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Aleksandar Milivojević
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Aleksandar Milivojević
Jonas Stelzig

Max-Planck-Institut für Mathematik
Vivatsgasse 7
53111 Bonn
Germany

Mathematisches Institut der
Ludwig-Maximilians-Universität München
Theresienstrasse 39
80993 München
Germany

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BIGRADED NOTIONS OF FORMALITY AND AEPPLI–BOTT–CHERN–MASSEY PRODUCTS

A. MILIVOJEVIĆ AND J. STELZIG

Abstract. We introduce and study notions of bigraded formality for the algebra of forms on a complex manifold, along with their relation to higher Aeppli–Bott–Chern–Massey products which extend the case of triple products studied by Angella–Tomassini. We show that these Aeppli–Bott–Chern–Massey products on complex manifolds pull back non-trivially to the blow-up along a complex submanifold, as long as their degree is less than the real codimension of the submanifold.

1. Introduction

From an early stage in the development of rational homotopy theory, there have been fruitful interactions with complex geometry: Deligne–Griffiths–Morgan–Sullivan famously proved that all compact Kähler manifolds are rationally formal [DGMS75], [Su77]. Neisendorfer–Taylor adapted the notions of models and formality to the holomorphic category [NT78] while giving some preference to the antiholomorphic differential $\bar{\partial}$ as necessitated by analogy to a singly-graded theory. In this article, we continue this adaptation by introducing notions of formality (and obstructions thereof) for bigraded bidifferential algebras which are symmetric in the two differentials.[1] This yields new complex-geometric invariants with a homotopy-theoretic flavor.

Two genuinely new cohomological invariants that arise when passing from simple complexes to double complexes – the Bott–Chern cohomology, forming an algebra, and the Aeppli cohomology, which is a module over the former – allow one to consider new, symmetric, Massey-like triple products, introduced by Angella–Tomassini in [AT15]. Generalizing these in a different direction than that taken by Tardini [Ta17, §4.4], we identify these Aeppli–Bott–Chern–Massey (ABC–Massey) triple products as one member of a sequence of higher ABC–Massey products, employing a relevant chain complex studied by Schweitzer and guided by a general framework for defining Massey–like products developed by Massey himself in the 1950's. In this framework, the ABC–Massey triple products studied in [AT15] can be thought of as associated to a 1–simplex, while the quadruple, quintuple, etc. products are associated to higher-dimensional simplices.

We then identify two homotopy–theoretic notions of formality for bigraded bidifferential algebras, one stronger than the other, which are obstructed by the presence of (some of) these ABC–Massey products. The relations between these, a metric–dependent formality condition akin to geometric formality, and ordinary (de Rham) formality are discussed through examples. We ask whether compact Kähler manifolds are formal under either of the two homotopy–theoretic notions of bigraded formality.

Babenko–Taimanov [BT00] showed that Massey products on symplectic manifolds are preserved under symplectic blow-up along a submanifold, as long as their degree is less than

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[1]The notion of models in this setup will be developed further in a forthcoming article by the second-named author.

twice the real codimension of the submanifold. We extend this to complex blow-ups and ABC–Massey products.

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2. Preliminaries

One of the basic objects we consider are double complexes, i.e. bigraded complex vector spaces with differentials, suggestively denoted ∂ and $\overline{\partial}$, of bidegree $(1,0)$ and $(0,1)$ respectively, such that $(\partial+\overline{\partial})^2=0$. We recall that (bounded) double complexes admit direct sum decompositions into well-understood indecomposable subcomplexes ("squares and zigzags"), see [KQ20], [Ste21].

Definition 2.1. A map of double complexes $\varphi : A \longrightarrow B$ is called an E_1–isomorphism if it induces an isomorphism in row and column cohomology.

Remark 2.2. In general, this notion of E_1–isomorphism is a stronger condition than inducing an isomorphism in column ("Dolbeault") cohomology only (see the notion of E_0-*quasi-isomorphism* in [CSLW20]). However, if both double complexes are equipped with a real structure, i.e. an antilinear involution σ such that $\sigma A^{p,q} = A^{q,p}$ and $\sigma\partial\sigma = \overline{\partial}$, and we consider maps compatible with this structure ($\sigma\varphi = \varphi\sigma$), then the condition of being an E_1–isomorphism is the same as inducing an isomorphism in column cohomology only. In particular, this applies to the case of $A = A_X$, $B = A_Y$ being the double complexes of forms on complex manifolds X, Y, and $\varphi = f^*$ induced by a holomorphic map $Y \xrightarrow{f} X$, or to the inclusion of forms $A_X^G \subseteq A_X$ invariant under a group acting by biholomorphisms.

Definition 2.3. A cohomological functor is a linear functor from the category of double complexes to the category of vector spaces which sends squares to the zero vector space.

Proposition 2.4. Let $f : A \to B$ be an E_1–isomorphism of bounded double complexes.

(1) The induced map $f^{\otimes n} : A^{\otimes n} \to B^{\otimes n}$ is an E_1–isomorphism.
(2) For any cohomological functor H, the induced map $H(f)$ is an isomorphism.

Proof. The first statement follows from the Künneth formula. The second is a special case of [Ste21, Prop. 12]. □

Remark 2.5. In particular, E_1–isomorphisms induce isomorphisms in Bott–Chern and Aeppli cohomology. Recall that the Bott–Chern cohomology H_{BC} and Aeppli cohomology H_A are given by

$$H_{BC} = \frac{\ker\partial \cap \ker\overline{\partial}}{\operatorname{im}\partial\overline{\partial}}, \quad H_A = \frac{\ker\partial\overline{\partial}}{\operatorname{im}\partial + \operatorname{im}\overline{\partial}}.$$

It will be shown in [Ste22] that, conversely, any map inducing an isomorphism in Bott–Chern and Aeppli cohomology is an E_1–isomorphism. Along the lines of Remark 2.2, a map induced by a holomorphic map of *compact* complex manifolds that induces an isomorphism on Bott–Chern cohomology automatically induces an isomorphism on Aeppli cohomology (and vice versa) by Serre duality [S07], and is hence an E_1–isomorphism.

The forms on a complex manifold have the additional structure of a graded–commutative product, where the graded–commutativity takes into consideration only the total degree, together with the differentials ∂ and $\overline{\partial}$ being (graded, again with respect to the total grading) derivations. We refer to such a structure as a *(graded–)commutative bigraded bidifferential algebra*, or **cbba** for short. A map of cbba's whose underlying map of double complexes is an E_1–isomorphism will be called a **weak equivalence** (this model–categorical terminology will be justified in [Ste22]).

There are augmented versions of all the above objects, and of Proposition 2.4; both will be relevant in the following sections.

3. Aeppli–Bott–Chern higher products

Recall the Aeppli–Bott–Chern–Massey (ABC–Massey) triple product, as defined in [AT15, Definition 2.1]. For Bott–Chern cohomology classes $a, b, c \in H_{BC}(X)$ such that $ab = bc = 0$, take representatives α, β, γ for a, b, c respectively, and choose forms x, y such that $\partial\overline{\partial}x = \alpha\beta$ and $\partial\overline{\partial}y = \beta\gamma$. Then the triple product is the coset in $H_A(X)/(aH_A(X) + cH_A(X))$ corresponding to the Aeppli cohomology class $[\alpha y - x\gamma] \in H_A(X)$. Note the different sign convention than in [AT15]; the results of loc. cit. carry through verbatim for this different sign convention.

We adapt Massey's spectral sequence construction of triple and higher order Massey products to the setting of complex manifolds. This construction is basepoint-dependent, and hence will give invariants of pointed complex manifolds. However, the construction of the Massey product as a differential in a certain spectral sequence reviewed below, informs one how to define the usual, basepoint-independent, "ad hoc" Massey products. In either case, we recover the ordinary and Dolbeault–Massey products [CT15], along with ABC–Massey triple products (see e.g. [TT14], [AT15]). This allows us to define higher ABC–Massey products in the spectral sequence setup. We give an explicit definition and formula for ad hoc quadruple ABC–Massey products, which can in principle be extended to quintuple and higher products. Generally the spectral sequence Massey products have a larger indeterminacy but also a larger domain of definition than the ad hoc ones.

