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SIMPLE COMPLEX TORI OF ALGEBRAIC DIMENSION 0

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Abstract. Using Galois theory, we construct explicitly (in all complex dimensions ≥ 2) an infinite family of simple g-dimensional complex tori T that enjoy the following properties.

- The Picard number of T is 0; in particular, the algebraic dimension of T is 0.
- If $T^{\vee}$ is the dual of T then $\mathrm{Hom}(T, T^{\vee}) = \{0\}$.
- The automorphism group $\mathrm{Aut}(T)$ of T is isomorphic to $\{\pm 1\} \times \mathbb{Z}^{g-1}$.
- The endomorphism algebra $\mathrm{End}^0(T)$ of T is isomorphic to a purely imaginary number field of degree $2g$.

1. Introduction

It is known that a "very general" complex torus T of complex dimension $\dim(T) = g \geq 2$ has the algebraic dimension $a(T) = 0$. But the explicit examples of such tori with $g > 2$ are very scarce. For $g = 2$ one may find the explicit examples of complex tori with algebraic dimension zero in [EF, Appendix] and [BL, Example 7.4]. (All the tori of complex dimension 1 have algebraic dimension 1.)

The aim of this paper is to provide explicit examples of *simple* complex tori T with $a(T) = 0$ in all complex dimensions $g \geq 2$.

The tori we construct have some interesting additional properties and may be viewed as non-algebraic analogues of abelian varieties of CM type, see [LangCM, pp. 12–13 and Th. 4.1 on p. 15]. They also played an important role in C. Voisin's construction of counterexamples to Kodaira's *algebraic approximation problem* [Vo04, Vo06], see also [GS]. (We discuss her results about tori in Remark 1.6 below.) We start with the following definitions.

Definition 1.1. A positive-dimensional complex torus X is called *simple* if $\{0\}$ and X are the only complex subtori of X (see, e.g., [BL, Chapter I, Section 7]).

Definition 1.2. A complex torus T of dimension $g \geq 2$ is called *special* if it enjoys the following properties.

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(a) T is simple and has algebraic dimension 0. In addition, its endomorphism algebra $\mathrm{End}^0(T) = \mathrm{End}(T) \otimes \mathbb{Q}$ is isomorphic to a purely imaginary number field of degree $2g$.
(b) The Picard number $\rho(T)$ of T is 0.
(c) If $T^\vee$ is the dual of T then $\mathrm{Hom}(T, T^\vee) = \{0\}$. In particular, complex tori T and $T^\vee$ are not isogenous.
(d) Let $\mathrm{Aut}(T)$ be the automorphism group of the complex Lie group T. Then $\mathrm{Aut}(T)$ is isomorphic to $\{1, -1\} \times \mathbb{Z}^{g-1}$. In particular, $\mathrm{Aut}(T)$ is an infinite commutative group, whose torsion subgroup is a cyclic group of order 2.

Our main result is the following

Theorem 1.3. *Let $g \geq 2$ be an integer and E a degree $2g$ number field that enjoys the following properties.*
(i) *E is purely imaginary;*
(ii) *E has no proper subfields except $\mathbb{Q}$.*

Choose any isomorphism of $\mathbb{R}$-algebras

$$\Psi : E_{\mathbb{R}} := E \otimes_{\mathbb{Q}} \mathbb{R} \to \oplus_{j=1}^{g} \mathbb{C} = \mathbb{C}^g \tag{1}$$

and a $\mathbb{Z}$-lattice Λ of rank $2g$ in $E \subset E_{\mathbb{R}}$. Isomorphism Ψ provides $E_{\mathbb{R}}$ with the structure of a g-dimensional complex vector space.

Then the complex torus $T = T_{E,\Psi,\Lambda} := E_{\mathbb{R}}/\Lambda$ is special and its endomorphism algebra $\mathrm{End}^0(T)$ is isomorphic to E.

We present explicit examples of such fields (see Sections 6, 7, 8) for all $g \geq 2$.

Remark 1.4. Some authors call number fields that enjoy the property (ii) of Theorem 1.3 *primitive*. One may view Proposition 2.1 below as a justification of this terminology.

