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Abstract

The goal of this note is to introduce D algebras, formulate a conjecture about these algebras, and prove this conjecture in several cases.

Introduction

Consider a polynomial ring $\mathbb{C}_n = \mathbb{C}[x_1, x_2, \dots, x_n]$ in n variables over the field $\mathbb{C}$ of complex numbers and an algebra $A = \mathbb{C}_n[y]$ where y is an algebraic function given by $P(x_1, x_2, \dots, x_n, y) = 0$. Denote by F the field of fractions of A. The partial derivatives $\partial_i = \frac{\partial}{\partial x_i}$ can be extended to the field F since $\partial_i(y) = -\frac{P_i}{P_y}$ where $P_i = \frac{\partial P}{\partial x_i},\ i = 1, 2, \dots n;\ P_y = \frac{\partial P}{\partial y}$, belong to F.

Definition. $\mathcal{D}(y)$ is the subalgebra of F generated by $\mathbb{C}_n,\ y$ and partial derivatives of y of arbitrary order.

In other words $\mathcal{D}(y)$ is a left module over the nth Weyl algebra $W_n = \mathbb{C}[x_1, x_2, \dots, x_n; \partial_1, \partial_2, \dots, \partial_n]$ which is the algebra generated by $\mathbb{C}_n$ and $W_n \cdot y$.

Conjecture. If $y \notin \mathbb{C}_n$ then $\mathcal{D}(y)$ cannot be embedded into a polynomial algebra.

Of course, $\mathcal{D}(y) = \mathbb{C}_n$ if $y \in \mathbb{C}_n$. From now on we assume that $y \notin \mathbb{C}_n$.
We'll prove Conjecture in several cases.

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Remark. The Jacobian Conjecture (JC, see [K]) follows immediately from this conjecture. Recall that the JC is the following statement. If $f_1, f_2, \dots, f_n \in \mathbb{C}[x_1, x_2, \dots, x_n]$ and the Jacobian, i.e. the determinant of the Jacobi matrix $\frac{\partial(f_1,f_2,\dots,f_n)}{\partial(x_1,x_2,\dots,x_n)}$, is 1 then $\mathbb{C}[f_1, f_2, \dots, f_n] = \mathbb{C}[x_1, x_2, \dots, x_n]$.

If there is a counterexample to JC then we may assume that, say, $x_1 \notin \mathbb{C}[f_1, f_2, \dots, f_n]$. Since $f_1, f_2, \dots, f_n$ are algebraically independent, x_1 is an algebraic function of $f_1, f_2, \dots, f_n$. Denote by $\mathrm{J}(g_1, \dots, g_n)$ the Jacobian of the corresponding Jacobi matrix. Then $\mathrm{J}(f_1, f_2, \dots, x_1) = \frac{\partial x_1}{\partial f_n}$. Similarly, $\mathrm{J}(f_1, \dots, f_{i-1}, x_1, f_{i+1}, \dots, f_n) = \frac{\partial x_1}{\partial f_i}$. Therefore all partial derivatives of x_1 belong to $\mathbb{C}_n$ and D-algebra $A(x_1)$ generated by x_1 belongs to a polynomial ring even if $x_1 \notin \mathbb{C}[f_1, f_2, \dots, f_n]$.

Section 1 $n = 1$.

Theorem 1 Let $P(x, y) = 0$ be an irreducible polynomial. Then the subalgebra $\mathcal{D}(y)$ is not embeddable into a polynomial algebra with one generator.
Proof. If $\mathcal{D}(y) \hookrightarrow \mathbb{C}[t]$ then we may assume that $x = x(t),\ y = y(t)$ where $x(t),\ y(t) \in \mathbb{C}[t]$. Let $d_x = \deg_t(x) > 0$ and $d_y = \deg_t(y) > 0$. If $f \in \mathbb{C}(t) \setminus \mathbb{C}$ is considered as a function of x then $f' = \frac{f_t}{x_t}$ since $f_t = f_x x_t$. Hence $\deg_t(f') = \deg_t(f) - \deg_t(x)$ if $f' \neq 0$. Therefore $\deg_t(y^{(i)}) = d_y - i d_x$ and since $d_x > 0$ either we'll get $y^{(i)}$ with negative degree (and then it is not a polynomial) or $y^{(i)} \in \mathbb{C}$ for some i (and then y is a polynomial in x contrary to our assumption). □

This approach does not work for the case of two (or more) variables since the notion of degree is not really well defined in polynomial rings with more than one variable: it is not stable under automorphisms. Say, if $x_1 = s,\ x_2 = t + s^3$ and $f = t$ then $\frac{\partial f}{\partial x_1} = 3s^2$, i.e. a degree may increase.

Section 2 The hypersurface $\mathcal{H}$ given by $P(x_1, x_2, \dots, x_n, y) = 0$ doesn't have singular points.

