Max-Planck-Institut für Mathematik
Bonn

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Preprint Series 2022 (8)

Date of submission: February 21, 2022

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MPIM 22-8

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To Marianne and Xenia, from Enno and Yuri, with all our love and gratitude.

This article is dedicated to the generalization of the operad of moduli spaces of curves to SUSY curves. SUSY curves are algebraic curves with additional supersymmetric or supergeometric structure. Here, we focus on the description of the relevant category of graphs and its combinatorics as well as the construction of dual graphs of SUSY curves and the supermodular operad taking values in a category of moduli spaces of SUSY curves with Neveu–Schwarz and Ramond punctures.

1 Introduction and summary

In the earlier works (cf. Losev and Manin 2004 and references therein), it was shown that operadic constructions developed in Getzler 1995 can be extended to include additional data on algebraic curves. In Borisov and Manin 2007 a general formalism of labeled graphs and generalized operads was developed. Here, we combine both techniques to construct an operad that encodes geometric properties of moduli spaces of SUSY curves with Neveu–Schwarz and Ramond punctures.

SUSY curves, also called super Riemann surfaces or super curves, are supergeometric generalizations of algebraic curves with spin structure that have been introduced in the context of superstring theory, see Friedan 1986. Their most distinctive feature for the context of this work is that families of SUSY curves may develop two types of nodes as was argued in Deligne 1987. The two types of nodes are now called Neveu–Schwarz and Ramond nodes and one also considers markings of those two types: Neveu–Schwarz punctures and Ramond punctures. Compact moduli spaces $\overline{\mathcal{M}}_{g,k_{NS},k_R}$ of stable SUSY curves of genus g with k_{NS} Neveu–Schwarz and k_R Ramond punctures have been constructed in Felder, Kazhdan, and Polishchuk 2020 as smooth Deligne–Mumford superstacks.

To achieve a combinatorial description of stable SUSY curves and the boundary strata of the moduli stack $\overline{\mathcal{M}}_{g,k_{NS},k_R}$, we introduce in Definition 2.3.1 below the category of SUSY graphs SGr. SUSY graphs are modular graphs with an additional coloring of the edges and tails to encode if nodes or markings are of Neveu–Schwarz or Ramond type. In Definition 3.3.1, we show how a stable SUSY curve has a stable SUSY graph as dual graph.

The main result of the paper is

Theorem 4.3.2. *The map*

$$\tau \mapsto \prod_{v \in V_\tau} \overline{\mathcal{M}}_{g(v),\#F_{\tau,NS}(v),\#F_{\tau,R}(v)}$$

can be extended to a symmetric monoidal functor $\mathsf{O}\colon \mathsf{SGr}^{\mathsf{st}} \to \mathsf{G}$ *such that* O *is an operad, that is, sends graftings to isomorphisms.*

Using this Theorem, one obtains a descriptions of the moduli space of SUSY curves with prescribed dual graph in terms of gluing data and its dimension. This is a generalization of a description of $\overline{\mathcal{M}}_{0,k_{NS},0}$ via trees and as a stratified superorbifold in Keßler, Sheshmani, and Yau 2020.

The structures, upon which we focus in this study, are partly motivated by the study of “phylogenetic trees” in Wu and Yau 2020. Y. Wu and S.-T. Yau describe combinatorially various patterns of degeneration of generically smooth projective curves of genus zero with marked points at the boundaries of moduli spaces of such curves.

The paper is organized as follows: We start the Section 2 with a systematic description of the formalism of graphs as a language for describing the combinatorics of degeneration of various types of algebraic/analytic curves and SUSY curves. The Section 3 recalls the notions of stable SUSY curve with Neveu–Schwarz and Ramond punctures from Felder, Kazhdan, and Polishchuk 2020 and describes how a stable SUSY graph is obtained from stable SUSY curves. Finally, Section 4 contains the main new results of this paper, the construction of the supermodular operad, its symmetry properties and relationship to the modular operad.

2 SUSY Graphs

2.1 Graphs, their morphisms and categories

We accept the general framework of Borisov and Manin 2007, stressing the difference between the formal definitions of *graphs* on the one side, and their *geometric realizations* on the other side.

A *graph* τ is a family of structures (just sets or structured sets and maps between them in the simplest cases) $(F_\tau, V_\tau, \partial_\tau, j_\tau)$. Elements of F_τ, resp. V_τ, are called *flags*, resp. *vertices* of τ. The map $\partial_\tau\colon F_\tau \to V_\tau$ associates to each flag a vertex, called its *boundary*.

The map $j_\tau\colon F_\tau \to F_\tau$ must satisfy the condition $j_\tau^2 = \mathrm{id}$, identical map of F_τ to itself. Flags fixed by j_τ are called tails; we denote the set of tails of τ by T_τ. Two-element orbits of j_τ are called edges of τ; we denote the set of edges of τ by E_τ.

For a discussion of their geometric meaning, and of many marginal (“degenerate”) cases, see Borisov and Manin 2007.

A *morphism of graphs* $h\colon \tau \to \sigma$ is a triple of maps

$$\left(h^F\colon F_\sigma \to F_\tau, \quad h_V\colon V_\tau \to V_\sigma, \quad j_h\colon F_\tau \setminus h^F(F_\sigma) \to F_\tau \setminus h^F(F_\sigma)\right),$$

where h^F is an injective contravariant map, h_V is a surjective covariant map, and j_h is an involution, satisfying a list of additional restrictions: cf. Borisov and Manin 2007, Definition 1.2.1.

Thus defined, graphs form a category Gr, with monoidal structure corresponding to the disjoint union $\coprod$ of geometric realizations of graphs.

Remark 2.1.1. The definition of Gr in this article is particularly suited for the study of dual graphs of curves and differs from other texts, for example, Bridson and Haefliger 1999. In particular, graphs in Gr can have tails, multiple but undirected edges, loops and disconnected components. Graph morphisms allow *graftings*, that is connecting two tails to form an edge, *contractions* of edges and *virtual contractions*, that is contraction of a pair of tails. But on the other hand, morphisms cannot break edges into tails and inclusions of subgraphs are not necessarily graph morphisms.

Remark 2.1.2. It is pretty clear, that in the definitions above one may replace (structured) sets by objects of a category, maps of sets by morphisms between objects of this category etc. Of course, one then should take care about various compatibility restrictions, lifted to the level of a (small) category.

2.2 Labeled graphs

In what follows we will need graphs with different labelings. Abstractly, a category of labeled graphs Γ comes with a functor $\psi\colon \Gamma \to \mathsf{Gr}$ satisfying several properties, see Borisov and Manin 2007, 1.3 Definition. As mentioned there, a labeled graph $\sigma \in \Gamma$ can be imagined as the underlying graph $\psi(\sigma)$ together with some additional data on vertices, flags or edges. Some examples:

Example 2.2.1 (genus labeled graphs, see Example 1.3.2c) in Borisov and Manin 2007)*.* A *genus labeling* on a graph $\tau = (F_\tau, V_\tau, \partial_\tau, j_\tau)$ is a map $g\colon V_\tau \to \mathbb{Z}_{\geq 0}$ A morphism of genus labeled graphs is a morphism of graphs such that the genus of a vertex in the image is given by the sum of the genera of the vertices in the preimage plus the number of contracted loops at that vertex.

The genus of a connected graph τ is given by

$$g_\tau = \sum_{v \in V_\tau} (g(v) - 1) + \#E_\tau + 1,$$

where $\#E_\tau$ is the cardinality of the set of edges of τ. The genus of disconnected graphs is the sum of the genera of its connected components.

A *tree* is a genus labeled graph of genus zero. We say that the genus labeled graph τ is *stable* if for every vertex $v \in V_\tau$ it holds $2g(v) - 2 + \#F_\tau(v) > 0$.

