w = \{[A,\Phi,\mathbf{x}] \in \bar{\mathcal{M}}_{\mathfrak{t}}/S^1 : \|\Phi\|_{L^2}^2 = \varepsilon\}, \tag{2.14}$$

where ε is a small positive constant; for generic ε, the link $\mathbf{L}_{\mathfrak{t},\kappa}^w$ is a smoothly-stratified, codimension-one subspace of $\bar{\mathcal{M}}_{\mathfrak{t}}/S^1$.

2.3. **Strata of Seiberg-Witten or reducible solutions.** We call a pair $(A,\Phi) \in \tilde{\mathcal{C}}_{\mathfrak{t}}$ *reducible* if the connection A on V respects a splitting,

$$V = W \oplus W \otimes L = W \otimes (\underline{\mathbb{C}} \oplus L), \tag{2.15}$$

for some spin^c structure $\mathfrak{s} = (\rho, W)$ and complex line bundle L, in which case $c_1(L) = c_1(\mathfrak{t}) - c_1(\mathfrak{s})$. A spin connection A on V is reducible with respect to the splitting (2.15) if and only if $\hat{A}$ is reducible with respect to the splitting $\mathfrak{g}_V \cong \underline{\mathbb{R}} \oplus L$, [17, Lemma 2.9]. If A is reducible, we can write $A = B \oplus B \otimes A_L$, where B is a spin connection on W and A_L is a unitary connection on L; then $\hat{A} = d_{\mathbb{R}} \oplus A_L$ and $A_L = A_\Lambda \otimes (B^{\det})^*$, where $B^{\det}$ is the connection on $\det(W^+)$ induced by B on W.

2.3.1. *Seiberg-Witten monopoles.* Given a spin^c structure $\mathfrak{s} = (\rho, W)$ on X, let $\mathcal{A}_\mathfrak{s}$ denote the affine space of L^2_k spin connections on W. Let $\mathcal{G}_\mathfrak{s}$ denote the group of L^2_{k+1} unitary automorphisms of W, commuting with $\mathbb{C}\ell(T^*X)$, which we identify with $L^2_{k+1}(X, S^1)$. We then define

$$\tilde{\mathcal{C}}_\mathfrak{s} = \mathcal{A}_\mathfrak{s} \times L^2_k(W^+) \quad \text{and} \quad \mathcal{C}_\mathfrak{s} = \tilde{\mathcal{C}}_\mathfrak{s}/\mathcal{G}_\mathfrak{s}, \tag{2.16}$$

where $\mathcal{G}_\mathfrak{s}$ acts on $\tilde{\mathcal{C}}_\mathfrak{s}$ by

$$(s, (B, \Psi)) \mapsto s(B, \Psi) = (B - (s^{-1}ds)\mathrm{id}_W, s\Psi). \tag{2.17}$$

We call a pair $(B, \Psi) \in \tilde{\mathcal{C}}_\mathfrak{s}$ a Seiberg-Witten monopole if

$$\begin{aligned} \mathrm{Tr}(F^+_B) - \tau\rho^{-1}(\Psi \otimes \Psi^*)_0 - F^+(A_\Lambda) &= 0, \\ D_B\Psi + \rho(\vartheta)\Psi &= 0, \end{aligned} \tag{2.18}$$

where $\mathrm{Tr} : \mathfrak{u}(W^+) \to i\mathbb{R}$ is defined by the trace on 2×2 complex matrices, $(\Psi \otimes \Psi^*)_0$ is the component of the section $\Psi \otimes \Psi^*$ of $i\mathfrak{u}(W^+)$ contained in $i\mathfrak{su}(W^+)$, $D_B : C^\infty(W^+) \to C^\infty(W^-)$ is the Dirac operator, and A_Λ is a unitary connection on a line bundle with first Chern class $\Lambda \in H^2(X; \mathbb{Z})$. The perturbations are chosen so that solutions to equation (2.18) are identified with reducible solutions to (2.9) when $c_1(\mathfrak{t}) = \Lambda$. Let $\tilde{M}_\mathfrak{s} \subset \tilde{\mathcal{C}}_\mathfrak{s}$ be the subspace cut out by equation (2.18) and denote the moduli space of Seiberg-Witten monopoles by $M_\mathfrak{s} = \tilde{M}_\mathfrak{s}/\mathcal{G}_\mathfrak{s}$.

2.3.2. *Seiberg-Witten invariants.* We let $\mathcal{C}^0_\mathfrak{s} \subset \mathcal{C}_\mathfrak{s}$ be the open subspace represented by pairs whose spinor components which are not identically zero and define a complex line bundle over $\mathcal{C}^0_\mathfrak{s} \times X$ by

$$\mathbb{L}_\mathfrak{s} = \tilde{\mathcal{C}}^0_\mathfrak{s} \times_{\mathcal{G}_\mathfrak{s}} \underline{\mathbb{C}}, \tag{2.19}$$

where $\underline{\mathbb{C}} = X \times \mathbb{C}$ and $s \in \mathcal{G}_\mathfrak{s}$ acts on $(B, \Psi) \in \tilde{\mathcal{C}}_\mathfrak{s}$ and $(x, \zeta) \in \underline{\mathbb{C}}$ by

$$((B, \Psi), (x, \zeta)) \mapsto (s(B, \Psi), (x, s(x)^{-1}\zeta)). \tag{2.20}$$

If $x \in H_0(X; \mathbb{Z})$ denotes the positive generator, we set

$$\mu_\mathfrak{s} = c_1(\mathbb{L}_\mathfrak{s})/x \in H^2(\mathcal{C}^0_\mathfrak{s}; \mathbb{Z}). \tag{2.21}$$

Equivalently, $\mu_\mathfrak{s}$ is the first Chern class of the S^1 base-point fibration over $\mathcal{C}^0_\mathfrak{s}$. If $b_1(X) = 0$, then $c_1(\mathbb{L}_\mathfrak{s}) = \mu_\mathfrak{s} \times 1$ by [17, Lemma 2.14].

For $b^+_2(X) > 0$ and generic Riemannian metrics on X, the space $M_\mathfrak{s}$ contains zero-section pairs if and only if $c_1(\mathfrak{s}) - \Lambda$ is a torsion class by [39, Proposition 6.3.1]. If $M_\mathfrak{s}$ contains no zero-section pairs then, for generic perturbations, it is a compact, oriented, smooth manifold of dimension

$$d_s(\mathfrak{s}) = \dim M_\mathfrak{s} = \tfrac{1}{4}(c_1(\mathfrak{s})^2 - 2\chi - 3\sigma). \tag{2.22}$$

Let $\tilde{X} = X \# \overline{\mathbb{CP}}^2$ denote the blow-up of X with exceptional class $e \in H_2(\tilde{X}; \mathbb{Z})$ and denote its Poincaré dual by $\mathrm{PD}[e] \in H^2(\tilde{X}; \mathbb{Z})$. Let $\mathfrak{s}^\pm = (\tilde{\rho}, \tilde{W})$ denote the spin^c structure on $\tilde{X}$ with $c_1(\mathfrak{s}^\pm) = c_1(\mathfrak{s}) \pm \mathrm{PD}[e]$ obtained by splicing the spin^c structure $\mathfrak{s} = (\rho, W)$ on X with the spin^c structure on $\overline{\mathbb{CP}}^2$ with first Chern class $\pm\mathrm{PD}[e]$. (See [18, §4.5] for an explanation of the relation between spin^c structures on X and $\tilde{X}$.) Now

$$c_1(\mathfrak{s}) \pm \mathrm{PD}[e] - \Lambda \in H^2(\tilde{X}; \mathbb{Z})$$

is not a torsion class and so — for $b_2^+(X) > 0$, generic Riemannian metrics on X and related metrics on the connected sum $\tilde{X}$ — the moduli spaces $M_{\mathfrak{s}^\pm}(\tilde{X})$ contain no zero-section pairs. Thus, for our choice of generic perturbations, the moduli spaces $M_{\mathfrak{s}^\pm}(\tilde{X})$ are compact, oriented, smooth manifolds, both of dimension $\dim M_{\mathfrak{s}}(X)$.

For $b_1(X) = 0$ and odd $b_2^+(X) > 1$, we define the *Seiberg-Witten invariant* by [18, §4.1]

$$SW_X(\mathfrak{s}) = \langle \mu_{\mathfrak{s}^+}^d, [M_{\mathfrak{s}^+}(\tilde{X})] \rangle = \langle \mu_{\mathfrak{s}^-}^d, [M_{\mathfrak{s}^-}(\tilde{X})] \rangle, \tag{2.23}$$

where $2d = d_s(\mathfrak{s}) = d_s(\mathfrak{s}^\pm)$. When $b_2^+(X) = 1$ the pairing on the right-hand side of definition (2.23) depends on the chamber in the positive cone of $H^2(\tilde{X};\mathbb{R})$ determined by the period point of the Riemannian metric on $\tilde{X}$. The definition of the Seiberg-Witten invariant for this case is also given in [18, §4.1]: we assume that the class $w_2(X) - \Lambda \pmod 2$ is good to avoid technical difficulties involved in chamber specification. Since $w \equiv w_2(X) - \Lambda \pmod 2$, this coincides with the constraint we use to define the Donaldson invariants in §2.5 when $b_2^+(X) = 1$. We refer to [18, Lemma 4.1 & Remark 4.2] for a comparison of the chamber structures required for the definition of Donaldson and Seiberg-Witten invariants when $b_2^+(X) = 1$.

2.3.3. *Reducible* SO(3) *monopoles.* If $\mathfrak{t} = (\rho, V)$ and $\mathfrak{s} = (\rho, W)$ with $V = W \oplus W \otimes L$, then there is an embedding

$$\iota : \tilde{\mathcal{C}}_{\mathfrak{s}} \hookrightarrow \tilde{\mathcal{C}}_{\mathfrak{t}}, \quad (B, \Psi) \mapsto (B \oplus B \otimes A_\Lambda \otimes B^{\det,*}, \Psi \oplus 0), \tag{2.24}$$

which is gauge-equivariant with respect to the homomorphism

$$\varrho : \mathcal{G}_{\mathfrak{s}} \hookrightarrow \mathcal{G}_{\mathfrak{t}}, \quad s \mapsto \mathrm{id}_W \otimes \begin{pmatrix} s & 0 \\ 0 & s^{-1} \end{pmatrix}. \tag{2.25}$$

According to [17, Lemma 3.13], the map (2.24) defines a topological embedding $\iota : M_{\mathfrak{s}}^0 \hookrightarrow \mathcal{M}_{\mathfrak{t}}$, where $M_{\mathfrak{s}}^0 = M_{\mathfrak{s}} \cap \mathcal{C}_{\mathfrak{s}}^0$ and an embedding of $M_{\mathfrak{s}}$ if $w_2(\mathfrak{t}) \neq 0$ or $b_1(X) = 0$; its image in $\mathcal{M}_{\mathfrak{t}}$ is represented by pairs which are reducible with respect to the splitting $V = W \oplus W \otimes L$. Henceforth, we shall not distinguish between $M_{\mathfrak{s}}$ and its image in $\mathcal{M}_{\mathfrak{t}}$ under this embedding.

2.3.4. *Circle actions.* When $V = W \oplus W \otimes L$ and $\mathfrak{t} = (\rho, V)$, the space $\tilde{\mathcal{C}}_{\mathfrak{t}}$ inherits a circle action defined by

$$S^1 \times V \to V, \quad (e^{i\theta}, \Psi \oplus \Psi') \mapsto \Psi \oplus e^{i\theta}\Psi', \tag{2.26}$$

where $\Psi \in C^\infty(W)$ and $\Psi' \in C^\infty(W \otimes L)$. With respect to the splitting $V = W \oplus W \otimes L$, the actions (2.26) and (2.8) are related by

$$\begin{pmatrix} 1 & 0 \\ 0 & e^{i2\theta} \end{pmatrix} = e^{i\theta} u, \text{ where } u = \begin{pmatrix} e^{-i\theta} & 0 \\ 0 & e^{i\theta} \end{pmatrix} \in \mathcal{G}_{\mathfrak{t}}, \tag{2.27}$$

and so, when we pass to the induced circle actions on the quotient $\mathcal{C}_{\mathfrak{t}} = \tilde{\mathcal{C}}_{\mathfrak{t}}/\mathcal{G}_{\mathfrak{t}}$, the actions (2.26) and (2.8) on $\mathcal{C}_{\mathfrak{t}}$ differ only in their multiplicity. Recall [17, Lemma 3.11] that the image in $\tilde{\mathcal{C}}_{\mathfrak{t}}$ of the map (2.24) contains all pairs which are fixed by the circle action (2.26).

2.3.5. *The virtual normal bundle of the Seiberg-Witten moduli space.* Suppose $\mathfrak{t} = (\rho, V)$ and $\mathfrak{s} = (\rho, W)$, with $V = W \oplus W \otimes L$, so we have a topological embedding $M_{\mathfrak{s}} \hookrightarrow \mathcal{M}_{\mathfrak{t}}$; we assume $M_{\mathfrak{s}}$ contains no zero-section monopoles. Recall from [17, §3.5] that there exist finite-rank, complex vector bundles,

$$\pi_\Xi : \Xi_{\mathfrak{t},\mathfrak{s}} \to M_{\mathfrak{s}} \quad \text{and} \quad \pi_N : N_{\mathfrak{t},\mathfrak{s}} \to M_{\mathfrak{s}}, \tag{2.28}$$

with $\Xi_{\mathfrak{t},\mathfrak{s}} \cong M_{\mathfrak{s}} \times \mathbb{C}^{r_\Xi}$, called the *obstruction bundles* and *virtual normal bundles* of $M_{\mathfrak{s}} \hookrightarrow \mathcal{M}_{\mathfrak{t}}$, respectively. For a small enough positive radius ε, there are a topological embedding [17, Theorem 3.21] of an open tubular neighborhood,

$$\boldsymbol{\gamma}_{\mathfrak{s}} : N_{\mathfrak{t},\mathfrak{s}}(\varepsilon) \hookrightarrow \mathcal{C}_{\mathfrak{t}}, \tag{2.29}$$

and a smooth section $\boldsymbol{\chi}_{\mathfrak{s}}$ of the pulled-back complex vector bundle,

$$\pi_N^* \Xi_{\mathfrak{t},\mathfrak{s}} \to N_{\mathfrak{t},\mathfrak{s}}(\varepsilon), \tag{2.30}$$

such that the tubular map yields a homeomorphism

$$\boldsymbol{\gamma}_{\mathfrak{s}} : \boldsymbol{\chi}_{\mathfrak{s}}^{-1}(0) \cap N_{\mathfrak{t},\mathfrak{s}}(\varepsilon) \cong \mathcal{M}_{\mathfrak{t}} \cap \boldsymbol{\gamma}_{\mathfrak{s}}(N_{\mathfrak{t},\mathfrak{s}}(\varepsilon)), \tag{2.31}$$

restricting to a diffeomorphism on the complement of $M_{\mathfrak{s}}$ and identifying $M_{\mathfrak{s}}$ with its image $\iota(M_{\mathfrak{s}}) \subset \mathcal{M}_{\mathfrak{t}}$. The image $\boldsymbol{\gamma}_{\mathfrak{s}}(N_{\mathfrak{t},\mathfrak{s}}(\varepsilon))$ is an open subset of a *virtual moduli space*, $\mathcal{M}_{\mathfrak{t},\mathfrak{s}}^{\mathrm{vir}} \subset \mathcal{C}_{\mathfrak{t}}$.

Our terminology is loosely motivated by that of [26] and [46], where the goal (translated to our setting) would be to construct a *virtual fundamental class* for $\mathcal{M}_{\mathfrak{t}}$, given by the cap product of the fundamental class of an ambient space containing $\mathcal{M}_{\mathfrak{t}}$ with the Euler class of a vector bundle over this ambient space, where $\mathcal{M}_{\mathfrak{t}}$ is the zero locus of a (possibly non-transversally vanishing) section. Here, $\mathcal{M}_{\mathfrak{t},\mathfrak{s}}^{\mathrm{vir}}$ plays the role of the ambient space and (the pushforward of) $\Xi_{\mathfrak{t},\mathfrak{s}}$ the vector bundle with zero section yielding (an open neighborhood in) $\mathcal{M}_{\mathfrak{t}}$. Then, $N_{\mathfrak{t},\mathfrak{s}}$ is the normal bundle of $M_{\mathfrak{s}} \hookrightarrow \mathcal{M}_{\mathfrak{t},\mathfrak{s}}^{\mathrm{vir}}$, while $[N_{\mathfrak{t},\mathfrak{s}}] - [\Xi_{\mathfrak{t},\mathfrak{s}}]$ would more properly be called the 'virtual normal bundle' of $M_{\mathfrak{s}} \hookrightarrow \mathcal{M}_{\mathfrak{t}}$, in the language of K-theory.

Recall that minus the index of the SO(3)-monopole elliptic deformation complex [17, Equations (2.47) & (3.35)] at a reducible solution can be written as

$$\dim \mathcal{M}_{\mathfrak{t}} = 2n_s(\mathfrak{t},\mathfrak{s}) + d_s(\mathfrak{s}), \tag{2.32}$$

where $d_s(\mathfrak{s})$ is the expected dimension of the Seiberg-Witten moduli space $M_{\mathfrak{s}}$ (see equation (2.22)), while $n_s(\mathfrak{t},\mathfrak{s}) = n_s'(\mathfrak{t},\mathfrak{s}) + n_s''(\mathfrak{t},\mathfrak{s})$ is minus the complex index of the normal deformation operator [17, Equations (3.71) & (3.72)]), with

$$\begin{aligned} n_s'(\mathfrak{t},\mathfrak{s}) &= -(c_1(\mathfrak{t}) - c_1(\mathfrak{s}))^2 - \tfrac{1}{2}(\chi + \sigma), \\ n_s''(\mathfrak{t},\mathfrak{s}) &= \tfrac{1}{8}(c_1(\mathfrak{s}) - 2c_1(\mathfrak{t}))^2 - \sigma). \end{aligned} \tag{2.33}$$

If r_Ξ is the complex rank of $\Xi_{\mathfrak{t},\mathfrak{s}} \to M_{\mathfrak{s}}$, and $r_N(\mathfrak{t},\mathfrak{s})$ is the complex rank of $N_{\mathfrak{t},\mathfrak{s}} \to M_{\mathfrak{s}}$, then

$$r_N(\mathfrak{t},\mathfrak{s}) = n_s(\mathfrak{t},\mathfrak{s}) + r_\Xi, \tag{2.34}$$

as we can see from the dimension relation (2.32) and the topological model (2.31).

The map (2.29) is circle equivariant when the circle acts trivially on $M_{\mathfrak{s}}$, by scalar multiplication on the fibers of $N_{\mathfrak{t},\mathfrak{s}}(\varepsilon)$, and by the action (2.26) on $\mathcal{C}_{\mathfrak{t}}$. The bundle (2.30) and section $\boldsymbol{\chi}_{\mathfrak{s}}$ are circle equivariant if the circle acts on $N_{\mathfrak{t},\mathfrak{s}}$ and the fibers of $\boldsymbol{\gamma}_{\mathfrak{s}}^* \Xi_{\mathfrak{t},\mathfrak{s}}$ by scalar multiplication.

Let $\tilde N_{\mathfrak{t},\mathfrak{s}} \to \tilde M_{\mathfrak{s}}$ be the pullback of $N_{\mathfrak{t},\mathfrak{s}}$ by the projection $\tilde M_{\mathfrak{s}} \to M_{\mathfrak{s}} = \tilde M_{\mathfrak{s}}/\mathcal{G}_{\mathfrak{s}}$, so $\tilde N_{\mathfrak{t},\mathfrak{s}}$ is a $\mathcal{G}_{\mathfrak{s}}$-equivariant bundle, where $\mathcal{G}_{\mathfrak{s}}$ acts on the base $\tilde M_{\mathfrak{s}}$ by the usual gauge group action (2.17) and the induced action on the total space,

$$\tilde N_{\mathfrak{t},\mathfrak{s}} \subset \tilde M_{\mathfrak{s}} \times L_k^2(\Lambda^1 \otimes_{\mathbb{R}} L) \oplus L^2(W^+ \otimes L) \subset \tilde M_{\mathfrak{s}} \times L_k^2(\Lambda^1 \otimes_{\mathbb{R}} \mathfrak{g}_V) \oplus L^2(V^+), \tag{2.35}$$

via the embedding (2.25) of $\mathcal{G}_{\mathfrak{s}}$ into $\mathcal{G}_{\mathfrak{t}}$ and the splittings $\mathfrak{g}_V \cong \underline{\mathbb{R}} \oplus L$ [17, Lemma 3.10] and $V = W \oplus W \otimes L$. Thus, $s \in \mathcal{G}_{\mathfrak{s}}$ acts by scalar multiplication by s^{-2} on sections of $\Lambda^1 \otimes_{\mathbb{R}} L$ and by s^{-1} on sections of $W^+ \otimes L$ [17, §3.5.4].

For a small enough positive ε, there is a smooth embedding [17, §3.5.4] of the open tubular neighborhood $\tilde{N}_{\mathfrak{t},\mathfrak{s}}(\varepsilon)$,

$$\tilde{\gamma}_{\mathfrak{s}} : \tilde{N}_{\mathfrak{t},\mathfrak{s}}(\varepsilon) \to \tilde{\mathcal{C}}_{\mathfrak{t}}, \tag{2.36}$$

which is gauge equivariant with respect to the preceding action of $\mathcal{G}_{\mathfrak{s}}$, and covers the topological embedding (2.29). The map (2.36) is circle equivariant, where the circle acts trivially on $\tilde{M}_{\mathfrak{s}}$, by scalar multiplication on the fibers of $\tilde{N}_{\mathfrak{t},\mathfrak{s}}(\varepsilon)$, and by the action (2.26) on $\mathcal{C}_{\mathfrak{t}}$. We note that the map (2.36) is also circle equivariant with respect to the action (2.8) on $\mathcal{C}_{\mathfrak{t}}$, if the circle acts on $\tilde{N}_{\mathfrak{t},\mathfrak{s}}$ by

$$(e^{i\theta}, (B, \Psi, \beta, \psi)) \mapsto \varrho(e^{i\theta})(B, \Psi, e^{2i\theta}\beta, e^{2i\theta}\psi) = (B, e^{i\theta}\Psi, \beta, e^{i\theta}\psi), \tag{2.37}$$

where $(B, \Psi) \in \tilde{M}_{\mathfrak{s}}$, $(\beta, \psi) \in L^2_k(\Lambda^1 \otimes_{\mathbb{R}} L) \oplus L^2(W^+ \otimes L)$, so $(B, \Psi, \beta, \psi) \in \tilde{N}_{\mathfrak{t},\mathfrak{s}}$, and $\varrho : \mathcal{G}_{\mathfrak{s}} \to \mathcal{G}_{\mathfrak{t}}$ is the homomorphism (2.25). This equivariance follows from the relation (2.27) between the actions (2.26) and (2.8).

We recall the following calculation of the Segre classes of the virtual normal bundle $N_{\mathfrak{t},\mathfrak{s}}$:

Lemma 2.4. [18, Lemma 4.11] *Assume $M_{\mathfrak{s}} \subset \mathcal{M}_{\mathfrak{t}}$ contains no zero-section pairs and let $\mu_{\mathfrak{s}} \in H^2(M_{\mathfrak{s}}; \mathbb{Z})$ be the cohomology class* (2.21)*. If $b_1(X) = 0$, then the Segre classes of the complex vector bundle $N_{\mathfrak{t},\mathfrak{s}} \to M_{\mathfrak{s}}$ are given by*

$$s_p(N_{\mathfrak{t},\mathfrak{s}}) = \mu_{\mathfrak{s}}^p \sum_{j=0}^{p} 2^j \binom{-n_s'}{j} \binom{-n_s''}{p-j}, \quad p = 0, 1, 2, \ldots \tag{2.38}$$

2.4. **Cohomology classes on the moduli space of SO(3) monopoles.** The identity (1.2) arises as an equality of pairings of suitable cohomology classes with a link in $\bar{\mathcal{M}}_{\mathfrak{t}}^{*,0}/S^1$ of the anti-self-dual moduli subspace and the links of the Seiberg-Witten moduli subspaces in $\bar{\mathcal{M}}_{\mathfrak{t}}/S^1$. We recall the definitions of these cohomology classes and their dual geometric representatives given in [18, §3].

The first kind of cohomology class is defined on $\mathcal{M}_{\mathfrak{t}}^*/S^1$, via the associated SO(3) bundle,

$$\mathbb{F}_{\mathfrak{t}} = \tilde{\mathcal{C}}_{\mathfrak{t}}^*/S^1 \times_{\mathcal{G}_{\mathfrak{t}}} \mathfrak{g}_V \to \mathcal{C}_{\mathfrak{t}}^*/S^1 \times X. \tag{2.39}$$

The group $\mathcal{G}_{\mathfrak{t}}$ acts diagonally in (2.39), with $\mathcal{G}_{\mathfrak{t}}$ acting on the left on $\mathfrak{g}_V$. We define [18, §3.1]

$$\mu_p : H_\bullet(X; \mathbb{R}) \to H^{4-\bullet}(\mathcal{C}_{\mathfrak{t}}^*/S^1; \mathbb{R}), \quad \beta \mapsto -\tfrac{1}{4}p_1(\mathbb{F}_{\mathfrak{t}})/\beta. \tag{2.40}$$

On restriction to $M_\kappa^w \hookrightarrow \mathcal{M}_{\mathfrak{t}}$, the cohomology classes $\mu_p(\beta)$ coincide with those used in the definition of Donaldson invariants [18, Lemma 3.1]. We assume $b_1(X) = 0$ and thus let

$$\mathbb{A}(X) = \mathrm{Sym}\left(H_{\mathrm{even}}(X; \mathbb{R})\right) \tag{2.41}$$

be the graded algebra, with $z = \beta_1\beta_2 \cdots \beta_r$ having total degree $\deg(z) = \sum_p (4 - i_p)$, when $\beta_p \in H_{i_p}(X; \mathbb{R})$. A point $x \in X$ gives a distinguished generator still called x in $\mathbb{A}(X)$ of degree four. Then μ_p extends in the usual way to a homomorphism of graded real algebras,

$$\mu_p : \mathbb{A}(X) \to \mathrm{Sym}(H^{\mathrm{even}}(\mathcal{C}_{\mathfrak{t}}^*/S^1; \mathbb{R})),$$

which preserves degrees. Next, we define a complex line bundle over $\mathcal{C}_{\mathfrak{t}}^{*,0}/S^1$,

$$\mathbb{L}_{\mathfrak{t}} = \mathcal{C}_{\mathfrak{t}}^{*,0} \times_{(S^1, \times -2)} \mathbb{C}, \tag{2.42}$$

where the circle action is given, for $[A, \Phi] \in \mathcal{C}_{\mathfrak{t}}^{*,0}$ and $\zeta \in \mathbb{C}$, by

$$([A, \Phi], \zeta) \mapsto ([A, e^{i\theta}\Phi], e^{2i\theta}\zeta). \tag{2.43}$$

Then we define the second kind of cohomology class on $\mathcal{M}_{\mathfrak{t}}^{*,0}/S^1$ by

$$\mu_c = c_1(\mathbb{L}_{\mathfrak{t}}) \in H^2(\mathcal{C}_{\mathfrak{t}}^{*,0}/S^1;\mathbb{R}). \tag{2.44}$$

For monomials $z \in \mathbb{A}(X)$, we constructed [18, §3.2] geometric representatives $\mathcal{V}(z)$ dual to $\mu_p(z)$ and $\mathcal{W}$ dual to μ_c, defined on $\mathcal{M}_{\mathfrak{t}}^{*}/S^1$ and $\mathcal{M}_{\mathfrak{t}}^{*,0}/S^1$, respectively; their closures in $\bar{\mathcal{M}}_{\mathfrak{t}}/S^1$ are denoted by $\bar{\mathcal{V}}(z)$ and $\bar{\mathcal{W}}$ [18, Definition 3.14]. When

$$\deg(z) + 2\eta = \dim(\mathcal{M}_{\mathfrak{t}}^{*,0}/S^1) - 1,$$

and $\deg(z) \geq \dim M_\kappa^w$ it follows that [18, §3.3] the intersection

$$\bar{\mathcal{V}}(z) \cap \bar{\mathcal{W}}^\eta \cap \bar{\mathcal{M}}_{\mathfrak{t}}^{*,0}/S^1, \tag{2.45}$$

is an oriented one-manifold (not necessarily connected) whose closure in $\bar{\mathcal{M}}_{\mathfrak{t}}/S^1$ can only intersect $(\bar{\mathcal{M}}_{\mathfrak{t}} - \mathcal{M}_{\mathfrak{t}})/S^1$ at points in $\mathcal{M}_{\mathfrak{t}}^{\mathrm{red}} \cong \cup(M_{\mathfrak{s}} \times \mathrm{Sym}^\ell(X))$, where the union is over $\ell \geq 0$ and $\mathfrak{s} \in \mathrm{Spin}^c(X)$ [18, Corollary 3.18].

2.5. **Donaldson invariants.** We first recall the definition [31, §2] of the Donaldson series when $b_1(X) = 0$ and $b_2^+(X) > 1$ is odd, so that, in this case, $\chi + \sigma \equiv 0 \pmod 4$. See also §3.4.2 in [18], especially for a definition of the Donaldson invariants when $b_2^+(X) = 1$. For any choice of $w \in H^2(X;\mathbb{Z})$, the Donaldson invariant is a linear function

$$D_X^w : \mathbb{A}(X) \to \mathbb{R}.$$

Let $\tilde{X} = X \# \overline{\mathbb{CP}}^2$ be the blow-up of X and let $e \in H_2(\tilde{X};\mathbb{Z})$ be the exceptional class, with Poincaré dual $\mathrm{PD}[e] \in H^2(\tilde{X};\mathbb{Z})$. If $z \in \mathbb{A}(X)$ is a monomial, we define $D_X^w(z) = 0$ unless

$$\deg(z) \equiv -2w^2 - \tfrac{3}{2}(\chi + \sigma) \pmod 8. \tag{2.46}$$

If $\deg(z)$ obeys equation (2.46), we let $\kappa \in \frac{1}{4}\mathbb{Z}$ be defined by

$$\deg(z) = 8\kappa - \tfrac{3}{2}(\chi + \sigma).$$

There exists an SO(3) bundle over $\tilde{X}$ with first Pontrjagin number $-4\kappa - 1$ and second Stiefel-Whitney class $w + \mathrm{PD}[e] \pmod 2$. One then defines the Donaldson invariant on monomials by

$$D_X^w(z) = \#\left(\bar{\mathcal{V}}(ze) \cap \bar{M}_{\kappa+1/4}^{w+\mathrm{PD}[e]}(\tilde{X})\right), \tag{2.47}$$

and extends to a real linear function on $\mathbb{A}(X)$. Note that $w + \mathrm{PD}[e] \pmod 2$ is good in the sense of Definition 2.3. If $w' \equiv w \pmod 2$, then [11]

$$D_X^{w'} = (-1)^{\frac{1}{4}(w'-w)^2} D_X^w. \tag{2.48}$$

The Donaldson series is a formal power series,

$$\mathbf{D}_X^w(h) = D_X^w((1 + \tfrac{1}{2}x)e^h), \quad h \in H_2(X;\mathbb{R}). \tag{2.49}$$

By equation (2.46), the series $\mathbf{D}_X^w$ is even if

$$-w^2 - \tfrac{3}{4}(\chi + \sigma) \equiv 0 \pmod 2,$$

and odd otherwise. A four-manifold has KM-simple type if for some w and all $z \in \mathbb{A}(X)$,

$$D_X^w(x^2 z) = 4D_X^w(z).$$

According to [31, Theorem 1.7], when X has KM-simple type the series $\mathbf{D}_X^w(h)$ is an analytic function of h and there are finitely many characteristic cohomology classes $K_1, \dots, K_m$ (the KM-basic classes) and constants $a_1, \dots, a_m$ (independent of w) so that

$$\mathbf{D}_X^w(h) = e^{\frac{1}{2}h\cdot h}\sum_{i=1}^r (-1)^{\frac{1}{2}(w^2+K_i\cdot w)} a_i e^{\langle K_i, h\rangle}.$$

Witten's conjecture (1.3) then relates the Donaldson and Seiberg-Witten series for four-manifolds of simple type.

When $b_2^+(X)=1$ the pairing on the right-hand side of definition (2.47) depends on the chamber in the positive cone of $H^2(\tilde X;\mathbb{R})$ determined by the period point of the Riemannian metric on $\tilde X$, just as in the case of Seiberg-Witten invariants described in §2.3.2. We refer to §3.4.2 in [18] for a detailed discussion of this case and, as in [18], we assume that the class w (mod 2) is good in order to avoid technical difficulties involved in chamber specification.

2.6. **Links and the cobordism.** Since the endpoints of the components of the one-manifold (2.45) either lie in M_κ^w or $M_\mathfrak{s}\times \operatorname{Sym}^\ell(X)$, for some $\mathfrak{s}$ and $\ell(\mathfrak{t},\mathfrak{s})\geq 0$ for which $\mathfrak{t}_\ell = \mathfrak{s}\oplus\mathfrak{s}\otimes L$, we have when $w_2(\mathfrak{t})$ is good in the sense of Definition 2.3 that

$$\#\left(\bar{\mathcal{V}}(z)\cap\bar{\mathcal{W}}^{n_a-1}\cap\bar{\mathbf{L}}_{\mathfrak{t},\kappa}^w\right) = -\sum_{\mathfrak{s}\in\operatorname{Spin}^c(X)}\#\left(\bar{\mathcal{V}}(z)\cap\bar{\mathcal{W}}^{n_a-1}\cap\bar{\mathbf{L}}_{\mathfrak{t},\mathfrak{s}}\right), \tag{2.50}$$

where $\bar{\mathbf{L}}_{\mathfrak{t},\kappa}^w$ is the link of $\bar M_\kappa^w$ in $\bar{\mathcal{M}}_\mathfrak{t}/S^1$ (see [17, Definition 3.7]) and where $\bar{\mathbf{L}}_{\mathfrak{t},\mathfrak{s}}$ is empty if $\ell(\mathfrak{t},\mathfrak{s})<0$ and is the boundary of an open neighborhood of the Seiberg-Witten stratum $M_\mathfrak{s}\times\operatorname{Sym}^\ell(X)$ in $\bar{\mathcal{M}}_\mathfrak{t}/S^1$ if $\ell=\ell(\mathfrak{t},\mathfrak{s})\geq 0$. By construction, the intersection of $\bar{\mathbf{L}}_{\mathfrak{t},\mathfrak{s}}$ with the top stratum of $\bar{\mathcal{M}}_\mathfrak{t}/S^1$ is a smooth manifold and the intersection of $\bar{\mathbf{L}}_{\mathfrak{t},\mathfrak{s}}$ with the geometric representatives is in this top stratum. The precise definition of $\bar{\mathbf{L}}_{\mathfrak{t},\mathfrak{s}}$ is given in [17, Definition 3.22] for $\ell=0$, in (3.79) for $\ell=1$, and will be given in [21] for $\ell>1$.

When $\deg(z)=\dim M_\kappa^w$, the intersection of the one-manifold (2.45) with the link $\bar{\mathbf{L}}_{\mathfrak{t},\kappa}^w$ is given by [18, Lemma 3.30]

$$2^{1-n_a}\#\left(\bar{\mathcal{V}}(z)\cap\bar{\mathcal{W}}^{n_a-1}\cap\bar{\mathbf{L}}_{\mathfrak{t},\kappa}^w\right) = \#\left(\bar{\mathcal{V}}(z)\cap\bar M_\kappa^w\right). \tag{2.51}$$

Applying this identity to the blow-up, $X\#\overline{\mathbb{CP}}^2$, when $n_a(\mathfrak{t})>0$, we recover the Donaldson invariant $D_X^w(z)$ on the right-hand side of (2.51) via definition (2.47).

Our proof of Theorem 1.1 relies on the following conjecture whose motivation is discussed in [14, p. 179]:

Conjecture 2.5. [14, Conjecture 3.1] When $b_1(X)=0$ and $\eta\geq 0$ is an integer for which $\deg(z)+2(\eta+1)=\dim\mathcal{M}_\mathfrak{t}$, the intersection number

$$\#\left(\bar{\mathcal{V}}(z)\cap\bar{\mathcal{W}}^\eta\cap\bar{\mathbf{L}}_{\mathfrak{t},\mathfrak{s}}\right)$$

is a multiple of $SW_X(\mathfrak{s})$ and thus vanishes if $SW_X(\mathfrak{s})=0$.

We need only assume this conjecture holds for $\ell(\mathfrak{t},\mathfrak{s})>1$, as we shall prove it here when $\ell(\mathfrak{t},\mathfrak{s})=1$; the relevant statement for $b_1(X)\geq 0$ is given as Conjecture 3.34 in [18], which we proved when $\ell(\mathfrak{t},\mathfrak{s})=0$ [18, Remark 4.15]. The identities (2.51) and (2.50) yield

$$2^{n_a-1}\#\left(\bar{\mathcal{V}}(z)\cap\bar M_\kappa^w\right) = -\sum_{\mathfrak{s}\in\operatorname{Spin}^c(X)}\#\left(\bar{\mathcal{V}}(z)\cap\bar{\mathcal{W}}^{n_a-1}\cap\bar{\mathbf{L}}_{\mathfrak{t},\mathfrak{s}}\right), \tag{2.52}$$

where, assuming Conjecture 2.5, the intersection numbers on the right-hand side are non-zero only when $SW_X(\mathfrak{s})\neq 0$ (and $\ell(\mathfrak{t},\mathfrak{s})\geq 0$). The hypotheses of Theorem 1.1 ensure that we only have $\ell(\mathfrak{t},\mathfrak{s}) = 0, 1$ in (2.52) when $SW_X(\mathfrak{s}) \neq 0$ and, since we addressed the case $\ell = 0$ in [18], the remainder of the present article concerns the case $\ell = 1$.

3. Gluing SO(3) monopoles

Our goal in this section is to construct a topological model for an open neighborhood in $\bar{\mathcal{M}}_{\mathfrak{t}}$, and thus a link, of the Seiberg-Witten stratum

$$(M_{\mathfrak{s}} \times \operatorname{Sym}^{\ell}(X)) \cap \bar{\mathcal{M}}_{\mathfrak{t}} \subset \bar{\mathcal{M}}_{\mathfrak{t}},$$

where $M_{\mathfrak{s}} \subset \mathcal{M}_{\mathfrak{t}_\ell}$ and $p_1(\mathfrak{t}_\ell) = p_1(\mathfrak{t}) + 4\ell$ with $\ell \geq 0$; in this article, we shall only carry out this construction when $\ell = 1$, whereas in [21] we consider the general case $\ell \geq 1$. For the remainder of this article, however, it will be more convenient to denote the pair $(\mathfrak{t}_\ell, \mathfrak{t})$ by $(\mathfrak{t}, \mathfrak{t}')$ in order to minimize the number of subscripts and simplify notation. Hence, for the remainder of the article, we shall denote the Seiberg-Witten stratum of interest by

$$(M_{\mathfrak{s}} \times \operatorname{Sym}^{\ell}(X)) \cap \bar{\mathcal{M}}_{\mathfrak{t}'} \subset \bar{\mathcal{M}}_{\mathfrak{t}'}, \tag{3.1}$$

where $M_{\mathfrak{s}} \subset \mathcal{M}_{\mathfrak{t}}$ and $p_1(\mathfrak{t}) = p_1(\mathfrak{t}') + 4\ell$ with $\mathfrak{t} = (\rho, V)$, $\mathfrak{t}' = (\rho', V')$, and $\ell \geq 0$.

The topological models for open neighborhoods in $\bar{\mathcal{M}}_{\mathfrak{t}'}$ of the strata (3.1) are constructed by gluing SO(3) monopoles. In §3.4 we describe the cut-and-paste process of splicing instantons from the four-sphere (corresponding to SO(3) monopoles with zero-spinor) onto a family of SO(3) monopoles over X parametrized by a tubular neighborhood $N_{\mathfrak{t},\mathfrak{s}}(\varepsilon)$ of $M_{\mathfrak{s}}$. The supporting technical details required to define this splicing process are discussed in §3.1, where we review the required Clifford module and spin group representation theory, in §3.2 where we describe the Clifford frame bundles and their structure groups, and in §3.3 where we describe the connections and spinors on the four-sphere to be spliced onto the four-manifold X. The space of gluing data and splicing map $\boldsymbol{\gamma}'$ are defined in §3.5, while in §3.7 we recall a gluing result (Theorem 3.8) from [19, 20] which asserts that the splicing map can be perturbed to a gluing map, thus giving a topological model for an open neighborhood of the stratum (3.1) in $\bar{\mathcal{M}}_{\mathfrak{t}'}$ when $\ell = 1$; the extension to the general case $\ell \geq 1$ is discussed in [21]. The obstruction bundle appearing in this model is described in §3.6. In particular, the model allows us to define in §3.8 a link $\bar{\mathbf{L}}_{\mathfrak{t}',\mathfrak{s}}$ of the stratum $M_{\mathfrak{s}} \times X$ in $\bar{\mathcal{M}}_{\mathfrak{t}'}$. The relationships among the orientations of the link and the associated moduli spaces are described in §3.9.

3.1. Clifford modules and representation theory.

We recall the representation theory we shall need for the complex Clifford algebra $\mathbb{C}\ell(\mathbb{R}^4)$. (In Lawson-Michelsohn [32], this algebra is denoted $\mathbb{C}\ell_4 = \mathrm{C}\ell_4 \otimes_{\mathbb{R}} \mathbb{C}$, with $\mathrm{C}\ell_4 = \mathrm{C}\ell_{4,0}$.) Recall from [32, Theorem I.4.3] that $\mathbb{C}\ell(\mathbb{R}^4) \cong M_4(\mathbb{C})$, where $M_d(\mathbb{C})$ denotes the algebra of complex $d \times d$ matrices. Then the natural representation of $M_4(\mathbb{C})$ on $\Delta = \mathbb{C}^4$ is the only irreducible complex representation of $M_4(\mathbb{C})$ [32, Theorems I.5.6–7], up to equivalence in the sense of [32, Definition I.5.1]. In particular, Δ is the only irreducible complex representation of $\mathbb{C}\ell(\mathbb{R}^4)$ [32, Theorems I.5.8].

If $\mathbb{V}$ is a complex vector space of dimension $4d$ and $\mathbb{C}\ell(\mathbb{R}^4)$-module, then $\mathbb{V}$ is isomorphic to $\Delta \oplus \cdots \oplus \Delta$ (d times), or $\Delta \otimes \mathbb{C}^d$, as a $\mathbb{C}\ell(\mathbb{R}^4)$-module by [32, Proposition I.5.4]. From the remarks surrounding [17, Equation (2.28)] for the case $d = 2$, the map

$$M_d(\mathbb{C}) \to \operatorname{End}_{\mathbb{C}\ell(\mathbb{R}^4)}(\Delta \otimes \mathbb{C}^d), \quad M \mapsto \mathrm{id}_\Delta \otimes M, \tag{3.2}$$

is an isomorphism of complex algebras.

Recall that a representation $\rho : \mathbb{C}\ell(\mathbb{R}^4) \to \operatorname{End}_{\mathbb{C}}(\mathbb{V})$ is *unitary*, in the sense implied by [32, Theorem I.5.16], when $\mathbb{V}$ is equipped with a Hermitian metric such that

$$\langle \rho(\alpha)\Phi, \rho(\alpha)\Psi \rangle = \langle \Phi, \Psi \rangle,$$

for all $\alpha \in \mathbb{R}^4$ with $|\alpha| = 1$ and $\Phi, \Psi \in \mathbb{V}$.

Lemma 3.1. *Let $\mathbb{V}$ be a complex vector space of dimension $4d$ and $\mathbb{C}\ell(\mathbb{R}^4)$-module. If $F_1, F_2 : \Delta \otimes \mathbb{C}^d \to \mathbb{V}$ are isomorphisms of $\mathbb{C}\ell(\mathbb{R}^4)$-modules, then*

$$F_1 = F_2 \circ (\mathrm{id}_\Delta \otimes M),$$

where $M \in \mathrm{GL}(d, \mathbb{C})$; if F_1, F_2 respect Hermitian metrics, then $M \in \mathrm{U}(d)$.

Proof. Given $\mathbb{C}\ell(\mathbb{R}^4)$-module isomorphisms F_1, F_2, then $F_2^{-1}F_1$ is a $\mathbb{C}\ell(\mathbb{R}^4)$-automorphism of the $\mathbb{C}\ell(\mathbb{R}^4)$-module $\Delta \otimes \mathbb{C}^d$ and so $F_2^{-1}F_1 = \mathrm{id}_\Delta \otimes M \in \mathrm{GL}(d, \mathbb{C})$ by equation (3.2). If F_1, F_2 respect Hermitian metrics, then $M^\dagger M = \mathrm{id}_{\mathbb{C}^d}$ and so $M \in \mathrm{U}(d)$. ☐

If $\mathbb{V}$ is a complex rank-eight vector space and complex $\mathbb{C}\ell(\mathbb{R}^4)$-module, then $\mathbb{V} = \mathbb{V}^+ \oplus \mathbb{V}^-$ and we have an associated complex determinant line $\det(\mathbb{V}^+) = \Lambda^4(\mathbb{V}^+)$. Given a complex rank-four vector space and $\mathbb{C}\ell(\mathbb{R}^4)$-module $\mathbb{W}$, there is an isomorphism $F : \mathbb{W} \to \Delta$ of $\mathbb{C}\ell(\mathbb{R}^4)$-modules (determined up to multiplication by some $e^{i\theta} \in S^1$). Setting $\mathbb{E} = \mathrm{Hom}_{\mathbb{C}\ell(\mathbb{R}^4)}(\mathbb{W}, \mathbb{V})$ (now determined up to multiplication by $e^{-i\theta} \in S^1$), we obtain an isomorphism of $\mathbb{C}\ell(\mathbb{R}^4)$-modules,

$$\mathbb{W} \otimes_{\mathbb{C}} \mathbb{E} \to \mathbb{V},$$

induced by the map $\Phi \otimes H \mapsto H(\Phi)$, where $\Phi \in \mathbb{W}$ and $H \in E$. We then obtain a square root of $\det(\mathbb{V}^+)$,

$$\det{}^{\frac{1}{2}}(\mathbb{V}^+) = \det(\mathbb{W}^+) \otimes_{\mathbb{C}} \det(\mathbb{E}) = \Lambda^2(\mathbb{W}^+) \otimes_{\mathbb{C}} \Lambda^2(\mathbb{E}), \tag{3.3}$$

which is independent of the choice of $\mathbb{W}$.

3.2. **Structure groups and associated bundles.** Given a Hermitian Clifford module (ρ, V) over (X, g), where V has complex rank $4d$, the Hermitian subbundles $V^\pm \subset V$ are defined as the ∓ 1 eigenspaces of $\rho(\mathrm{vol})$ on V. The determinant bundles $\det(V^\pm)$ are canonically isomorphic via an isomorphism $\det(\rho(\alpha)) : \det(V^+) \to \det(V^-)$, $\alpha \in C^\infty(U, T^*X)$ with $|\alpha| = 1$, which is independent of the choice of local one-forms [48, §5.2].

We let $\mathrm{Fr}(T^*X) \to X$ be the principal $\mathrm{SO}(4)$ bundle with fiber $\mathrm{Isom}^+_{\mathbb{R}}(\mathbb{R}^4, T^*X|_x)$ over $x \in X$ given by the space of orientation-preserving isometries, with $\mathrm{SO}(4)$ action induced by the standard action on $\mathbb{R}^4$, and analogously define $\mathrm{Fr}(TX)$.

Next, we define the frame bundles for the Clifford modules. Given $\Delta \otimes_{\mathbb{C}} \mathbb{C}^d$ with its $\mathbb{C}\ell(\mathbb{R}^4)$-module structure induced by the Clifford map $\rho_0 : \mathbb{R}^4 \to \mathrm{End}_{\mathbb{C}}(\Delta \otimes_{\mathbb{C}} \mathbb{C}^d)$, and V_x with its $\mathbb{C}\ell(T^*X|_x)$-module structure induced by the Clifford map $\rho : T^*X|_x \to \mathrm{End}_{\mathbb{C}}(V_x)$, for each $x \in X$ we let $\mathrm{Isom}_{\mathbb{C}\ell}(\Delta \otimes_{\mathbb{C}} \mathbb{C}^d, V_x)$ be the space of unitary isomorphisms $\tilde F : \Delta \otimes \mathbb{C}^d \to V_x$ which are Clifford-module isomorphisms with respect to a Clifford-algebra isomorphism $F : \mathbb{C}\ell(\mathbb{R}^4) \to \mathbb{C}\ell(T^*X|_x)$ determined by a frame $F : \mathbb{R}^4 \to T^*X|_x$. Thus each $\tilde F \in \mathrm{Isom}_{\mathbb{C}\ell}(\Delta \otimes_{\mathbb{C}} \mathbb{C}^d, V_x)$ covers a frame $F \in \mathrm{Isom}^+_{\mathbb{R}}(\mathbb{R}^4, T^*X|_x)$ and corresponding Clifford-algebra isomorphism yielding a commutative diagram, for all $\alpha \in \mathbb{C}\ell(\mathbb{R}^4)$,

$$\begin{array}{ccc} \Delta \otimes_{\mathbb{C}} \mathbb{C}^d & \xrightarrow{\tilde F} & V_x \\ {\scriptstyle \rho_0(\alpha)}\downarrow & & \downarrow{\scriptstyle \rho(F\alpha)} \\ \Delta \otimes_{\mathbb{C}} \mathbb{C}^d & \xrightarrow{\tilde F} & V_x \end{array}$$

Hence, we define the Clifford-module frame bundle for V,

$$\pi : \mathrm{Fr}_{\mathbb{C}\ell(T^*X)}(V) \to \mathrm{Fr}(T^*X), \tag{3.4}$$

to have fibers $\mathrm{Isom}_{\mathbb{C}\ell}(\Delta\otimes_{\mathbb{C}}\mathbb{C}^d, V_x)$, where $\tilde F\in\mathrm{Isom}_{\mathbb{C}\ell}(\Delta\otimes_{\mathbb{C}}\mathbb{C}^d, V_x)$ and $\pi(\tilde F)=F$ obey

$$\tilde F\rho_0(\alpha)\tilde F^{-1}=\rho(F\alpha),\quad \alpha\in\mathbb{C}\ell(\mathbb{R}^4). \tag{3.5}$$

If $\tilde F_1,\tilde F_2\in\mathrm{Fr}_{\mathbb{C}\ell(T^*X)}(V_x)$ both lie in the fiber of the projection (3.4) over $F\in\mathrm{Fr}(T^*X|_x)$, then $\tilde F_2^{-1}\tilde F_1$ is a Hermitian $\mathbb{C}\ell(\mathbb{R}^4)$-module automorphism of $\Delta\otimes_{\mathbb{C}}\mathbb{C}^d$ and hence $\tilde F_2^{-1}\tilde F_1=\mathrm{id}_\Delta\otimes M$, for some $M\in\mathrm{U}(d)$ by Lemma 3.1. Therefore, we have

Lemma 3.2. *The space $\mathrm{Fr}_{\mathbb{C}\ell(T^*X)}(V)$ is a principal $\mathrm{U}(d)$ bundle over $\mathrm{Fr}(T^*X)$.*

We now consider the case $d=1$ and let (ρ,W) be a spin^c structure over X.

Lemma 3.3. *The space $\mathrm{Fr}_{\mathbb{C}\ell(T^*X)}(W)$ is a principal $\mathrm{Spin}^c(4)$ bundle over X.*

Proof. If $\tilde F_1,\tilde F_2\in\mathrm{Fr}_{\mathbb{C}\ell(T^*X)}(W_x)$ lie in the fibers over $F_1,F_2\in\mathrm{Fr}(T^*X|_x)$ of the projection (3.4), respectively, then $F_2^{-1}F_1=R\in\mathrm{SO}(4)$ is an isometric automorphism of $\mathbb{R}^4$ while $\tilde F_2^{-1}\tilde F_1=\tilde R\in\mathrm{U}(4)$ is a Hermitian automorphism of Δ, yielding a commutative diagram

$$\begin{array}{ccc}\Delta & \xrightarrow{\tilde R} & \Delta\\ {\scriptstyle\rho_0(\alpha)}\downarrow & & \downarrow{\scriptstyle\rho_0(R\alpha)}\\ \Delta & \xrightarrow{\tilde R} & \Delta\end{array}$$

Thus, just as in (3.5), these automorphisms obey

$$\tilde R\rho_0(\alpha)\tilde R^{-1}=\rho_0(R\alpha),\quad \alpha\in\mathbb{C}\ell(\mathbb{R}^4). \tag{3.6}$$

Because R has determinant one, it preserves the volume form. Thus identity (3.6) implies that $\tilde R$ preserves the splitting $\Delta=\Delta^+\oplus\Delta^-$ and hence, we may write

$$\tilde R=(\tilde R_+,\tilde R_-)\in\mathrm{U}(2)\times\mathrm{U}(2)\subset\mathrm{U}(4).$$

Identity (3.6) then implies that $\tilde R_-\rho_0(\alpha)\tilde R_+^{-1}=\rho_0(R\alpha)$ on Δ^+, for $\alpha\in\mathbb{R}^4$, and since the isomorphism $\det(\rho_0(\alpha)):\Lambda^2(\Delta^+)\cong\Lambda^2(\Delta^-)$ of complex lines is independent of the choice of unit-norm $\alpha\in\mathbb{R}^4$, we must have

$$\det(\tilde R_+)=\det(\tilde R_-).$$

Recall that $\mathrm{Spin}(4)=\mathrm{SU}(2)\times\mathrm{SU}(2)$. Thus,

$$\begin{aligned}\tilde R&\in\{(M,N)\in\mathrm{U}(2)\times\mathrm{U}(2):\det(M)=\det(N)\}\\&=(\mathrm{SU}(2)\times\mathrm{SU}(2)\times S^1)/\{\pm1\}\\&=(\mathrm{Spin}(4)\times S^1)/\{\pm1\}\\&=\mathrm{Spin}^c(4).\end{aligned}\tag{3.7}$$

Therefore, $\mathrm{Fr}_{\mathbb{C}\ell(T^*X)}(W)\to X$ is a principal $\mathrm{Spin}^c(4)$ bundle, as claimed. □

For $A\in\mathrm{Spin}^c(4)$, $\mathrm{Ad}(A)$ denotes an element of $\mathrm{SO}(\mathrm{End}(\Delta))$. However, we note that $\mathrm{Ad}(A)$ preserves $\rho(\mathbb{R}^4)\subset\mathrm{End}(\Delta)$ and thus defines a linear transformation of $\mathbb{R}^4$ which is an element of $\mathrm{SO}(4)$. This defines a homomorphism

$$\mathrm{Ad}^c:\mathrm{Spin}^c(4)\to\mathrm{SO}(4).$$

By [32, Corollary I.5.19] the homomorphism Ad^c is equal to the projection $\mathrm{Spin}^c(4)\to\mathrm{SO}(4)$ implicit in the description of $\mathrm{Spin}^c(4)$ in (3.7), recalling that $\mathrm{Spin}(4)=(\mathrm{SU}(2)\times$

$\mathrm{SU}(2))/\{\pm 1\}$. The S^1 factor in the presentation (3.7) of $\mathrm{Spin}^c(4)$ is the kernel of the homomorphism Ad^c. Since $\mathrm{Fr}_{\mathbb{C}\ell(T^*X)}(W)/S^1 \cong \mathrm{Fr}(T^*X)$ by Lemma 3.2, we have

$$\mathrm{Fr}_{\mathbb{C}\ell(T^*X)}(W) \times_{(\mathrm{Spin}^c(4),\mathrm{Ad}^c)} \mathbb{R}^4 \cong T^*X. \tag{3.8}$$

From the description of $\mathrm{Spin}^c(4)$ in (3.7), we see that there is a homomorphism,

$$\det{}^c : \mathrm{Spin}^c(4) \to S^1, \quad (M,N) \mapsto \det(M) = \det(N),$$

and thus,

$$\mathrm{Fr}_{\mathbb{C}\ell(T^*X)}(W) \times_{(\mathrm{Spin}^c(4),\det^c)} \mathbb{C} \cong \det(W^+). \tag{3.9}$$

This completes our discussion of the Clifford frame bundle for W and its associated vector bundles.