This is a trivial observation: $\mathcal{H}$ doesn't have singular points if the system $P = 0,\ \partial_1(P) = 0, \dots, \partial_n(P) = 0,\ P_y = 0$ doesn't have a solution. Then one can find polynomials $a_0, a_1, \dots, a_n, a_y \in \mathbb{C}_n[y]$ such that

$a_0 P + \sum_i a_i \partial_i(P) + a_y P_y = 1$. Hence $\frac{a_0 P}{P_y} - \sum_i a_i \partial_i(y) + a_y = \frac{1}{P_y} \in \mathcal{D}(y)$. In this case $\mathcal{D}(y) = \mathbb{C}[x_1, \ldots, x_{n-1}, x_n, y][\frac{1}{P_y}]$ contains a nontrivial unit and cannot be embedded into a polynomial ring.

On the other hand it shows that it is sufficient to find a *non-singular* (and non-polynomial) element in $\mathcal{D}(y)$ to prove the conjecture. In all examples which I was able to compute $\mathcal{D}(y)$ contains such an element.

Regularity conjecture. If $\mathcal{D}(y)$ is generated by a non-polynomial function then it contains a non-singular non-polynomial element.

Section 3. y is a rational function.

Theorem 2 If $y = pq^{-1}$ where $p,\ q \in \mathbb{C}_n$ are relatively prime polynomials then $\mathcal{D}(y) = \mathbb{C}[x_1, \ldots, x_n,\ q^{-1}]$.

The Theorem will be proved by induction on n. The base of induction when $n = 1$ is obvious. In this case p and q are relatively prime and we can find $a, b \in \mathbb{C}[x]$ such that $ap + bq = 1$. Therefore $apq^{-1} + b = q^{-1}$ and $\mathcal{D}(y) = \mathbb{C}[x, q^{-1}]$ since $\mathbb{C}[x, q^{-1}]$ is stable under $\frac{d}{dx}$.

If we replace $\mathbb{C}[x_1, \ldots, x_{n-1}, x_n]$ by $\mathbb{C}(x_n)[x_1, \ldots, x_{n-1}]$ then by induction $\mathcal{D}(y) \ni \frac{p_n(x_n)}{q}$ where $p_n \in \mathbb{C}[x_n] \setminus 0$. Similarly $\mathcal{D}(y) \ni \frac{p_i(x_i)}{q}$ where $p_i(x_i) \in \mathbb{C}[x_i]$ is a nonzero polynomial.

Denote by J_i the ideal $\mathcal{D}(y)q^i \cap \mathbb{C}_n$ of $\mathbb{C}_n$. Since $J_1 \supset J_i \supset (J_1)^i$ the set S of common zeros of the polynomials from J_i does not depend on i.

The ideal J_1 contains all $p_i(x_i)$, therefore S consists of a finite number of points. The claim of the Theorem is that S is empty. If S is not empty we can assume that S contains the point $(0, \ldots, 0)$.

Denote by I the augmentation ideal of $\mathbb{C}_n$.

Lemma 1. The ideal $J_1 \supset u_1 I^d$ where u_1 is a polynomial, $u_1(0, \ldots, 0) \neq 0$, and $d > 0$ is a natural number.
Proof. We can write $p_i = x_i^{k_i} q_i(x_i)$ where $k_i \geq 0$ and $q_i(0) \neq 0$. Therefore $x_j^{k_j} \prod_i q_i(x_i) \in J_1$ and $I^k \prod_i q_i(x_i) \subset J_1$ for $k = \sum_i k_i - n + 1$ since any monomial with the total degree at least $\sum_i k_i - n + 1$ contains one of the monomials $x_j^{k_j}$ as a factor. $\square$

Remark. $J_s \supset I^{sk}(\prod_i q_i(x_i))^s$ since $J_s \supset (J_1)^s$.

Since $J_s \supset uI^d$ for some $d > 0$ and a polynomial u which is not zero in the origin there exists minimal d_s for which $J_s \supset u_s I^{d_s}$ where u_s is a polynomial not equal to zero in the origin.

Lemma 2. $d_s = d_1$ for all s.
Proof. Since $J_{s+1} \subset J_s$ the sequence d_s is non-decreasing. Denote d_1 by d and $d_a \geq d$ by e (here a is a natural number). Consider monomials $m_1(j) = x_1^j \mu_1(j) \in I^d,\ m_2(j) = x_1^{\beta-j}\mu_2(j) \in I^e,\quad j = 0,1,...,d,\ \beta \geq d,$ where $\mu_1(j),\ \mu_2(j)$ are monomials in $\mathbb{C}[x_2,\ldots,x_n]$ such that $\deg(\mu_1(j)) = d-j,\ \deg(\mu_2(j)) = e - \beta + j$ and $\mu_1(j)\mu_2(j) = \mu$ doesn't depend on j.