Example 2.2.2 (colored graphs, see Example 1.3.2.d) in Borisov and Manin 2007)*.* A *colored graph* is a graph where edges and half-edges are assigned a color. More precisely, an I-coloring on $\tau = (F_\tau, V_\tau, \partial_\tau, j_\tau)$ is a map $c\colon F_\tau \to I$ to some fixed-finite set I whose elements are called colors such that $c \circ j_\tau = c$. Morphisms $h\colon \tau \to \sigma$ of I-colored graphs preserve the color of edges, that is $c_\tau \circ h^F \;= c_\sigma$ and $c_\tau \circ j_h = c_\tau|_{F_\tau \setminus h^F(F_\sigma)}$.

Example 2.2.3 (k-labeled graphs). Let $k \in \mathbb{Z}_{\geq 0}$ be a non-negative integer. A *k-labeling* on a graph τ is given by a bijective map $l_\tau\colon \{1, \dots, k\} \to T_\tau$, that is, a numbering of the tails. A morphism $h\colon \tau \to \sigma$ between k-labeled graphs either

- is bijective on tails and preserves the labeling, that is, $l_\tau \circ h^F|_{T_\sigma} = l_\sigma$
- or if h grafts or virtually contracts pairs of tails, the labeling l_σ is obtained from l_τ by deleting the labels of tails not in the image and renumbering consecutively while preserving the order.

Note that the symmetric group S_k acts on k-labeled graphs by renumbering.

Definition 2.2.4. Let MGr be the category of *modular graphs* consisting of the graphs with a genus labeling and a k-labeling of tails. The morphisms in MGr are compatible with the genus labeling and the k-labeling in the sense of Example 2.2.1 and Example 2.2.3.

We furthermore consider the full subcategories of stable graphs $\mathsf{MGr}^{\mathsf{st}}$, modular trees MTr and stable modular trees $\mathsf{MTr}^{\mathsf{st}}$.

2.3 Graphs relevant to the encoding of SUSY curves

Definition 2.3.1. Choose a pair, consisting of a non-negative integer k_{NS} and an even non-negative integer k_R. A (k_{NS}, k_R)–SUSY graph is a graph $\tau = (F_\tau, V_\tau, \partial_\tau, j_\tau)$, endowed with

(i) a genus labeling $g_\tau\colon V_\tau \to \mathbb{Z}_{\geq 0}$;

(ii) a coloring $c_\tau\colon F_\tau \to \{NS, R\}$ such that for every vertex $v \in V_\tau$ the number of adjacent flags with color R must be even;

(iii) two separate labelings of NS-tails and R-tails: bijections

$$
\begin{aligned}
l_{\tau,NS}\colon \{1, \dots, k_{NS}\} &\to T_{\tau,NS} \coloneqq F_{\tau,NS} \cap T_\tau, \\
l_{\tau,R}\colon \{1, \dots, k_R\} &\to T_{\tau,R} \coloneqq F_{\tau,R} \cap T_\tau.
\end{aligned}
$$

The coloring c_τ induces partitions of the set of all flags $F_\tau = F_{\tau,NS} \cup F_{\tau,R}$, the set of flags adjacent to a vertex v $F_\tau(v) = F_{\tau,NS}(v) \cup F_{\tau,R}(v)$, tails $T_\tau = T_{\tau,NS} \cup T_{\tau,R}$ and edges $E_\tau = E_{\tau,NS} \cup E_{\tau,R}$ into those with color NS and those with color R. We denote elements of $F_{\tau,NS}$ Neveu–Schwarz flags and elements of $F_{\tau,R}$ Ramond flags which we sometimes abbreviate as NS-flags and R-flags respectively. Similarly for edges and tails. If we do not specify the numbers k_{NS} and k_R we just use the short-hand SUSY graph.

Definition 2.3.2. A morphism of SUSY graphs is a morphism between graphs that preserves the genus labeling as well as the coloring of flags and is compatible with the labeling of Neveu–Schwarz tails and Ramond tails, compare Examples 2.2.1, 2.2.2, 2.2.3.

Notice that the number of Neveu–Schwarz tails (resp. Ramond tails) of the image of a morphisms of SUSY graphs may be lower than the number of Neveu–Schwarz tails (resp. Ramond tails) of the domain. Morphisms between SUSY graphs can graft tails of the same color and virtually contract pairs of tails of the same color.

Definition 2.3.3. We denote the category of SUSY graphs by SGr. The category SGr is a symmetric monoidal category with respect to the disjoint union of SUSY graphs.

We denote the full subcategory of SGr where objects are stable graphs by $\mathsf{SGr}^{\mathsf{st}}$. The full subcategory of SGr where objects are trees is denoted by STr and the category of stable SUSY trees by $\mathsf{STr}^{\mathsf{st}}$.

There is a forgetful functor $\mathsf{F}\colon \mathsf{SGr} \to \mathsf{MGr}$ that restricts to forgetful functors $\mathsf{SGr}^{\mathsf{st}} \to \mathsf{MGr}^{\mathsf{st}}$, $\mathsf{STr} \to \mathsf{MTr}$ and $\mathsf{STr}^{\mathsf{st}} \to \mathsf{MTr}^{\mathsf{st}}$. The functor F is constructed as follows: The SUSY graph τ with genus labeling g_τ, coloring c_τ and labelings $l_{\tau,NS}$, $l_{\tau,R}$ is sent to the modular graph σ which coincides with τ as a genus labeled graph. The labeling l_σ of the tails is defined as $l_\sigma|_{T_{\tau,NS}} = l_{\tau,NS}$ and $l_\sigma|_{T_{\tau,R}} = k_{NS} + l_{\tau,R}$. With this convention, any morphism $h\colon \tau \to \sigma$ between SUSY graphs is also a morphism between the modular graphs $\mathsf{F}(\mathsf{h})\colon \mathsf{F}(\tau) \to \mathsf{F}(\sigma)$.

2.4 Directed and oriented SUSY graphs

Denote by $\mathbb{N}_0$ the family of subsets of natural numbers $\{0\}$, $\{0,1\}$, $\{0,1,2\}$, $\dots$, $\{0,1,2,\dots,n\}$, $\dots$.

Definition 2.4.1. We say that a SUSY graph τ is *oriented and directed* if it is equipped with

(i) Another partition of F_τ into two disjoint subsets: a map $F_\tau \to \{in, out\}$ such that halves of any edge get different labels. In the geometric realization, the flag labeled by in (resp. out) is oriented towards (resp. outwards) its vertex. Such a labeling determines an orientation of τ, and τ is then called an oriented graph.

(ii) A binary relation among vertices of an oriented graph "v is higher than v'", such that each flag becomes oriented downwards with respect to this relation: from a higher vertex to a lower one.

(iii) For all v, a choice of bijection $F_\tau(v)$ with an element of $\mathbb{N}_0$, such that all flags of this set, not labeled by 0, are either inputs, or outputs, so that 0 labels the only output, resp. input.

A corolla is a graph, having one vertex. In the Definition 2.4.1 of oriented and directed SUSY graph above, property (ii) implies that only one tail is oriented towards (or outwards) the vertex, in which case all other tails are oriented outwards (resp. towards). So, in particular, a geometric realization of an oriented and directed SUSY corolla can be embedded into a real plane, with coordinates (x,y), so that the vertex becomes $(0,0)$, the 0-tail goes down along the y-axis, and the structural ordering of all other tails agrees with their ordering with respect to clock-wise ordering.

Lemma 2.4.2. A geometric realization of an oriented SUSY graph τ can me embedded into real plane in such a way that each corolla of it is embedded as above.

Proof. It is sufficient to consider τ with connected geometric realization, having at least two different vertices (one-vertex graphs are corollas, and this case was already treated above). Choose in it the highest vertex, and consider the only oriented flag leading from this vertex downwards. It is half of an edge, oriented downwards. We may cut this edge, and τ will be the result of grafting a corolla and a SUSY graph with a smaller amount of flags. This allows us to make an inductive step. □

Note that there are SUSY graphs which cannot be oriented and directed: For example the graph consisting of four vertices of genus zero and a Neveu–Schwarz edge from every vertex to every other.