We now determine the structure group of the bundle $\mathrm{Fr}_{\mathbb{C}\ell(T^*X)}(V)$ over X, where (ρ, V) is a complex-rank eight Hermitian Clifford module over X. Suppose $\tilde F_1, \tilde F_2 \in \mathrm{Fr}_{\mathbb{C}\ell(T^*X)}(V_x)$ lie in the fibers over $F_1, F_2 \in \mathrm{Fr}(T^*X|_x)$ of the projection (3.4), so that $F_2^{-1}F_1 = R = \mathrm{Ad}^c(\tilde R) \in \mathrm{SO}(4)$, for some $\tilde R \in \mathrm{Spin}^c(4)$. Therefore, according to (3.5), we have

$$\tilde F_2 \rho_0(\alpha) \tilde F_2^{-1} = \rho(F_2\alpha), \quad \alpha \in \mathbb{C}\ell(\mathbb{R}^4),$$

and hence, replacing α by $R\alpha$ and applying identity (3.6), we see that

$$\begin{aligned}
\tilde F_2 \tilde R \rho_0(\alpha)(\tilde F_2 \tilde R)^{-1} &= \tilde F_2 \rho_0(R\alpha)\tilde F_2^{-1} \\
&= \rho(F_2 R\alpha) \\
&= \rho(F_1\alpha) \quad \alpha \in \mathbb{C}\ell(\mathbb{R}^4),
\end{aligned}$$

so that $\tilde F_2 \tilde R \in \mathrm{Fr}_{\mathbb{C}\ell(T^*X)}(V_x)$ also lies in the fiber of the projection (3.4) over $F_1 \in \mathrm{Fr}(T^*X|_x)$. Thus, $(\tilde F_2\tilde R)^{-1}\tilde F_1 = \tilde R^{-1}\tilde F_2^{-1}\tilde F_1$ is a Hermitian $\mathbb{C}\ell(\mathbb{R}^4)$-module automorphism of $\Delta \otimes_{\mathbb{C}} \mathbb{C}^2$ and hence, by Lemma 3.1, we have

$$(\tilde R^{-1} \otimes \mathrm{id}_{\mathbb{C}^2}) \circ \tilde F_2^{-1}\tilde F_1 = \mathrm{id}_\Delta \otimes M,$$

for some $M \in \mathrm{U}(2)$. Therefore,

$$\tilde F_1 = \tilde F_2 \circ (\tilde R \otimes M).$$

Hence, any two elements of the fiber of $\mathrm{Fr}_{\mathbb{C}\ell(T^*X)}(V)$ over $x \in X$ differ by an element of

$$\begin{aligned}
\mathrm{Spin}^u(4) &= (\mathrm{Spin}^c(4) \times \mathrm{U}(2))/S^1 \\
&= \mathrm{Spin}(4) \times_{\{\pm 1\}} \mathrm{U}(2), \\
&= (\mathrm{SU}(2) \times \mathrm{SU}(2)) \times_{\{\pm 1\}} (\mathrm{SU}(2) \times_{\{\pm 1\}} S^1),
\end{aligned} \tag{3.10}$$

noting that $\mathrm{U}(2) = (\mathrm{SU}(2) \times S^1)/\{\pm 1\}$. Therefore, we have

Lemma 3.4. *The space* $\mathrm{Fr}_{\mathbb{C}\ell(T^*X)}(V)$ *is a principal* $\mathrm{Spin}^u(4)$ *bundle over* X.

The composition of projection onto the factor S^1 of $\mathrm{Spin}^u(4)$ in the presentation (3.10) and squaring defines a homomorphism

$$\det{}^u : \mathrm{Spin}^u(4) \to S^1.$$

Equivalently, writing elements of $\mathrm{Spin}^u(4)$ as $\tilde R \otimes M$, with $\tilde R \in \mathrm{Spin}^c(4)$ and $M \in \mathrm{U}(2)$ using the first isomorphism in (3.10), this homomorphism is given by

$$\det{}^u : \mathrm{Spin}^u(4) \to S^1, \quad \tilde R \otimes M \mapsto \det{}^c(\tilde R)\det(M). \tag{3.11}$$

Definition (3.11), the identification (3.9) of the line bundle associated to $\det^c$, and the definition (3.3) of the line bundle $\det^{\frac{1}{2}}(V^+)$ imply that

$$\mathrm{Fr}_{\mathbb{C}\ell(T^*X)}(V)\times_{(\mathrm{Spin}^u(4),\det^u)}\mathbb{C}\cong\det{}^{\frac{1}{2}}(V^+). \tag{3.12}$$

The homomorphisms

$$\begin{aligned}\mathrm{Ad}^u_{\mathrm{SO}(4)}&:\mathrm{Spin}^u(4)\to\mathrm{SO}(4),\quad \tilde{R}\otimes M\mapsto\mathrm{Ad}^c(\tilde{R}),\\ \mathrm{Ad}^u_{\mathrm{SO}(3)}&:\mathrm{Spin}^u(4)\to\mathrm{SO}(3),\quad \tilde{R}\otimes M\mapsto\mathrm{Ad}(M),\end{aligned} \tag{3.13}$$

induce bundle isomorphisms

$$\begin{aligned}\mathrm{Fr}_{\mathbb{C}\ell(T^*X)}(V)\times_{(\mathrm{Spin}^u(4),\mathrm{Ad}^u_{\mathrm{SO}(4)})}\mathbb{R}^4&\cong T^*X,\\ \mathrm{Fr}_{\mathbb{C}\ell(T^*X)}(V)\times_{(\mathrm{Spin}^u(4),\mathrm{Ad}^u_{\mathrm{SO}(3)})}\mathfrak{su}(2)&\cong\mathfrak{g}_V.\end{aligned} \tag{3.14}$$

The first isomorphism in (3.14) follows from the argument giving the isomorphism (3.8). To prove the second isomorphism in (3.14), observe that one has a bundle isomorphism

$$\mathrm{Fr}_{\mathbb{C}\ell(T^*X)}(V)\times_{(\mathrm{Spin}^u(4),\mathrm{Ad})}\mathfrak{su}(\Delta\otimes_{\mathbb{C}}\mathbb{C}^2)\cong\mathfrak{su}(V), \tag{3.15}$$

defined via the adjoint representation,

$$\mathrm{Ad}:\mathrm{Spin}^u(4)\to\mathrm{SO}(\mathfrak{su}(\Delta\otimes_{\mathbb{C}}\mathbb{C}^2)),\quad U\mapsto U(\,\cdot\,)U^\dagger.$$

Recall from [17, Equation (2.18)] that the SO(3) subbundle $\mathfrak{g}_V\subset\mathfrak{su}(V)$ is characterized as the span of the sections which commute with the sections $\rho(\omega)$ of $\mathrm{End}_{\mathbb{C}}(V)$ for all $\omega\in\mathbb{C}\ell(T^*X)$. In the decomposition,

$$\mathfrak{su}(\Delta\otimes_{\mathbb{C}}\mathbb{C}^2)\cong\mathfrak{su}(\Delta)\oplus\mathfrak{su}(\Delta)\otimes\mathfrak{su}(2)\oplus\mathfrak{su}(2), \tag{3.16}$$

one has a similar characterization of the subspace

$$\mathfrak{su}(2)\cong\mathrm{id}_\Delta\otimes\mathfrak{su}(2)\subset\mathfrak{su}(\Delta\otimes_{\mathbb{C}}\mathbb{C}^2). \tag{3.17}$$

Because any frame $\tilde{F}\in\mathrm{Fr}_{\mathbb{C}\ell(T^*X)}(V_x)$ defines an isomorphism of Hermitian Clifford modules, $\Delta\otimes_{\mathbb{C}}\mathbb{C}^2\cong V_x$, the induced isomorphism $\mathfrak{su}(\Delta\otimes_{\mathbb{C}}\mathbb{C}^2)\cong\mathfrak{su}(V_x)$ preserves the splitting (3.16) and restricts to an isomorphism $\mathfrak{su}(2)\cong\mathfrak{g}_V|_x$. Hence, the isomorphism (3.15) gives an isomorphism of subbundles,

$$\mathrm{Fr}_{\mathbb{C}\ell(T^*X)}(V)\times_{(\mathrm{Spin}^u(4),\mathrm{Ad})}\mathfrak{su}(2)\cong\mathfrak{g}_V. \tag{3.18}$$

For $\tilde{R}\in\mathrm{Spin}^c(4)$ and $M\in\mathrm{U}(2)$, one can write

$$\mathrm{Ad}(\tilde{R}\otimes M)=\mathrm{Ad}^c(\tilde{R})\oplus(\mathrm{Ad}^c(\tilde{R})\otimes\mathrm{Ad}(M))\oplus\mathrm{Ad}(M)$$

with respect to the decomposition (3.16), the restrictions of $\mathrm{Ad}(\tilde{R}\otimes M)$ and $\mathrm{Ad}^u_{\mathrm{SO}(3)}(\tilde{R}\otimes M)$ to the subspace $\mathfrak{su}(2)$ in (3.17) agree, so the isomorphism (3.18) yields the second bundle isomorphism in (3.14).

3.3. **Anti-self-dual and spin connections over the four-sphere.** Assume $S^4\subset\mathbb{R}^5$ has the standard embedding, with round metric of radius one. Up to equivalence, there is a unique spin^c structure $(\rho,\mathbf{W})$ over S^4 and unitary connection on $\mathbf{W}$ which is spin with respect to the Levi-Civita connection on T^*S^4 and Clifford map $\rho:T^*S^4\to\mathrm{End}_{\mathbb{C}}(\mathbf{W})$. The sphere S^4 has north pole $n=(0,0,0,0,1)$, south pole $s=(0,0,0,0,-1)$, and local parametrizations

$$\begin{aligned}\varphi_n&:\mathbb{R}^4\to S^4\setminus\{s\},\quad x\mapsto\varphi_n(x),\\ \varphi_s&:\mathbb{R}^4\to S^4\setminus\{n\},\quad y\mapsto\varphi_s(y),\end{aligned} \tag{3.19}$$

defined by inverse stereographic projection from the south and north poles, respectively.

Let $\mathbf{E} \to S^4$ be a Hermitian, rank-two bundle with $c_2(\mathbf{E}) = k \geq 1$ and set $\mathbf{V} = \mathbf{W} \otimes \mathbf{E}$, yielding a complex rank-eight, Hermitian Clifford module $(\rho, \mathbf{V})$. We may fix isomorphisms of complex line bundles $\det(\mathbf{W}^\pm) \cong S^4 \times \mathbb{C}$ and $\det(\mathbf{E}) \cong S^4 \times \mathbb{C}$.

Let $\mathcal{A}_k(S^4)$ be the space of spin connections on $\mathbf{V}$, which induce the product connection on $\det(\mathbf{V}^+) \cong S^4 \times \mathbb{C}$, let $\mathcal{G}_k$ be the group of determinant-one, unitary automorphisms of $\mathbf{E}$, identify $\mathcal{G}_k$ with the group of spin^u automorphisms of $\mathbf{V}$ via $u \mapsto \mathrm{id}_\mathbf{W} \otimes u$, and set

$$\mathcal{B}_k^s(S^4) = \mathcal{A}_k(S^4) \times_{\mathcal{G}_k} \mathrm{Fr}(\mathfrak{g}_\mathbf{V}|_s) \tag{3.20}$$

where $\mathrm{Fr}(\mathfrak{g}_\mathbf{V})$ denotes the frame bundle for the $\mathrm{SO}(3)$ bundle $\mathfrak{g}_\mathbf{V}$. This is a principal $\mathrm{SO}(3)$ bundle over the quotient space $\mathcal{B}_k(S^4)$ of unframed spin connections on $\mathbf{V}$.

A point $[\mathbf{A}] \in \mathcal{B}_k(S^4)$ has an associated 'center' and 'scale' defined by [51, pp. 343–344]

$$\begin{aligned} z[\mathbf{A}] &= \frac{1}{8\pi^2 k}\int_{\mathbb{R}^4} x|\varphi_n^* F(\hat{\mathbf{A}})|^2\,d^4x, \\ \lambda^2[\mathbf{A}] &= \frac{1}{8\pi^2 k}\int_{\mathbb{R}^4} |x - z[\mathbf{A}]|^2|\varphi_n^* F(\hat{\mathbf{A}})|^2\,d^4x. \end{aligned} \tag{3.21}$$

Let $M_k^s(S^4) \subset \mathcal{B}_k^s(S^4)$ be the moduli space of pairs consisting of frames for the fiber $\mathfrak{g}_\mathbf{V}|_s$ and 'instanton' connections on $\mathbf{V}$ — the spin connections $\mathbf{A}$ on $\mathbf{V}$ corresponding to anti-self-dual $\mathrm{SO}(3)$ connections $\hat{\mathbf{A}}$ on $\mathfrak{g}_\mathbf{V} = \mathfrak{su}(\mathbf{E})$. Let $M_k^\natural(S^4) \subset M_k^s(S^4)$ denote the subspace of 'mass-centered' instantons, so $z[\mathbf{A}] = 0$ for $[\mathbf{A}] \in M_k^\natural(S^4)$ and similarly define $M_k^{s,\natural}(S^4)$. Let

$$M_k^\natural(S^4, \delta) \subset M_k^\natural(S^4) \tag{3.22}$$

denote the subspace of points $[\mathbf{A}] \in M_k^\natural(S^4)$ with scales $\lambda[\mathbf{A}]$ in $(0, \delta]$, where δ is a small positive constant, and similarly define $M_k^{s,\natural}(S^4, \delta)$. Hence, points in $M_\kappa^{s,\natural}(S^4, \delta)$ have connections with curvature density with center of mass at and energy concentrated near the north pole.

Now suppose $k = 1$. Because there is, up to gauge transformation and rescaling, a unique mass-centered anti-self-dual connection on $\mathfrak{g}_\mathbf{V}$, there is diffeomorphism [19, §3.1], [51, §3],

$$M_1^{s,\natural}(S^4, \delta) \cong (0, \delta] \times \mathrm{SO}(3). \tag{3.23}$$

The Uhlenbeck compactification $\bar{M}_1^{s,\natural}(S^4, \delta)$ of $M_1^{s,\natural}(S^4, \delta)$ is obtained by adjoining the ideal point $\{n\} \times [\Theta] \in S^4 \times M_0(S^4)$, where Θ is the tensor product of the Levi-Civita connection on $\mathbf{W}$ with the product connection on $S^4 \times \mathbb{C}^2$, and forgetting the frame for $\mathfrak{g}_\mathbf{V}|_s$. Thus, we have a homeomorphism

$$\bar{M}_1^{s,\natural}(S^4, \delta) \cong c(\mathrm{SO}(3)), \tag{3.24}$$

where $c(\mathrm{SO}(3))$ is the cone on $\mathrm{SO}(3)$.

An element $R \in \mathrm{SO}(4)$ acts by rotation on $\mathbb{R}^4$ and hence on $S^4 \subset \mathbb{R}^5$ via the fixed stereographic projection from the south pole,

$$S^4 \setminus \{s\} \cong \mathbb{R}^4, \tag{3.25}$$

where the north pole is identified with the origin in $\mathbb{R}^4$. The group $\mathrm{SO}(4)$ in turn acts by pullback via (3.25) on sections of and spin connections on $\mathbf{V} \to S^4$.

3.4. **Splicing Clifford modules, connections, and spinors.** In this section we define the process of splicing instantons from the four-sphere onto $\mathrm{SO}(3)$ monopoles over X, together with the required gluing data choices.

3.4.1. *Splicing Clifford modules.* Let $B(x_0, r_0)$ be a geodesic ball in X with center $x_0 \in X$ and radius r_0. Choose a frame $F \in \mathrm{Fr}(T^*X|_{x_0})$, that is, an isomorphism of oriented real inner-product spaces,

$$F : \mathbb{R}^4 \cong T^*X|_{x_0}. \tag{3.26}$$

This determines a dual isomorphism $\mathbb{R}^4 \cong TX|_{x_0}$ and an isomorphism of Clifford algebras,

$$F : \mathbb{C}\ell(\mathbb{R}^4) \cong \mathbb{C}\ell(T^*X|_{x_0}). \tag{3.27}$$

Extend the frame (3.26) by parallel translation along radial geodesics for the Levi-Civita connection on (X, g) to a local frame for T^*X (and dual frame for TX) over $B(x_0, r_0)$ and geodesic, normal coordinates on this ball, yielding a geodesic, normal coordinate chart

$$\varphi_{x_0}^{-1} : B(x_0, r_0) \subset X \to \mathbb{R}^4, \tag{3.28}$$

and trivializations

$$T^*X|_{B(x_0,r_0)} \cong B(x_0, r_0) \times \mathbb{R}^4 \quad \text{and} \quad \mathbb{C}\ell(T^*X)|_{B(x_0,r_0)} \cong B(x_0, r_0) \times \mathbb{C}\ell(\mathbb{R}^4), \tag{3.29}$$

induced by parallel translation with respect to the Levi-Civita connection along radial geodesics of the frame $F \in \mathrm{Fr}(T^*X|_{x_0})$.

Let (ρ, V) be a complex-rank eight, Hermitian-Clifford module over (X, g), equipped with unitary connection A which is spin with respect to the Levi-Civita connection on T^*X and induces the fixed connection $2A_\Lambda$ on the fixed determinant line bundle $\det(V^+)$, where $\Lambda = \det^{\frac{1}{2}}(V^+)$. Choose a frame $\tilde{F} \in \mathrm{Fr}(V_{x_0})$ in the fiber of the bundle (3.4) over F, so $\tilde{F}$ is an isomorphism of Hermitian-Clifford modules with respect to the isomorphism (3.27),

$$\tilde{F} : \Delta \otimes_{\mathbb{C}} \mathbb{C}^2 \cong V_{x_0}. \tag{3.30}$$

It will be useful to note the

Lemma 3.5. *Let (U, ρ) be a $\mathbb{C}\ell(T^*Y)$-module, of arbitrary complex rank, over an oriented, Riemannian manifold (Y, g). Then parallel translation with respect to a spin connection on U gives $\mathbb{C}\ell(T^*Y)$-module isomorphisms of the fibers of $U \to Y$, with respect to the isomorphisms of the fibers of $T^*Y \to Y$ given by parallel translation with respect to the Levi-Civita connection on T^*Y.*

Proof. Suppose γ is a smooth curve in Y with initial point y_0. Let $\alpha \in \Omega^1(Y, \mathbb{C})$, let $\Phi \in C^\infty(Y, U)$, and let ∇ be a spin connection on U. Because ∇ is a Clifford module derivation of $C^\infty(Y, U)$, we have (see equation (2.2))

$$\nabla_{\dot\gamma}(\rho(\alpha)\Phi) = \rho(\nabla_{\dot\gamma}\alpha)\Phi + \rho(\alpha)\nabla_{\dot\gamma}\Phi,$$

where ∇ also denotes the Levi-Civita connection on T^*X. Thus, if α and Φ are parallel along γ with respect to ∇, then $\rho(\alpha)\Phi$ must be parallel along γ with respect to ∇. Let P_y denote parallel translation, with respect to the Levi-Civita connection on T^*Y or spin connection on U, along γ from y_0 to a point $y \in Y$. Therefore, $\alpha_y = P_y\alpha_0$ and $\Phi_y = P_y\Phi_0$. The sections of U over $\gamma \subset Y$ given by

$$\rho_y(P_y\alpha_0)P_y\Phi_0 \quad \text{and} \quad P_y(\rho_0(\alpha_0)\Phi_0), \quad y \in \gamma,$$

agree when $y = y_0$ and are parallel along the curve γ from y_0. Hence, parallel translation along curves in Y with respect to spin connections commutes with Clifford multiplication and thus yields Clifford-module isomorphisms of the fibers of $U \to Y$. □

Therefore, given a spin connection A on V and a choice of frame $\tilde{F} : \Delta \otimes_{\mathbb{C}} \mathbb{C}^2 \cong V_{x_0}$, parallel translation with respect to A along radial geodesics in X from x_0 yields a Clifford-module isomorphism,

$$V|_{B(x_0,r_0)} \cong B(x_0,r_0) \times \Delta \otimes_{\mathbb{C}} \mathbb{C}^2, \tag{3.31}$$

with respect to the Clifford algebra isomorphism (3.29). The trivialization (3.31) also induces a trivialization

$$\mathfrak{g}_V|_{B(x_0,r_0)} \cong B(x_0,r_0) \times \mathfrak{su}(2). \tag{3.32}$$

This completes our discussion of the Clifford modules and bundles over X.

Over S^4, we have the standard spin^c structure $(\rho, \mathbf{W})$ with Clifford multiplication $\rho : T^*S^4 \to \mathrm{End}_{\mathbb{C}}(\mathbf{W})$ and unitary connection which is spin with respect to the Levi-Civita connection on T^*S^4 for the standard round metric of radius one. We may fix, once and for all, a unit-norm $\mathbb{C}\ell(T^*S|_n)$-frame $\Delta \cong \mathbf{W}|_n$ for $\mathbf{W}|_n$ and a trivialization defined by parallel radial translation which is a Clifford-module isomorphism (see Lemma 3.5),

$$\mathbf{W}|_{S^4\setminus\{s\}} \cong \mathbb{R}^4 \times \Delta, \tag{3.33}$$

with respect to the Clifford algebra isomorphism

$$\mathbb{C}\ell(T^*S^4)|_{S^4\setminus\{s\}} \cong S^4 \setminus \{s\} \times \mathbb{C}\ell(\mathbb{R}^4). \tag{3.34}$$

A spin connection $\mathbf{A}$ on $\mathbf{V} = \mathbf{W} \otimes_{\mathbb{C}} \mathbf{E} \to S^4$ and a choice of frame $\mathfrak{su}(2) \cong \mathfrak{g}_{\mathbf{V}|_s}$, that is, an isomorphism of oriented real inner product spaces, define a trivialization of SO(3) bundles,

$$\mathfrak{g}_{\mathbf{V}}|_{S^4\setminus\{n\}} \cong \mathbb{R}^4 \times \mathfrak{su}(2), \tag{3.35}$$

via parallel translation with respect to the SO(3) connection $\hat{\mathbf{A}}$ on $\mathfrak{g}_{\mathbf{V}}$. Up to an ambiguity corresponding to the action of $\{\pm 1\}$, the frame $\mathfrak{su}(2) \cong \mathfrak{su}(\mathbf{E})|_s = \mathfrak{g}_{\mathbf{V}|_s}$ lifts to a frame $\mathbb{C}^2 \cong \mathbf{E}_s$, an isomorphism of complex inner product spaces. This ambiguity will vanish when we take the quotient by the gauge group. Thus, up to this ambiguity, we obtain an isomorphism lifting (3.35),

$$\mathbf{E}|_{S^4\setminus\{n\}} \cong \mathbb{R}^4 \times \mathbb{C}^2, \tag{3.36}$$

coinciding with the trivialization defined by the isomorphism $\mathbb{C}^2 \cong \mathbf{E}_s$ and radial parallel translation with respect to the unique lift of $\hat{\mathbf{A}}$ to an SU(2) connection on $\mathbf{E}$.

The isomorphisms (3.33) and (3.36) combine to give a Clifford-module isomorphism

$$\mathbf{V}|_{S^4\setminus\{n,s\}} \cong \mathbb{R}^4 \setminus \{0\} \times \Delta \otimes_{\mathbb{C}} \mathbb{C}^2, \tag{3.37}$$

with respect to the Clifford algebra isomorphism (3.34).

We then define a spin^u structure $\mathfrak{t}' = (\rho, V')$ over (X, g) by setting

$$V' = (V|_{X\setminus\{x_0\}}) \# (\mathbf{V}|_{S^4\setminus\{s\}}), \tag{3.38}$$

where the bundles V and $\mathbf{V}$ are identified via the trivializations (3.31) and (3.37) and the embedding

$$\varphi_n \circ \varphi_{x_0}^{-1} : B(x_0, r_0) \to S^4 \setminus \{s\}, \tag{3.39}$$

identifying a ball in X centered at x_0 with a ball in S^4 centered at the north pole. Because these trivializations are Clifford module isomorphisms, the bundle V' has a $\mathbb{C}\ell(T^*X)$-module structure. The Clifford module (3.38) induces an associated SO(3)-bundle $\mathfrak{g}_{V'}$,

$$\mathfrak{g}_{V'} = (\mathfrak{g}_V|_{X\setminus\{x_0\}}) \# (\mathfrak{g}_{\mathbf{V}}|_{S^4\setminus\{s\}}), \tag{3.40}$$

where the bundles $\mathfrak{g}_V$ and $\mathfrak{g}_{\mathbf{V}}$ are identified over small annuli centered at $x \in X$ and $n \in S^4$ by the trivializations (3.32) and (3.35) and the embedding (3.39). The characteristic classes of $\mathfrak{t} = (\rho, V)$ and $\mathfrak{t}' = (\rho, V')$ are related by

$$p_1(\mathfrak{g}_{V'}) = p_1(\mathfrak{g}_V) - 4, \quad c_1(V') = c_1(V), \quad \text{and} \quad w_2(\mathfrak{g}_{V'}) = w_2(\mathfrak{g}_V), \tag{3.41}$$

as we shall later exploit, often without comment.

We give a more detailed description of where the bundles V' and $\mathfrak{g}_{V'}$ are spliced below. Define a smooth cutoff function $\chi_{x_0,\varepsilon} : X \to [0,1]$ by setting

$$\chi_{x_0,\varepsilon}(x) := \chi(\operatorname{dist}(x, x_0)/\varepsilon), \tag{3.42}$$

where $\chi : \mathbb{R} \to [0,1]$ is a smooth function such that $\chi(t) = 1$ for $t \geq 1$ and $\chi(t) = 0$ for $t \leq 1/2$. Thus, we have

$$\chi_{x_0,\varepsilon}(x) = \begin{cases} 1 & \text{for } x \in X - B(x_0, \varepsilon), \\ 0 & \text{for } x \in B(x_0, \varepsilon/2). \end{cases}$$

The orientation-preserving embedding (3.39) therefore identifies the annulus $\varphi_n(\Omega(0, \frac{1}{2}\sqrt{\lambda}, 2\sqrt{\lambda}))$ in S^4, where

$$\Omega\left(0, \tfrac{1}{2}\sqrt{\lambda}, 2\sqrt{\lambda}\right) := \left\{x \in \mathbb{R}^4 : \tfrac{1}{2}\sqrt{\lambda} < |x| < 2\sqrt{\lambda}\right\} \subset \mathbb{R}^4,$$

with the annulus in X,

$$\Omega\left(x_0, \tfrac{1}{2}\sqrt{\lambda}, 2\sqrt{\lambda}\right) := \left\{x \in X : \tfrac{1}{2}\sqrt{\lambda} < \operatorname{dist}_g(x, x_0) < 2\sqrt{\lambda}\right\} \subset X.$$

Hence, the spliced bundle (3.38) is defined explicitly by setting

$$V' = \begin{cases} V & \text{over } X - B(x_0, \frac{1}{2}\sqrt{\lambda}), \\ \mathbf{V} & \text{over } B(x_0, 2\sqrt{\lambda}). \end{cases}$$

The bundles V and $\mathbf{V}$ are identified over the annulus $\Omega(x_0, \frac{1}{2}\sqrt{\lambda}, 2\sqrt{\lambda})$ in X via the orientation-preserving diffeomorphism (3.39) identifying the annulus $\Omega(x_0, \frac{1}{2}\sqrt{\lambda}, 2\sqrt{\lambda})$ with the corresponding annulus in S^4 and the bundle map defined by the trivializations (3.31) and (3.37).

Similarly, the spliced SO(3) bundle (3.40) is defined explicitly by setting

$$\mathfrak{g}_{V'} = \begin{cases} \mathfrak{g}_V & \text{over } X - B(x_0, \frac{1}{2}\sqrt{\lambda}), \\ \mathfrak{g}_{\mathbf{V}} & \text{over } B(x_0, 2\sqrt{\lambda}). \end{cases}$$

The bundles $\mathfrak{g}_V$ and $\mathfrak{g}_{\mathbf{V}}$ are identified over the annulus $\Omega(x_0, \frac{1}{2}\sqrt{\lambda}, 2\sqrt{\lambda}\})$ in X via the diffeomorphism (3.39) and the SO(3) bundle map defined by the trivializations (3.32) and (3.35).

3.4.2. *Splicing connections and cutting off background spinors.* To construct a spliced spin connection on the Clifford module V', we choose the following data:

- An SO(3) connection $\hat{A}$ on $\mathfrak{g}_V \to X$,
- An SO(4) frame $F \in \operatorname{Fr}(T^*X|_{x_0})$, and induced chart $B(x_0, r_0) \subset X \to \mathbb{R}^4$ as in (3.28),
- An SO(3) frame for $\operatorname{Fr}(\mathfrak{g}_V|_{x_0})$, and corresponding trivialization $\mathfrak{g}_V|_{B(x_0,r_0)} \cong B(x_0, r_0) \times \mathfrak{su}(2)$ as in (3.32),
- An SO(3) connection $\hat{\mathbf{A}}$ on $\mathfrak{g}_{\mathbf{V}} \to S^4$, and associated scale parameter $\lambda \in (0, \delta]$,
- An SO(3) frame for $\mathfrak{g}_{\mathbf{V}}|_s$, and corresponding trivialization $\mathfrak{g}_{\mathbf{V}}|_{S^4 \setminus \{n\}} \cong S^4 \setminus \{n\} \times \mathfrak{su}(2)$ as in (3.35), together with the fixed chart $S^4 \setminus \{s\} \cong \mathbb{R}^4$ as in (3.19).

This yields a spliced connection $\hat{A}'$ on the spliced SO(3) bundle $\mathfrak{g}_{V'}$ over X, as in (3.40).

We give a more detailed description of where the SO(3) connections are spliced below. We first define a cut-off SO(3) connection on the bundle $\mathfrak{g}_V$ over X by setting

$$\chi_{x_0,4\sqrt{\lambda}}\hat{A} := \begin{cases} \hat{A} & \text{over } X - B(x_0, 4\sqrt{\lambda}), \\ \Gamma + \chi_{x_0,4\sqrt{\lambda}}\sigma_0^*\hat{A} & \text{over } \Omega(x_0, 2\sqrt{\lambda}, 4\sqrt{\lambda}), \\ \Gamma & \text{over } B(x_0, 2\sqrt{\lambda_0}), \end{cases} \tag{3.43}$$

where Γ denotes the product connection on $B(x_0, r_0) \times \mathrm{SO}(3)$ and σ_0 is the section of $\mathfrak{g}_V$ over $B(x_0, r_0)$ defined by the trivialization (3.32).

We define a cut-off SO(3) connection on the bundle $\mathfrak{g}_{\mathbf{V}}$ over S^4, with mass center at the north pole, by setting

$$(1-\chi_{n,\sqrt{\lambda}/2})\hat{\mathbf{A}} := \begin{cases} \hat{\mathbf{A}} & \text{over } \varphi_n(B(0, \tfrac{1}{4}\sqrt{\lambda})), \\ \Gamma + (1-\chi_{n,\sqrt{\lambda}/2})\tau_0^*\hat{\mathbf{A}} & \text{over } \varphi_n(\Omega(0, \tfrac{1}{4}\sqrt{\lambda}, \tfrac{1}{2}\sqrt{\lambda})), \\ \Gamma & \text{over } S^4 - \varphi_n(B(0, \tfrac{1}{2}\sqrt{\lambda})), \end{cases} \tag{3.44}$$

where Γ denotes the product connection on $(S^4 - \{n\}) \times \mathrm{SO}(3)$ and τ_0 is the section of $\mathfrak{g}_{\mathbf{V}}$ over $S^4 - \{n\}$ defined by the trivialization (3.35).

Hence, we define a spliced connection $\hat{A}'$ on $\mathfrak{g}_{V'}$ by setting

$$\hat{A}' := \begin{cases} \hat{A} & \text{over } X - B(x_0, 4\sqrt{\lambda}), \\ \Gamma + \chi_{x_0,4\sqrt{\lambda}}\hat{A} + (1-\chi_{x_0,\sqrt{\lambda}/2})\hat{\mathbf{A}} & \text{over } \Omega(x_0, \tfrac{1}{4}\sqrt{\lambda}, 4\sqrt{\lambda}), \\ \hat{\mathbf{A}} & \text{over } B(x_0, \tfrac{1}{4}\sqrt{\lambda}), \end{cases} \tag{3.45}$$

where the cut-off connections are defined as above; the bundle and annulus identifications are understood.

According to Lemma 2.11 in [17], we then obtain a unitary connection A' on V' defined by the SO(3) connection $\hat{A}'$ on $\mathfrak{g}_{V'}$ and by the

- Frame $\tilde{F} \in \mathrm{Fr}_{\mathbb{C}\ell(T^*X)}(V_{x_0})$ covering $F \in \mathrm{Fr}(T^*X|_{x_0})$, and corresponding Clifford-module isomorphism $V|_{B(x_0,r_0)} \cong B(x_0, r_0) \times \Delta \otimes_{\mathbb{C}} \mathbb{C}^2$ as in (3.31),
- Fixed U(1) connection $2A_\Lambda$ on $\det(V'^+)$, and the
- Requirement that A' be spin with respect to the Levi-Civita connection on T^*X for the metric g and Clifford map $\rho' : T^*X \to \mathrm{End}_{\mathbb{C}}(V')$.

Finally, given a section Φ of $V \to X$, we obtain a section Φ' of $V' \to X$ by cutting off Φ in a ball around $x_0 \in X$, with radius determined by λ.

More explicitly, we define the spinor Φ' on the bundle V' over X by setting

$$\Phi' \equiv \chi_{x_0,8\lambda^{1/3}}\Phi := \begin{cases} \Phi & \text{over } X - B(x_0, 8\lambda^{1/3}), \\ \chi_{x_0,8\lambda^{1/3}}\Phi & \text{over } \Omega(x_0, 4\lambda^{1/3}, 8\lambda^{1/3}), \\ 0 & \text{over } B(x_0, 4\lambda^{1/3}). \end{cases} \tag{3.46}$$

The reason for the different choice of annulus radii is explained in [19].

3.4.3. *Splicing in spinors from the four-sphere.* To construct a spliced section of the Clifford module V', given a spinor over the four-sphere, we choose the following data:

- A spin connection A on $V \to X$,
- An SO(4) frame $F \in \mathrm{Fr}(T^*X|_{x_0})$ and induced chart $B(x_0, r_0) \subset X \to \mathbb{R}^4$ as in (3.28),
- A frame $\tilde{F} \in \mathrm{Fr}_{\mathbb{C}\ell(T^*X)}(V_{x_0})$ covering $F \in \mathrm{Fr}(T^*X|_{x_0})$, and corresponding Clifford-module isomorphism $V|_{B(x_0,r_0)} \cong B(x_0, r_0) \times \Delta \otimes_{\mathbb{C}} \mathbb{C}^2$ as in (3.31),

- A spin connection $\mathbf{A}$ on $\mathbf{V} \to S^4$, and associated scale parameter $\lambda \in (0, \delta]$,
- An oriented, orthonormal frame for $\mathbf{E}_s$, and corresponding trivialization $\mathbf{E}|_{S^4 \setminus \{n\}} \cong S^4 \setminus \{n\} \times \mathbb{C}^2$, together with the fixed chart $S^4 \setminus \{s\} \cong \mathbb{R}^4$ as in (3.19), fixed unit-norm $\mathbb{C}\ell(T^*S^4|_n)$-frame for $\mathbf{W}|_n$, and fixed Clifford-module isomorphism $\mathbf{W}|_{S^4\setminus\{s\}} \cong S^4 \setminus \{s\} \times \Delta$. The bundle isomorphisms combine to give a Clifford-module isomorphism $\mathbf{V}|_{S^4\setminus\{n,s\}} \cong S^4 \setminus \{n, s\} \times \Delta \otimes \mathbb{C}^2$, which we employ in conjunction with the chart $S^4 \setminus \{s\} \cong \mathbb{R}^4$.

Spinors Ψ on $\mathbf{V} \to S^4$ are then spliced from $\mathbb{R}^4 \cong S^4 \setminus \{s\}$ onto the ball $B(x_0, r_0) \subset X$, via the preceding data, with the north pole $n \in S^4$ being identified with $x_0 \in X$.

More explicitly, we define the spinor Ψ' on the bundle V' over X by setting

$$
\begin{aligned}
\Psi' &:= (1 - \chi_{n,\sqrt{\lambda}/2})\Psi \\
&:= \begin{cases} \Psi & \text{over } \varphi_n(B(0, \tfrac{1}{4}\sqrt{\lambda})), \\ (1 - \chi_{n,\sqrt{\lambda}/2})\Psi & \text{over } \varphi_n(\Omega(0, \tfrac{1}{4}\sqrt{\lambda}, \tfrac{1}{2}\sqrt{\lambda})), \\ 0 & \text{over } X - B(x_0, \tfrac{1}{2}\sqrt{\lambda}), \end{cases}
\end{aligned} \tag{3.47}
$$

where the identification of the bundles V over the annulus $\Omega(x_0, \frac{1}{4}\sqrt{\lambda}, \frac{1}{2}\sqrt{\lambda})$ in X and the bundle $\mathbf{V}$ over the annulus $\varphi_n(\Omega(0, \frac{1}{4}\sqrt{\lambda}, \frac{1}{2}\sqrt{\lambda}))$ in S^4, together with these annuli, are described following the definition (3.38) of V'.

3.5. **Space of gluing data and the definition of the splicing map.** The patching constructions described in §3.4 yield a pre-splicing map

$$\tilde{\boldsymbol{\gamma}}' : \tilde{N}_{\mathfrak{t},\mathfrak{s}}(\varepsilon) \times \mathrm{Fr}(\mathfrak{g}_V) \times_X \mathrm{Fr}(T^*X) \times \mathrm{Fr}(\mathfrak{g}_{\mathbf{V}}|_s) \times \tilde{M}_1^\natural(S^4, \delta) \to \tilde{\mathcal{C}}_{\mathfrak{t}'}, \tag{3.48}$$

where

$$\tilde{M}_1^\natural(S^4, \delta) \subset \mathcal{A}_1(S^4)$$

is the preimage of the subspace $M_1^\natural(S^4, \delta)$ under the projection $\mathcal{A}_1(S^4) \to \mathcal{B}_1(S^4)$.

The map (3.48) is equivariant with respect to the

1. Diagonal action of $\mathcal{G}_\mathfrak{s}$ on $\tilde{N}_{\mathfrak{t},\mathfrak{s}}(\varepsilon) \times \mathrm{Fr}(\mathfrak{g}_V)$, where $\mathcal{G}_\mathfrak{s}$ acts on $\tilde{N}_{\mathfrak{t},\mathfrak{s}}$ as described following equation (2.35) and on $\mathrm{Fr}(\mathfrak{g}_V)$ by the inclusion (2.25) of $\mathcal{G}_\mathfrak{s}$ in $\mathcal{G}_\mathfrak{t}$,
2. Diagonal action of $\mathcal{G}_1$ on $\mathrm{Fr}(\mathfrak{g}_{\mathbf{V}}|_s) \times \tilde{M}_1^\natural(S^4, \delta)$,
3. Diagonal action of $\mathrm{SO}(4)$ on $\mathrm{Fr}(T^*X) \times \tilde{M}_1^\natural(S^4, \delta)$ (where the action of $\mathrm{SO}(4)$ is as described following (3.25)),
4. Diagonal action of $\mathrm{SO}(3)$ on $\mathrm{Fr}(\mathfrak{g}_V) \times \mathrm{Fr}(\mathfrak{g}_{\mathbf{V}}|_s)$.

Therefore, if we define the space

$$\tilde{\mathrm{Gl}}_\mathfrak{t}(\delta) = \left(\mathrm{Fr}(\mathfrak{g}_V) \times_X \mathrm{Fr}(T^*X) \times \mathrm{Fr}(\mathfrak{g}_{\mathbf{V}}|_s) \times \tilde{M}_1^\natural(S^4, \delta)\right) / (\mathrm{SO}(3) \times \mathrm{SO}(4)), \tag{3.49}$$

where $\mathrm{SO}(3) \times \mathrm{SO}(4)$ act as described prior to (3.49), the pre-splicing map (3.48) descends to a map

$$\tilde{\boldsymbol{\gamma}}' : \tilde{N}_{\mathfrak{t},\mathfrak{s}}(\varepsilon) \times \tilde{\mathrm{Gl}}_\mathfrak{t}(\delta) \to \tilde{\mathcal{C}}^*_{\mathfrak{t}'}. \tag{3.50}$$

The map (3.50) is gauge equivariant and thus, if we define

$$
\begin{aligned}
\mathrm{Gl}_\mathfrak{t}(\delta) &= \tilde{\mathrm{Gl}}_\mathfrak{t}(\delta)/\mathcal{G}_1 \\
&= \left(\mathrm{Fr}(\mathfrak{g}_V) \times_X \mathrm{Fr}(T^*X) \times M_1^{s,\natural}(S^4, \delta)\right) / (\mathrm{SO}(3) \times \mathrm{SO}(4)),
\end{aligned} \tag{3.51}
$$

(recalling that $M_1^{s,\natural}(S^4,\delta) = \mathrm{Fr}(\mathfrak{g}_{\mathbf{V}}|_s) \times_{\mathcal{G}_1} \tilde{M}_1^{\natural}(S^4,\delta)$), the pre-splicing map descends to the gauge group quotients and gives the *splicing map*

$$\boldsymbol{\gamma}' : \mathcal{M}^{\mathrm{vir}}_{\mathfrak{t}',\mathfrak{s}} \to \mathcal{C}_{\mathfrak{t}'}, \tag{3.52}$$

where we have defined

$$\mathcal{M}^{\mathrm{vir}}_{\mathfrak{t}',\mathfrak{s}} = \tilde{N}_{\mathfrak{t},\mathfrak{s}}(\varepsilon) \times_{\mathcal{G}_\mathfrak{s}} \mathrm{Gl}_\mathfrak{t}(\delta). \tag{3.53}$$

We refer to the space (3.53) as a *virtual moduli space*; it is defined by the stabilizing bundle used to construct $N_{\mathfrak{t},\mathfrak{s}}$. It can be shown [19, 20] that $\boldsymbol{\gamma}'$ is a smooth embedding, provided ε and δ are sufficiently small.

By analogy with the definition of the Uhlenbeck compactification in §2.1.2, we define

$$\bar{\mathrm{Gl}}_\mathfrak{t}(\delta) = \left(\mathrm{Fr}(\mathfrak{g}_V) \times_X \mathrm{Fr}(T^*X) \times \bar{M}_1^{s,\natural}(S^4,\delta)\right) / (\mathrm{SO}(3) \times \mathrm{SO}(4)). \tag{3.54}$$

Points in $\bar{\mathrm{Gl}}_\mathfrak{t}(\delta) - \mathrm{Gl}_\mathfrak{t}(\delta)$ then correspond to the cone point in (3.24). The action of $\mathrm{SO}(4) \times \mathrm{SO}(3)$ in (3.54) is trivial on these cone points, so $\bar{\mathrm{Gl}}_\mathfrak{t}(\delta) - \mathrm{Gl}_\mathfrak{t}(\delta) = X$. The cone completion of $\mathrm{Gl}_\mathfrak{t}(\delta)$ then gives an 'Uhlenbeck compactification'

$$\bar{\mathcal{M}}^{\mathrm{vir}}_{\mathfrak{t}',\mathfrak{s}} = \tilde{N}_{\mathfrak{t},\mathfrak{s}}(\varepsilon) \times_{\mathcal{G}_\mathfrak{s}} \bar{\mathrm{Gl}}_\mathfrak{t}(\delta) \tag{3.55}$$

of the space $\mathcal{M}^{\mathrm{vir}}_{\mathfrak{t}',\mathfrak{s}}$, where $\mathcal{G}_\mathfrak{s}$ acts trivially on $\bar{\mathrm{Gl}}_\mathfrak{t}(\delta) - \mathrm{Gl}_\mathfrak{t}(\delta)$.

The splitting $\mathfrak{g}_V \cong \underline{\mathbb{R}} \oplus L$, induced by the splitting $V = W \oplus W \otimes L$, and the circle action given in (2.26) induce a circle action on $\mathrm{Fr}(\mathfrak{g}_V)$ and thus on $\bar{\mathrm{Gl}}_\mathfrak{t}(\delta)$ by scalar multiplication on L. If we write this action as $(e^{i\theta}, \mathbf{g}) \mapsto e^{i\theta}\mathbf{g}$, where $e^{i\theta} \in S^1$ and $\mathbf{g} \in \bar{\mathrm{Gl}}_\mathfrak{t}(\delta)$, then the description of the action of $\mathcal{G}_\mathfrak{s}$ on $\mathfrak{g}_V$ following equation (2.35) implies that

$$\varrho(e^{i\theta})\mathbf{g} = e^{-2i\theta}\mathbf{g}, \quad \text{for } e^{i\theta} \in S^1 \text{ and } \mathbf{g} \in \bar{\mathrm{Gl}}_\mathfrak{t}(\delta). \tag{3.56}$$

If we set

$$\bar{\mathcal{C}}_{\mathfrak{t}'} = \bigsqcup_{\ell=0}^{\infty} (\mathcal{C}_{\mathfrak{t}'_\ell} \times \mathrm{Sym}^\ell(X)),$$

and extend the splicing map $\boldsymbol{\gamma}'$ to a map

$$\boldsymbol{\gamma}' : \bar{\mathcal{M}}^{\mathrm{vir}}_{\mathfrak{t}',\mathfrak{s}} \to \bar{\mathcal{C}}_{\mathfrak{t}'}, \tag{3.57}$$

by setting $\boldsymbol{\gamma}'$ equal to $\boldsymbol{\gamma}_\mathfrak{s} \times \mathrm{id}_X : N_{\mathfrak{t},\mathfrak{s}}(\varepsilon) \times X \to \mathcal{C}_\mathfrak{t} \times X$ on the cone points, where $\boldsymbol{\gamma}_\mathfrak{s}$ is defined in (2.29), then this extension is continuous with respect to the Uhlenbeck topology. The extended splicing map (3.57) is S^1-equivariant if the circle acts by the action (2.8) on $\bar{\mathcal{C}}_{\mathfrak{t}'}$ and diagonally with weight two on each factor in the product $\tilde{N}_{\mathfrak{t},\mathfrak{s}} \times \bar{\mathrm{Gl}}_\mathfrak{t}(\delta)$:

Lemma 3.6. *The following two circle actions on $\tilde{N}_{\mathfrak{t},\mathfrak{s}}(\varepsilon) \times_{\mathcal{G}_\mathfrak{s}} \bar{\mathrm{Gl}}_\mathfrak{t}(\delta)$ are equivalent:*

1. *The action* (2.37) *on $\tilde{N}_{\mathfrak{t},\mathfrak{s}}(\varepsilon)$ and the trivial action on $\bar{\mathrm{Gl}}_\mathfrak{t}(\delta)$, and*
2. *The diagonal action with weight two on the fibers of $\tilde{N}_{\mathfrak{t},\mathfrak{s}}(\varepsilon)$ and with weight two on $\bar{\mathrm{Gl}}_\mathfrak{t}(\delta)$.*

Furthermore, the extended splicing and gluing maps

$$\boldsymbol{\gamma}, \boldsymbol{\gamma}' : \tilde{N}_{\mathfrak{t},\mathfrak{s}}(\varepsilon) \times_{\mathcal{G}_\mathfrak{s}} \bar{\mathrm{Gl}}_\mathfrak{t}(\delta) \to \bar{\mathcal{C}}_{\mathfrak{t}'}$$

are circle-equivariant if the circle acts on $\bar{\mathcal{C}}_{\mathfrak{t}'}$ by the action (2.8) *and on the domain by either of the above equivalent actions.*

Proof. The embedding $\tilde{N}_{\mathfrak{t},\mathfrak{s}}(\varepsilon) \to \tilde{\mathcal{C}}_{\mathfrak{t}}$ is S^1-equivariant if the circle acts on $\tilde{\mathcal{C}}_{\mathfrak{t}}$ by the action (2.8) and on $\tilde{N}_{\mathfrak{t},\mathfrak{s}}(\varepsilon)$ by the composition of multiplication with weight two on the fibers of $\tilde{N}_{\mathfrak{t},\mathfrak{s}}(\varepsilon)$ and the circle action (2.37). The splicing map is S^1-equivariant if the circle acts by scalar multiplication on the sections both before and after cutting off the sections. Hence, the splicing map $\boldsymbol{\gamma}'$ is S^1-equivariant if the circle acts on $\mathcal{C}_{\mathfrak{t}'}$ by the action (2.8) and by the action (1) in the lemma statement on the domain, namely: for $(B,\Psi,\eta) \in \tilde{N}_{\mathfrak{t},\mathfrak{s}}$ and $\mathbf{g} \in \mathrm{Gl}_{\mathfrak{t}}(\delta)$, by

$$[(B,\Psi,\eta),\mathbf{g}] \mapsto [\varrho(e^{i\theta})(B,\Psi,e^{i2\theta}\eta),\mathbf{g}].$$

Because $\mathcal{G}_{\mathfrak{s}}$ acts diagonally in the definition of $\tilde{N}_{\mathfrak{t},\mathfrak{s}}(\varepsilon) \times_{\mathcal{G}_{\mathfrak{s}}} \bar{\mathrm{Gl}}_{\mathfrak{t}}(\delta)$, the action (1) on the domain is equal to the circle action given by scalar multiplication with weight two on the fibers of $\tilde{N}_{\mathfrak{t},\mathfrak{s}}(\varepsilon)$ and by the action of the constant S^1 in $\mathcal{G}_{\mathfrak{s}}$ with weight negative one on the factor $\bar{\mathrm{Gl}}_{\mathfrak{t}}(\delta)$. This last action is equal to the action (2) on the domain in the statement of the lemma by the equation (3.56) comparing the action of the constant S^1 subgroup of $\mathcal{G}_{\mathfrak{s}}$ on $\bar{\mathrm{Gl}}_{\mathfrak{t}}(\delta)$ with the standard action:

$$[\varrho(e^{i\theta})(B,\Psi,e^{i2\theta}\eta),\mathbf{g}] = [(B,\Psi,e^{i2\theta}\eta),\varrho(e^{-i\theta})\mathbf{g}] = [(B,\Psi,e^{i2\theta}\eta),e^{2i\theta}\mathbf{g}].$$

This completes the proof. □

We note that there is a projection map,

$$\tilde{N}_{\mathfrak{t},\mathfrak{s}}(\varepsilon) \times_{\mathcal{G}_{\mathfrak{s}}\times S^1} \bar{\mathrm{Gl}}_{\mathfrak{t}}(\delta) \to M_{\mathfrak{s}} \times X, \tag{3.58}$$

defined via the projection maps $\bar{\mathrm{Gl}}_{\mathfrak{t}}(\delta) \to X$ (given by the fiber bundle structure in (3.54)) and $N_{\mathfrak{t},\mathfrak{s}} \to M_{\mathfrak{s}}$.

3.6. **The obstruction bundle.** The SO(3)-monopole obstruction bundle is a finite-rank, S^1-equivariant subbundle,

$$\begin{array}{ccc} \Upsilon_{\mathfrak{t}',\mathfrak{s}} & \xrightarrow{\varphi} & \mathfrak{V}_{\mathfrak{t}'} \\ \downarrow & & \downarrow \\ \tilde{N}_{\mathfrak{t},\mathfrak{s}}(\varepsilon) \times_{\mathcal{G}_{\mathfrak{s}}} \mathrm{Gl}_{\mathfrak{t}}(\delta) & \xrightarrow{\gamma'} & \mathcal{C}^{*,0}_{\mathfrak{t}'} \end{array} \tag{3.59}$$

of the S^1-equivariant vector bundle

$$\mathfrak{V}_{\mathfrak{t}'} = \tilde{\mathcal{C}}^{*,0}_{\mathfrak{t}'} \times_{\mathcal{G}_{\mathfrak{t}'}} L^2_{k-1}(\Lambda^+ \otimes \mathfrak{g}_{V'} \oplus V'^-) \to \mathcal{C}^{*,0}_{\mathfrak{t}'}. \tag{3.60}$$

The circle actions in (3.60) are given by the diagonal circle action, with the action (2.8) on $\tilde{\mathcal{C}}^{*,0}_{\mathfrak{t}'}$, the trivial action on $L^2_{k-1}(\Lambda^+ \otimes \mathfrak{g}_{V'})$, and by scalar multiplication on $L^2_{k-1}(V')$. As described in [19], the fiber $\Upsilon_{[(B,\Psi,\eta),\mathbf{g}]}$ over a point $[(B,\Psi,\eta),\mathbf{g}] = (\boldsymbol{\gamma}')^{-1}([A',\Phi'])$ in the base of the obstruction bundle, $\Upsilon_{\mathfrak{t}',\mathfrak{s}}$, is given by

$$\Upsilon_{[(B,\Psi,\eta),\mathbf{g}]} \cong \Upsilon^s_{[B,\Psi,\eta]} \oplus \Upsilon^i_{[\mathbf{A}]}, \tag{3.61}$$

where $\Upsilon^s_{[B,\Psi,\eta]}$ is a fiber of the background obstruction bundle (see §3.6.1), and $\Upsilon^i_{[\mathbf{A}]}$ is a fiber of the instanton obstruction bundle (see §3.6.2), and $[\mathbf{A}]$ is a point in $M^{s,\natural}_k(S^4,\delta)$ associated with $[\mathbf{g}] \in \mathrm{Gl}_{\mathfrak{t}}(\delta)$.

3.6.1. *Background or Seiberg-Witten component of the obstruction bundle.* In [19], the Seiberg-Witten component of the SO(3)-monopole obstruction bundle was identified with the subbundle constructed in [17, §3.5.2], namely $\pi_N^*\Xi_{\mathfrak{t},\mathfrak{s}} \cong N_{\mathfrak{t},\mathfrak{s}}(\varepsilon)\times\mathbb{C}^{r_\Xi}$ where $\pi_N : N_{\mathfrak{t},\mathfrak{s}} \to M_{\mathfrak{s}}$ is the projection, and there is an embedding of S^1-equivariant vector bundles,

$$
\begin{array}{ccc}
\pi_N^*\Xi_{\mathfrak{t},\mathfrak{s}} & \longrightarrow & \mathfrak{V}_{\mathfrak{t}} \\
\downarrow & & \downarrow \\
N_{\mathfrak{t},\mathfrak{s}}(\varepsilon) & \longrightarrow & \mathcal{C}_{\mathfrak{t}}
\end{array}
\tag{3.62}
$$

The bundle map $\pi_N^*\Xi_{\mathfrak{t},\mathfrak{s}} \to \mathfrak{V}_{\mathfrak{t}}$ in (3.62) is S^1-equivariant if the circle acts diagonally on $N_{\mathfrak{t},\mathfrak{s}}(\varepsilon)\times\mathbb{C}^{r_\Xi}$ by scalar multiplication with weight two on both the fibers of $N_{\mathfrak{t},\mathfrak{s}}$ and on $\mathbb{C}^{r_\Xi}$, while the circle acts on $\mathfrak{V}_{\mathfrak{t}}$ by the action described following (3.60).

The cut-and-paste construction in §3.5 defines an embedding of vector bundles,

$$
\begin{array}{ccc}
\Upsilon^s_{\mathfrak{t}',\mathfrak{s}} & \xrightarrow{\varphi_s} & \mathfrak{V}_{\mathfrak{t}'} \\
\downarrow & & \downarrow \\
\tilde N_{\mathfrak{t},\mathfrak{s}}(\varepsilon)\times_{\mathcal{G}_{\mathfrak{s}}}\mathrm{Gl}_{\mathfrak{t}}(\delta) & \xrightarrow{\gamma'} & \mathcal{C}_{\mathfrak{t}'}
\end{array}
\tag{3.63}
$$

where $\Upsilon^s_{\mathfrak{t}',\mathfrak{s}}$ is the pullback by the projection $\tilde N_{\mathfrak{t},\mathfrak{s}}(\varepsilon)\times_{\mathcal{G}_{\mathfrak{s}}}\mathrm{Gl}_{\mathfrak{t}}(\delta)\to N_{\mathfrak{t},\mathfrak{s}}(\varepsilon)$ of the obstruction bundle $\pi_N^*\Xi_{\mathfrak{t},\mathfrak{s}}$ appearing in (2.30), so that

$$
\Upsilon^s_{\mathfrak{t}',\mathfrak{s}} \cong \mathbb{C}^{r_\Xi}\times\tilde N_{\mathfrak{t},\mathfrak{s}}(\varepsilon)\times_{\mathcal{G}_{\mathfrak{s}}}\mathrm{Gl}_{\mathfrak{t}}(\delta).
$$

Because this cut-and-paste construction is circle equivariant, the embedding (3.63) has the same circle-equivariance properties as the embedding (3.62), where the circle acts on $\mathfrak{V}_{\mathfrak{t}'}$ as previously described and diagonally on $\mathbb{C}^{r_\Xi}\times\tilde N_{\mathfrak{t},\mathfrak{s}}(\varepsilon)\times_{\mathcal{G}_{\mathfrak{s}}}\mathrm{Gl}_{\mathfrak{t}}(\delta)$, with weight two on $\mathbb{C}^{r_\Xi}$ and on $\tilde N_{\mathfrak{t},\mathfrak{s}}(\varepsilon)\times_{\mathcal{G}_{\mathfrak{s}}}\mathrm{Gl}_{\mathfrak{t}}(\delta)$ with the action given in Lemma 3.6.

To describe the intersection of the image of the extended gluing map, associated with the extended splicing map (3.57) (see §3.7) with $\bar{\mathcal{M}}_{\mathfrak{t}'}$, we need to extend the background obstruction bundle over the space of extended gluing data (3.54). Thus, we define the extension by

$$
\bar\Upsilon^s_{\mathfrak{t}',\mathfrak{s}} \cong \mathbb{C}^{r_\Xi}\times\tilde N_{\mathfrak{t},\mathfrak{s}}(\varepsilon)\times_{\mathcal{G}_{\mathfrak{s}}}\bar{\mathrm{Gl}}_{\mathfrak{t}}(\delta).
\tag{3.64}
$$

The restriction of $\bar\Upsilon^s_{\mathfrak{t}',\mathfrak{s}}$ to the stratum $N_{\mathfrak{t},\mathfrak{s}}(\varepsilon)\times X$ of the space of extended gluing data is given by the pullback of the bundle in the left-hand column of (3.62) to $N_{\mathfrak{t},\mathfrak{s}}(\varepsilon)\times X$ and is embedded into $\mathfrak{V}_{\mathfrak{t}}$ via the diagram (3.62).