Remark 1.5. Suppose that $g \geq 2$ and a degree $2g$ number field E enjoys the properties (i)-(ii) of Theorem 1.3. Let Γ be an integer lattice of rank $2g$ in E and $T_0 = T_{E,\Psi,\Gamma}$ the corresponding complex torus of dimension g. If Λ is any subgroup of finite index in Γ then it is also an integer lattice of rank $2g$ in $E \subset E_{\mathbb{R}}$. By Theorem 1.3, all complex tori $T = T_{E,\Psi,\Lambda}$ are special and $\mathrm{End}^0(T) \cong E$. On the other hand, the set of of all tori $T_{E,\Psi,\Lambda}$ is precisely the *isogeny class* of T_0 (up to an isomorphism). Let $\mathscr{X}_g \to B_g$ be a versal family of complex tori of dimension g that was constructed in [BL, Sect. 10]. (Every complex torus of dimension g appears as its fiber.) Its base B_g is a *homogeneous* $\mathrm{GL}_{2g}(\mathbb{R})$-space. Each isogeny class is a $\mathrm{GL}_{2g}(\mathbb{Q})$-*orbit* in B_g, which is a dense subset of B_g , because $\mathrm{GL}_{2g}(\mathbb{Q})$ is a dense subgroup of $\mathrm{GL}_{2g}(\mathbb{R})$. Therefore each isogeny class is dense in the *moduli space* $B_g/\mathrm{GL}_{2g}(\mathbb{Z})$ of complex tori of dimension g. This implies that the subset of all g-dimensional *special* tori is dense in the moduli space.

Remark 1.6. Let $T = V/\Gamma$ be a complex torus of dimension $g \geq 2$ where V is a g-dimensional complex vector space and Γ is a discrete lattice of rank $2g$ in V. Let ϕ_T be a holomorphic endomorphism of the complex Lie group T and ϕ_Γ is the endomorphism of Γ induced by ϕ_T. Let $f(x) \in \mathbb{Z}[x]$ be the characteristic polynomial of ϕ_Γ, which is monic of degree $2g$. Suppose that the polynomial $f(x)$ is separable, has no real roots and its Galois group $\mathrm{Gal}(f)$ over $\mathbb{Q}$ is the full symmetric group $\mathbf{S}_{2g}$. Such a pair (T, ϕ_T) is called a *scenic torus* in [GS, Sect. 3, p. 271]. C. Voisin [Vo04, Sect. 1] proved that a scenic T is *not* algebraic and its Picard number is 0. It follows from Theorem 1.3 that T is actually special. Indeed, let E be the $\mathbb{Q}$-subalgebra of $\mathrm{End}^0(T)$ generated by ϕ_T. The conditions on $f(x)$ and $\mathrm{Gal}(f)$ imply that $f(x)$ is irreducible and $E \cong \mathbb{Q}[x]/f(x)\mathbb{Q}[x]$ is a purely imaginary number field of degree $2g$. The condition on $\mathrm{Gal}(f)$ implies (thanks to Example 2.3 below) that E has no proper subfields except $\mathbb{Q}$. Thus all conditions of Theorem 1.3 are met.

The proof of Theorem 1.3 is based on results of [OZ]. Properties (b), (c), (d) of Definition 1.2 are consequences of the following assertions concerning the **endomorphism algebra**

$$\mathrm{End}^0(T) = \mathrm{End}(T) \otimes \mathbb{Q}$$

of T. Recall [OZ] that $\mathrm{End}^0(T)$ is a finite-dimensional (not necessarily semisimple) $\mathbb{Q}$-algebra.

Proposition 1.7. *Let T be a complex torus of dimension $g \geq 2$. Suppose that $\mathrm{End}^0(T)$ is a degree $2g$ number field that does not contain a subfield of degree g. Then*

(a) *T is a simple complex torus of algebraic dimension 0;*
(b) *The Picard number $\rho(T)$ of T is 0, i.e., its Néron-Severi group $\mathrm{NS}(T) = \{0\}$.*

Proposition 1.8. *Let T be a complex torus of dimension $g \geq 2$. Suppose that $\mathrm{End}^0(T)$ is a degree $2g$ number field that does not contain a proper subfield except $\mathbb{Q}$.*

If $T^\vee$ is the dual of T then $\mathrm{Hom}(T, T^\vee) = \{0\}$. In particular, T is not isogenous to $T^\vee$.