2.5 Stable SUSY trees

Recall that stable SUSY trees are SUSY graphs of genus zero such that every vertex has at least three adjacent flags. The forgetful functor $\mathsf{F}\colon \mathsf{STr}^{\mathsf{st}} \to \mathsf{MTr}^{\mathsf{st}}$ sends a stable SUSY tree to a stable modular tree. Conversely, we will now show that a modular tree can be given the structure of a SUSY graph by coloring the tails. The color of the edges can be inferred using the parity condition at every vertex.

Lemma 2.5.1. Let σ be stable modular tree with k numbered tails and $k = k_{NS} + k_R$ for non-negative integer k_{NS} and a non-negative even integer k_R. There is a unique stable (k_{NS}, k_R)-SUSY tree τ such that $\mathsf{F}(\tau) = \sigma$.

Proof. The stable SUSY graph τ coincides with the modular graph σ as a genus labeled graph. We need to construct the partition of the flags of τ into Neveu–Schwarz and Ramond flags as well as the labeling of the Neveu–Schwarz and Ramond tails from the labeling l_σ of the tails. The partition of the tails $T_\tau = T_{\tau,NS} \cup T_{\tau,R}$ is given by

$$T_{\tau,NS} = \{t \in T_\tau \mid l_\sigma(t) \leq k_{NS}\} \qquad T_{\tau,R} = \{t \in T_\tau \mid l_\sigma(t) > k_{NS}\}$$

The numbering of the Neveu–Schwarz and Ramond tails of τ is induced from the numbering of the tails of σ as follows:

$$l_{\tau,NS} = l_\sigma|_{T_{\tau,NS}}, \qquad l_{\tau,R} = l_\sigma|_{T_{\tau,R}} - k_{NS}.$$

The partition of the edges is obtained from the parity condition by induction over the number of edges: If the tree τ has no edges there is nothing to show. Every tree τ with at least one edge has a vertex v which is the boundary of precisely one edge e. Stability implies that v has at least two tails. If the number of Ramond tails bounding to v is even the edge e needs to be be a Neveu–Schwarz edge. If the number of Ramond tails bounding v is odd the edge e needs to be an Ramond edge. We can now proceed by inductively considering the tree τ' obtained from tau by cutting the edge e and deleting the vertex v as well as its adjacent flags. □

3 Stable SUSY curves with punctures and their dual graph

In this section we recall the notions of stable SUSY curves with punctures and their respective moduli spaces. Furthermore we give the construction of the dual graph of a super curve with punctures as a SUSY graph.

The definition of super Riemann surfaces or SUSY curves appeared first in the context of super string theory, see, for example, Friedan 1986. SUSY curves are superschemes of dimension $1|1$ with an additional structure and generalize, in many aspects, algebraic curves to supergeometry. Particular examples of SUSY curves can be constructed from algebraic curves together with a spinor bundle.

Early studies of the moduli spaces of SUSY curves are LeBrun and Rothstein 1988; Crane and Rabin 1988. It was argued in Deligne 1987 that families of SUSY curves may degenerate in SUSY curves with two distinct types of nodes, called Neveu–Schwarz (NS) nodes and Ramond (R) nodes. While Neveu–Schwarz nodes are transversal intersections of SUSY curves, Ramond nodes have an additional degeneration of the spinor bundle.

The role of marked points on purely even algebraic curves is played by "punctures" on a SUSY curve. Punctures are likewise divided into two classes, called Neveu–Schwarz and Ramond punctures. For a family of stable SUSY curves $M \to B$, Neveu–Schwarz punctures are given by sections $B \to M$, whereas Ramond punctures are relative Cartier divisors $R \subset M$ such that the projection $R \to B$ is smooth of dimension $0|1$ together with additional structural conditions.

For modern accounts on stable SUSY curves with punctures and their moduli spaces we refer to Felder, Kazhdan, and Polishchuk 2020; Ott and Voronov 2019.

3.1 SUSY curves with punctures

We assume familiarity with algebraic supergeometry as, for example, in Manin 1988. The following definition of SUSY curve with punctures is taken from Felder, Kazhdan, and Polishchuk 2020, Def. 2.3 in Sec. 2.2.

Definition 3.1.1. A be a super Riemann surface over the base B with k_{NS} Neveu–Schwarz punctures and k_R Ramond punctures is a tuple $(M, \{s_i\}, \mathcal{R}, \mathcal{D})$ where

- $\pi\colon M \to B$ is a smooth, proper morphism of superschemes of relative dimension $1|1$ generic fibers,
- $s_i\colon B \to M$ for $i = 1, \dots, k_{NS}$ are sections of π such that their reductions are distinct, called Neveu–Schwarz punctures,
- $\mathcal{R}$ is an unramified relative effective Cartier divisor of codimension $0|1$ in M of degree k_R whose labeled components r_i, $i = 1, \dots, k_R$ are called the Ramond punctures
- the line bundle $\mathcal{D}$ is a subbundle $\mathcal{D} \subset \mathcal{T}_M$ of rank $0|1$ such that the commutator of vector fields induces an isomorphism

$$\mathcal{D} \otimes \mathcal{D} \to \left({}^{\mathcal{T}_M}\!/\!_{\mathcal{D}} \right)(-\mathcal{R}).$$

Considering the quotient of the structure sheaf of M by the ideal of nilpotent elements one obtains the reduction $i_{red}\colon M_{red} \to M$. The family $M_{red} \to B_{red}$ is a smooth and proper family of curves over B_{red}. The reduction of the Neveu–Schwarz punctures s_i yields marked points of M_{red}. The reduction of the Ramond punctures yields a divisor $\mathcal{R}_{red}$ which are equivalent to further marked points of M_{red}. That is, we can see $M_{red} \to B_{red}$ as a smooth and proper family of curves with $k_{NS} + k_R$ marked points.

But it is also interesting to make a distinction between the Neveu–Schwarz punctures and the Ramond punctures in the reduced case: The pullback $S = i^*\mathcal{D}$ is a spinor bundle on M with a degeneration over $\mathcal{R}_{red}$, or more precisely

$$S \otimes S = \mathcal{T}_{M_{red}}(-\mathcal{R}_{red}) \tag{3.1.2}$$

The family $M_{red} \to B_{red}$ together with the spinor bundle S as in Equation (3.1.2) is called a spin curve with k_{NS} punctures of type 0 and k_R punctures of type 1 in Abramovich and Jarvis 2003.

A SUSY curve over $B = \mathbb{C}^{0|0}$ is equivalent to the data of $(M_{red}, S, \{s_{i,red}\}, \mathcal{R}_{red})$ satisfying Equation (3.1.2).

If M is irreducible of genus g the Theorem of Riemann Roch implies

$$2\deg S = \deg \mathcal{T}_{M_{red}} - \deg \mathcal{R}_{red} = 2 - 2g - \deg \mathcal{R}_{red}.$$

that is, the number of Ramond punctures needs to be even.

Remark 3.1.3. SUSY curves are sometimes also called super Riemann surfaces or super curves, especially if the number of Ramond punctures is zero. We will use in this work exclusively the name SUSY curve.

3.2 Stable SUSY curves with punctures

Intuitively, a stable SUSY curve is a generalization of SUSY curves with punctures to include nodes such that its reduced space is a stable curve. To make this precise, it is necessary to reformulate the non-integrability condition of $\mathcal{D}$ as was argued in Deligne 1987.

Dualizing the exact sequence $0 \to \mathcal{D} \to \mathcal{T}_{M/B} \to \mathcal{T}_{M/B}/\mathcal{D} = \mathcal{D}^{\otimes 2}(R) \to 0$, we get $0 \to \mathcal{D}^{-2}(-R) \to \Omega_{M/B} \to \mathcal{D}^{-1} \to 0$. This shows that $\mathcal{D}^{-1}(-R)$ is isomorphic to the Berezinian of $\Omega_{M/B}$, denoted $\omega_{M/B} = \operatorname{Ber}\Omega_{M/B}$, and in turn produces the derivation $\delta\colon \mathcal{O}_M \to \omega_{M/B}(R)$, trivial on lifts of $\mathcal{O}_B$. The data of δ is equivalent to the data of the line bundle $\mathcal{D}$.