Then $m_1(j)u_1 \in J_1,\ m_2(j)u_a \in J_a$ and

$$\mathcal{D}(y) \ni a_j = m_2(j)\frac{u_a}{q^a}\frac{\partial^d}{\partial x_1^d}(m_1(j)\frac{u_1}{q}) = Mu_a q^{-a}\sum_{i=0}^{d}\frac{j!}{(j-i)!}\binom{d}{i}x_1^{-i}(u_1q^{-1})^{(d-i)}$$

where $M = m_1m_2 = x^\beta\mu$, for $j = 0,1,...,d$.

We got $d+1$ linear equations

$$Mu_aq^{-a}\sum_{i=0}^{d}\binom{j}{i}i!\binom{d}{i}x_1^{-i}(u_1q^{-1})^{(d-i)} = a_j$$

for $d+1$ elements $i!\binom{d}{i}Mu_aq^{-a}x_1^{-i}(u_1q^{-1})^{(d-i)},\ i = 0,1,...,d$ with the matrix $\{a_{i,j} = \binom{j}{i}\}$ which is triangular with 1's on the diagonal. Therefore all $Mu_aq^{-a}x_1^{-i}(u_1q^{-1})^{(d-i)} \in \mathcal{D}(y)$. If we take $i = d$ then $Mx_1^{-d}u_1u_aq^{-a-1} \in \mathcal{D}(y)$ which means that $J_{a+1} \ni x_1^{\beta-d}\mu u_1 u_a$ where $\mu \in \mathbb{C}[x_2,\ldots,x_n]$ is an arbitrary monomial of the total degree $d+e-\beta$. Hence $\deg(x_1^{\beta-d}\mu) = e$ i.e. $I^e \subset J_{a+1}$. Therefore $d(a+1) = d(a)$. □

Lemma 3. $d = 0$.
Proof. By the previous Lemma $\mathcal{D}(y) \supset I^d u_a q^{-a}$ for any natural number a. Hence A contains $(I^du_aq^{-a})'$ where f' denotes a partial derivative. If $d > 0$ then $(I^du_aq^{-a})' = I^{d-1}u_aq^{-a} + I^du_a'q^{-a} + I^du_aq'q^{-a-1}$. Thus $u_au_{a+1}[I^{d-1}u_aq^{-a} + I^du_a'q^{-a} - I^du_aq'q^{-a-1}] \subset \mathcal{D}(y)$ and we can conclude that $I^{d-1}u_a^2u_{a+1} \subset J_a$ which means that $d = 0$. □

We see that contrary to our assumption $(0,\ldots,0)$ is not in S. This proves the Theorem. □

This proof of the Theorem doesn't depend on the algebraic closeness of $\mathbb{C}$. A proof for a Jacobian extension which uses closeness can be found in [An].

Remark. In this case $\mathcal{D}(y)$ has a nontrivial unit and is not embeddable into a polynomial ring. Of course, q^{-1} is a non-polynomial non-singular function.

Corollary. If $\mathbb{C}(x_1, x_2, \ldots, x_n, y)$ is a Galois extension of $\mathbb{C}(x_1, x_2, \ldots, x_n)$ and $y \notin \mathbb{C}(x_1, x_2, \ldots, x_n$ then $\mathcal{D}(y)$ is not embeddable into a polynomial ring.
Explanation. All roots y_i of $P(x_1, x_2, \ldots, x_n, y)$ are in $\mathbb{C}(x_1, x_2, \ldots, x_n)[y]$. We can find $\Delta \in \mathbb{C}[x_1, x_2, \ldots, x_n]$ so that all $\Delta y_i \in \mathcal{D}(y)$. If $z = \Delta y$ is not integral over $\mathbb{C}[x_1, x_2, \ldots, x_n]$ then, since all coefficients of the irreducible relation $Q(x_1, x_2, \ldots, x_n, z) = 0$ as a monic polynomial in z belong to $\mathcal{D}(y)$, $\mathcal{D}(y)$ contains a rational function.

If z is integral then we can take a partial derivative of z of a sufficiently large order and obtain an element which is not integral. □

Section 4. Jacobian situation.

Consider now a counterexample to the JC: $\mathbb{C}[x_1, x_2, \ldots, x_n] \supsetneqq \mathbb{C}[f_1, f_2, \ldots, f_n]$ where $\mathrm{J}(f_1, f_2, \ldots, f_n) = 1$. Assume that $x_1 \notin \mathbb{C}(f_1, f_2, \ldots, f_n)$. For any (affine) point $\mathcal{P} = (a_1, a_2, \ldots, a_n)$ the linear components of $f_1, f_2, \ldots, f_n$ at this point are algebraically independent. Hence the irreducible dependence $P(f_1, f_2, \ldots, f_n, x) = 0$ between $f_1, f_2, \ldots, f_n$ and $x = x_1$ doesn't have a singular point at $f_1(\mathcal{P}), f_2(\mathcal{P}), \ldots, f_n(\mathcal{P})$. It seems that we have a regular hypersurface given by $P(f_1, f_2, \ldots, f_n, x) = 0$ and a contradiction mentioned in Section 2. Of course, we are far from proving JC. The problem is that we can have a point at infinity which under the map given by $f_1, f_2, \ldots, f_n$ becomes an affine point. Hence a singularity may appear only if one of the x_i's is infinity. We can conclude that if $x_2, \ldots, x_n$ are integral over $\mathbb{C}[x_1, x_2, \ldots, x_n]$ then a counterexample is not possible.