Proposition 1.9. *Let T be a complex torus of positive dimension. Suppose that the endomorphism algebra $\mathrm{End}^0(T)$ is a purely imaginary number field of degree $2s$ that does not contain roots of unity except $\{1, -1\}$. Let $\mathrm{Aut}(T)$ be the automorphism group of the complex Lie group T.*

Then $\mathrm{Aut}(T)$ is isomorphic to $\{\pm 1\} \times \mathbb{Z}^{s-1}$. In particular, $\mathrm{Aut}(T)$ is commutative and its torsion subgroup is a cyclic group of order 2.

As a by-product we get examples of poor manifolds for any dimension.

The notion of a poor manifold was introduced in [BZ20]. It is a complex compact connected manifold containing neither rational curves nor analytic subsets of codimension 1 (and, *a fortiori,* having algebraic dimension 0).

It was proven in [BZ20] that for a $\mathbb{P}^1$−bundle X over a poor manifold Y the group $\mathrm{Bim}(X)$ of its bimeromorphic selfmaps coincides with the group $\mathrm{Aut}(X)$ of its biholomorphic automorphisms; the latter has the commutative identity component $\mathrm{Aut}_0(X)$ and the order of any finite subgroup of the quotient $\mathrm{Aut}(X)/\mathrm{Aut}_0(X)$ is bounded by a constant depending on X only.

As it was mentioned in [BZ20], a complex torus T has algebraic dimension $a(T) = 0$ if and only if it is poor. There exists an explicit example of a $K3$ surface that does *not* contain analytic subsets of codimension 1 ([McM]) and therefore is poor. We prove the following

Theorem 1.10. *Let T be a complex torus of dimension $g \geq 2$. Suppose that $\mathrm{End}^0(T)$ contains a a degree $2g$ number field E with the same 1 such that E does not contain a CM subfield.*

Then T has algebraic dimension 0 and therefore is poor. In addition, there exist a simple complex torus S and a positive integer r such that T is isogenous to the self-product S^r of S.

Remark 1.11. Let us note an additional property of special tori. The notion of the invariant Brauer group $\mathrm{Br}_T(T)$ of a complex toris T was introduced in [OSVZ] (see also [Cao]). This group is a finite abelian group of exponent 2.

We claim that $\mathrm{Br}_T(T) = \{0\}$ if T is *special*. Indeed, $\mathrm{Br}_T(T)$ is isomorphic to a subquotient of $\mathrm{Hom}(T, T^\vee)$ [OSVZ, Sect. 3.3, displayed formula (13) and Prop. 3.19]. Since $\mathrm{Hom}(T, T^\vee) = \{0\}$ for *special* T, the group $\mathrm{Br}_T(T)$ is also $\{0\}$.

The paper is organized as follows. In Section 2 we give some background. Section 3 contains proofs of main results. In Section 4 we prove Theorem 1.10. In Sections 5, 6, 7, 8, 9 we present a plenty of explicit examples of certain number fields that give rise to special tori. (Notice that explicit examples of simple complex 2-dimensional tori T with $a(T) = 0$ and Picard number 0 were given in [EF, Appendix] and [BL, Example 7.4] in terms of their period lattices.)

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2. A construction from the Galois Theory

As usual, $\mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C}$ stand for the ring of integers and fields of rational, real, and complex numbers respectively. We write $\bar{\mathbb{Q}}$ for an algebraic closure of $\mathbb{Q}$.

Let us recall the properties of a purely imaginary number field E.

We may view it as $E = \mathbb{Q}(\alpha)$, where $\alpha \in E$ and there is an irreducible over $\mathbb{Q}$ polynomial $f(x) \in \mathbb{Q}[x]$ of degree $2g$ such that $f(\alpha) = 0$. The property of E to be *purely imaginary* means that $f(x)$ has *no* real roots in $\mathbb{C}$. Let $\alpha_1, \overline{\alpha_1} \dots, \alpha_g, \overline{\alpha_g}$ be roots of $f(x)$ (here $\overline{\alpha}_j$ stands for the complex conjugate of α_j). There are $2g$ field embeddings $E \hookrightarrow \mathbb{C}$, namely, two for every $j, 1 \le j \le g$:

$$\sigma_j : 1 \to 1, \alpha \to \alpha_j$$

and

$$\overline{\sigma_j} : 1 \to 1, \alpha \to \overline{\alpha_j}.$$

For every choice of $g-$tuple $(\tau_1, \ldots, \tau_g)$, where each τ_j is either σ_j or $\overline{\sigma_j}$ we define an injective $\mathbb{Q}$-algebra homomorphism