Definition 3.2.1. Consider a superscheme B. A family of *stable SUSY curves with punctures over* B is a tuple $(M, \{s_i\}, \mathcal{R}, \delta)$ consisting of

- a proper, flat and relatively Cohen–Macaulay superscheme $\pi\colon M \to B$
- $s_i\colon B \to M$ for $i = 1, \dots, k_{NS}$ are sections of π such that their reductions are different, called Neveu–Schwarz punctures,

- $\mathcal{R}$ is an unramified relative effective Cartier divisor of codimension $0|1$ in M of degree k_R whose labeled components r_i, $i = 1, \dots, k_R$ are called the Ramond punctures
- A derivation $\delta\colon \mathcal{O}_M \to \omega_{M/B}(\mathcal{R})$, trivial on lifts of $\mathcal{O}_B$ to M.

These data must satisfy the following conditions:

(i) M contains an open fiberwise dense subset U, such that U/B is smooth of relative dimension $1|1$, and all s_i and r_i are contained in U.

(ii) The tuple $(U, \{s_i\}, \mathcal{R}, \delta)$ is equivalent to a SUSY curve in the sense of Definition 3.1.1.

(iii) The reduction $M_{red} \to B_{red}$ is a stable family of marked curves.

Notice that, in general, M can be reducible and consist of several components. Each irreducible component M_i contains a fiberwise dense open subset $U_i = M_i \cup U$ which carries the structure of an open SUSY curve with punctures. The SUSY curve structure on U_i can be completed uniquely to a smooth SUSY curve with punctures $\tilde{M}_i$ on M_i. We call the punctures of $\tilde{M}_i$ special points of M_i. A special point of M_i is either a puncture of M or represents a node of M. Every special point is either a Neveu–Schwarz special point or a Ramond special point. The number of Ramond special points on an irreducible component is even.

The reduction M_{red} of a stable SUSY curve M is a stable algebraic curve of genus g with marked points $s_1, \dots s_{k_{NS}}$ and marked points $r_1, \dots r_{k_R}$. The pullback $S = i_{red}^* \mathcal{D}$ of $\mathcal{D}$ along the reduction map $i_{red}\colon M_{red} \to M$ is a locally free sheaf of rank one with a morphism

$$S \otimes S \to \mathcal{T}_{M_{red}}(-\sum_i r_i) \tag{3.2.2}$$

which is an isomorphism away from the nodes.

3.3 The dual graph of a stable SUSY curve with punctures

To a given algebraic curve with marked points one associates a dual graph, see Kontsevich and Manin 1994; Arbarello, Cornalba, and Griffiths 2011, Chapter X, §2; Combe and Manin 2019. We are going to generalize the concept of dual graph to SUSY curves by adding additional labels to edges and half-edges as follows:

Definition 3.3.1. Let $(M, \{s_i\}, \mathcal{R}, \mathcal{D})$ be a stable SUSY curve with k_{NS} Neveu–Schwarz punctures and k_R Ramond punctures over B such that $B_{red} = pt$. The dual graph of $(M, \{s_i\}, \mathcal{R}, \mathcal{D})$ consists of a (k_{NS}, k_R)-SUSY graph $\tau \in \mathsf{SGr}^{\mathsf{st}}$ where

(i) The set of vertices V_τ is the set of irreducible components of M.

(ii) The vertex $v \in V_\tau$ representing the irreducible component M_i has an adjacent flag for each special point of M_i. This determines F_τ and ∂_τ.

(iii) The map j_τ maps flags associated to punctures to itself and flags associated to nodes to the flag that represents the same node on the intersecting irreducible component.

(iv) The genus labeling sends the vertex v representing the irreducible component M_i to the genus of M_i. The stability of the SUSY curve implies that τ is a stable graph.

(v) The flags F_τ are partitioned into Neveu–Schwarz flags $F_{\tau,NS}$ and Ramond flags $F_{\tau,R}$ according to the special point they represent. As the number of Ramond special points on each irreducible component is even, we have that for every vertex v the number $\#F_{\tau,R}(v)$ of Ramond flags at v is even.

(vi) The labeling $l_{\tau,NS}$ (resp. $l_{\tau,R}$) sends the tail corresponding to the i-th Neveu–Schwarz puncture (resp. Ramond puncture) to i.

For different B and B' such that $B_{red} = pt$ the dual graph is invariant under base change. In particular, we can assume without loss of generality that we consider the dual graph of a super Riemann surface over $B = \mathbb{C}$, that is equivalently, a spin curve.

Remark 3.3.2. The dual graph of a SUSY curve of genus zero is a SUSY tree. More generally, the genus of a stable SUSY curve and its dual graph coincide.

The Definition 3.3.1 of dual graphs of SUSY curves is compatible with the definition of dual graphs of algebraic nodal curves. Indeed, the definition of dual graphs of algebraic nodal curves coincides with the points (i)–(iv) of Definition 3.3.1 and has a simplified version of (vi). This yields:

Proposition 3.3.3. *Let M be a stable SUSY curve with k_{NS} Neveu–Schwarz and k_R Ramond punctures. Its reduction M_{red} is a stable nodal curve with $k = k_{NS} + k_R$ marked points obtained from the reduction of Neveu–Schwarz and Ramond punctures. The marked points are labeled in the order of the punctures of M with Neveu–Schwarz punctures followed by Ramond punctures. Then for the dual graph τ of M and the dual graph σ of M_{red} it holds $\sigma = \mathsf{F}(\tau)$.*

4 The supermodular operads

In this section, we construct the supermodular operads.

Recall from Borisov and Manin 2007 that a generalized operad is a symmetric monoidal functor $F\colon \mathsf{\Gamma} \to \mathsf{G}$ from a category of graphs to a "ground category" G that maps graftings to isomorphisms. All operads in this sections are generalized operads in the sense of Borisov and Manin 2007.

The supermodular operads we will construct are supergeometric generalizations of the modular operad

$$\begin{aligned} \mathsf{o}\colon \mathsf{MGr}^{\mathsf{st}} &\to \mathsf{M} \\ \sigma &\mapsto \prod_{v \in V_\sigma} \overline{M}_{g_\sigma(v), \#F_\sigma(v)} \end{aligned}$$

that sends stable modular graphs to products of moduli stacks $\overline{M}_{g_\sigma(v),\#F_\sigma(v)}$ of stable algebraic curves with genus $g_\sigma(v)$ and $\#F_\sigma(v)$ marked points. Central to the proof of functoriality of o are the gluing maps that allow to glue stable curves along marked points and yield maps

$$\overline{M}_{g,k+1} \times \overline{M}_{g',k'+1} \to \overline{M}_{g+g',k+k'} \qquad\qquad \overline{M}_{g,k+2} \to \overline{M}_{g+1,k}$$

The operad o is known to encode interesting data about the geometry and topology of the compact moduli stack $\overline{M}_{g,k}$. For further information about the modular operads, see Getzler and Kapranov 1998; Manin 1999; Combe and Manin 2019; Combe, Manin, and Marcolli n.d.

The challenge in the construction of the supermodular operads is that SUSY curves cannot be uniquely glued along Ramond punctures. This is why we first introduce a restricted supermodular operad $\mathsf{O}^\mathsf{r}\colon \mathsf{SGr}^{\mathsf{st},\mathsf{r}} \to \mathsf{G}^\mathsf{r}$ that only needs to glue along Neveu–Schwarz punctures. In a second step, we generalize the notion of morphism in G^r to obtain a category G that can accommodate an operad $\mathsf{O}\colon \mathsf{SGr}^{\mathsf{st}} \to \mathsf{G}$

In Section 4.1 we recall the necessary background on moduli stacks of stable SUSY curves with punctures and gluings. The restricted supermodular operad O^r is constructed in Section 4.2 whereas the operad O is constructed in Section 4.3. It is then shown in Section 4.4 that this operad is bi-symmetric with respect to permutations of the labelings of tails.