3.6.2. *Instanton component of the obstruction bundle.* In [19], the instanton components of the SO(3)-monopole gluing obstruction space were identified with cokernels of the Dirac operators,

$$
D_{\mathbf{A}} : C^\infty(S^4,\mathbf{V}^+)\to C^\infty(S^4,\mathbf{V}^-),
$$

where the induced connections $\hat{\mathbf{A}}$ on the SO(3) bundle $\mathfrak{g}_{\mathbf{V}}$ over S^4 are anti-self-dual. Because the standard round metric on S^4 has positive scalar curvature, one has $\operatorname{Ker} D_{\mathbf{A}} = \{0\}$ and as $\operatorname{Index}_{\mathbb{C}} D_{\mathbf{A}} = -1$ (see, for example, [16, p. 314]), the cokernel bundles,

$$
\operatorname{Coker}\tilde{\mathbf{D}}_{\mathbf{V}}\to\tilde M_1^{s,\natural}(S^4,\delta)\quad\text{and}\quad\operatorname{Coker}\mathbf{D}_{\mathbf{V}}\to M_1^{s,\natural}(S^4,\delta),
\tag{3.65}
$$

are complex line bundles with fibers $\operatorname{Coker}\mathbf{D}_{\mathbf{V}}|_{\mathbf{A}} = \operatorname{Coker} D_{\mathbf{A}}$. The total space of the bundle $\operatorname{Coker}\mathbf{D}_{\mathbf{V}}$ is identified by the diffeomorphism

$$
\operatorname{Coker}\mathbf{D}_{\mathbf{V}}\cong(\mathrm{U}(2)\times(0,\delta])\times_{S^1}\mathbb{C}\cong\mathrm{SU}(2)\times_{\{\pm1\}}\mathbb{C}\times(0,\delta],
\tag{3.66}
$$

where S^1 acts *diagonally* on $\mathrm{U}(2)\times\mathbb{C}$. The bundle $\operatorname{Coker}\mathbf{D}_{\mathbf{V}}\to M_1^{s,\natural}(S^4,\delta)$ is non-trivial, but torsion. There is an action of $\mathrm{Spin}^u(4)$ on $\operatorname{Coker}\mathbf{D}_{\mathbf{V}}$, covering the action of $\mathrm{SO}(4)\times\mathrm{SO}(3)$ on $M_1^{s,\natural}(S^4,\delta)$: this action is most easily understood using the trivializations (3.37) of $\mathbf{V}|_{S^4\setminus\{n,s\}}$. The section Ψ, in this trivialization, is a map

$$\mathbb{R}^4-\{0\}\to\Delta\otimes_{\mathbb{C}}\mathbb{C}^2.$$

Then $\tilde R\otimes M\in\mathrm{Spin}^u(4)$ acts by

$$\Psi\mapsto(\tilde R\otimes M)\circ\Psi\circ\mathrm{Ad}^c(\tilde R)^{-1}.$$

The procedure for splicing spinors over S^4 onto X defines a map (see §3.4.3)

$$\tilde{\boldsymbol{\varphi}}_i:\tilde N_{\mathfrak{t},\mathfrak{s}}\times\mathrm{Fr}_{\mathbb{C}\ell(T^*X)}(V)\times\operatorname{Coker}\tilde{\mathbf{D}}_{\mathbf{V}}\to\tilde{\mathcal{C}}_{\mathfrak{t}'}\times\Gamma(\Lambda^+\otimes\mathfrak{g}_{V'}\oplus V'^-). \tag{3.67}$$

The map (3.67) is invariant under the diagonal action of $\mathrm{Spin}^u(4)$ on $\mathrm{Fr}_{\mathbb{C}\ell(T^*X)}(V)\times\operatorname{Coker}\tilde{\mathbf{D}}_{\mathbf{V}}$. Note that this $\mathrm{Spin}^u(4)$ action covers the diagonal $\mathrm{SO}(4)$ action on the component $\mathrm{Fr}(T^*X)\times\tilde M_1^{\natural}(S^4,\delta)$ of the base and the diagonal $\mathrm{SO}(3)$ action on $\mathrm{Fr}(\mathfrak{g}_V)\times\mathrm{Fr}(\mathfrak{g}_{\mathbf{V}}|_s)$. In addition, the map (3.67) is equivariant with respect to the action of $\mathcal{G}_{\mathfrak{t}'}$ on the image and

- The diagonal action of $\mathcal{G}_{\mathfrak{s}}$ on $\tilde N_{\mathfrak{t},\mathfrak{s}}\times\mathrm{Fr}_{\mathbb{C}\ell(T^*X)}(V)$, where $\mathcal{G}_{\mathfrak{s}}$ acts on $\mathrm{Fr}_{\mathbb{C}\ell(T^*X)}(V)$ by the homomorphism $\mathcal{G}_{\mathfrak{s}}\hookrightarrow\mathcal{G}_{\mathfrak{t}}$ in (2.25), covering the diagonal action of $\mathcal{G}_{\mathfrak{s}}$ on $\tilde N_{\mathfrak{t},\mathfrak{s}}\times\mathrm{Fr}(\mathfrak{g}_V)$,
- The action of $\mathcal{G}_1$ on $\operatorname{Coker}\tilde{\mathbf{D}}_{\mathbf{V}}$, covering its diagonal action on the component $\tilde M_1^{s,\natural}(S^4,\delta)$ of the base.

Finally, the map (3.67) is S^1-equivariant with respect to the circle action $\mathfrak{V}_{\mathfrak{t}'}$ described after (3.60) and the diagonal circle action on $\tilde N_{\mathfrak{t},\mathfrak{s}}\times\operatorname{Coker}\mathbf{D}_A$, where the circle acts on $\operatorname{Coker}\mathbf{D}_A$ by scalar multiplication and on $\tilde N_{\mathfrak{t},\mathfrak{s}}$ by the action (2.37). We define a vector bundle over $\tilde N_{\mathfrak{t},\mathfrak{s}}\times_{\mathcal{G}_{\mathfrak{s}}}\mathrm{Gl}_{\mathfrak{t}}(\delta)$ by setting

$$\Upsilon^i_{\mathfrak{t}',\mathfrak{s}}=\tilde N_{\mathfrak{t},\mathfrak{s}}\times_{\mathcal{G}_{\mathfrak{s}}}\mathrm{Fr}_{\mathbb{C}\ell(T^*X)}(V)\times_{\mathrm{Spin}^u(4)}\operatorname{Coker}\mathbf{D}_{\mathbf{V}}, \tag{3.68}$$

where $\mathrm{Spin}^u(4)$ acts diagonally on $\mathrm{Fr}_{\mathbb{C}\ell(T^*X)}(V)\times\operatorname{Coker}\mathbf{D}_{\mathbf{V}}$ and trivially on $\tilde N_{\mathfrak{t},\mathfrak{s}}$, while $\mathcal{G}_{\mathfrak{s}}$ acts diagonally on $\tilde N_{\mathfrak{t},\mathfrak{s}}\times\mathrm{Fr}_{\mathbb{C}\ell(T^*X)}(V)$ as described above and trivially on $\operatorname{Coker}\mathbf{D}_{\mathbf{V}}$. We then have the

Lemma 3.7. *The map $\tilde{\boldsymbol{\varphi}}_i$ descends to an embedding of S^1-equivariant vector bundles,*

$$\begin{array}{ccc}\Upsilon^i_{\mathfrak{t}',\mathfrak{s}} & \xrightarrow{\varphi_i} & \mathfrak{V}_{\mathfrak{t}'}\\ \downarrow & & \downarrow\\ \tilde N_{\mathfrak{t},\mathfrak{s}}\times_{\mathcal{G}_{\mathfrak{s}}}\mathrm{Gl}_{\mathfrak{t}}(\delta) & \xrightarrow{\gamma'} & \mathcal{C}_{\mathfrak{t}'}\end{array} \tag{3.69}$$

where the group actions are as described in the preceding paragraph.

3.7. **Construction of the gluing map.** The Uhlenbeck compactification (3.55) of the virtual moduli space model $\mathcal{M}^{\mathrm{vir}}_{\mathfrak{t}',\mathfrak{s}}$ has a smooth, circle invariant stratification (where the strata are manifolds with boundary),

$$\begin{aligned}\bar{\mathcal{M}}^{\mathrm{vir}}_{\mathfrak{t}',\mathfrak{s}}&=\tilde N_{\mathfrak{t},\mathfrak{s}}(\varepsilon)\times_{\mathcal{G}_{\mathfrak{s}}}\mathrm{Gl}_{\mathfrak{t}}(\delta)\\ &\sqcup(N_{\mathfrak{t},\mathfrak{s}}(\varepsilon)-M_{\mathfrak{s}})\times X\\ &\sqcup M_{\mathfrak{s}}\times X.\end{aligned} \tag{3.70}$$

If r_N is the complex rank of $N_{\mathfrak{t},\mathfrak{s}} \to M_\mathfrak{s}$ and $d_s(\mathfrak{s}) = \dim M_\mathfrak{s}$, then because the fiber of $\mathrm{Gl}_\mathfrak{t}(\delta) \to X$ is four-dimensional, the dimensions of the strata in (3.70) are given by (see the rank and dimension relations (2.32) and (2.34))

$$\begin{aligned}
\dim\left(\tilde N_{\mathfrak{t},\mathfrak{s}}(\varepsilon) \times_{\mathcal{G}_\mathfrak{s}} \mathrm{Gl}_\mathfrak{t}(\delta)\right) &= 2r_N + d_s(\mathfrak{s}) + 8,\\
\dim\left((N_{\mathfrak{t},\mathfrak{s}}(\varepsilon) - M_\mathfrak{s}) \times X\right) &= 2r_N + d_s(\mathfrak{s}) + 4,\\
\dim\left(M_\mathfrak{s} \times X\right) &= d_s(\mathfrak{s}) + 4.
\end{aligned}$$

The restriction of the splicing map $\boldsymbol{\gamma}'$ to the top stratum $\mathcal{M}^{\mathrm{vir}}_{\mathfrak{t}',\mathfrak{s}}$ takes values in $\mathcal{C}^{*,0}_{\mathfrak{t}'}$; on the middle stratum, $(N_{\mathfrak{t},\mathfrak{s}}(\varepsilon) - M_\mathfrak{s}) \times X$, the splicing map $\boldsymbol{\gamma}'$ restricts to the map $\boldsymbol{\gamma}_{\mathfrak{t},\mathfrak{s}} \times \mathrm{id}_X$, which takes values in $\mathcal{C}^{*,0}_\mathfrak{t} \times X$; on the lowest stratum, $M_\mathfrak{s} \times X$, the splicing map $\boldsymbol{\gamma}'$ restricts to the identity map on $M_\mathfrak{s} \times X$. The stratum $M_\mathfrak{s} \times X$ is the fixed point set of the circle action described in Lemma 3.6.

The splicing map $\boldsymbol{\gamma}'$ may be deformed S^1-equivariantly to a 'gluing map' $\boldsymbol{\gamma}$ with the properties described in the following

Theorem 3.8. [19, 20] *For small enough positive ε and δ, there is a topological embedding,*

$$\boldsymbol{\gamma} : \bar{\mathcal{M}}^{\mathrm{vir}}_{\mathfrak{t}',\mathfrak{s}} \to \bar{\mathcal{C}}_{\mathfrak{t}'} = \mathcal{C}_{\mathfrak{t}'} \sqcup (\mathcal{C}_\mathfrak{t} - M_\mathfrak{s}) \times X \sqcup M_\mathfrak{s} \times X, \tag{3.71}$$

restricting to a smooth embedding of the top stratum of (3.70) *into $\mathcal{C}^{*,0}_{\mathfrak{t}'}$, the smooth embedding $\boldsymbol{\gamma}_{\mathfrak{t},\mathfrak{s}} \times \mathrm{id}_X$ of the middle stratum into $\mathcal{C}^{*,0}_\mathfrak{t} \times X$, and the identity map on the lowest stratum, $M_\mathfrak{s} \times X \subset \mathcal{C}^0_\mathfrak{t} \times X$. There is a smooth, circle-equivariant section $\boldsymbol{\chi}_i$ of the instanton obstruction bundle* (3.68),

$$\Upsilon^i_{\mathfrak{t}',\mathfrak{s}} \to \mathcal{M}^{\mathrm{vir}}_{\mathfrak{t}',\mathfrak{s}},$$

and a continuous, circle-equivariant section $\boldsymbol{\chi}_s$ of the background obstruction bundle (3.64),

$$\bar\Upsilon^s_{\mathfrak{t}',\mathfrak{s}} \to \bar{\mathcal{M}}^{\mathrm{vir}}_{\mathfrak{t}',\mathfrak{s}},$$

which is smooth when restricted to each stratum of $\bar{\mathcal{M}}^{\mathrm{vir}}_{\mathfrak{t}',\mathfrak{s}}$ in (3.70) *such that, if $\boldsymbol{\chi} = \boldsymbol{\chi}_s \oplus \boldsymbol{\chi}_i$, then*

$$\boldsymbol{\gamma}(\mathcal{M}^{\mathrm{vir}}_{\mathfrak{t}',\mathfrak{s}} \cap \boldsymbol{\chi}^{-1}(0)) = \boldsymbol{\gamma}(\mathcal{M}^{\mathrm{vir}}_{\mathfrak{t}',\mathfrak{s}}) \cap \mathcal{M}_{\mathfrak{t}'},$$

and

$$\boldsymbol{\gamma}\left(((N_{\mathfrak{t},\mathfrak{s}}(\varepsilon) - M_\mathfrak{s}) \times X) \cap \boldsymbol{\chi}_s^{-1}(0)\right) = \boldsymbol{\gamma}((N_{\mathfrak{t},\mathfrak{s}}(\varepsilon) - M_\mathfrak{s}) \times X) \cap \bar{\mathcal{M}}_{\mathfrak{t}'}.$$

The sections $\boldsymbol{\chi}_s \oplus \boldsymbol{\chi}_i$ and $\boldsymbol{\chi}_s$ of the vector bundles,

$$\bar\Upsilon^s_{\mathfrak{t}',\mathfrak{s}} \oplus \Upsilon^i_{\mathfrak{t}',\mathfrak{s}} \to \mathcal{M}^{\mathrm{vir}}_{\mathfrak{t}',\mathfrak{s}} \quad \textit{and} \quad \bar\Upsilon^s_{\mathfrak{t}',\mathfrak{s}} \to (N_{\mathfrak{t},\mathfrak{s}} - M_\mathfrak{s}) \times X,$$

respectively, vanish transversely.

We shall formally extend the section $\boldsymbol{\chi}_i$ over the lower strata of $\bar{\mathcal{M}}^{\mathrm{vir}}_{\mathfrak{t}',\mathfrak{s}}$ by setting it equal to zero on the lower strata. This is done so that we may discuss $\boldsymbol{\chi}^{-1}(0)$ as a subspace of $\bar{\mathcal{M}}^{\mathrm{vir}}_{\mathfrak{t}',\mathfrak{s}}$; we make no assumptions about the continuity or transversality of this formal extension of $\boldsymbol{\chi}_i$ over the lower strata.

3.8. **Link of a level-one Seiberg-Witten stratum.** We define the *Seiberg-Witten* and *instanton components* of the *virtual link* of the stratum $M_\mathfrak{s} \times X$ in $\bar{\mathcal{M}}^{\mathrm{vir}}_{\mathfrak{t}',\mathfrak{s}}/S^1$ by

$$\begin{aligned}
\bar{\mathbf{L}}^{\mathrm{vir},s}_{\mathfrak{t}',\mathfrak{s}} &= \left(\partial\tilde N_{\mathfrak{t},\mathfrak{s}}(\varepsilon) \times_{\mathcal{G}_\mathfrak{s}} \bar{\mathrm{Gl}}_\mathfrak{t}(\delta)\right)/S^1,\\
\mathbf{L}^{\mathrm{vir},i}_{\mathfrak{t}',\mathfrak{s}} &= \left(\tilde N_{\mathfrak{t},\mathfrak{s}}(\varepsilon) \times_{\mathcal{G}_\mathfrak{s}} \partial\bar{\mathrm{Gl}}_\mathfrak{t}(\delta)\right)/S^1,
\end{aligned} \tag{3.72}$$

with (see (3.51) and (3.54) for the definitions of $\mathrm{Gl}_{\mathfrak{t}}(\delta)$ and $\bar{\mathrm{Gl}}_{\mathfrak{t}}(\delta)$, respectively),

$$\partial\bar{\mathrm{Gl}}_{\mathfrak{t}}(\delta) = \left(\mathrm{Fr}(\mathfrak{g}_V) \times_X \mathrm{Fr}(T^*X) \times \lambda^{-1}(\delta) \cap M_1^{s,\natural}(S^4)\right)/(\mathrm{SO}(3)\times\mathrm{SO}(4)), \tag{3.73}$$

and where the circle action on $\bar{\mathcal{M}}^{\mathrm{vir}}_{\mathfrak{t}',\mathfrak{s}}$ is induced by the circle action (2.37) on $\tilde{N}_{\mathfrak{t},\mathfrak{s}}$ and the trivial action on $\bar{\mathrm{Gl}}_{\mathfrak{t}}(\delta)$ (as described in the first action in Lemma 3.6). The map

$$\lambda : \mathcal{B}_1(S^4) \to (0,\infty)$$

appearing in the gluing-data boundary (3.73) is defined by the scale definition (3.21).

We define the *virtual link* of the stratum $M_{\mathfrak{s}} \times X$ in $\bar{\mathcal{M}}^{\mathrm{vir}}_{\mathfrak{t}',\mathfrak{s}}/S^1$ by setting

$$\bar{\mathbf{L}}^{\mathrm{vir}}_{\mathfrak{t}',\mathfrak{s}} = \bar{\mathbf{L}}^{\mathrm{vir},s}_{\mathfrak{t}',\mathfrak{s}} \cup \mathbf{L}^{\mathrm{vir},i}_{\mathfrak{t}',\mathfrak{s}}, \tag{3.74}$$

We let

$$\mathbf{L}^{\mathrm{vir}}_{\mathfrak{t}',\mathfrak{s}} = \bar{\mathbf{L}}^{\mathrm{vir}}_{\mathfrak{t}',\mathfrak{s}} \cap \mathcal{M}^{\mathrm{vir}}_{\mathfrak{t}',\mathfrak{s}}/S^1 \quad \text{and} \quad \mathbf{L}^{\mathrm{vir},s}_{\mathfrak{t}',\mathfrak{s}} = \bar{\mathbf{L}}^{\mathrm{vir},s}_{\mathfrak{t}',\mathfrak{s}} \cap \mathcal{M}^{\mathrm{vir}}_{\mathfrak{t}',\mathfrak{s}}/S^1 \tag{3.75}$$

denote the intersection of these subspaces with the top stratum $\mathcal{M}^{\mathrm{vir}}_{\mathfrak{t}',\mathfrak{s}}/S^1$ of $\bar{\mathcal{M}}^{\mathrm{vir}}_{\mathfrak{t}',\mathfrak{s}}/S^1$. Note that the top level, $\mathbf{L}^{\mathrm{vir}}_{\mathfrak{t}',\mathfrak{s}}$, of the virtual link is only a topological and not a smooth manifold because of the 'edge':

$$\mathbf{L}^{\mathrm{vir},i}_{\mathfrak{t}',\mathfrak{s}} \cap \mathbf{L}^{\mathrm{vir},s}_{\mathfrak{t}',\mathfrak{s}}. \tag{3.76}$$

Let $\mathbf{L}^{\mathrm{low}}_{\mathfrak{t}',\mathfrak{s}}$ be the intersection of $\bar{\mathbf{L}}^{\mathrm{vir}}_{\mathfrak{t}',\mathfrak{s}}$ with the union of the lower levels of $\bar{\mathcal{M}}^{\mathrm{vir}}_{\mathfrak{t}',\mathfrak{s}}/S^1$:

$$\begin{aligned}\mathbf{L}^{\mathrm{low}}_{\mathfrak{t}',\mathfrak{s}} &= \bar{\mathbf{L}}^{\mathrm{vir}}_{\mathfrak{t}',\mathfrak{s}} - \mathbf{L}^{\mathrm{vir}}_{\mathfrak{t}',\mathfrak{s}} \\ &= \bar{\mathbf{L}}^{\mathrm{vir}}_{\mathfrak{t}',\mathfrak{s}} \cap \left(\bar{\mathcal{M}}^{\mathrm{vir}}_{\mathfrak{t}',\mathfrak{s}} - \mathcal{M}^{\mathrm{vir}}_{\mathfrak{t}',\mathfrak{s}}\right)/S^1 \\ &= \partial N_{\mathfrak{t},\mathfrak{s}}(\varepsilon)/S^1 \times X,\end{aligned} \tag{3.77}$$

We then have a stratification:

$$\bar{\mathbf{L}}^{\mathrm{vir}}_{\mathfrak{t}',\mathfrak{s}} = \mathbf{L}^{\mathrm{vir}}_{\mathfrak{t}',\mathfrak{s}} \sqcup \mathbf{L}^{\mathrm{low}}_{\mathfrak{t}',\mathfrak{s}}. \tag{3.78}$$

In this article, where $\ell(\mathfrak{t}',\mathfrak{s}) = 1$, we see from (3.77) that $\mathbf{L}^{\mathrm{low}}_{\mathfrak{t}',\mathfrak{s}}$ is a closed, smooth manifold.

We define the *link* of $(M_{\mathfrak{s}} \times X) \cap \bar{\mathcal{M}}_{\mathfrak{t}'}/S^1$ in $\bar{\mathcal{M}}_{\mathfrak{t}'}/S^1$ and its top stratum by

$$\bar{\mathbf{L}}_{\mathfrak{t}',\mathfrak{s}} = \boldsymbol{\gamma}\left(\boldsymbol{\chi}^{-1}(0) \cap \bar{\mathbf{L}}^{\mathrm{vir}}_{\mathfrak{t}',\mathfrak{s}}\right) \quad \text{and} \quad \mathbf{L}_{\mathfrak{t}',\mathfrak{s}} = \bar{\mathbf{L}}_{\mathfrak{t}',\mathfrak{s}} \cap \mathcal{M}^{*,0}_{\mathfrak{t}'}/S^1. \tag{3.79}$$

For generic choices of the parameters ε and δ (which we shall henceforth assume) defining $\mathbf{L}^{\mathrm{vir}}_{\mathfrak{t}',\mathfrak{s}}$, the subspaces

$$\mathbf{L}_{\mathfrak{t}',\mathfrak{s}} \cap \mathbf{L}^{\mathrm{vir},i}_{\mathfrak{t}',\mathfrak{s}} \quad \text{and} \quad \mathbf{L}_{\mathfrak{t}',\mathfrak{s}} \cap \mathbf{L}^{\mathrm{vir},s}_{\mathfrak{t}',\mathfrak{s}}$$

will be smooth submanifolds of $\mathbf{L}^{\mathrm{vir},i}_{\mathfrak{t}',\mathfrak{s}}$ and $\mathbf{L}^{\mathrm{vir},s}_{\mathfrak{t}',\mathfrak{s}}$ respectively, transverse to the common boundary (3.76). The following lemma is the first step in showing that the intersection numbers on the right-hand-side of (2.50) are well-defined.

Lemma 3.9. *Assume $w \in H^2(X;\mathbb{Z})$ is such that $w \pmod 2$ is good in the sense of Definition 2.3. Given a Riemannian metric on X and a pair $(\mathfrak{t}',\mathfrak{s})$ with $\ell(\mathfrak{t}',\mathfrak{s}) = 1$ and $w_2(\mathfrak{t}') \equiv w \pmod 2$, there are positive constants ε_0 and δ_0 such that the following hold for all generic choices of $\varepsilon \le \varepsilon_0$ and $\delta \le \delta_0$ defining $\bar{\mathbf{L}}^{\mathrm{vir}}_{\mathfrak{t}',\mathfrak{s}}$.*

- *The link $\bar{\mathbf{L}}_{\mathfrak{t}',\mathfrak{s}}$ is disjoint from $\bar{M}^w_\kappa$ and $\bar{\mathcal{M}}^{\mathrm{red}}_{\mathfrak{t}'}$ in the stratification (2.13) of $\bar{\mathcal{M}}_{\mathfrak{t}'}/S^1$.*

- *For all* $z \in \mathbb{A}(X)$, *the intersection,*

$$\bar{\mathcal{V}}(z) \cap \bar{\mathcal{W}}^{\eta} \cap \bar{\mathbf{L}}_{\mathfrak{t}',\mathfrak{s}} \tag{3.80}$$

is contained in the top stratum $\mathbf{L}_{\mathfrak{t}',\mathfrak{s}}$ *of* $\bar{\mathbf{L}}_{\mathfrak{t}',\mathfrak{s}} \subset \bar{\mathcal{M}}_{\mathfrak{t}'}/S^1$*, and is disjoint from the image of the edge* (3.76) *under the gluing map* $\boldsymbol{\gamma}$*.*

Proof. Because $w \pmod 2$ is good, the union of strata $\bar{M}^w_\kappa$ is disjoint from the union of strata $\bar{\mathcal{M}}^{\text{red}}_{\mathfrak{t}'}$ (see remarks in §2.2). Hence, for sufficiently small parameters δ and ε in the definition of $\bar{\mathbf{L}}^{\text{vir}}_{\mathfrak{t}',\mathfrak{s}}$, the link $\bar{\mathbf{L}}_{\mathfrak{t}',\mathfrak{s}}$ is disjoint from these strata.

The geometric representatives $\bar{\mathcal{V}}(z)\cap\bar{\mathcal{W}}^{\eta}$ do not intersect the lower levels $(\bar{\mathcal{M}}_{\mathfrak{t}'} - \mathcal{M}_{\mathfrak{t}'})/S^1$ of $\bar{\mathcal{M}}_{\mathfrak{t}'}/S^1$ except at points in $\bar{\mathcal{M}}^{\text{red}}_{\mathfrak{t}'}$ or at points in $\bar{M}^w_\kappa$ by [18, Corollary 3.18]. Therefore, the intersection (3.80) is contained in the top stratum, $\mathbf{L}_{\mathfrak{t}',\mathfrak{s}}$.

The geometric representatives are transverse to $\mathcal{M}^{*,0}_{\mathfrak{t}'}/S^1$. Hence, the intersection

$$\boldsymbol{\gamma}^{-1}(\bar{\mathcal{V}}(z) \cap \bar{\mathcal{W}}^{\eta}) \cap \boldsymbol{\chi}^{-1}(0),$$

is a smooth submanifold of $\mathcal{M}^{\text{vir},*}_{\mathfrak{t}',\mathfrak{s}}/S^1$. For generic values of ε and δ, the preceding intersection will be transverse to the edge and thus, by dimension-counting, disjoint from the edge. □

Because the intersection (3.80) is contained in the locus of smooth points of $\bar{\mathbf{L}}_{\mathfrak{t}',\mathfrak{s}}$, it will be possible to define an intersection number once we have discussed the orientation of the link.

We note that the construction of the link in this section applies to links $\bar{\mathbf{L}}_{\mathfrak{t}',\mathfrak{s}}$ of $M_{\mathfrak{s}} \times \operatorname{Sym}^{\ell}(X)$ when $\ell(\mathfrak{t}',\mathfrak{s}) > 1$. The main difference is that the additional dilation parameters needed to describe a neighborhood of the strata $M_{\mathfrak{s}} \times \operatorname{Sym}^{\ell}(X)$ result in more complicated 'edges' in the boundaries of this neighborhood. However, for generic choices of dilation parameters, these edges will be smooth submanifolds of codimension-one or greater and the argument in Lemma 3.9 will show that the geometric representatives are disjoint from these edges.

3.9. **Orientations.** We now discuss the orientations of the moduli spaces and links appearing in the SO(3)-monopole cobordism — in particular links of level-one Seiberg-Witten moduli spaces — with respect to which the pairings are defined. Orientations for moduli spaces and links of top-level Seiberg-Witten moduli spaces were discussed in [18, §2].

An orientation for $\mathcal{M}_{\mathfrak{t}'}$ determines one for $\mathbf{L}_{\mathfrak{t}',\mathfrak{s}}$ through the convention introduced in [18, Equations (2.16), (2.16) & (2.25)] by considering $\mathbf{L}_{\mathfrak{t}',\mathfrak{s}}$ as a boundary of $(\mathcal{M}_{\mathfrak{t}'} - \boldsymbol{\gamma}(\mathcal{M}^{\text{vir}}_{\mathfrak{t}',\mathfrak{s}}))/S^1$. Specifically, at a point $[A,\Phi] \in \mathbf{L}_{\mathfrak{t}',\mathfrak{s}}$, if

- $\vec{r} \in T\mathcal{M}^{*,0}_{\mathfrak{t}'}$ is an outward-pointing radial vector with respect to the open neighborhood $\mathcal{M}_{\mathfrak{t}'} \cap \boldsymbol{\gamma}(\mathcal{M}^{\text{vir}}_{\mathfrak{t}',\mathfrak{s}})$ and complementary to the tangent space of $\mathbf{L}_{\mathfrak{t}',\mathfrak{s}}$,
- $v_{S^1} \in T\mathcal{M}^{*,0}_{\mathfrak{t}'}$ is tangent to the orbit of $[A,\Phi]$ under the (free) circle action (where $S^1 \subset \mathbb{C}$ has its usual orientation), and
- $\lambda_{\mathcal{M}} \in \det(T\mathcal{M}^{*,0}_{\mathfrak{t}'})$ is an orientation for $T\mathcal{M}_{\mathfrak{t}'}$ at $[A,\Phi]$,

then we define an orientation λ_L for $T\mathbf{L}_{\mathfrak{t}',\mathfrak{s}}$ at $[A,\Phi]$ by

$$\lambda_{\mathcal{M}} = -v_{S^1} \wedge \vec{r} \wedge \tilde{\lambda}_L, \tag{3.81}$$

where the lift $\tilde{\lambda}_L \in \Lambda^{\max-2}(T\mathcal{M}^{*,0}_{\mathfrak{t}'})$ at $[A,\Phi]$ of $\lambda_L \in \det(T\mathbf{L}_{\mathfrak{t}',\mathfrak{s}}) \subset \Lambda^{\max-1}(T(\mathcal{M}^{*,0}_{\mathfrak{t}'}/S^1))$, obeys $\pi_*\tilde{\lambda}_L = \lambda_L$, if $\pi : \mathcal{M}_{\mathfrak{t}'} \to \mathcal{M}_{\mathfrak{t}'}/S^1$ is the quotient map.

Definition 3.10. If O is an orientation for $\mathcal{M}_{\mathfrak{t}'}$, we call the orientation for $\mathbf{L}_{\mathfrak{t}',\mathfrak{s}}$ related to O by equation (3.81) the *boundary orientation defined by O*.

We note that the orientation convention (3.81) can be applied more generally to define an orientation for a quotient $\partial M/S^1$, given an orientation for a manifold with boundary M with a free circle action.

We now introduce an orientation for $\mathbf{L}_{\mathfrak{t}',\mathfrak{s}}$, based on a choice of orientation for the component $\mathbf{L}_{\mathfrak{t}',\mathfrak{s}}^{\mathrm{vir},i}$ (see (3.72)), which is useful for cohomological computations. First, observe that the gluing map $\boldsymbol{\gamma}$ identifies a relatively open subspace of $\mathbf{L}_{\mathfrak{t}',\mathfrak{s}}$ with the zero-locus of the obstruction section $\boldsymbol{\chi}$ in $\mathbf{L}_{\mathfrak{t}',\mathfrak{s}}^{\mathrm{vir},i}$. Because this section vanishes transversely, the normal bundle of the zero locus in $\mathbf{L}_{\mathfrak{t}',\mathfrak{s}}^{\mathrm{vir},i}$ is identified with the obstruction bundle, $\Upsilon_{\mathfrak{t}',\mathfrak{s}}/S^1$, which has a complex orientation. If this normal bundle is given the complex orientation, orientations of $\mathbf{L}_{\mathfrak{t}',\mathfrak{s}}$ are thus determined by orientations for $\mathbf{L}_{\mathfrak{t}',\mathfrak{s}}^{\mathrm{vir},i}$. From its definition (3.72), we see that $\mathbf{L}_{\mathfrak{t}',\mathfrak{s}}^{\mathrm{vir},i}$ can be viewed as a complex disk bundle:

$$\mathbf{L}_{\mathfrak{t}',\mathfrak{s}}^{\mathrm{vir},i} = \tilde{N}_{\mathfrak{t},\mathfrak{s}}(\varepsilon) \times_{\mathcal{G}_{\mathfrak{s}}\times S^1} \partial\bar{\mathrm{G}}\mathrm{l}_{\mathfrak{t}}(\delta) \to \tilde{M}_{\mathfrak{s}} \times_{\mathcal{G}_{\mathfrak{s}}\times S^1} \partial\bar{\mathrm{G}}\mathrm{l}_{\mathfrak{t}}(\delta). \tag{3.82}$$

The circle action in (3.82), described as the second circle action in Lemma 3.6, is trivial on $\tilde{M}_{\mathfrak{s}}$. Because the circle acts trivially on $\tilde{M}_{\mathfrak{s}}$ and $\mathcal{G}_{\mathfrak{s}}$ acts trivially on the quotient $\partial\bar{\mathrm{G}}\mathrm{l}_{\mathfrak{t}}(\delta)/S^1$, we can identify the base of the bundle (3.82) with the following product, a compact manifold:

$$\begin{aligned} \mathbf{BL}_{\mathfrak{t}',\mathfrak{s}}^{\mathrm{vir},i} &:= \tilde{M}_{\mathfrak{s}} \times_{\mathcal{G}_{\mathfrak{s}}} \partial\bar{\mathrm{G}}\mathrm{l}_{\mathfrak{t}}(\delta)/S^1 \\ &\cong M_{\mathfrak{s}} \times \partial\bar{\mathrm{G}}\mathrm{l}_{\mathfrak{t}}(\delta)/S^1. \end{aligned} \tag{3.83}$$

We may further assume without loss of generality that the point $[A,\Phi] \in \mathbf{L}_{\mathfrak{t}',\mathfrak{s}} \subset \mathcal{M}_{\mathfrak{t}'}/S^1$ — at which we choose to compare orientations — is identified, via the gluing map $\boldsymbol{\gamma} : \mathcal{M}_{\mathfrak{t}',\mathfrak{s}}^{\mathrm{vir}} \to \mathcal{C}_{\mathfrak{t}'}^{*,0}$, with a point

$$\boldsymbol{\gamma}^{-1}([A,\Phi]) = [[B,\Psi,0],[\mathbf{g}]],$$

in the base $\mathbf{BL}_{\mathfrak{t}',\mathfrak{s}}^{\mathrm{vir},i}$ of the bundle (3.82), where $[B,\Psi,0] \in M_{\mathfrak{s}} \subset N_{\mathfrak{t},\mathfrak{s}}(\varepsilon)$ and $\mathbf{g} \in \partial\bar{\mathrm{G}}\mathrm{l}_{\mathfrak{t}}(\delta)$. The commutative diagram (3.89) shows that the fiber of the disk bundle (3.82) over the point $[[B,\Psi,0],[\mathbf{g}]]$ is identified with the fiber of $N_{\mathfrak{t},\mathfrak{s}}(\varepsilon)$ over $[B,\Psi]$. We therefore have an isomorphism of tangent spaces at this point:

$$T_{\boldsymbol{\gamma}^{-1}([A,\Phi])}\mathbf{L}_{\mathfrak{t}',\mathfrak{s}}^{\mathrm{vir},i} \cong T_0(N_{\mathfrak{t},\mathfrak{s}}|_{[B,\Psi]}) \oplus T_{[B,\Psi]}M_{\mathfrak{s}} \oplus T_{[\mathbf{g}]}(\partial\bar{\mathrm{G}}\mathrm{l}_{\mathfrak{t}}(\delta)/S^1). \tag{3.84}$$

We define the *standard orientation* for $\partial\bar{\mathrm{G}}\mathrm{l}_{\mathfrak{t}}(\delta)/S^1$ by applying the convention (3.81) to an orientation for $\mathrm{Gl}_{\mathfrak{t}}(2\delta)$, considering $\partial\bar{\mathrm{G}}\mathrm{l}_{\mathfrak{t}}(\delta)/S^1$ as a boundary of $\mathrm{Gl}_{\mathfrak{t}}(2\delta)/S^1 - \mathrm{Gl}_{\mathfrak{t}}(\delta)/S^1$. From its definition, we see that the space $\mathrm{Gl}_{\mathfrak{t}}(2\delta)$ is a locally-trivial fiber-bundle over X with fiber $(0,2\delta)\times\mathrm{SO}(3)$, so the orientation for X and an orientation for $(0,2\delta)\times\mathrm{SO}(3)$ determine an orientation for $\mathrm{Gl}_{\mathfrak{t}}(2\delta)$. The standard orientation for $\partial\bar{\mathrm{G}}\mathrm{l}_{\mathfrak{t}}(\delta)/S^1$ will then be given through the convention (3.81) by taking the orientation of $\mathrm{Gl}_{\mathfrak{t}}(2\delta)$ induced by the orientation on $(0,2\delta)\times\mathrm{SO}(3)$ given by identifying its tangent spaces with the space of quaternions, $\mathbb{H}$, with the tangent spaces to $\mathrm{SO}(3)$ being identified with $\mathrm{Im}(\mathbb{H})$. (This agrees with the convention given in [11, §3(c) & §3(d)].)

Definition 3.11. The *standard orientation* for $\mathbf{L}_{\mathfrak{t}',\mathfrak{s}}^{\mathrm{vir},i}$ is determined, through the isomorphism (3.84), by the

- Standard orientation for $\partial\bar{\mathrm{G}}\mathrm{l}_{\mathfrak{t}}(\delta)/S^1$,
- Complex orientation of the fibers of the bundle (3.82), and the
- Orientation of $M_{\mathfrak{s}}$ induced by a homology orientation Ω [39, §6.6].

Definition 3.12. The *standard orientation* for $\mathbf{L}_{\mathfrak{t}',\mathfrak{s}}$ is determined by the

- Standard orientation for $\mathbf{L}^{\mathrm{vir},i}_{\mathfrak{t}',\mathfrak{s}}$,
- Complex orientation of the normal bundle of $\boldsymbol{\chi}^{-1}(0)\cap\mathbf{L}^{\mathrm{vir},i}_{\mathfrak{t}',\mathfrak{s}}$ in $\mathbf{L}^{\mathrm{vir},i}_{\mathfrak{t}',\mathfrak{s}}$, and the
- Diffeomorphism $\boldsymbol{\gamma}:\boldsymbol{\chi}^{-1}(0)\cap\mathcal{M}^{\mathrm{vir}}_{\mathfrak{t}',\mathfrak{s}}\to\mathcal{M}_{\mathfrak{t}'}\cap\boldsymbol{\gamma}(\mathcal{M}^{\mathrm{vir}}_{\mathfrak{t}',\mathfrak{s}})$.

We now relate the two orientations of $\mathbf{L}_{\mathfrak{t}',\mathfrak{s}}$ which we defined above:

Lemma 3.13. *Suppose that the spin^u structure $\mathfrak{t}$ admits a splitting $\mathfrak{t}=\mathfrak{s}\oplus\mathfrak{s}\otimes L$. Then the boundary orientation for $\mathbf{L}_{\mathfrak{t}',\mathfrak{s}}$ defined by the orientation $O^{\mathrm{asd}}(\Omega,c_1(L))$ for $\mathcal{M}_{\mathfrak{t}'}$ (see* [18, Definition 2.3]*) agrees with the standard orientation for $\mathbf{L}_{\mathfrak{t}',\mathfrak{s}}$.*

Proof. The standard orientation for $\mathbf{L}_{\mathfrak{t}',\mathfrak{s}}\subset\mathcal{M}_{\mathfrak{t}'}/S^1$ is defined, through the diffeomorphism

$$\boldsymbol{\gamma}:\boldsymbol{\chi}^{-1}(0)\cap\mathcal{M}^{\mathrm{vir}}_{\mathfrak{t}',\mathfrak{s}}\to\mathcal{M}_{\mathfrak{t}'}\cap\boldsymbol{\gamma}(\mathcal{M}^{\mathrm{vir}}_{\mathfrak{t}',\mathfrak{s}}),$$

by applying the convention (3.81) to the submanifold $\boldsymbol{\chi}^{-1}(0)\cap\mathcal{M}^{\mathrm{vir}}_{\mathfrak{t}',\mathfrak{s}}$ of $\mathcal{M}^{\mathrm{vir}}_{\mathfrak{t}',\mathfrak{s}}$, while the orientation $O^{\mathrm{asd}}(\Omega,c_1(L))$ is defined by applying it to $\mathcal{M}_{\mathfrak{t}'}$. Thus, to compare the two orientations of $\mathbf{L}_{\mathfrak{t}',\mathfrak{s}}$, it suffices to compare the orientations of $\boldsymbol{\chi}^{-1}(0)\cap\mathcal{M}^{\mathrm{vir}}_{\mathfrak{t}',\mathfrak{s}}$ and $\mathcal{M}_{\mathfrak{t}'}$ which induce these orientations of $\mathbf{L}_{\mathfrak{t}',\mathfrak{s}}$.

An orientation for $\mathcal{M}_{\mathfrak{t}'}$ is given by an orientation for the index bundle of the deformation operator $\mathcal{D}$ defined in [17, Equation (2.62)], whose kernel gives the tangent spaces of $\mathcal{M}^{*,0}_{\mathfrak{t}'}$. Suppose that the point $(A,\Phi)\in\tilde{\mathcal{M}}_{\mathfrak{t}'}$ at which we do the orientation comparisons is obtained by gluing a *framed* spin^u connection $\mathbf{A}\in\tilde{M}^{s,\natural}_1(S^4,\delta)$ onto the background pair $(B,\Psi)\in\tilde{M}_{\mathfrak{s}}$ at a point $x\in X$:

$$(A,\Phi)=(B,\Psi)\#(\mathbf{A},0). \tag{3.85}$$

Note that we abuse notation here and omit explicit mention of the frame for $\mathfrak{g}_{\mathbf{V}}|_s$. Then the excision argument [12, §7.1] yields the isomorphism

$$\det\mathcal{D}_{[A,\Phi]}\cong\det\mathcal{D}_{[B,\Psi]}\otimes\det\mathcal{D}_{[\mathbf{A},0]}. \tag{3.86}$$

There are isomorphisms (see [18, Equations (2.5), (2.11), and (2.26)]),

$$\begin{aligned}\det\mathcal{D}_{[B,\Psi]}&\cong\det(N_{\mathfrak{t},\mathfrak{s}}|_{[B,\Psi]})\otimes\det(T_{[B,\Psi]}M_{\mathfrak{s}})\otimes\det(\Xi_{\mathfrak{t},\mathfrak{s}}|_{[B,\Psi]})^*,\\ \det\mathcal{D}_{[\mathbf{A},0]}&\cong\det(T_{[\mathbf{A}]}M^s_1(S^4,\delta))\otimes\det(\operatorname{Coker}D_{\mathbf{A}})^*\\ &\cong\det(T_{[\mathbf{A}]}M^{s,\natural}_1(S^4,\delta))\otimes\det(T_xX)\otimes\det(\operatorname{Coker}D_{\mathbf{A}})^*,\end{aligned} \tag{3.87}$$

where in the last isomorphism we have used the identification $M^s_1(S^4,\delta)\cong M^{s,\natural}_1(S^4,\delta)\times\mathbb{R}^4$ and the identification of a ball in $\mathbb{R}^4$ with a ball around x in X via the gluing map. Lemma 2.6 in [18] implies that the orientation $O^{\mathrm{asd}}(\Omega,c_1(L))$ of $\det\mathcal{D}_{[A,\Phi]}$ is given through the isomorphisms (3.86) and (3.87) by the complex orientations of $N_{\mathfrak{t},\mathfrak{s}}|_{[B,\Psi]}$, $(\Xi_{\mathfrak{t},\mathfrak{s}}|_{[B,\Psi]})^*$, and $(\operatorname{Coker}D_{\mathbf{A}})^*$, the orientation of $T_{[B,\Psi]}M_{\mathfrak{s}}$ determined by the homology orientation Ω [39, §6.6], the standard orientation of $T_{[\mathbf{A}]}M^{s,\natural}_1(S^4,\delta)$ defined in [11, p. 413], and the orientation for T_xX given by that of X.

We now describe the orientation of $\boldsymbol{\chi}^{-1}(0)\cap\mathcal{M}^{\mathrm{vir}}_{\mathfrak{t}',\mathfrak{s}}$ inducing the standard orientation of $\mathbf{L}_{\mathfrak{t}',\mathfrak{s}}$. First, the orientation of $\boldsymbol{\chi}^{-1}(0)\cap\mathcal{M}^{\mathrm{vir}}_{\mathfrak{t}',\mathfrak{s}}$ is induced by one for $\mathcal{M}^{\mathrm{vir}}_{\mathfrak{t}',\mathfrak{s}}$ by identifying the normal bundle of $\boldsymbol{\chi}^{-1}(0)\cap\mathcal{M}^{\mathrm{vir}}_{\mathfrak{t}',\mathfrak{s}}$ in $\mathcal{M}^{\mathrm{vir}}_{\mathfrak{t}',\mathfrak{s}}$ with the obstruction bundle $\Upsilon_{\mathfrak{t}',\mathfrak{s}}$ and using the complex orientation of this obstruction bundle. With $[A,\Phi]$ as in (3.85), then by (3.61), (3.63), and (3.68) there is an isomorphism of obstruction bundle fibers,

$$\Upsilon_{\mathfrak{t}',\mathfrak{s}}|_{\gamma^{-1}([A,\Phi])}\cong\operatorname{Coker}D_{\mathbf{A}}\oplus\Xi_{\mathfrak{t},\mathfrak{s}}|_{[B,\Psi]}. \tag{3.88}$$

Hence, the complex orientation of the obstruction bundle matches the complex orientation of the factors $\operatorname{Coker} D_{\mathbf{A}}$ and $\Xi_{\mathfrak{t},\mathfrak{s}}|_{[B,\Psi]}$ in (3.87).

We now compare the tangent space $T_{\boldsymbol{\gamma}^{-1}([A,\Phi])}\mathcal{M}^{\mathrm{vir}}_{\mathfrak{t}',\mathfrak{s}}$ with the remaining factors on the right-hand-sides of the isomorphisms (3.87). The moduli space $\mathcal{M}^{\mathrm{vir}}_{\mathfrak{t}',\mathfrak{s}}$ can also be written as the disk bundle appearing on the left-hand-side of the following diagram:

$$\begin{tikzcd} \tilde{N}_{\mathfrak{t},\mathfrak{s}}(\varepsilon) \times_{\mathcal{G}_{\mathfrak{s}}} \mathrm{Gl}_{\mathfrak{t}}(\delta) \arrow[r] \arrow[d, "\pi_{N,\mathrm{Gl}}"'] & N_{\mathfrak{t},\mathfrak{s}}(\varepsilon) \times X \arrow[d, "\pi_{N,X}"'] \\ \tilde{M}_{\mathfrak{s}} \times_{\mathcal{G}_{\mathfrak{s}}} \mathrm{Gl}_{\mathfrak{t}}(\delta) \arrow[r, "\pi_{\mathrm{Gl}}"] & M_{\mathfrak{s}} \times X \end{tikzcd} \tag{3.89}$$

We further suppose, without loss of generality, that $[A,\Phi] \in \mathbf{L}_{\mathfrak{t}',\mathfrak{s}}$ corresponds, via the gluing map $\boldsymbol{\gamma}$, to a point $\boldsymbol{\gamma}^{-1}([A,\Phi])$ in the base of the bundle on the left-hand-side of the diagram (3.89), so

$$\pi_{\mathrm{Gl}}(\boldsymbol{\gamma}^{-1}([A,\Phi])) = ([B,\Psi],x).$$

The fiber of the projection π_{Gl} in diagram (3.89) is $M_1^{s,\natural}(S^4,\delta)$, while the same diagram identifies the fiber of the projection $\pi_{N,\mathrm{Gl}}$ with the fiber $N_{\mathfrak{t},\mathfrak{s}}(\varepsilon)|_{[B,\Psi]}$ of the projection $\pi_{N,X}$. Thus,

$$\begin{aligned} T_{\boldsymbol{\gamma}^{-1}([A,\Phi])}\mathcal{M}^{\mathrm{vir}}_{\mathfrak{t}',\mathfrak{s}} &\cong N_{\mathfrak{t},\mathfrak{s}}|_{[B,\Psi]} \oplus T_{\boldsymbol{\gamma}^{-1}([A,\Phi])}(\tilde{M}_{\mathfrak{s}} \times_{\mathcal{G}_{\mathfrak{s}}} \mathrm{Gl}_{\mathfrak{t}}(\delta)) \\ &\cong N_{\mathfrak{t},\mathfrak{s}}|_{[B,\Psi]} \oplus T_{[B,\Psi]}M_{\mathfrak{s}} \oplus T_{[\mathbf{A}]}M_1^{s,\natural}(S^4,\delta) \oplus T_xX. \end{aligned} \tag{3.90}$$

If we compare the isomorphisms (3.90) with (3.84), we see that the standard orientation of $\mathbf{L}_{\mathfrak{t}',\mathfrak{s}}$ is induced, through the convention (3.81), by the orientation of $\boldsymbol{\chi}^{-1}(0) \cap \mathcal{M}^{\mathrm{vir}}_{\mathfrak{t}',\mathfrak{s}}$ given by the complex orientation of the normal bundle of $\boldsymbol{\chi}^{-1}(0) \cap \mathcal{M}^{\mathrm{vir}}_{\mathfrak{t}',\mathfrak{s}}$ and the orientation of $\mathcal{M}^{\mathrm{vir}}_{\mathfrak{t}',\mathfrak{s}}$ defined through the isomorphism (3.90) by the complex orientation of $N_{\mathfrak{t},\mathfrak{s}}|_{[B,\Psi]}$, the orientation of $T_{[B,\Psi]}M_{\mathfrak{s}}$ determined by the homology orientation Ω, the standard orientation of $T_{[\mathbf{A}]}M_1^{s,\natural}(S^4,\delta)$, and the orientation of T_xX. This orientation matches the orientation $O^{\mathrm{asd}}(\Omega,c_1(L))$ as described in the paragraph following (3.87). This completes the proof. □

We shall work with a fixed orientation $O^{\mathrm{asd}}(\Omega,w)$ of $\mathcal{M}_{\mathfrak{t}'}$ in the sum (2.52) and thus we include the following lemma on how the orientations change as the spin^c structure $\mathfrak{s}$ varies.

Lemma 3.14. *If $\mathfrak{t}$ is a spin^u structure on X, let $\mathfrak{t}'$ be the spliced spin^u structure (3.38). Let Ω be a homology orientation and let w be an integral lift of $w_2(\mathfrak{t}')$. If $\mathfrak{t}$ admits a splitting $\mathfrak{t} = \mathfrak{s} \oplus \mathfrak{s} \otimes L$, then the standard orientation for $\mathbf{L}_{\mathfrak{t}',\mathfrak{s}}$ and the boundary orientation for $\mathbf{L}_{\mathfrak{t}',\mathfrak{s}}$ defined through the orientation $O^{\mathrm{asd}}(\Omega,w)$ for $\mathcal{M}_{\mathfrak{t}'}$ differ by a factor of*

$$(-1)^{o_{\mathfrak{t}'}(w,\mathfrak{s})}, \quad \textit{where} \quad o_{\mathfrak{t}'}(w,\mathfrak{s}) = \tfrac{1}{4}(w - c_1(L))^2. \tag{3.91}$$

Proof. The result follows from Lemma 3.13 and the identity

$$O^{\mathrm{asd}}(\Omega,w) = (-1)^{\frac{1}{4}(w-c_1(L))^2} O^{\mathrm{asd}}(\Omega,c_1(L))$$

given in [18, Lemma 2.6]. □

4. Pullback and extension of cohomology classes. Euler classes of obstruction bundles

In §4.1 we compute the pullbacks of the cohomology classes $\mu_p(\beta)$ and μ_c to $\mathcal{M}^{\mathrm{vir}}_{\mathfrak{t}',\mathfrak{s}}/S^1$ with respect to the gluing map $\boldsymbol{\gamma} : \mathcal{M}^{\mathrm{vir}}_{\mathfrak{t}',\mathfrak{s}}/S^1 \to \mathcal{C}^{*,0}_{\mathfrak{t}'}/S^1$. When no confusion can arise, we denote the pullbacks $\boldsymbol{\gamma}^*\mu_p(\beta)$ and $\boldsymbol{\gamma}^*\mu_c$ by $\mu_p(\beta)$ and μ_c, respectively. We then observe that these pullbacks are the restrictions of cohomology classes $\bar{\mu}_p(\beta)$ and $\bar{\mu}_c$ on $\bar{\mathcal{M}}^{\mathrm{vir},*}_{\mathfrak{t}',\mathfrak{s}}/S^1$, where $\bar{\mathcal{M}}^{\mathrm{vir},*}_{\mathfrak{t}',\mathfrak{s}}$ is defined in (4.1). In §4.2 we calculate Euler classes of the obstruction bundle (3.59) over $\mathcal{M}^{\mathrm{vir}}_{\mathfrak{t}',\mathfrak{s}}/S^1$. We will see that the background component (3.63) of the obstruction bundle (3.59) (and hence its Euler class) extends over $\bar{\mathcal{M}}^{\mathrm{vir},*}_{\mathfrak{t}',\mathfrak{s}}/S^1$ while the Euler class of the instanton component (3.69) extends over $\bar{\mathcal{M}}^{\mathrm{vir},*}_{\mathfrak{t}',\mathfrak{s}}/S^1$ as a rational cohomology class. Although we only perform the calculations relevant to the stratum

$$(M_{\mathfrak{s}} \times \mathrm{Sym}^{\ell}(X)) \cap \bar{\mathcal{M}}_{\mathfrak{t}'} \subset \bar{\mathcal{M}}_{\mathfrak{t}'}$$

when $\ell(\mathfrak{t}',\mathfrak{s}) = 1$ in the present article, we shall usually indicate the nature of the changes required to address the general case $\ell \geq 1$.

4.1. **Pullbacks of cohomology classes.** By analogy with our definition (following (2.13)) of $\bar{\mathcal{M}}^*_{\mathfrak{t}}$ as the subspace of $\bar{\mathcal{M}}_{\mathfrak{t}}$ represented by ideal SO(3) monopoles with reducible associated SO(3) connections, we define

$$\bar{\mathcal{M}}^{\mathrm{vir},*}_{\mathfrak{t}',\mathfrak{s}} = \bar{\mathcal{M}}^{\mathrm{vir}}_{\mathfrak{t}',\mathfrak{s}} - (M_{\mathfrak{s}} \times X), \tag{4.1}$$

together with an inclusion map

$$\iota : \mathcal{M}^{\mathrm{vir}}_{\mathfrak{t}',\mathfrak{s}}/S^1 \to \bar{\mathcal{M}}^{\mathrm{vir},*}_{\mathfrak{t}',\mathfrak{s}}/S^1. \tag{4.2}$$

The circle action on $\bar{\mathcal{M}}^{\mathrm{vir}}_{\mathfrak{t}',\mathfrak{s}}$, given in Lemma 3.6, is free on $\bar{\mathcal{M}}^{\mathrm{vir},*}_{\mathfrak{t}',\mathfrak{s}}$ and trivial on $M_{\mathfrak{s}} \times X$.

Definition 4.1. Let ν be the first Chern class of the circle bundle

$$\bar{\mathcal{M}}^{\mathrm{vir},*}_{\mathfrak{t}',\mathfrak{s}} \to \bar{\mathcal{M}}^{\mathrm{vir},*}_{\mathfrak{t}',\mathfrak{s}}/S^1, \tag{4.3}$$

where the circle acts diagonally on $\bar{\mathcal{M}}^{\mathrm{vir}}_{\mathfrak{t}',\mathfrak{s}} = \tilde{N}_{\mathfrak{t},\mathfrak{s}}(\varepsilon) \times_{\mathcal{G}_{\mathfrak{s}}} \bar{\mathrm{Gl}}_{\mathfrak{t}}(\delta)$, acting on $\tilde{N}_{\mathfrak{t},\mathfrak{s}}(\varepsilon)$ by scalar multiplication on the fibers and on $\bar{\mathrm{Gl}}_{\mathfrak{t}}(\delta)$ by the action described before equation (3.56).

Recall that $\mu_c \in H^2(\mathcal{C}^{*,0}_{\mathfrak{t}'}/S^1;\mathbb{Z})$ is the first Chern class (2.44) of the line bundle $\mathbb{L}_{\mathfrak{t}'} \to \mathcal{C}^{*,0}_{\mathfrak{t}'}/S^1$ defined in (2.42); we now identify its pullback to $\mathcal{M}^{\mathrm{vir}}_{\mathfrak{t}',\mathfrak{s}}/S^1$.

Lemma 4.2. *Let μ_c be as in the preceding paragraph, let ν be as in Definition 4.1, and let ι be the inclusion* (4.2). *Then*

$$\boldsymbol{\gamma}^*\mu_c = -\iota^*\nu.$$

Proof. From (2.44) and (2.42), the cohomology class μ_c is the first Chern class of

$$\mathbb{L}_{\mathfrak{t}'} = \mathcal{C}^{*,0}_{\mathfrak{t}'} \times_{(S^1,\times -2)} \mathbb{C}.$$

By Lemma 3.6, the gluing map $\boldsymbol{\gamma} : \mathcal{M}^{\mathrm{vir}}_{\mathfrak{t}',\mathfrak{s}} \to \mathcal{C}^{*,0}_{\mathfrak{t}'}$ is circle-equivariant when the circle acts on $\mathcal{C}^{*,0}_{\mathfrak{t}'}$ by the action (2.8) and on $\mathcal{M}^{\mathrm{vir}}_{\mathfrak{t}',\mathfrak{s}}$ by the action in Definition 4.1, but with multiplicity two. Thus,

$$\boldsymbol{\gamma}^*\mathbb{L}_{\mathfrak{t}'} = \mathcal{M}^{\mathrm{vir}}_{\mathfrak{t}',\mathfrak{s}} \times_{(S^1,-1)} \mathbb{C},$$

and the conclusion follows. □

Lemma 4.2 and its proof translate easily to the case $\ell \geq 1$.