$$\Psi : E \hookrightarrow \oplus_{j=1}^{g} \mathbb{C} = \mathbb{C}^g, E \ni e \mapsto (\tau_1(e), \ldots, \tau_g(e)) \in \mathbb{C}^g \tag{2}$$

that extends by $\mathbb{R}$-linearity to a homomorphism $\Psi : E_{\mathbb{R}} \to \mathbb{C}^g$ of $\mathbb{R}$-algebras (we keep the notation Ψ). Actually, Ψ is an isomorphism of $\mathbb{R}$-algebras. Indeed, let $\{\beta_1, \ldots, \beta_{2g}\}$ be a basis of the $2g$-dimensional $\mathbb{Q}$-vector space E. It is proven in [LangCM, Proof of Th. 4.1 on pp. 15–16] that the $2g$-element set

$$\{\Psi(\beta_1), \ldots, \Psi(\beta_{2g})\} \subset \mathbb{C}^g$$

is linearly independent over $\mathbb{R}$. It follows that the image $\Psi(E_{\mathbb{R}})$ has $\mathbb{R}$-dimension $2g$. Since

$$\dim_{\mathbb{R}}(E_{\mathbb{R}}) = 2g = \dim_{\mathbb{R}}(\mathbb{C}^g),$$

$\Psi : E_{\mathbb{R}} \to \mathbb{C}^g$ is an *isomorphism* of $\mathbb{R}$-algebras. There are precisely 2^g isomorphisms of $\mathbb{R}$-algebras $E_{\mathbb{R}}$ and $\mathbb{C}^g$ of the form $\Psi = (\tau_1, \ldots, \tau_g)$, where τ_i are defined in (2). We will use these isomorphisms in order to construct complex tori $E_{\mathbb{R}}/\Gamma$ with needed properties where Γ is a discrete lattice of maximal rank in E. We will need the following elementary construction from Galois theory.

Let $n \ge 3$ be an integer and $f(x) \in \mathbb{Q}[x]$ a degree n *irreducible* polynomial. This means that the $\mathbb{Q}$-algebra

$$K_f = \mathbb{Q}[x]/f(x)\mathbb{Q}[x]$$

is a degree n number field. Let $\mathscr{R}_f \subset \bar{\mathbb{Q}}$ be the n-element set of roots of $f(x)$. If $\alpha \in \mathscr{R}_f$ then there is an isomorphism of $\mathbb{Q}$-algebras

$$\Phi_\alpha : K_f = \mathbb{Q}[x]/f(x)\mathbb{Q}[x] \cong \mathbb{Q}(\alpha), x \mapsto \alpha \tag{3}$$

where $\mathbb{Q}(\alpha)$ is the subfield of $\bar{\mathbb{Q}}$ generated by α. Clearly, K_f (and hence, $\mathbb{Q}(\alpha)$) is *purely imaginary* if and only if $f(x)$ has *no* real roots.

Let $\mathbb{Q}(\mathscr{R}_f) \subset \bar{\mathbb{Q}}$ be the splitting field of $f(x)$, i.e., the subfield of $\bar{\mathbb{Q}}$ generated by $\mathscr{R}_f$. Then $\mathbb{Q}(\mathscr{R}_f) \subset \bar{\mathbb{Q}}$ is a finite Galois extension of $\mathbb{Q}$ containing $\mathbb{Q}(\alpha)$. We write $G = \mathrm{Gal}(f)$ for the Galois group $\mathrm{Gal}(\mathbb{Q}(\mathscr{R}_f)/\mathbb{Q})$ of $\mathbb{Q}(\mathscr{R}_f)/\mathbb{Q}$, which may be viewed as a certain subgroup of the group $\mathrm{Perm}(\mathscr{R}_f)$ of permutations of $\mathscr{R}_f$. The irreducibility of $f(x)$ means that $\mathrm{Gal}(f)$ is a *transitive* permutation subgroup of $\mathrm{Perm}(\mathscr{R}_f)$. Let us consider the stabilizer subgroup

$$G_\alpha = \{\sigma \in G \mid \sigma(\alpha) = \alpha\} \subset G. \tag{4}$$

Clearly, G_α coincides with the Galois group $\mathrm{Gal}(\mathbb{Q}(\mathscr{R}_f)/\mathbb{Q}(\alpha))$ of Galois extension $\mathbb{Q}(\mathscr{R}_f)/\mathbb{Q}(\alpha)$. If one starts to vary α in $\mathscr{R}_f$ then all the subgroups G_α constitute a conjugacy class in G.