4.1 Moduli spaces of stable SUSY curves

In Felder, Kazhdan, and Polishchuk 2020 it was proved (Theorem A), that for each pair (k_{NS}, k_R) as in Definition 2.3.1 above, the functor of families of stable SUSY curves of genus g with respective numbers of punctures is represented by a smooth and proper Deligne–Mumford superstack $\overline{\mathcal{M}}_{g,k_{NS},k_R}$. The reader can find a categorical background of stacks in Olsson 2016. Basic geometric definitions of the theory of superstacks are presented in Bruzzo and Ruipérez 2019.

The stack $\overline{\mathcal{M}}_{g,k_{NS},k_R}$ is endowed with a *boundary Cartier divisor* $\Delta = \Delta_{NS} + \Delta_R$, with normal crossings, see Section 8 of Felder, Kazhdan, and Polishchuk 2020. The boundary divisor Δ_{NS} encodes stable SUSY curves with at least one Neveu–Schwarz node and the boundary divisor Δ_R encodes stable SUSY curves with at least one Ramond node.

Stable SUSY curves can be glued along punctures of the same type, as was worked out in Felder, Kazhdan, and Polishchuk 2020, Section 8. Two Neveu–Schwarz punctures can be uniquely glued. Gluing a stable SUSY curve of genus g with $k_{NS}+1$ Neveu–Schwarz punctures and k_R Ramond punctures to a stable SUSY curve of genus g' and $k'_{NS}+1$ Neveu–Schwarz punctures and k'_R Ramond punctures along the Neveu–Schwarz punctures with numbers j and j' respectively yields a stable SUSY curve of genus $g+g'$ with $k_{NS}+k'_{NS}$ Neveu–Schwarz punctures and $k_R+k'_R$ Ramond punctures. On the level of moduli spaces this gluing yields an embedding of codimension $1|0$, see Felder, Kazhdan, and Polishchuk 2020, Lemma 8.10:

$$gl_{NS}(j,j')\colon \overline{\mathcal{M}}_{g,k_{NS}+1,k_R} \times \overline{\mathcal{M}}_{g',k'_{NS}+1,k'_R} \to \overline{\mathcal{M}}_{g+g',k_{NS}+k'_{NS},k_R+k'_R} \tag{4.1.1}$$

Gluing the two Neveu–Schwarz punctures with label j and j' of a stable SUSY curve of genus g with $k_{NS}+2$ Neveu–Schwarz punctures and k_R Ramond punctures yields a stable SUSY curve of genus $g+1$ and k_{NS} Neveu–Schwarz punctures and k_R-Ramond punctures. On the level of moduli space this gluing yields an embedding:

$$gl_{NS}(j,j')\colon \overline{\mathcal{M}}_{g,k_{NS}+2,k_R} \to \overline{\mathcal{M}}_{g+1,k_{NS},k_R} \tag{4.1.2}$$

Similar gluing maps are not defined uniquely around Ramond punctures because Ramond punctures are of dimension $0|1$. According to Felder, Kazhdan, and Polishchuk 2020, Lemma 8.11, there is a principal fiber bundle $\mathcal{P}_j \to \overline{\mathcal{M}}_{g,k_{NS},k_R}$ that parametrize preferred coordinate systems of the j-th Ramond puncture. The structure group of $\mathcal{P}_j$ is of the form $\mathbb{Z}_2 \times \mathbb{C}^{0|1}$. It follows from Felder, Kazhdan, and Polishchuk 2020, Lemma 8.13 that in order to glue two Ramond punctures j and j' one needs to choose an isomorphism between $\mathcal{P}_j$ and $[\mathrm{i}]_*\mathcal{P}_{j'}$, where $[\mathrm{i}]_*\mathcal{P}_{j'}$ is the rescaling of the fibers of $\mathcal{P}_{j'}$ by the complex unity i. Hence there are bundles of isometries and gluing maps

$$\begin{array}{ccc} \operatorname{Iso}(\mathcal{P}_j, [\mathrm{i}]_*\mathcal{P}_{j'}) & \xrightarrow{gl_R(j,j')} & \overline{\mathcal{M}}_{g+1,k_{NS},k_R} \\ \big\downarrow{\scriptstyle \pi_{jj'}} & & \\ \overline{\mathcal{M}}_{g,k_{NS},k_R+2} & & \end{array} \tag{4.1.3}$$

$$\begin{array}{ccc} \operatorname{Iso}(\mathcal{P}_j, [\mathrm{i}]_*\mathcal{P}_{j'}) & \xrightarrow{gl_R(j,j')} & \overline{\mathcal{M}}_{g+g',k_{NS}+k'_{NS},k_R+k'_R} \\ \big\downarrow{\scriptstyle \pi_{jj'}} & & \\ \overline{\mathcal{M}}_{g,k_{NS},k_R+1} \times \overline{\mathcal{M}}_{g',k'_{NS},k'_R+1} & & \end{array} \tag{4.1.4}$$

In both cases the gluing maps $gl_R(j,j')$ are embeddings, see Felder, Kazhdan, and Polishchuk 2020, Lemma 8.14.

The reduced space $\left(\overline{\mathcal{M}}_{g,k_{NS},k_R}\right)_{red}$ is the moduli stack of pairs (M_{red}, S) consisting of an algebraic stable curve M_{red} together with a twisted spinor bundle satisfying (3.2.2). We denote the moduli stack of pairs (M_{red}, S) by $\overline{M}^{spin}_{gk_{NS},k_R} = \left(\overline{\mathcal{M}}_{g,k_{NS},k_R}\right)_{red}$. The moduli space $\overline{M}^{spin}_{g,k_{NS},k_R}$ is a bundle over $\overline{M}_{g,k_{NS}+k_R}$ where the map

$$\pi\colon \overline{M}^{spin}_{gk_{NS},k_R} \to \overline{M}_{g,k_{NS}+k_R}$$

forgets the spinor bundle, that is, sends (M_{red}, S) to M_{red}. In the case of genus zero the map π is an isomorphism for all pairs (k_{NS}, k_R).

For further information about moduli superspaces, see Donagi and Witten 2015; Bruzzo and Ruipérez 2019; Codogni and Viviani 2019; Keßler, Sheshmani, and Yau 2020.

4.2 Restricted Operad

In this section we will construct a restricted operad that allows gluing of Neveu–Schwarz tails and (virtual) contraction of Neveu–Schwarz edges only. As the gluing maps along

Neveu–Schwarz punctures are unique, we can proceed similar to the construction of the classical operad o.

Definition 4.2.1. Let G^{r} be the full subcategory of the category of Deligne–Mumford super stacks whose objects are finite products of compact moduli stacks $\overline{\mathcal{M}}_{g,k_{NS},k_R}$.

The category G^{r} is a symmetric monoidal category with respect to the product of stacks.

Definition 4.2.2. Let $\mathsf{SGr}^{\mathsf{st,r}}$ be the *category of stable SUSY graphs with restricted morphism* which has the same objects as $\mathsf{SGr}^{\mathsf{st}}$ but allows only morphisms $h\colon \tau \to \sigma$ such that $h^F|_{F_{\sigma,R}}\colon F_{\sigma,R} \to F_{\tau,R}$ is bijective and under this identification $j_\tau|_{F_{\tau,R}} = j_\sigma|_{F_{\sigma,R}}$.

That is, morphisms in $\mathsf{SGr}^{\mathsf{st,r}}$ do not graft Ramond tails or contract Ramond edges.