3.1. Eilenberg–Moore spectral sequence.

We fix an augmented abstract simplicial complex K, i.e. a collection of subsets (including the empty set) of some fixed set such that for every $\sigma \in K$ and $\tau \subseteq \sigma$, we also have $\tau \in K$. We will later only consider the case that K is the standard simplex, i.e. $K = \Delta^n$ = all subsets of $\{0, ..., n\}$, but the construction works in this greater generality and gives, in principle, additional invariants.

We further fix a linear functor S from double complexes to simple complexes such that its composition with taking cohomology is a cohomological functor. On a first reading, the reader may want to assume that this associates to a double complex A its total complex. This choice of A will recover the Massey products as presented in [M58, §§3–4].

Now let A be an augmented cbba and write $A^+ = \ker(A \to \mathbb{C})$ for its augmentation ideal. This is a cbba without unit. The principal example we have in mind is $A = A_X$ for a complex manifold and the augmentation given by the restriction of forms to some basepoint $x \in X$. We may now form a new triple complex $C^{*,*}_*(A, K)$ as follows: For every fixed $p \in \mathbb{Z}$, define

$$C_p(A,K) := \bigoplus_{\sigma\in K, |\sigma|=p} \underbrace{A^+ \otimes ... \otimes A^+}_{p+2 \text{ times}} \tag{1}$$

(where we consider the empty set to have cardinality -1). Each $C_p(A, K)$ inherits the structure of a double complex from A, by using the usual induced differentials and gradings on tensor products and direct sums of double complexes. This gives the upper grading on $C^{*,*}_*(A, K)$. The differential δ in the direction of the lower grading is given as follows: Denote the summand belonging to a given p-simplex σ by $C_p(A,K)_\sigma$. If σ_i is the face obtained by omitting the i-th vertex, then on elementary tensors in $C_p(A,K)_\sigma$, one sets

$$\delta(a_0 \otimes ... \otimes a_n) = \sum_{\sigma_i \in K} (-1)^i a_0 \otimes ... \otimes a_i \cdot a_{i+1} \otimes ... \otimes a_n$$

and extends linearly. This defines a map of double complexes $C_p(A,K) \to C_{p-1}(A,K)$ and satisfies $\delta^2 = 0$.

Now we may apply S to each $C_p^{*,*}(A,K)$ to obtain a simple complex $C_p^*(A,K,S)$ and, consequently, a double complex $C_*^*(A,K,S)$ (with commuting differentials instead of anti-commuting ones). Note that the "horizontal" differential δ is of degree -1 and the vertical differential is of degree 1.

Definition 3.1. The **Eilenberg–Moore spectral sequence** $\{EM(A,K,S)_r^{*,*}, d_r\}_{r\geq 1}$ is the spectral sequence associated with the filtration of $C_*^*(A,K,S)$ by horizontal degree. Its differentials are of degree $|d_r| = (-r, -r+1)$ and the spaces on the first page are given by

$$EM(A,K,S)_1^{p,q} = H^q(C_p^*(X,K,S)) = \bigoplus_{\sigma\in K, |\sigma|=p} H^q(S(A^+ \otimes \cdots \otimes A^+)).$$

with differential d_1 induced by δ.

Remark 3.2. By construction, this spectral sequence is functorial for maps of augmented cbba's, and by (the augmented version of) Proposition 2.4, a weak equivalence of augmented cbba's gives rise to isomorphic spectral sequences.

Example 3.3. ([M58]) Let S be the total-complex functor and $K = \Delta^n$ the standard simplex. Then for $n = 0$, The first page looks as follows:

$$H(A_{tot}^+) \xleftarrow{d_1} H(A_{tot}^+) \otimes H(A_{tot}^+),$$

where we suppress the vertical grading in the notation and the differential is given by multiplication. For $n = 1$, we obtain as first page

$$H(A_{tot}^+) \xleftarrow{d_1} H(A_{tot}^+)^{\otimes 2} \oplus H(A_{tot}^+)^{\otimes 2} \xleftarrow{d_1} H(A_{tot}^+)^{\otimes 3},$$

where the first map is given by $(a\otimes b, c\otimes d) \mapsto ab + cd$ and the second one by $(a\otimes b\otimes c) \mapsto (ab\otimes c, -a\otimes bc)$. In particular, for elementary tensors $[\alpha]\otimes[\beta]\otimes[\gamma]$ with $\alpha\beta = dx$, $\beta\gamma = dy$ (and hence $d_1([\alpha]\otimes[\beta]\otimes[\gamma]) = 0$, the second differential $d_2([\alpha]\otimes[\beta]\otimes[\gamma]) = [x\gamma - (-1)^{|\alpha|}\alpha y]$ is given by the formula for the ordinary triple Massey product. In general, for arbitrary n, the longest possible differential will (on elementary tensors) be given by the formula for $n+2$–fold Massey products.

Motivated by this, we define:

Definition 3.4. Let $n \geq 2$. An n–**fold Massey product in the spectral sequence sense** (with respect to S) in the augmented algebra A is an element in the image of the map

$$d_{n-1} : EM(A, \Delta^{n-2}, S)_{n-1}^{n-2,*} \to EM(A, \Delta^{n-2}, S)_{n-1}^{-1,*-n+2}.$$

Remark 3.5. Note that by definition, such a Massey product is a subset of $H(S(A^+))$ (namely, a coset for the space generated by the images of all lower page differentials). By Remark 3.2, Massey products can be pulled back and are invariants of a cbba up to weak equivalence.

Example 3.6. Forgetting the ∂-differential in A, i.e. taking S to be the column complex functor, we obtain a spectral sequence version of the Dolbeault–Massey products [CT15] (which are obtained by the corresponding ad hoc procedure, and are the natural Massey products in Neisendorfer–Taylor's theory [NT78]).

As a more substantial example, we will now generalize ABC–Massey triple products to arbitrary length. Our generalization will be of a different nature than the one considered by Tardini [Ta17, §4.4], where an odd number of Bott–Chern classes is taken in to produce an Aeppli class.

To this end, we recall [S07, 4.b] the Schweitzer complex $S_{p,q}(A)$ associated to a double complex A and a fixed bidegree (p,q), which presents $H_A^{p-1,q-1}$ and $H_{BC}^{p,q}$ as the cohomology of a singly-graded complex. The complex is given by

$$\xrightarrow{d} A^{p-2,q-1} \oplus A^{p-1,q-2} \xrightarrow{d} A^{p-1,q-1} \xrightarrow{\partial\bar{\partial}} A^{p,q} \xrightarrow{d} A^{p,q+1} \oplus A^{p+1,q} \xrightarrow{d}$$

where d denotes the map induced by restricting or projecting the de Rham differential in the obvious way.

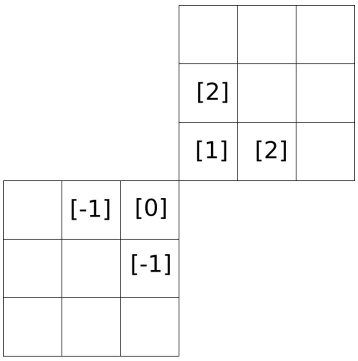

FIGURE 1. The indices in the Schweitzer complex.