The fact that a counterexample doesn't exist in the Galois case is well known, see [Ab], [An], [C], [R], [V], [W]. In these papers integrality condition is discussed as well (in a stronger form that all elements of $\mathbb{C}_n$ are integral over $\mathbb{C}[f_1, f_2, \ldots, f_n]$).

Section 5. Cubic algebraic functions.

The proof of the conjecture for algebraic functions of one variable didn't provide a non-singular element. The first interesting case is a a cubic function $a_0y^3 = a_1y^2 + a_2y + a_3$ since a quadratic function gives a Galois extension.

In this section we'll consider several examples of algebraic function given by $y^3 - a_2y - a_3 = 0,\ a_i \in \mathbb{C}[x]$. Then $y' = \frac{a_2'y+a_3'}{3y^2-a_2}$.
Straightforward computations show that

$$y' = \frac{(3a_2'a_3 - 2a_2a_3')(3y^2 - 2a_2) + (-2a_2^2a_2' + 9a_3a_3')y}{27a_3^2 - 4a_2^3}.$$

Let us start with
Example 1.$y^3 - xy - x^2 = 0$, i.e. $a_2 = x,\ a_3 = x^2$. In this case y has only one singular point $(0,0)$ and $y' = \frac{-x^2(3y^2-2x)+(-2x^2+18x^3)y}{27x^4-4x^3} = \frac{-(3y^2-2x)+(-2+18x)y}{27x^2-4x}$. Hence $\mathcal{D}(y) \ni \frac{3y^2+2y}{x},\ \frac{3y^3+2y^2}{x},\ \frac{y^2}{x},\ \frac{y}{x}$. So $v = \frac{y}{x} \in \mathcal{D}(y)$ and $x^3v^3 - x^2v - x^2 = 0$, i.e. $xv^3 - v - 1 = 0$ and v is a non-singular function since the system $3xv^2 - 1 = 0,\ v^3 = 0$ doesn't have a solution.

Here is a more complicated example.
Example 2. $a_2 = x^3,\ a_3 = x^4$. Then
$y' = \frac{x^6(3y^2-2x^3)+(-6x^8+36x^7)y}{27x^8-4x^9}$, $\frac{3y^2+36xy}{x^2} \in \mathcal{D}(y),\ \frac{y^2}{x} \in \mathcal{D}(y)$. We cannot conclude yet that $\frac{y}{x} \in \mathcal{D}(y)$.

Denote $y^{(i)} = r_{i,0}y^2 + r_{i,1}y + r_{i,2}$. Then
$y^{(i+1)} = r_{i,0}'y^2 + r_{i,1}'y + r_{i,2}' + (2r_{i,0}y + r_{i,1})(r_{1,0}y^2 + r_{1,1}y + r_{1,2}) =$
$2r_{i,0}r_{1,0}(a_2y+a_3)+(r_{i,0}'+2r_{i,0}r_{1,1}+r_{i,1}r_{1,0})y^2+(r_{i,1}'+2r_{i,0}r_{1,2}+r_{i,1}r_{1,1})y+r_{i,2}'+$
$r_{i,1}r_{1,2} = (r_{i,0}' + 2r_{i,0}r_{11} + r_{i,1}r_{1,0})y^2 + (r_{i,1}' + 2r_{i,0}r_{1,2} + r_{i,1}r_{1,1} + 2r_{i,0}r_{1,0}a_2)y +$
$r_{i,2}' + r_{i,1}r_{1,2} + 2r_{i,0}r_{1,0}a_3$
and
$r_{i+1,0} = r_{i,0}' + 2r_{i,0}r_{1,1} + r_{i,1}r_{1,0}$
$r_{i+1,1} = r_{i,1}' + 2r_{i,0}r_{1,2} + r_{i,1}r_{1,1} + 2r_{i,0}r_{1,0}a_2,$
$r_{i+1,2} = r_{i,2}' + r_{i,1}r_{1,2} + 2r_{i,0}r_{1,0}a_3$

In particular,
$r_{2,0} = r_{1,0}' + 2r_{1,0}r_{1,1} + r_{1,1}r_{1,0} = r_{1,0}' + 3r_{1,0}r_{1,1}$
$r_{2,1} = r_{1,1}' + 2r_{1,0}r_{1,2} + r_{1,1}r_{1,1} + 2r_{1,0}r_{1,0}a_2 = r_{1,1}' + 2r_{1,0}r_{1,2} + r_{1,1}^2 + 2r_{1,0}^2a_2,$