In this section we want to prove the following proposition:

Proposition 4.2.3. *The map*

$$\tau \mapsto \prod_{v\in V_\tau} \overline{\mathcal{M}}_{g(v),\#F_{\tau,NS}(v),\#F_{\tau,R}(v)}$$

can be extended to a symmetric monoidal functor $\mathsf{O}^{\mathsf{r}}\colon \mathsf{SGr}^{\mathsf{st,r}} \to \mathsf{G}^{\mathsf{r}}$ *such that* O^{r} *is an operad, that is, sends graftings to isomorphisms.*

As graftings are bijective on vertices, one can already see that if the functor O^{r} exists, it sends graftings to isomorphisms. But in order to construct the functor, we need some preparation:

Definition 4.2.4. Let σ be a SUSY graph. For every vertex v of σ we denote by $\rho_\sigma(v)$ the *Ramond-connected component* of v. The SUSY graph $\rho_\sigma(v)$ consists of all vertices $\{v'\}$ that can be reached from v in one or more steps via Ramond edges together with all flags of σ connected to any of the v'. The map $j_{\rho_\sigma(v)}$ is given by the restriction of $j_{\sigma,R}$ to $V_{\rho_\sigma(v),R}$ and $j_{\rho_\sigma(v),NS} = \mathrm{id}$.

We denote the collection of Ramond-connected components by $R_\sigma = \{\rho_\sigma(v)\}$. Given a SUSY graph σ the *Neveu–Schwarz total grafting* to be the morphism

$$\circ_{\sigma,NS}\colon \coprod_{\rho\in R_\sigma} \rho \to \sigma$$

in $\mathsf{SGr}^{\mathsf{st,r}}$ which is bijective on vertices and flags and grafts all Neveu–Schwarz edges.

Lemma 4.2.5. Given a morphism $h\colon \tau \to \sigma$ in the category $\mathsf{SGr}^{\mathsf{st,r}}$ there is a commutative diagram of the form

$$\begin{array}{ccc} \coprod_{\rho\in R_\sigma} \tau_\rho & \xrightarrow{\coprod h_\rho} & \coprod_{\rho\in R_\sigma} \rho \\ \downarrow{\scriptstyle n} & & \downarrow{\scriptstyle \circ_{\sigma,NS}} \\ \tau & \xrightarrow{\quad h \quad} & \sigma \end{array} \tag{4.2.6}$$

in $\mathsf{SGr}^{\mathsf{st,r}}$ where

- the graphs τ_ρ are defined by

$$V_{\tau_\rho} = \{v \in V_\tau \mid h^V(v) \in V_\rho \subset V_\sigma\} \qquad F_{\tau_\rho} = \{f \in F_\tau \mid h^V(\partial_\tau f) \in V_\rho\}$$
$$\partial_{\tau_\rho} = \partial_\tau|_{V_{\tau_\rho}} \qquad j_{\tau_\rho} = j_\tau|_{\tau_\rho}$$

- the morphism $h_\rho\colon \tau_\rho \to \rho$ is given by

$$h_{\rho,V} = h_V|_{V_{\tau_\rho}} \qquad h_\rho^F = h^F|_{F_\rho} \qquad j_{h_\rho} = j_h|_{F_{\tau_\rho}}$$

  and

- the morphism n is given by the grafting of τ_ρ.

We call the commutative diagram (4.2.6) the *Neveu–Schwarz atomization*. The maps h_ρ are contractions in the category $\mathsf{SGr}^{\mathsf{st,r}}$, that is, they contract Neveu–Schwarz edges.

Now we are prepared for the

Proof of Proposition 4.2.3. By the existence of the Neveu–Schwarz atomization, see Lemma 4.2.5, we can write any morphism $h\colon \tau \to \sigma$ in $\mathsf{SGr}^{\mathsf{st,r}}$ as a composition of a contraction and a grafting in $\mathsf{SGr}^{\mathsf{st,r}}$.

If h is a grafting, it is bijective on the vertices and flags and hence the we define

$$\mathsf{O}^{\mathsf{r}}(h) = \mathrm{id}\colon \mathsf{O}^{\mathsf{r}} = \prod_{v\in V_\tau} \overline{\mathcal{M}}_{g(v),\#F_{\tau,NS}(v),\#F_{\tau,R}(v)} \to \mathsf{O}^{\mathsf{r}} = \prod_{w\in V_\sigma} \overline{\mathcal{M}}_{g(w),\#F_{\sigma,NS}(w),\#F_{\sigma,R}(w)}.$$

If h is a contraction, we may without loss of generality assume that h contracts a single edge because any contraction can be written as a composition of contractions of single edges. This single contracted edge can be either an edge connecting two different vertices or a loop.

If h contracts the edge e, consisting of the flags f_1 and f_2, connecting the distinct vertices v_1 and $v_2 \in V_\tau$, both v_1 and v_2 are mapped by h_V to the same $w_1 \in V_\sigma$. It holds $g(w_1) = g(v_1) + g(v_2)$ as well as $\#F_{\sigma,NS}(w_1) = \#F_{\tau,NS}(v_1) + \#F_{\tau,NS}(v_2) - 2$ and $\#F_{\sigma,R}(w_1) = \#F_{\tau,R}(v_1) + \#F_{\tau,R}(v_2)$. We consider the gluing map

$$\begin{aligned} gl_{NS}(f_1,f_2)\colon \overline{\mathcal{M}}_{g_\tau(v_1),\#F_{\tau,NS}(v_1),\#F_{\tau,R}(v_1)} \times \overline{\mathcal{M}}_{g_\tau(v_2),\#F_{\tau,NS}(v_2),\#F_{\tau,R}(v_2)}& \\ \to \overline{\mathcal{M}}_{g_\sigma(w_1),\#F_{\sigma,NS}(w_1),\#F_{\sigma,R}(w_1)}&. \end{aligned}$$

see Equation (4.1.1). The restriction of h_V to $V_\tau \setminus \{v_1, v_2\} \to V_\sigma \setminus \{w\}$ and the restriction of h^F to

$$F_\sigma \setminus \{f \mid \partial_\sigma f = w\} \to F_\tau \setminus \{f|\partial_\tau f \neq v_1, v_2\}$$

is bijective. Hence we define

$$\begin{aligned} \mathsf{O}^{\mathsf{r}}(h) = \mathrm{id} \times gl_{NS}(f_1,f_2)\colon \prod_{v\in V_\tau\setminus\{v_1,v_2\}} &\overline{\mathcal{M}}_{g_\tau(v),\#F_{\tau,NS}(v),\#F_{\tau,R}(v)} \\ \times \overline{\mathcal{M}}_{g_\tau(v_1),\#F_{\tau,NS}(v_1),\#F_{\tau,R}(v_1)} &\times \overline{\mathcal{M}}_{g_\tau(v_2),\#F_{\tau,NS}(v_2),\#F_{\tau,R}(v_2)} \\ \to \prod_{w\in V_\sigma\setminus\{w_1\}} \overline{\mathcal{M}}_{g_\sigma(w),\#F_{\sigma,NS}(w),\#F_{\sigma,R}(w)} &\times \overline{\mathcal{M}}_{g_\sigma(w_1),\#F_{\sigma,NS}(w_1),\#F_{\sigma,R}(w_1)}. \end{aligned}$$

