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MOST HITCHIN REPRESENTATIONS ARE STRONGLY DENSE

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Abstract. We prove that generic Hitchin representations are strongly dense: every pair of non commuting elements in their image generate a Zariski-dense subgroup of $\mathrm{SL}_n(\mathbb{R})$. The proof uses a theorem of Rapinchuk, Benyash-Krivetz and Chernousov, to show that the set of Hitchin representations is Zariski-dense in the variety of representations of a surface group in $\mathrm{SL}_n(\mathbb{R})$.

1. Introduction

Following Breuillard, Green, Guralnick and Tao [2], we say that a subgroup $\Gamma \subset \mathrm{SL}_n(\mathbb{R})$ is *strongly dense* if any pair of non-commuting elements of Γ generate a Zariski-dense subgroup of $\mathrm{SL}_n(\mathbb{R})$. They proved that, among many other semisimple algebraic groups, the group $\mathrm{SL}_n(\mathbb{R})$ contains a strongly dense non abelian free subgroup [2, Theorem 4.5]. In this note, we extend the Breuillard, Green, Guralnick and Tao result to certain (discrete and) faithful representations of surface groups of genus at least two into $\mathrm{SL}_n(\mathbb{R})$.

To describe this more carefully, we introduce some background and terminology. For fixed $g \geqslant 2$, and base field k, the set of representations of the surface group $\pi_1(\Sigma_g)$ to $\mathrm{SL}_n(k)$ is denoted by $\mathrm{Hom}(\pi_1(\Sigma_g), \mathrm{SL}_n(k))$ and is naturally an affine subvariety of $k^{2g\cdot n^2}$ known as the *representation variety*. In the case of $k = \mathbb{R}$, those representations of interest to us, the Hitchin representations, are of particular geometric importance and can be defined as follows.

The *Teichmüller representations* in $\mathrm{Hom}(\pi_1(\Sigma_g), \mathrm{SL}_n(\mathbb{R}))$ are those obtained by composing any faithful and discrete representation $\pi_1(\Sigma_g) \to \mathrm{SL}_n(\mathbb{R})$ with an irreducible representation $\mathrm{SL}_2(\mathbb{R}) \to \mathrm{SL}_n(\mathbb{R})$. The Hitchin representations are those that lie in the same (topological) connected component of $\mathrm{Hom}(\pi_1(\Sigma_g), \mathrm{SL}_n(\mathbb{R}))$ as a Teichmüller representation. Note that, depending on the parity of n, there may be more than one such component, but we simply choose one and denote it by HIT_n. [1]

We say that a representation is strongly dense if its image is a strongly dense subgroup of $\mathrm{SL}_n(\mathbb{R})$, and we say that a subset of $\mathrm{Hom}(\pi_1(\Sigma_g), \mathrm{SL}_n(\mathbb{R}))$ is *generic* if its complement consists of a countable union of proper subvarieties. The main result of this note is:

Theorem 1.1. *Let $g \geqslant 2$ and $n \geqslant 3$. Then the set of strongly dense representations of $\pi_1(\Sigma_g)$ is generic in HIT_n.*

It is known that all the representations in HIT_n are faithful and discrete (see [5, Theorem 1.5]), so this provides the representations promised in the first paragraph. We note that the result of Theorem 1.1 was obtained recently in [6] in the case of $n = 3$ by direct geometric methods.

[1] We note that a *Hitchin component* more usually refers to a connected component of the *character variety* $X(\pi_1(\Sigma_g), \mathrm{SL}_n(\mathbb{R}))$ and the notation Hit_n is frequently used, but in this note it will be technically simpler to work at the level of representations.

To prove Theorem 1.1 we prove the following result, which seems independently interesting, and uses a result of Rapinchuk, Benyash-Krivetz and Chernousov [8], that $\mathrm{Hom}(\pi_1(\Sigma_g), \mathrm{SL}_n(\mathbb{C}))$ is an irreducible subvariety of $\mathbb{C}^{2g\cdot n^2}$; in fact, it is Zariski and classically connected.