$r_{2,2} = r'_{1,2} + r_{1,1}r_{1,2} + 2r_{1,0}r_{1,0}a_3 = r'_{1,2} + r_{1,1}r_{1,2} + 2r_{1,0}^2 a_3$

In our case $r_{1,0} = \frac{3}{27x^2-4x^3},\ r_{1,1} = \frac{-6x+36}{27x-4x^2},\ r_{1,2} = \frac{-2x}{27-4x}$ and
$r_{2,0} = r'_{1,0} + 2r_{1,0}r_{1,1} + r_{1,1}r_{1,0} = (\frac{3}{27x^2-4x^3})' + 3\frac{3}{27x^2-4x^3}\frac{-6x+36}{27x-4x^2} =$
$\frac{3(12x^2-54x)}{(27x^2-4x^3)^2} + \frac{-54x+324}{x(27x-4x^2)^2} = 9\frac{4x^2-18x-6x^2+36x}{(27x^2-4x^3)^2} = 18\frac{9x-x^2}{(27x^2-4x^3)^2}$

$r_{2,1} = (\frac{-6x+36}{27x-4x^2})' + 2\frac{3}{27x^2-4x^3}\frac{-2x}{27-4x} + (\frac{-6x+36}{27x-4x^2})^2 + 2(\frac{3}{27x^2-4x^3})^2x^3 =$
$-6\frac{27x-4x^2-(x-6)(27-8x)}{(27x-4x^2)^2} - \frac{12x}{(27x-4x^2)^2} + 36(\frac{x-6}{27x-4x^2})^2 + \frac{18x}{(27x-4x^2)^2} =$
$-6[\frac{162-48x+4x^2}{(27x-4x^2)^2} - \frac{x}{(27x-4x^2)^2} - 6\frac{(x-6)^2}{(27x-4x^2)^2}] = -6\frac{-54+23x-2x^2}{(27x-4x^2)^2}.$

$r_{2,2} = (\frac{-2x}{27-4x})' + \frac{-6x+36}{27x-4x^2}\frac{-2x}{27-4x} + 2(\frac{3}{27x^2-4x^3})^2x^4 =$
$-2\frac{27-4x-x(-4)}{(27-4x)^2} + \frac{12x-72}{(27-4x)^2} + 2(\frac{3}{27-4x})^2 = \frac{12x-108}{(27-4x)^2}.$

Hence $18\frac{x(9-x)}{x^4(27-4x)^2}y^2 + 6\frac{54-23x+2x^2}{x^2(27-4x)^2}y + \frac{12(x-9)}{(27-4x)^2} \in \mathcal{D}(y)$,
$3\frac{x(9-x)}{x^4}y^2 + \frac{54-23x+2x^2}{x^2}y \in \mathcal{D}(y),\ 3\frac{x(9-x)}{x^3}y^2 + \frac{54-23x+2x^2}{x}y \in \mathcal{D}(y)$,
$\frac{9-x+18xy}{x^2}y^2 \in \mathcal{D}(y)$ and $\frac{9+18xy}{x^2}y^2 \in \mathcal{D}(y)$ since $\frac{y^2}{x} \in \mathcal{D}(y)$. As we saw above, $\frac{3y^2+36xy}{x^2} \in \mathcal{D}(y)$. Thus $\frac{y}{x} \in \mathcal{D}(y)$.

For $v = x^{-1}y$ we have $v^3 - xv - x = 0$ and v is a regular function since the system $3v^2 - x = 0,\ v + 1 = 0,\ v^3 - xv - x = 0$ doesn't have solutions.

Similar approach allows to check that $x^{-1}y \in \mathcal{D}(y)$ if y is given by $y^3 - x^iy - x^j = 0$ and show that in any of these cases $\mathcal{D}(y)$ contains a non-singular function.

Example 3. $y^3 = (3+x^2)y + 2 + x^2;\quad a_2 = 3 + x^2,\ a_3 = 2 + x^2.$
Singularities: $3y^2 = 3 + x^2,\ 2xy + 2x = 0,\ 2(3+x^2)y + 3(2+x^2) = 0$
$x = 0,\ y = -1;\ x \neq 0$ then $2y + 2 = 0,\ y = -1,\ x^2 = 0$
$(0, -1)$ is the singular point.

$\Delta = 27(2+x^2)^2 - 4(3+x^2)^3 = -9x^4 - 4x^6$
$L = 6x(2+x^2) - 4x(3+x^2) = 2x^3$
$y' = \frac{2x^3(3y^2-6-2x^2)-(6x^3+4x^5)y}{-9x^4-4x^6} = -\frac{2(3y^2-6-2x^2)-(6+4x^2)y}{9x+4x^3}$

$r_{1,0} = \frac{-6}{x(9+4x^2)},\ r_{1,1} = \frac{2(3+2x^2)}{x(9+4x^2)},\ r_{1,2} = \frac{4(3+x^2)}{x(9+4x^2)}$