If h contracts a loop e at the vertex $v_1 \in V_\tau$ consisting of the flags f_1 and f_2, the map h_V is bijective and the restriction of h^F to

$$F_\sigma \setminus \{f \mid \partial_\sigma f = w_1\} \to F_\tau \setminus \{f \mid \partial_\tau f = v_1\}$$

is bijective and it holds $g_\sigma(w_1) = g_\tau(v_1) + 1$ as well as $\#F_{\sigma,NS}(w_1) = \#F_{\tau,NS}(v_1) - 2$ and $\#F_{\sigma,R}(w_1) = \#F_{\tau,R}(v_1)$. Using the gluing map

$$gl_{NS}(f_1, f_2)\colon \overline{\mathcal{M}}_{g_\tau(v_1),\#F_{\tau,NS}(v_1),\#F_{\tau,R}(v_1)} \to \overline{\mathcal{M}}_{g_\sigma(w_1),\#F_{\sigma,NS}(w_1),\#F_{\sigma,R}(w_1)}$$

from Equation (4.1.2) we define

$$\begin{aligned}\mathsf{O}^{\mathsf{r}}(h) = \mathrm{id} \times gl(f_1, f_2)\colon &\prod_{v \in V_\tau \setminus \{v_1\}} \overline{\mathcal{M}}_{g_\tau(v),\#F_{\tau,NS}(v),\#F_{\tau,R}(v)} \times \overline{\mathcal{M}}_{g_\tau(v_1),\#F_{\tau,NS}(v_1),\#F_{\tau,R}(v_1)} \\ &\to \prod_{w \in V_\sigma \setminus \{w_1\}} \overline{\mathcal{M}}_{g_\sigma(w),\#F_{\sigma,NS}(w),\#F_{\sigma,R}(w)} \times \overline{\mathcal{M}}_{g_\sigma(w_1),\#F_{\sigma,NS}(w_1),\#F_{\sigma,R}(w_1)}.\end{aligned}$$

With those definitions O^{r} is obviously a symmetric monoidal functor that sends graftings to isomorphisms. □

4.3 Operad with Ramond gluing data

In the construction of the operad O^{r} we had to restrict to the category $\mathsf{SGr}^{\mathsf{st,r}}$ instead of the category $\mathsf{SGr}^{\mathsf{st}}$ effectively forbidding grafting of Ramond tails and the contraction of Ramond edges. This was needed because the gluing of Ramond punctures is not unique and but rather an additional parameter is needed for the identification of the Ramond nodes with each other.

We now want to define an operad $\mathsf{O}\colon \mathsf{SGr}^{\mathsf{st}} \to \mathsf{G}$ which also encodes grafting of Ramond tails and contractions of Ramond edges. Heuristically, we will achieve this by adding the gluing diagrams (4.1.3) and (4.1.4) as morphisms to G.

Definition 4.3.1. Let M and M' be objects of G^{r}. We say that a triangle of the form

$$\begin{array}{ccc} & P & \\ {\scriptstyle \pi}\swarrow & & \searrow{\scriptstyle f} \\ M & & M' \end{array}$$

is a weak map from M to M' if P is a fiber bundle over M. Weak maps can be composed in an obvious way that is also associative.

Let G be the category whose objects are objects of G^{r}, that is products of moduli stacks of punctured SUSY curves. Morphisms in G are weak maps in the above sense.

One can verify that the category G is a symmetric monoidal category with respect to the product of stacks.

We now prove the following theorem:

Theorem 4.3.2. *The map*

$$\tau \mapsto \prod_{v\in V_\tau} \overline{\mathcal{M}}_{g(v),\#F_{\tau,NS}(v),\#F_{\tau,R}(v)}$$

can be extended to a symmetric monoidal functor $\mathsf{O}\colon \mathsf{SGr}^{\mathsf{st}} \to \mathsf{G}$ *such that* O *is an operad, that is, sends graftings to isomorphisms.*

Proof. As argued in the proof of Proposition 4.2.3, every morphism of SUSY graphs can be decomposed into graftings and contractions. The case of graftings and contractions of Neveu–Schwarz edges has been treated there. Here, it remains to treat the case of contraction of a Ramond loop and contraction of a Ramond edge between different vertices. Both cases yield a weak morphism in G.

Let us first treat the case of a contraction $h\colon \tau \to \sigma$ of a Ramond edge e connecting the distinct vertices v_1 and $v_2 \in V_2$. Both v_1 and v_2 are mapped to the same vertex $w_1 \in V_\sigma$ and we denote the two flags of e by f_1 and f_2. Consider the projection

$$p_e\colon \mathsf{O}(\tau) \to \overline{\mathcal{M}}_{g_\tau(v_1),\#F_{\tau,NS}(v_1),\#F_{\tau,R}(v_1)} \times \overline{\mathcal{M}}_{g_\tau(v_2),\#F_{\tau,NS}(v_2),\#F_{\tau,R}(v_2)}$$

and the gluing map

$$\begin{array}{ccc} \operatorname{Iso}(\mathcal{P}_{f_1}, [\mathrm{i}]_*\mathcal{P}_{f_2}) & \xrightarrow{gl_R(f_1,f_2)} & \overline{\mathcal{M}}_{g_\sigma(w_1),\#F_{\sigma,NS}(w_1),\#F_{\sigma,R}(w_1)} \\ \Big\downarrow{\scriptstyle \pi_e} & & \\ \overline{\mathcal{M}}_{g_\tau(v_1),\#F_{\tau,NS}(v_1),\#F_{\tau,R}(v_1)} \times \overline{\mathcal{M}}_{g_\tau(v_2),\#F_{\tau,NS}(v_2),\#F_{\tau,R}(v_2)} & & \end{array}$$

where $g(w_1) = g(v_1) + g(v_2)$ as well as $\#F_{\sigma,NS}(w_1) = \#F_{\tau,NS}(v_1) + \#F_{\tau,NS}(v_2)$ and $\#F_{\sigma,R}(w_1) = \#F_{\tau,R}(v_1) + \#F_{\tau,R}(v_2) - 2$. Then $\mathsf{O}(h)$ is given by the weak map

$$\begin{array}{ccc} p_e^* \operatorname{Iso}(\mathcal{P}_{f_1}, [\mathrm{i}]_*\mathcal{P}_{f_2}) & \xrightarrow{(\mathrm{id},gl_R(f_1,f_2))} & \mathsf{O}(\sigma) \\ \Big\downarrow & & \\ \mathsf{O}(\tau) & & \end{array}$$

Similarly, if $h\colon \tau \to \sigma$ is the contraction of the loop e at the vertex v_1, consider the projection

$$p_{v_1}\colon \mathsf{O}(\tau) \to \overline{\mathcal{M}}_{g_\tau(v_1),\#F_{\tau,NS}(v_1),\#F_{\tau,R}(v_1)}$$

and the gluing map

$$\begin{array}{ccc} \operatorname{Iso}(\mathcal{P}_{f_1}, [\mathrm{i}]_*\mathcal{P}_{f_2}) & \xrightarrow{gl_R(f_1,f_2)} & \overline{\mathcal{M}}_{g_\sigma(w_1),\#F_{\sigma,NS}(w_1),\#F_{\sigma,R}(w_1)} \\ \Big\downarrow{\scriptstyle \pi_e} & & \\ \overline{\mathcal{M}}_{g_\tau(v_1),\#F_{\tau,NS}(v_1),\#F_{\tau,R}(v_1)} & & \end{array}$$

where $g_\sigma(w_1) = g_\tau(v_1) + 1$ as well as $\#F_{\sigma,NS}(w_1) = \#F_{\tau,NS}(v_1)$ and $\#F_{\sigma,R}(w_1) = \#F_{\tau,R}(v_1) - 2$. Then $\mathsf{O}(h)$ is given by the weak map

$$\begin{array}{ccc} p_{v_1}^* \operatorname{Iso}(\mathcal{P}_{f_1}, [\mathrm{i}]_* \mathcal{P}_{f_2}) & \xrightarrow{(\mathrm{id}, gl_R(f_1,f_2))} & \mathsf{O}(\sigma) \\ \downarrow & & \\ \mathsf{O}(\tau) & & \end{array}$$

□

By construction, the restriction of $\mathsf{O}\colon \mathsf{SGr}^{\mathsf{st}} \to \mathsf{G}$ to $\mathsf{SGr}^{\mathsf{st,r}}$ yields the operad O^{r} from Section 4.2.