Theorem 1.2. *For all $n \geqslant 2$, the set HIT_n is Zariski-dense in the affine algebraic set $\mathrm{Hom}(\pi_1(\Sigma_g), \mathrm{SL}_n(\mathbb{C}))$.*

The case $n = 2$ was already essentially observed in [3, Chapter 3].

As we describe below, Theorem 1.1 follows from Theorem 1.2 together with [2] and the fact that surface groups are residually free [1].

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2. Proofs.

The proof of Theorem 1.2 rests on the following observation:

Suppose that V is an irreducible complex affine variety defined by real polynomials. If V has a smooth real point, then $V(\mathbb{R})$ is Zariski dense in V.

Although apparently well-known and natural, we could not locate a proof in the literature, and so we decided to include a proof for completeness.

Lemma 2.1. *Let $N \geqslant 1$; let $V(\mathbb{C}) \subset \mathbb{C}^N$ be an irreducible affine variety, defined over $\mathbb{R}$, of dimension n. Let H be an open subset of $V(\mathbb{R})$, for the usual, Hausdorff topology of $\mathbb{R}^N$, and suppose H contains a smooth point x_0 of $V(\mathbb{C})$. Then H is Zariski-dense in $V(\mathbb{R})$, for its structure as a real affine variety in $\mathbb{R}^N$.*

Proof. In fact we will prove that H is Zariski-dense in $V(\mathbb{C})$: any (complex) polynomial function on $\mathbb{C}^N$ vanishing identically on H, vanishes on $V(\mathbb{C})$. By restricting to real polynomials, this implies the lemma.

Let $F\colon \mathbb{C}^N \to \mathbb{C}^{N-n}$ be a polynomial map, with real coefficients, defining V. Since x_0 is a smooth point of $V(\mathbb{C})$, or equivalently, of $V(\mathbb{R})$, the Jacobian matrix $JF(x_0)$ has rank $N-n$. Up to permuting the coordinates, we may suppose that the minor of $JF(x_0)$ corresponding to the $N-n$ last coordinates is invertible. Then the inverse function theorem yields a neighborhood $U \times V$ of x_0, and a map $\phi\colon U \to V$ with $U \subset \mathbb{C}^n$ and $V \subset \mathbb{C}^{N-n}$, such that $V(\mathbb{C}) \cap (U \times V) = F^{-1}(\{0\}) \cap (U \times V)$ is the graph of ϕ.

This map ϕ is obtained as the limit of an explicit fixed point process which is defined only in terms of F: it follows that ϕ is holomorphic, as a uniform limit of holomorphic maps, and that the restriction of ϕ to the reals is the solution to the same inverse function problem, hence (up to taking smaller neighborhoods) for all $t \in U$ we have $t \in \mathbb{R}^n$ if and only if $\phi(t) \in \mathbb{R}^{N-n}$. See *e.g.* [7, Paragraph 1.3].

Now let $P\colon \mathbb{C}^N \to \mathbb{C}$ be a polynomial function vanishing identically on H. Then the map $U \to \mathbb{C}$, $t \mapsto P(t, \phi(t))$ is holomorphic, and it vanishes identically on $U \cap \mathbb{R}^n$. Such a map vanishes identically on U: this can be checked by induction on n, where both the base step and the induction step use the principle of isolated zeroes of holomorphic maps of one variable. □

Proof of Theorem 1.2. As noted in §1, $R(\mathbb{C}) = \mathrm{Hom}(\pi_1(\Sigma_g), \mathrm{SL}_n(\mathbb{C}))$ is an affine subvariety of $\mathbb{C}^{2g \cdot n^2}$, and it was proved in [8, Theorem 3] to be irreducible of dimension $(2g-1)(n^2-1)$.

The set HIT_n is, by definition, a (topological) connected component of $X_{\mathbb{R}}$, which is a real algebraic variety, and hence HIT_n is open. We claim that it contains smooth points of $R(\mathbb{R})$, or equivalently, of $R(\mathbb{C})$: in fact, we will show that all its points are regular.

Indeed, by a result of Goldman [4, Propositon 1.2], at each point ρ of $R(\mathbb{R})$, the dimension of the Zariski tangent space at ρ equals $(2g-1)(n^2-1)+\dim(\zeta(\rho(\pi_1\Sigma_g)))$, where $\zeta(\rho(\pi_1\Sigma_g))$ is the centralizer of the image group $\rho(\pi_1\Sigma_g)$ in $\mathrm{SL}_n(\mathbb{R})$.