The following assertion is certainly well known but we failed to find a suitable reference.

Proposition 2.1. *The following conditions are equivalent.*
(i) *K_f has no proper subfields except $\mathbb{Q}$.*
(ii) *$\mathbb{Q}(\alpha)$ has no proper subfields except $\mathbb{Q}$.*
(iii) *G_α is a maximal subgroup in G.*
(iv) *G is a primitive permutation subgroup of* $\mathrm{Perm}(\mathscr{R}_f)$.

Remark 2.2.
(1) A transitive permutation group G is *primitive* if and only if the stabilizer of a point is a maximal subgroup of G [Pa, Prop. 3.4 on p. 15].
(2) Every 2-transitive permutation group is primitive [Pa, Prop. 3.8 on p. 18].

Proof of Proposition 2.1. It follows from (3) that (i) and (ii) are equivalent. It follows from Remark 2.2(1) that (iii) and (iv) are equivalent.

Let us prove that (ii) and (iii) are equivalent. Let H be a subgroup of G that contains G_α. Let us consider the subfield of H-invariants

$$F := \mathbb{Q}(\mathscr{R}_f)^H = \{e \in \mathbb{Q}(\mathscr{R}_f) \mid \sigma(e) = e \ \forall \sigma \in H\} \subset \mathbb{Q}(\mathscr{R}_f).$$

Clearly F is contained $\mathbb{Q}(\mathscr{R}_f)^{G_\alpha} = \mathbb{Q}(\alpha)$.

There is a bijection between the set of subfields $\mathbb{Q}(\mathscr{R}_f)$ and the set of the subgroups of G (see e.g., [Lang, Chapter VI, Theorem 1.1]). If H is neither G_α nor G (i.e., G_α is *not* maximal) then F is neither $\mathbb{Q}(\alpha)$ nor $\mathbb{Q}(\mathscr{R}_f)^G = \mathbb{Q}$. This means if (iii) does not hold then (ii) does not hold as well.

Conversely, let F be a field that lies strictly between $\mathbb{Q}(\alpha)$ and $\mathbb{Q}$. Then the Galois group $H := \mathrm{Gal}(\mathbb{Q}(\mathscr{R}_f)/F)$ is a proper subgroup of G that contains G_α but does *not* coincide with it. Hence G_α is *not* maximal. This means that if (ii) does not hold then (iii) does not hold as well. This ends the proof. □

Example 2.3. Suppose that $n \geq 4$. Let $\mathrm{Alt}(\mathscr{R}_f)$ be the only index two subgroup of $\mathrm{Perm}(\mathscr{R}_f)$, which is isomorphic to the **alternating group $\mathbf{A}_n$**. Then both $\mathrm{Perm}(\mathscr{R}_f)$ and $\mathrm{Alt}(\mathscr{R}_f)$ are doubly transitive permutation groups [Pa] and therefore are primitive. It follows from Proposition 2.1 that if $\mathrm{Gal}(f)$ coincides with either $\mathrm{Perm}(\mathscr{R}_f)$ or $\mathrm{Alt}(\mathscr{R}_f)$ then K_f does not contain a proper subfield except $\mathbb{Q}$. In other words, K_f does not contain a proper subfield except $\mathbb{Q}$ if $\mathrm{Gal}(f)$ is isomorphic either to the full symmetric group $\mathbf{S}_n$ or to the alternating group $\mathbf{A}_n$. (The case of $\mathbf{S}_n$ was discussed earlier in [LO, Sect. 3, p. 51].)

3. Proofs of main results

If X is a complex torus then its endomorphism algebra $\mathrm{End}^0(X) = \mathrm{End}(X) \otimes \mathbb{Q}$ will be denoted also by $D(X)$ in order to be consistent with the notation in [OZ].

In the following definition, **smCM** is short for *sufficiently many Complex Multiplications*: this terminology is inspired by a similar notion for abelian varieties introduced by F. Oort [O].

Definition 3.1. Let T be a positive-dimensional complex torus and E a number field of degree $2\dim(T)$. We say that T is a **smCM**-torus or a **smCM**-torus with multiplication by E if there is a $\mathbb{Q}$-algebra embedding $E \hookrightarrow D(T)$ that sends $1 \in E$ to the identity automorphism of E.