Note that the cohomology of this complex at the $A^{p,q}$ entry is precisely $H_{BC}^{p,q}(A)$, and at $A^{p-1,q-1}$ it is $H_A^{p-1,q-1}(A)$. Indexing the entries so that $A^{p-1,q-1}$ is at index 0 and $A^{p,q}$ is at index 1, let us denote by $H^i_{S_{p,q}}(A)$ the cohomology at index i; in this convention, $H_A^{p-1,q-1}(A) = H^0_{S_{p,q}}(A)$, $H_{BC}^{p,q}(A) = H^1_{S_{p,q}}(A)$ and

$$H^{-1}_{S_{p,q}}(A) = \frac{\ker\left(\mathrm{pr}\circ d : A^{p-2,q-1} \oplus A^{p-1,q-2} \longrightarrow A^{p-1,q-1}\right)}{\mathrm{im}\left(\mathrm{pr}\circ d : A^{p-3,q-1} \oplus A^{p-2,q-2} \oplus A^{p-1,q-3} \longrightarrow A^{p-2,q-1} \oplus A^{p-1,q-2}\right)}.$$

Let us explain how triple ABC–Massey products arise in this setup. The part of the first page of the Eilenberg–Moore spectral sequence $EM(A, \Delta^1, S_{p,q})$ relevant for us looks as follows:

$$
\begin{array}{ccccc}
H_{BC}^{p,q}(A^+) & \longleftarrow & H_{BC}^{p,q}((A^+)^{\otimes 2}) \oplus H_{BC}^{p,q}((A^+)^{\otimes 2}) & \longleftarrow & H_{BC}^{p,q}((A^+)^{\otimes 3}) \\
 & & & & \\
H_A^{p-1,q-1}(A^+) & \longleftarrow & H_A^{p-1,q-1}((A^+)^{\otimes 2}) \oplus H_A^{p-1,q-1}((A^+)^{\otimes 2}) & \longleftarrow & H_A^{p-1,q-1}((A^+)^{\otimes 3})
\end{array}
$$

Now pick three Bott–Chern classes a, b, c whose degrees sum up to (p, q). There is a natural map $(H_{BC}(A^+))^{\otimes 3} \to H_{BC}((A^+)^{\otimes 3})$, so their tensor product defines an element in $H_{BC}^{p,q}((A^+)^{\otimes 3})$. If $ab = bc = 0$, this element will be d_1-closed. The second differential, i.e. the triple ABC–Massey product is represented by a zigzag:

$$
\begin{array}{ccccc}
S^1_{p,q}(A^+) & & S^1_{p,q}((A^+)^{\otimes 2}) \oplus S^1_{p,q}((A^+)^{\otimes 2}) & \longleftarrow & S^1_{p,q}((A^+)^{\otimes 3}) \\
 & & \uparrow{\scriptstyle \partial\bar\partial} & & \\
S^0_{p,q}(A^+) & \longleftarrow & S^0_{p,q}((A^+)^{\otimes 2}) \oplus S^0_{p,q}((A^+)^{\otimes 2}) & & S^0_{p,q}((A^+)^{\otimes 3})
\end{array}
$$

If $a = [\alpha], b = [\beta], c = [\gamma]$ and $ab = [\partial\bar\partial x], \beta\gamma = \partial\bar\partial y$, such a zigzag is given by

$$
\begin{array}{ccccc}
 & & (\alpha\beta \otimes \gamma, -\alpha \otimes \beta\gamma) & \longleftarrow & \alpha \otimes \beta \otimes \gamma \\
 & & \uparrow & & \\
x\gamma - \alpha y & \longleftarrow & (x \otimes \gamma, -\alpha \otimes y) & &
\end{array}
$$

So $d_2(a \otimes b \otimes c)$ is represented by the usual triple ABC–Massey product.

Now we describe the quadruple ABC–Massey products. They will be represented by a zigzag:

$$
\begin{array}{ccccccc}
S^1_{p,q}(A^+) & & (S^1_{p,q}((A^+)^{\otimes 2})^{\oplus 3} & & (S^1_{p,q}((A^+)^{\otimes 3})^{\oplus 3} & \longleftarrow & S^1_{p,q}((A^+)^{\otimes 4}) \\
 & & & & \uparrow & & \\
S^0_{p,q}(A^+) & & (S^0_{p,q}((A^+)^{\otimes 2})^{\oplus 3} & \longleftarrow & (S^0_{p,q}((A^+)^{\otimes 3})^{\oplus 3} & & S^0_{p,q}((A^+)^{\otimes 4}) \\
 & & \uparrow & & & & \\
S^{-1}_{p,q}(A^+) & \longleftarrow & (S^{-1}_{p,q}((A^+)^{\otimes 2})^{\oplus 3} & & (S^{-1}_{p,q}((A^+)^{\otimes 3})^{\oplus 3} & & S^{-1}_{p,q}((A^+)^{\otimes 4})
\end{array}
$$

If we start with classes $a, b, c, d \in H_{BC}$ of pure bidegree, with representatives $\alpha, \beta, \gamma, \delta$ such that $|\alpha\beta\gamma\delta| = (p, q)$ and assume there are pure bidegree elements $x, y, z, \eta, \eta', \xi, \xi'$ such that

$$
\partial\bar\partial x = \alpha\beta,\ \partial\bar\partial y = \beta\gamma,\ \partial\bar\partial z = \gamma\delta \text{ and } x\gamma - \alpha y = \partial\eta + \bar\partial\eta',\ y\delta - \beta z = \partial\xi + \bar\partial\xi' \tag{2}
$$

such a zigzag may be represented as

$$
\begin{array}{ccccccc}
 & & & & \begin{pmatrix} \alpha\beta \otimes \gamma \otimes \delta \\ -\alpha \otimes \beta\gamma \otimes \delta \\ \alpha \otimes \beta \otimes \gamma\delta \end{pmatrix} & \longleftarrow & \alpha \otimes \beta \otimes \gamma \otimes \delta \\
 & & & & \uparrow & & \\
 & & \begin{pmatrix} \alpha \otimes (y\delta - \beta z) \\ \alpha\beta \otimes z - x \otimes \gamma\delta \\ (x\gamma - \alpha y) \otimes \delta \end{pmatrix} & \longleftarrow & \begin{pmatrix} x \otimes \gamma \otimes \delta \\ -\alpha \otimes y \otimes \delta \\ \alpha \otimes \beta \otimes z \end{pmatrix} & & \\
 & & \uparrow & & & & \\
\begin{matrix} (-1)^{|\alpha|}\alpha(\xi + \xi') \\ -(\partial x)z + (-1)^{|x|+1} x\bar\partial z \\ +(\eta + \eta')\delta \end{matrix} & \longleftarrow & \begin{pmatrix} (-1)^{|\alpha|}\alpha \otimes (\xi + \xi') \\ -\left(\partial x \otimes z + (-1)^{|x|} x \otimes \bar\partial z\right) \\ (\eta + \eta') \otimes \delta \end{pmatrix} & & & &
\end{array}
$$

The quadruple ABC–Massey product of a, b, c, d is then represented by the class

$$[(-1)^{|\alpha|}\alpha(\xi + \xi') - (\partial x)z + (-1)^{|x|+1}x\bar{\partial}z + (\eta + \eta')\delta]$$

in (a quotient of) $H^{-1}_{S_{p,q}}(A^+)$. One can continue this construction to obtain formulas for the "higher" ABC–Massey products.

Remark 3.7. Note that we could have taken our initial data in a different cohomology group of the Schweitzer complex of a tensor power of A^+, beside Bott–Chern cohomology. Hence we in fact obtain a doubly–indexed family of Massey–like products in the spectral sequence sense; one index for the "length" of the product (that we may think of as the number of inputs), and one index for the cohomology group of the Schweitzer complex our inital data lives in.

Three somewhat distinct behaviors emerge among the doubly–indexed family of products in the spectral sequence sense: The upper "half" of the Schweitzer complex (i.e. with index ≥ 1) has a map of complexes to the total de Rham complex, while the lower half (i.e. with index ≤ 0) receives a map from the total de Rham complex. Consequently, if we start in high degree in the upper half of the Schweitzer complex and form sufficiently short Massey products, these products map to the ordinary de Rham Massey products (but they live in finer groups). Similarly, we may map de Rham Massey products to ABC–Massey products starting in the lower part of the Schweitzer complex. On the other hand, those products that cross the $\partial\bar{\partial}$-"bottleneck" in the Schweitzer complex, i.e. with input in the upper half and output in the lower half, are genuinely new phenomena that do not seem to be related in any straightforward way to ordinary Massey products.

We also remark that one can map Bott–Chern classes to the cohomology at any index in the appropriate Schweitzer complex, and hence, if one prefers to start with a pure tensor of Bott-Chern classes, this is possible at any stage.