Let τ be a connected SUSY graph and let $c\colon \tau \to \sigma$ be the full contraction. That is, σ is a corolla of genus $g(\tau)$ with k_{NS} Neveu–Schwarz flags and k_R Ramond flags. Then $\mathsf{O}(c)$ is given by a diagram of the form

$$\begin{array}{ccc} P & \xrightarrow{gl} & \overline{\mathcal{M}}_{g(\tau),k_{NS},k_R} \\ \downarrow{\scriptstyle \pi} & & \\ \prod_{v \in V_\tau} \overline{\mathcal{M}}_{g(v),\#F_{\tau,NS}(v),\#F_{\tau,R}(v)} & & \end{array}$$

The image of gl in $\overline{\mathcal{M}}_{g(\tau),k_{NS},k_R}$ is $\overline{\mathcal{M}}_\tau$, the closed moduli space of SUSY curves of dual graph τ. Here $\overline{\mathcal{M}}_\tau$ is described as product over the moduli spaces of the vertices together with the gluing data contained in P.

We can use the dimension of the bundle P to calculate the dimension of $\overline{\mathcal{M}}_\tau$. The fiber dimension of P is $0|\#E_{\tau,R}$. The base space has the even dimension

$$\sum_{v \in V_\tau} (3g_\tau(v) - 3 + \#F_{\tau,NS}(v) + \#F_{\tau,R})$$

which coincides with the even dimension of P. Using the formula for the genus of the graph and that the number of flags coincides with the number of markings plus twice the numbers of edges

$$\begin{aligned} \sum_{v \in V_\tau} (g_\tau(v) - 1) &= g_\tau - 1 - \#E_\tau \\ \sum_{v \in V_\tau} \#F_{\tau,NS}(v) &= \#F_{\tau,NS} = k_{NS} + 2\#E_{\tau,NS} \\ \sum_{v \in V_\tau} \#F_{\tau,R}(v) &= \#F_{\tau,R} = k_R + 2\#E_{\tau,R} \end{aligned}$$

Hence for the even dimension of P we have

$$\begin{aligned} d_e(P) &= \sum_{v \in V_\tau} (3g_\tau(v) - 3 + \#F_{\tau,NS}(v) + \#F_{\tau,R}) \\ &= 3g_\tau - 3 - 3\#E_\tau + k_{NS} + 2\#E_{\tau,NS} + k_R + 2\#E_{\tau,R} \\ &= 3g_\tau - 3 + k_{NS} + k_R - \#E_\tau \end{aligned}$$

Similarly for the odd dimension of P, taking into account the fiber dimension

$$\begin{aligned} d_o(P) &= \sum_{v\in V_\tau}\left(2g_\tau(v) - 2 + \#F_{\tau,NS}(v) + \frac{1}{2}\#F_{\tau,R}\right) + \#E_{\tau,R} \\ &= 2g_\tau - 2 - 2\#E_\tau + k_{NS} + 2\#E_{\tau,NS} + \frac{1}{2}k_R + \#E_{\tau,R} + \#E_{\tau,R} \\ &= 2g_\tau - 2 + k_{NS} + \frac{1}{2}k_R \end{aligned}$$

As all gluing maps are embeddings, it follows that $\overline{\mathcal{M}}_\tau \subset \overline{\mathcal{M}}_{g_\tau,k_{NS},k_R}$ is a subspace of codimension $\#E_\tau|0$.

The supermodular operad gives rise to another operad: Let $\mathsf{G}_{\mathsf{red}}$ be the image of G under the reduction functor Red that sends every superspace to its reduced space. That is $\mathsf{G}_{\mathsf{red}}$ is the full subcategory of smooth Deligne–Mumford stacks whose objects are products of $\overline{M}^{spin}_{g,k_{NS},k_R}$. Then $\mathsf{O}_{\mathsf{red}} \coloneqq \mathsf{Red}\circ\mathsf{O}\colon \mathsf{SGr}^{\mathsf{st}} \to \mathsf{G}_{\mathsf{red}}$ is an operad because reductions sends isomorphisms to isomorphism. This gives a new operad for moduli spaces of spin curves.

In the case of genus zero, we have isomorphisms $\overline{M}^{spin}_{g,k_{NS},k_R} \simeq \overline{M}_{g,k_{NS}+k_R}$ and hence an equivalence of categories $\mathsf{G}_{\mathsf{red}} = \mathsf{M}$. Using this equivalence we have an equality of functors

$$\begin{array}{ccc} \mathsf{STr}^{\mathsf{st}} & \xrightarrow{\mathsf{O}} & \mathsf{G} \\ & \searrow^{\mathsf{O}_{\mathsf{red}}} & \downarrow{\scriptstyle \mathsf{Red}} \\ \downarrow{\scriptstyle \mathsf{F}} & & \mathsf{G}_{\mathsf{red}} \\ & & \| \\ \mathsf{MTr}^{\mathsf{st}} & \xrightarrow{\mathsf{o}} & \mathsf{M} \end{array} \qquad (4.3.3)$$

In the case of genus different from zero, there are projection maps $\pi\colon \overline{M}^{spin}_{g,k_{NS},k_R} \to \overline{M}_{g,k_{NS}+k_R}$ such that for every $g \in \mathsf{G}$ we have an element $\pi(g) \in \mathsf{M}$ but this map π is not a functor. Hence, in the case of arbitrary genus the above commutative diagram only holds for objects of the respective categories.

4.4 Permutations of punctures and symmetric operad

The symmetric group S_k acts on algebraic stable curves with k marked points by renumbering the marked points. This action descends to an action on the moduli stack $\overline{M}_{g,k}$ of algebraic stable curves with k marked points. If $k \geq 3$ all automorphisms of $\overline{M}_{0,k}$ are obtained in this way, and if $k \geq 5$ the automorphisms are in bijection with elements of S_n, see Combe and Manin 2019 and references therein.

Here we want to define analogously an action of $S_{k_{NS}} \times S_{k_R}$ on the moduli space of SUSY curves with punctures.

Definition 4.4.1. We define the group of bi-permutations $S_{k_{NS},k_R} = S_{k_{NS}} \times S_{k_R}$. The group S_{k_{NK},k_R} acts on $\mathsf{SGr}_{k_{NS},k_R}$ by renumbering the NS- and R-tails respectively. That is, the element $s = (s_{NS}, s_R) \in S_{k_{NS},k_R}$ acts on the labeling of tails of (k_{NS}, k_R)-SUSY graphs by

$$l_{s\tau,NS} = s_{NS} \circ l_{\tau,NS}, \qquad\qquad l_{s\tau,R} = s_R \circ l_{\tau,R}.$$

Definition 4.4.2. Let $s = (s_{NS}, s_R) \in S_{k_{NS},k_R}$ be a bi-permutation and $(M, \{s_i\}, \mathcal{R}, \delta)$ be a stable super curve with k_{NS} Neveu–Schwarz punctures and k_R Ramond punctures. We denote by $s(M, \{s_i\}, \mathcal{R}, \delta)$ the stable super curve obtained from $(M, \{s_i\}, \mathcal{R}, \delta)$ by renumbering the Neveu–Schwarz punctures by s_{NS} and the Ramond divisors by s_R. This renumbering of the punctures yields an automorphism of the moduli stack $\overline{\mathcal{M}}_{g,k_{NS},k_R}$.

From the definitions, the following is immediate:

Lemma 4.4.3. The operad $\mathsf{O}\colon \mathsf{SGr}^{\mathsf{st}} \to \mathsf{G}$ commutes with the action of the group S_{k_{NS},k_R} of bi-permutations.

Hence, S_{k_{NS},k_R} yields an automorphism of the moduli space $\overline{\mathcal{M}}_\tau$ for any (k_{NS}, k_R)-SUSY graph τ.

Recall that the modular operad $\mathsf{o}\colon \mathsf{MTr}^{\mathsf{st}} \to \mathsf{M}$ is symmetric, that is commutes with the action of the symmetric group S_k. When viewing the element $s \in S_{k_{NS},k_R}$ as an element of $S_{k_{NS}+k_R}$ the S_{k_{NS},k_R}-symmetry of the supermodular operad O is compatible with the $S_{k_{NS}+k_R}$ symmetry of the modular operad o, compare diagram (4.3.3).

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