The following assertion is contained in [OZ, Corollary 1.7 on p. 15]. [1]

Proposition 3.2. *Let T be a positive-dimensional* **smCM**-*torus with multiplication by a number field E.*

Then there are a simple complex torus S and a positive integer r such that

(1) *r divides $2\dim(T)$;*
(2) *T is isogenous to S^r;*
(3)
$$[D(S) : \mathbb{Q}] = 2\dim(S); \tag{5}$$
(4) *the field E contains a subfield,* $\mathrm{HDG}(T)$, *that is isomorphic to $D(S)$, and*
$$r = \frac{2\dim(T)}{\dim_{\mathbb{Q}}(\mathrm{HDG}(T))}. \tag{6}$$

The next lemma is an almost immediate corollary of Proposition 3.2.

Lemma 3.3. *Let T be a positive-dimensional complex torus with smCM by a field $E \subset D(T)$. Suppose that at least one of the following conditions holds.*

(i) *$D(T) = E$.*
(ii) *E has no proper subfields except $\mathbb{Q}$.*

Then T is a simple torus and $D(T) = E$.

Proof of Lemma 3.3. By Proposition 3.2, there are a simple complex torus S and a positive integer r with properties (1-4) of Proposition 3.2.

This implies that $D(T)$ is isomorphic to the matrix algebra $\mathrm{Mat}_r(D(S))$ of size r over $D(S)$. In particular, $D(T)$ is not a field if $r > 1$. This implies readily that in case (i) of Lemma 3.3 $r = 1$ and therefore T is isogenous to simple S and therefore is simple itself; by assumption, $D(T) = E$.

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[1] There is a typo in the assertion 2 of this Corollary. Namely, one should read in the displayed formula $[D(S) : \mathbb{Q}]$ (not $[E : \mathbb{Q}]$).

Let us do the case (ii). The absence of intermediate subfields in E implies that either $D(S) = \mathbb{Q}$ or $D(S) \cong E$. In light of (5), $[D(S) : \mathbb{Q}]$ is *even*, which implies that $D(S) \cong E$ and, therefore,

$$\dim_{\mathbb{Q}}(\mathrm{HDG}(T)) = [E : \mathbb{Q}] = [D(S) : \mathbb{Q}] = 2g = 2\dim(T). \tag{7}$$

It follows that $\dim_{\mathbb{Q}}(\mathrm{HDG}(T)) = 2\dim(T)$. Now (6) implies that $r = 1$, hence, T is isogenous to simple S and, therefore, is a simple torus itself. In addition,

$$D(T) \cong D(S) \cong E. \tag{8}$$

So, the $\mathbb{Q}$-algebra $D(T)$ is isomorphic to its subfield E and therefore coincides with E. □

Lemma 3.4. *Let T be a simple complex torus of positive dimension g such that its endomorphism algebra $D(T)$ is a degree $2g$ number field E that is not CM. Then $a(T) = 0$.*

Proof. Every complex torus T admits a maximal quotient abelian variety T_a such that $\dim T_a = a(T)$ ([BL, Ch. 2, Sect. 6]). The (connected) kernel of the surjective homomorphism $T \to T_a$ is a (complex) subtorus of T. Thus, if T is simple, either it is an abelian variety or $a(T) = 0$. Suppose T is an abelian variety. Then Albert's classification of endomorphism algebras of simple complex abelian varieties [Mum, Section 1, Application I] implies that $E = D(T)$ has degree $[E : \mathbb{Q}] \le 2g$; if the equality holds then E is a CM field. Since E has degree $2g$ but is not a CM field, we get a contradiction that proves that $a(T) = 0$. □

Remark 3.5. If T is a torus with smCM by a field E, $g = \dim(T) > 1$, and condition (ii) of Lemma 3.3 holds then E is not a CM field, because a degree $2g$ CM field contains a (totally) real subfield of degree g.

Proof of Proposition 1.7. We are given that $E = D(T)$ is a number field of degree $2g$, hence T is a smCM torus and condition (i) of Lemma 3.3 holds. Thus T is a *simple* complex torus of positive dimension g. The absence of degree g subfields in E implies that E is *not* a CM field (see Remark 3.5). It follows from Lemma 3.4 that T has algebraic dimension 0.