$r_{2,0} = (\frac{-6}{x(9+4x^2)})' + 3\frac{-6}{x(9+4x^2)}\frac{2(3+2x^2)}{x(9+4x^2)} =$

$\frac{18(3+4x^2)}{x^2(9+4x^2)^2} - 36\frac{(3+2x^2)}{x^2(9+4x^2)^2} = \frac{-54}{x^2(9+4x^2)^2}$

$r_{2,1} = (\frac{2(3+2x^2)}{x(9+4x^2)})' + 2\frac{-6}{x(9+4x^2)}\frac{4(3+x^2)}{x(9+4x^2)} + (\frac{2(3+2x^2)}{x(9+4x^2)})^2 + 2(\frac{-6}{x(9+4x^2)})^2(3+x^2) =$
$2\frac{4x^2(9+4x^2)-3(3+2x^2)(3+4x^2)}{x^2(9+4x^2)^2} - 48\frac{3+x^2}{x^2(9+4x^2)^2} + 4\frac{(3+2x^2)^2}{x^2(9+4x^2)^2} + 72\frac{3+x^2}{x^2(9+4x^2)^2} =$
$2\frac{4x^2(9+4x^2)-3(9+18x^2+8x^4)}{x^2(9+4x^2)^2} + 4\frac{9+12x^2+4x^4}{x^2(9+4x^2)^2} + 24\frac{3+x^2}{x^2(9+4x^2)^2} =$
$-2\frac{27+18x^2+8x^4}{x^2(9+4x^2)^2} + 4\frac{9+12x^2+4x^4}{x^2(9+4x^2)^2} + 24\frac{3+x^2}{x^2(9+4x^2)^2} =$
$\frac{-54-36x^2-16x^4}{x^2(9+4x^2)^2} + \frac{36+48x^2+16x^4}{x^2(9+4x^2)^2} + \frac{72+24x^2}{x^2(9+4x^2)^2} = \frac{54+36x^2}{x^2(9+4x^2)^2}$

$r_{2,2} = (\frac{4(3+x^2)}{x(9+4x^2)})' + \frac{2(3+2x^2)}{x(9+4x^2)}\frac{4(3+x^2)}{x(9+4x^2)} + 2(\frac{-6}{x(9+4x^2)})^2(2+x^2) =$
$4\frac{2x^2(9+4x^2)-3(3+x^2)(3+4x^2)}{x^2(9+4x^2)^2} + 8\frac{(3+2x^2)(3+x^2)}{x^2(9+4x^2)^2} + 72\frac{2+x^2}{x^2(9+4x^2)^2} =$
$4\frac{2x^2(9+4x^2)-3(9+15x^2+4x^4)}{x^2(9+4x^2)^2} + 8\frac{9+9x^2+2x^4}{x^2(9+4x^2)^2} + 72\frac{2+x^2}{x^2(9+4x^2)^2} =$
$-4\frac{27+27x^2+4x^4}{x^2(9+4x^2)^2} + 8\frac{9+9x^2+2x^4}{x^2(9+4x^2)^2} + 72\frac{2+x^2}{x^2(9+4x^2)^2} = \frac{108+36x^2}{x^2(9+4x^2)^2}$

$r_{2,0} = \frac{-54}{x^2(9+4x^2)^2},\ r_{2,1} = 18\frac{3+2x^2}{x^2(9+4x^2)^2},\ r_{2,2} = 36\frac{3+x^2}{x^2(9+4x^2)^2}$

$$y'' = \frac{9}{x(9+4x^2)}y'$$

$(\frac{y'}{(9+4x^2)^k})' = \frac{y'}{(9+4x^2)^k}\frac{9}{x(9+4x^2)} - k\frac{8xy'}{(9+4x^2)^{k+1}} = \frac{(9-8kx^2)y'}{x(9+4x^2)^{k+1}} \Rightarrow \frac{y'}{(9+4x^2)^{k+1}} \in \mathcal{D}(y)$

Therefore $\frac{1}{(9+4x^2)} \in \mathcal{D}(y)$. Indeed, y' satisfies an equation $\alpha_0 y'^3 + \alpha_1 y'^2 + \alpha_2 y' + \alpha_3 = 0,\ \alpha_i \in \mathbb{C}[x]$ where $\alpha_3 \neq 0$. Hence $\frac{\alpha_3}{(9+4x^2)^k} \in \mathcal{D}(y)$ for any k and $\frac{1}{(9+4x^2)} \in \mathcal{D}(y)$.