3.2. **Ad hoc Massey products.** The ABC–Massey products introduced above are associated with an augmented cbba. This is in contrast with the more common ad hoc definitions of (triple ABC–) Massey products, which do not need an augmentation, as for instance in [M58, Section 2], [K66] (resp. [AT15], [Ta17]). The ad hoc version always starts with a pure tensor of classes as input and outputs a subset of cohomology, which, apart from the triple product case, is generally not the coset of a linear subspace. The product is then said to vanish if zero is contained in this subset.

There is, in principle, a straightforward way of obtaining such an unaugmented ad hoc version from the explicit description of the differentials in the spectral sequence version. Since the amount of necessary notation becomes unwieldy, let us only sketch this for triple and quadruple products, taking our cues from our examples discussed in the previous section and [K66]:

Definition 3.8. Let A be a cbba and $a, b, c, d \in H_{BC}(A)$ classes of pure bidegree with representatives $\alpha, \beta, \gamma, \delta$ and write $(p_\alpha, q_\alpha) = |\alpha|$, etc.

(1) A **defining system** for the ad hoc ABC–Massey quadruple product $\langle a, b, c, d\rangle$ is a collection of elements $x, y, z, \tilde{\eta}, \tilde{\xi}$ such that

$$x \in S^0_{|\alpha\beta|}(A),\ y \in S^0_{|\beta\gamma|}(A),\ z \in S^0_{|\gamma\delta|}(A),$$
$$\tilde{\eta} = \eta + \eta' \in S^{-1}_{|\alpha\beta\gamma|}(A) = A^{|\alpha\beta\gamma|-(2,1)} \oplus A^{|\alpha\beta\gamma|-(1,2)},$$
$$\tilde{\xi} = \xi + \xi' \in S^{-1}_{|\beta\gamma\delta|}(A) = A^{|\beta\gamma\delta|-(2,1)} \oplus A^{|\beta\gamma\delta|-(1,2)},$$

satisfying Equation (2) above.

(2) The **ad hoc quadruple ABC–Massey product** $\langle a, b, c, d\rangle$ is the subset of $H^{-1}_{S_{|\alpha\beta\gamma\delta|}}(A)$ given by the collection of all classes
$$[(-1)^{|\alpha|}\alpha\tilde{\xi} - (\partial x)z + (-1)^{|x|+1}x\bar{\partial}z + \tilde{\eta}\delta]$$
determined by a defining system.

Another choice for the classes in the ad hoc quadruple product would be
$$[(-1)^{|\alpha|}\alpha\tilde{\xi} + (\bar{\partial}x)z + (-1)^{|x|}x\partial z + \tilde{\eta}\delta].$$
Note that the difference between the representatives of this and the previous choice is $d(xz)$, hence the two representatives give the same class in $H^{-1}_{S_{|\alpha\beta\gamma\delta|}}(A)$.

Remark 3.9. As seen above, if one makes the analogous definitions in order to define an ad hoc triple ABC–Massey product, one recovers the ABC–Massey product $\langle a, b, c\rangle$ of [AT15]. One can also pursue defining ad hoc quintuple and higher products, but we do not do so here.

The functoriality and invariance properties in this ad hoc setup need to be proven with more care:

Proposition 3.10. Let $\varphi : A \to B$ be a map of cbba's.
(1) (Functoriality) For $a, b, c \in H_{BC}(A)$, one has $\varphi\langle a, b, c\rangle \subseteq \langle\varphi(a), \varphi(b), \varphi(c)\rangle$.
(2) (Invariance under weak equivalences) If φ is a weak equivalence, $a, b, c \in H_{BC}(A)$, $a', b', c' \in H_{BC}(B)$, there are equalities of sets
$$\varphi\langle a, b, c\rangle = \langle\varphi(a), \varphi(b), \varphi(c)\rangle \subseteq H_A(B)$$
and
$$\varphi^{-1}\langle a', b', c'\rangle = \langle\varphi^{-1}(a'), \varphi^{-1}(b'), \varphi^{-1}(c')\rangle \subseteq H_A(A).$$
In particular, admitting a non-trivial Massey product is invariant under weak equivalences.

Proof. Given any defining system for the $\langle a, b, c\rangle$, its image under φ is a defining system for $\langle\varphi(a), \varphi(b), \varphi(c)\rangle$, hence the functoriality assertion. For the first equality of part (2), we have to show the other inclusion. Let $a = [\alpha], b = [\beta], c = [\gamma]$ and $x, y \in B$ with $\partial\bar{\partial}x = \varphi(\alpha\beta)$ and $\partial\bar{\partial}y = \varphi(\beta\gamma)$, such that $[\varphi(\alpha)y - x\varphi(\gamma)] \in \langle\varphi(a), \varphi(b), \varphi(c)\rangle$. Then we want to find $\varphi(x'), \varphi(y') \in \operatorname{im}\varphi$ such that $\partial\bar{\partial}x' = \alpha\beta$ and $\partial\bar{\partial}y' = \beta\gamma$ such that $[\varphi(\alpha)y - x\varphi(\gamma)] = [\varphi(\alpha y') - \varphi(x'\gamma)] \in \varphi\langle a, b, c\rangle$. First, since $H_{BC}(A) \cong H_{BC}(B)$, we may pick *some* primitives $\partial\bar{\partial}\varphi(\tilde{x}) = \varphi(\alpha\beta)$ and $\partial\bar{\partial}(\tilde{y}) = \varphi(\beta\gamma)$. Now, $\varphi(\tilde{x}) - x$ and $\varphi(\tilde{y}) - y$ are $\partial\bar{\partial}$-closed and therefore define Aeppli-classes. Because φ is a weak equivalence, we may choose elements $\tilde{x}', \tilde{y}' \in A$ such that $[\varphi(\tilde{x}) - x] = [\varphi(\tilde{x}')]$ and $[\varphi(\tilde{y}) - y] = [\varphi(\tilde{y}')]$. Then, setting $x' = \tilde{x}' + \tilde{x}$ and $y' = \tilde{y}' + \tilde{y}$ does the job, as
$$[\varphi(\alpha)y - x\varphi(\gamma)] = [\varphi(\alpha)(y - \varphi(\tilde{y}) + \varphi(\tilde{y})) - (x - \varphi(\tilde{x}) + \varphi(\tilde{x}'))\varphi(\gamma)] = [\varphi(\alpha)\varphi(y') - \varphi(x')\varphi(\gamma)].$$
The second assertion follows from the first since we may write $a' = \varphi(a), b' = \varphi(b), c' = \varphi(c)$ for some $a, b, c \in H(A)$. □

Remark 3.11. The same kind of arguments (with more cumbersome notation) show that Proposition 3.10 remains valid for quadruple ABC–Massey products, for Tardini's [Ta17] Massey products, and also for usual Massey products and quasi-isomorphisms.

Remark 3.12. We would like to emphasize that neither invariant, the Massey products in the spectral sequence sense or the ad hoc sense, is finer than the other. The spectral sequence products have a large indeterminacy (e.g. *any* product of positive-degree cohomology classes will represent a trivial triple product in the spectral sequence sense), but also a larger domain of definition, as the input data is not restricted to pure tensors. One can find a discussion on this (for ordinary Massey products) in [P17].

Example 3.13. We exhibit an example of a non-vanishing ad hoc quadruple ABC–Massey product, i.e. with four Bott–Chern classes as input. Consider a complex nilmanifold with model

$$(\Lambda(x, \bar{x}, y, \bar{y}, z, \bar{z}, w, \bar{w}), dz = xy, dw = xz)\,.$$

That is, the above is the algebra of invariant forms on a complex parallelizable nilmanifold together with its bigrading; the inclusion of this into all forms is a weak equivalence by [Sa76]; see also the work of Console–Fino, Rollenske and others, summarized in [R11].