Suppose that $\mathrm{NS}(T) \ne \{0\}$. Then there exists a holomorphic line bundle $\mathscr{L}$ on T, whose first Chern class $c_1(\mathscr{L}) \ne 0$. Then $\mathscr{L}$ gives rise to a *nonzero* morphism of complex tori

$$\phi_{\mathscr{L}} : T \to T^{\vee}$$

where the g-dimensional complex torus $T^{\vee} = \mathrm{Pic}^0(T)$ is the dual of T (see [BL, Ch. 2, Sect. 3]).

Since T is simple and both T and $T^{\vee}$ have the same dimension g, the *nonzero* morphism $\phi_{\mathscr{L}}$ is an *isogeny* of complex tori. This means that T is a *nondegenerate* complex torus [BL, Ch. 2, Prop. 3.1] in the terminology of [BL]. Since T is simple, $\mathscr{L}$ is a "polarization" on T (see [BL, Proposition 1.7, Ch. 2, Sect. 1]).

Let

$$\mathrm{End}^0(T) \to \mathrm{End}^0(T),\ u \mapsto u'$$

be the *Rosati involution* attached to $\mathscr{L}$ [BL, Ch. 2, Sect. 3]. If it is nontrivial then the subalgebra of its invariants is a degree g subfield of the field $E = \mathrm{End}^0(T)$ (see [Lang, Theorem 1.8, Chapter VI]). However, by our assumption, such a subfield does not exist. This implies that the Rosati involution is the identity map. It follows from [BL, Ch. 5, Prop. 1.2, last assertion] that $2g = [E : \mathbb{Q}]$ divides g, which is nonsense. The obtained contradiction implies that $c_1(\mathscr{L})$ is always 0, i.e., $\mathrm{NS}(T) = \{0\}$. □

Proof of Proposition 1.8. Let us present complex torus T as the quotient

$$T = V/\Gamma$$

where V is a g-dimensional complex vector space and Γ a discrete additive subgroup of rank $2g$. Let

$$\Gamma_{\mathbb{Q}} := \Gamma \otimes \mathbb{Q},\ \Gamma_{\mathbb{R}} := \Gamma \otimes \mathbb{R}$$

be $2g$-dimensional $\mathbb{Q}$- and $\mathbb{R}$-vector spaces, respectively.

Note that $V \cong \mathbb{C}^g$ coincides with $\Gamma_{\mathbb{R}}$ endowed with complex structure. Namely, there is

$$J \in \mathrm{End}_{\mathbb{R}}(\Gamma_{\mathbb{R}}),$$

which is multiplication by $\mathbf{i} = \sqrt{-1}$ in the $\mathbb{C}$-vector space V.

Moreover,

$$\mathrm{End}_{\mathbb{R}}(V) = \mathrm{End}_{\mathbb{R}}(\Gamma_{\mathbb{R}}),\ \mathrm{End}_{\mathbb{C}}(V) = \{u \in \mathrm{End}_{\mathbb{R}}(\Gamma_{\mathbb{R}})\ \mid uJ = Ju\}.$$

We have

$$J^2 = -1,\ J^{-1} = -J. \tag{9}$$

It is known ([Ha, Proposition 5.2.11]) that $\mathrm{End}(T) \subset \mathrm{End}(\Gamma)$ and

$$\mathrm{End}(T) \otimes_{\mathbb{Z}} \mathbb{R} = \mathrm{End}^0(T) \otimes_{\mathbb{Q}} \mathbb{R} = \{u \in \mathrm{End}_{\mathbb{R}}(\Gamma_{\mathbb{R}}) \mid uJ = Ju\}. \tag{10}$$

In particular, the $2g$-dimensional $\mathbb{Q}$-vector space $\Gamma_{\mathbb{Q}}$ carries the natural structure of a faithful $\mathrm{End}^0(T)$-module. Recall that $E = \mathrm{End}^0(T)$ is a number field of degree $2g$. Hence, $\Gamma_{\mathbb{Q}}$ becomes the one-dimensional E-vector space and therefore E coincides with its own centralizer in $\mathrm{End}_{\mathbb{Q}}(\Gamma_{\mathbb{Q}})$. This implies that if we put

$$E_{\mathbb{R}} = E \otimes_{\mathbb{Q}} \mathbb{R} \subset \mathrm{End}_{\mathbb{Q}}(\Gamma_{\mathbb{Q}}) \otimes_{\mathbb{Q}} \mathbb{R} = \mathrm{End}_{\mathbb{R}}(\Gamma_{\mathbb{R}})$$

then $\Gamma_{\mathbb{R}}$ becomes the free $E_{\mathbb{R}}$-module of rank 1 and therefore $E_{\mathbb{R}}$ coincides with its own centralizer in $\mathrm{End}_{\mathbb{R}}(\Gamma_{\mathbb{R}})$. This implies that $J \in E_{\mathbb{R}}$.