Since $(\frac{y'}{x})' = \frac{y'}{x}\frac{9}{x(9+4x^2)} - \frac{y'}{x^2} = \frac{-4y'}{9+4x^2}$ we can conclude that $\mathcal{D}(y) = \mathbb{C}[x, \frac{1}{9+4x^2}, y, \frac{y'}{x}]$

In example 3 the second derivative is proportional to the first derivative with a rational coefficient. In example 2 it is not the case though $r_{2,0}r_{1,2} - r_{1,0}r_{2,2} = 0$. This is always true:
$r_{2,0}r_{1,2} - r_{1,0}r_{2,2} = (r'_{1,0} + 3r_{1,0}r_{1,1})r_{1,2} - r_{1,0}(r'_{1,2} + r_{1,1}r_{1,2} + 2r_{1,0}^2 a_3) =$
$(r'_{1,0} + 2r_{1,0}r_{1,1})r_{1,2} - r_{1,0}(r'_{1,2} + 2r_{1,0}^2 a_3)$
Since $y' = \frac{(3a'_2a_3 - 2a_2a'_3)(3y^2 - 2a_2) + (-2a_2^2a'_2 + 9a_3a'_3)y}{27a_3^2 - 4a_2^3}$,
$r_{1,0} = \frac{3L}{\Delta},\ r_{1,1} = \frac{\Delta'}{6\Delta},\ r_{1,2} = \frac{-2a_2L}{\Delta}$ where $\Delta = 27a_3^2 - 4a_2^3,\ L = 3a'_2a_3 - 2a_2a'_3$.

Thus $(r'_{1,0} + 2r_{1,0}r_{1,1})r_{1,2} - r_{1,0}(r'_{1,2} + 2r^2_{1,0}a_3) =$
$[(\frac{3L}{\Delta})' + 2\frac{3L}{\Delta}\frac{\Delta'}{6\Delta}]\frac{-2a_2L}{\Delta} - \frac{3L}{\Delta}[(\frac{-2a_2L}{\Delta})' + 2(\frac{3L}{\Delta})^2a_3] =$
$[(\frac{3L'}{\Delta} - \frac{3L\Delta'}{\Delta^2}) + 2\frac{3L}{\Delta}\frac{\Delta'}{6\Delta}]\frac{-2a_2L}{\Delta} - \frac{3L}{\Delta}[(\frac{-2a'_2L}{\Delta} - \frac{2a_2L'}{\Delta} + \frac{2a_2L\Delta'}{\Delta^2}) + 2(\frac{3L}{\Delta})^2a_3] =$
$-\frac{2a_2L^2\Delta'}{\Delta^3} - \frac{3L}{\Delta}[(\frac{-2a'_2L}{\Delta}) + 2(\frac{3L}{\Delta})^2a_3] = -\frac{2L^2}{\Delta^3}(a_2\Delta' - 3a'_2\Delta + 27La_3)$ and
$a_2\Delta' - 3a'_2\Delta + 27La_3 = a_2(27a_3^2 - 4a_2^3)' - 3a'_2(27a_3^2 - 4a_2^3) + 27(3a'_2a_3 - 2a_2a'_3)a_3 =$
$a_2(54a_3a'_3 - 12a_2^2a'_2) - 3a'_2(27a_3^2 - 4a_2^3) + 27(3a'_2a_3 - 2a_2a'_3)a_3 = 0$

To check when the derivatives are proportional we should find when $r_{2,0}r_{1,1} - r_{1,0}r_{2,1} = 0$.
$r_{2,0}r_{1,1} - r_{1,0}r_{2,1} = (r'_{1,0} + 3r_{1,0}r_{1,1})r_{1,1} - r_{1,0}(r'_{1,1} + 2r_{1,0}r_{1,2} + r^2_{1,1} + 2r^2_{1,0}a_2) =$
$r'_{1,0}r_{1,1} - r_{1,0}(r'_{1,1} + 2r_{1,0}r_{1,2} - 2r^2_{1,1} + 2r^2_{1,0}a_2) =$
$(\frac{3L'}{\Delta} - \frac{3L\Delta'}{\Delta^2})\frac{\Delta'}{6\Delta} - \frac{3L}{\Delta}[(\frac{\Delta'}{6\Delta})' + 2\frac{3L}{\Delta}(\frac{-2a_2L}{\Delta}) - 2(\frac{\Delta'}{6\Delta})^2 + 2(\frac{3L}{\Delta})^2a_2] =$
$\frac{L'\Delta'}{2\Delta^2} - \frac{L\Delta'^2}{2\Delta^3} - \frac{3L}{\Delta}[(\frac{\Delta''}{6\Delta} - \frac{\Delta'^2}{6\Delta^2}) - 12\frac{a_2L^2}{\Delta^2} - \frac{\Delta'^2}{18\Delta^2} + 18\frac{a_2L^2}{\Delta^2}] =$
$\frac{L'\Delta'}{2\Delta^2} + \frac{L\Delta'^2}{6\Delta^3} - \frac{L\Delta''}{2\Delta^2} - 18\frac{a_2L^3}{\Delta^3} = \frac{3(L'\Delta' - L\Delta'')\Delta + L\Delta'^2 - 108a_2L^3}{6\Delta^3}$