Consider the Bott–Chern classes represented (uniquely) by $x, x\bar{y}, y\bar{x}$. Note that $x^2 = 0$, $x(x\bar{y}) = 0$, $(x\bar{y})(y\bar{x}) = 0$, and that $(x\bar{y})(y\bar{x}) = \partial\overline{\partial}(z\bar{z})$. The triple ABC–Massey product $\langle x, x\bar{y}, y\bar{x}\rangle$ vanishes, as we have $xz\bar{z} = \partial(w\bar{z})$. The triple product $\langle x, x, x\bar{y}\rangle$ also vanishes, trivially.

The quadruple ABC–Massey product $\langle x, x, x\bar{y}, y\bar{x}\rangle$ is thus well defined, and is represented by the $H^{-1}_{S_{4,2}}$ class $[xw\bar{z}]$. To verify that this quadruple product is indeed non-vanishing, we show that no other choice of primitive elements can yield the trivial class. First of all, notice that for degree reasons, 0 is the only possible choice of $\partial\overline{\partial}$–primitive for x^2 and for $x(x\bar{y})$. A primitive for $(x\bar{y})(y\bar{x})$ can be anything in the affine space

$$z\bar{z} + \operatorname{span}(x\bar{x}, x\bar{y}, x\bar{z}, x\bar{w}, y\bar{x}, y\bar{y}, y\bar{z}, y\bar{w}, z\bar{x}, z\bar{y}, w\bar{x}, w\bar{y})\,.$$

Therefore the possible representatives for $\langle x, x\bar{y}, \bar{x}y\rangle$ lie in the affine space

$$xz\bar{z} + \operatorname{span}(xy\bar{x}, xy\bar{y}, xy\bar{z}, xy\bar{w}, xz\bar{x}, xz\bar{y}, xw\bar{x}, xw\bar{y})\,.$$

Since $\overline{\partial}$ vanishes on $(2,0)$ forms, any representative of the quadruple ABC–Massey product is thus in

$$x\,(w\bar{z} + \operatorname{span}(x\bar{x}, x\bar{y}, x\bar{z}, x\bar{w}, y\bar{x}, y\bar{y}, y\bar{z}, y\bar{w}, z\bar{x}, z\bar{y}, w\bar{x}, w\bar{y}, z\bar{z}, z\bar{w}))\,.$$

No differential from $S^{-2}_{4,2}$ contains a term of $xw\bar{z}$, and hence we are done.

This manifold is a holomorphic analogue of the filiform nilmanifold, which is the simplest example of a nilmanifold with a non-vanishing quadruple Massey product. Indeed, our complex nilmanifold is a holomorphic torus bundle over the Iwasawa bundle, while the filiform nilmanifold is a circle bundle over the Heisenberg nilmanifold. (Here we refer to “the” filiform and Heisenberg nilmanifold, though there are various lattices in the appropriate simply connected nilpotent Lie group one could choose, resulting in different homotopy types; however, the rational homotopy theoretic minimal model is unique.)

Remark 3.14. Our definitions of ABC–Massey products can be adapted to the almost complex setting by using the double complexes $(A_X)_s$ and $(A_X)_q$, associated with an almost complex manifold X in [CPS21], and the ensuing notions of Schweitzer complexes.

4. Bigraded notions of formality

In usual rational homotopy theory, a commutative differential graded algebra (cdga) is called *formal* if it is connected by a chain of quasi-isomorphisms to a cdga with trivial differential (whose underlying algebra is consequently isomorphic to the cohomology of the original cdga). A singly-graded complex additively splits into dots and lines (i.e. zigzags of length two), where the latter do not contribute to cohomology. Hence we may interpret formality of a cdga as the existence of a chain of quasi-isomorphisms to a cdga with either only dots, or (equivalently) with no shapes that do not contribute to cohomology. This motivates the following two notions of formality in the bigraded setting, where we wish to consider the differentials ∂ and $\overline{\partial}$ on equal footing:

Definition 4.1. A cbba A is called:

(1) **weakly formal**, if it is connected by a chain of weak equivalences to a cbba H which satisfies $\partial_H \bar{\partial}_H = 0$.
(2) **strongly formal**, if it is connected by a chain of weak equivalences to a cbba H which satisfies $\partial_H = \bar{\partial}_H = 0$.

We use the same terminology if A is augmented and the weak equivalences preserve augmentations.

A strongly formal cbba is one that is connected by a chain of weak equivalences to a cbba whose underlying complex consists only of dots, while a weakly formal one is connected to a cbba which additively has no squares.

Note that strong formality implies weak formality and the $\partial\bar{\partial}$-lemma (the latter being an (additive) E_1–isomorphism invariant of double complexes). In the definition of strong formality, one may thus take H to be $H_{BC}(A)$. Conversely if A satisfies the $\partial\bar{\partial}$-lemma, weak and strong formality are equivalent.

Remark 4.2. Since we are not assuming the existence of real structures on our cbba's, these notions are a priori unrelated to the notions of *Dolbeault formality* and *strict (Dolbeault) formality* considered in [NT78, p.187], as in both cases the weak equivalences considered (the former being in the category of bigraded algebras with a single, $\bar{\partial}$, differential, and the latter being in the category of cbba's) require only an isomorphism on $\bar{\partial}$-cohomology. If we were to require all considered cbba's to have a real structure, and the maps between them to preserve this structure (e.g. if the chain of weak equivalences were induced by holomorphic maps of complex manifolds), then the notions of strict formality and strong formality would coincide. Without these additional requirements, the two notions are different; see Example 4.9.

As an immediate consequence of Remark 3.5, Proposition 3.10 and Remark 3.11, we have the following, where we mean Massey products in the spectral sequence sense for augmented formality and in the ad hoc sense for unaugmented formality:

Proposition 4.3. On a strongly formal manifold all ABC–Massey products vanish. On a weakly formal manifold, all ABC–Massey products that cross the $\partial\bar{\partial}$-bottleneck (c.f. Remark 3.7) vanish.

In ordinary rational homotopy theory, and in the Dolbeault homotopy theory of [NT78], formality is equivalent to augmented formality for cohomologically connected spaces (e.g. compact complex manifolds in the latter), essentially due to the fact that the construction of a cofibrant model does not involve adding generators in degree 0 (or bidegree $(0,0)$ for the latter). From now on, we will consider only the unaugmented notions of formality, unless explicitly stated otherwise.

Taking metrics into account, recall that a compact Riemannian manifold is called geometrically formal if products of harmonic forms are again harmonic. Considering the Bott–Chern and Aeppli harmonic forms (see [S07, Section 2]), which we denote by $\mathcal{H}_{BC}$ and $\mathcal{H}_A$, respectively, one can ask for the additional condition that the collection of harmonic forms is closed under the differential:

Definition 4.4. A connected compact complex manifold is called **ABC-geometrically formal** if it admits a Hermitian metric such that following two equivalent conditions are satisfied:

(1) The space $\mathcal{H}_A + \mathcal{H}_{BC}$ is closed under multiplication and $\partial, \bar{\partial}$.
(2) There exists a bigraded, bidifferential subalgebra $\mathcal{H} \subseteq A_X$ which is closed under $*$ and satisfies $\partial\bar{\partial} = 0$, such that the inclusion is a weak equivalence.

Let us prove that these two conditions are indeed equivalent: If condition 1 holds, set $\mathcal{H} := \mathcal{H}_A + \mathcal{H}_B$. Since

$$\omega \in \mathcal{H}_{BC} \quad \Leftrightarrow \quad \begin{cases} \partial\omega = 0 \\ \bar\partial\omega = 0 \\ \partial\bar\partial * \omega = 0 \end{cases} \quad \Leftrightarrow \quad *\omega \in \mathcal{H}_A,$$

this space is closed under $*$ and satisfies $\partial\bar\partial = 0$ by definition. It is a bigraded subalgebra by assumption. Since $H_{BC}(\mathcal{H}) = \mathcal{H}_{BC}(X)$ and $H_A(\mathcal{H}) = H_A(X)$, the inclusion is a weak equivalence by Remark 2.5.