Next, we identify the pullbacks to $\mathcal{M}^{\mathrm{vir}}_{\mathfrak{t}',\mathfrak{s}}$ of the cohomology classes $\mu_p(\beta)$ on $\mathcal{C}^{*,0}_{\mathfrak{t}'}$. Let

$$\pi_{\mathfrak{s}} : \bar{\mathcal{M}}^{\mathrm{vir},*}_{\mathfrak{t}',\mathfrak{s}} \to M_{\mathfrak{s}} \quad \text{and} \quad \pi_X : \bar{\mathcal{M}}^{\mathrm{vir},*}_{\mathfrak{t}',\mathfrak{s}} \to X \tag{4.4}$$

denote the restrictions of the components of the projection $\bar{\mathcal{M}}^{\text{vir}}_{\mathfrak{t}',\mathfrak{s}} \to M_{\mathfrak{s}} \times X$ given in (3.58) to $\bar{\mathcal{M}}^{\text{vir},*}_{\mathfrak{t}',\mathfrak{s}}$. We define some additional projections:

$$
\begin{aligned}
\pi_{\mathcal{M}} &: \bar{\mathcal{M}}^{\text{vir},*}_{\mathfrak{t}',\mathfrak{s}} \times X \to \bar{\mathcal{M}}^{\text{vir},*}_{\mathfrak{t}',\mathfrak{s}}, \\
\pi_{X,2} &: \bar{\mathcal{M}}^{\text{vir},*}_{\mathfrak{t}',\mathfrak{s}} \times X \to X, \\
\pi_{\mathfrak{s},1} = \pi_{\mathfrak{s}} \circ \pi_{\mathcal{M}} &: \bar{\mathcal{M}}^{\text{vir},*}_{\mathfrak{t}',\mathfrak{s}} \times X \to M_{\mathfrak{s}},
\end{aligned}
\tag{4.5}
$$

We shall use the same notation for the projections when the space $\bar{\mathcal{M}}^{\text{vir},*}_{\mathfrak{t}',\mathfrak{s}}$ above is replaced, for example, by its circle quotient, $\bar{\mathcal{M}}^{\text{vir},*}_{\mathfrak{t}',\mathfrak{s}}/S^1$.

Recall from definition (2.39) that

$$\mathbb{F}_{\mathfrak{t}'} = \tilde{\mathcal{C}}^*_{\mathfrak{t}'}/S^1 \times_{\mathcal{G}_{\mathfrak{t}'}} \mathfrak{g}_{V'}$$

is a universal SO(3) bundle over $\mathcal{C}^*_{\mathfrak{t}'}/S^1 \times X$. We now define an analogous SO(3) bundle over $\mathcal{M}^{\text{vir}}_{\mathfrak{t}',\mathfrak{s}}/S^1 \times X$. Using 'cl' for convenience here to indicate the 'Uhlenbeck compactification' implicit when $\text{Gl}_{\mathfrak{t}}(\delta)$ is replaced by $\bar{\text{Gl}}_{\mathfrak{t}}(\delta)$, we denote

$$\text{cl}\tilde{\mathcal{M}}^{\text{vir},*}_{\mathfrak{t}',\mathfrak{s}} = \tilde{N}_{\mathfrak{t},\mathfrak{s}}(\varepsilon) \times \bar{\text{Gl}}_{\mathfrak{t}}(\delta) - \left(\tilde{M}_{\mathfrak{s}} \times X\right), \tag{4.6}$$

so that $\bar{\mathcal{M}}^{\text{vir},*}_{\mathfrak{t}',\mathfrak{s}} = \text{cl}\tilde{\mathcal{M}}^{\text{vir},*}_{\mathfrak{t}',\mathfrak{s}}/\mathcal{G}_{\mathfrak{s}}$, where $\mathcal{G}_{\mathfrak{s}}$ acts diagonally on $\tilde{N}_{\mathfrak{t},\mathfrak{s}}(\varepsilon) \times \bar{\text{Gl}}_{\mathfrak{t}}(\delta)$. We set

$$\bar{\mathbb{F}}^{\text{vir},*}_{\mathfrak{t}',\mathfrak{s}} = \text{cl}\tilde{\mathcal{M}}^{\text{vir},*}_{\mathfrak{t}',\mathfrak{s}} \times_{\mathcal{G}_{\mathfrak{s}} \times S^1} (\underline{\mathbb{R}} \oplus L) \to \bar{\mathcal{M}}^{\text{vir},*}_{\mathfrak{t}',\mathfrak{s}}/S^1 \times X, \tag{4.7}$$

where $\mathcal{G}_{\mathfrak{s}}$ acts diagonally on $\text{cl}\tilde{\mathcal{M}}^{\text{vir},*}_{\mathfrak{t}',\mathfrak{s}} \times \mathfrak{g}_V$ — acting on $\mathfrak{g}_V \cong \underline{\mathbb{R}} \oplus L$ by multiplication with weight negative two on L and on $\text{cl}\tilde{\mathcal{M}}^{\text{vir},*}_{\mathfrak{t}',\mathfrak{s}}$ by the action described above; the circle acts diagonally on $\text{cl}\tilde{\mathcal{M}}^{\text{vir},*}_{\mathfrak{t}',\mathfrak{s}} \times \mathfrak{g}_V$ — by scalar multiplication on $\tilde{N}_{\mathfrak{t},\mathfrak{s}}(\varepsilon)$, by the action on $\bar{\text{Gl}}_{\mathfrak{t}}(\delta)$ described before equation (3.56), and on $\mathfrak{g}_V$ by scalar multiplication with weight one on L.

Lemma 4.3 below compares the restriction of the bundle $\bar{\mathbb{F}}^{\text{vir},*}_{\mathfrak{t}',\mathfrak{s}}$ with the restriction of the pulled-back SO(3) bundle $(\boldsymbol{\gamma} \times \text{id}_X)^*\mathbb{F}_{\mathfrak{t}'}$ to the complement of $(\pi_X \times \text{id}_X)^{-1}(\Delta)$ in $\mathcal{M}^{\text{vir}}_{\mathfrak{t}',\mathfrak{s}}/S^1 \times X$, where $\Delta \subset X \times X$ is the diagonal. We restrict the pullback of the universal bundle to the complement of the subspace $(\pi_X \times \text{id}_X)^{-1}(\Delta)$ because the splicing process at a point $x \in X$ only identifies the restrictions of the bundles $\mathfrak{g}_V$ and $\mathfrak{g}_{V'}$ to $X \setminus \{x\}$.

Lemma 4.3. *Suppose the spin^u structure $\mathfrak{t} = (\rho, V)$ admits a splitting $\mathfrak{t} = \mathfrak{s} \oplus \mathfrak{s} \otimes L$, so that $\mathfrak{g}_V \cong \underline{\mathbb{R}} \oplus L$, where $\underline{\mathbb{R}} = X \times \mathbb{R}$. If $\mathcal{O}$ denotes the complement of $(\pi_X \times \text{id}_X)^{-1}(\Delta)$ in $\mathcal{M}^{\text{vir}}_{\mathfrak{t}',\mathfrak{s}}/S^1 \times X$, then there is an isomorphism of* SO(3) *bundles:*

$$(\boldsymbol{\gamma} \times \text{id}_X)^*\mathbb{F}_{\mathfrak{t}'}|_{\mathcal{O}} \cong \bar{\mathbb{F}}^{\text{vir},*}_{\mathfrak{t}',\mathfrak{s}}\Big|_{\mathcal{O}}.$$

Proof. Let $\mathfrak{t}' = (\rho, V')$ be the spin^u structure (3.38) obtained by splicing, over a neighborhood of $x \in X$, the spin^u structure $\mathfrak{t} = (\rho, V)$ over X with the spin^u structure $(\rho, \mathbf{V})$ over S^4, where $-\frac{1}{4}p_1(\mathfrak{g}_{\mathbf{V}}) = 1$. We thus obtain an associated SO(3) bundle $\mathfrak{g}_{V'}$ as in (3.40) and a bundle isomorphism,

$$\iota_{V,V'} : \mathfrak{g}_V|_{X \setminus \{x\}} \to \mathfrak{g}_{V'}|_{X \setminus \{x\}}.$$

Hence, if $\tilde{\mathcal{O}} \subset \tilde{N}_{\mathfrak{t},\mathfrak{s}}(\varepsilon) \times \tilde{\text{Gl}}_{\mathfrak{t}}(\delta) \times X$ is the pre-image of $\mathcal{O}$ under the obvious projection, we have a bundle map

$$\tilde{\boldsymbol{\gamma}}' \times \iota_{V,V'} : \left(\tilde{N}_{\mathfrak{t},\mathfrak{s}}(\varepsilon) \times \tilde{\text{Gl}}_{\mathfrak{t}}(\delta) \times \mathfrak{g}_V\right)\Big|_{\tilde{\mathcal{O}}} \to \tilde{\mathcal{C}}^*_{\mathfrak{t}'} \times \mathfrak{g}_{V'}, \tag{4.8}$$

where $\tilde{\boldsymbol{\gamma}}'$ is the pre-splicing map (3.50). The map $\tilde{\boldsymbol{\gamma}}'$ is gauge equivariant with respect to the action of $\mathcal{G}_{\mathfrak{s}} \times \mathcal{G}_1$ on the domain and the action of $\mathcal{G}_{\mathfrak{t}'}$ on the range. The group $\mathcal{G}_1$ of gauge transformations over S^4 (see §3.3) acts trivially on the restriction of $\mathfrak{g}_V$ to the complement of the splicing point $x \in X$. From definitions (3.51) and (4.6) we have

$$\tilde{N}_{\mathfrak{t},\mathfrak{s}}(\varepsilon) \times \tilde{\mathrm{Gl}}_{\mathfrak{t}}(\delta)/\mathcal{G}_1 = \iota^* \mathrm{cl}\tilde{\mathcal{M}}^{\mathrm{vir},*}_{\mathfrak{t}',\mathfrak{s}},$$

(where ι is the inclusion in (4.2)) and so the bundle map (4.8) descends to a bundle map on gauge-group quotients,

$$\boldsymbol{\gamma}' \times \iota_{V,V'} : \left(\mathrm{cl}\tilde{\mathcal{M}}^{\mathrm{vir},*}_{\mathfrak{t}',\mathfrak{s}} \times_{\mathcal{G}_{\mathfrak{s}}} \mathfrak{g}_V\right)\Big|_{\mathcal{O}} \to \tilde{\mathcal{C}}^*_{\mathfrak{t}'} \times_{\mathcal{G}_{\mathfrak{t}'}} \mathfrak{g}_{V'}. \tag{4.9}$$

By Lemma 3.6, the bundle map (4.9) is circle-equivariant with respect to the circle action on the domain induced by the action (2.37) on the factor $\tilde{N}_{\mathfrak{t},\mathfrak{s}}(\varepsilon)$ in $\mathrm{cl}\tilde{\mathcal{M}}^{\mathrm{vir},*}_{\mathfrak{t}',\mathfrak{s}}$ together with the trivial action on $\mathrm{Gl}_{\mathfrak{t}}(\delta)$ and the circle action on $\tilde{\mathcal{C}}^{*,0}_{\mathfrak{t}'}$ induced by (2.8). The bundle map (4.9) thus descends to a bundle map on circle quotients,

$$\boldsymbol{\gamma}' \times \iota_{V,V'} : \left(\mathrm{cl}\tilde{\mathcal{M}}^{\mathrm{vir},*}_{\mathfrak{t}',\mathfrak{s}}/S^1 \times_{\mathcal{G}_{\mathfrak{s}}} \mathfrak{g}_V\right)\Big|_{\mathcal{O}} \to \mathbb{F}_{\mathfrak{t}'}. \tag{4.10}$$

The argument in the proof of Lemma 3.6 which shows that the two circle actions on $\mathcal{M}^{\mathrm{vir}}_{\mathfrak{t}',\mathfrak{s}}$ (described in its hypothesis) are equal then implies that the circle action on $\mathrm{cl}\tilde{\mathcal{M}}^{\mathrm{vir},*}_{\mathfrak{t}',\mathfrak{s}} \times \mathfrak{g}_V$ induced by the circle action (2.37) on the factor $\tilde{N}_{\mathfrak{t},\mathfrak{s}}(\varepsilon)$ in $\mathrm{cl}\tilde{\mathcal{M}}^{\mathrm{vir},*}_{\mathfrak{t}',\mathfrak{s}} \times \mathfrak{g}_V$ and trivial actions on $\mathrm{Gl}_{\mathfrak{t}}(\delta)$ and on $\mathfrak{g}_V$ is equal to the twice the circle action described following (4.7). The multiplicity of this action does not affect the quotient, so the bundle given as the domain of the map (4.10) is isomorphic to $\bar{\mathbb{F}}^{\mathrm{vir},*}_{\mathfrak{t}',\mathfrak{s}}$ and hence the restrictions of $(\boldsymbol{\gamma}' \times \mathrm{id}_X)^*\mathbb{F}_{\mathfrak{t}'}$ and $\bar{\mathbb{F}}^{\mathrm{vir},*}_{\mathfrak{t}',\mathfrak{s}}$ to $\mathcal{O}$ are isomorphic. Finally, because the gluing map $\boldsymbol{\gamma}$ and the splicing map $\boldsymbol{\gamma}'$ are S^1-equivariantly homotopic, there is a bundle isomorphism $(\boldsymbol{\gamma} \times \mathrm{id}_X)^*\mathbb{F}_{\mathfrak{t}'} \cong (\boldsymbol{\gamma}' \times \mathrm{id}_X)^*\mathbb{F}_{\mathfrak{t}'}$. This completes the proof of the lemma. □

Since $\mu_p(\beta) = -\frac{1}{4}p_1(\mathbb{F}_{\mathfrak{t}'})/\beta$, our next task is to compare the Pontrjagin classes of the bundles $\iota^*\bar{\mathbb{F}}^{\mathrm{vir},*}_{\mathfrak{t}',\mathfrak{s}}$ and $(\boldsymbol{\gamma} \times \mathrm{id}_X)^*\mathbb{F}_{\mathfrak{t}'}$ over $\mathcal{M}^{\mathrm{vir}}_{\mathfrak{t}',\mathfrak{s}} \times X$:

Lemma 4.4. *Continue the hypotheses of Lemma 4.3. Let* $\mathrm{PD}[\Delta]$ *be the Poincaré dual of the diagonal* $\Delta \subset X \times X$. *Then, on* $\mathcal{M}^{\mathrm{vir}}_{\mathfrak{t}',\mathfrak{s}} \times X$, *we have*

$$(\boldsymbol{\gamma} \times \mathrm{id}_X)^* p_1(\mathbb{F}_{\mathfrak{t}'}) = (\iota \times \mathrm{id}_X)^* \left(p_1(\bar{\mathbb{F}}^{\mathrm{vir},*}_{\mathfrak{t}',\mathfrak{s}}) - 4(\pi_X \times \mathrm{id}_X)^* \mathrm{PD}[\Delta]\right), \tag{4.11}$$

where ι *is the inclusion* (4.2).

Proof. The restriction of the difference,

$$(\boldsymbol{\gamma} \times \mathrm{id}_X)^* p_1(\mathbb{F}_{\mathfrak{t}'}) - (\iota \times \mathrm{id}_X)^* p_1(\bar{\mathbb{F}}^{\mathrm{vir},*}_{\mathfrak{t}',\mathfrak{s}}), \tag{4.12}$$

to the subspace $\mathcal{O}$ defined in Lemma 4.3 vanishes by that lemma. Therefore, by considering the exact sequence of the pair,

$$\left(\mathcal{M}^{\mathrm{vir}}_{\mathfrak{t}',\mathfrak{s}}/S^1 \times X, \mathcal{O}\right), \tag{4.13}$$

we see that the difference (4.12) lies in the image of the homomorphism

$$H^4\left(\mathcal{M}^{\mathrm{vir}}_{\mathfrak{t}',\mathfrak{s}}/S^1 \times X, \mathcal{O}; \mathbb{R}\right) \to H^4\left(\mathcal{M}^{\mathrm{vir}}_{\mathfrak{t}',\mathfrak{s}}/S^1 \times X; \mathbb{R}\right) \tag{4.14}$$

appearing in the exact sequence of the pair (4.13). Because the map $\pi_X \times \mathrm{id}_X$ is transverse to Δ, the image of the homomorphism (4.14) (see [5, p. 69] and [10, Proposition VIII.11.10]) is generated by $(\pi_X \times \mathrm{id}_X)^* \mathrm{PD}[\Delta]$. The difference (4.12) is therefore a multiple of $(\pi_X \times$

$\mathrm{id}_X)^*\,\mathrm{PD}[\Delta]$. One can calculate this multiple by evaluating the difference (4.12) on a chain which intersects $(\pi_X \times \mathrm{id}_X)^{-1}(\Delta)$ transversely at a single point. Such computations are carried out in the proof of [23, Theorem III.6.1] and in [12, §5.3], giving equation (4.11) $\square$

We see from (4.7) that the cohomology class on the right-hand side of (4.11) is defined on $\bar{\mathcal{M}}^{\mathrm{vir},*}_{\mathfrak{t}',\mathfrak{s}}/S^1\times X$ and thus extends the class $(\boldsymbol{\gamma}\times\mathrm{id}_X)^*p_1(\mathbb{F}_{\mathfrak{t}'})$, which is defined on $\mathcal{M}^{\mathrm{vir}}_{\mathfrak{t}',\mathfrak{s}}/S^1\times X$.

Definition 4.5. We define extensions of cohomology classes from $\mathcal{M}^{\mathrm{vir}}_{\mathfrak{t}',\mathfrak{s}}/S^1$ to $\bar{\mathcal{M}}^{\mathrm{vir},*}_{\mathfrak{t}',\mathfrak{s}}/S^1$:

- Let $\bar{\mu}_c = -\nu$ be the extension of μ_c given by Lemma 4.2.
- For $\beta \in H_\bullet(X;\mathbb{R})$, let $\bar{\mu}_p(\beta)$ be the extension of (the pullback by $\boldsymbol{\gamma}$ of) the cohomology class $\mu_p(\beta) = -\frac{1}{4}p_1(\mathbb{F}_{\mathfrak{t}'})/\beta$ given by replacing $(\boldsymbol{\gamma}\times\mathrm{id}_X)^*p_1(\mathbb{F}_{\mathfrak{t}'})$ with the cohomology class on the right-hand side of (4.11).

In order to use equation (4.11) to compute $(\boldsymbol{\gamma}\times\mathrm{id}_X)^*p_1(\mathbb{F}_{\mathfrak{t}'})$, we must identify the Pontrjagin class $p_1(\bar{\mathbb{F}}^{\mathrm{vir},*}_{\mathfrak{t}',\mathfrak{s}})$. The bundle $\bar{\mathbb{F}}^{\mathrm{vir},*}_{\mathfrak{t}',\mathfrak{s}}$ admits a reduction $\bar{\mathbb{F}}^{\mathrm{vir},*}_{\mathfrak{t}',\mathfrak{s}} \cong \underline{\mathbb{R}} \oplus \bar{\mathbb{L}}^{\mathrm{vir},*}_{\mathfrak{t}',\mathfrak{s}}$, where $\underline{\mathbb{R}} = \bar{\mathcal{M}}^{\mathrm{vir},*}_{\mathfrak{t}',\mathfrak{s}}/S^1\times X\times\mathbb{R}$ and

$$\bar{\mathbb{L}}^{\mathrm{vir},*}_{\mathfrak{t}',\mathfrak{s}} = \mathrm{cl}\tilde{\mathcal{M}}^{\mathrm{vir},*}_{\mathfrak{t}',\mathfrak{s}} \times_{\mathcal{G}_\mathfrak{s}\times S^1} L \to \bar{\mathcal{M}}^{\mathrm{vir},*}_{\mathfrak{t}',\mathfrak{s}}/S^1\times X. \tag{4.15}$$

The actions of S^1 and $\mathcal{G}_\mathfrak{s}$ in the definition of the line bundle $\bar{\mathbb{L}}^{\mathrm{vir},*}_{\mathfrak{t}',\mathfrak{s}}$ are described in Lemma 4.3. Since $p_1(\bar{\mathbb{F}}^{\mathrm{vir},*}_{\mathfrak{t}',\mathfrak{s}}) = c_1(\bar{\mathbb{L}}^{\mathrm{vir},*}_{\mathfrak{t}',\mathfrak{s}})^2$, it suffices to compute $c_1(\bar{\mathbb{L}}^{\mathrm{vir},*}_{\mathfrak{t}',\mathfrak{s}})$ and for this purpose we shall use the following technical tool:

Lemma 4.6. [17, Lemma 3.27] *If Q_i, for $i = 1,2$, are circle bundles over a topological space M and $L_i = Q_i\times_{S^1}\mathbb{C}$ are the associated complex line bundles, and $k\in\mathbb{Z}$, then the following hold:*

1. *If V is a complex vector bundle over M, and $e^{i\theta}\in S^1$ acts on the fiber product $Q_1\times_M V$ by $e^{i\theta}\cdot(q_1,v) = (e^{i\theta}q_1, e^{ik\theta}v)$, then*
$$(Q_1\times_M V)/S^1 \cong L_1^{-k}\otimes V.$$
2. *If $e^{i\theta}\in S^1$ acts on the fiber product $Q_1\times_M Q_2$ by $e^{i\theta}\cdot(q_1,q_2) = (e^{i\theta}q_1, e^{ik\theta}q_2)$, then the first Chern class of the circle bundle $(Q_1\times_M Q_2)/S^1\to M$ is*
$$c_1((Q_1\times_M Q_2)/S^1) = c_1(Q_2) - kc_1(Q_1),$$
where the circle action on $(Q_1\times_M Q_2)/S^1$ is induced by the circle action on Q_2 of weight one.

Recall that $\mu_\mathfrak{s}\in H^2(M_\mathfrak{s};\mathbb{Z})$ is the first Chern class of the base-point fibration over $M_\mathfrak{s}$ (see the remark following equation (2.21)); when there is no ambiguity we may write $\mu_\mathfrak{s}$ for the pullback $\pi_\mathfrak{s}^*\mu_\mathfrak{s}$ to the total space $\bar{\mathcal{M}}^{\mathrm{vir}}_{\mathfrak{t}',\mathfrak{s}}/S^1$. Recall from §2.3.1 that if $b_1(X) = 0$ then $\mu_\mathfrak{s}\times 1 = c_1(\mathbb{L}_\mathfrak{s})$, where $\mathbb{L}_\mathfrak{s}\to\mathcal{C}^0_\mathfrak{s}\times X$ is the line bundle (2.19). Moreover, $c_1(L) = c_1(\mathfrak{t}) - c_1(\mathfrak{s})$ in the definition (4.15) of the line bundle $\bar{\mathbb{L}}^{\mathrm{vir},*}_{\mathfrak{t}',\mathfrak{s}}$.

In the following we will use $\mathbb{L}_\mathfrak{s}$ to denote the restriction of the bundle $\mathbb{L}_\mathfrak{s}\to\mathcal{C}^0_\mathfrak{s}\times X$ to the subspace $M_\mathfrak{s}\times X$.

Lemma 4.7. *Continue the hypotheses of Lemma 4.3. Let*

$$\mathbb{L}_\nu \to \bar{\mathcal{M}}^{\mathrm{vir},*}_{\mathfrak{t}',\mathfrak{s}}/S^1$$

be the complex line bundle associated to the circle bundle (4.3). *Then there is an isomorphism of complex line bundles over* $\bar{\mathcal{M}}^{\mathrm{vir},*}_{\mathfrak{t}',\mathfrak{s}}/S^1 \times X$*:*

$$\bar{\mathbb{L}}^{\mathrm{vir},*}_{\mathfrak{t}',\mathfrak{s}} \cong \pi^*_{\mathcal{M}}\mathbb{L}^{-1}_{\nu} \otimes (\pi_{\mathfrak{s},1} \times \mathrm{id}_X)^*\mathbb{L}^{\otimes 2}_{\mathfrak{s}} \otimes \pi^*_{X,2}L. \tag{4.16}$$

Proof. The argument yielding equation (3.68) in [17] implies that if $\mathcal{G}_{\mathfrak{s}}$ acts diagonally on $\tilde{M}_{\mathfrak{s}} \times L$ with weight negative two on L, then we have an isomorphism of line bundles over $M_{\mathfrak{s}} \times X$:

$$\tilde{M}_{\mathfrak{s}} \times_{\mathcal{G}_{\mathfrak{s}}} L \cong \mathbb{L}^{\otimes 2}_{\mathfrak{s}} \otimes \pi^*_{X,\mathfrak{s}}L, \tag{4.17}$$

where $\pi_{X,\mathfrak{s}} : M_{\mathfrak{s}} \times X \to X$ is the projection. By the preceding isomorphism and the definition (4.5) of $\pi_{\mathfrak{s}} \times \mathrm{id}_X$, we have an isomorphism of line bundles over $\bar{\mathcal{M}}^{\mathrm{vir},*}_{\mathfrak{t}',\mathfrak{s}}/S^1 \times X$:

$$(\pi_{\mathfrak{s}} \times \mathrm{id}_X)^*(\mathbb{L}^{\otimes 2}_{\mathfrak{s}} \otimes \pi^*_{X,\mathfrak{s}}L) \cong \left(\bar{\mathcal{M}}^{\mathrm{vir},*}_{\mathfrak{t}',\mathfrak{s}}/S^1 \times X\right) \times_{M_{\mathfrak{s}}\times X} \left(\tilde{M}_{\mathfrak{s}} \times_{\mathcal{G}_{\mathfrak{s}}} L\right).$$

Consequently, from this isomorphism we see that, as bundles over $\bar{\mathcal{M}}^{\mathrm{vir},*}_{\mathfrak{t}',\mathfrak{s}}/S^1 \times X$,

$$\begin{aligned}&\left(\bar{\mathcal{M}}^{\mathrm{vir},*}_{\mathfrak{t}',\mathfrak{s}} \times X\right) \times_{\bar{\mathcal{M}}^{\mathrm{vir},*}_{\mathfrak{t}',\mathfrak{s}}/S^1\times X} (\pi_{\mathfrak{s},1} \times \mathrm{id}_X)^*(\mathbb{L}^{\otimes 2}_{\mathfrak{s}} \otimes \pi^*_{X,\mathfrak{s}}L)\\ &\quad\cong \left(\bar{\mathcal{M}}^{\mathrm{vir},*}_{\mathfrak{t}',\mathfrak{s}} \times X\right) \times_{M_{\mathfrak{s}}\times X} \left(\tilde{M}_{\mathfrak{s}} \times_{\mathcal{G}_{\mathfrak{s}}} L\right).\end{aligned} \tag{4.18}$$

Now, consider the circle quotient of the bundle on the right-hand side of the preceding isomorphism,

$$\left(\bar{\mathcal{M}}^{\mathrm{vir},*}_{\mathfrak{t}',\mathfrak{s}} \times X \times_{M_{\mathfrak{s}}\times X} (\tilde{M}_{\mathfrak{s}} \times_{\mathcal{G}_{\mathfrak{s}}} L)\right)/S^1 \to \bar{\mathcal{M}}^{\mathrm{vir},*}_{\mathfrak{t}',\mathfrak{s}}/S^1 \times X, \tag{4.19}$$

where the circle acts diagonally on the factors $\bar{\mathcal{M}}^{\mathrm{vir},*}_{\mathfrak{t}',\mathfrak{s}}$ and L. The definition of $\mathbb{L}_{\nu}$ as the line bundle associated to the circle bundle (4.3), Lemma 4.6 and the isomorphism (4.18) then yield the bundle isomorphisms over $\bar{\mathcal{M}}^{\mathrm{vir},*}_{\mathfrak{t}',\mathfrak{s}} \times X$:

$$\begin{aligned}&\left((\bar{\mathcal{M}}^{\mathrm{vir},*}_{\mathfrak{t}',\mathfrak{s}} \times X) \times_{M_{\mathfrak{s}}\times X} (\tilde{M}_{\mathfrak{s}} \times_{\mathcal{G}_{\mathfrak{s}}} L)\right)/S^1\\ &\quad\cong \pi^*_{\mathcal{M}}\mathbb{L}^{-1}_{\nu} \otimes (\pi_{\mathfrak{s},1} \times \mathrm{id}_X)^*(\mathbb{L}^{\otimes 2}_{\mathfrak{s}} \otimes \pi^*_{X,\mathfrak{s}}L)\\ &\quad\cong \pi^*_{\mathcal{M}}\mathbb{L}^{-1}_{\nu} \otimes (\pi_{\mathfrak{s},1} \times \mathrm{id}_X)^*\mathbb{L}^{\otimes 2}_{\mathfrak{s}} \otimes \pi^*_{X,2}L.\end{aligned} \tag{4.20}$$

One can check that the map defined below is an isomorphism of line bundles over $\bar{\mathcal{M}}^{\mathrm{vir},*}_{\mathfrak{t}',\mathfrak{s}} \times X$,

$$\bar{\mathbb{L}}^{\mathrm{vir},*}_{\mathfrak{t}',\mathfrak{s}} = \mathrm{cl}\tilde{\mathcal{M}}^{\mathrm{vir},*}_{\mathfrak{t}',\mathfrak{s}} \times_{\mathcal{G}_{\mathfrak{s}}\times S^1} L \cong \left((\bar{\mathcal{M}}^{\mathrm{vir},*}_{\mathfrak{t}',\mathfrak{s}} \times X) \times_{M_{\mathfrak{s}}\times X} (\tilde{M}_{\mathfrak{s}} \times_{\mathcal{G}_{\mathfrak{s}}} L)\right)/S^1, \tag{4.21}$$

given for $(B,\Psi) \in \tilde{M}_{\mathfrak{s}}$, $(B,\Psi,\eta) \in \tilde{N}_{\mathfrak{t},\mathfrak{s}}(\varepsilon)$, $\mathbf{g} \in \mathrm{Gl}_{\mathfrak{t}}(\delta)$, and $\zeta \in L|_x$ by

$$[(B,\Psi,\eta),\mathbf{g},\zeta] \mapsto (([(B,\Psi,\eta),\mathbf{g}],x),[B,\Psi,\zeta]).$$

Therefore, the isomorphisms (4.21) and (4.20) give the desired isomorphism (4.16). □

Corollary 4.8. *Continue the hypotheses of Lemma 4.7. Then, on* $\bar{\mathcal{M}}^{\mathrm{vir},*}_{\mathfrak{t}',\mathfrak{s}} \times X$*, we have*

$$c_1(\bar{\mathbb{L}}^{\mathrm{vir},*}_{\mathfrak{t}',\mathfrak{s}}) = 2(\pi_{\mathfrak{s},1} \times \mathrm{id}_X)^*c_1(\mathbb{L}_{\mathfrak{s}}) - \pi^*_{\mathcal{M}}\nu + \pi^*_{X,2}c_1(L). \tag{4.22}$$

Lemma 4.4, equation (4.22), and the relation $p_1(\bar{\mathbb{F}}^{\mathrm{vir},*}_{\mathfrak{t}',\mathfrak{s}}) = c_1(\bar{\mathbb{L}}^{\mathrm{vir},*}_{\mathfrak{t}',\mathfrak{s}})^2$ then yield:

Corollary 4.9. *Continue the hypotheses of Lemma 4.3. Let* $\mathrm{PD}[\Delta]$ *be the Poincaré dual of the diagonal* $\Delta \subset X \times X$, *let* ν *be the class in Definition 4.1, and let* ι *be the inclusion* (4.2)*. Then, on* $\mathcal{M}^{\mathrm{vir}}_{\mathfrak{t}',\mathfrak{s}}/S^1$, *we have*

$$
\begin{aligned}
(\boldsymbol{\gamma} \times \mathrm{id}_X)^* p_1(\mathbb{F}_{\mathfrak{t}}) = (\iota \times \mathrm{id}_X)^* &\left(2(\pi_{\mathfrak{s},1} \times \mathrm{id}_X)^* c_1(\mathbb{L}_{\mathfrak{s}}) - \pi_{\mathcal{M}}^*\nu + \pi_{X,2}^* c_1(L)\right)^2 \\
&- 4(\iota \times \mathrm{id}_X)^*(\pi_X \times \mathrm{id}_X)^* \,\mathrm{PD}[\Delta].
\end{aligned} \tag{4.23}
$$

The equation (4.23) for $(\boldsymbol{\gamma} \times \mathrm{id}_X)^* p_1(\mathbb{F}_{\mathfrak{t}'})$ yields the following formulae for the pullback of $\mu_p(\beta) = -\frac{1}{4} p_1(\mathbb{F}_{\mathfrak{t}'})/\beta$ to $\mathcal{M}^{\mathrm{vir}}_{\mathfrak{t}',\mathfrak{s}}/S^1$:

Lemma 4.10. *Continue the hypotheses of Lemma 4.3. Let* ι *be the inclusion* (4.2)*, let* $x \in H_0(X;\mathbb{Z})$ *be the positive generator, and let* $h \in H_2(X;\mathbb{R})$*. Let* $\nu, \pi_{\mathfrak{s}}^*\mu_{\mathfrak{s}} \in H^2(\bar{\mathcal{M}}^{\mathrm{vir},*}_{\mathfrak{t}',\mathfrak{s}}/S^1;\mathbb{Z})$ *be the classes given by Definition 4.1 and equation* (2.21)*, respectively, while* $\pi_{\mathfrak{s}}$ *is the projection* (4.4)*. Then, on* $\mathcal{M}^{\mathrm{vir}}_{\mathfrak{t}',\mathfrak{s}}/S^1$, *we have*

$$
\begin{aligned}
\mu_p(x) &= -\tfrac{1}{4}\iota^*(2\pi_{\mathfrak{s}}^*\mu_{\mathfrak{s}} - \nu)^2 + \iota^*\pi_X^* \,\mathrm{PD}[x] \in H^4(\mathcal{M}^{\mathrm{vir}}_{\mathfrak{t}',\mathfrak{s}}/S^1;\mathbb{R}), \\
\mu_p(h) &= \tfrac{1}{2}\iota^*\langle c_1(\mathfrak{s}) - c_1(\mathfrak{t}), h\rangle(2\pi_{\mathfrak{s}}^*\mu_{\mathfrak{s}} - \nu) + \iota^*\pi_X^* \,\mathrm{PD}[h] \in H^2(\mathcal{M}^{\mathrm{vir}}_{\mathfrak{t}',\mathfrak{s}}/S^1;\mathbb{R}).
\end{aligned} \tag{4.24}
$$

Proof. If $\beta \in H_\bullet(X;\mathbb{R})$, then $\mathrm{PD}[\Delta]/\beta = \mathrm{PD}[\beta]$ by [28, Theorem 30.6]. Thus,

$$
(\pi_X \times \mathrm{id}_X)^* \,\mathrm{PD}[\Delta]/\beta = \pi_X^* \,\mathrm{PD}[\beta].
$$

The assertions now follow from Corollary 4.9 and standard computations (see, for example, [12, §5.1.4]). □

Finally, we note that the results of §4.1, bearing on cohomology classes on a neighborhood $M_{\mathfrak{s}} \times X$, will extend (with appropriate modifications, though these do not cause undue difficulty) to the case of a neighborhood of $M_{\mathfrak{s}} \times \mathrm{Sym}^\ell(X)$ when $\ell > 1$. For example, one must replace the diagonal $\Delta \subset X \times X$ appearing in Lemma 4.3 with an incidence subset,

$$
\{(\mathbf{x}, x) \in \mathrm{Sym}^\ell(X) \times X : x \in |\mathbf{x}|\},
$$

where $|\mathbf{x}| \subset X$ is the support of a point $\mathbf{x} \in \mathrm{Sym}^\ell(X)$.

4.2. **Euler class of the obstruction bundle.** We first compute the Euler class of the background component $\bar{\Upsilon}^s_{\mathfrak{t}',\mathfrak{s}}/S^1 \to \bar{\mathcal{M}}^{\mathrm{vir},*}_{\mathfrak{t}',\mathfrak{s}}/S^1$ of the obstruction bundle defined in (3.64).

Lemma 4.11. *Continue the hypotheses of Lemma 4.3. Let* ν *be the class in Definition 4.1. Let* $\bar{\Upsilon}^s_{\mathfrak{t}',\mathfrak{s}}/S^1 \to \bar{\mathcal{M}}^{\mathrm{vir},*}_{\mathfrak{t}',\mathfrak{s}}/S^1$ *be the extended Seiberg-Witten obstruction bundle* (3.64)*, and let* r_Ξ *denote its complex rank. Then*

$$
e(\bar{\Upsilon}^s_{\mathfrak{t}',\mathfrak{s}}/S^1) = (-\nu)^{r_\Xi} \in H^{2r_\Xi}(\bar{\mathcal{M}}^{\mathrm{vir},*}_{\mathfrak{t}',\mathfrak{s}}/S^1;\mathbb{Z}). \tag{4.25}
$$

Proof. The S^1 quotient of the bundle $\bar{\Upsilon}^s_{\mathfrak{t}',\mathfrak{s}}$ by the action described at the end of §3.6.1 can be written as

$$
\bar{\Upsilon}^s_{\mathfrak{t}',\mathfrak{s}}/S^1 \cong \bar{\mathcal{M}}^{\mathrm{vir},*}_{\mathfrak{t}',\mathfrak{s}} \times_{(S^1,-1)} \mathbb{C}^{r_\Xi}, \tag{4.26}
$$

with the factor of negative one appearing because the action is diagonal, as explained in Lemma 4.6. Note that the circle action described in Lemma 3.6 is twice the action in the definition of ν (see Definition 4.1), so the weight two action on $\mathbb{C}^{r_\Xi}$ described at the end of §3.6.1 becomes a weight one action here. Equation (4.25) then follows immediately from equation (4.26) and the definition of ν. □

We now compute the Euler class of the instanton component (3.68) of the obstruction bundle when $\ell(\mathfrak{t}', \mathfrak{s}) = 1$; more work is required to extend this calculation to the case $\ell(\mathfrak{t}', \mathfrak{s}) > 1$. For the proof of the following lemma, it is convenient to define

$$\partial^i \mathcal{M}^{\text{vir}}_{\mathfrak{t}',\mathfrak{s}} = \tilde{N}_{\mathfrak{t},\mathfrak{s}}(\varepsilon) \times_{\mathcal{G}_\mathfrak{s}} \partial \bar{\mathrm{Gl}}_\mathfrak{t}(\delta) \subset \mathcal{M}^{\text{vir}}_{\mathfrak{t}',\mathfrak{s}}, \tag{4.27}$$

so that $\mathbf{L}^{\text{vir},i}_{\mathfrak{t}',\mathfrak{s}} = \partial^i \mathcal{M}^{\text{vir}}_{\mathfrak{t}',\mathfrak{s}}/S^1$.

Lemma 4.12. *Let ν be the class in Definition 4.1. Then the Euler class of the instanton obstruction bundle, $\Upsilon^i_{\mathfrak{t}',\mathfrak{s}}/S^1 \to \mathcal{M}^{\text{vir}}_{\mathfrak{t}',\mathfrak{s}}/S^1$ defined in* (3.68), *is given as an element of rational cohomology by*

$$e(\Upsilon^i_{\mathfrak{t}',\mathfrak{s}}/S^1) = \tfrac{1}{2}\left(\pi_X^* c_1(\mathfrak{t}) - \nu\right) \in H^2(\mathcal{M}^{\text{vir}}_{\mathfrak{t}',\mathfrak{s}}/S^1; \mathbb{Q}). \tag{4.28}$$

Proof. Because $\mathcal{M}^{\text{vir}}_{\mathfrak{t}',\mathfrak{s}}/S^1$ retracts onto $\partial^i \mathcal{M}^{\text{vir}}_{\mathfrak{t}',\mathfrak{s}}/S^1 = \mathbf{L}^{\text{vir},i}_{\mathfrak{t}',\mathfrak{s}}$, as one can see from the definition (3.72) of $\mathbf{L}^{\text{vir},i}_{\mathfrak{t}',\mathfrak{s}}$, it suffices to compute the Euler class of the restriction of $e(\Upsilon^i_{\mathfrak{t}',\mathfrak{s}}/S^1)$ to $\partial^i \mathcal{M}^{\text{vir}}_{\mathfrak{t}',\mathfrak{s}}/S^1$. It is easier to compute the Euler class of the bundle $(\Upsilon^i_{\mathfrak{t}',\mathfrak{s}}/S^1)^{\otimes 2}$. The observation (following from (3.66)) that

$$(\operatorname{Coker} \mathbf{D})^{\otimes 2} \cong \mathrm{SU}(2) \times_{\{\pm \mathrm{id}\}} (\mathbb{C} \otimes \mathbb{C}) = \mathrm{SO}(3) \times \mathbb{C},$$

Lemma 3.7, and the description of $\Upsilon^i_{\mathfrak{t}',\mathfrak{s}}$ in (3.68) imply that

$$(\Upsilon^i_{\mathfrak{t}',\mathfrak{s}})^{\otimes 2}\big|_{\partial^i \mathcal{M}^{\text{vir}}_{\mathfrak{t}',\mathfrak{s}}} \cong \tilde{N}_{\mathfrak{t},\mathfrak{s}}(\varepsilon) \times_{\mathcal{G}_\mathfrak{s}} \mathrm{Fr}_{\mathbb{C}\ell(T^*X)}(V) \times_{\mathrm{Spin}^u(4)} (\mathrm{SO}(3) \times \mathbb{C}).$$

By the description of the circle action on the bundle $\Upsilon^i_{\mathfrak{t}',\mathfrak{s}}$ prior to its definition (3.68), the preceding isomorphism is circle-equivariant if the circle acts on the factor $\tilde{N}_{\mathfrak{t},\mathfrak{s}}(\varepsilon)$ by the action (2.37) and with weight two on the fiber $\mathbb{C}$ and trivially on the remaining factors; the weight of the circle action on the fiber $\mathbb{C}$ is two because $\Upsilon^i_{\mathfrak{t}',\mathfrak{s}}$ appears in the tensor-product square on the left-hand side. The action of $\mathrm{Spin}^u(4)$ on the vector bundle $(\operatorname{Coker} \mathbf{D})^{\otimes 2} \to M_1^{s,\natural}(S^4, \delta)$ is given, for $(\hat{M}, \zeta) \in \mathrm{SO}(3) \times \mathbb{C} = (\operatorname{Coker} \mathbf{D})^{\otimes 2}$ and $\tilde{U} \in \mathrm{Spin}^u(4)$, by

$$(\hat{M}, \zeta) \mapsto (\mathrm{Ad}(\tilde{U})\hat{M}, \det{}^u(\tilde{U})^{-1}\zeta), \tag{4.29}$$

where the homomorphisms $\det^u : \mathrm{Spin}^u(4) \to S^1$ and $\mathrm{Ad} = (\mathrm{Ad}^u, \mathrm{Ad}^\mathfrak{g}) : \mathrm{Spin}^u(4) \to \mathrm{SO}(4) \times \mathrm{SO}(3)$ are defined in (3.11) and (3.13), respectively. To see this, observe that

1. The action of $\mathrm{Spin}^u(4)$ on $\operatorname{Coker} \mathbf{D}$ covers the action of $\mathrm{SO}(4) \times \mathrm{SO}(3)$ on $M_1^{s,\natural}(S^4, \delta)$,
2. The central S^1 in $\mathrm{Spin}^u(4)$ acts on $\operatorname{Coker} \mathbf{D}$ by scalar multiplication with weight negative one.

Property (1) implies that the action of $\mathrm{Spin}^u(4)$ on the component $M_1^{s,\natural}(S^4, \delta)$ of

$$(\operatorname{Coker} \mathbf{D})^{\otimes 2} = M_1^{s,\natural}(S^4, \delta) \times \mathbb{C}$$

is given by the projection $\mathrm{Ad} : \mathrm{Spin}^u \to \mathrm{SO}(4) \times \mathrm{SO}(3)$ and the action of $\mathrm{SO}(4) \times \mathrm{SO}(3)$ on $M_1^{s,\natural}(S^4, \delta)$. Property (2) implies that the central S^1 acts on $(\operatorname{Coker} \mathbf{D})^{\otimes 2}$ by scalar multiplication on the fibers with weight negative two, just as in the definition of the homomorphism $\det^u$ before equation (3.11). Any group action satisfying the above two properties will differ from the action (4.29) by a representation of $\mathrm{Spin}(4) \subset \mathrm{Spin}^u(4)$ on $\mathbb{C}$. However, by [7, Proposition 5.1] there are no such non-trivial representations. Hence, these two properties characterize the above group action. Equations (3.14) and (3.12) yield the bundle isomorphisms,

$$\begin{aligned}\mathrm{Fr}_{\mathbb{C}\ell(T^*X)}(V) \times_{(\mathrm{Spin}^u(4),\mathrm{Ad})} \mathrm{SO}(3) &\cong \mathrm{Fr}(T^*X) \times_X \mathrm{Fr}(\mathfrak{g}_V) \times_{\mathrm{SO}(4)\times\mathrm{SO}(3)} \mathrm{SO}(3)\\ &= \partial\bar{\mathrm{Gl}}_{\mathfrak{t}}(\delta),\\ \mathrm{Fr}_{\mathbb{C}\ell(T^*X)}(V) \times_{(\mathrm{Spin}^u(4),\det^u)} \mathbb{C} &\cong \det^{\frac{1}{2}}(V^+),\end{aligned}$$

and so

$$\mathrm{Fr}_{\mathbb{C}\ell(T^*X)}(V) \times_{\mathrm{Spin}^u(4)} (\mathrm{SO}(3)\times\mathbb{C}) \cong \partial\bar{\mathrm{Gl}}_{\mathfrak{t}}(\delta) \times_X \det^{\frac{1}{2}}(V^+).$$

The preceding isomorphism is $\mathcal{G}_{\mathfrak{s}}$-equivariant, where $\mathcal{G}_{\mathfrak{s}}$ acts on $\mathrm{Fr}_{\mathbb{C}\ell(T^*X)}(V)$ as in the description of the instanton obstruction bundle (3.68), if $\mathcal{G}_{\mathfrak{s}}$ acts trivially on $\det^{\frac{1}{2}}(V^+)$ and on $\partial\bar{\mathrm{Gl}}_{\mathfrak{t}}(\delta)$ by the standard action on $\mathrm{Fr}(\mathfrak{g}_V)$. (The action is trivial on $\det^{\frac{1}{2}}(V^+)$ because elements of $\varrho(\mathcal{G}_{\mathfrak{s}}) \subset \mathcal{G}_{\mathfrak{t}}$ act on the fiber of $\mathrm{Fr}_{\mathbb{C}\ell(T^*X)}(V)$ by elements of $\mathrm{Spin}^u(4)$ which are in the kernel of the homomorphism $\det^u : \mathrm{Spin}^u(4) \to S^1$.) Therefore,

$$(\Upsilon^i_{\mathfrak{t}',\mathfrak{s}})^{\otimes 2}|_{\partial^i\mathcal{M}^{\mathrm{vir}}_{\mathfrak{t}',\mathfrak{s}}} \cong \tilde{N}_{\mathfrak{t},\mathfrak{s}}(\varepsilon) \times_{\mathcal{G}_{\mathfrak{s}}} \partial\bar{\mathrm{Gl}}_{\mathfrak{t}}(\delta) \times_X \det^{\frac{1}{2}}(V^+). \tag{4.30}$$

The bundle isomorphism (4.30) is circle-equivariant and so, from the definition (4.27) of $\partial^i\mathcal{M}^{\mathrm{vir}}_{\mathfrak{t}',\mathfrak{s}}$, we see that the isomorphism (4.30) descends to an isomorphism

$$(\Upsilon^i_{\mathfrak{t}',\mathfrak{s}}/S^1)^{\otimes 2}|_{\partial^i\mathcal{M}^{\mathrm{vir}}_{\mathfrak{t}',\mathfrak{s}}/S^1} \cong \left(\partial^i\mathcal{M}^{\mathrm{vir}}_{\mathfrak{t}',\mathfrak{s}} \times_X \det^{\frac{1}{2}}(V^+)\right)/S^1, \tag{4.31}$$

where the circle acts with weight two on the fibers of $\det^{\frac{1}{2}}(V^+)$ and the action (2.37) on the factor $\tilde{N}_{\mathfrak{t}',\mathfrak{s}}(\varepsilon)$ in $\partial^i\mathcal{M}^{\mathrm{vir}}_{\mathfrak{t}',\mathfrak{s}}$. By the equivalence of the circle actions on $\mathcal{M}^{\mathrm{vir}}_{\mathfrak{t}',\mathfrak{s}}$ given by Lemma 3.6 and because the circle action of Lemma 3.6 is twice the action in the Definition 4.1, the circle quotient on the right-hand side of (4.31) is equivalent to one where the circle acts diagonally by scalar multiplication on the fibers of $\det^{\frac{1}{2}}(V^+)$ and by the action of Definition 4.1 on $\partial^i\mathcal{M}^{\mathrm{vir}}_{\mathfrak{t}',\mathfrak{s}} \subset \mathcal{M}^{\mathrm{vir}}_{\mathfrak{t}',\mathfrak{s}}$. The desired equation (4.28) for the Euler class of $\Upsilon^i_{\mathfrak{t}',\mathfrak{s}}/S^1$ then follows from the isomorphism (4.31), the preceding description of the circle quotient on the right-hand side of (4.31), and Lemma 4.6. □

5. Intersection numbers and cohomology

In this section we prove the topological results necessary to compute the intersection number

$$\#\left(\bar{\mathcal{V}}(z) \cap \bar{\mathcal{W}}^\eta \cap \bar{\mathbf{L}}_{\mathfrak{t}',\mathfrak{s}}\right) \tag{5.1}$$

where $\bar{\mathbf{L}}_{\mathfrak{t}',\mathfrak{s}}$ is the link of $M_{\mathfrak{s}} \times \mathrm{Sym}^\ell(X)$ in $\bar{\mathcal{M}}_{\mathfrak{t}'}$, with $\ell(\mathfrak{t}',\mathfrak{s}) = 1$ and $\deg(z) + 2(\eta+1) = \dim\mathcal{M}_{\mathfrak{t}'}$.

We solve this problem in two stages. In §5.1 we prove that the intersection number (5.1) is equal to the pairing of certain cohomology classes with a homology class $[\bar{\mathbf{L}}^{\mathrm{vir}}_{\mathfrak{t}',\mathfrak{s}}]$ which can be understood as a fundamental class of $\bar{\mathbf{L}}^{\mathrm{vir}}_{\mathfrak{t}',\mathfrak{s}}$; see Proposition 5.2. In §5.2 we compute enough of the cohomology ring of $\bar{\mathbf{L}}^{\mathrm{vir}}_{\mathfrak{t}',\mathfrak{s}}$ to allow us to compute this pairing.

Since the gluing map $\boldsymbol{\gamma} : \bar{\mathcal{M}}^{\mathrm{vir}}_{\mathfrak{t}',\mathfrak{s}} \to \bar{\mathcal{C}}_{\mathfrak{t}'}$ is a circle-equivariant embedding of stratified spaces (a homeomorphism preserving strata and restricting to a diffeomorphism on smooth strata), when no confusion can arise we shall follow the convention stated in the introduction to §4. Thus, we shall not explicitly distinguish in this section between cohomology classes on $\bar{\mathcal{C}}_{\mathfrak{t}'}$ and their pullbacks to $\bar{\mathcal{M}}^{\mathrm{vir}}_{\mathfrak{t}',\mathfrak{s}}$ or between cycles or geometric representatives in $\bar{\mathcal{C}}_{\mathfrak{t}'}$ and their pre-images in $\bar{\mathcal{M}}^{\mathrm{vir}}_{\mathfrak{t}',\mathfrak{s}}$. For example, we shall simply write $\mathcal{V}(\beta)$ and $\mu_p(\beta)$ for the pre-image

$\gamma^{-1}(\mathcal{V}(\beta))$ or pullback $\gamma^*\mu_p(\beta)$ in $\mathcal{M}^{\text{vir}}_{\mathfrak{t}',\mathfrak{s}}$ of the corresponding geometric representative or cohomology class on $\gamma(\mathcal{M}^{\text{vir}}_{\mathfrak{t}',\mathfrak{s}}) \subset \mathcal{C}^{*,0}_{\mathfrak{t}'}$.

It is worth mentioning where we use the various properties of the gluing maps and obstruction sections described in Theorem 3.8, as we shall occasionally exploit these properties without comment later this section. The fact that γ is a homeomorphism from $\bar{\mathcal{M}}^{\text{vir}}_{\mathfrak{t}',\mathfrak{s}}$ into $\bar{\mathcal{C}}_{\mathfrak{t}'}$ ensures that the image of γ contains an open neighborhood of $\bar{\mathcal{V}}(z)\cap\bar{\mathcal{W}}\cap\bar{\mathbf{L}}_{\mathfrak{t}',\mathfrak{s}}$ in $\bar{\mathcal{M}}_{\mathfrak{t}'}$. The gluing map is a smooth embedding of the top stratum $\mathcal{M}^{\text{vir}}_{\mathfrak{t}',\mathfrak{s}}$ into $\mathcal{C}^{*,0}_{\mathfrak{t}'}$ and, when orientations are properly taken into account, orientation-preserving; thus, the pre-images in $\mathcal{M}^{\text{vir}}_{\mathfrak{t}',\mathfrak{s}}$ of oriented, transverse intersections in $\mathcal{M}_{\mathfrak{t}'}$ are again appropriately oriented, and transverse. We use continuity of γ on $\bar{\mathcal{M}}^{\text{vir}}_{\mathfrak{t}',\mathfrak{s}}$ in arguments involving extensions of certain cocycles on $\mathbf{L}_{\mathfrak{t}',\mathfrak{s}}$ to $\bar{\mathbf{L}}_{\mathfrak{t}',\mathfrak{s}}$; see the introduction to §5.1.5 for precise statements.

Transversality and smoothness of the obstruction section χ is used in Lemma 5.9, where we use the fact that $\gamma(\chi^{-1}(0))$ is a submanifold of $\mathcal{M}_{\mathfrak{t}'}$ of the appropriate 'multiplicity'. Continuity of the section χ_s on $\bar{\mathcal{M}}^{\text{vir}}_{\mathfrak{t}',\mathfrak{s}}$ is used in the proofs of Lemmas 5.17 and 5.18 and in writing the decomposition (5.44) of a certain relative Euler class.

5.1. **Duality.** The goal of this subsection is to prove Proposition 5.2.

Definition 5.1. Let $\bar{e}_s$ and $\bar{e}_i$ be the extensions of the Euler classes $e(\Upsilon^s_{\mathfrak{t}',\mathfrak{s}}/S^1)$ and $e(\Upsilon^i_{\mathfrak{t}',\mathfrak{s}}/S^1)$ from $\mathcal{M}^{\text{vir}}_{\mathfrak{t}',\mathfrak{s}}/S^1$ to $\bar{\mathcal{M}}^{\text{vir},*}_{\mathfrak{t}',\mathfrak{s}}/S^1$ defined by Lemmas 4.11 and 4.12 respectively, and set $\bar{e} = \bar{e}_s \smile \bar{e}_i$.

Proposition 5.2. *Assume $w \in H^2(X;\mathbb{Z})$ is such that $w \pmod 2$ is good in the sense of Definition 2.3. Suppose $(\mathfrak{t}',\mathfrak{s})$ is a pair with $\ell(\mathfrak{t}',\mathfrak{s}) = 1$ and $w_2(\mathfrak{t}') \equiv w \pmod 2$. Let $[\bar{\mathbf{L}}^{\text{vir}}_{\mathfrak{t}',\mathfrak{s}}] \in H_{\max}(\bar{\mathbf{L}}^{\text{vir}}_{\mathfrak{t}',\mathfrak{s}};\mathbb{Z})$ be the homology class defined in (5.9). Let $z \in \mathbb{A}(X)$ and $\eta \in \mathbb{N}$ satisfy $\deg(z) + 2(\eta+1) = \dim \mathcal{M}_{\mathfrak{t}'}$. Let $\bar{\mu}_p(z)$ and $\bar{\mu}_c$ be the extensions of $\mu_p(z)$ and μ_c in Definition 4.5 from $\mathcal{M}^{\text{vir}}_{\mathfrak{t}',\mathfrak{s}}/S^1$ to $\bar{\mathcal{M}}^{\text{vir},*}_{\mathfrak{t}',\mathfrak{s}}/S^1$. Give $\mathbf{L}_{\mathfrak{t}',\mathfrak{s}}$ the standard orientation described in §3.9. Then,*

$$\#\left(\bar{\mathcal{V}}(z)\cap\bar{\mathcal{W}}^{\eta}\cap\bar{\mathbf{L}}_{\mathfrak{t}',\mathfrak{s}}\right) = \left\langle \bar{\mu}_p(z)\smile\bar{\mu}_c^{\eta}\smile\bar{e}, [\bar{\mathbf{L}}^{\text{vir}}_{\mathfrak{t}',\mathfrak{s}}]\right\rangle. \tag{5.2}$$

The corresponding result [18, Lemma 3.31] for the level-zero Seiberg-Witten stratum, $M_{\mathfrak{s}} \subset \mathcal{M}_{\mathfrak{t}}/S^1$, followed trivially from the definition of a geometric representative (see [18, Definition 3.4] or [31, p. 588]) because the link of $M_{\mathfrak{s}} \subset \mathcal{M}_{\mathfrak{t}}/S^1$ is a smooth, compact manifold without boundary whose fundamental class can be represented by a smooth cycle intersecting the geometric representatives transversely.