$L'\Delta' - L\Delta'' =$
$(3a''_2a_3 - 2a_2a''_3 + a'_2a'_3)(54a_3a'_3 - 12a_2^2a'_2) - (3a'_2a_3 - 2a_2a'_3)(54a_3a''_3 - 12a_2^2a''_2 +$
$54a'^2_3 - 24a_2a'^2_2) =$
$6[(3a''_2a_3 - 2a_2a''_3)(9a_3a'_3 - 2a_2^2a'_2) - (3a'_2a_3 - 2a_2a'_3)(9a_3a''_3 - 2a_2^2a''_2) +$
$a'_2a'_3(9a_3a'_3 - 2a_2^2a'_2) - (3a'_2a_3 - 2a_2a'_3)(9a'^2_3 - 4a_2a'^2_2)] =$
$6[(27a_3^2a'_3a''_2 - 6a_2^2a_3a'_2a''_2) - (18a_2a_3a'_3a''_3 - 4a_2^3a'_2a''_3) -$
$(27a_3^2a'_2a''_3 - 6a_2^2a_3a'_2a''_2) + (18a_2a_3a'_3a''_3 - 4a_2^3a'_3a''_2) -$
$(9a_3a'_2a'^2_3 - 2a_2^2a'^2_2a'_3) - (27a_3a'_2a'^2_3 - 12a_2a_3a'^3_2) + (18a_2a'^3_3 - 8a_2^2a'^2_2a'_3)] =$
$6(27a_3^2a'_3a''_2 + 4a_2^3a'_2a''_3 - 27a_3^2a'_2a''_3 - 4a_2^3a'_3a''_2 -$
$18a_3a'_2a'^2_3 - 10a_2^2a'^2_2a'_3 + 12a_2a_3a'^3_2 + 18a_2a'^3_3) =$
$6[(27a_3^2 - 4a_2^3)(a'_3a''_2 - a'_2a''_3) - 18a_3a'_2a'^2_3 - 10a_2^2a'^2_2a'_3 + 12a_2a_3a'^3_2 + 18a_2a'^3_3]$

$\Delta'^2 - 108a_2L^2 =$
$36(9a_3a'_3 - 2a_2^2a'_2)^2 - 108a_2(3a'_2a_3 - 2a_2a'_3)^2 =$
$36[81a_3^2a'^2_3 - 36a_2^2a_3a'_2a'_3 + 4a_2^4a'^2_2 - 3a_2(9a_3^2a'^2_2 - 12a_2a_3a'_2a'_3 + 4a_2^2a'^2_3)] =$
$36(27a_3^2 - 4a_2^3)(3a'^2_3 - a_2a'^2_2)$

$3(L'\Delta' - L\Delta'')\Delta + L\Delta'^2 - 108a_2L^3 =$
$18[(27a_3^2 - 4a_2^3)(a'_3a''_2 - a'_2a''_3) - 18a_3a'_2a'^2_3 - 10a_2^2a'^2_2a'_3 + 12a_2a_3a'^3_2 + 18a_2a'^3_3]\Delta +$
$36L\Delta(3a'^2_3 - a_2a'^2_2) =$
$18(a'_3a''_2 - a'_2a''_3)\Delta^2 + 18(-18a_3a'_2a'^2_3 - 10a_2^2a'^2_2a'_3 + 12a_2a_3a'^3_2 + 18a_2a'^3_3)\Delta +$

$36L\Delta(3a_3'^2 - a_2a_2'^2)$

$-18a_3a_2'a_3'^2 - 10a_2^2a_2'^2a_3' + 12a_2a_3a_2'^3 + 18a_2a_3'^3 + 2L(3a_3'^2 - a_2a_2'^2) =$
$-18a_3a_2'a_3'^2 - 10a_2^2a_2'^2a_3' + 12a_2a_3a_2'^3 + 18a_2a_3'^3 + 2(3a_2'a_3 - 2a_2a_3')(3a_3'^2 - a_2a_2'^2) =$
$-18a_3a_2'a_3'^2 - 10a_2^2a_2'^2a_3' + 12a_2a_3a_2'^3 + 18a_2a_3'^3 + 2(9a_3a_2'a_3'^2 - 6a_2a_3'^3 - 3a_2a_3a_2'^3 + 2a_2^2a_2'^2a_3') =$
$-6a_2^2a_2'^2a_3' + 6a_2a_3a_2'^3 + 6a_2a_3'^3 = 6a_2(a_3a_2'^3 - a_2a_2'^2a_3' + a_3'^3)$

$3(L'\Delta' - L\Delta'')\Delta + L\Delta'^2 - 108a_2L^3 =$
$18[(a_3'a_2'' - a_2'a_3'')\Delta + 6a_2(a_3a_2'^3 - a_2a_2'^2a_3' + a_3'^3)]\Delta$

Hence $r_{2,0}r_{1,1} - r_{1,0}r_{2,1} = 0$ when

$$3(a_3'a_2'' - a_2'a_3'')\Delta + a_2(a_3a_2'^3 - a_2a_2'^2a_3' + a_3'^3) = 0.$$

(If $\Delta = 0$ then y is a polynomial.)

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