Conversely, if condition 2 holds, $\mathcal{H}$ has to be finite dimensional, since in any decomposition into indecomposable complexes, there can be no squares (since $\partial\bar\partial = 0$) and every zigzag has to occur with the same multiplicity as in A_X (where the multiplicities are known to be finite). In particular, $\mathcal{H}$ is a closed subspace and we have an orthogonal projection $A_X \to \mathcal{H}$. Because $\mathcal{H}$ is closed under $*$ and is a bigraded subspace, this projection is a map of (bi)complexes: Indeed, pick any orthonormal basis $\{h_i\}$ for $\mathcal{H}$. Then the projection of any element in $\omega \in A_X$ is written as $\mathrm{pr}_{\mathcal{H}}(\omega) = \sum \langle \omega, h_i\rangle h_i$. Thus:

$$(d \circ \mathrm{pr}_{\mathcal{H}})(\omega) = d\left(\sum \langle \omega, h_i\rangle h_i\right) = \sum_{i,j} \langle \omega, h_i\rangle \langle dh_i, h_j\rangle h_j$$

$$(\mathrm{pr}_{\mathcal{H}} \circ d)(\omega) = \sum_i \langle \omega, d^* h_i\rangle h_i = \sum_{i,j} \langle \omega, h_j\rangle \langle d^* h_i, h_j\rangle h_i$$

and both terms coincide. In particular, this implies that the Bott–Chern and Aeppli Laplacians have block-diagonal form with respect to the splitting $A_X = \mathcal{H} \oplus \mathcal{H}^\perp$ and consequently $\mathcal{H}_{BC} = \mathcal{H}_{BC} \cap \mathcal{H} \oplus \mathcal{H}_{BC} \cap \mathcal{H}^\perp$. Since $H_{BC}(\mathcal{H}) = H_{BC}(X)$, this implies $\mathcal{H}_{BC} \subseteq \mathcal{H}$ and similarly $\mathcal{H}_A \subseteq \mathcal{H}$. On the other hand, if $\omega \in \mathcal{H}$, necessarily $\partial\bar\partial\omega = 0$, i.e. it gives rise to an Aeppli class. The harmonic component ω^A lies in $\mathcal{H}$, so $\omega' = \omega - \omega^A \in \mathcal{H} \cap (\operatorname{im}\partial + \operatorname{im}\bar\partial) = \partial\mathcal{H} + \bar\partial\mathcal{H}$, where for the last equality we use that an element in $\mathcal{H}$ which gives the zero class in $H_A(X)$ has already a primitive in $\mathcal{H}$ as $H_A(\mathcal{H}) \cong H_A(X)$. But then, because $\partial\bar\partial = 0$ in $\mathcal{H}$, we have $\partial\omega' = \bar\partial\omega' = 0$ and $\partial\bar\partial * \omega' = 0$, i.e. $\omega' \in \mathcal{H} \cap \mathcal{H}_{BC}$. This completes the proof.

Remark 4.5. Note that the multiplicative structure was not used in proving the equivalence of the two conditions in Definition 4.4. So we also have the following statement: the space $\mathcal{H}_A + \mathcal{H}_{BC}$ is closed under ∂ and $\bar\partial$ if and only if there exists a bigraded subcomplex $\mathcal{H} \subseteq A_X$ which is closed under $*$ and satisfies $\partial\bar\partial = 0$, such that the inclusion is an (additive) E_1–isomorphism. Note also that this equivalence implies that the subspace in the second condition has to coincide with $\mathcal{H}_{BC} + \mathcal{H}_A$, and in particular must be real.

Remark 4.6. Note that the inclusion $\mathcal{H} \subseteq A_X$ is automatically compatible with the augmentation given by the choice of any basepoint. In particular, ABC-geometric formality implies weak formality in the augmented sense.

Remark 4.7. (Relation to other notions of geometric formality) ABC-geometric formality implies geometric Bott–Chern formality in the sense of Angella–Tomassini [AT15], i.e. the Bott–Chern harmonics form an algebra. Indeed, the product of two Bott–Chern harmonic forms will of course be closed under ∂ and $\bar\partial$, as well as under $(\partial\bar\partial)^*$ by the assumption of ABC-geometric formality.

If we further assume our manifold satisfies the $\partial\bar\partial$–lemma, then ABC-geometric formality implies Dolbeault geometric formality [TT14] and ordinary geometric formality. Indeed, e.g. for the latter, since the Bott–Chern harmonics and Aeppli harmonics coincide, they are closed under $\partial, \bar\partial, \partial^*, \bar\partial^*$ and hence de Rham harmonic, and vice versa.

Example 4.8. We show on the example of the Kodaira–Thurston surface KT that ABC-geometric formality, and hence also weak formality, does not imply the $\partial\overline{\partial}$–lemma nor de Rham formality. A bigraded model for this complex manifold is given by

$$(\Lambda(x, \bar{x}, y, \bar{y}), dy = x\bar{x}),$$

where x and y are in bidegree $(1, 0)$. The inclusion of this cbba into the forms on KT is a weak equivalence.

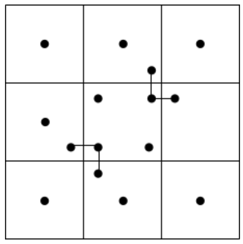

FIGURE 2. The double complex of the Kodaira–Thurston surface.

With respect to the obvious (diagonal) metric, we see that this algebra satisfies the second condition in Definition 4.4. Being a non-trivial nilmanifold, KT is not formal (nor is it Dolbeault formal). A concrete non-vanishing (Dolbeault–) Massey triple product is $\langle x, \bar{x}, \bar{x}\rangle$, represented by $y\bar{x}$.

Example 4.9. We consider the Hopf surface, which is weakly formal, de Rham formal (as the underlying smooth manifold is $S^1 \times S^3$), and does not satisfy the $\partial\overline{\partial}$–lemma (as the first Betti number is odd). A model, i.e. a cbba with a weak equivalence to the cbba of forms, for the Hopf surface is given by [Ste22]

$$\left(\Lambda(x, \bar{x}, y, z, \partial z, \overline{\partial} z), dx = -d\bar{x} = y, \partial\overline{\partial} z = iy^2\right),$$

where x is in bidegree $(1, 0)$ and y, z are in $(1, 1)$. This model maps to

$$\left(\Lambda(x, \bar{x}, y)/(y^2), dx = -d\bar{x} = y\right)$$

by sending $z, \partial z, \overline{\partial} z$ to 0. This map is a weak equivalence and the latter cbba satisfies $\partial\overline{\partial} \equiv 0$, so the Hopf surface is weakly formal. It is also strictly formal, so in particular Dolbeault formal, in the sense of Neisendorfer–Taylor [NT78, p. 197].

Example 4.10. Consider the Calabi–Eckmann complex structure on $S^3 \times S^3$, with a global basis $\{\phi^1, \phi^2, \phi^3\}$ of $(1, 0)$–forms satisfying

$$\begin{aligned} d\phi^1 &= i\phi^1\phi^3 + i\phi^1\overline{\phi^3}, \\ d\phi^2 &= \phi^2\phi^3 - \phi^2\overline{\phi^3}, \\ d\phi^3 &= -i\phi^1\overline{\phi^1} + \phi^2\overline{\phi^2}, \end{aligned}$$

see [TT17, §3]. Note that the inclusion of the algebra generated by these elements and their conjugates into all forms is an injection on $H_{\overline{\partial}}$ since it is a map of algebras satisfying Serre duality, with respect to the invariant volume form $\phi^1\phi^2\phi^3\overline{\phi^1\phi^2\phi^3}$. From knowledge of the Hodge numbers of $S^3 \times S^3$ [B78], the map is also a surjection on $H_{\overline{\partial}}$, and hence, since it preserves real structures (see Remark 2.2), it is a weak equivalence. With respect to the diagonal metric, the subalgebra generated by $\phi^1\overline{\phi^1}, \phi^2\overline{\phi^2}, \phi^3\overline{\phi^3}$ and their ∂ and $\overline{\partial}$-derivatives satisfies the second condition of Definition 4.4, and so in particular this manifold is weakly formal.

As shown in [TT17, §3], there is a small deformation of this complex manifold that carries a non-trivial ABC–Massey triple product. In particular, by Proposition 4.3, weak formality is not stable under (small) deformation.