By contrast, the link of the level-one Seiberg-Witten stratum,

$$(M_{\mathfrak{s}}\times X)\cap\bar{\mathcal{M}}_{\mathfrak{t}'}/S^1 \subset \bar{\mathcal{M}}_{\mathfrak{t}'}/S^1,$$

can have non-empty intersection with a lower level of $\bar{\mathcal{M}}_{\mathfrak{t}'}/S^1$. This raises two difficulties which prevent an immediate translation of the level-zero argument in [18] to the level-one case here:

1. The obstruction section χ does not vanish transversely on the lower level $\mathbf{L}^{\text{low}}_{\mathfrak{t}',\mathfrak{s}}$ of $\bar{\mathbf{L}}^{\text{vir}}_{\mathfrak{t}',\mathfrak{s}} = \mathbf{L}^{\text{vir}}_{\mathfrak{t}',\mathfrak{s}} \sqcup \mathbf{L}^{\text{low}}_{\mathfrak{t}',\mathfrak{s}}$ (see (3.78)), so it is not immediately obvious that the link $\bar{\mathbf{L}}_{\mathfrak{t}',\mathfrak{s}} = \chi^{-1}(0)\cap\bar{\mathbf{L}}^{\text{vir}}_{\mathfrak{t}',\mathfrak{s}}$ defines a homology class.
2. It is not obvious that the closure $\bar{\mathcal{V}}(\beta)$ in the compactification $\bar{\mathcal{M}}^{\text{vir}}_{\mathfrak{t}',\mathfrak{s}}/S^1$ of the geometric representative $\mathcal{V}(\beta)$ in $\mathcal{M}^{\text{vir}}_{\mathfrak{t}',\mathfrak{s}}/S^1$ defines a geometric representative for the cohomology class $\bar{\mu}_p(\beta)$. (See Definition 5.6 for a review of the concept of a geometric representative for a cohomology class.)

The intersection number on the left-hand side of (5.2) is a count with sign of the points in the transverse intersection of the geometric representatives and the zero-locus, $\mathbf{L}_{\mathfrak{t}',\mathfrak{s}}$, of the section $\boldsymbol{\chi}$ of the obstruction bundle over $\mathbf{L}^{\mathrm{vir}}_{\mathfrak{t}',\mathfrak{s}}$. We emphasize that this intersection is contained in the top stratum of $\bar{\mathbf{L}}^{\mathrm{vir}}_{\mathfrak{t}',\mathfrak{s}}$; see Lemma 5.4. As discussed in the following paragraphs, the geometric representatives and the obstruction section define a cohomology class on $\mathbf{L}^{\mathrm{vir}}_{\mathfrak{t}',\mathfrak{s}}$ with compact support in the manifold-with-boundary $\mathbf{L}^{\mathrm{vir},i}_{\mathfrak{t}',\mathfrak{s}}$ (defined in (3.72)). This compactly supported cohomology class is Poincaré dual to the intersection

$$\bar{\mathcal{V}}(z) \cap \bar{\mathcal{W}}^{\eta} \cap \bar{\mathbf{L}}_{\mathfrak{t}',\mathfrak{s}}. \tag{5.3}$$

Therefore, the intersection number (5.2) is equal to the pairing of this compactly supported cohomology class with the relative fundamental class $[\mathbf{L}^{\mathrm{vir},i}_{\mathfrak{t}',\mathfrak{s}}, \partial\mathbf{L}^{\mathrm{vir},i}_{\mathfrak{t}',\mathfrak{s}}]$ of $\mathbf{L}^{\mathrm{vir},i}_{\mathfrak{t}',\mathfrak{s}}$.

The first step in defining the compactly supported cohomology class is to define a representative of the cohomology class $\mu_p(z) \smile \mu_c^{\eta}$ with specified support. In §5.1.3 we observe that the geometric representatives $\mathcal{V}(\beta)$ and $\mathcal{W}$ define singular cocycles c_β and $c_{\mathcal{W}}$, respectively, in the cohomology classes $\mu_p(\beta)$ and μ_c. The cocycle c_β has support on $\mathcal{V}(\beta) \cap \mathcal{M}^{\mathrm{vir}}_{\mathfrak{t}'\mathfrak{s}}/S^1$, in the sense that c_β restricts to zero on any singular chain not intersecting $\mathcal{V}(\beta)$. Therefore, c_β defines a relative cohomology class, $[c_\beta]$, which maps to $\mu_p(\beta)$ under the homomorphism

$$H^{\deg(\beta)}(\mathcal{M}^{\mathrm{vir}}_{\mathfrak{t}'\mathfrak{s}}/S^1, \mathcal{M}^{\mathrm{vir}}_{\mathfrak{t}'\mathfrak{s}}/S^1 - \mathcal{V}(\beta) \cap \mathcal{M}^{\mathrm{vir}}_{\mathfrak{t}'\mathfrak{s}}/S^1; \mathbb{R}) \to H^{\deg(\beta)}(\mathcal{M}^{\mathrm{vir}}_{\mathfrak{t}'\mathfrak{s}}/S^1; \mathbb{R}),$$

given by the exact sequence of the pair. We say a relative cohomology class $[c] \in H^\bullet(X, A; \mathbb{R})$ is a representative of $\mu \in H^\bullet(X; \mathbb{R})$ if $\jmath_A^*[c] = \mu$, where $\jmath_A : (X, \varnothing) \to (X, A)$ is the inclusion map. Thus, $[c_\beta]$ is a representative of $\mu_p(\beta)$. Similar comments apply to the support of the cocycle $c_{\mathcal{W}}$. We will write $c(z,\eta)$ for the cup-product of the cocycles defined by the geometric representatives appearing in the left-hand side of equation (5.2). The cocycle $c(z,\eta)$ defines a relative cohomology class, $[c(z,\eta)]$, supported on the intersection of the geometric representatives.

The *relative Euler class* [29] of the obstruction bundle and section, $e(\Upsilon_{\mathfrak{t}',\mathfrak{s}}/S^1, \boldsymbol{\chi})$, has support on the zero-locus in $\mathcal{M}^{\mathrm{vir}}_{\mathfrak{t}',\mathfrak{s}}$ of $\boldsymbol{\chi}$. In §5.1.4 (see (5.20)), we show that the cup-product $[c(z,\eta)] \smile e(\Upsilon_{\mathfrak{t}',\mathfrak{s}}/S^1, \boldsymbol{\chi})$ has compact support in the top level $\mathbf{L}^{\mathrm{vir}}_{\mathfrak{t}',\mathfrak{s}}$ of $\bar{\mathbf{L}}^{\mathrm{vir}}_{\mathfrak{t}',\mathfrak{s}}$. Note that because the geometric representatives can intersect the lower strata of $\bar{\mathbf{L}}^{\mathrm{vir}}_{\mathfrak{t}',\mathfrak{s}}$, the class $[c(z,\eta)]$ does not have compact support in $\mathbf{L}^{\mathrm{vir},i}_{\mathfrak{t}',\mathfrak{s}}$; it is only the cup-product $[c(z,\eta)] \smile e(\Upsilon_{\mathfrak{t}',\mathfrak{s}}/S^1, \boldsymbol{\chi})$ which has this compact support. In Lemma 5.9, we then prove that this cup-product is Poincaré dual to the intersection (5.3) and hence the intersection number (5.1) is equal to the pairing

$$\left\langle [c(z,\eta)] \smile e(\Upsilon_{\mathfrak{t}',\mathfrak{s}}/S^1, \boldsymbol{\chi}), \left[\mathbf{L}^{\mathrm{vir},i}_{\mathfrak{t}',\mathfrak{s}}, \partial\mathbf{L}^{\mathrm{vir},i}_{\mathfrak{t}',\mathfrak{s}}\right] \right\rangle. \tag{5.4}$$

Computing the pairing of the relative classes in (5.4) directly does not seem practical. The representatives $[c_\beta]$ of $\mu_p(\beta)$, $[c_{\mathcal{W}}]$ of μ_c, and $e(\Upsilon_{\mathfrak{t}',\mathfrak{s}}/S^1, \boldsymbol{\chi})$ of $e(\Upsilon_{\mathfrak{t}',\mathfrak{s}}/S^1)$ in the cup-product in (5.4) are elements of different cohomology rings (compare (5.15) and (5.18)), so it is not possible to compute their product using only the algebraic structure of the cohomology ring $H^\bullet(\mathbf{L}^{\mathrm{vir},i}_{\mathfrak{t}',\mathfrak{s}}, \partial\mathbf{L}^{\mathrm{vir},i}_{\mathfrak{t}',\mathfrak{s}}; \mathbb{R})$. We overcome this difficulty by replacing the pairing (5.4) with a pairing with relative cohomology classes in $H^{\max}(\bar{\mathbf{L}}^{\mathrm{vir}}_{\mathfrak{t}',\mathfrak{s}}, \bar{\mathbf{L}}^{\mathrm{vir},s}_{\mathfrak{t}',\mathfrak{s}}; \mathbb{R})$. There is an isomorphism

$$\jmath^* : H^{\max}(\bar{\mathbf{L}}^{\mathrm{vir}}_{\mathfrak{t}',\mathfrak{s}}, \bar{\mathbf{L}}^{\mathrm{vir},s}_{\mathfrak{t}',\mathfrak{s}}; \mathbb{R}) \cong H^{\max}(\bar{\mathbf{L}}^{\mathrm{vir}}_{\mathfrak{t}',\mathfrak{s}}; \mathbb{R}),$$

because $\bar{\mathbf{L}}^{\mathrm{vir},s}_{\mathfrak{t}',\mathfrak{s}}$ retracts onto the codimension-four subspace $\mathbf{L}^{\mathrm{low}}_{\mathfrak{t}',\mathfrak{s}}$. Thus, a pairing with relative cohomology classes in $H^\bullet(\bar{\mathbf{L}}^{\mathrm{vir}}_{\mathfrak{t}',\mathfrak{s}}, \bar{\mathbf{L}}^{\mathrm{vir},s}_{\mathfrak{t}',\mathfrak{s}};\mathbb{R})$ will depend only on the image of the relative cohomology classes under $\jmath^*$, and thus only on the absolute cohomology classes which these relative cohomology classes represent. These absolute cohomology classes are all elements of the ring $H^\bullet(\bar{\mathbf{L}}^{\mathrm{vir}}_{\mathfrak{t}',\mathfrak{s}};\mathbb{R})$, allowing us to compute products in the algebra of that single ring.

To replace the pairing (5.4) with one involving the compactification, we construct extensions of the cocycles c_β and $c_{\mathcal{W}}$ from $\mathbf{L}^{\mathrm{vir}}_{\mathfrak{t}',\mathfrak{s}}$ to $\bar{\mathbf{L}}^{\mathrm{vir}}_{\mathfrak{t}',\mathfrak{s}}$ in §5.1.5. (This extension process for cocycles is simpler than the corresponding process for geometric representatives.) We find exact cocycles $\delta\theta_\beta$ and $\delta\theta_{\mathcal{W}}$ such that $c_\beta + \delta\theta_\beta = \iota^*\bar{c}_\beta$ and $c_{\mathcal{W}} + \delta\theta_{\mathcal{W}} = \iota^*\bar{c}_{\mathcal{W}}$, where $\bar{c}_\beta$ and $\bar{c}_{\mathcal{W}}$ are cocycles on $\bar{\mathbf{L}}^{\mathrm{vir}}_{\mathfrak{t}',\mathfrak{s}}$ and ι is the inclusion map (4.2). By keeping track of the support of $\delta\theta_\beta$ and $\delta\theta_{\mathcal{W}}$, we can prove that replacing c_β and $c_{\mathcal{W}}$ with $\iota^*\bar{c}_\beta$ and $\iota^*\bar{c}_{\mathcal{W}}$ does not change the pairing (5.4). The construction of $\bar{c}_\beta$ and $\bar{c}_{\mathcal{W}}$ implies that they represent the cohomology classes on the right-hand-side of identity (5.2), namely $\bar{\mu}_p(\beta)$ and $\bar{\mu}_c$.

Having applied this extension method to the cocycle $c(z,\eta)$, we next apply it to the relative Euler class in the pairing (5.4). In §5.1.6, we describe a homotopy of the obstruction section $\boldsymbol{\chi}$ which does not change the pairing (5.4). We then recognize the relative Euler class of the obstruction bundle and this homotoped obstruction section as the restriction of a relative Euler class on $\bar{\mathbf{L}}^{\mathrm{vir}}_{\mathfrak{t}',\mathfrak{s}}$.

These extensions accomplish the goal of replacing the relative cohomology classes in (5.4) with relative cohomology classes in $H^\bullet(\bar{\mathbf{L}}^{\mathrm{vir}}_{\mathfrak{t}',\mathfrak{s}}, \bar{\mathbf{L}}^{\mathrm{vir}}_{\mathfrak{t}',\mathfrak{s}};\mathbb{R})$. Proposition 5.2 then follows, by the argument sketched above, from Lemma 5.9 and our computations in §5.1.5 of the cohomology classes which the extended cocycles represent.

The preceding argument contains, albeit implicitly, solutions to the two previously mentioned problems preventing a direct translation of [18, Corollary 3.11]. The homotopy of the obstruction section should define a deformation of $\bar{\mathbf{L}}_{\mathfrak{t}',\mathfrak{s}}$ in $\bar{\mathbf{L}}^{\mathrm{vir}}_{\mathfrak{t}',\mathfrak{s}}$ to a subspace whose intersection with the lower strata of $\bar{\mathbf{L}}^{\mathrm{vir}}_{\mathfrak{t}',\mathfrak{s}}$ is sufficiently regular to define a homology class. Similarly, changing the cocycle c_β to $c_\beta+\delta\theta_\beta$ should be equivalent to changing the geometric representative $\mathcal{V}(\beta)$ (by a cobordism) to a geometric representative whose closure is a geometric representative for $\bar{\mu}_p(\beta)$. We believe that the cohomological formulations given here are easier to construct than these deformations of $\bar{\mathbf{L}}_{\mathfrak{t}',\mathfrak{s}}$ and the geometric representatives.

Employing the techniques of virtual fundamental classes developed in [8], [35], [26], [46] [47], and [49] to construct a homology class representing $\bar{\mathbf{L}}_{\mathfrak{t}',\mathfrak{s}}$ — and thus eliminating the first of the two problems discussed above — would not simplify the proof of Proposition 5.2. While those techniques are well developed for the moduli space of pseudoholomorphic curves, applying that theory to the moduli spaces in this article would require additional discussion. For example, we would have to construct an extension of the obstruction bundle $\Upsilon_{\mathfrak{t}',\mathfrak{s}}/S^1$ over an appropriate compactification of $\mathcal{M}^{\mathrm{vir}}_{\mathfrak{t}',\mathfrak{s}}/S^1$; this would require further work as the instanton component $\Upsilon^i_{\mathfrak{t}',\mathfrak{s}}/S^1$ does not extend from $\mathcal{M}^{\mathrm{vir}}_{\mathfrak{t}',\mathfrak{s}}/S^1$ to the relatively simple 'cone' compactification $\bar{\mathcal{M}}^{\mathrm{vir}}_{\mathfrak{t}',\mathfrak{s}}/S^1$ in (3.55) which we currently employ. In contrast, the method used here — involving a relative Euler class and a homotopy of the obstruction section rather than constructing a homology class for $\bar{\mathbf{L}}_{\mathfrak{t}',\mathfrak{s}}$ — does not require any elaborate technical apparatus. Moreover, defining a virtual fundamental class would only solve the first of the two previously mentioned problems — defining a fundamental class for $\bar{\mathbf{L}}_{\mathfrak{t}',\mathfrak{s}}$; it would still be necessary to construct the extensions of the cocycles c_β and $c_{\mathcal{W}}$ and to prove that they represent the cohomology classes $\bar{\mu}_p(\beta)$ and $\bar{\mu}_c$. As the construction of the extended cocycles $\bar{c}_\beta$ and $\bar{c}_{\mathcal{W}}$ and the proof that they represent the cohomology classes $\bar{\mu}_p(\beta)$

and $\bar{\mu}_c$ make up the bulk of the proof of Proposition 5.2, a construction of a fundamental class for $\bar{\mathbf{L}}_{\mathfrak{t}',\mathfrak{s}}$ would not shorten this section appreciably.

5.1.1. *The fundamental class of the virtual link.* We begin by defining the homology class $[\bar{\mathbf{L}}^{\text{vir}}_{\mathfrak{t}',\mathfrak{s}}] \in H_{\max}(\bar{\mathbf{L}}^{\text{vir}}_{\mathfrak{t}',\mathfrak{s}};\mathbb{Z})$ referred to in Proposition 5.2. We refer to this homology class as the 'fundamental class' of the virtual link, although a precise definition of a fundamental class for a stratified space will not be necessary for this article. The existence and uniqueness of this class is an easy application of the exact sequence of the pair $(\bar{\mathbf{L}}^{\text{vir}}_{\mathfrak{t}',\mathfrak{s}}, \bar{\mathbf{L}}^{\text{vir},s}_{\mathfrak{t}',\mathfrak{s}})$, where $\bar{\mathbf{L}}^{\text{vir},s}_{\mathfrak{t}',\mathfrak{s}}$ is the neighborhood (3.72) of the lower level of $\bar{\mathbf{L}}^{\text{vir}}_{\mathfrak{t}',\mathfrak{s}}$. The definition of $[\bar{\mathbf{L}}^{\text{vir}}_{\mathfrak{t}',\mathfrak{s}}]$ will extend from the case of $\ell(\mathfrak{t}',\mathfrak{s})=1$ to the general case of $\ell(\mathfrak{t}',\mathfrak{s})\geq 1$ as it only requires the existence of a neighborhood of $\mathbf{L}^{\text{low}}_{\mathfrak{t}',\mathfrak{s}}$ in $\bar{\mathbf{L}}^{\text{vir}}_{\mathfrak{t}',\mathfrak{s}}$ which retracts onto $\mathbf{L}^{\text{low}}_{\mathfrak{t}',\mathfrak{s}}$. We note, however, that the construction of this neighborhood is more difficult in the general case.

By definition (3.72) of $\bar{\mathbf{L}}^{\text{vir},s}_{\mathfrak{t}',\mathfrak{s}}$ there is a deformation retraction of $\bar{\mathbf{L}}^{\text{vir},s}_{\mathfrak{t}',\mathfrak{s}}$ onto $\mathbf{L}^{\text{low}}_{\mathfrak{t}',\mathfrak{s}}$ (see (3.77)) induced by the deformation retraction $\bar{\mathrm{Gl}}_{\mathfrak{t}'}(\delta)\to X$ (given by shrinking the scale to zero). We have an inclusion map of pairs:

$$j : \left(\bar{\mathbf{L}}^{\text{vir}}_{\mathfrak{t}',\mathfrak{s}},\varnothing\right) \to \left(\bar{\mathbf{L}}^{\text{vir}}_{\mathfrak{t}',\mathfrak{s}}, \bar{\mathbf{L}}^{\text{vir},s}_{\mathfrak{t}',\mathfrak{s}}\right). \tag{5.5}$$

Because $\bar{\mathbf{L}}^{\text{vir},s}_{\mathfrak{t}',\mathfrak{s}}$ retracts onto $\mathbf{L}^{\text{low}}_{\mathfrak{t}',\mathfrak{s}}$ and because $\mathbf{L}^{\text{low}}_{\mathfrak{t}',\mathfrak{s}}$ has codimension four in $\bar{\mathbf{L}}^{\text{vir}}_{\mathfrak{t}',\mathfrak{s}}$, the inclusion (5.5) induces an isomorphism,

$$j_* : H_{\max}(\mathbf{L}^{\text{vir}}_{\mathfrak{t}',\mathfrak{s}};\mathbb{Z}) \cong H_{\max}(\mathbf{L}^{\text{vir}}_{\mathfrak{t}',\mathfrak{s}}, \mathbf{L}^{\text{vir},s}_{\mathfrak{t}',\mathfrak{s}};\mathbb{Z}). \tag{5.6}$$

Let

$$\left[\mathbf{L}^{\text{vir},i}_{\mathfrak{t}',\mathfrak{s}}, \partial\mathbf{L}^{\text{vir},i}_{\mathfrak{t}',\mathfrak{s}}\right] \in H_{\max}(\mathbf{L}^{\text{vir},i}_{\mathfrak{t}',\mathfrak{s}}, \partial\mathbf{L}^{\text{vir},i}_{\mathfrak{t}',\mathfrak{s}};\mathbb{Z}) \tag{5.7}$$

be the relative fundamental class [50, p. 303] of the manifold-with-boundary $\mathbf{L}^{\text{vir},i}_{\mathfrak{t}',\mathfrak{s}}$. By the construction (3.72) of $\mathbf{L}^{\text{vir},i}_{\mathfrak{t}',\mathfrak{s}}$ and $\bar{\mathbf{L}}^{\text{vir},s}_{\mathfrak{t}',\mathfrak{s}}$, the boundary of $\mathbf{L}^{\text{vir},i}_{\mathfrak{t}',\mathfrak{s}}$ lies in $\mathbf{L}^{\text{vir},s}_{\mathfrak{t}',\mathfrak{s}}$ and so there is an inclusion map

$$\bar{j}_{\mathbf{L}^i} : \left(\mathbf{L}^{\text{vir},i}_{\mathfrak{t}',\mathfrak{s}}, \partial\mathbf{L}^{\text{vir},i}_{\mathfrak{t}',\mathfrak{s}}\right) \to \left(\bar{\mathbf{L}}^{\text{vir}}_{\mathfrak{t}',\mathfrak{s}}, \bar{\mathbf{L}}^{\text{vir},s}_{\mathfrak{t}',\mathfrak{s}}\right). \tag{5.8}$$

Using the isomorphism (5.6), we then define $[\bar{\mathbf{L}}^{\text{vir}}_{\mathfrak{t}',\mathfrak{s}}]$ to be the unique homology class satisfying

$$(\bar{j}_{\mathbf{L}^i})_* \left[\mathbf{L}^{\text{vir},i}_{\mathfrak{t}',\mathfrak{s}}, \partial\mathbf{L}^{\text{vir},i}_{\mathfrak{t}',\mathfrak{s}}\right] = j_*[\bar{\mathbf{L}}^{\text{vir}}_{\mathfrak{t}',\mathfrak{s}}]. \tag{5.9}$$

5.1.2. *Deforming the geometric representatives.* The intersection of the geometric representatives used to define the Donaldson invariants in [31] with the lower strata can only be easily understood on the set of triples $[A,\Phi,\mathbf{x}]$ where the sets of points representing $\mathbf{x}\in\mathrm{Sym}^\bullet(X)$ do not intersect the 'suitable neighborhoods' [31] used to define the geometric representatives. In this section we prove that the intersection number on the right-hand side of (5.2) is equal to one defined by geometric representatives where suitable neighborhoods are replaced by tubular neighborhoods, simplifying our cohomological computations in §5.1.5.

For $\beta\in H_\bullet(X;\mathbb{R})$, let T_β be a smooth submanifold of X with fundamental class $[T_\beta]=\beta$. The geometric representative $\mathcal{V}(\beta)$ was defined in [18, §3.2] by pulling back a geometric representative from the quotient space of connections over a 'suitable' neighborhood U_β of T_β. Recall that a *suitable neighborhood* of T_β was defined in [18, Definition 3.8] or [31, p. 589] as a smoothing of the union of a tubular neighborhood of T_β and a set of loops $\{\gamma_i\}$, where the loops γ_i generate $H_1(X;\mathbb{Z}/2\mathbb{Z})$, are mutually disjoint, and are transverse to T_β. We need in [18] to use a suitable neighborhood rather than a tubular neighborhood because if $H^1(X;\mathbb{Z}/2\mathbb{Z}) \neq 0$, there could be a point $[A,0] \in \mathcal{M}_{\mathfrak{t}'}$ such that the restriction of the induced SO(3) connection $\hat{A}$ on $\mathfrak{g}_V$ to a tubular neighborhood of T_β would be reducible [31, p. 586] even if $\hat{A}$ is not globally reducible. We note that the assumption that $w \pmod 2$ is good (as defined prior to (2.14)) implies that there are no reducible, zero-section pairs in $\bar{\mathcal{M}}_{\mathfrak{t}'}$; Lemmas 5.3, 5.4, and 5.5 rely on this constraint on w. The following lemma shows that when defining geometric representatives near the strata of reducible pairs (but not zero-section pairs) it suffices to use tubular rather than suitable neighborhoods of T_β:

Lemma 5.3. *Assume $w \in H^2(X;\mathbb{Z})$ is such that $w \pmod 2$ is good. Given a Riemannian metric on X and a pair $(\mathfrak{t}',\mathfrak{s})$ with $\ell(\mathfrak{t}',\mathfrak{s}) = 1$ and $w_2(\mathfrak{t}') \equiv w \pmod 2$, there are positive constants ε_0 and δ_0 such that for all positive $\varepsilon \leq \varepsilon_0$ and $\delta \leq \delta_0$ used in the definition of $\mathcal{M}^{\text{vir}}_{\mathfrak{t}',\mathfrak{s}}$, the following holds. For $\beta \in H_\bullet(X;\mathbb{R})$, let T_β be the submanifold of X with $[T_\beta] = \beta$ and let $\nu(\beta)$ be a tubular neighborhood of T_β. Then, for any $[A,\Phi,\mathbf{x}] \in \boldsymbol{\gamma}(\bar{\mathcal{M}}^{\text{vir},*}_{\mathfrak{t}',\mathfrak{s}})$, the restriction of $\hat{A}$ to $\nu(\beta)$ is not reducible.*

Proof. Because $w_2(\mathfrak{t}') = w_2(\mathfrak{t})$ is good, there are no zero-section pairs in $M_{\mathfrak{s}}$. Thus, if the parameters ε and δ in the definition of $\mathcal{M}^{\text{vir}}_{\mathfrak{t}',\mathfrak{s}}$ are sufficiently small, then the neighborhood $\boldsymbol{\gamma}(\bar{\mathcal{M}}^{\text{vir},*}_{\mathfrak{t}',\mathfrak{s}})$ is disjoint from $\bar{M}^w_\kappa$. Hence, for any point $[A,\Phi,\mathbf{x}] \in \boldsymbol{\gamma}(\bar{\mathcal{M}}^{\text{vir},*}_{\mathfrak{t}',\mathfrak{s}})$, the section Φ is not identically zero and $\hat{A}$ is not a reducible connection. By [18, Theorem 3.10] or [16, Theorem 5.10], the restriction of $\hat{A}$ to any open subspace of X cannot be reducible. $\square$

Lemma 5.3 implies that there is a geometric representative, $\mathcal{V}'(\beta)$ for $\mu_p(\beta)$, pulled back from the quotient space of irreducible SO(3) connections, $\mathcal{B}^*(\nu(\beta))$, over the tubular neighborhood $\nu(\beta)$. This geometric representative is then constructed by the same methods used to construct $\mathcal{V}(\beta)$ in [31]. The only difference between $\mathcal{V}(\beta)$ and $\mathcal{V}'(\beta)$ is that $\mathcal{V}(\beta)$ is pulled back from $\mathcal{B}^*(U_\beta)$ rather than $\mathcal{B}^*(\nu(\beta))$.

Henceforth, we assume that the $z \in \mathbb{A}(X)$ considered in Proposition 5.2 is a monomial, $z = \beta_1\beta_2\cdots\beta_m$. We can then choose the submanifold T_{β_i} defining the geometric representative $\mathcal{V}'(\beta_i)$, the points $x_1,\dots,x_\eta$ defining the geometric representatives $\mathcal{W}$, and their tubular neighborhoods so that for any point $x \in X$ we have

$$\sum_{\{\beta_i:\ x\in\nu'(\beta_i)\}} \deg(\beta_i) + \sum_{\{x_j:\ x\in\nu'(x_j)\}} 4 \leq 4, \tag{5.10}$$

where $\nu'(\beta_i)$ is a tubular neighborhood of T_{β_i} with $\nu(\beta_i) \Subset \nu'(\beta_i)$; compare [31, Equation (2.7)]. Let $\mathcal{V}'(z) = \cap_i \mathcal{V}'(\beta_i)$ and let $\bar{\mathcal{V}}'(z)$ denote the Uhlenbeck closure of $\mathcal{V}'(z)$. The following lemma shows we can define an intersection number of these new geometric representatives with $\bar{\mathbf{L}}_{\mathfrak{t}',\mathfrak{s}}$.

Lemma 5.4. *Assume $w \in H^2(X;\mathbb{Z})$ is such that $w \pmod 2$ is good. Suppose $(\mathfrak{t}',\mathfrak{s})$ is a pair with $\ell(\mathfrak{t}',\mathfrak{s}) = 1$ and $w_2(\mathfrak{t}') \equiv w \pmod 2$. Then the following hold:*

1. *The intersection, $\bar{\mathcal{V}}'(z) \cap \bar{\mathcal{W}}^\eta \cap \bar{\mathbf{L}}_{\mathfrak{t}',\mathfrak{s}}$, is contained in the top stratum $\mathbf{L}_{\mathfrak{t}',\mathfrak{s}}$ of $\bar{\mathbf{L}}_{\mathfrak{t}',\mathfrak{s}} \subset \bar{\mathcal{M}}_{\mathfrak{t}'}/S^1$ and is disjoint from the edge (3.76).*
2. *The following intersection is empty,*

$$\bar{\mathcal{V}}'(z) \cap \bar{\mathcal{W}}^\eta \cap \boldsymbol{\gamma}\left(\boldsymbol{\chi}_s^{-1}(0) \cap \mathbf{L}^{\text{low}}_{\mathfrak{t}',\mathfrak{s}}\right) = \varnothing.$$

Proof. Inequality (5.10) and the dimension-counting argument used in the proof of [18, Corollary 3.18] imply that the intersection $\bar{\mathcal{V}}'(z) \cap \bar{\mathcal{W}}^\eta$ is disjoint from the lower level, $(\bar{\mathcal{M}}_{\mathfrak{t}'} - \mathcal{M}_{\mathfrak{t}'}) \cap \boldsymbol{\gamma}(\bar{\mathcal{M}}^{\text{vir},*}_{\mathfrak{t}',\mathfrak{s}})/S^1$. The first assertion then follows from the proof of Lemma 3.9.

The image $\boldsymbol{\gamma}(\boldsymbol{\chi}_s^{-1}(0)\cap\mathbf{L}^{\text{low}}_{\mathfrak{t}',\mathfrak{s}})$ is the intersection of $\boldsymbol{\gamma}(\bar{\mathbf{L}}^{\text{vir}}_{\mathfrak{t}',\mathfrak{s}})$ with the union of the lower levels, $(\bar{\mathcal{M}}_{\mathfrak{t}'}-\mathcal{M}_{\mathfrak{t}'})/S^1$. Therefore, the second assertion follows from the first and the fact that $\boldsymbol{\gamma}$ preserves strata on $\bar{\mathcal{M}}^{\text{vir}}_{\mathfrak{t}',\mathfrak{s}}$. □

We now prove that replacing $\bar{\mathcal{V}}(z)$ with $\bar{\mathcal{V}}'(z)$ does not change the intersection number in (5.2).

Lemma 5.5. *Assume $w\in H^2(X;\mathbb{Z})$ is such that $w \pmod 2$ is good. Suppose $(\mathfrak{t}',\mathfrak{s})$ is a pair with $\ell(\mathfrak{t}',\mathfrak{s})=1$ and $w_2(\mathfrak{t}')\equiv w \pmod 2$. Then, for $z\in\mathbb{A}(X)$ and $\eta\in\mathbb{N}$ as in Proposition 5.2, we have:*

$$\#\left(\bar{\mathcal{V}}(z)\cap\bar{\mathcal{W}}^{\eta}\cap\bar{\mathbf{L}}_{\mathfrak{t}',\mathfrak{s}}\right)=\#\left(\bar{\mathcal{V}}'(z)\cap\bar{\mathcal{W}}^{\eta}\cap\bar{\mathbf{L}}_{\mathfrak{t}',\mathfrak{s}}\right). \tag{5.11}$$

Proof. Both sides of (5.11) are linear in z, so we may assume without loss of generality that $z=\beta_1\cdots\beta_m$ for $\beta_i\in H_\bullet(X;\mathbb{R})$.

First, we note that the intersection numbers on both sides of (5.11) do not change if we decrease the parameter δ defining $\mathbf{L}^{\text{vir}}_{\mathfrak{t}',\mathfrak{s}}$ (to another positive generic value) because of the obvious cobordism defined by this change of parameter. Thus, in the proof we may assume that the parameter δ is as small as desired.

There is a cobordism

$$\mathcal{H}\subset\boldsymbol{\gamma}(\mathcal{M}^{\text{vir}}_{\mathfrak{t}',\mathfrak{s}}/S^1)\times[0,1],$$

with boundaries given by

$$\left(\mathcal{V}(\beta)\cap\boldsymbol{\gamma}(\mathcal{M}^{\text{vir}}_{\mathfrak{t}',\mathfrak{s}}/S^1)\right)\times\{0\}\quad\text{and}\quad\left(\mathcal{V}'(\beta)\cap\boldsymbol{\gamma}(\mathcal{M}^{\text{vir}}_{\mathfrak{t}',\mathfrak{s}}/S^1)\right)\times\{1\}.$$

Such a cobordism exists because $\mathcal{V}(\beta)$ and $\mathcal{V}'(\beta)$ are defined by pullbacks (by the appropriate restriction maps) of zero loci of sections of the same line bundle, for $\beta\in H_2(X;\mathbb{R})$, or by the degeneracy locus of sections of the same vector bundle for $\beta\in H_0(X)$. If $z=z_1\beta_1$, where $z_1\in\mathbb{A}(X)$ and $\beta_1\in H_\bullet(X;\mathbb{R})$, we replace $\bar{\mathcal{V}}(z)$ with $\bar{\mathcal{V}}(z_1)\cap\bar{\mathcal{V}}'(\beta_1)$ in the intersection (5.11) as follows. By perturbing the cobordism $\mathcal{H}$, we can assume that $\mathcal{H}$ is transverse to

$$\left(\bar{\mathcal{V}}(z_1)\cap\bar{\mathcal{W}}^{\eta}\cap\bar{\mathbf{L}}_{\mathfrak{t}',\mathfrak{s}}\cap\boldsymbol{\gamma}(\mathbf{L}^{\text{vir},i}_{\mathfrak{t}',\mathfrak{s}})\right)\times[0,1]. \tag{5.12}$$

The dimension-counting arguments in the proof of Corollary 3.18 in [18] show that the closure $\bar{\mathcal{H}}$ in $\boldsymbol{\gamma}(\bar{\mathcal{M}}^{\text{vir}}_{\mathfrak{t}',\mathfrak{s}}/S^1)\times[0,1]$ of the cobordism $\mathcal{H}$ will not intersect

$$\left(\bar{\mathcal{V}}(z_1)\cap\bar{\mathcal{W}}^{\eta}\cap\left(\bar{\mathbf{L}}_{\mathfrak{t}',\mathfrak{s}}-\mathbf{L}_{\mathfrak{t}',\mathfrak{s}}\right)\right)\times[0,1].$$

Then, for a sufficiently small parameter δ, the space $\bar{\mathcal{H}}$ will not intersect

$$\left(\bar{\mathcal{V}}(z_1)\cap\bar{\mathcal{W}}^{\eta}\cap\bar{\mathbf{L}}_{\mathfrak{t}',\mathfrak{s}}\cap\boldsymbol{\gamma}(\bar{\mathbf{L}}^{\text{vir},s}_{\mathfrak{t}',\mathfrak{s}})\right)\times[0,1].$$

Therefore the intersection $\bar{\mathcal{H}}\cap\left(\bar{\mathcal{V}}(z_1)\cap\bar{\mathcal{W}}^{\eta}\cap\bar{\mathbf{L}}_{\mathfrak{t}',\mathfrak{s}}\times[0,1]\right)$ is contained in the transverse intersection of $\mathcal{H}$ with the space (5.12). Hence, this last intersection is a family of smooth, compact, oriented one-dimensional submanifolds with one boundary given by the set of points in the intersection on the left-hand side of (5.11) and the other boundary given by the set of points in the intersection

$$\bar{\mathcal{V}}(z_1)\cap\bar{\mathcal{V}}'(\beta_1)\cap\bar{\mathcal{W}}^{\eta}\cap\bar{\mathbf{L}}_{\mathfrak{t}',\mathfrak{s}}.$$

We now repeat this process with each β_i in the product z until $\bar{\mathcal{V}}(z)$ has been replaced by $\bar{\mathcal{V}}'(z)$. This completes the proof. □

From this point onwards, we shall work with the geometric representative $\bar{\mathcal{V}}'(z)$ defined by tubular rather than suitable neighborhoods. By choosing generic parameters ε and δ in the definition of $\bar{\mathbf{L}}^{\text{vir}}_{\mathfrak{t}',\mathfrak{s}}$ (see (3.72)), we can also ensure that these geometric representatives are transverse to $\mathbf{L}^{\text{vir},i}_{\mathfrak{t}',\mathfrak{s}}$. To simplify notation, we shall omit the primes, writing $\bar{\mathcal{V}}(z)$ for $\bar{\mathcal{V}}'(z)$, as the original geometric representatives defined by suitable neighborhoods will not reappear in the remainder of this article.

5.1.3. *Geometric representatives and cocycles.* We now define cocycles dual to the geometric representatives $\mathcal{V}(\beta)$ and $\mathcal{W}$. We first recall the following definition:

Definition 5.6. [31, p 588]. Let Z be a smoothly stratified space. A *geometric representative* for a real cohomology class μ of dimension d on Z is a closed, smoothly stratified subspace $\mathcal{V}$ of Z together with a real coefficient q, the *multiplicity*, satisfying

1. The intersection $Z_0 \cap \mathcal{V}$ of $\mathcal{V}$ with the top stratum Z_0 of Z has codimension d in Z_0 and has an oriented normal bundle.
2. The intersection of $\mathcal{V}$ with all strata of Z other than the top stratum has codimension 2 or more in $\mathcal{V}$.
3. The pairing of μ with a homology class h of dimension d is obtained by choosing a smooth singular cycle σ representing h whose intersection with all strata of $\mathcal{V}$ has codimension $\dim Z_0 - d$ in that stratum of $\mathcal{V}$, and then taking q times the count (with signs) of the intersection points between the cycle and the top stratum of $\mathcal{V}$:
$$\langle \mu, h\rangle = q \cdot \#((Z_0 \cap \mathcal{V}) \cap \sigma).$$

The intersection of the geometric representative with the top stratum with real coefficient, $(\mathcal{V} \cap Z_0, q)$, defines a singular cocycle c representing the restriction of the cohomology class μ to Z_0 with the properties described in the following lemma.

Lemma 5.7. *Let Z_0 be a smooth manifold and let $(\mathcal{V}, q)$ be a geometric representative for a real cohomology class μ on Z_0. Then, there is a singular cocycle c on Z_0 which represents the cohomology class μ and which vanishes when restricted to $Z_0 - \mathcal{V}$.*

Proof. For any smooth manifold W, let $\Delta^\infty_\bullet(W)$ denote the chain complex of smooth singular chains [6, p. 291] and let $S^\bullet_\infty(W;\mathbb{R}) = \operatorname{Hom}(\Delta^\infty_\bullet(W),\mathbb{R})$ be the complex of smooth singular cochains. For any smooth submanifold $Y \subset W$, we define the complex of smooth singular, relative cochains by
$$S^\bullet_\infty(W,Y;\mathbb{R}) = \operatorname{Hom}(\Delta^\infty_\bullet(W)/\Delta^\infty_\bullet(Y),\mathbb{R}).$$
We will write $H^\bullet_\infty(W;\mathbb{R})$ and $H^\bullet_\infty(W,Y;\mathbb{R})$ for the homology of the complexes $S^\bullet_\infty(W;\mathbb{R})$ and $S^\bullet_\infty(W,Y;\mathbb{R})$, respectively. Thus, $H^\bullet_\infty(W;\mathbb{R})$ is the smooth singular cohomology of W and $H^\bullet_\infty(W,Y;\mathbb{R})$ is the smooth singular, relative cohomology of (W,Y). By the de Rham theorem (see the discussion in [6, p. 291]), there is a functorial isomorphism $H^p(W;\mathbb{R}) \cong H^\bullet_\infty(W;\mathbb{R})$. By applying the Five Lemma to the long exact sequences of the pair (W,Y) in singular and smooth singular cohomology, we obtain an isomorphism $H^\bullet(W,Y;\mathbb{R}) \cong H^\bullet_\infty(W,Y;\mathbb{R})$.

We first define a smooth singular cocycle, that is, a closed $c' \in S^\bullet_\infty(Z_0;\mathbb{R})$, which represents the cohomology class μ. For any smooth singular chain σ of dimension equal to that of μ, set

$$\langle c', \sigma\rangle = q \cdot \#((\mathcal{V} \cap Z_0) \cap \sigma'), \tag{5.13}$$

where σ' is any smooth singular chain in Z_0 which is homologous in Z_0 to σ and transverse to $\mathcal{V}$. By definition of $\mathcal{V}$, if σ is a smooth singular cycle in Z_0 which represents a homology class h, we then have

$$\langle c', \sigma \rangle = \langle \mu, h \rangle,$$

so c' is a smooth singular cocycle representing the cohomology class μ on Z_0. (By the de Rham isomorphism, it does not matter whether we consider μ as a singular cohomology class or a smooth singular cohomology class.) Because the restriction of c' to $Z_0 - \mathcal{V}$ vanishes, the cocycle c' defines an element $[c']$ of the smooth singular, relative cohomology $H^\bullet_\infty(Z_0, Z_0 - \mathcal{V}; \mathbb{R})$ which maps to μ under the homomorphism $H^\bullet_\infty(Z_0, Z_0 - \mathcal{V}; \mathbb{R}) \to H^\bullet_\infty(Z_0; \mathbb{R})$. Let $\alpha \in H^\bullet(Z_0, Z_0 - \mathcal{V}; \mathbb{R})$ be the element of relative singular cohomology given by the image of $[c']$ under the isomorphism $H^\bullet_\infty(Z_0, Z_0 - \mathcal{V}; \mathbb{R}) \cong H^\bullet(Z_0, Z_0 - \mathcal{V}; \mathbb{R})$ and let $\hat{c} \in S^\bullet(Z_0, Z_0 - \mathcal{V}; \mathbb{R})$ be any representative of α. Then the element $c \in S^\bullet(Z_0; \mathbb{R})$ given by the image of $\hat{c}$ under the homomorphism $S^\bullet(Z_0, Z_0 - \mathcal{V}; \mathbb{R}) \to S^\bullet(Z_0; \mathbb{R})$ will satisfy the conclusion of the lemma. □

For $\beta \in H_\bullet(X; \mathbb{R})$, we let c_β be the cocycle on $\mathcal{M}^{\mathrm{vir}}_{\mathfrak{t}',\mathfrak{s}}/S^1$ representing $\mu_p(\beta)$, defined by Lemma 5.7 and the geometric representative $\mathcal{V}(\beta)$; we let $c_{\mathcal{W}}$ be the cocycle on $\mathcal{M}^{\mathrm{vir}}_{\mathfrak{t}',\mathfrak{s}}/S^1$ representing μ_c, defined as in Lemma 5.7, by the geometric representative $\mathcal{W}$. Note that by considering the smooth manifold Z_0 in Lemma 5.7 to be the image of $\mathcal{M}^{\mathrm{vir}}_{\mathfrak{t}',\mathfrak{s}}/S^1$ under the composition of the gluing map and the restriction map $r_{\nu(\beta)}$, we can assume the cocycle c_β is pulled back from $\mathcal{B}^*(\nu(\beta))$.

5.1.4. *Relative Euler classes and intersection numbers.* We next prove that the intersection number in equation (5.2) is equal to a pairing of a relative cohomology class with the relative fundamental class $[\mathbf{L}^{\mathrm{vir},i}_{\mathfrak{t}',\mathfrak{s}}, \partial \mathbf{L}^{\mathrm{vir},i}_{\mathfrak{t}',\mathfrak{s}}]$ of $\mathbf{L}^{\mathrm{vir},i}_{\mathfrak{t}',\mathfrak{s}}$.

Definition 5.8. If $e(V)$ is the Euler class of a real-rank r, oriented vector bundle V a CW complex over M and $\chi : U \subset M \to V$ is a continuous, local section of V, then the *relative Euler class* [29] of the pair (V, χ) is a relative cohomology class,

$$e(V, \chi) \in H^r(M, U - \chi^{-1}(0); \mathbb{Z}),$$

satisfying the relation

$$j^*_\chi e(V, \chi) = e(V), \tag{5.14}$$

defined by the inclusion map:

$$j_\chi : (M, \varnothing) \to (M, U - \chi^{-1}(0)).$$

The relative Euler class $e(\Upsilon_{\mathfrak{t}',\mathfrak{s}}/S^1, \boldsymbol{\chi})$ of the obstruction bundle $\Upsilon_{\mathfrak{t}',\mathfrak{s}}/S^1$ and obstruction section $\boldsymbol{\chi}$ of Theorem 3.8 thus defines a relative cohomology class

$$e(\Upsilon_{\mathfrak{t}',\mathfrak{s}}/S^1, \boldsymbol{\chi}) \in H^{2r_\Xi+2}(\mathbf{L}^{\mathrm{vir}}_{\mathfrak{t}',\mathfrak{s}}, \mathbf{L}^{\mathrm{vir}}_{\mathfrak{t}',\mathfrak{s}} - \boldsymbol{\chi}^{-1}(0); \mathbb{Z}). \tag{5.15}$$

Recall that r_Ξ is the complex rank of the background component of the obstruction bundle, $\bar{\Upsilon}^s_{\mathfrak{t}',\mathfrak{s}}/S^1$, defined in §3.6.1. For $z = \beta_1 \cdots \beta_m \in \mathbb{A}(X)$, define a subspace

$$\mathcal{K}(z, \eta) = \mathcal{V}(z) \cap \mathcal{W}^\eta \cap \mathbf{L}^{\mathrm{vir}}_{\mathfrak{t}',\mathfrak{s}} \subset \mathbf{L}^{\mathrm{vir}}_{\mathfrak{t}',\mathfrak{s}},$$

with closure

$$\bar{\mathcal{K}}(z, \eta) = \bar{\mathcal{V}}(z) \cap \bar{\mathcal{W}}^\eta \cap \bar{\mathbf{L}}^{\mathrm{vir}}_{\mathfrak{t}',\mathfrak{s}} \subset \bar{\mathbf{L}}^{\mathrm{vir}}_{\mathfrak{t}',\mathfrak{s}}. \tag{5.16}$$

The cocycle on $\mathcal{M}^{\mathrm{vir}}_{\mathfrak{t}',\mathfrak{s}}$,

$$c(z, \eta) := c_{\beta_1} \smile \cdots \smile c_{\beta_m} \smile c^\eta_{\mathcal{W}}, \tag{5.17}$$

vanishes on the complement of $\mathcal{K}(z,\eta)$ in $\mathcal{M}^{\mathrm{vir}}_{\mathfrak{t}',\mathfrak{s}}$, and so defines a relative cohomology class

$$[c(z,\eta)] \in H^{\deg(z)+2\eta}(\mathbf{L}^{\mathrm{vir}}_{\mathfrak{t}',\mathfrak{s}}, \mathbf{L}^{\mathrm{vir}}_{\mathfrak{t}',\mathfrak{s}} - \mathcal{K}(z,\eta);\mathbb{R}). \tag{5.18}$$

By Lemma 5.4 and the continuity of the gluing map $\boldsymbol{\gamma}$, the following intersection is empty:

$$\bar{\mathcal{K}}(z,\eta) \cap \mathbf{L}^{\mathrm{low}}_{\mathfrak{t}',\mathfrak{s}} \cap \boldsymbol{\chi}^{-1}(0) = \varnothing.$$

Therefore, noting that the link definition (3.72) gives

$$\bar{\mathbf{L}}^{\mathrm{vir},s}_{\mathfrak{t}',\mathfrak{s}} = \mathbf{L}^{\mathrm{vir},s}_{\mathfrak{t}',\mathfrak{s}} \sqcup \left(\partial N_{\mathfrak{t},\mathfrak{s}}(\varepsilon)/S^1 \times X\right),$$

and that $\bar{\mathbf{L}}^{\mathrm{vir},s}_{\mathfrak{t}',\mathfrak{s}}$ retracts onto the lower level $\partial N_{\mathfrak{t},\mathfrak{s}}(\varepsilon)/S^1 \times X$ of $\bar{\mathbf{L}}^{\mathrm{vir},s}_{\mathfrak{t}',\mathfrak{s}}$ by shrinking the instanton scale δ, we can also arrange that the following intersection is empty by taking δ to be sufficiently small:

$$\mathcal{K}(z,\eta) \cap \mathbf{L}^{\mathrm{vir},s}_{\mathfrak{t}',\mathfrak{s}} \cap \boldsymbol{\chi}^{-1}(0) = \varnothing. \tag{5.19}$$

Because cup product of relative cohomology is a map [50, p. 251],

$$H^i(Y,A)\otimes H^j(Y,B) \to H^{i+j}(Y, A\cup B),$$

and (5.19) yields

$$\left(\mathbf{L}^{\mathrm{vir}}_{\mathfrak{t}',\mathfrak{s}} - \boldsymbol{\chi}^{-1}(0)\right) \cup \left(\mathbf{L}^{\mathrm{vir}}_{\mathfrak{t}',\mathfrak{s}} - \mathcal{K}(z,\eta)\right) \supset \mathbf{L}^{\mathrm{vir},s}_{\mathfrak{t}',\mathfrak{s}},$$

we see from (5.15) and (5.18) that the cup product $[c(z,\eta)] \smile e(\Upsilon_{\mathfrak{t}',\mathfrak{s}}/S^1,\boldsymbol{\chi})$ vanishes on $\mathbf{L}^{\mathrm{vir},s}_{\mathfrak{t}',\mathfrak{s}}$ and therefore we have

$$[c(z,\eta)] \smile e(\Upsilon_{\mathfrak{t}',\mathfrak{s}}/S^1,\boldsymbol{\chi}) \in H^{\max}\left(\mathbf{L}^{\mathrm{vir}}_{\mathfrak{t}',\mathfrak{s}}, \mathbf{L}^{\mathrm{vir},s}_{\mathfrak{t}',\mathfrak{s}};\mathbb{R}\right). \tag{5.20}$$

The inclusion

$$\iota : \mathbf{L}^{\mathrm{vir}}_{\mathfrak{t}',\mathfrak{s}} \to \bar{\mathbf{L}}^{\mathrm{vir}}_{\mathfrak{t}',\mathfrak{s}}, \tag{5.21}$$

induces an isomorphism of relative homology by excision,

$$\iota_* : H_\bullet\left(\mathbf{L}^{\mathrm{vir}}_{\mathfrak{t}',\mathfrak{s}}, \mathbf{L}^{\mathrm{vir},s}_{\mathfrak{t}',\mathfrak{s}};\mathbb{R}\right) \cong H_\bullet\left(\bar{\mathbf{L}}^{\mathrm{vir}}_{\mathfrak{t}',\mathfrak{s}}, \bar{\mathbf{L}}^{\mathrm{vir},s}_{\mathfrak{t}',\mathfrak{s}};\mathbb{R}\right). \tag{5.22}$$

Hence, we can express the intersection number in (5.2) as a pairing of relative homology and cohomology classes:

Lemma 5.9. *If $[\mathbf{L}^{\mathrm{vir},i}_{\mathfrak{t}',\mathfrak{s}}, \partial\mathbf{L}^{\mathrm{vir},i}_{\mathfrak{t}',\mathfrak{s}}]$ is the relative fundamental class of $\mathbf{L}^{\mathrm{vir},i}_{\mathfrak{t}',\mathfrak{s}}$, if $\bar{\jmath}_{\mathbf{L}^i}$ is the inclusion map* (5.8)*, and if ι_* is the excision isomorphism* (5.22)*, then*

$$\begin{aligned} &\#\left(\bar{\mathcal{V}}(z)\cap\bar{\mathcal{W}}^\eta\cap\bar{\mathbf{L}}_{\mathfrak{t}',\mathfrak{s}}\right) \\ &= \left\langle [c(z,\eta)] \smile e\left(\Upsilon_{\mathfrak{t}',\mathfrak{s}}/S^1, \boldsymbol{\chi}|_{\mathbf{L}^{\mathrm{vir},s}_{\mathfrak{t}',\mathfrak{s}}}\right), \iota_*^{-1}(\bar{\jmath}_{\mathbf{L}^i})_*\left[\mathbf{L}^{\mathrm{vir},i}_{\mathfrak{t}',\mathfrak{s}}, \partial\mathbf{L}^{\mathrm{vir},i}_{\mathfrak{t}',\mathfrak{s}}\right]\right\rangle. \end{aligned} \tag{5.23}$$

Proof. Let

$$\jmath_{\mathbf{L}^i} : \left(\mathbf{L}^{\mathrm{vir},i}_{\mathfrak{t}',\mathfrak{s}}, \partial\mathbf{L}^{\mathrm{vir},i}_{\mathfrak{t}',\mathfrak{s}}\right) \to \left(\mathbf{L}^{\mathrm{vir}}_{\mathfrak{t}',\mathfrak{s}}, \mathbf{L}^{\mathrm{vir},s}_{\mathfrak{t}',\mathfrak{s}}\right)$$

be the inclusion map of pairs. The equality $\bar{\jmath}_{\mathbf{L}^i} = \iota\circ\jmath_{\mathbf{L}^i}$ implies that

$$\iota_*^{-1}(\bar{\jmath}_{\mathbf{L}^i})_*\left[\mathbf{L}^{\mathrm{vir},i}_{\mathfrak{t}',\mathfrak{s}}, \partial\mathbf{L}^{\mathrm{vir},i}_{\mathfrak{t}',\mathfrak{s}}\right] = (\jmath_{\mathbf{L}^i})_*\left[\mathbf{L}^{\mathrm{vir},i}_{\mathfrak{t}',\mathfrak{s}}, \partial\mathbf{L}^{\mathrm{vir},i}_{\mathfrak{t}',\mathfrak{s}}\right]$$

is the image of the relative fundamental class of $\mathbf{L}^{\mathrm{vir},i}_{\mathfrak{t}',\mathfrak{s}}$ under the inclusion $\jmath_{\mathbf{L}^i}$. By (5.19), the intersection on the left-hand-side of (5.23) is a finite collection of points in the interior of $\mathbf{L}^{\mathrm{vir},i}_{\mathfrak{t}',\mathfrak{s}}$ (with the multiplicities of the geometric representatives). The obstruction section $\boldsymbol{\chi}$ vanishes transversely on $\mathbf{L}^{\mathrm{vir},i}_{\mathfrak{t}',\mathfrak{s}}$ by Theorem 3.8 and the assumption that the parameters

ε and δ in the definition of the link are generic, so the relative fundamental class of the manifold-with-boundary $\mathbf{L}_{\mathfrak{t}',\mathfrak{s}} - \mathbf{L}^{\mathrm{vir},s}_{\mathfrak{t}',\mathfrak{s}}$ is given by

$$e\left(\Upsilon_{\mathfrak{t}',\mathfrak{s}}/S^1, \boldsymbol{\chi}|_{\mathbf{L}^{\mathrm{vir},s}_{\mathfrak{t}',\mathfrak{s}}}\right) \cap (\jmath_{\mathbf{L}^i})_* \left[\mathbf{L}^{\mathrm{vir},i}_{\mathfrak{t}',\mathfrak{s}}, \partial\mathbf{L}^{\mathrm{vir},i}_{\mathfrak{t}',\mathfrak{s}}\right]. \tag{5.24}$$

The geometric representatives intersect $\mathbf{L}_{\mathfrak{t}',\mathfrak{s}} - \mathbf{L}^{\mathrm{vir},s}_{\mathfrak{t}',\mathfrak{s}}$ transversely so, by the definition of a geometric representative and the cocycle $c(z,\eta)$, the intersection number on the left-hand-side of (5.23) is given by evaluating $[c(z,\eta)]$ on the relative fundamental class (5.24), yielding (5.23). □

5.1.5. *Extending the cocycles.* To rewrite the pairing in identity (5.23) as a pairing of absolute cohomology classes via the isomorphism $\jmath_*$ in (5.6), we must first eliminate the excision isomorphism ι_*. For this reason, we now show how the cocycles c_β and $c_{\mathcal{W}}$ are, modulo exact cocycles, the restrictions of cocycles on $\bar{\mathbf{L}}^{\mathrm{vir}}_{\mathfrak{t}',\mathfrak{s}}$. Because the cocycle c_β on $\mathbf{L}^{\mathrm{vir}}_{\mathfrak{t}',\mathfrak{s}}$ represents the cohomology class $\mu_p(\beta)$, which extends to $\bar{\mu}_p(\beta)$ on $\bar{\mathbf{L}}^{\mathrm{vir}}_{\mathfrak{t}',\mathfrak{s}}$, by Definition 4.5, we can always find cocycles in the class $\mu_p(\beta)$ which extend to $\bar{\mathbf{L}}^{\mathrm{vir}}_{\mathfrak{t}',\mathfrak{s}}$ and represent $\bar{\mu}_p(\beta)$. However, we emphasize that the changes of the cocycles c_β and $c_{\mathcal{W}}$ which we construct here have the property that they do not alter the pairing in (5.23), which is false for arbitrary choices of cocycles in the cohomology classes $\mu_p(\beta)$ or μ_c.