We will make use of the following facts proved by Labourie (see [5, Theorem 1.5 and Paragraph 10]). First, if $\rho \in \mathrm{HIT}_n$, then ρ is irreducible, and second, for all nonidentity elements $\gamma \in \pi_1(\Sigma_g)$, the matrix $\rho(\gamma)$ is diagonalizable with pairwise distinct real eigenvalues.

Fix such a γ_0; by conjugating the image of ρ in $\mathrm{SL}_n(\mathbb{R})$, we may suppose that $\rho(\gamma_0)$ is diagonal. Let ξ be an element of $\zeta(\rho(\pi_1(\Sigma_g)))$. Since ξ commutes with $\rho(\gamma_0)$, it is also diagonal, and if λ is an eigenvalue of ξ, the matrix $\xi - \lambda I$ also commutes with $\rho(\pi_1(\Sigma_g))$. Hence $\ker(\xi - \lambda I)$ is invariant by $\rho(\pi_1(\Sigma_g))$. However, ρ is irreducible, and so this implies that ξ is a scalar matrix, that is to say, $\xi = \pm I$.

Thus, the Zariski tangent space at any representation $\rho \in \mathrm{HIT}_n$ has minimal dimension, $(2g-1)(n^2-1)$, in other words, these are regular points of the varieties $R(\mathbb{R})$ and $R(\mathbb{C})$.

The result now follows from Lemma 2.1. □

Proof of Theorem 1.1. For every pair of non commuting elements $a, b \in \pi_1(\Sigma_g)$, let $\mathrm{Bad}(a,b)$ denote the subset of $\mathrm{Hom}(\pi_1(\Sigma_g), \mathrm{SL}_n(\mathbb{R}))$ consisting of representations ρ such that $\rho(a)$ and $\rho(b)$ do not generate a Zariski-dense subgroup of $\mathrm{SL}_n(\mathbb{R})$, and let $\mathrm{Good}(a,b)$ denote its complement.

The proof will be complete once we know that for every pair of non commuting elements $a, b \in \pi_1(\Sigma_g)$, the set $\mathrm{Bad}(a,b) \cap \mathrm{HIT}_n$ is Zariski-closed, and that it is a proper subset of HIT_n.

The fact that the sets $\mathrm{Bad}(a,b)$ are Zariski-closed follows from [2, Theorem 4.1].

Now let us check that $\mathrm{Bad}(a,b) \cap \mathrm{HIT}_n$ is a proper subset of HIT_n, or equivalently, that $\mathrm{Good}(a,b) \cap \mathrm{HIT}_n$ is nonempty. Since $\mathrm{Good}(a,b)$ is Zariski-open, and since HIT_n is Zariski-dense in $\mathrm{Hom}(\pi_1(\Sigma_g), \mathrm{SL}_n(\mathbb{R}))$ by Theorem 1.2, it suffices to check that $\mathrm{Good}(a,b)$ is nonempty.

By [2, Theorem 4.5], there exists a strongly dense representation $\rho_0 \colon F_2 \to \mathrm{SL}_n(\mathbb{R})$. Let $a, b \in \pi_1(\Sigma_g)$ be a pair of non commuting elements. Since $\pi_1(\Sigma_g)$ is residually free (see Baumslag [1]) and $[a,b] \neq 1$, there exists a surjective morphism ψ from $\pi_1(\Sigma_g)$ onto a free group F, such that $\psi([a,b]) \neq 1$. By composing ψ with an injective morphism $F \to F_2$, this yields a morphism $\varphi \colon \pi_1(\Sigma_g) \to F_2$ such that $\varphi([a,b]) \neq 1$. Thus, $\varphi(a)$ and $\varphi(b)$ do not commute, hence $\rho_0(\varphi(a))$ and $\rho_0(\varphi(b))$ generate a Zariski dense subgroup of $\mathrm{SL}_n(\mathbb{R})$. In other words, $\rho_0 \circ \varphi$ lies in $\mathrm{Good}(a,b)$, so this set is non empty. □

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