Example 4.11. We give a non-trivial example of a strongly formal manifold. Consider the full flag manifold $SU(3)/(U(1)\times U(1))$. The details of the calculation to follow will be given in [MW22]. There is a left-invariant complex structure J on $SU(3)$ descending to $SU(3)/(U(1)\times U(1))$, such that in a certain basis $\phi^1,\phi^2,\phi^3,\phi^4$ of left-invariant $(1,0)$–forms on $SU(3)$, we have that the subalgebra of forms that descend to the quotient is generated by

$$\alpha_2 = -\tfrac{i}{2}\phi^2\overline{\phi^2},\ \alpha_2' = -\tfrac{i}{2}\phi^3\overline{\phi^3},\ \alpha_2'' = -\tfrac{i}{2}\phi^4\overline{\phi^4},$$
$$\alpha_3 = -\tfrac{1}{2}\left(\phi^2\phi^4\overline{\phi^3} + \phi^3\overline{\phi^2\phi^4}\right),\ \alpha_3' = \tfrac{i}{2}\left(\phi^3\overline{\phi^2\phi^4} - \phi^2\phi^4\overline{\phi^3}\right)$$

with $d\alpha_2' = -d\alpha_2 = -d\alpha_2'' = \frac{1}{2}\left(\beta+\overline{\beta}\right)$, $d\alpha_3 = 0$, and $d\alpha_3' = 4(\alpha_2\alpha_2'' - \alpha_2\alpha_2' - \alpha_2'\alpha_2'')$, where β denotes the $(2,1)$ form $-\alpha_3 + i\alpha_3'$. Denoting $\gamma = 4i(\alpha_2\alpha_2'' - \alpha_2\alpha_2' - \alpha_2'\alpha_2'')$, the invariant subalgebra is represented by the double complex in Figure 3.

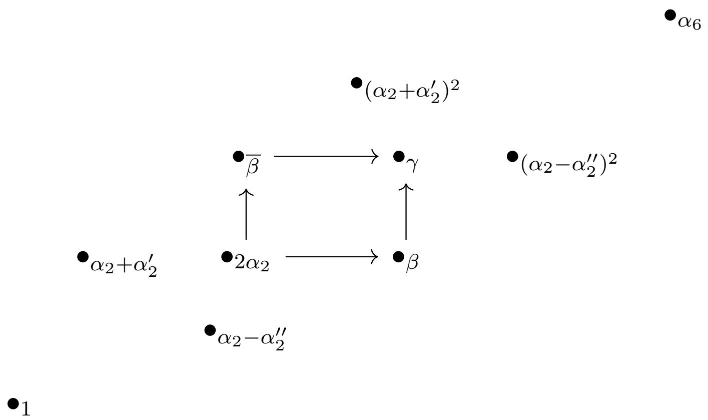

FIGURE 3. The left-invariant double complex for $SU(3)/(U(1)\times U(1))$

We can model this complex by

$$\left(\Lambda(x,y,z,w), \partial\overline{\partial}z = xy - x^2 - y^2, \partial\overline{\partial}w = x^3\right),$$

where x,y,z are in degree $(1,1)$ and w is in $(2,2)$, by mapping

$$x \mapsto \alpha_2 + \alpha_2',\ y \mapsto \alpha_2 - \alpha_2'',\ z \mapsto \alpha_2,\ w \mapsto 0.$$

This map is a weak equivalence. On the other hand, the inclusion of the full complex above into the complex of all forms on the flag manifold is also a weak equivalence. Indeed, $(\alpha_2+\alpha_2') + (\alpha_2 - \alpha_2'')$ yields a left-invariant Kähler form; by a classical result of Chevalley–Eilenberg, the inclusion of the complex into all forms induces an isomorphism on de Rham cohomology, so by the degeneracy of the Frölicher spectral sequence our claim follows.

Hence our manifold is weakly formal. Since it also satisfies the $\partial\overline{\partial}$–lemma, it is equivalently strongly formal. Notice that it is not ABC–geometrically formal with respect to any $SU(3)$-invariant metric, as otherwise the products of its degree two part would span a three–dimensional space.

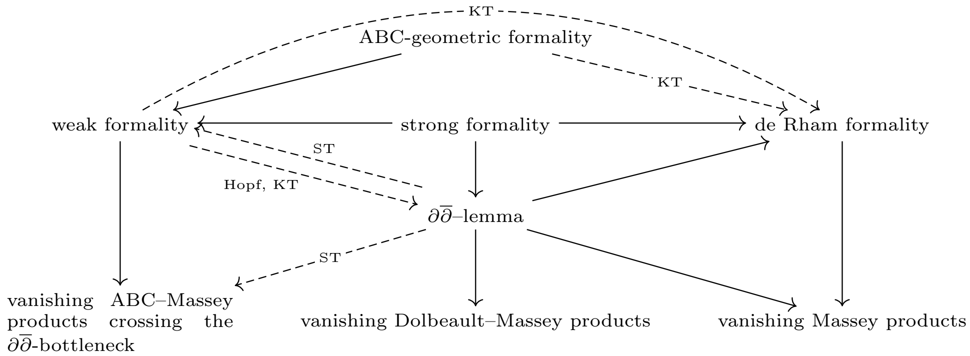

FIGURE 4. Relations between various discussed notions. A dotted arrow means that the source property does not imply the target property, with a counterexample labelling the arrow. KT denotes the Kodaira–Thurston surface, ST the manifold constructed by Sferruzza–Tomassini, and Hopf denotes the Hopf surface.

Example 4.12. Sferruzza and Tomassini have constructed an example of a compact complex threefold which satisfies the $\partial\bar{\partial}$–lemma, yet carries a non-vanishing triple ABC–Massey product [ST22], and hence is not weakly formal. For the convenience of the reader, we give a simple algebraic version of their example: Consider the cbba of left-invariant forms on the Iwasawa manifold:

$$\left(\Lambda(\varphi^1,\varphi^2,\varphi^3,\bar{\varphi}^1,\bar{\varphi}^2,\bar{\varphi}^3), d\varphi^3 = -\varphi^1\varphi^3, d\bar{\varphi}^3 - \bar{\varphi}^1\bar{\varphi}^2\right)$$

Now define A to be the sub-cbba generated by $\varphi^1\bar{\varphi}^1$, $\varphi^2\bar{\varphi}^2$, $\varphi^3\bar{\varphi}^3$, i.e. as a bigraded vector space:

$$\begin{aligned}
A^{0,0} &= \mathbb{C},\\
A^{1,1} &= \operatorname{span}(\varphi^1\bar{\varphi}^1,\ \varphi^2\bar{\varphi}^2,\ \varphi^3\bar{\varphi}^3)\\
A^{2,2} &= \operatorname{span}(\varphi^1\varphi^2\bar{\varphi}^1\bar{\varphi}^2,\ \varphi^2\varphi^3\bar{\varphi}^2\bar{\varphi}^3,\ \varphi^1\varphi^3\bar{\varphi}^1\bar{\varphi}^3)\\
A^{2,1} &= \operatorname{span}(\varphi^1\varphi^2\bar{\varphi}^3) \quad A^{1,2} = \operatorname{span}(\varphi^3\bar{\varphi}^1\bar{\varphi}^2)\\
A^{3,3} &= \operatorname{span}(\varphi^1\varphi^2\varphi^3\bar{\varphi}^1\bar{\varphi}^2\bar{\varphi}^3)
\end{aligned}$$

with all other $A^{p,q} = 0$ and only nontrivial differential $\partial\bar{\partial}\varphi^3\bar{\varphi}^3 = \varphi^1\varphi^2\bar{\varphi}^1\bar{\varphi}^2$. Then A satisfies the $\partial\bar{\partial}$-lemma, but $\langle\varphi^1\bar{\varphi}^1, \varphi^1\bar{\varphi}^1, \varphi^2\bar{\varphi}^2\rangle \neq 0$.

We collect the relations between the various notions of formality in Figure 4.

The existence of the Sferruzza–Tomassini manifold leads us to the following question:

Question 4.13. Are compact Kähler manifolds strongly formal?