For any subset $U \subset X$, we define

$$\mathcal{I}(U) = \bar{\mathbf{L}}^{\mathrm{vir}}_{\mathfrak{t}',\mathfrak{s}} \cap \boldsymbol{\gamma}^{-1}\left(\{[A,\Phi,\mathbf{x}] : |\mathbf{x}| \cap U \neq \varnothing\}\right), \tag{5.25}$$

where $|\mathbf{x}| \subset X$ denotes the support of $\mathbf{x} \in \mathrm{Sym}^\ell(X)$; since $\ell = 1$ here, the set on the right-hand side above is simply $\{[A,\Phi,x] : x \in U\}$. Because the cocycles c_β and $c_{\mathcal{W}}$ are pulled back from $\mathcal{B}^*(\nu(\beta))$ and $\mathcal{C}^{*,0}_{\mathfrak{t}}(\nu(x))/S^1$, respectively, by the composition of the gluing map and the restriction map, the cocycles c_β and $c_{\mathcal{W}}$ naturally extend over $\bar{\mathbf{L}}^{\mathrm{vir}}_{\mathfrak{t}',\mathfrak{s}} - \mathcal{I}(\nu(\beta))$ and $\bar{\mathbf{L}}^{\mathrm{vir}}_{\mathfrak{t}',\mathfrak{s}} - \mathcal{I}(\nu(x))$, respectively. (Note that we use the continuity of the gluing map with respect to Uhlenbeck limits here.) To construct the extension of c_β to $\bar{\mathbf{L}}^{\mathrm{vir}}_{\mathfrak{t}',\mathfrak{s}}$, we first show in Lemma 5.12 that the cohomology class represented by c_β on $\bar{\mathbf{L}}^{\mathrm{vir}}_{\mathfrak{t}',\mathfrak{s}} - \mathcal{I}(\nu(\beta))$ is equal to the restriction of $\bar{\mu}_p(\beta)$ to this subspace. The proof of Lemma 5.12 uses some cohomological computations appearing in Lemmas 5.10 and 5.11 and the continuity of the gluing map mentioned above. Then, in Lemma 5.14 we describe the perturbation, in the form of an exact cocycle, necessary to extend c_β from $\bar{\mathbf{L}}^{\mathrm{vir}}_{\mathfrak{t}',\mathfrak{s}} - \mathcal{I}(\nu(\beta))$ to a cocycle $\bar{c}_\beta$ on $\bar{\mathbf{L}}^{\mathrm{vir}}_{\mathfrak{t}',\mathfrak{s}}$. The construction of $\bar{c}_\beta$ will show that it represents the cohomology class $\bar{\mu}_p(\beta)$.

We begin by proving the following lemma about the cohomology of the link. While the proofs of Lemmas 5.10 and 5.11 apply only to the link of the stratum $M_{\mathfrak{s}} \times \mathrm{Sym}^\ell(X)$ when $\ell = 1$, the results should be true for any $\ell \geq 1$. However, the proofs for larger ℓ may be difficult so it may be easier to prove Lemma 5.12 by a direct, if tedious, analysis of the bundles discussed there.

Lemma 5.10. *If $\pi_X : \mathbf{L}^{\mathrm{vir}}_{\mathfrak{t}',\mathfrak{s}} \to X$ is the projection map, then the induced map on cohomology is injective:*

$$\pi_X^* : H^\bullet(X;\mathbb{R}) \to H^\bullet(\mathbf{L}^{\mathrm{vir}}_{\mathfrak{t}',\mathfrak{s}};\mathbb{R}).$$

Proof. The definition of $\mathbf{L}^{\mathrm{vir}}_{\mathfrak{t}',\mathfrak{s}}$ in (3.75) and the existence of a retraction from the uncompactified gluing data space $\mathrm{Gl}_{\mathfrak{t}}(\delta)$ to the boundary $\partial\bar{\mathrm{G}}\mathrm{l}_{\mathfrak{t}}(\delta) \subset \mathrm{Gl}_{\mathfrak{t}}(\delta)$ imply that $\mathbf{L}^{\mathrm{vir}}_{\mathfrak{t}',\mathfrak{s}}$ deformation-retracts to $\mathbf{L}^{\mathrm{vir},i}_{\mathfrak{t}',\mathfrak{s}}$. In turn, $\mathbf{L}^{\mathrm{vir},i}_{\mathfrak{t}',\mathfrak{s}}$ deformation-retracts to the space $\mathbf{BL}^{\mathrm{vir},i}_{\mathfrak{t}',\mathfrak{s}}$ defined in (3.83). These retractions commute with π_X, so it suffices to prove the lemma for the restriction of the projection π_X to $\mathbf{BL}^{\mathrm{vir},i}_{\mathfrak{t}',\mathfrak{s}}$. This restriction of π_X can be written as the composition of the projections

$$\pi_1 : M_{\mathfrak{s}} \times \partial\bar{\mathrm{G}}\mathrm{l}_{\mathfrak{t}}(\delta)/S^1 \to M_{\mathfrak{s}} \times X \quad \text{and} \quad M_{\mathfrak{s}} \times X \to X.$$

The projection $M_{\mathfrak{s}} \times X \to X$ induces an injective homomorphism on cohomology, so we need only verify that π_1 induces an injective homomorphism on cohomology. The map π_1 is the product of the identity on $M_{\mathfrak{s}}$ and the projection

$$\partial\bar{\mathrm{G}}\mathrm{l}_{\mathfrak{t}}(\delta)/S^1 \to X.$$

As will be discussed the proof of Lemma 5.24, the space $\partial\bar{\mathrm{G}}\mathrm{l}_{\mathfrak{t}}(\delta)/S^1$ can be identified with the projectivization of a complex rank-two vector bundle $F \to X$. By [5, Equation (20.7)], the projection map $\mathbb{P}(F) \to X$ induces an injective homomorphism on cohomology. This proves the lemma. □

The following lemma will be used to identify the cohomology classes which the extensions of the cocycles represent; the result should be true for the links of ideal Seiberg-Witten moduli spaces $M_{\mathfrak{s}} \times \mathrm{Sym}^\ell(X)$ defined in [21] for $\ell \geq 1$. The difficulties in extending the proof of Lemma 5.11 to the case $\ell > 1$ lie first in the use of Lemma 5.10 (which is only proven for $\ell = 1$) and secondly in applying the Thom isomorphism theorem (as is done below, before (5.29)) to the analogue of $\nu(\beta)$ in $\mathrm{Sym}^\ell(X)$.

Lemma 5.11. *For $\beta \in H_\bullet(X;\mathbb{R})$, let T_β be a smooth submanifold of X with $[T_\beta] = \beta$ and let $\nu(\beta)$ be a tubular neighborhood of T_β in X. Then for $p = \deg(\beta)$, the following direct sum of restriction maps is injective:*

$$H^p(\bar{\mathbf{L}}^{\mathrm{vir}}_{\mathfrak{t}',\mathfrak{s}} - \mathcal{I}(\nu(\beta));\mathbb{R}) \to H^p(\mathbf{L}^{\mathrm{vir}}_{\mathfrak{t}',\mathfrak{s}};\mathbb{R}) \oplus H^p(\bar{\mathbf{L}}^{\mathrm{vir}}_{\mathfrak{t}',\mathfrak{s}} - \pi_X^{-1}(\nu(\beta));\mathbb{R}). \tag{5.26}$$

Proof. To prove the homomorphism (5.26) is injective, we first observe that it appears in the Meyer-Vietoris sequence for the open cover

$$\bar{\mathbf{L}}^{\mathrm{vir}}_{\mathfrak{t}',\mathfrak{s}} - \mathcal{I}(\nu(\beta)) = \mathbf{L}^{\mathrm{vir}}_{\mathfrak{t}',\mathfrak{s}} \cup \left(\bar{\mathbf{L}}^{\mathrm{vir}}_{\mathfrak{t}',\mathfrak{s}} - \pi_X^{-1}(\nu(\beta))\right).$$

The intersection of these two open sets is $\mathbf{L}^{\mathrm{vir}}_{\mathfrak{t}',\mathfrak{s}} - \pi_X^{-1}(\nu(\beta))$. Hence, the homomorphism (5.26) will be injective if the restriction map

$$H^{p-1}(\mathbf{L}^{\mathrm{vir}}_{\mathfrak{t}',\mathfrak{s}};\mathbb{R}) \to H^{p-1}(\mathbf{L}^{\mathrm{vir}}_{\mathfrak{t}',\mathfrak{s}} - \pi_X^{-1}(\nu(\beta));\mathbb{R}) \tag{5.27}$$

is surjective. As noted in the proof of Lemma 5.10, there is a deformation retraction from $\mathbf{L}^{\mathrm{vir}}_{\mathfrak{t}',\mathfrak{s}}$ to $\mathbf{BL}^{\mathrm{vir},i}_{\mathfrak{t}',\mathfrak{s}}$ which commutes with π_X. Thus, proving the map (5.27) is surjective is equivalent to proving that the following restriction map is surjective:

$$H^{p-1}(\mathbf{BL}^{\mathrm{vir},i}_{\mathfrak{t}',\mathfrak{s}};\mathbb{R}) \to H^{p-1}(\mathbf{BL}^{\mathrm{vir},i}_{\mathfrak{t}',\mathfrak{s}} - \pi_X^{-1}(\nu(\beta));\mathbb{R}), \tag{5.28}$$

The surjectivity of the map (5.28) follows from a discussion of the following diagram in which the vertical maps come from the long exact sequences of the pairs:

$$\begin{array}{ccc}
H^{p-1}(X-\nu(\beta);\mathbb{R}) & \xrightarrow{\pi_X^*} & H^{p-1}(\mathbf{BL}^{\mathrm{vir},i}_{\mathfrak{t}',\mathfrak{s}} - \pi_X^{-1}(\nu(\beta));\mathbb{R}) \\
{\scriptstyle \delta^*_{X,\beta}}\downarrow & & {\scriptstyle \delta^*_{\mathbf{L},\beta}}\downarrow \\
H^p(X, X-\nu(\beta);\mathbb{R}) & \xrightarrow{\pi_X^*} & H^p(\mathbf{BL}^{\mathrm{vir},i}_{\mathfrak{t}',\mathfrak{s}}, \mathbf{BL}^{\mathrm{vir},i}_{\mathfrak{t}',\mathfrak{s}} - \pi_X^{-1}(\nu(\beta));\mathbb{R}) \\
{\scriptstyle \jmath^*_{X,\beta}}\downarrow & & {\scriptstyle \jmath^*_{\mathbf{L},\beta}}\downarrow \\
H^p(X;\mathbb{R}) & \xrightarrow{\pi_X^*} & H^p(\mathbf{BL}^{\mathrm{vir},i}_{\mathfrak{t}',\mathfrak{s}};\mathbb{R})
\end{array}$$

Because $\pi_X^{-1}(T_\beta)$ is a smooth, codimension-p submanifold of $\mathbf{BL}^{\mathrm{vir},i}_{\mathfrak{t}',\mathfrak{s}}$, with the Thom class of its normal bundle given by the pullback of the Thom class $\pi_X^* \operatorname{Th}(\beta)$ of the normal bundle of T_β in X, we see that the relative cohomology

$$H^p(\mathbf{BL}^{\mathrm{vir},i}_{\mathfrak{t}',\mathfrak{s}}, \mathbf{BL}^{\mathrm{vir},i}_{\mathfrak{t}',\mathfrak{s}} - \pi_X^{-1}(\nu(\beta)); \mathbb{R}), \tag{5.29}$$

is generated by $\pi_X^* \operatorname{Th}(\beta)$. By Lemma 5.10, the following class is non-zero:

$$\jmath^*_{\mathbf{L},\beta}\pi^*_X \operatorname{Th}(\beta) = \pi^*_X \jmath^*_{X,\beta} \operatorname{Th}(\beta) = \pi^*_X \operatorname{PD}[\beta].$$

Therefore, the generator $\pi^*_X \operatorname{Th}(\beta)$ of the relative cohomology (5.29) is not in the image of $\delta^*_{\mathbf{L},\beta}$, implying that $\delta^*_{\mathbf{L},\beta}$ is the zero map. Hence, the map (5.28) is surjective. As discussed previously, this implies that the map (5.26) is injective. □

Lemma 5.11 gives the following identity between the cohomology class represented by the cocycle c_β and the restriction of the cohomology class $\bar\mu_p(\beta)$:

Lemma 5.12. *For $\beta \in H_\bullet(X;\mathbb{R})$, let c_β be the cocycle on $\bar{\mathbf{L}}^{\mathrm{vir}}_{\mathfrak{t}',\mathfrak{s}} - \mathcal{I}(\nu(\beta))$ defined at the beginning of §5.1.5 and consider the inclusion map*

$$\iota_\beta : \bar{\mathbf{L}}^{\mathrm{vir}}_{\mathfrak{t}',\mathfrak{s}} - \mathcal{I}(\nu(\beta)) \to \bar{\mathbf{L}}^{\mathrm{vir}}_{\mathfrak{t}',\mathfrak{s}}. \tag{5.30}$$

Let $[c_\beta] \in H^{\deg(\beta)}(\bar{\mathbf{L}}^{\mathrm{vir}}_{\mathfrak{t}',\mathfrak{s}} - \mathcal{I}(\nu(\beta));\mathbb{R})$ denote the cohomology class which c_β represents and let $\bar\mu_p(\beta) \in H^{\deg(\beta)}(\bar{\mathbf{L}}^{\mathrm{vir}}_{\mathfrak{t}',\mathfrak{s}};\mathbb{R})$ be the restriction from $\bar{\mathcal{M}}^{\mathrm{vir},*}_{\mathfrak{t}',\mathfrak{s}}/S^1$ to $\bar{\mathbf{L}}^{\mathrm{vir}}_{\mathfrak{t}',\mathfrak{s}}$ of the class in Definition 4.5. Then,

$$[c_\beta] = \iota^*_\beta \bar\mu_p(\beta). \tag{5.31}$$

Proof. By the definition of c_β in §5.1.3, the restrictions of the cohomology classes in (5.31) to $\mathbf{L}^{\mathrm{vir}}_{\mathfrak{t}',\mathfrak{s}}$ coincide. The identity (5.31) will then follow from Lemma 5.11 if we can prove that the difference of the two cohomology classes is in the kernel of the restriction map given by the second component of the map (5.26):

$$\iota_1^* : H^{\deg(\beta)}(\bar{\mathbf{L}}^{\mathrm{vir}}_{\mathfrak{t}',\mathfrak{s}} - \mathcal{I}(\nu(\beta));\mathbb{R}) \to H^{\deg(\beta)}(\bar{\mathbf{L}}^{\mathrm{vir}}_{\mathfrak{t}',\mathfrak{s}} - \pi_X^{-1}(\nu(\beta));\mathbb{R}).$$

First, we describe the image of $\bar\mu_p(\beta)$ under this map. Because there is an inclusion

$$\left(\bar{\mathbf{L}}^{\mathrm{vir}}_{\mathfrak{t}',\mathfrak{s}} - \pi_X^{-1}(\nu(\beta))\right) \times T_\beta \to \left(\bar{\mathbf{L}}^{\mathrm{vir}}_{\mathfrak{t}',\mathfrak{s}} \times X\right) - (\pi_X \times \mathrm{id}_X)^{-1}(\Delta),$$

the restriction of $(\pi_X \times \mathrm{id}_X)^* \operatorname{PD}[\Delta]/\beta$ to $\bar{\mathbf{L}}^{\mathrm{vir}}_{\mathfrak{t}',\mathfrak{s}} - \pi_X^{-1}(\nu(\beta))$ vanishes. The expression for $\bar\mu_p(\beta)$ in Definition 4.5 then implies that

$$\iota_1^*\iota_\beta^*\bar\mu_p(\beta) = -\tfrac{1}{4}\iota_1^*\iota_\beta^*(p_1(\mathbb{F}^{\mathrm{vir}}_{\mathfrak{t}',\mathfrak{s}})/\beta). \tag{5.32}$$

We now consider the restriction of c_β. Because the geometric representative $\mathcal{V}(\beta)$ is pulled back from the quotient space $\mathcal{B}^*(\nu(\beta))$ of irreducible SO(3) connections over $\nu(\beta)$ by the map

$$r_{\nu(\beta)} \circ \boldsymbol{\gamma} : \bar{\mathbf{L}}^{\mathrm{vir}}_{\mathfrak{t}',\mathfrak{s}} - \mathcal{I}(\nu(\beta)) \to \mathcal{B}^*(\nu(\beta)),$$

the cocycle c_β is pulled back from a cocycle $c_{T,\beta}$ on $\mathcal{B}^*(\nu(\beta))$. By the construction [31, pp. 588-592] of the geometric representatives, the cocycle $c_{T,\beta}$ represents the cohomology class $-\frac{1}{4}p_1(\mathbb{F}_\beta)/\beta$ defined by the universal SO(3) bundle:

$$\mathbb{F}_\beta := \mathcal{A}^*(\nu(\beta)) \times_{\mathcal{G}} \mathfrak{g}_{V'}|_{T_\beta} \to \mathcal{B}^*(\nu(\beta)) \times T_\beta$$

We conclude that

$$[\iota_1^* c_\beta] = -\tfrac{1}{4}\iota_1^*(r_{\nu(\beta)} \circ \boldsymbol{\gamma})^*(p_1(\mathbb{F}_\beta)/\beta). \tag{5.33}$$

To compare the cohomology classes (5.32) and (5.33), we compare the bundles $\mathbb{F}_\beta$ and $\bar{\mathbb{F}}^{\mathrm{vir},*}_{\mathfrak{t}',\mathfrak{s}}$. If $\iota_T : T_\beta \to X$ is the inclusion map, then there is an isomorphism

$$((\iota_\beta \circ \iota_1) \times \iota_T)^* \bar{\mathbb{F}}^{\mathrm{vir},*}_{\mathfrak{t}',\mathfrak{s}} \cong ((r_{\nu(\beta)} \circ \boldsymbol{\gamma} \circ \circ \iota_1) \times \mathrm{id}_{T_\beta})^* \mathbb{F}_\beta. \tag{5.34}$$

(The existence of this isomorphism follows from the method used to obtain the corresponding isomorphism in Lemma 4.3.) The isomorphism (5.34) then implies that

$$\iota_1^*(r_{\nu(\beta)} \circ \boldsymbol{\gamma})^*(p_1(\mathbb{F}_\beta)/\beta) = \iota_1^* \iota_\beta^*(p_1(\bar{\mathbb{F}}^{\mathrm{vir},*}_{\mathfrak{t}',\mathfrak{s}})/\beta)$$

which, together with the identities (5.32) and (5.33), yields the desired result, (5.31). □

A similar argument gives the

Lemma 5.13. *Let $c_{\mathcal{W}}$ be the cocycle defined on $\mathbf{L}^{\mathrm{vir}}_{\mathfrak{t}',\mathfrak{s}} - \mathcal{I}(\nu(x))$ which represents μ_c when restricted to $\mathbf{L}^{\mathrm{vir}}_{\mathfrak{t}',\mathfrak{s}}$. Consider the inclusion map*

$$\iota_{\mathcal{W}} : \bar{\mathbf{L}}^{\mathrm{vir}}_{\mathfrak{t}',\mathfrak{s}} - \mathcal{I}(\nu(x)) \to \bar{\mathbf{L}}^{\mathrm{vir}}_{\mathfrak{t}',\mathfrak{s}}. \tag{5.35}$$

If $\bar{\mu}_c \in H^2(\bar{\mathbf{L}}^{\mathrm{vir}}_{\mathfrak{t}',\mathfrak{s}};\mathbb{R})$ denotes the restriction from $\bar{\mathcal{M}}^{\mathrm{vir},*}_{\mathfrak{t}',\mathfrak{s}}$ to $\bar{\mathbf{L}}^{\mathrm{vir}}_{\mathfrak{t}',\mathfrak{s}}$ of the class in Definition 4.5, then

$$[c_{\mathcal{W}}] = \iota_{\mathcal{W}}^* \bar{\mu}_c. \tag{5.36}$$

We now construct the modification of the cocycle c_β on $\mathbf{L}^{\mathrm{vir}}_{\mathfrak{t}',\mathfrak{s}}$ necessary to extend it to a cocycle on $\bar{\mathbf{L}}^{\mathrm{vir}}_{\mathfrak{t}',\mathfrak{s}}$. For any space A, let $S^p(A;\mathbb{R})$ be the module of real, singular p-cochains on A. For any pair (A,B), let $S^p(A,B;\mathbb{R})$ be the module of real, singular p-cochains on A, vanishing on p-chains in B.

Lemma 5.14. *For $\beta \in H_\bullet(X;\mathbb{R})$, let $\mathcal{U}(\beta) \subset \bar{\mathbf{L}}^{\mathrm{vir}}_{\mathfrak{t}',\mathfrak{s}}$ be any open neighborhood of $\mathcal{I}(\nu(\beta)$ satisfying $\mathcal{I}(\nu(\beta)) \Subset \mathcal{U}(\beta)$. Let ι_β be the inclusion map (5.30). Then, there is a cochain*

$$\theta_\beta \in S^{\deg(\beta)-1}(\bar{\mathbf{L}}^{\mathrm{vir}}_{\mathfrak{t}',\mathfrak{s}} - \mathcal{I}(\nu(\beta)), \bar{\mathbf{L}}^{\mathrm{vir}}_{\mathfrak{t}',\mathfrak{s}} - \mathcal{U}(\beta);\mathbb{R}), \tag{5.37}$$

and a cocycle $\bar{c}_\beta$ representing the class $\bar{\mu}_p(\beta) \in H^\bullet(\bar{\mathbf{L}}^{\mathrm{vir}}_{\mathfrak{t}',\mathfrak{s}};\mathbb{R})$ in Definition 4.5 such that

$$\iota_\beta^* \bar{c}_\beta = c_\beta + \delta\theta_\beta.$$

Proof. Let $\bar{c}'_\beta$ be any cocycle in the cohomology class $\bar{\mu}_p(\beta)$. By Lemma 5.12, there is a cochain θ_0 of degree $\deg(\beta) - 1$ on $\mathbf{L}^{\mathrm{vir}}_{\mathfrak{t}',\mathfrak{s}} - \mathcal{I}(\nu(\beta))$ such that

$$\iota_\beta^* \bar{c}'_\beta = c_\beta + \delta\theta_0. \tag{5.38}$$

Because $\mathcal{I}(\nu(\beta)) \Subset \mathcal{U}(\beta)$, there is an open subspace $\mathcal{U}'(\beta) \subset \bar{\mathbf{L}}^{\mathrm{vir}}_{\mathfrak{t}',\mathfrak{s}}$ with

$$\mathcal{I}(\nu(\beta)) \subset \mathcal{U}'(\beta) \Subset \mathcal{U}(\beta).$$

Hence, the following intersection is empty:

$$\left(\bar{\mathbf{L}}^{\mathrm{vir}}_{\mathfrak{t}',\mathfrak{s}} - \mathcal{U}(\beta)\right) \cap \mathcal{U}'(\beta) = \varnothing.$$

This implies that the map

$$\begin{array}{c} S^p\left(\bar{\mathbf{L}}^{\mathrm{vir}}_{\mathfrak{t}',\mathfrak{s}} - \mathcal{I}(\nu(\beta)), \bar{\mathbf{L}}^{\mathrm{vir}}_{\mathfrak{t}',\mathfrak{s}} - \mathcal{U}(\beta);\mathbb{R}\right) \oplus S^p\left(\bar{\mathbf{L}}^{\mathrm{vir}}_{\mathfrak{t}',\mathfrak{s}} - \mathcal{I}(\nu(\beta)), \mathcal{U}'(\beta);\mathbb{R}\right) \\ \downarrow \\ S^p(\bar{\mathbf{L}}^{\mathrm{vir}}_{\mathfrak{t}',\mathfrak{s}} - \mathcal{I}(\nu(\beta)), \left(\bar{\mathbf{L}}^{\mathrm{vir}}_{\mathfrak{t}',\mathfrak{s}} - \mathcal{U}(\beta)\right) \cap \mathcal{U}'(\beta);\mathbb{R}) \end{array}$$

which is surjective because the pairs appearing in this diagram are excisive couples; see [50, Theorem 4.6.3] and [50, p. 218], is actually a map to the space of absolute cochains,

$$S^p(\bar{\mathbf{L}}^{\mathrm{vir}}_{\mathfrak{t}',\mathfrak{s}} - \mathcal{I}(\nu(\beta)), \varnothing; \mathbb{R}).$$

Thus, we can write $\theta_0 = \theta_\beta + \theta_p$, where

$$\begin{aligned}\theta_\beta &\in S^{\deg(\beta)-1}\left(\bar{\mathbf{L}}^{\mathrm{vir}}_{\mathfrak{t}',\mathfrak{s}} - \mathcal{I}(\nu(\beta)), \bar{\mathbf{L}}^{\mathrm{vir}}_{\mathfrak{t}',\mathfrak{s}} - \mathcal{U}(\beta); \mathbb{R}\right),\\ \theta_p &\in S^{\deg(\beta)-1}\left(\bar{\mathbf{L}}^{\mathrm{vir}}_{\mathfrak{t}',\mathfrak{s}} - \mathcal{I}(\nu(\beta)), \mathcal{U}'(\beta); \mathbb{R}\right).\end{aligned}$$

Because θ_p is supported on the complement of $\mathcal{U}'(\beta)$ in $\bar{\mathbf{L}}^{\mathrm{vir}}_{\mathfrak{t}',\mathfrak{s}} - \mathcal{I}(\nu(\beta))$, we see that θ_p defines a cochain of degree $\deg(\beta)-1$ on $\bar{\mathbf{L}}^{\mathrm{vir}}_{\mathfrak{t}',\mathfrak{s}}$ (by extending θ_p by zero) and if we set

$$\bar{c}_\beta = \bar{c}'_\beta - \delta\theta_p,$$

then $\bar{c}_\beta$ also represents the cohomology class $\bar{\mu}_p(\beta)$ and equation (5.38) yields

$$\begin{aligned}\iota^*_\beta \bar{c}_\beta &= \iota^*_\beta \bar{c}'_\beta - \delta\theta_p\\ &= c_\beta + \delta(\theta_\beta + \theta_p) - \delta\theta_p\\ &= c_\beta + \delta\theta_\beta.\end{aligned}$$

This completes the proof. □

A similar argument yields:

Lemma 5.15. *Let $\mathcal{U}(x) \subset \bar{\mathbf{L}}^{\mathrm{vir}}_{\mathfrak{t}',\mathfrak{s}}$ be any open neighborhood of $\mathcal{I}(\nu(x)$ satisfying $\mathcal{I}(\nu(x)) \Subset \mathcal{U}(x)$. Let $\iota_{\mathcal{W}}$ be the inclusion map (5.35). Then, there is a cochain*

$$\theta_{\mathcal{W}} \in S^1(\bar{\mathbf{L}}^{\mathrm{vir}}_{\mathfrak{t}',\mathfrak{s}} - \mathcal{I}(\nu(x)), \bar{\mathbf{L}}^{\mathrm{vir}}_{\mathfrak{t}',\mathfrak{s}} - \mathcal{U}(x); \mathbb{R}) \tag{5.39}$$

and a cocycle $\bar{c}_{\mathcal{W}}$ in the cohomology class $\bar{\mu}_c \in H^2(\bar{\mathbf{L}}^{\mathrm{vir}}_{\mathfrak{t}',\mathfrak{s}}; \mathbb{Z})$ in Definition 4.5 such that

$$\iota^*_{\mathcal{W}} \bar{c}_{\mathcal{W}} = c_{\mathcal{W}} + \delta\theta_{\mathcal{W}}.$$

The construction of $\bar{c}_\beta$ and $\bar{c}_{\mathcal{W}}$ in Lemmas 5.14 and 5.15 yields the following useful corollary:

Corollary 5.16. *For $\beta \in H_\bullet(X;\mathbb{R})$, let $\bar{c}_\beta$ and $\bar{c}_{\mathcal{W}}$ be the cocycles defined in Lemmas 5.14 and 5.15 respectively. Then, as elements of $H^\bullet(\bar{\mathbf{L}}^{\mathrm{vir}}_{\mathfrak{t}',\mathfrak{s}}; \mathbb{R})$,*

$$[\bar{c}_\beta] = \bar{\mu}_p(\beta) \quad \text{and} \quad [\bar{c}_{\mathcal{W}}] = \bar{\mu}_c. \tag{5.40}$$

5.1.6. *Eliminating the excision isomorphism.* The next step in the proof of Proposition 5.2 is to eliminate the excision isomorphism ι_* from the pairing in (5.23). We do this by presenting the relative cohomology classes as restrictions (that is, the image under ι^*) of relative cohomology classes on $\bar{\mathbf{L}}^{\mathrm{vir}}_{\mathfrak{t}',\mathfrak{s}}$ and using the adjoint relation between ι_* and ι^*.

The relative Euler class is not the restriction of a relative cohomology class on $\bar{\mathbf{L}}^{\mathrm{vir}}_{\mathfrak{t}',\mathfrak{s}}$ because the obstruction bundle $\Upsilon_{\mathfrak{t}',\mathfrak{s}}/S^1$ does not extend over $\mathbf{L}^{\mathrm{low}}_{\mathfrak{t}',\mathfrak{s}}$. We use the decomposition $\Upsilon_{\mathfrak{t}',\mathfrak{s}} = \Upsilon^s_{\mathfrak{t}',\mathfrak{s}} \oplus \Upsilon^i_{\mathfrak{t}',\mathfrak{s}}$ of the obstruction bundle, with background component $\Upsilon^s_{\mathfrak{t}',\mathfrak{s}}/S^1$ which extends over $\mathbf{L}^{\mathrm{low}}_{\mathfrak{t}',\mathfrak{s}}$ (yielding $\bar{\Upsilon}^s_{\mathfrak{t}',\mathfrak{s}}/S^1$ in (3.64)), and an instanton component (which does not extend), $\Upsilon^i_{\mathfrak{t}',\mathfrak{s}}/S^1$, to overcome this difficulty. We observe in Lemma 5.17 that we can take advantage of this decomposition of $\Upsilon_{\mathfrak{t}',\mathfrak{s}}/S^1$ to perturb the section $\boldsymbol{\chi}$ without changing the intersection number in (5.23). Recall from (5.7) and (5.22) that the relative homology class in the pairing (5.41) below is contained in $H_{\max}(\mathbf{L}^{\mathrm{vir}}_{\mathfrak{t}',\mathfrak{s}}, \mathbf{L}^{\mathrm{vir},s}_{\mathfrak{t}',\mathfrak{s}}; \mathbb{Z})$.

Lemma 5.17. *Let $\chi = \chi_s \oplus \chi_i$ be the obstruction section provided by Theorem 3.8. Then the cup product in* (5.41) *is a cohomology class in $H^{\max}(\mathbf{L}^{\mathrm{vir}}_{\mathfrak{t}',\mathfrak{s}}, \mathbf{L}^{\mathrm{vir},s}_{\mathfrak{t}',\mathfrak{s}};\mathbb{Z})$ and so the following pairing is well-defined:*

$$\left\langle [c(z,\eta)] \smile e\left(\Upsilon_{\mathfrak{t}',\mathfrak{s}}/S^1, \chi_s|_{\mathbf{L}^{\mathrm{vir},s}_{\mathfrak{t}',\mathfrak{s}}}\right), \iota_*^{-1}(\jmath_{\mathbf{L}^i})_*\left[\mathbf{L}^{\mathrm{vir},i}_{\mathfrak{t}',\mathfrak{s}}, \partial\mathbf{L}^{\mathrm{vir},i}_{\mathfrak{t}',\mathfrak{s}}\right]\right\rangle. \tag{5.41}$$

Moreover, we can replace the section χ_s in the definition of the relative Euler class in the pairing (5.41) *with the section χ, so the pairing* (5.41) *is equal to the pairing on the right-hand side of equation* (5.23)*:*

$$\begin{aligned}&\left\langle [c(z,\eta)] \smile e\left(\Upsilon_{\mathfrak{t}',\mathfrak{s}}/S^1, \chi|_{\mathbf{L}^{\mathrm{vir},s}_{\mathfrak{t}',\mathfrak{s}}}\right), \iota_*^{-1}(\jmath_{\mathbf{L}^i})_*\left[\mathbf{L}^{\mathrm{vir},i}_{\mathfrak{t}',\mathfrak{s}}, \partial\mathbf{L}^{\mathrm{vir},i}_{\mathfrak{t}',\mathfrak{s}}\right]\right\rangle \\ &= \left\langle [c(z,\eta)] \smile e\left(\Upsilon_{\mathfrak{t}',\mathfrak{s}}/S^1, \chi_s|_{\mathbf{L}^{\mathrm{vir},s}_{\mathfrak{t}',\mathfrak{s}}}\right), \iota_*^{-1}(\jmath_{\mathbf{L}^i})_*\left[\mathbf{L}^{\mathrm{vir},i}_{\mathfrak{t}',\mathfrak{s}}, \partial\mathbf{L}^{\mathrm{vir},i}_{\mathfrak{t}',\mathfrak{s}}\right]\right\rangle.\end{aligned} \tag{5.42}$$

Proof. Because $\chi^{-1}(0) \subset \chi_s^{-1}(0)$, there is an inclusion map of pairs:

$$\jmath_s : \left(\mathbf{L}^{\mathrm{vir}}_{\mathfrak{t}',\mathfrak{s}}, \mathbf{L}^{\mathrm{vir},s}_{\mathfrak{t}',\mathfrak{s}} - \chi_s^{-1}(0)\right) \to \left(\mathbf{L}^{\mathrm{vir}}_{\mathfrak{t}',\mathfrak{s}}, \mathbf{L}^{\mathrm{vir},s}_{\mathfrak{t}',\mathfrak{s}} - \chi^{-1}(0)\right).$$

The sections χ and χ_s are homotopic through non-vanishing sections on $\mathbf{L}^{\mathrm{vir},s}_{\mathfrak{t}',\mathfrak{s}} - \chi_s^{-1}(0)$ (by the homotopy $\chi_s \oplus t\chi_i$, for $t \in [0,1]$) and so the relative Euler classes are equal,

$$\jmath_s^* e\left(\Upsilon_{\mathfrak{t}',\mathfrak{s}}/S^1, \chi|_{\mathbf{L}^{\mathrm{vir},s}_{\mathfrak{t}',\mathfrak{s}}}\right) = e\left(\Upsilon_{\mathfrak{t}',\mathfrak{s}}/S^1, \chi_s|_{\mathbf{L}^{\mathrm{vir},s}_{\mathfrak{t}',\mathfrak{s}}}\right) \tag{5.43}$$

as elements of

$$H^{2r_\Xi+2}\left(\mathbf{L}^{\mathrm{vir}}_{\mathfrak{t}',\mathfrak{s}}, \mathbf{L}^{\mathrm{vir},s}_{\mathfrak{t}',\mathfrak{s}} - \chi_s^{-1}(0);\mathbb{Z}\right).$$

For δ sufficiently small, the intersection $\bar{\mathcal{K}}(z,\eta) \cap \bar{\mathbf{L}}^{\mathrm{vir},s}_{\mathfrak{t}',\mathfrak{s}} \cap \chi_s^{-1}(0)$ is empty by Lemma 5.4 and the continuity of the gluing map and obstruction section χ_s with respect to Uhlenbeck limits. Because $[c(z,\eta)]$ is supported on $\mathcal{K}(z,\eta)$ while the relative Euler class in (5.43) is supported on the complement of $\chi_s^{-1}(0)$, the cup-product in (5.41) thus defines an element of

$$H^{\max}\left(\mathbf{L}^{\mathrm{vir}}_{\mathfrak{t}',\mathfrak{s}}, \mathbf{L}^{\mathrm{vir},s}_{\mathfrak{t}',\mathfrak{s}};\mathbb{R}\right),$$

(compare the argument giving (5.20)), so the pairing (5.41) is well-defined. We can then write

$$\begin{aligned}&\left\langle [c(z,\eta)] \smile e\left(\Upsilon_{\mathfrak{t}',\mathfrak{s}}/S^1, \chi|_{\mathbf{L}^{\mathrm{vir},s}_{\mathfrak{t}',\mathfrak{s}}}\right), \iota_*^{-1}(\jmath_{\mathbf{L}^i})_*\left[\mathbf{L}^{\mathrm{vir},i}_{\mathfrak{t}',\mathfrak{s}}, \partial\mathbf{L}^{\mathrm{vir},i}_{\mathfrak{t}',\mathfrak{s}}\right]\right\rangle \\ &= \left\langle [c(z,\eta)] \smile \jmath_s^* e\left(\Upsilon_{\mathfrak{t}',\mathfrak{s}}/S^1, \chi|_{\mathbf{L}^{\mathrm{vir},s}_{\mathfrak{t}',\mathfrak{s}}}\right), \iota_*^{-1}(\jmath_{\mathbf{L}^i})_*\left[\mathbf{L}^{\mathrm{vir},i}_{\mathfrak{t}',\mathfrak{s}}, \partial\mathbf{L}^{\mathrm{vir},i}_{\mathfrak{t}',\mathfrak{s}}\right]\right\rangle \\ &= \left\langle [c(z,\eta)] \smile e\left(\Upsilon_{\mathfrak{t}',\mathfrak{s}}/S^1, \chi_s|_{\mathbf{L}^{\mathrm{vir},s}_{\mathfrak{t}',\mathfrak{s}}}\right), \iota_*^{-1}(\jmath_{\mathbf{L}^i})_*\left[\mathbf{L}^{\mathrm{vir},i}_{\mathfrak{t}',\mathfrak{s}}, \partial\mathbf{L}^{\mathrm{vir},i}_{\mathfrak{t}',\mathfrak{s}}\right]\right\rangle,\end{aligned}$$

using equation (5.43) in the last line. This proves the desired identity (5.42). □

The advantage of the relative Euler class $e(\Upsilon_{\mathfrak{t}',\mathfrak{s}}/S^1, \chi_s)$ over $e(\Upsilon_{\mathfrak{t}',\mathfrak{s}}/S^1, \chi)$ is that it can be written as a cup-product,

$$e\left(\Upsilon_{\mathfrak{t}',\mathfrak{s}}/S^1, \chi_s|_{\mathbf{L}^{\mathrm{vir},s}_{\mathfrak{t}',\mathfrak{s}}}\right) = \iota^* e\left(\bar{\Upsilon}^s_{\mathfrak{t}',\mathfrak{s}}/S^1, \chi_s|_{\bar{\mathbf{L}}^{\mathrm{vir},s}_{\mathfrak{t}',\mathfrak{s}}}\right) \smile e\left(\Upsilon^i_{\mathfrak{t}',\mathfrak{s}}/S^1\right), \tag{5.44}$$

where ι is the inclusion map (5.21). Note that the continuity of the obstruction map χ_s with respect to Uhlenbeck limits is required to define the first relative Euler class appearing on the right-hand-side of (5.44) as a continuous section is needed to define a relative Euler class. The relative cohomology class and the absolute cohomology class on the right-hand-side of equation (5.44) are both restrictions of classes on $\bar{\mathbf{L}}^{\text{vir}}_{\mathfrak{t}',\mathfrak{s}}$. We now show that we can also replace the relative cohomology class $[c(z,\eta)]$ with one which extends over $\bar{\mathbf{L}}^{\text{vir}}_{\mathfrak{t}',\mathfrak{s}}$ without changing the pairing (5.41).

Let $z=\beta_1\cdots\beta_m\in\mathbb{A}(X)$ be the monomial appearing in Proposition 5.2. We replace $c^{\eta}_{\mathcal{W}}$ with

$$c_{\mathcal{W},1}\smile\cdots\smile c_{\mathcal{W},\eta},$$

as it will be necessary to distinguish between the factors of this cup-product, which are defined via specific choices described below. Let $\nu'(\beta_i)$ and $\nu'(x_j)$, $j=1,\dots,\eta$, be tubular neighborhoods of T_{β_i} and the point x_j defining $c_{\mathcal{W},j}$, respectively. Assume these neighborhoods satisfy both $\nu(\beta_i)\Subset\nu'(\beta_i)$ and $\nu(x_j)\Subset\nu'(x_j)$ and the intersection condition (5.10). Therefore, we can find open subspaces $\mathcal{U}(\beta_i)$ and $\mathcal{U}(x_j)$ in $\bar{\mathbf{L}}^{\text{vir}}_{\mathfrak{t}',\mathfrak{s}}$ such that

$$\mathcal{I}(\nu(\beta))\Subset\mathcal{U}(\beta)\quad\text{and}\quad\mathcal{I}(\nu(x_j))\Subset\mathcal{U}(x_j),$$

and such that the intersection

$$\mathcal{U}(\beta_{i_1})\cap\cdots\cap\mathcal{U}(\beta_{i_k})\cap\mathcal{U}(x_{j_1})\cap\cdots\cap\mathcal{U}(x_{j_r}),\tag{5.45}$$

is empty unless

$$\sum_{p=1}^{k}\deg(\beta_{i_p})+4r\leq 4.\tag{5.46}$$

Then let

$$\iota^*_{\beta_i}\bar{c}_{\beta_i}=c_{\beta_i}+\delta\theta_{\beta_i}\quad\text{and}\quad\iota^*_{\mathcal{W}}\bar{c}_{\mathcal{W},j}=c_{\mathcal{W},j}+\delta\theta_{\mathcal{W},j},\tag{5.47}$$

be the cocycles constructed in Lemmas 5.14 and 5.15 respectively, where $\bar{c}_{\beta_i}$ and $\bar{c}_{\mathcal{W},j}$ are cocycles on $\bar{\mathbf{L}}^{\text{vir}}_{\mathfrak{t}',\mathfrak{s}}$, $\delta\theta_{\beta_i}$ is supported in $\mathcal{U}(\beta_i)$, and $\delta\theta_{\mathcal{W},j}$ is supported in $\mathcal{U}(x_j)$. Define

$$\bar{c}(z,\eta)=\bar{c}_{\beta_1}\smile\cdots\smile\bar{c}_{\beta_m}\smile\bar{c}_{\mathcal{W},1}\smile\cdots\smile\bar{c}_{\mathcal{W},\eta},\tag{5.48}$$

so $\iota^*\bar{c}(z,\eta)=c(z,\eta)$. We now show that we can replace $c(z,\eta)$ with $\iota^*\bar{c}(z,\eta)$ without changing the pairing (5.41).

Lemma 5.18. *Let $z=\beta_1\cdots\beta_m\in\mathbb{A}(X)$ be the monomial appearing in Proposition 5.2. Let ι be the inclusion map (5.21). Then we can replace the class $[c(z,\eta)]$ in the pairing (5.41) by the class $\iota^*[\bar{c}(z,\eta)]$:*

$$\begin{aligned}&\left\langle[c(z,\eta)]\smile e\left(\Upsilon_{\mathfrak{t}',\mathfrak{s}}/S^1,\boldsymbol{\chi}_s|_{\mathbf{L}^{\text{vir},s}_{\mathfrak{t}',\mathfrak{s}}}\right),\iota_*^{-1}(\jmath_{\mathbf{L}^i})_*\left[\mathbf{L}^{\text{vir},i}_{\mathfrak{t}',\mathfrak{s}},\partial\mathbf{L}^{\text{vir},i}_{\mathfrak{t}',\mathfrak{s}}\right]\right\rangle\\&=\left\langle\iota^*[\bar{c}(z,\eta)]\smile e\left(\Upsilon_{\mathfrak{t}',\mathfrak{s}}/S^1,\boldsymbol{\chi}_s|_{\mathbf{L}^{\text{vir},s}_{\mathfrak{t}',\mathfrak{s}}}\right),\iota_*^{-1}(\jmath_{\mathbf{L}^i})_*\left[\mathbf{L}^{\text{vir},i}_{\mathfrak{t}',\mathfrak{s}},\partial\mathbf{L}^{\text{vir},i}_{\mathfrak{t}',\mathfrak{s}}\right]\right\rangle.\end{aligned}\tag{5.49}$$

Proof. By construction and (5.47), the cocycles $c(z,\eta)$ and $\iota^*\bar{c}(z,\eta)$ differ by cocycles of the form (up to a re-ordering which will be seen to be irrelevant)

$$c(z',\eta-r)\smile\delta\theta_{i_1}\smile\cdots\smile\delta\theta_{i_k}\smile\delta\theta_{j_1}\smile\cdots\smile\delta\theta_{j_r},\tag{5.50}$$

where $z=z'\beta_{i_1}\cdots\beta_{i_k}$. Note that we assume $k>0$ or $r>0$, so we actually have a difference term in (5.50) and not just $\iota^*\bar{c}(z,\eta)$. Because the intersection (5.45) is empty unless the condition (5.46) holds and because $\delta\theta_{\beta_i}$ and $\delta\theta_{\mathcal{W},j}$ have support in $\mathcal{U}(\beta_i)$ and $\mathcal{U}(x_j)$ respectively, the term (5.50) vanishes unless condition (5.46) holds. Consequently,

$$
\begin{aligned}
\deg(z') + 2(\eta - r) &\geq \deg(z) + 2\eta - \left(\sum_{\mu=1}^{k} \deg(\beta_i) + 4r\right) \\
&\geq \dim \mathcal{M}_{\mathfrak{t}'} - 6 \\
&= \dim \mathcal{M}_{\mathfrak{t}} \\
&= \dim\left(\boldsymbol{\chi}_{\mathfrak{s}}^{-1}(0) \cap \mathbf{L}^{\text{low}}_{\mathfrak{t}',\mathfrak{s}}\right) + 2.
\end{aligned} \tag{5.51}
$$

The inequality (5.51) implies that the following intersection is empty:

$$\bar{\mathcal{K}}(z', \eta - r) \cap \mathbf{L}^{\text{low}}_{\mathfrak{t}',\mathfrak{s}} \cap \boldsymbol{\chi}_s^{-1}(0) = \varnothing.$$

Hence, because of the continuity of the gluing map and of the obstruction section $\boldsymbol{\chi}_s$ on $\bar{\mathbf{L}}^{\text{vir},s}_{\mathfrak{t}',\mathfrak{s}}$ and by the same argument used to establish (5.19), the following intersection is also empty for δ sufficiently small:

$$\mathcal{K}(z', \eta - r) \cap \mathbf{L}^{\text{vir},s}_{\mathfrak{t}',\mathfrak{s}} \cap \boldsymbol{\chi}_s^{-1}(0) = \varnothing.$$

By the reasoning which gave (5.20), the preceding equality implies that the cup-product

$$[c(z', \eta - r)] \smile e\left(\bar{\Upsilon}^s_{\mathfrak{t}',\mathfrak{s}}/S^1, \boldsymbol{\chi}_s|_{\mathbf{L}^{\text{vir},s}_{\mathfrak{t}',\mathfrak{s}}}\right)$$

vanishes on $\mathbf{L}^{\text{vir},s}_{\mathfrak{t}',\mathfrak{s}}$ for δ sufficiently small. If we assume that the neighborhoods $\mathcal{U}(\beta_i)$ and $\mathcal{U}(x_j)$ are contained in $\mathbf{L}^{\text{vir},s}_{\mathfrak{t}',\mathfrak{s}}$ for all i and j, the assumption that $k > 0$ or $r > 0$ implies that the difference term (5.50) is supported in $\mathbf{L}^{\text{vir},s}_{\mathfrak{t}',\mathfrak{s}}$ and hence the cup-product of the difference term with the relative Euler class vanishes. Therefore identity (5.49) holds. □

We now have all the ingredients we need to conclude the proof of Proposition 5.2 at our disposal:

Proof of Proposition 5.2. Equations (5.23), (5.42), (5.44), and (5.49) yield the identity:

$$
\begin{aligned}
&\#\left(\bar{\mathcal{V}}(z) \cap \bar{\mathcal{W}}^{\eta} \cap \bar{\mathbf{L}}_{\mathfrak{t}',\mathfrak{s}}\right) \\
&= \left\langle \iota^*\left([\bar{c}(z,\eta)] \smile e\left(\bar{\Upsilon}^s_{\mathfrak{t}',\mathfrak{s}}/S^1, \boldsymbol{\chi}_s|_{\bar{\mathbf{L}}^{\text{vir},s}_{\mathfrak{t}',\mathfrak{s}}}\right) \smile \bar{e}_i\right), \iota_*^{-1}(\jmath_{\mathbf{L}^i})_*\left[\mathbf{L}^{\text{vir},i}_{\mathfrak{t}',\mathfrak{s}}, \partial \mathbf{L}^{\text{vir},i}_{\mathfrak{t}',\mathfrak{s}}\right]\right\rangle \\
&= \left\langle [\bar{c}(z,\eta)] \smile e\left(\bar{\Upsilon}^s_{\mathfrak{t}',\mathfrak{s}}/S^1, \boldsymbol{\chi}_s|_{\bar{\mathbf{L}}^{\text{vir},s}_{\mathfrak{t}',\mathfrak{s}}}\right) \smile \bar{e}_i, (\jmath_{\mathbf{L}^i})_*\left[\mathbf{L}^{\text{vir},i}_{\mathfrak{t}',\mathfrak{s}}, \partial \mathbf{L}^{\text{vir},i}_{\mathfrak{t}',\mathfrak{s}}\right]\right\rangle.
\end{aligned} \tag{5.52}
$$

Because $\bar{c}(z,\eta)$ is a cocycle representing the cohomology class $\bar{\mu}_p(z) \smile \bar{\mu}_c^{\eta}$ on $\bar{\mathcal{M}}^{\text{vir},*}_{\mathfrak{t}',\mathfrak{s}}/S^1$ by Corollary 5.16 and because of the relation (5.14) between relative and absolute Euler classes and the definition of $\bar{e}_s$ given by Lemma 4.11, we see that

$$
\begin{aligned}
&\jmath^*\left([\bar{c}(z,\eta)] \smile e\left(\bar{\Upsilon}^s_{\mathfrak{t}',\mathfrak{s}}/S^1, \boldsymbol{\chi}_s|_{\bar{\mathbf{L}}^{\text{vir},s}_{\mathfrak{t}',\mathfrak{s}}}\right) \smile \bar{e}_i\right) \\
&= \bar{\mu}_p(z) \smile \bar{\mu}_c^{\eta} \smile \bar{e}_s \smile \bar{e}_i,
\end{aligned} \tag{5.53}
$$

where $\jmath$ is the inclusion map (5.5). Combining the expression (5.52) for the intersection number, equation (5.53), and the definition of $[\bar{\mathbf{L}}^{\text{vir}}_{\mathfrak{t}',\mathfrak{s}}]$ in (5.9) then completes the proof of Proposition 5.2. □

5.2. **Cohomological results.** In this section we prove some results about the cohomology ring of $\mathbf{L}^{\text{vir}}_{\mathfrak{t}',\mathfrak{s}}$ which we shall need to compute the right-hand side of equation (5.2) and thus complete the proof of Theorem 6.1. In Lemma 5.19, we compute the Poincaré dual of the submanifold of $\mathbf{L}^{\text{vir}}_{\mathfrak{t}',\mathfrak{s}}$,

$$\begin{aligned}\mathbf{BL}^{\text{vir},i}_{\mathfrak{t}',\mathfrak{s}} &= \tilde{M}_{\mathfrak{s}} \times_{\mathcal{G}_{\mathfrak{s}}\times S^1} \partial\bar{\mathrm{G}}\mathrm{l}_{\mathfrak{t}}(\delta) \\ &\subset \tilde{N}_{\mathfrak{t},\mathfrak{s}}(\varepsilon) \times_{\mathcal{G}_{\mathfrak{s}}\times S^1} \partial\bar{\mathrm{G}}\mathrm{l}_{\mathfrak{t}}(\delta) = \mathbf{L}^{\text{vir},i}_{\mathfrak{t}',\mathfrak{s}} \subset \mathbf{L}^{\text{vir}}_{\mathfrak{t}',\mathfrak{s}},\end{aligned} \tag{5.54}$$

and in Propositions 5.20 and 5.21 we give a formula for division by this Poincaré dual. Dividing the cohomology class in the pairing on the right-side of identity (5.2) by the Poincaré dual of the fundamental class of $\mathbf{BL}^{\text{vir},i}_{\mathfrak{t}',\mathfrak{s}}$ reduces the pairing with $[\bar{\mathbf{L}}^{\text{vir}}_{\mathfrak{t}',\mathfrak{s}}]$ to one with $[\mathbf{BL}^{\text{vir},i}_{\mathfrak{t}',\mathfrak{s}}]$. Because, as noted in equation (3.83), the space $\mathbf{BL}^{\text{vir},i}_{\mathfrak{t}',\mathfrak{s}}$ is a product

$$\mathbf{BL}^{\text{vir},i}_{\mathfrak{t}',\mathfrak{s}} = M_{\mathfrak{s}} \times \partial\bar{\mathrm{G}}\mathrm{l}_{\mathfrak{t}}(\delta)/S^1, \tag{5.55}$$

the result of pairing with the fundamental class $[\mathbf{BL}^{\text{vir},i}_{\mathfrak{t}',\mathfrak{s}}]$ is equal to a product of pairings with $[M_{\mathfrak{s}}]$ and with $[\partial\bar{\mathrm{G}}\mathrm{l}_{\mathfrak{t}}(\delta)/S^1]$. The former gives the Seiberg-Witten invariant, while the latter has been previously computed in [34] (see Lemma 5.24).

A similar technique would work for the link of $M_{\mathfrak{s}} \times \mathrm{Sym}^{\ell}(X)$, for $\ell \geq 2$, although the computations of Lemma 5.24 for the analogue of the space $\partial\bar{\mathrm{G}}\mathrm{l}_{\mathfrak{t}}(\delta)/S^1$ when $\ell \geq 3$ appear very difficult, though the case of $\ell = 2$ can be deduced from the work of [34].

Both the computation of the Poincaré dual of $\mathbf{BL}^{\text{vir},i}_{\mathfrak{t}',\mathfrak{s}}$ and our formula for division by this Poincaré dual use equivariant cohomology, which we now briefly review.

As customary, $\mathrm{ES}^1 \to \mathrm{BS}^1$ denotes the universal circle bundle, where BS^1 is homotopy equivalent to $\mathbb{CP}^\infty$. For any pair of topological spaces (X, Y) with a circle action,

$$\begin{aligned}H^\bullet_{S^1}(X;\mathbb{Z}) &= H^\bullet(\mathrm{ES}^1 \times_{S^1} X;\mathbb{Z}),\\ H^\bullet_{S^1}(X,Y;\mathbb{Z}) &= H^\bullet(\mathrm{ES}^1 \times_{S^1} X, \mathrm{ES}^1 \times_{S^1} Y;\mathbb{Z}),\end{aligned}$$

are the equivariant cohomology of X and relative equivariant cohomology of (X, Y), respectively [1, §2].