Either compact Kähler manifolds are strongly formal, in which case ABC–Massey products can tell the difference between Kähler manifolds and non–Kähler manifolds satisfying the $\partial\bar{\partial}$–lemma, or we have potential invariants for Kähler manifolds.

Since all Kähler nilmanifolds are biholomorphic to tori [BC06, Theorem 2], one could consider looking at Kähler solvmanifolds for a negative answer to the above question. A complicating factor is that all such manifolds are finitely covered by tori [Ar04, Corollary 1], and hence their non-formality could not be detected by a non-vanishing ABC–Massey triple product, see Remark 5.3.

On the positive side, we have the following:

Proposition 4.14. Hermitian symmetric spaces are strongly formal (even ABC-geometrically formal).

Proof. On Hermitian symmetric spaces, the cohomology can be computed from the sub-cbba of invariant forms, but it is known that $d \equiv 0$ on such forms. (This is the same proof as for usual (geometric) formality of Riemannian symmetric spaces, see e.g. [Ko01].) ☐

5. Massey products under blow-ups

The following theorem allows one to construct new non-formal manifolds from existing ones:

Theorem 5.1. Let $Z \subseteq X$ a complex submanifold of complex codimension $k \geq 2$ and $\pi : \tilde{X} \to X$ the blow-up of X along Z. If m is a non-trivial ABC–Massey product (either in the ad hoc or the spectral sequence sequence) on X, living in total degree $< 2k$, then also $\pi^* m$ is non-trivial.

Proof. Recall [Ste21] that additively, there is the following formula for the cohomology of $\tilde{X}$:

$$H_{BC}(X) \oplus \bigoplus_{i=1}^{k-1} H_{BC}(Z)[i] \cong H_{BC}(\tilde{X}). \tag{3}$$

Here $[i]$ denotes an up-right shift in bidegree by (i, i). The induced ring structure up to degree $2k$ on the left hand side is simple to describe: Both summands have a natural $H_{BC}(Z)$-module structure (via the identity, resp. restriction) and elements in $H_{BC}(Z)$ multiply according to the multiplication in $H_{BC}(Z)$ with appropriate degree shift. In particular, if we truncate before degree $2k$, the right summand (which corresponds to $\ker \pi_*$ under the isomorphism), is an ideal. The analogous formulae hold for $H_A(\tilde{X})$ with its structure as an $H_{BC}(\tilde{X})$-module.

The main idea is to construct a partial (bigraded) relative model for the blow-up, which retracts to A_X. More precisely, we will construct a map $\varphi : \mathcal{M} \to A_{\tilde{X}}$ with the following properties:

(1) As algebras, $\mathcal{M} = A_X \otimes \Lambda V$ for some bigraded vector space V.
(2) The ideal $I(V)$ generated by V is a differential ideal (i.e. $\mathcal{M} = A_X \oplus I(V)$ as complexes).
(3) On A_X, $\varphi = \pi^*$.
(4) The map
$$H(\varphi) : H(\mathcal{M}_{\tilde{X}}) \to H(\tilde{X})$$
is an isomorphism in total degrees $< 2k$, for $H = H_{BC}$ and $H = H_A$.

With this setup, we obtain a commutative diagram of cbba's:

$$\begin{array}{ccccc} A_X & \longrightarrow & \mathcal{M}_{\tilde{X}} & \longrightarrow & A_X \\ & \searrow^{\pi^*} & \downarrow & & \\ & & A_{\tilde{X}} & & \end{array}$$

in which the composition of the two maps (inclusion and projection) in the first row is the identity and the vertical map induces an isomorphism in cohomology up to degree k. This implies the claim by functoriality of Massey products. Note that if we had chosen basepoints of X and $\tilde{X}$, the above is compatible with the induced augmentations.

Let us indicate how to construct such a model. Since this is essentially a special case of the general construction of (bigraded, relative) models that will be carried out in [Ste22], we allow ourselves to be brief. First, pick a bigraded generating set $\{\mathfrak{b}_i\}$ for $H_{BC}(Z)$ and a collection of linearly independent classes $\{\mathfrak{a}_j\}$ that span a complement to the image of the natural map $H_{BC}(Z) \to H_A(Z)$. Now pick representatives b_i, a_j for the classes $j_*\pi^*\mathfrak{b}_i$ and $j_*\pi^*\mathfrak{a}_\mathfrak{i}$, where $j : E \to \tilde{X}$ is the inclusion of the exceptional divisor into $\tilde{X}$. We may choose the unique class in $H^{0,0}_{BC}(Z)$ to be [1], which maps to $[\theta]$. Let us denote by θ a representative for that class. Then define V' to be the vector space generated by the symbols (of pure bidegree) $x_i, y_j, \partial y_j, \bar{\partial} y_j$ and set $\mathcal{M}' := A_X \otimes \Lambda V'$ and define a differential to be as indicated by the symbols on the y_j and zero on all other generators (i.e. $dx_i = d\partial y_j = d\bar{\partial} y_j = 0$). Then there is a well-defined map $\varphi' : \mathcal{M}' \to A_{\tilde{X}}$ by sending $x_i \mapsto b_i, y_i \mapsto a_j$. By 3, the induced maps $H(\varphi')$ will be surjective and the pair $(\mathcal{M}', \varphi')$ satisfies all conditions except maybe 4. Therefore, consider

$$C = \ker H_{BC}(\varphi') : H^{<2k}_{BC}(\mathcal{M}') \to H^{<2k}_{BC}(\tilde{X})$$

Under the identification $H_{BC}(\mathcal{M}') = H_{BC}(X) \oplus H_{BC}(I(V'))$ and the formula 3, we see that, since we are in degrees less than $2k$, $C = \{0\} \oplus C'$. (In fact, in degree $2k$, the class θ^k will create a relation between both summands). Therefore, in order to kill these relations, we may pick (generators c_i for) classes in $\ker \pi_*$, and then define V'' to be the vector space given by the symbols $z_i, \partial z_i, \bar{\partial} z_i$. Then set $\mathcal{M}'' := \mathcal{M}' \otimes \Lambda V''$, with differentials as indicated by the symbols and $\partial\bar{\partial} z_i = c_i$. We may then extend φ' to $\varphi'' : \mathcal{M}'' \to A_{\tilde{X}}$ by sending z_i to some $\partial\bar{\partial}$-primitive for $\varphi'(c_i)$. The map $H(\mathcal{M}') \to H(\mathcal{M}'')$ has the property that all elements in C map to 0 and all elements in $\ker H_A(\varphi')$ map to elements in the image of $H_{BC}(\mathcal{M}'') \to H_A(\mathcal{M}'')$. Repeating this process if necessary, we obtain our desired pair (V, φ) and hence $\mathcal{M}$. □

Remark 5.2. Essentially the same proof (with some obvious simplifications) shows the statement for ordinary Massey products in the de Rham cohomology of any blow-up of a submanifold with almost complex normal bundle (c.f [M84]), in particular for symplectic blow-ups. This had previously been obtained by Babenko–Taimanov [BT00].

Remark 5.3. In [Ta10], Taylor shows that a non-trivial triple Massey product pulls back non-trivially under a non-zero degree map of rational Poincaré duality spaces (in particular, closed orientable manifolds). An immediate adaptation of his argument to compact complex manifolds, where Aeppli and Bott–Chern cohomology are paired non-degenerately under Serre duality [S07], gives the following: Let $Y \to X$ be a non-zero degree holomorphic map of compact complex manifolds. A non-vanishing ABC–Massey triple product on X pulls back to a non-vanishing ABC–Massey triple product on Y. By the same argument, Tardini's Aeppli–Bott–Chern Massey products [Ta17] likewise pull back non-trivially. □

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Max-Planck-Institut für Mathematik, Vivatsgasse 7, 53111 Bonn
Email address: `milivojevic@mpim-bonn.mpg.de`

Mathematisches Institut der Ludwig-Maximilians-Universität München, Theresienstrasse 39, 80993 München
Email address: `jonas.stelzig@math.lmu.de`