Lemma 5.19. *There is a continuous map*

$$\pi_{N,\nu} : \bar{\mathcal{M}}^{\text{vir},*}_{\mathfrak{t}',\mathfrak{s}}/S^1 \to N_{\mathfrak{t},\mathfrak{s}} \times_{S^1} \mathrm{ES}^1, \tag{5.56}$$

such that if $\mathrm{Th}_{S^1}(N_{\mathfrak{t},\mathfrak{s}}) \in H^\bullet_{S^1}(N_{\mathfrak{t},\mathfrak{s}}, N^0_{\mathfrak{t},\mathfrak{s}};\mathbb{Z})$ *is the Thom class* [37, Theorem 9.1], [5, §6] *of the bundle*

$$N_{\mathfrak{t},\mathfrak{s}} \times_{S^1} \mathrm{ES}^1 \to M_{\mathfrak{s}} \times \mathrm{BS}^1, \tag{5.57}$$

where $N^0_{\mathfrak{t},\mathfrak{s}}$ *denotes the complement of the zero-section, and we have an inclusion map,*

$$j_{\mathbf{BL}} : \left(\bar{\mathbf{L}}^{\text{vir}}_{\mathfrak{t}',\mathfrak{s}}, \varnothing\right) \to \left(\bar{\mathbf{L}}^{\text{vir}}_{\mathfrak{t}',\mathfrak{s}}, \bar{\mathbf{L}}^{\text{vir}}_{\mathfrak{t}',\mathfrak{s}} - \mathbf{BL}^{\text{vir},i}_{\mathfrak{t}',\mathfrak{s}}\right), \tag{5.58}$$

then the following hold:

1. *For any* $\omega \in H^{d_s(\mathfrak{s})+6}(\bar{\mathbf{L}}^{\text{vir}}_{\mathfrak{t}',\mathfrak{s}};\mathbb{Z})$, *where* $\dim \mathbf{BL}^{\text{vir},i}_{\mathfrak{t}',\mathfrak{s}} = d_s(\mathfrak{s}) + 6$, *we have*

$$\left\langle \omega \smile j^*_{\mathbf{BL}} \pi^*_{N,\nu} \mathrm{Th}_{S^1}(N_{\mathfrak{t},\mathfrak{s}}), [\bar{\mathbf{L}}^{\text{vir}}_{\mathfrak{t}',\mathfrak{s}}] \right\rangle = \left\langle \omega, [\mathbf{BL}^{\text{vir},i}_{\mathfrak{t}',\mathfrak{s}}] \right\rangle. \tag{5.59}$$

2. *If* $\mathbf{c} \in H^2(\mathrm{BS}^1;\mathbb{Z})$ *is the universal first Chern class and*
$$\pi_B : N_{\mathfrak{t},\mathfrak{s}} \times_{S^1} \mathrm{ES}^1 \to \mathrm{BS}^1,$$
is the projection and $\nu \in H^2(\bar{\mathcal{M}}^{\mathrm{vir}}_{\mathfrak{t}',\mathfrak{s}}/S^1;\mathbb{Z})$ *is the cohomology class of Definition 4.1, then*
$$\pi^*_{N,\nu}\pi^*_B\mathbf{c} = \nu.$$
3. *If* $\mu_{\mathfrak{s}} \in H^2(M_{\mathfrak{s}};\mathbb{Z})$ *is as defined in* (2.21) *and*
$$\pi_M : N_{\mathfrak{t},\mathfrak{s}} \times_{S^1} \mathrm{ES}^1 \to M_{\mathfrak{s}},$$
is the projection, then
$$\pi^*_{N,\nu}\pi^*_M\mu_{\mathfrak{s}} = \pi^*_{\mathfrak{s}}\mu_{\mathfrak{s}}.$$

Proof. We define a bundle map π_N by the obvious projection so that the following diagram commutes:
$$\begin{array}{ccc}
\tilde{N}_{\mathfrak{t},\mathfrak{s}} \times_{\mathcal{G}_{\mathfrak{s}}} \bar{\mathrm{G}}\mathrm{l}_{\mathfrak{t}}(\delta) & \xrightarrow{\pi_N} & N_{\mathfrak{t},\mathfrak{s}} \\
\downarrow & & \downarrow \\
\tilde{M}_{\mathfrak{s}} \times_{\mathcal{G}_{\mathfrak{s}}} \bar{\mathrm{G}}\mathrm{l}_{\mathfrak{t}}(\delta) & \xrightarrow{\pi_{\mathfrak{s}}} & M_{\mathfrak{s}}
\end{array} \tag{5.60}$$
By construction, the projection π_N is S^1-equivariant where the circle acts by scalar multiplication on the fibers of $N_{\mathfrak{t},\mathfrak{s}}$ and by the action in Definition 4.1 defining ν on the fibers of the bundle on the left-hand-side of the diagram. Let ι_ν be the classifying map for the circle bundle defining ν, covered by the bundle map $\tilde{\iota}_\nu$, so that the following diagram commutes:
$$\begin{array}{ccc}
\bar{\mathcal{M}}^{\mathrm{vir},*}_{\mathfrak{t}',\mathfrak{s}} & \xrightarrow{\tilde{\iota}_\nu} & \mathrm{ES}^1 \\
\downarrow & & \downarrow \\
\bar{\mathcal{M}}^{\mathrm{vir},*}_{\mathfrak{t}',\mathfrak{s}}/S^1 & \xrightarrow{\iota_\nu} & \mathrm{BS}^1
\end{array} \tag{5.61}$$
By construction of the map ι_ν, we have $\iota^*_\nu\mathbf{c} = \nu$. Then, because the maps π_N and $\tilde{\iota}_\nu$ are S^1-equivariant, the product $\pi_N \times \tilde{\iota}_\nu$ defines the map $\pi_{N,\nu}$ on the circle quotients:
$$\pi_{N,\nu} : \bar{\mathcal{M}}^{\mathrm{vir},*}_{\mathfrak{t}',\mathfrak{s}}/S^1 \to N_{\mathfrak{t},\mathfrak{s}} \times_{S^1} \mathrm{ES}^1\,.$$
Observe that the intersection of the pre-image under $\pi_{N,\nu}$ of the zero-section of the bundle (5.57) with $\bar{\mathbf{L}}^{\mathrm{vir}}_{\mathfrak{t}',\mathfrak{s}}$ is the base space (5.54):
$$\bar{\mathbf{L}}^{\mathrm{vir}}_{\mathfrak{t}',\mathfrak{s}} \cap \pi^{-1}_{N,\nu}(M_{\mathfrak{s}} \times \mathrm{BS}^1) = \mathbf{BL}^{\mathrm{vir},i}_{\mathfrak{t}',\mathfrak{s}}. \tag{5.62}$$
Hence, the map $\pi_{N,\nu}$ induces a map on relative cohomology,
$$\pi^*_{N,\nu} : H^\bullet_{S^1}(N_{\mathfrak{t},\mathfrak{s}}, N^0_{\mathfrak{t},\mathfrak{s}};\mathbb{Z}) \to H^\bullet(\bar{\mathbf{L}}^{\mathrm{vir}}_{\mathfrak{t}',\mathfrak{s}}, \bar{\mathbf{L}}^{\mathrm{vir}}_{\mathfrak{t}',\mathfrak{s}} - \mathbf{BL}^{\mathrm{vir},i}_{\mathfrak{t}',\mathfrak{s}};\mathbb{Z}),$$
where $N^0_{\mathfrak{t},\mathfrak{s}}$ denotes the complement of the zero-section. Because the restriction of $\pi_{N,\nu}$ to $\mathbf{L}^{\mathrm{vir},i}_{\mathfrak{t}',\mathfrak{s}}$ defines a bundle map,
$$\begin{array}{ccc}
\tilde{N}_{\mathfrak{t},\mathfrak{s}}(\varepsilon) \times_{\mathcal{G}_{\mathfrak{s}} \times S^1} \partial\bar{\mathrm{G}}\mathrm{l}_{\mathfrak{t}}(\delta) & \xrightarrow{\pi_{N,\nu}} & N_{\mathfrak{t},\mathfrak{s}} \times_{S^1} \mathrm{ES}^1 \\
{\scriptstyle \pi_{\mathrm{Gl}}}\downarrow & & {\scriptstyle \pi_N \times \pi_B}\downarrow \\
\tilde{M}_{\mathfrak{s}} \times_{\mathcal{G}_{\mathfrak{s}} \times S^1} \partial\bar{\mathrm{G}}\mathrm{l}_{\mathfrak{t}}(\delta) & \xrightarrow{\pi_{\mathfrak{s}} \times \iota_\nu} & M_{\mathfrak{s}} \times \mathrm{BS}^1
\end{array} \tag{5.63}$$
the Thom class of the normal bundle of $\mathbf{BL}^{\mathrm{vir},i}_{\mathfrak{t}',\mathfrak{s}}$ in $\mathbf{L}^{\mathrm{vir},i}_{\mathfrak{t}',\mathfrak{s}}$ is given by $\pi^*_{N,\nu}\,\mathrm{Th}_{S^1}(N_{\mathfrak{t},\mathfrak{s}})$. The relative fundamental class of the manifold with boundary $\mathbf{L}^{\mathrm{vir},i}_{\mathfrak{t}',\mathfrak{s}}$ is equal to $[\mathbf{L}^{\mathrm{vir},i}_{\mathfrak{t}',\mathfrak{s}}, \partial\mathbf{L}^{\mathrm{vir},i}_{\mathfrak{t}',\mathfrak{s}}]$,

as noted in (5.7). Hence, because the Thom class of the normal bundle of a submanifold is equal to the Poincaré dual of that submanifold [6, p. 371], the fundamental class of $\mathbf{BL}^{\mathrm{vir},i}_{\mathfrak{t}',\mathfrak{s}}$ is given by

$$[\mathbf{BL}^{\mathrm{vir},i}_{\mathfrak{t}',\mathfrak{s}}] = j_1^*\pi_{N,\nu}^* \operatorname{Th}_{S^1}(N_{\mathfrak{t},\mathfrak{s}}) \cap \left[\mathbf{L}^{\mathrm{vir},i}_{\mathfrak{t}',\mathfrak{s}}, \partial\mathbf{L}^{\mathrm{vir},i}_{\mathfrak{t}',\mathfrak{s}}\right], \tag{5.64}$$

where $\left[\mathbf{L}^{\mathrm{vir},i}_{\mathfrak{t}',\mathfrak{s}}, \partial\mathbf{L}^{\mathrm{vir},i}_{\mathfrak{t}',\mathfrak{s}}\right]$ is defined following (5.6) and

$$j_1 : (\bar{\mathbf{L}}^{\mathrm{vir}}_{\mathfrak{t}',\mathfrak{s}}, \bar{\mathbf{L}}^{\mathrm{vir},s}_{\mathfrak{t}',\mathfrak{s}}) \to (\bar{\mathbf{L}}^{\mathrm{vir}}_{\mathfrak{t}',\mathfrak{s}}, \bar{\mathbf{L}}^{\mathrm{vir}}_{\mathfrak{t}',\mathfrak{s}} - \mathbf{BL}^{\mathrm{vir},i}_{\mathfrak{t}',\mathfrak{s}})$$

is the inclusion of pairs. Observe that

$$j_{\mathbf{BL}} = j_1 \circ j, \tag{5.65}$$

where $j : (\bar{\mathbf{L}}^{\mathrm{vir}}_{\mathfrak{t}',\mathfrak{s}}, \varnothing) \to (\bar{\mathbf{L}}^{\mathrm{vir}}_{\mathfrak{t}',\mathfrak{s}}, \bar{\mathbf{L}}^{\mathrm{vir},s}_{\mathfrak{t}',\mathfrak{s}})$ is the inclusion (5.5). Thus, for any $\omega \in H^\bullet(\bar{\mathbf{L}}^{\mathrm{vir}}_{\mathfrak{t}',\mathfrak{s}};\mathbb{Z})$, we have

$$\begin{aligned}
&\langle \omega \smile j_{\mathbf{BL}}^*\pi_{N,\nu}^* \operatorname{Th}_{S^1}(N_{\mathfrak{t},\mathfrak{s}}), [\bar{\mathbf{L}}^{\mathrm{vir}}_{\mathfrak{t}',\mathfrak{s}}]\rangle \\
&= \langle j^*\left(\omega \smile j_1^*\pi_{N,\nu}^* \operatorname{Th}_{S^1}(N_{\mathfrak{t},\mathfrak{s}})\right), [\bar{\mathbf{L}}^{\mathrm{vir}}_{\mathfrak{t}',\mathfrak{s}}]\rangle \quad \text{(By (5.65) \& [50, Statement 5.6.8])} \\
&= \langle \omega \smile j_1^*\pi_{N,\nu}^* \operatorname{Th}_{S^1}(N_{\mathfrak{t},\mathfrak{s}}), j_*[\bar{\mathbf{L}}^{\mathrm{vir}}_{\mathfrak{t}',\mathfrak{s}}]\rangle \\
&= \left\langle \omega, j_1^*\pi_{N,\nu}^* \operatorname{Th}_{S^1}(N_{\mathfrak{t},\mathfrak{s}}) \cap \left[\mathbf{L}^{\mathrm{vir},i}_{\mathfrak{t}',\mathfrak{s}}, \partial\mathbf{L}^{\mathrm{vir},i}_{\mathfrak{t}',\mathfrak{s}}\right]\right\rangle \quad \text{(By Equation (5.9))} \\
&= \left\langle \omega, [\mathbf{BL}^{\mathrm{vir},i}_{\mathfrak{t}',\mathfrak{s}}]\right\rangle, \quad \text{(By Equation (5.64))}
\end{aligned}$$

which proves equation (5.59) and thus Assertion (1).

Because the diagram (5.61) commutes, we observe that $\pi_B \circ \pi_{N,\nu} = \iota_\nu$; this observation and the fact that $\iota_\nu^*\mathbf{c} = \nu$ proves Assertion (2).

Because the diagram (5.60) commutes, we have $\pi_M \circ \pi_{N,\nu} = \pi_{\mathfrak{s}}$ and this proves Assertion (3), completing the proof of the lemma. □

We next give an explicit formula for division by an equivariant Thom class in terms of the Segre classes of the vector bundle; this division process is also referred to as *equivariant localization* — see [4, Theorem 7.13]. Recall that the total Segre class $s(N) = s_0(N) + s_1(N) + \cdots$, where $s_i(N) \in H^{2i}(M;\mathbb{Z})$, of a complex vector bundle N over a topological space M is the formal inverse of the total Chern class $c(N) = 1 + c_1(N) + c_2(N) + \cdots$, where $c_i(N) \in H^{2i}(M;\mathbb{Z})$, so that $s(N)c(N) = 1$ [24, p. 69], [18, Lemma 4.10].

Proposition 5.20. *Let $\pi_M : N \to M$ be a complex vector bundle of rank r over a topological space M with a fundamental class of dimension d. Suppose $\pi_B : \mathrm{ES}^1 \times_{S^1} N \to \mathrm{BS}^1$ and $\pi_M : \mathrm{ES}^1 \times_{S^1} N \to M$ are the projection maps, where the circle acts diagonally on $N \times \mathrm{ES}^1$ by scalar multiplication on the fibers of N and by the standard action on ES^1. Let N^0 denote the complement of the zero-section and let $\operatorname{Th}_{S^1}(N) \in H^{2r}_{S^1}(N, N^0;\mathbb{Z})$ be the Thom class of*

$$\pi_B \times \pi_M : \mathrm{ES}^1 \times_{S^1} N \to \mathrm{BS}^1 \times M. \tag{5.66}$$

If $\mathbf{c} \in H^2(\mathrm{BS}^1;\mathbb{Z})$ denotes the universal first Chern class, $0 \le k \le [d/2]$ is an integer, and $\alpha \in H^{d-2k}(M;\mathbb{Z})$, then for integers $m \ge k + r$, one has

$$\pi_B^*\mathbf{c}^m \smile \pi_M^*\alpha = \iota_n^* \operatorname{Th}_{S^1}(N) \smile \left(\sum_{j=0}^{k} (-1)^{r+j}\pi_B^*\mathbf{c}^{m-r-j} \smile \pi_M^*(\alpha \smile s_j(N))\right), \tag{5.67}$$

where $\iota_N : (N, \varnothing) \to (N, N^0)$ *is the inclusion of pairs, and the pushforward formula,*

$$\pi_{B,*}(\mathbf{c}^m \smile \pi_M^* \alpha) = (-1)^{k+r} \mathbf{c}^{m-r-k} \langle \alpha \smile s_k(N), [M] \rangle. \tag{5.68}$$

Proof. By the splitting principle we may suppose that $N = \oplus_{i=1}^r L_i$, with $y_i = c_1(L_i)$, and so Lemma 4.6 implies that

$$j^* \operatorname{Th}_{S^1}(N) = \prod_{i=1}^r (-\pi_B^* \mathbf{c} + \pi_M^* y_i) = \sum_{i=0}^r (-1)^{r-i} \pi_B^* \mathbf{c}^{r-i} \pi_M^* c_i(N). \tag{5.69}$$

The negative sign above arises because the circle action is diagonal, as explained in Lemma 4.6. The bundle N has total Segre class $s(N) = \prod_{i=1}^r (1+y_i)^{-1}$, and thus

$$\begin{aligned}
\frac{1}{\operatorname{Th}_{S^1}(N)} &= \prod_{i=1}^r \frac{1}{(-\pi_B^* \mathbf{c} + \pi_M^* y_i)} \\
&= (-\pi_B^* \mathbf{c})^{-r} \prod_{i=1}^r \frac{1}{(1 - \pi_M^* y_i / \pi_B^* \mathbf{c})} \\
&= \sum_{j=0}^\infty (-\pi_B^* \mathbf{c})^{-r-j} \pi_M^* s_j(N).
\end{aligned}$$

This expression for the formal inverse of $\operatorname{Th}_{S^1}(N)$ yields

$$\frac{\pi_B^* \mathbf{c}^m \smile \pi_M^* \alpha}{\operatorname{Th}_{S^1}(N)} = \sum_{j=0}^k (-1)^{r+j} \pi_B^* \mathbf{c}^{m-r-j} \smile \pi_M^* (\alpha \smile s_j(N)), \tag{5.70}$$

and equation (5.67) follows. Because the pushforward $\pi_{B,*}$ is the composition of division by the Thom class $\operatorname{Th}_{S^1}(N)$ and integration over M [5, §6], we have

$$\begin{aligned}
\pi_{B,*}(\pi_B^* \mathbf{c}^m \smile \pi_M^* \alpha) &= \left(\frac{\pi_B^* \mathbf{c}^m \smile \pi_M^* \alpha}{\operatorname{Th}_{S^1}(N)} \right) / [M] \\
&= \left(\sum_{j=0}^k (-1)^{r+j} \pi_B^* \mathbf{c}^{m-r-j} \smile \pi_M^* (\alpha \smile s_j(N)) \right) / [M],
\end{aligned}$$

which yields equation (5.68), since M has dimension d. □

Lemmas 4.2, 4.10, 4.11, and 4.12 show how to express the cohomology classes in (5.2) in terms of the cohomology classes ν, $\mu_{\mathfrak{s}}$, and classes pulled back from X. The next proposition reduces the pairing in (5.2) of products of these latter classes with $[\mathbf{L}_{\mathfrak{t}',\mathfrak{s}}^{\mathrm{vir}}]$ to a pairing with the fundamental class of the submanifold (5.54).

Proposition 5.21. *Let* $d_s = \dim M_{\mathfrak{s}}$ *and let* r_N *denote the rank of the complex vector bundle* $N_{\mathfrak{t},\mathfrak{s}}$ *over* $M_{\mathfrak{s}}$. *Assume* $b_1(X) = 0$ *and* $b_2^+(X)$ *is odd. For integers* $0 \le i \le d = \frac{1}{2} d_s$ *and* $0 \le k \le 2$ *and any class* $\alpha \in H^{2k}(X; \mathbb{Z})$, *we have*

$$\begin{aligned}
&\left\langle \nu^{d+r_N+3-k-i} \smile \pi_{\mathfrak{s}}^* \mu_{\mathfrak{s}}^i \smile \pi_X^* \alpha, [\bar{\mathbf{L}}_{\mathfrak{t}',\mathfrak{s}}^{\mathrm{vir}}] \right\rangle \\
&= \sum_{j=0}^{d-i} (-1)^{r_N+j} \left\langle \nu^{d+3-i-j-k} \smile \pi_{\mathfrak{s}}^* (\mu_{\mathfrak{s}}^i \smile s_j(N_{\mathfrak{t},\mathfrak{s}})) \smile \pi_X^* \alpha, [\mathbf{BL}_{\mathfrak{t}',\mathfrak{s}}^{\mathrm{vir},i}] \right\rangle,
\end{aligned} \tag{5.71}$$

where $\nu \in H^2(\bar{\mathbf{L}}^{\mathrm{vir}}_{\mathfrak{t}',\mathfrak{s}};\mathbb{Z})$ is the cohomology class of Definition 4.1, $\mu_{\mathfrak{s}} \in H^2(M_{\mathfrak{s}};\mathbb{Z})$ is the restriction of the cohomology class defined in (2.21), and $\pi_X : \bar{\mathbf{L}}^{\mathrm{vir}}_{\mathfrak{t}',\mathfrak{s}} \to X$ and $\pi_{\mathfrak{s}} : \bar{\mathbf{L}}^{\mathrm{vir}}_{\mathfrak{t}',\mathfrak{s}} \to M_{\mathfrak{s}}$ are the restrictions of the projections defined in (4.4).

Proof. Applying the equality $\nu = \iota_\nu^*\mathbf{c}$ from Lemma 5.19 and using the abbreviations $A = d + r_N + 3 - k$, $T = \mathrm{Th}_{S^1}(N_{\mathfrak{t},\mathfrak{s}})$, and $s_j = s_j(N_{\mathfrak{t},\mathfrak{s}})$ yields

$$
\begin{aligned}
&\langle \nu^{A-i} \smile \pi_{\mathfrak{s}}^*\mu_{\mathfrak{s}}^i \smile \pi_X^*\alpha, [\bar{\mathbf{L}}^{\mathrm{vir}}_{\mathfrak{t}',\mathfrak{s}}]\rangle \\
&= \langle \pi_{N,\nu}^*\left(\pi_B^*\mathbf{c}^{A-i} \smile \pi_{\mathfrak{s}}^*\mu_{\mathfrak{s}}^i\right) \smile \pi_X^*\alpha, [\bar{\mathbf{L}}^{\mathrm{vir}}_{\mathfrak{t}',\mathfrak{s}}]\rangle \\
&= \sum_{j=0}^{d-i}(-1)^{r_N+j}\langle \pi_{N,\nu}^*\left(\pi_B^*\mathbf{c}^{A-i-j} \smile \pi_{\mathfrak{s}}^*(\mu_{\mathfrak{s}}^i \smile s_j) \smile j^*T\right) \smile \pi_X^*\alpha, [\bar{\mathbf{L}}^{\mathrm{vir}}_{\mathfrak{t}',\mathfrak{s}}]\rangle \\
&\qquad \text{(By Equation (5.67))} \\
&= \sum_{j=0}^{d-i}(-1)^{r_N+j}\langle (j^*\pi_{N,\nu}^*T) \smile \nu^{A-r_N-i-j} \smile \pi_{\mathfrak{s}}^*(\mu_{\mathfrak{s}}^i \smile s_j) \smile \pi_X^*\alpha, [\bar{\mathbf{L}}^{\mathrm{vir}}_{\mathfrak{t}',\mathfrak{s}}]\rangle \\
&= \sum_{j=0}^{d-i}(-1)^{r_N+j}\left\langle \nu^{A-r_N-i-j} \smile \pi_{\mathfrak{s}}^*(\mu_{\mathfrak{s}}^i \smile s_j) \smile \pi_X^*\alpha, [\mathbf{BL}^{\mathrm{vir},i}_{\mathfrak{t}',\mathfrak{s}}]\right\rangle
\end{aligned}
\tag{5.72}
$$

where the last equality follows from equation (5.59). This proves the desired formula (5.71). □

The next step is to use the product decomposition (5.55) of $\mathbf{BL}^{\mathrm{vir},i}_{\mathfrak{t}',\mathfrak{s}}$ to reduce the pairing on the right-hand side of equation (5.71) to a sum of products of pairings with $[M_{\mathfrak{s}}]$ and pairings with $[\partial\bar{\mathrm{Gl}}_{\mathfrak{t}}(\delta)/S^1]$. To accomplish this we must write the cohomology classes appearing in (5.71), namely $\mu_{\mathfrak{s}}$, ν, and $\pi_X^*\alpha$, as pullbacks of cohomology classes on the factors $M_{\mathfrak{s}}$ and $\partial\bar{\mathrm{Gl}}_{\mathfrak{t}}(\delta)/S^1$ of the product; of these three cohomology classes, only ν is not obviously such a pullback.

Definition 5.22. Let $\nu_{\mathfrak{t}} \in H^2(\partial\bar{\mathrm{Gl}}_{\mathfrak{t}}(\delta)/S^1;\mathbb{R})$ be the first Chern class of the circle bundle

$$\partial\bar{\mathrm{Gl}}_{\mathfrak{t}}(\delta) \to \partial\bar{\mathrm{Gl}}_{\mathfrak{t}}(\delta)/S^1,$$

where the circle action is described in the paragraph preceding equation (3.56). We shall also let $\nu_{\mathfrak{t}}$ denote the pullback of this class to the product $M_{\mathfrak{s}} \times \partial\bar{\mathrm{Gl}}_{\mathfrak{t}}(\delta)/S^1$, where it is the first Chern class of the circle bundle

$$M_{\mathfrak{s}} \times \partial\bar{\mathrm{Gl}}_{\mathfrak{t}}(\delta) \to M_{\mathfrak{s}} \times \partial\bar{\mathrm{Gl}}_{\mathfrak{t}}(\delta)/S^1.$$

In the following lemma, we obtain the desired decomposition of ν by comparing this class with $\nu_{\mathfrak{t}}$:

Lemma 5.23. *Let ν, $\nu_{\mathfrak{t}}$ be the first Chern classes in Definitions 4.1 and 5.22, respectively. Let $\pi_{\mathfrak{s}} : M_{\mathfrak{s}} \times \partial\bar{\mathrm{Gl}}_{\mathfrak{t}}(\delta)/S^1 \to M_{\mathfrak{s}}$ be the projection. Then the restriction of ν to $M_{\mathfrak{s}} \times \partial\bar{\mathrm{Gl}}_{\mathfrak{t}}(\delta)/S^1$ is given by*

$$\nu|_{M_{\mathfrak{s}}\times\partial\bar{\mathrm{Gl}}_{\mathfrak{t}}(\delta)/S^1} = \nu_{\mathfrak{t}} + 2c_1(\mathbb{L}_{\mathfrak{s}}), \tag{5.73}$$

and if $b_1(X) = 0$, this simplifies to

$$\nu|_{M_{\mathfrak{s}}\times\partial\bar{\mathrm{Gl}}_{\mathfrak{t}}(\delta)/S^1} = \nu_{\mathfrak{t}} + 2\mu_{\mathfrak{s}}.$$

Proof. To prove the lemma, we must compare the second bundle in Definition 5.22 with the circle bundle

$$\tilde{M}_{\mathfrak{s}} \times_{\mathcal{G}_{\mathfrak{s}}} \partial\bar{\mathrm{Gl}}_{\mathfrak{t}}(\delta) \to \tilde{M}_{\mathfrak{s}} \times_{\mathcal{G}_{\mathfrak{s}}\times S^1} \partial\bar{\mathrm{Gl}}_{\mathfrak{t}}(\delta). \tag{5.74}$$

The circle action in (5.74) is trivial on $\tilde{M}_{\mathfrak{s}}$ and given by the action described prior to (3.56) on $\partial\bar{\mathrm{Gl}}_{\mathfrak{t}}(\delta) \subset \bar{\mathrm{Gl}}_{\mathfrak{t}}(\delta)$; observe that the action of the group $\mathcal{G}_{\mathfrak{s}}$ in the bundle (5.74) is defined in (3.56), for $s \in \mathcal{G}_{\mathfrak{s}}$, $(B,\Psi) \in \tilde{M}_{\mathfrak{s}}$, and $\mathbf{g} \in \partial\bar{\mathrm{Gl}}_{\mathfrak{t}}(\delta)$, by

$$(s,(B,\Psi,\mathbf{g})) \mapsto (s_*(B,\Psi), s^{-2}\mathbf{g}).$$

Let $\mathbb{L}^1_{\mathfrak{s}}$ be the unit sphere of the bundle (2.19), so

$$\mathbb{L}^1_{\mathfrak{s}} = \tilde{M}_{\mathfrak{s}} \times_{\mathcal{G}_{\mathfrak{s}}} (X \times S^1),$$

with the action (2.20) of $\mathcal{G}_{\mathfrak{s}}$. Consider the bundle

$$\left(\mathbb{L}^1_{\mathfrak{s}} \times_X \partial\bar{\mathrm{Gl}}_{\mathfrak{t}}(\delta)\right)/S^1 \to M_{\mathfrak{s}} \times \partial\bar{\mathrm{Gl}}_{\mathfrak{t}}(\delta)/S^1, \tag{5.75}$$

where the action of $e^{i\gamma} \in S^1$ is given for $[B,\Psi,x,e^{i\theta}] \in \mathbb{L}^1_{\mathfrak{s}}|_x$ and $\mathbf{g} \in \partial\bar{\mathrm{Gl}}_{\mathfrak{t}}(\delta)|_x$ by

$$\left([B,\Psi,x,e^{i\theta}],\mathbf{g}\right) \mapsto \left([B,\Psi,x,e^{i(\theta+\gamma)}], e^{-2i\gamma}\mathbf{g}\right).$$

By Lemma 4.6, the first Chern class of the bundle (5.75) is $\nu_{\mathfrak{t}} + 2c_1(\mathbb{L}_{\mathfrak{s}})$. To prove (5.73), it suffices to show that the bundles (5.75) and (5.74) are isomorphic. Because $\mathcal{G}_{\mathfrak{s}}$ acts anti-diagonally in the definition of $\mathbb{L}_{\mathfrak{s}}$, the map

$$\begin{gathered}\left(\mathbb{L}^1_{\mathfrak{s}} \times_X \partial\bar{\mathrm{Gl}}_{\mathfrak{t}}(\delta)\right)/S^1 \to \tilde{M}_{\mathfrak{s}} \times_{\mathcal{G}_{\mathfrak{s}}} \partial\bar{\mathrm{Gl}}_{\mathfrak{t}}(\delta), \\ \left[[B,\Psi,x,e^{i\theta}],\mathbf{g}\right] \mapsto \left[(B,\Psi), e^{2i\theta}\mathbf{g}\right]\end{gathered}$$

is a well-defined bundle isomorphism. The final statement of the lemma follows from the fact that $c_1(\mathbb{L}_{\mathfrak{s}}) = \mu$ [17, Lemma 2.24]. □

As discussed in the paragraph preceding Definition 5.22, the results of Lemma 5.23 show how the relevant cohomology classes on $\mathbf{BL}^{\mathrm{vir},i}_{\mathfrak{t}',\mathfrak{s}}$ pull back from either $M_{\mathfrak{s}}$ or $\partial\bar{\mathrm{Gl}}_{\mathfrak{t}}(\delta)/S^1$. Pairing the appropriate cohomology class with the fundamental class $[M_{\mathfrak{s}}]$ yields the Seiberg-Witten invariant. The required pairings with $[\partial\bar{\mathrm{Gl}}_{\mathfrak{t}}(\delta)/S^1]$ are given by the following lemma:

Lemma 5.24. *Let $\nu_{\mathfrak{t}}$ be the characteristic class in Definition 5.22. Let $\partial\bar{\mathrm{Gl}}_{\mathfrak{t}}(\delta)/S^1$ have the standard orientation as defined in the paragraph following* (3.84). *If $x \in H_0(X;\mathbb{Z})$ is the positive generator and $h \in H_2(X;\mathbb{R})$, then*

$$\begin{aligned}\langle \nu_{\mathfrak{t}} \smile \pi_X^* \mathrm{PD}[x], [\partial\bar{\mathrm{Gl}}_{\mathfrak{t}}(\delta)/S^1]\rangle &= 2, \\ \langle \nu_{\mathfrak{t}}^2 \smile \pi_X^* \mathrm{PD}[h], [\partial\bar{\mathrm{Gl}}_{\mathfrak{t}}(\delta)/S^1]\rangle &= -4\langle (c_1(\mathfrak{s}) - c_1(\mathfrak{t})) \smile \mathrm{PD}[h], [X]\rangle, \\ \langle \nu_{\mathfrak{t}}^3, [\partial\bar{\mathrm{Gl}}_{\mathfrak{t}}(\delta)/S^1]\rangle &= 6(c_1(\mathfrak{s}) - c_1(\mathfrak{t}))^2 + 2c_1^2(X).\end{aligned} \tag{5.76}$$

Proof. In [34, §3], a rank-two, Hermitian vector bundle, $F \to X'$, where $\pi : X' \to X$ is a finite-degree cover is constructed with $F \to \pi^* \mathrm{Gl}_{\mathfrak{t}}$ a branched cover of degree negative two. (Similar bundles are constructed in [54, 42, 33].) The circle action on F induced by the action of $\mathbb{C}^*$ and the inclusion $S^1 \subset \mathbb{C}^*$ maps to twice the circle action on $\pi^* \mathrm{Gl}_X$. This implies that the map on quotients, $\mathbb{P}(F) \to \pi^* \partial\bar{\mathrm{Gl}}_{\mathfrak{t}}(\delta)/S^1$, has degree negative one and that the class $\nu_{\mathfrak{t}}$ pulls back to $2h$ where h is the first Chern class of the circle bundle $F/\mathbb{R}^* \to \mathbb{P}(F)$. Pulling back the cohomology classes in (5.76) to $\mathbb{P}(F)$ and applying the computation of the Segre classes of F in [34, §3] then yield the formulas in (5.76). □

Using Poincaré duality, we can use Lemma 5.24 to compute further pairings needed in our proof of Theorem 6.1. For example, if $\alpha \in H^2(X;\mathbb{R})$, then

$$\langle \nu_{\mathfrak{t}} \smile \pi_X^*(\alpha \smile \mathrm{PD}[h]), [\partial \bar{\mathrm{G}}\mathrm{l}_{\mathfrak{t}}(\delta)/S^1]\rangle = \langle \nu_{\mathfrak{t}} \smile \pi_X^* \mathrm{PD}[x], [\partial \bar{\mathrm{G}}\mathrm{l}_{\mathfrak{t}}(\delta)/S^1]\rangle \langle \alpha, h\rangle, \tag{5.77}$$

using

$$\begin{aligned} \alpha \smile \mathrm{PD}[h] &= \langle \alpha \smile \mathrm{PD}[h], [X]\rangle \,\mathrm{PD}[x] \\ &= \langle \alpha, \mathrm{PD}[h] \cap [X]\rangle \,\mathrm{PD}[x] \\ &= \langle \alpha, h\rangle \,\mathrm{PD}[x], \end{aligned}$$

by Poincaré duality.

Remark 5.25. In comparing the formulas of this and related articles with computations of wall-crossing formulas for Donaldson invariants, the cohomology class $\xi = c_1(Q_\xi) \in H^2(X;\mathbb{Z})$ of [30, 34] defining a reduction $\mathfrak{g}_V \cong \underline{\mathbb{R}} \oplus L$ is equal to the cohomology class $c_1(L) = c_1(\mathfrak{t}) - c_1(\mathfrak{s}) = c_1(\mathfrak{t}') - c_1(\mathfrak{s})$ appearing in this article and in [17, 18, 14].

6. Intersection with link of level-one Seiberg-Witten moduli space

In this section we prove Theorem 1.1 by computing a formula, given in Theorem 6.1, for intersection number (5.1) and applying this formula to the cobordism sum (2.52). We divide this task into three parts. Section 6.1 is devoted to the proof of Theorem 6.1. Because the Donaldson invariants are defined using a moduli space of anti-self-dual connections on $X\#\overline{\mathbb{CP}}^2$ (see (2.47)), we present a blow-up formula for the link intersection numbers in §6.2. Finally, in §6.3, we apply the results of [14] and Theorem 6.1 to the cobordism formula (2.52) to prove Theorem 1.1.

6.1. **Algebraic computations and completion of proof of Theorem 6.1.** With the results of §5, the proof of the following theorem is essentially algebraic. Recall that a spin^u structure over X splits, $\mathfrak{t} = \mathfrak{s} \oplus \mathfrak{s}'$, if and only if $(c_1(\mathfrak{t}) - c_1(\mathfrak{s}))^2 = p_1(\mathfrak{t})$ [18, Lemma 3.32]. Hence, a Seiberg-Witten stratum $M_{\mathfrak{s}} \times \mathrm{Sym}^\ell(X)$ is contained in the level $\mathcal{M}_{\mathfrak{t}_\ell} \times \mathrm{Sym}^\ell(X)$ of the space of ideal SO(3) monopoles enclosing $\bar{\mathcal{M}}_{\mathfrak{t}'}$ if and only if $(c_1(\mathfrak{t}')-c_1(\mathfrak{s}))^2 = p_1(\mathfrak{t}')+4\ell$; in this situation, $\mathbf{L}_{\mathfrak{t}',\mathfrak{s}}$ is the link (3.79) of the stratum $M_{\mathfrak{s}} \times \mathrm{Sym}^\ell(X)$ in $\bar{\mathcal{M}}_{\mathfrak{t}'}/S^1$.

Theorem 6.1. *Let X be a four-manifold with $b_1(X) = 0$, odd $b_2^+(X) \geq 1$. Suppose X has a generic Riemannian metric and a spin^u structure $\mathfrak{t}'$, where $w_2(\mathfrak{t}')$ is* good *in the sense of Definition 2.3. Let $\mathfrak{s}$ be a spin^c structure over X for which $(c_1(\mathfrak{t}') - c_1(\mathfrak{s}))^2 = p_1(\mathfrak{t}') + 4$. Let δ, m, and η be non-negative integers satisfying*

$$0 \leq m \leq [\delta/2] \quad \textit{and} \quad 2(\delta+\eta) = \dim(\mathcal{M}_{\mathfrak{t}'}/S^1) - 1. \tag{6.1}$$

If $x \in H_0(X;\mathbb{Z})$ is the positive generator, $h \in H_2(X;\mathbb{R})$, and $\mathbf{L}_{\mathfrak{t}',\mathfrak{s}}$ has the standard orientation defined prior to Lemma 3.13, then

$$\begin{aligned} &\#\left(\bar{\mathcal{V}}(h^{\delta-2m}x^m) \cap \bar{\mathcal{W}}^\eta \cap \mathbf{L}_{\mathfrak{t}',\mathfrak{s}}\right) \\ &= (-1)^{m+1+d_s(\mathfrak{s})/2} 2^{-\delta} 2^{d_s(\mathfrak{s})/2} P^{a,b}_{d_s(\mathfrak{s})/2}(0) \langle \mu_{\mathfrak{s}}^{d_s(\mathfrak{s})/2}, [M_{\mathfrak{s}}]\rangle \Big(a_0 \langle c_1(\mathfrak{s}) - c_1(\mathfrak{t}'), h\rangle^{\delta-2m} \\ &\quad + b_0 \langle c_1(\mathfrak{s}) - c_1(\mathfrak{t}'), h\rangle^{\delta-2m-1} \langle c_1(\mathfrak{t}'), h\rangle \\ &\quad + a_1 \langle c_1(\mathfrak{s}) - c_1(\mathfrak{t}'), h\rangle^{\delta-2m-2} Q_X(h,h)\Big), \end{aligned} \tag{6.2}$$

where all terms on the right which would have a negative exponent are omitted,

$$
\begin{aligned}
a_0 &= 3(c_1(\mathfrak{s}) - c_1(\mathfrak{t}'))^2 + c_1^2(X) + 2(c_1(\mathfrak{s}) - c_1(\mathfrak{t}')) \cdot c_1(\mathfrak{t}') + 4(\delta - 2m) - 4m, \\
b_0 &= 2(\delta - 2m)\frac{P^{a-1,b+1}_{d_s(\mathfrak{s})/2}(0)}{P^{a,b}_{d_s(\mathfrak{s})/2}(0)}, \\
a_1 &= 4\binom{\delta - 2m}{2},
\end{aligned}
$$

and $P^{a,b}_{d_s(\mathfrak{s})/2}(0)$ *is the constant coefficient of the Jacobi polynomial* (1.7) *with*

$$
\begin{aligned}
a &= \eta - \tfrac{1}{2}d_s(\mathfrak{s}) + 1, \\
b &= \tfrac{1}{2}(2\delta - d_a(\mathfrak{t}') - d_s(\mathfrak{s})) - \tfrac{1}{4}(\chi + \sigma),
\end{aligned}
\tag{6.3}
$$

where $d_a(\mathfrak{t}')$ *is given by equation* (2.11)*. If* $d_s(\mathfrak{s}) = 0$*, then* $P^{a,b}_{d_s(\mathfrak{s})/2}(0) = 1$.

Proof. According to Proposition 5.2, the intersection number on the left-hand side of equation (6.2) is equal to

$$
\begin{aligned}
\mathcal{P} :&= \#\left(\bar{\mathcal{V}}(h^{\delta-2m}x^m) \cap \bar{\mathcal{W}}^\eta \cap \mathbf{L}_{\mathfrak{t}',\mathfrak{s}}\right) \\
&= \langle \boldsymbol{\gamma}^*\bar{\mu}_p(h^{\delta-2m}x^m) \smile \boldsymbol{\gamma}^*\bar{\mu}_c^\eta \smile \bar{e}_s \smile \bar{e}_i, [\bar{\mathbf{L}}^{\mathrm{vir}}_{\mathfrak{t}',\mathfrak{s}}]\rangle,
\end{aligned}
\tag{6.4}
$$

where $\mathbf{L}^{\mathrm{vir}}_{\mathfrak{t}',\mathfrak{s}}$ has the standard orientation defined in §3.9. Writing

$$
d = \tfrac{1}{2}d_s(\mathfrak{s}) \quad \text{and} \quad \mathfrak{b} = \langle c_1(\mathfrak{s}) - c_1(\mathfrak{t}'), h\rangle,
\tag{6.5}
$$

we apply Lemmas 4.10, 4.11 and 4.12 to expand the cup product in the pairing (6.4) to give

$$
\begin{aligned}
\mathcal{P} = \Big\langle &\left(\tfrac{1}{2}\mathfrak{b}(2\mu_{\mathfrak{s}} - \nu) + \mathrm{PD}[h]\right)^{\delta-2m} \smile \left(-\tfrac{1}{4}(2\mu_{\mathfrak{s}} - \nu)^2 + \mathrm{PD}[x]\right)^m \\
&\smile (-\nu)^{\eta + r_\Xi}\left(\tfrac{1}{2}c_1(\mathfrak{t}') - \tfrac{1}{2}\nu\right), [\bar{\mathbf{L}}^{\mathrm{vir}}_{\mathfrak{t}',\mathfrak{s}}]\Big\rangle,
\end{aligned}
\tag{6.6}
$$

where, for notational simplicity, we omit explicit mention of pullbacks. Denoting

$$
C = (-1)^{m+\eta+r_\Xi+1}2^{-\delta-1},
\tag{6.7}
$$

using the fact that the classes $\mathrm{PD}[x], \mathrm{PD}[h], c_1(\mathfrak{t}')$ are pulled back from X, and multiplying out, we see that equation (6.6) becomes

$$
\begin{aligned}
\mathcal{P} = C\Big\langle &\mathfrak{b}^{\delta-2m}(2\mu_{\mathfrak{s}} - \nu)^\delta\nu^{\eta+r_\Xi+1} - \mathfrak{b}^{\delta-2m}(2\mu_{\mathfrak{s}} - \nu)^\delta\nu^{\eta+r_\Xi}c_1(\mathfrak{t}') \\
&+ 2(\delta - 2m)\mathfrak{b}^{\delta-2m-1}(2\mu_{\mathfrak{s}} - \nu)^{\delta-1}\nu^{\eta+r_\Xi+1}\,\mathrm{PD}[h] \\
&- 2(\delta - 2m)\mathfrak{b}^{\delta-2m-1}(2\mu_{\mathfrak{s}} - \nu)^{\delta-1}\nu^{\eta+r_\Xi}\,\mathrm{PD}[h]c_1(\mathfrak{t}') \\
&+ 4\binom{\delta - 2m}{2}\mathfrak{b}^{\delta-2m-2}(2\mu_{\mathfrak{s}} - \nu)^{\delta-2}\nu^{\eta+r_\Xi+1}Q_X(h,h)\,\mathrm{PD}[x] \\
&-4m\mathfrak{b}^{\delta-2m}(2\mu_{\mathfrak{s}} - \nu)^{\delta-2}\nu^{\eta+r_\Xi+1}\,\mathrm{PD}[x], [\bar{\mathbf{L}}^{\mathrm{vir}}_{\mathfrak{t}',\mathfrak{s}}]\Big\rangle.
\end{aligned}
\tag{6.8}
$$

If r_N denotes the rank of the complex vector bundle $N_{\mathfrak{t},\mathfrak{s}} \to M_{\mathfrak{s}}$, then (see equations (2.32) and (2.34))

$$
\begin{aligned}
\delta + \eta &= \tfrac{1}{2}\dim(\mathcal{M}_{\mathfrak{t}'}/S^1) - 1 \\
&= \tfrac{1}{2}\dim(\mathcal{M}_{\mathfrak{t}}/S^1) + 2 \quad \text{(By Equation (2.11))} \\
&= d + r_N - r_\Xi + 2.
\end{aligned}
\tag{6.9}
$$

The identity (6.9) and the fact that $\mu_{\mathfrak{s}}^i = 0$ when $i > d$, because the class $\mu_{\mathfrak{s}}$ is pulled back from $M_{\mathfrak{s}}$, imply that equation (6.8) can be rewritten as

$$
\begin{aligned}
\mathcal{P} = (-1)^\delta C \sum_{i=0}^{d} \Big\langle & \mathfrak{b}^{\delta-2m}\binom{\delta}{i} 2^i(-1)^i \mu_{\mathfrak{s}}^i \nu^{d+r_N+3-i} \\
& - \mathfrak{b}^{\delta-2m}\binom{\delta}{i} 2^i(-1)^i \mu_{\mathfrak{s}}^i \nu^{d+r_N+2-i} c_1(\mathfrak{t}') \\
& - 2(\delta-2m)\mathfrak{b}^{\delta-2m-1}\binom{\delta-1}{i} 2^i(-1)^i \mu_{\mathfrak{s}}^i \nu^{d+r_N+2-i}\,\mathrm{PD}[h] \\
& + 2(\delta-2m)\mathfrak{b}^{\delta-2m-1}\binom{\delta-1}{i} 2^i(-1)^i \mu_{\mathfrak{s}}^i \nu^{d+r_N+1-i}\,\mathrm{PD}[h] c_1(\mathfrak{t}') \\
& + 4\binom{\delta-2m}{2}\mathfrak{b}^{\delta-2m-2}\binom{\delta-2}{i} 2^i(-1)^i \mu_{\mathfrak{s}}^i \nu^{d+r_N+1-i} Q_X(h,h)\,\mathrm{PD}[x] \\
& -4m\mathfrak{b}^{\delta-2m}\binom{\delta-2}{i} 2^i(-1)^i \mu_{\mathfrak{s}}^i \nu^{d+r_N+1-i}\,\mathrm{PD}[x], [\bar{\mathbf{L}}_{\mathfrak{t}',\mathfrak{s}}^{\mathrm{vir}}] \Big\rangle,
\end{aligned}
\tag{6.10}
$$

where it is understood that binomial coefficients $\binom{\delta}{i}$ are by definition zero when $i > \delta$. By applying the formula (5.71) for division by the Poincaré dual of $[\mathbf{BL}_{\mathfrak{t}',\mathfrak{s}}^{\mathrm{vir},i}] = [M_{\mathfrak{s}} \times \partial\bar{\mathrm{G}}\mathrm{l}_{\mathfrak{t}}(\delta)/S^1]$, we see that equation (6.10) yields

$$
\begin{aligned}
\mathcal{P} = (-1)^{\delta+r_N} C \sum_{i=0}^{d}\sum_{j=0}^{d-i} \Big\langle & \mathfrak{b}^{\delta-2m}\binom{\delta}{i} 2^i(-1)^{i+j} \nu^{d+3-i-j} \mu_{\mathfrak{s}}^i s_j(N_{\mathfrak{t},\mathfrak{s}}) \\
& - \mathfrak{b}^{\delta-2m}\binom{\delta}{i} 2^i(-1)^{i+j} \nu^{d+2-i-j} \mu_{\mathfrak{s}}^i s_j(N_{\mathfrak{t},\mathfrak{s}}) c_1(\mathfrak{t}') \\
& - 2(\delta-2m)\mathfrak{b}^{\delta-2m-1}\binom{\delta-1}{i} 2^i(-1)^{i+j} \nu^{d+2-i-j} \mu_{\mathfrak{s}}^i s_j(N_{\mathfrak{t},\mathfrak{s}})\,\mathrm{PD}[h] \\
& + 2(\delta-2m)\mathfrak{b}^{\delta-2m-1}\binom{\delta-1}{i} 2^i(-1)^{i+j} \nu^{d+1-i-j} \mu_{\mathfrak{s}}^i s_j(N_{\mathfrak{t},\mathfrak{s}})\,\mathrm{PD}[h] c_1(\mathfrak{t}') \\
& + 4\binom{\delta-2m}{2}\mathfrak{b}^{\delta-2m-2}\binom{\delta-2}{i} 2^i(-1)^{i+j} \nu^{d+1-i-j} \mu_{\mathfrak{s}}^i s_j(N_{\mathfrak{t},\mathfrak{s}}) Q_X(h,h)\,\mathrm{PD}[x] \\
& -4m\mathfrak{b}^{\delta-2m}\binom{\delta-2}{i} 2^i(-1)^{i+j} \nu^{d+1-i-j} \mu_{\mathfrak{s}}^i s_j(N_{\mathfrak{t},\mathfrak{s}})\,\mathrm{PD}[x], [M_{\mathfrak{s}} \times \partial\bar{\mathrm{G}}\mathrm{l}_{\mathfrak{t}}(\delta)/S^1] \Big\rangle.
\end{aligned}
\tag{6.11}
$$

We then make the substitution $\nu = 2\mu_{\mathfrak{s}} + \nu_{\mathfrak{t}}$ from equation (5.73) and expand the powers of ν in (6.11) as binomial sums. For $\alpha \in H^{2k}(X;\mathbb{R})$, the equality

$$
\begin{aligned}
&\Big\langle \nu_{\mathfrak{t}}^{d+3-k-i-j-\ell} \smile \mu_{\mathfrak{s}}^{i+\ell} \smile s_j(N_{\mathfrak{t},\mathfrak{s}}) \smile \pi_X^*\alpha, [M_{\mathfrak{s}} \times \partial\bar{\mathrm{G}}\mathrm{l}_{\mathfrak{t}}(\delta)/S^1] \Big\rangle \\
&\quad = \begin{cases} \langle \mu_{\mathfrak{s}}^{d-j} \smile s_j(N_{\mathfrak{t},\mathfrak{s}}), [M_{\mathfrak{s}}]\rangle \langle \nu_{\mathfrak{t}}^{3-k} \smile \pi_X^*\alpha, [\partial\bar{\mathrm{G}}\mathrm{l}_{\mathfrak{t}}(\delta)/S^1]\rangle, & \text{if } \ell = d-i-j, \\ 0, & \text{if } \ell \neq d-i-j, \end{cases}
\end{aligned}
\tag{6.12}
$$

implies that only one term from each of the binomial expansions of the powers of ν in equation (6.11) will not vanish. Therefore, equation (6.11) yields

$$
\begin{aligned}
\mathcal{P} = (-1)^{\delta+r_N} C \sum_{i=0}^{d}\sum_{j=0}^{d-i} S_j \langle \mu_{\mathfrak{s}}^d, [M_{\mathfrak{s}}]\rangle \Bigg\langle & \mathfrak{b}^{\delta-2m}\binom{\delta}{i}\binom{d+3-i-j}{d-i-j}(-1)^{i+j}2^{d-j}\nu_{\mathfrak{t}}^3 \\
& - \mathfrak{b}^{\delta-2m}\binom{\delta}{i}\binom{d+2-i-j}{d-i-j}(-1)^{i+j}2^{d-j}\nu_{\mathfrak{t}}^2 c_1(\mathfrak{t}') \\
& - 2(\delta-2m)\mathfrak{b}^{\delta-2m-1}\binom{\delta-1}{i}\binom{d+2-i-j}{d-i-j}(-1)^{i+j}2^{d-j}\nu_{\mathfrak{t}}^2 \operatorname{PD}[h] \\
& + 2(\delta-2m)\mathfrak{b}^{\delta-2m-1}\binom{\delta-1}{i}\binom{d+1-i-j}{d-i-j}(-1)^{i+j}2^{d-j}\nu_{\mathfrak{t}} \operatorname{PD}[h] c_1(\mathfrak{t}') \\
& + 4\binom{\delta-2m}{2}\mathfrak{b}^{\delta-2m-2}\binom{\delta-2}{i}\binom{d+1-i-j}{d-i-j}(-1)^{i+j}2^{d-j} Q_X(h,h)\nu_{\mathfrak{t}} \operatorname{PD}[x] \\
& -4m\mathfrak{b}^{\delta-2m}\binom{\delta-2}{i}\binom{d+1-i-j}{d-i-j}(-1)^{i+j}2^{d-j}\nu_{\mathfrak{t}} \operatorname{PD}[x], [\partial\bar{\mathrm{G}}\mathrm{l}_{\mathfrak{t}}(\delta)/S^1]\Bigg\rangle ,
\end{aligned}
\tag{6.13}
$$

where we use Lemma 2.4 to see that

$$\Big\langle \mu_{\mathfrak{s}}^{d-j} s_j(N_{\mathfrak{t},\mathfrak{s}}), [M_{\mathfrak{s}}]\Big\rangle = S_j \langle \mu_{\mathfrak{s}}^d, [M_{\mathfrak{s}}]\rangle,$$

and the constants S_j are defined by

$$S_j = \sum_{k=0}^{j} 2^k \binom{-n_s'}{k}\binom{-n_s''}{j-k}. \tag{6.14}$$

See [18, Lemma 4.16] for identities involving binomial coefficients and extensions of the definition of $\binom{n}{p}$ to the case $n \le 0$. Before applying Lemma 5.24 to compute each of the pairings with $[\partial\bar{\mathrm{G}}\mathrm{l}_{\mathfrak{t}}(\delta)/S^1]$ on the right-hand side of equation (6.13), we simplify the combinatorial factors appearing in each term in (6.13) using the identities, for $v = 0, \dots, 3$,

$$
\begin{aligned}
& \sum_{i=0}^{d}\sum_{j=0}^{d-i}(-1)^{i+j}2^{d-j}\binom{\delta-v}{i}\binom{d+3-v-i-j}{d-i-j}S_j \\
& = \sum_{i=0}^{d}\sum_{j=0}^{d-i}\sum_{k=0}^{j}(-1)^{i+j}2^{d-j+k}\binom{\delta-v}{i}\binom{d+3-v-i-j}{d-i-j}\binom{-n_s'}{k}\binom{-n_s''}{j-k} \\
& \qquad \text{(By Equation (6.14))} \\
& = 2^d P_d^{a,b}(0) \quad \text{(By Equation (6.19))},
\end{aligned}
\tag{6.15}
$$

where $P_d^{a,b}(\zeta)$ is the Jacobi polynomial (1.7), with constants

$$a = 3 + n_s' + n_s'' - \delta \quad \text{and} \quad b = \delta - n_s' - 4 - d. \tag{6.16}$$

Because the constant on the left-hand side of equation (6.15) is independent of v, we can factor it out of all the terms in equation (6.13) except from the fourth term (the one containing $\operatorname{PD}[h]c_1(\mathfrak{t}')$), where the binomial factors can be rewritten as

$$\binom{(\delta+1)-2}{i}\binom{d+1-i-j}{d-i-j},$$

and thus, by replacing δ by $\delta+1$ in the definition of a and b in (6.16), the combinatorial expression from the fourth term of equation (6.13) corresponding to equation (6.15) will give the Jacobi polynomial $P_d^{a-1,b+1}(0)$. Hence, by substituting (6.15), equation (6.13) takes the shape

$$
\begin{aligned}
\mathcal{P} &= (-1)^{\delta+r_N} C 2^d P_d^{a,b}(0)\langle \mu_{\mathfrak{s}}^d, [M_{\mathfrak{s}}]\rangle \Big\langle \mathfrak{b}^{\delta-2m}(\nu_{\mathfrak{t}}^3 - \nu_{\mathfrak{t}}^2 c_1(\mathfrak{t}')) \\
&\quad - 2(\delta-2m)\mathfrak{b}^{\delta-2m-1}\nu_{\mathfrak{t}}^2 \operatorname{PD}[h] + 2(\delta-2m)\frac{P_d^{a-1,b+1}(0)}{P_d^{a,b}(0)}\mathfrak{b}^{\delta-2m-1}\nu_{\mathfrak{t}} \operatorname{PD}[h] c_1(\mathfrak{t}') \\
&\quad +4\binom{\delta-2m}{2}\mathfrak{b}^{\delta-2m-2} Q_X(h,h)\nu_{\mathfrak{t}} \operatorname{PD}[x] - 4m\mathfrak{b}^{\delta-2m}\nu_{\mathfrak{t}} \operatorname{PD}[x], [\partial\bar{\mathrm{G}}\mathrm{l}_{\mathfrak{t}}(\delta)/S^1]\Big\rangle .
\end{aligned}
\tag{6.17}
$$

Lemma 5.24 and equation (5.77) provide formulae for the pairings:

$$
\begin{gathered}
\langle \nu_{\mathfrak{t}} \operatorname{PD}[x], [\partial\bar{\mathrm{G}}\mathrm{l}_{\mathfrak{t}}(\delta)/S^1]\rangle, \\
\langle \nu_{\mathfrak{t}} \operatorname{PD}[h] c_1(\mathfrak{t}'), [\partial\bar{\mathrm{G}}\mathrm{l}_{\mathfrak{t}}(\delta)/S^1]\rangle, \\
\langle \nu_{\mathfrak{t}}^2 \operatorname{PD}[h], [\partial\bar{\mathrm{G}}\mathrm{l}_{\mathfrak{t}}(\delta)/S^1]\rangle, \\
\langle \nu_{\mathfrak{t}}^3, [\partial\bar{\mathrm{G}}\mathrm{l}_{\mathfrak{t}}(\delta)/S^1]\rangle.
\end{gathered}
$$

Hence, Lemma 5.24 and the definition (6.7) of C imply that equation (6.17) can be written as

$$
\begin{aligned}
\mathcal{P} &= (-1)^{\delta+\eta+r_N+r_\Xi+m+1} 2^{d-\delta-1} P_d^{a,b}(0)\langle \mu_{\mathfrak{s}}^d, [M_{\mathfrak{s}}]\rangle \\
&\quad \times \Big(\mathfrak{b}^{\delta-2m}\left(6(c_1(\mathfrak{s}) - c_1(\mathfrak{t}'))^2 + 2c_1^2(X)\right. \\
&\quad \left. +4(c_1(\mathfrak{s}) - c_1(\mathfrak{t}'))\cdot c_1(\mathfrak{t}') + 8(\delta-2m) - 8m\right) \\
&\quad + 4(\delta-2m)\mathfrak{b}^{\delta-2m-1}\frac{P_d^{a-1,b+1}(0)}{P_d^{a,b}(0)}\langle c_1(\mathfrak{t}'), h\rangle \\
&\quad +8\mathfrak{b}^{\delta-2m-2}\binom{\delta-2m}{2} Q_X(h,h)\Big) .
\end{aligned}
\tag{6.18}
$$

We write the power of 2 as $2^{d_s(\mathfrak{s})/2}2^{-\delta}2^{-1}$, since $d = d_s(\mathfrak{s})/2$ by definition (6.5), with the factor 2^{-1} being absorbed into the coefficients a_0, b_0, a_1, whose resulting formulae agree by inspection with those following equation (6.2).

Equation (6.9) implies that the power of (-1) in equation (6.18) simplifies to

$$
(-1)^{d_s(\mathfrak{s})/2+m+1},
$$

which agrees with the power of (-1) in (6.2).

Finally, we simplify the combinatorial coefficients $P_d^{a,b}(0)$ in (6.18). Because

$$
\begin{aligned}
d + n_s'(\mathfrak{t},\mathfrak{s}) + n_s''(\mathfrak{t},\mathfrak{s}) &= \tfrac{1}{2}\dim \mathcal{M}_{\mathfrak{t}} \quad \text{(By Equation (2.32))} \\
&= \tfrac{1}{2}\dim \mathcal{M}_{\mathfrak{t}'} - 3 \quad \text{(By Equation (2.11))} \\
&= \delta + \eta - 2 \quad \text{(By Equation (6.1))},
\end{aligned}
$$

we have, for a as defined in (6.16),

$$
\begin{aligned}
n_s'' + n_s' - \delta + 3 &= \eta - d + 1 \\
&= a,
\end{aligned}
$$

agreeing with the claimed formula for a in (6.3). Then, using

$$\begin{aligned} n'_s &= \tfrac{1}{2}d_a(\mathfrak{t}) + \tfrac{1}{4}(\chi+\sigma) \quad \text{(By Equation (2.33))} \\ &= \tfrac{1}{2}d_a(\mathfrak{t}') - 4 + \tfrac{1}{4}(\chi+\sigma), \end{aligned}$$

and $d_s(\mathfrak{s}) = 2d$, we have, for b as defined in (6.16),

$$\begin{aligned} \delta - 4 - n'_s - d &= \tfrac{1}{2}(2\delta - d_a(\mathfrak{t}') - d_s(\mathfrak{s})) - \tfrac{1}{4}(\chi+\sigma) \\ &= b, \end{aligned}$$

agreeing with the claimed formula for b in (6.3). This completes the proof of Theorem 6.1. □

It remains to prove the combinatorial identity (6.15).

Lemma 6.2. *For integers A, M, N, d with $d \geq 0$, and $v = 0, \dots, 3$, and $P_d^{a,b}(\zeta)$ the Jacobi polynomial* (1.7)*, we have*

$$2^d P_d^{a,b}(0) = \sum_{i=0}^{d}\sum_{j=0}^{d-i}\sum_{k=0}^{j}(-1)^{i+j}2^{d-j+k}\binom{A-v}{i}\binom{d+3-v-i-j}{d-i-j}\binom{M}{k}\binom{N}{j-k}, \tag{6.19}$$

where the constants a and b are given by

$$a = 3 - N - A - M \quad \textit{and} \quad b = A + M - 4 - d.$$

Proof. For $r \in \mathbb{R}$ and $\ell \in \mathbb{N}$ it is convenient to define

$$(r)_\ell := r(r+1)\cdots(r+\ell-1).$$

We then recall the following identities (see [18, Lemma 4.16]):

$$\binom{r}{\ell} = \frac{(-1)^\ell(-r)_\ell}{\ell!} \quad \text{and} \quad (r)_\ell = (-1)^\ell(1-r-\ell)_\ell. \tag{6.20}$$

The preceding identities yield

$$\begin{aligned} \binom{d+3-v-i-j}{d-i-j} &= \frac{(-1)^{d-i-j}(-d-3+i+j+v)_{d-i-j}}{(d-i-j)!} \\ &= \frac{(4-v)_{d-i-j}}{(d-i-j)!} \\ &= (-1)^{d-i-j}\binom{v-4}{d-i-j}. \end{aligned} \tag{6.21}$$

Let $\mathcal{S}$ denote right-hand-side of equation (6.19) and observe that, using equation (6.21), we can write $\mathcal{S}$ in the form

$$\mathcal{S} = \sum_{i=0}^{d}\sum_{j=0}^{d-i}\sum_{k=0}^{j}(-1)^d 2^{d-j+k}\binom{A-v}{i}\binom{v-4}{d-i-j}\binom{M}{k}\binom{N}{j-k}. \tag{6.22}$$

With the substitution $u = j - k$, equation (6.22) becomes

$$\begin{aligned} \mathcal{S} &= (-2)^d \sum_{i=0}^{d}\sum_{u=0}^{d-i}\sum_{j=u}^{d-i} 2^{-u}\binom{A-v}{i}\binom{v-4}{d-i-j}\binom{M}{j-u}\binom{N}{u} \\ &= (-2)^d \sum_{i=0}^{d}\sum_{u=0}^{d-i} 2^{-u}\binom{A-v}{i}\binom{N}{u}\sum_{j'=0}^{d-i-u}\binom{v-4}{d-i-u-j'}\binom{M}{j'} \end{aligned} \tag{6.23}$$

where we have set $j' = j - u$. Applying the Vandermonde convolution identity [18, Lemma 4.16 (5)] to the sum over j' in the right-hand side of (6.23), we see that

$$\begin{aligned}
\mathcal{S} &= (-2)^d \sum_{i=0}^{d}\sum_{u=0}^{d-i} 2^{-u}\binom{A-v}{i}\binom{N}{u}\binom{v+M-4}{d-i-u} \\
&= (-2)^d \sum_{u=0}^{d} 2^{-u}\binom{N}{u}\sum_{i=0}^{d-u}\binom{A-v}{i}\binom{v+M-4}{d-i-u} \quad \text{(Reversing summation order)} \\
&= (-2)^d \sum_{u=0}^{d} 2^{-u}\binom{N}{u}\binom{A+M-4}{d-u} \quad \text{(By [18, Lemma 4.16(5)])}.
\end{aligned} \tag{6.24}$$

We now express this sum in terms of the hypergeometric function [27, §9.10]

$$ {}_2F_1(a,b;c;\zeta) = \sum_{k=0}^{\infty}\frac{(a)_k(b)_k}{(c)_k k!}\zeta^k, \quad \zeta\in\mathbb{C}.$$

Using the first identity in (6.20), equation (6.24) yields

$$\mathcal{S} = (-2)^d \sum_{u=0}^{d} 2^{-u}\frac{(-1)^u(-N)_u}{u!}\frac{(-1)^{d-u}(4-A-M)_{d-u}}{(d-u)!}. \tag{6.25}$$

Then applying the identities (see [18, Lemma 4.16])

$$(a)_{d-u} = \frac{(-1)^u(a)_d}{(1-a-d)_u} \quad\text{and}\quad (d-u)! = \frac{d!}{(-1)^u(-d)_u} \tag{6.26}$$

to equation (6.25) gives

$$\begin{aligned}
\mathcal{S} &= 2^d\sum_{u=0}^{d}2^{-u}\frac{(-N)_u(-1)^u(4-A-M)_d(-1)^u(-d)_u}{u!(A+M-3-d)_u d!} \\
&= \frac{2^d(4-A-M)_d}{d!}\sum_{r=0}^{d}\frac{(-d)_u(-N)_u}{(A+M-3-d)_u u!}2^{-u} \\
&= \frac{2^d(4-A-M)_d}{d!}{}_2F_1(-d,-N;A+M-3-d;\tfrac{1}{2}) \\
&= \frac{(-2)^d(4-A-M)_d}{(A+M-3-d)_d}P_d^{3-N-A-M,A+M-4-d}(0) \quad \text{(By [18, Equation (4.40)])}.
\end{aligned} \tag{6.27}$$

Finally, we note that

$$\frac{(-2)^d(4-A-M)_d}{(A+M-3-d)_d} = 2^d$$

by the second identity in (6.20). This completes the proof of the lemma. □

6.2. **The blow-up formula for level-one Seiberg-Witten link pairings.** To apply Theorem 6.1 and express the link pairing (6.2) in terms of Seiberg-Witten invariants of X and thus express, via (2.52), the Donaldson invariants of X in terms of Seiberg-Witten invariants of X, we must apply the SO(3)-monopole cobordism to the blow-up, $\tilde{X} = X\#\overline{\mathbb{CP}}^2$, of X in view of our definitions (2.47) and (2.23) of the four-manifold invariants. Our definitions of the gauge-theoretic four-manifold invariants incorporate the blow-up formula because the classes $w + \mathrm{PD}[e] \pmod 2$ and $c_1(\mathfrak{s}) \pm \mathrm{PD}[e] - \Lambda \pmod 2$ are always good in the sense of Definition 2.3, for arbitrary w, $c_1(\mathfrak{s})$, and Λ.

Following the discussion in [18, §4.5], every spin^c structure on $\tilde{X}$ is given by $\mathfrak{s}\#\mathfrak{s}_{2k-1}$, the result of splicing a spin^c structure $\mathfrak{s}$ on X and the spin^c structure $\mathfrak{s}_{2k-1}$ on $\overline{\mathbb{CP}}^2$ with $c_1(\mathfrak{s}_{2k-1}) = (2k-1)\,\mathrm{PD}[e]$, $k \in \mathbb{Z}$. Then,

$$c_1(\mathfrak{s}\#\mathfrak{s}_{2k-1}) = c_1(\mathfrak{s}) + (2k-1)\,\mathrm{PD}[e]. \tag{6.28}$$

The dimensions of the Seiberg-Witten moduli spaces are related by

$$d_s(\mathfrak{s}\#\mathfrak{s}_{2k-1}) = d_s(\mathfrak{s}) - k(k-1). \tag{6.29}$$

For the cases $k = 1$ or 0, we denote $\mathfrak{s}^+ = \mathfrak{s}\#\mathfrak{s}_1$ and $\mathfrak{s}^- = \mathfrak{s}\#\mathfrak{s}_{-1}$, so that

$$c_1(\mathfrak{s}^\pm) = c_1(\mathfrak{s}) \pm \mathrm{PD}[e]. \tag{6.30}$$

If $d_s(\mathfrak{s}\#\mathfrak{s}_{2k-1}) \geq 0$, then $SW_X(\mathfrak{s}\#\mathfrak{s}_{2k-1}) = SW_X(\mathfrak{s})$ according to the blow-up formula [22, Theorem 1.4], [43, Theorem 3.2] (see the statement given in [18, Theorem 4.20]).

Recall from [18, Lemma 4.19] that given a spin^u structure $\mathfrak{t}$ on X, there is a spin^u structure $\tilde{\mathfrak{t}}$ on $\tilde{X}$ such that

$$c_1(\tilde{\mathfrak{t}}) = c_1(\mathfrak{t}), \quad p_1(\tilde{\mathfrak{t}}) = p_1(\mathfrak{t}) - 1, \quad w_2(\tilde{\mathfrak{t}}) \equiv w_2(\mathfrak{t}) + \mathrm{PD}[e] \pmod 2. \tag{6.31}$$

We then have the following extension of Lemma 4.19 in [18].

Lemma 6.3. *Continue the notation of the preceding paragraph. Then the following hold for each integer $\ell \geq 0$:*

1. *There is a bijection between the*
   - *Set of strata $\iota(M_{\mathfrak{s}}(X)) \times \mathrm{Sym}^\ell(X)$ in $I\mathcal{M}_{\mathfrak{t}}(X)$, and the*
   - *Set of families of strata $\iota(M_{\mathfrak{s}_{k^\pm}}(X)) \times \mathrm{Sym}^{\ell-j}(X)$ in $I\mathcal{M}_{\tilde{\mathfrak{t}}}(\tilde{X})$, where $\mathfrak{s}_{k^\pm} = \mathfrak{s}\#\mathfrak{s}_{2k^\pm-1}$ and $j, k^\pm \in \mathbb{Z}_{\geq 0}$ obey $j = k^\pm(k^\pm - 1) \leq \min\{\ell, d_s(\mathfrak{s})\}$.*
2. *The preceding correspondence further restricts to a bijection between the*
   - *Set of strata in level ℓ with $SW_X(\mathfrak{s}) \neq 0$, and the*
   - *Set of families of strata in levels $\ell - j$ with $SW_X(\mathfrak{s}_{k^\pm}) \neq 0$, where $j, k^\pm \in \mathbb{Z}_{\geq 0}$ obey $j = k^\pm(k^\pm - 1) \leq \min\{\ell, d_s(\mathfrak{s})\}$.*
3. *If X has SW-simple type then so has $\tilde{X}$ and the preceding correspondence further restricts to a bijection between the*
   - *Set of strata in level ℓ with $SW_X(\mathfrak{s}) \neq 0$, and the*
   - *Set of pairs of strata in level ℓ with $SW_X(\mathfrak{s}^\pm) \neq 0$.*

Proof. From [18, Lemma 3.32] there is a splitting $\mathfrak{t}_\ell = \mathfrak{s} \oplus \mathfrak{s}'$ if and only if $(c_1(\mathfrak{s}) - c_1(\mathfrak{t}))^2 = p_1(\mathfrak{t}) + 4\ell$, recalling that $\mathfrak{t}_\ell$ is a spin^u structure with $p_1(\mathfrak{t}_\ell) = p_1(\mathfrak{t}) + 4\ell$, $c_1(\mathfrak{t}_\ell) = c_1(\mathfrak{t})$, and $w_2(\mathfrak{t}_\ell) = w_2(\mathfrak{t})$. Hence, $M_{\mathfrak{s}}$ is contained in the level $\mathcal{M}_{\mathfrak{t}_\ell} \times \mathrm{Sym}^\ell(X)$ if and only if

$$\ell = \ell(\mathfrak{t}, \mathfrak{s}) := \tfrac{1}{4}((c_1(\mathfrak{s}) - c_1(\mathfrak{t}))^2 - p_1(\mathfrak{t})). \tag{6.32}$$

Substituting (6.28) and (6.31) into (6.32), we see that

$$\ell(\tilde{\mathfrak{t}}, \mathfrak{s}\#\mathfrak{s}_{2k-1}) = \ell(\mathfrak{t}, \mathfrak{s}) - k(k-1) \leq \ell(\mathfrak{t}, \mathfrak{s}).$$

Assertion (1) follows from the preceding formula relating levels and the relation (6.29) between the dimensions of the Seiberg-Witten moduli spaces. Assertions (2) and (3) follow from the blow-up formula [22, 43] and the definition of SW-simple type. □

Remark 6.4. If $\ell = 0$ or 1 in Cases (1) or (2) of Lemma 6.3, then we must have $j = 0$, $k^+ = 1$, $k^- = 0$, and $\ell(\mathfrak{t}, \mathfrak{s}) = \ell(\tilde{\mathfrak{t}}, \mathfrak{s}^\pm)$. Hence, for $\ell \in \{0, 1\}$ and $\mathfrak{s} \in \mathrm{Spin}^c(X)$, there is an injective map from the

- Set of strata, $M_{\mathfrak{s}} \times \mathrm{Sym}^\ell(X)$, in $I\mathcal{M}_{\mathfrak{t}}$ with $SW_X(\mathfrak{s}) \neq 0$, to the

- Set of pairs of strata, $M_{\mathfrak{s}^\pm} \times \mathrm{Sym}^\ell(\tilde{X})$, in $I\mathcal{M}_{\tilde{\mathfrak{t}}}$ with $SW_X(\mathfrak{s}^\pm) \neq 0$.

Furthermore, if all $\mathfrak{s} \in \mathrm{Spin}^c(X)$ with $\ell(\mathfrak{t}, \mathfrak{s}) \geq 2$ have $SW_X(\mathfrak{s}) = 0$, then the preceding map is bijective and all $\tilde{\mathfrak{s}} \in \mathrm{Spin}^c(\tilde{X})$ with $\ell(\tilde{\mathfrak{t}}, \tilde{\mathfrak{s}}) \geq 2$ have $SW_{\tilde{X}}(\tilde{\mathfrak{s}}) = 0$.

Proposition 6.5. *Continue the hypotheses of Theorem 6.1 leading to equation* (6.2)*, except we omit the requirement that $M_{\mathfrak{s}}$ contains no zero-section pairs. Let the links $\mathbf{L}_{\tilde{\mathfrak{t}}',\mathfrak{s}^\pm}$ have the standard orientation. Then, for k even,*

$$
\begin{aligned}
&(-1)^{o_{\tilde{\mathfrak{t}}'}(w+\mathrm{PD}[e],\mathfrak{s}^+)}\ \#\left(\bar{\mathcal{V}}(h^{\delta-2m-k}e^{k+1}x^m) \cap \bar{\mathcal{W}}^\eta \cap \mathbf{L}_{\tilde{\mathfrak{t}}',\mathfrak{s}^+}\right) \\
&+(-1)^{o_{\tilde{\mathfrak{t}}'}(w+\mathrm{PD}[e],\mathfrak{s}^-)}\ \#\left(\bar{\mathcal{V}}(h^{\delta-2m-k}e^{k+1}x^m) \cap \bar{\mathcal{W}}^\eta \cap \mathbf{L}_{\tilde{\mathfrak{t}}',\mathfrak{s}^-}\right) \\
&= (-1)^{o_{\mathfrak{t}}(w,\mathfrak{s})+m+1-d_s(\mathfrak{s})/2} 2^{-\delta} 2^{d_s(\mathfrak{s})/2} P^{a,b}_{d_s(\mathfrak{s})/2}(0) SW_X(\mathfrak{s}) \\
&\quad \times \Big(\tilde{a}_0 \langle c_1(\mathfrak{s}) - c_1(\mathfrak{t}'), h\rangle^{\delta-2m-k} \\
&\quad + \tilde{b}_0 \langle c_1(\mathfrak{s}) - c_1(\mathfrak{t}'), h\rangle^{\delta-2m-k-1} \langle c_1(\mathfrak{t}'), h\rangle \\
&\quad + \tilde{a}_1 \langle c_1(\mathfrak{s}) - c_1(\mathfrak{t}'), h\rangle^{\delta-2m-k-2} Q_X(h,h)\Big),
\end{aligned}
\tag{6.33}
$$

where the sign change factor $o_{\mathfrak{t}}(w,\mathfrak{s})$ is defined by (3.91)*, the coefficients $\tilde{a}_0, \tilde{b}_0, \tilde{a}_1$ are related to those of Theorem 6.1 by*

$$
\tilde{a}_0 = a_0 - 4\binom{k+1}{2}, \quad \tilde{b}_0 = b_0 - 2k\frac{P^{a-1,b+1}_{d_s(\mathfrak{s})/2}(0)}{P^{a,b}_{d_s(\mathfrak{s})/2}(0)}, \quad \text{and} \quad \tilde{a}_1 = 4\binom{\delta-2m-k}{2},
$$

and $P^{a,b}_{d_s(\mathfrak{s})/2}(0)$ is given by (1.7)*. If k is odd, the sum on the left-hand side of* (6.33) *is zero.*

Proof. From [18, Equation (4.55)] we see that $o_{\tilde{\mathfrak{t}}'}(w + \mathrm{PD}[e], \mathfrak{s}^+) = o_{\mathfrak{t}'}(w, \mathfrak{s}) - 1$ and that $o_{\tilde{\mathfrak{t}}'}(w + \mathrm{PD}[e], \mathfrak{s}^-) = o_{\mathfrak{t}'}(w, \mathfrak{s})$, so the left-hand side of (6.33) is $(-1)^{o_{\mathfrak{t}'}(w,\mathfrak{s})-1}$ times

$$
\#\left(\bar{\mathcal{V}}(h^{\delta-2m-k}e^{k+1}x^m) \cap \bar{\mathcal{W}}^\eta \cap \mathbf{L}_{\tilde{\mathfrak{t}}',\mathfrak{s}^+}\right) - \#\left(\bar{\mathcal{V}}(h^{\delta-2m-k}e^{k+1}x^m) \cap \bar{\mathcal{W}}^\eta \cap \mathbf{L}_{\tilde{\mathfrak{t}}',\mathfrak{s}^-}\right),
\tag{6.34}
$$

where the links $\mathbf{L}_{\tilde{\mathfrak{t}}',\mathfrak{s}^\pm}$ have the standard orientation. Suppose $h_i = h$, for $0 \leq i \leq \delta - 2m - k$ and $h_i = e$ for $\delta - 2m - k < i \leq \delta + 1 - 2m$. Applying the polarization identity [23, p. 396] to the formula (6.2) to compute the pairings

$$
\#\left(\bar{\mathcal{V}}(h^{\delta-2m-k}e^{k+1}x^m) \cap \bar{\mathcal{W}}^\eta \cap \mathbf{L}_{\tilde{\mathfrak{t}}',\mathfrak{s}^\pm}\right),
$$

and noting that $\dim\mathcal{M}_{\tilde{\mathfrak{t}}'}(\tilde{X}) = \dim\mathcal{M}_{\mathfrak{t}'}(X) + 2$ by equation (2.11), gives (using $d = d_s(\mathfrak{s})/2 = d_s(\mathfrak{s}^\pm)/2$ for brevity)

$$\begin{aligned}
&\#\left(\bar{\mathcal{V}}(h^{\delta-2m-k}e^{k+1}x^m)\cap\bar{\mathcal{W}}^\eta\cap\mathbf{L}_{\tilde{\mathfrak{t}}',\mathfrak{s}^\pm}\right)\\
&= (-1)^{m+1+d}2^{-\delta-1}2^d P_d^{a^\pm,b^\pm}(0)\langle\mu^d_{\mathfrak{s}^\pm},M_{\mathfrak{s}^\pm}(\tilde{X})\rangle\left(a_0^\pm\prod_{i=1}^{\delta+1-2m}\langle c_1(\mathfrak{s}^\pm)-c_1(\tilde{\mathfrak{t}}'),h_i\rangle\right.\\
&\quad+2\frac{P_d^{a^\pm-1,b^\pm+1}(0)}{P_d^{a^\pm,b^\pm}(0)}\sum_{j=1}^{\delta+1-2m}\prod_{\substack{i=1\\ i\neq j}}^{\delta+1-2m}\langle c_1(\mathfrak{s}^\pm)-c_1(\tilde{\mathfrak{t}}'),h_i\rangle\langle c_1(\tilde{\mathfrak{t}}'),h_j\rangle\\
&\quad\left.+4\sum_{1\le j<l\le\delta+1-2m}\prod_{\substack{i=1\\ i\neq j,l}}^{\delta+1-2m}\langle c_1(\mathfrak{s}^\pm)-c_1(\tilde{\mathfrak{t}}'),h_i\rangle Q_{\tilde{X}}(h_j,h_l)\right),
\end{aligned}\tag{6.35}$$

where, using $c_1(\mathfrak{s}^\pm) = c_1(\mathfrak{s}) \pm \mathrm{PD}[e]$ and $c_1^2(\tilde{X}) = c_1^2(X) - 1$, and $c_1(\tilde{\mathfrak{t}}') = c_1(\mathfrak{t}')$,

$$\begin{aligned}
a_0^\pm &= 3\left(c_1(\mathfrak{s}^\pm)-c_1(\tilde{\mathfrak{t}}')\right)^2 + c_1^2(\tilde{X}) + 2\left(c_1(\mathfrak{s}^\pm)-c_1(\tilde{\mathfrak{t}}')\right)\cdot c_1(\tilde{\mathfrak{t}}')\\
&\qquad + 4(\delta+1-2m) - 4m\\
&= 3(c_1(\mathfrak{s})-c_1(\mathfrak{t}'))^2 - 3 + c_1^2(X) - 1 + 2(c_1(\mathfrak{s})-c_1(\mathfrak{t}'))\cdot c_1(\mathfrak{t}')\\
&\qquad + 4(\delta-2m) - 4m + 4\\
&= 3(c_1(\mathfrak{s})-c_1(\mathfrak{t}'))^2 + c_1^2(X) + 2(c_1(\mathfrak{s})-c_1(\mathfrak{t}'))\cdot c_1(\mathfrak{t}')\\
&\qquad + 4(\delta-2m) - 4m,
\end{aligned}\tag{6.36}$$

and where $a^\pm, b^\pm$ are given by

$$\begin{aligned}
a^\pm &= \eta - d + 1\\
&= a,\\
b^\pm &= \tfrac{1}{2}(2(\delta+1) - d_a(\tilde{\mathfrak{t}}') - d_s(\mathfrak{s}^\pm)) - \tfrac{1}{4}(\chi(\tilde{X}) + \sigma(\tilde{X}))\\
&= \tfrac{1}{2}(2\delta - d_a(\mathfrak{t}') - d_s(\mathfrak{s})) - \tfrac{1}{4}(\chi+\sigma)\\
&= b.
\end{aligned}\tag{6.37}$$

Note that the factor of $(\delta + 1 - 2m)$ in the coefficient $b_0/2$ and the coefficient $a_1/4$ in equation (6.2) are absorbed by the application of the polarization identity [23, p. 396]. Using $c_1(\tilde{\mathfrak{t}}') = c_1(\mathfrak{t}')$ and the identities

$$\begin{aligned}
Q_{\tilde{X}}(e,e) &= -1,\\
Q_{\tilde{X}}(e,h) &= 0,\\
\langle c_1(\mathfrak{s}^\pm)-c_1(\tilde{\mathfrak{t}}'),e\rangle &= \mp 1,\\
\langle c_1(\mathfrak{s}^\pm)-c_1(\tilde{\mathfrak{t}}'),h\rangle &= \langle c_1(\mathfrak{s})-c_1(\mathfrak{t}'),h\rangle,\\
\langle c_1(\tilde{\mathfrak{t}}'),e\rangle &= 0,\\
\langle c_1(\tilde{\mathfrak{t}}'),h\rangle &= \langle c_1(\mathfrak{t}'),h\rangle,
\end{aligned}$$

we simplify the terms on the right-hand side of equation (6.35) (recall that $h_i = h \in H_2(X;\mathbb{R})$, $1 \leq i \leq \delta - k - 2m$, and $h_i = e$ for $\delta - k - 2m < i \leq \delta + 1 - 2m$) to give:

$$\prod_{i=1}^{\delta+1-2m} \langle c_1(\mathfrak{s}^{\pm}) - c_1(\tilde{\mathfrak{t}}'), h_i\rangle = (\mp 1)^{k+1}\langle c_1(\mathfrak{s}) - c_1(\mathfrak{t}'), h\rangle^{\delta-k-2m}, \tag{6.38}$$

and

$$\begin{aligned}\sum_{j=1}^{\delta+1-2m} \prod_{\substack{i=1\\ i\neq j}}^{\delta+1-2m} &\langle c_1(\mathfrak{s}^{\pm}) - c_1(\tilde{\mathfrak{t}}'), h_i\rangle\langle c_1(\tilde{\mathfrak{t}}'), h_j\rangle \\ &= (\mp 1)^{k+1}(\delta - 2m - k)\langle c_1(\mathfrak{s}) - c_1(\mathfrak{t}'), h\rangle^{\delta-k-2m-1}\langle c_1(\mathfrak{t}'), h\rangle,\end{aligned} \tag{6.39}$$

and

$$\begin{aligned}\sum_{1\leq j<l\leq \delta+1-2m} \prod_{\substack{i=1\\ i\neq j,l}}^{\delta+1-2m} &\langle c_1(\mathfrak{s}^{\pm}) - c_1(\tilde{\mathfrak{t}}'), h_i\rangle Q_{\tilde{X}}(h_j, h_l) \\ &= (\mp 1)^{k+1}\binom{\delta - 2m - k}{2}\langle c_1(\mathfrak{s}) - c_1(\mathfrak{t}'), h\rangle^{\delta-k-2m-2} Q_X(h,h) \\ &\quad + (\mp 1)^{k-1}\binom{k+1}{2}\langle c_1(\mathfrak{s}) - c_1(\mathfrak{t}'), h\rangle^{\delta-k-2m} Q_{\tilde{X}}(e,e).\end{aligned} \tag{6.40}$$

Thus, if $k+1$ is *even*, the terms (6.38), (6.39), and (6.40) will *cancel* when we substitute them into equation (6.35) and subtract the results in (6.34), so in this case the difference (6.34) and hence the pairing (6.33) are zero, as claimed. If $k+1$ is *odd,* the terms (6.38), (6.39), and (6.40) will *add* when we substitute them into equation (6.35) and subtract the results in (6.34). Hence, equation (6.33) follows from (6.34) together with the preceding substitutions and substitution of equation (6.36) for the coefficient $a_0^{\pm}$ and equation (6.37) for $a^{\pm}, b^{\pm}$. Note that the factor of 2 obtained by adding like terms and the coefficient $2^{-\delta-1}$ in (6.35) yields the coefficient $2^{-\delta}$ in (6.33) and that the factors $(-1)^{k+1} = -1$ and $(-1)^{o_{\mathfrak{t}'}(w,\mathfrak{s})-1}$ (mentioned before (6.34)) yield the factor $(-1)^{o_{\mathfrak{t}'}(w,\mathfrak{s})}$ appearing on the right-hand-side of (6.33). □

6.3. **Proofs of main theorems.** We now apply the computation of Theorem 6.1 to the sum (2.52) to prove Theorem 1.5 and hence Theorem 1.1.

Proof of Theorem 1.4. We shall derive equation (1.8) from the basic cobordism identity (2.52) applied to the blow-up $\tilde{X} = X\#\overline{\mathbb{CP}}^2$, the definition (2.47) for the Donaldson invariant, the blow-up formula (6.33) for level-one Seiberg-Witten link pairings, and the formula from [18, Equation (4.52)] for level-zero Seiberg-Witten link pairings.

Since $\deg(z) = 2\delta$ obeys condition (2.46), we can choose an integer p such that $p \equiv w^2 \pmod 4$ and

$$\delta = -p - \tfrac{3}{4}(\chi + \sigma). \tag{6.41}$$

According to the last paragraph of [17, §2.1.3], there is a spin^u structure $\mathfrak{t}'$ over X with $c_1(\mathfrak{t}') = \Lambda$, $w_2(\mathfrak{t}') \equiv w \pmod 2$, and $p_1(\mathfrak{t}) = p$. Let $\tilde{\mathfrak{t}}'$ be the related spin^u structure over

$\tilde{X} = X\#\overline{\mathbb{CP}}^2$ defined prior to Lemma 6.3. From equation (2.11) we see that

$$\begin{aligned} n_a(\tilde{\mathfrak{t}}') &= \tfrac{1}{4}(p_1(\tilde{\mathfrak{t}}') + \Lambda^2 - \sigma(\tilde{X})) \\ &= \tfrac{1}{4}(p_1(\mathfrak{t}') + \Lambda^2 - \sigma) \\ &= \tfrac{1}{4}(i(\Lambda) - \delta) \\ &= n_a(\mathfrak{t}'). \end{aligned} \tag{6.42}$$

Therefore, the hypothesis that $\delta < i(\Lambda)$ is equivalent to $n_a(\tilde{\mathfrak{t}}') = n_a(\mathfrak{t}') > 0$ and hence the moduli space $M^{w+\mathrm{PD}[e]}_{\kappa+1/4}(\tilde{X})$ has real codimension greater than or equal to two in $\mathcal{M}_{\tilde{\mathfrak{t}}'}(\tilde{X})$. Moreover, $w_2(\tilde{\mathfrak{t}}') = w + \mathrm{PD}[e] \pmod 2$ is good and so the cobordism formula (2.52) applies.

In general, by Remark 3.36 in [18], the Seiberg-Witten stratum $M_{\mathfrak{s}} \times \mathrm{Sym}^\ell(X)$ corresponding to a splitting $\mathfrak{t}'_\ell = \mathfrak{s} \oplus \mathfrak{s}'$ lies in level

$$\ell(\mathfrak{t},\mathfrak{s}) = \tfrac{1}{4}(\delta - r(\Lambda, c_1(\mathfrak{s}))),$$

of the space of ideal SO(3) monopoles $I\mathcal{M}_{\mathfrak{t}'}(X)$, since $2\delta = d_a(\mathfrak{t}') = \dim M^w_\kappa(X)$. By hypothesis, $\delta = r(\Lambda) + 4$ and $r(\Lambda) \leq r(\Lambda, c_1(\mathfrak{s}))$ for all $\mathfrak{s}$ with $SW_X(\mathfrak{s}) \neq 0$ (by definition (1.6) of $r(\Lambda)$), so our choice of δ implies that

$$0 \leq \ell(\mathfrak{t},\mathfrak{s}) \leq 1,$$

if $M_{\mathfrak{s}} \times \mathrm{Sym}^\ell(X) \subset I\mathcal{M}_{\mathfrak{t}'}$ and $SW_X(\mathfrak{s}) \neq 0$.

By hypothesis, X is 'effective' in the sense of Definition 1.3 and so the only non-zero Seiberg-Witten contributions to $D^w_X(h^{\delta-2m}x^m)$ arise from moduli spaces $M_{\mathfrak{s}}(X)$ with $SW_X(\mathfrak{s}) \neq 0$ contained in levels $\ell(\mathfrak{t},\mathfrak{s}) = 0$ or 1 of $I\mathcal{M}_{\mathfrak{t}'}(X)$ or, equivalently, from moduli spaces $M_{\mathfrak{s}^\pm}(\tilde{X})$ contained in levels $\ell(\tilde{\mathfrak{t}},\mathfrak{s}^\pm) = 0$ or 1 of $I\mathcal{M}_{\tilde{\mathfrak{t}}'}(\tilde{X})$, again when $SW_X(\mathfrak{s}) \neq 0$ by Remark 6.4.

Therefore, the cobordism identity (2.52) applied to the moduli space $\bar{\mathcal{M}}_{\tilde{\mathfrak{t}}'}(\tilde{X})/S^1$ and the definition (2.47) of $D^w_X(h^{\delta-2m}x^m)$ give

$$\begin{aligned} &D^w_X(h^{\delta-2m}x^m) \\ &= -2^{1-n_a(\mathfrak{t}')} \sum_{\mathfrak{s}\in\mathrm{Spin}^c(X)} \Big((-1)^{o_{\tilde{\mathfrak{t}}}(w+\mathrm{PD}[e],\mathfrak{s}^+)} \# \left(\bar{\mathcal{V}}(h^{\delta-2m}ex^m) \cap \bar{\mathcal{W}}^{n_a(\mathfrak{t})-1} \cap \mathbf{L}_{\tilde{\mathfrak{t}}',\mathfrak{s}^+} \right) \\ &\qquad + (-1)^{o_{\tilde{\mathfrak{t}}}(w+\mathrm{PD}[e],\mathfrak{s}^-)} \# \left(\bar{\mathcal{V}}(h^{\delta-2m}ex^m) \cap \bar{\mathcal{W}}^{n_a(\mathfrak{t})-1} \cap \mathbf{L}_{\tilde{\mathfrak{t}}',\mathfrak{s}^-} \right) \Big). \end{aligned} \tag{6.43}$$

By the remarks in the preceding paragraphs the sum over Spin^c(X) has potentially non-zero terms when

- $\ell(\mathfrak{t},\mathfrak{s}) = 0$ or, equivalently, $r(\Lambda, c_1(\mathfrak{s})) = \delta = r(\Lambda) + 4$, and
- $\ell(\mathfrak{t},\mathfrak{s}) = 1$ or, equivalently, $r(\Lambda, c_1(\mathfrak{s})) = \delta - 4 = r(\Lambda)$.

Recall that the links $\mathbf{L}_{\tilde{\mathfrak{t}},\mathfrak{s}^\pm}$ are empty by definition when $\ell(\tilde{\mathfrak{t}}',\mathfrak{s}^\pm) < 0$. Substituting [18, Equation (4.52)] into equation (6.43) to compute the terms with $\ell(\mathfrak{t},\mathfrak{s}) = 0$ and substituting the blow-up formulas (6.33) into equation (6.43) to compute the terms with $\ell(\mathfrak{t},\mathfrak{s})=1$ yields

$$
\begin{aligned}
D_X^w(h^{\delta-2m}x^m) &= 2^{1-n_a(\mathfrak{t}')-\delta} \sum_{\substack{\mathfrak{s}\in\mathrm{Spin}^c(X)\\ r(\Lambda,c_1(\mathfrak{s}))=\delta}} (-1)^{o_\mathfrak{t}(w,c_1(\mathfrak{s}))+m+d_s(\mathfrak{s})/2} \\
&\quad \times 2^{d_s(\mathfrak{s})/2} P^{a-1,b}_{d_s(\mathfrak{s})/2}(0) SW_X(\mathfrak{s}) \langle c_1(\mathfrak{s})-\Lambda, h\rangle^{\delta-2m} \\
&\quad + 2^{1-n_a(\mathfrak{t}')-\delta} \sum_{\substack{\mathfrak{s}\in\mathrm{Spin}^c(X)\\ r(\Lambda,c_1(\mathfrak{s}))=\delta-4}} (-1)^{o_\mathfrak{t}(w,c_1(\mathfrak{s}))+m+d_s(\mathfrak{s})/2} \\
&\quad \times 2^{d_s(\mathfrak{s})/2} P^{a,b}_{d_s(\mathfrak{s})/2}(0) SW_X(\mathfrak{s}) \Big(a_0 \langle c_1(\mathfrak{s})-\Lambda, h\rangle^{\delta-2m} \\
&\quad + b_0 \langle c_1(\mathfrak{s})-\Lambda, h\rangle^{\delta-2m-1} \langle \Lambda, h\rangle \\
&\quad + a_1 \langle c_1(\mathfrak{s})-\Lambda, h\rangle^{\delta-2m-2} Q_X(h,h) \Big),
\end{aligned}
\tag{6.44}
$$

where the coefficients a_0, b_0, a_1 are as given in the statements of Theorems 6.1 and 1.5 (since they coincide with those of Proposition 6.5 when $k=0$). From equation (6.42) for $n_a(\mathfrak{t}')$, we see that

$$
\begin{aligned}
1-n_a(\mathfrak{t}')-\delta &= 1-\tfrac{1}{4}(i(\Lambda)-\delta)-\delta \\
&= 1-\tfrac{1}{4}i(\Lambda)-\tfrac{3}{4}\delta,
\end{aligned}
\tag{6.45}
$$

and so the power of 2 in equation (6.44) matches that in equation (1.8). Finally, via [18, Equation (4.62)], we see that

$$
m+o_\mathfrak{t}(w,\mathfrak{s}) \equiv m+\tfrac{1}{2}(\sigma-w^2)+\tfrac{1}{2}(w^2+c_1(\mathfrak{s})\cdot(w-\Lambda)) \pmod 2,
$$

and so the power of (-1) in equation (6.44) also matches that in equation (1.8). Thus, substituting the preceding formulas for powers of 2 and (-1) into equation (6.44) yields equation (1.8).

Lastly, we simplify the expressions given in Theorem 6.1 for the constants a and b in $P^{a,b}_{d_s(\mathfrak{s})/2}(0)$. From equation (6.3) and the fact that $d_a(\mathfrak{t}')=2\delta$, we have

$$
b=-\tfrac{1}{2}d_s(\mathfrak{s})-\tfrac{1}{4}(\chi+\sigma).
$$

Similarly, equation (6.1) for η, the hypothesis that $\delta=r(\Lambda)+4$, and equation (2.11) for $\dim\mathcal{M}_{\mathfrak{t}'}$ yield

$$
\begin{aligned}
\eta &= \tfrac{1}{2}(\dim(\mathcal{M}_{\mathfrak{t}'}/S^1)-1)-\delta \\
&= \tfrac{1}{2}(d_a(\mathfrak{t}')+2n_a(\mathfrak{t}')-2)-\delta \\
&= n_a(\mathfrak{t}')-1,
\end{aligned}
$$

so equation (6.42) for $n_a(\mathfrak{t}')$ then implies that

$$
\eta=\tfrac{1}{4}(i(\Lambda)-\delta)-1,
\tag{6.46}
$$

and thus equation (6.3) gives

$$
a=\tfrac{1}{4}(i(\Lambda)-\delta)-\tfrac{1}{2}d_s(\mathfrak{s}).
$$

This completes the proof of Theorem 1.4. □

Proof of Theorem 1.5. As $\Lambda\in B^\perp$ and X has SW-simple type by hypothesis, equations (1.5) and (1.10) imply that

$$
i(\Lambda)+r(\Lambda)=2c(X).
$$

Since $\delta = r(\Lambda) + 4$, this gives

$$i(\Lambda) = 2c(X) - r(\Lambda) = 2c(X) - \delta + 4. \tag{6.47}$$

In particular, $r(\Lambda, c_1(\mathfrak{s})) = r(\Lambda)$ and thus $\ell(\mathfrak{t}, \mathfrak{s}) = 1$ for all $\mathfrak{s} \in \mathrm{Spin}^c(X)$ with $SW_X(\mathfrak{s}) \neq 0$.

We simplify the expression for the power of 2 in equation (1.8), noting that we now have $d_s(\mathfrak{s}) = 0$ whenever $SW_X(\mathfrak{s}) \neq 0$ since X has SW-simple type. Substituting the equation (6.47) for $i(\Lambda)$ yields

$$1 - \tfrac{1}{4}i(\Lambda) - \tfrac{3}{4}\delta = -\tfrac{1}{2}(c(X) + \delta),$$

matching the power of 2 in equation (1.11).

The Jacobi polynomial constants $P^{a,b}_{d_s(\mathfrak{s})/2}$ are equal to 1 when $d_s(\mathfrak{s}) = 0$, irrespective of the values of a, b. The assertions concerning the power of (-1) and the coefficients a_0, b_0, a_1 follow immediately from the fact that $c_1(\mathfrak{s}) \in B$ and $\Lambda \in B^\perp$. This completes the proof of Theorem 1.5. $\square$

In addition to the hypotheses of Theorem 1.5, the hypotheses of Theorem 1.1 requires that X be abundant in the sense of §1.

Lemma 6.6. [14, Lemma 2.2] *If X is an abundant four-manifold, then the following hold:*

1. *There are classes $\Lambda_0, \Lambda_1 \in B^\perp$ with $\Lambda_0 \equiv \Lambda_1 \pmod 2$ such that $\Lambda_0^2 = -(\chi + \sigma)$ and $\Lambda_1^2 = 4 - (\chi + \sigma)$.*
2. *There is a class $\Lambda \in 2B^\perp$ with $\Lambda^2 = -(\chi + \sigma)$ if $-(\chi + \sigma) \equiv 0 \pmod 8$ and $\Lambda^2 = 4 - (\chi + \sigma)$ if $-(\chi + \sigma) \equiv 4 \pmod 8$.*

Proof. For Assertion (1), the abundance condition implies that there are classes $e_1, e_2 \in B^\perp$ with $e_1 \cdot e_1 = e_2 \cdot e_2 = 0$ and $e_1 \cdot e_2 = 1$. Let $t = \frac{1}{4}(\chi + \sigma)$. Then the classes $\Lambda_0 = e_1 - 2te_2$ and $\Lambda_1 = e_1 + (2 - 2t)e_2$ will do.

For Assertion (2), the class $\Lambda_2 = 2e_1 - te_2$ will satisfy the conclusion when $t \equiv 0 \pmod 2$, so we can take $\Lambda = \Lambda_2$, while $\Lambda_3 = 2e_1 + (1 - t)e_2$ will work when $t \equiv 1 \pmod 2$, and we can take $\Lambda = \Lambda_3$. $\square$

Remark 6.7. Although the statement of Lemma 6.6 is identical to that of Lemma 2.2 in [14], the definitions of the classes Λ_0, Λ_1 here differs slightly from those of [14]. Contrary to the assertion in the proof of Lemma 2.2 in [14], the classes Λ_0, Λ_1 defined in [14] do not satisfy $\Lambda_0 \equiv \Lambda_1 \pmod 2$. The proof given here corrects that error in the proof of [14, Lemma 2.2].

Assertion (1) is only used in the proof of [14, Theorem 1.1] — when applying the sign-change formula for Donaldson invariants — and not in the proof of [14, Theorem 1.3], which is the only part of that article where Assertion (2) is used.

In the present article, Assertion (2) is not required and we only use Assertion (1) to prove the existence of the class Λ in Theorem 1.1.

Proof of Theorem 1.1. The existence of $\Lambda \in B^\perp$ and $w \in H^2(X;\mathbb{Z})$ with $\Lambda^2 = 4 - (\chi + \sigma)$ and $w - \Lambda$ characteristic follows immediately from Assertion (1) of Lemma 6.6. Take $\Lambda = \Lambda_1$ and set $w = \Lambda + v$ where v is characteristic.

We shall apply Theorem 1.5 and the vanishing result in [14, Theorem 1.1]. By hypothesis, $\Lambda \in B^\perp$ and so $\Lambda^2 \equiv 0 \pmod 2$ (see first paragraph of proof of Theorem 1.1 in [18, §4.6]). Since $\chi + \sigma$ is even, it is convenient to write $\Lambda^2 = 2j - (\chi + \sigma)$, where $j \in \mathbb{Z}$, so equations (1.5) and (1.10) would then take the form

$$r(\Lambda) = c(X) - 2j \quad \text{and} \quad i(\Lambda) = c(X) + 2j.$$

The constraints $\delta \leq r(\Lambda)+4$ and $\delta < i(\Lambda)$ in Theorem 1.5 are thus equivalent to (see [14, Figure 1]):

$$\delta \leq c(X) - 2j + 4 \quad \text{and} \quad \delta < c(X) + 2j.$$

Therefore, when using Theorem 1.5 to compute $D^w_X(h^{\delta-2m}x^m)$, we can choose $\delta \in \mathbb{N}$ no larger than $\delta = c(X)$, with $\Lambda^2 = 2-(\chi+\sigma)$ when $j=1$, or $\Lambda^2 = 4-(\chi+\sigma)$ when $j=2$; as we shall see from equation (6.53), we have $D^w_X(h^{c(X)+1-2m}x^m) = 0$, while (6.51) shows that the term in $\mathbf{SW}^w_X(h)$ of degree $c(X)+1$ in h is also zero.

We first observe that both sides of the second identity in (1.2) vanish in sufficiently low degree. Theorem 1.1 in [14] implies that, for v characteristic,

$$\sum_{\mathfrak{s}\in \mathrm{Spin}^c(X)} (-1)^{\frac{1}{2}(v^2+v\cdot c_1(\mathfrak{s}))} SW_X(\mathfrak{s})\langle c_1(\mathfrak{s}), h\rangle^d = 0, \tag{6.48}$$

if $d < c(X)-2$ or (see [14, §2]) if $d \not\equiv c(X) \pmod 2$, since the series has parity (using $v^2 \equiv \sigma \pmod 8$)

$$\begin{aligned} -v^2 - \tfrac{3}{4}(\chi+\sigma) &\equiv -\sigma - \tfrac{3}{4}(\chi+\sigma) \pmod 8 \\ &\equiv -\tfrac{1}{4}(7\chi+11\sigma) \pmod 4 \\ &= c(X). \end{aligned} \tag{6.49}$$

By hypothesis of Theorem 1.1, $w-\Lambda$ is characteristic and so we may write

$$w = v + \Lambda, \tag{6.50}$$

for some $v \equiv w_2(X) \pmod 2$. Since $\Lambda \in B^\perp$, then equation (6.48) implies that

$$\sum_{\mathfrak{s}\in \mathrm{Spin}^c(X)} (-1)^{\frac{1}{2}(w^2+w\cdot c_1(\mathfrak{s}))} SW_X(\mathfrak{s})\langle c_1(\mathfrak{s}), h\rangle^d = 0, \tag{6.51}$$

if $d < c(X)-2$ or if $d \not\equiv c(X) \pmod 2$. In particular, aside from the fact that w need not be characteristic, the vanishing result for the Seiberg-Witten series in equation (1.2) restates Theorem 1.1 in [14].

According to the hypothesis of Theorem 1.1, we have

$$\Lambda^2 = 4 - (\chi+\sigma), \tag{6.52}$$

so $r(\Lambda) = c(X)-4$ and $i(\Lambda) = c(X)+4$. Because v is characteristic we have $v\cdot\Lambda \equiv \Lambda^2 \pmod 2$ and as $\Lambda^2 \equiv 0 \pmod 4$, then $w^2 \equiv (v+\Lambda)^2 \equiv v^2 \pmod 4$. Therefore, equations (6.49) and condition (2.46) show that

$$D^w_X(h^{\delta-2m}x^m) = 0, \quad \text{when} \quad \delta \not\equiv c(X) \pmod 4. \tag{6.53}$$

(Note that the alternative solutions to the $r(\Lambda)$ and $i(\Lambda)$ constraints yielding $\delta \leq c(X)$, namely those with $\Lambda^2 = 2-(\chi+\sigma)$, would yield $\delta \equiv c(X)+2 \pmod 4$ and so for that choice of Λ^2 we could choose δ no larger than $c(X)-2$.) Hence, from definition (2.49), the potentially non-zero terms in the Donaldson series $\mathbf{D}^w_X(h)$ take the form

$$\frac{1}{(c(X)-4i)!} D^w_X(h^{c(x)-4i}) \quad \text{and} \quad \frac{1}{2(c(X)-4i-2)!} D^w_X(h^{c(x)-4i-2}x).$$

Because $r(\Lambda) = c(X)-4$, Theorem 1.4(a) in [18] gives

$$D^w_X(h^{\delta-2m}x^m) = 0 \quad \text{for } \delta < c(X)-4, \tag{6.54}$$

while Theorem 1.4(b) in [18] (with $\delta = c(X) - 4$) yields

$$
\begin{aligned}
D^w_X(h^{c(X)-4-2m}x^m) = 2^{2-c(X)}(-1)^{m+1} &\sum_{\mathfrak{s}\in\mathrm{Spin}^c(X)} (-1)^{\frac{1}{2}(w^2+c_1(\mathfrak{s})\cdot w)} \\
&\times SW_X(\mathfrak{s})\langle c_1(\mathfrak{s}) - \Lambda, h\rangle^{c(X)-4-2m}.
\end{aligned} \tag{6.55}
$$

Replacing the terms $\langle c_1(\mathfrak{s}) - \Lambda, h\rangle^{c(X)-4-2m}$ in equation (6.55) by their binomial expansions and applying equation (6.51) gives

$$
D^w_X(h^{c(X)-4}) = 0 = D^w_X(h^{c(X)-6}x). \tag{6.56}
$$

Equations (6.53), (6.54), and (6.56) then imply that the terms of $\mathbf{D}^w_X(h)$ of degree less than $c(X) - 2$ in h are zero, yielding the vanishing result for the Donaldson series stated in (1.2). From equation (6.51) the terms of $\mathbf{SW}^w_X(h)$ of degree less than $c(X) - 2$ in h are also zero.

Thus, to obtain the second identity in (1.2), it suffices to prove that

$$
D^w_X(h^{c(X)-2}x) = 2^{3-c(X)} \sum_{\mathfrak{s}\in\mathrm{Spin}^c(X)} (-1)^{\frac{1}{2}(w^2+w\cdot c_1(\mathfrak{s}))} SW_X(\mathfrak{s})\langle c_1(\mathfrak{s}), h\rangle^{c(X)-2}, \tag{6.57}
$$

and, noting that $\frac{1}{2}\frac{c(X)!}{(c(X)-2)!} = \binom{c(X)}{2}$,

$$
\begin{aligned}
D^w_X(h^{c(X)}) = 2^{2-c(X)} &\sum_{\mathfrak{s}\in\mathrm{Spin}^c(X)} (-1)^{\frac{1}{2}(w^2+w\cdot c_1(\mathfrak{s}))} SW_X(\mathfrak{s}) \\
&\times \left(\langle c_1(\mathfrak{s}), h\rangle^{c(X)} + \binom{c(X)}{2} \langle c_1(\mathfrak{s}), h\rangle^{c(X)-2} Q_X(h,h) \right).
\end{aligned} \tag{6.58}
$$

Since $\Lambda^2 = 4 - (\chi + \sigma)$, $\delta = c(X)$, $c(X) = -\frac{1}{4}(7\chi + 11\sigma)$, and $c_1^2(X) = 2\chi + 3\sigma$ by definition (1.13), the coefficients in equation (1.11) for $D^w_X(h^{\delta-2m}x^m)$ (with $w - \Lambda$ characteristic) become

$$
\begin{aligned}
a_0 &= 4c_1^2(X) + \Lambda^2 + 4c(X) - 12m \\
&= 4 - 12m, \\
b_0 &= 2(c(X) - 2m), \\
a_1 &= 4\binom{c(X) - 2m}{2}.
\end{aligned} \tag{6.59}
$$

According to equation (6.51), when we expand the terms

$$
(-1)^{\frac{1}{2}(w^2+c_1(\mathfrak{s})\cdot w)} SW_X(\mathfrak{s})\langle c_1(\mathfrak{s}) - \Lambda, h\rangle^d
$$

in equation (1.11) into a binomial sum of terms of the form

$$
(-1)^{\frac{1}{2}(w^2+c_1(\mathfrak{s})\cdot w)} SW_X(\mathfrak{s})(-1)^k \binom{d}{k} \langle c_1(\mathfrak{s}), h\rangle^{d-k} \langle \Lambda, h\rangle^k
$$

and sum over $\mathfrak{s} \in \mathrm{Spin}^c(X)$, the sums of terms with $d - k < c(X) - 2$ or $d - k \not\equiv c(X) \pmod 2$ will vanish. Therefore, applying equation (6.51) to the right-hand side of equation (1.11) with $\delta = c(X)$ and $m = 1$, only the terms of degree $\delta - 2$ in h will be non-zero. Hence, only the term with coefficient a_0 will be non-zero and equation (6.51) implies that it only contributes

$$
a_0\langle c_1(\mathfrak{s}), h\rangle^{c(X)-2}. \tag{6.60}
$$

Since $\delta = c(X)$ and $a_0 = -8$ by equation (6.59), the power of 2 the right-hand side of equation (1.11) reduces to $-c(X) + 3$. As $m = 1$, and $a_0 = -8$, and $w^2 \equiv \sigma \pmod 8$

(as $w \equiv w_2(X) \pmod 2$), the power of (-1) in the right-hand side of (1.11) reduces to $\frac{1}{2}(w^2 + w \cdot c_1(\mathfrak{s}))$. Therefore, by these remarks and observation (6.60) we obtain equation (6.57) for $D^w_X(h^{c(X)-2}x)$.

Next, applying equation (6.51) to the right-hand side of equation (1.11) with $\delta = c(X)$ and $m = 0$, only the terms of degree δ in h will be non-zero. In particular, equation (6.51) implies that the term in the right-hand side of equation (1.11) with coefficient a_0 contributes only

$$a_0 \left(\langle c_1(\mathfrak{s}), h\rangle^{c(X)} + \binom{c(X)}{2} \langle c_1(\mathfrak{s}), h\rangle^{c(X)-2} \langle \Lambda, h\rangle^2 \right). \tag{6.61}$$

Similarly the term on the right in equation (1.11) with coefficient b_0 contributes only

$$-b_0(c(X)-1)\langle c_1(\mathfrak{s}), h\rangle^{c(X)-2}\langle \Lambda, h\rangle^2. \tag{6.62}$$

Finally, the term on the right in equation (1.11) with coefficient a_1 contributes only

$$a_1 \langle c_1(\mathfrak{s}), h\rangle^{c(X)-2} Q_X(h,h). \tag{6.63}$$

Hence, as $a_0 = 4$, $b_0 = 2c(X)$, and $a_1 = 4\binom{c(X)}{2}$ by equation (6.59) when $m = 0$, noting that $\delta = c(X)$ and $w^2 \equiv \sigma \pmod 8$, and combining observations (6.61), (6.62), and (6.63), we see that equation (1.11) yields

$$\begin{aligned} D^w_X(h^{c(X)}) &= 2^{-c(X)} \sum_{\mathfrak{s}\in \mathrm{Spin}^c(X)} (-1)^{\frac{1}{2}(w^2+w\cdot c_1(\mathfrak{s}))} SW_X(\mathfrak{s}) \\ &\quad \times \Bigg(4\langle c_1(\mathfrak{s}), h\rangle^{c(X)} + 4\binom{c(X)}{2} \langle c_1(\mathfrak{s}), h\rangle^{c(X)-2}\langle \Lambda, h\rangle^2 \\ &\quad - 2c(X)(c(X)-1)\langle c_1(\mathfrak{s}), h\rangle^{c(X)-2}\langle \Lambda, h\rangle^2 \\ &\quad +4\binom{c(X)}{2} \langle c_1(\mathfrak{s}), h\rangle^{c(X)-2} Q_X(h,h) \Bigg) \\ &= 2^{2-c(X)} \sum_{\mathfrak{s}\in \mathrm{Spin}^c(X)} (-1)^{\frac{1}{2}(w^2+w\cdot c_1(\mathfrak{s}))} SW_X(\mathfrak{s}) \\ &\quad \times \left(\langle c_1(\mathfrak{s}), h\rangle^{c(X)} + \binom{c(X)}{2} \langle c_1(\mathfrak{s}), h\rangle^{c(X)-2} Q_X(h,h) \right), \end{aligned}$$

which proves equation (6.58) for $D^w_X(h^{c(X)})$. Thus, by the remark preceding equation (6.57), this proves the second identity in (1.2) and completes the proof of Theorem 1.1. $\square$

Remark 6.8. If one attempted to compute D^w_X, where w is characteristic, instead of $D^{w+\Lambda}_X$, the requirement that $\Lambda - w$ also be characteristic would imply that $\Lambda \equiv 0 \pmod 2$ so $\Lambda = 2\Lambda_2$ for some $\Lambda_2 \in H^2(X;\mathbb{Z})$. If B is non-empty, then $c_1(\mathfrak{s}) \cdot \Lambda = 2(c_1(\mathfrak{s}) \cdot \Lambda_2) = 0$ implies $\Lambda_2^2 \equiv 0 \pmod 2$ so $\Lambda^2 \equiv 0 \pmod 8$.

If $\chi + \sigma \equiv 4 \pmod 8$, there exists a class $\Lambda \in 2B^\perp$ with $\Lambda^2 = 4 - (\chi+\sigma)$ by Lemma 6.6. In this case, Theorem 1.1 would hold for D^w_X with w characteristic. In the case $\chi + \sigma \equiv 0 \pmod 8$, the identity

$$c(X) - r(\Lambda) = \Lambda^2 + (\chi + \sigma)$$

and the requirement that $\Lambda^2 \equiv 0 \pmod 8$ would imply that $c(X) \equiv r(\Lambda) \pmod 8$. Now the Seiberg-Witten moduli spaces appear in level ℓ, where

$$\ell = \tfrac{1}{4}(\delta - r(\Lambda)).$$

Thus, if we wished to compute degree $\delta = c(X)$ Donaldson invariants in the case $\chi + \sigma \equiv 0 \pmod 8$, we would need $\ell \equiv 0 \pmod 2$. If $\ell = 0$, the relations $\delta \leq r(\Lambda)$ and $\delta < i(\Lambda)$ in [14, Theorem 2.1] imply that $\delta < c(X)$. The cases $\ell \geq 2$ have not yet been computed, though work is in progress [19, 20, 21] and some partial results have been reported by Kronheimer and Mrowka [40]. In the meantime, if $\chi + \sigma \equiv 0 \pmod 8$, we cannot yet compute Donaldson invariants D^w_X of degree $c(X)$ with this methods of this article in full generality.

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