Lemma 3.6. *Let $B : \Gamma \times \Gamma \to \mathbb{Z}$ be a $\mathbb{Z}$-bilinear form. Let us extend it by $\mathbb{R}$-linearity to the $\mathbb{R}$-bilinear form*

$$\Gamma_{\mathbb{R}} \times \Gamma_{\mathbb{R}} \to \mathbb{R},$$

which we continue to denote by B. Suppose that

$$B(Jv_1, Jv_2) = B(v_1, v_2)\ \forall v_1, v_2 \in \Gamma_{\mathbb{R}}. \tag{11}$$

Then $B \equiv 0$.

Proof of Lemma 3.6. Clearly,

$$B(\Gamma_{\mathbb{Q}}, \Gamma_{\mathbb{Q}}) \subset \mathbb{Q}. \tag{12}$$

The J-invariance of B means that

$$B(Jv_1, v_2) = B(v_1, J^{-1}v_2) = -B(v_1, Jv_2) \ \forall v_1, v_2 \in \Gamma_{\mathbb{R}} \tag{13}$$

because $J^{-1} = -J$ (since $J^2 = -1$). It follows that the $\mathbb{R}$-vector subspace

$$E_{\mathbb{R}}^{-} = \{u \in E_{\mathbb{R}} \mid B(u(v_1), v_2) = -B(v_1, u(v_2)) \ \forall v_1, v_2 \in \Gamma_{\mathbb{R}}\}$$

of $E_{\mathbb{R}}$ is *not* zero. In light of (12), there is a nonzero $\mathbb{Q}$-vector subspace E^{-} of E such that

$$E_{\mathbb{R}}^{-} = E^{-} \otimes_{\mathbb{Q}} \mathbb{R}.$$

Clearly, $E^{-} = E_{\mathbb{R}}^{-} \bigcap E$ and

$$E^{-} = \{u \in E \mid B(u(v_1), v_2) = -B(v_1, u(v_2)) \ \forall v_1, v_2 \in \Gamma_{\mathbb{Q}}.\}$$

Let u_{-} be a nonzero element of E^{-}. Clearly,

$$u_{-} \notin \mathbb{Q} \subset E.$$

On the other hand,

$$u_{+} := u_{-}^2 \in E$$

also does *not* lie in $\mathbb{Q}$, because otherwise $\mathbb{Q} + \mathbb{Q} \cdot u_{-}$ is a quadratic subfield of E, which does *not* contain quadratic subfields. (Recall that $[E : \mathbb{Q}] = 2g > 2$.) Notice that

$$B(u_{+}(v_1), v_2) = B(v_1, u_{+}(v_2)) \ \forall v_1, v_2 \in \Gamma_{\mathbb{Q}}.$$

Let us consider

$$E^{+} = \{u \in E_{\mathbb{R}} \mid B(u(v_1), v_2) = B(v_1, u(v_2)) \ \forall v_1, v_2 \in \Gamma_{\mathbb{R}}\}.$$

Clearly, E^{+} is a subfield of E that contains u_{+} and therefore does *not* coincide with $\mathbb{Q}$. This implies that $E^{+} = E$. It follows that for all $u \in E_{\mathbb{R}}$

$$B(u(v_1), v_2) = B(v_1, u(v_2)) \ \forall v_1, v_2 \in \Gamma_{\mathbb{R}}.$$

Since $J \in E_{\mathbb{R}}$, it follows from (13) that

$$B(Jv_1, v_2) = 0 \ \forall v_1, v_2 \in \Gamma_{\mathbb{R}}.$$

Since J is an automorphism of $\Gamma_{\mathbb{R}}$, we get $B \equiv 0$. □

We continue to prove Proposition 1.8. Let us recall a description of the dual complex torus $T^{\vee}$ of T ([BL, Ch. 1, Sect. 4], [Ke, Sect. 1.4]). Namely, $T^{\vee} = V^{\vee}/\Gamma^{\vee}$ where $V^{\vee}$ is the complex vector space of all $\mathbb{C}$-antilinear maps $l : V \to \mathbb{C}$ while

$$\Gamma^{\vee} = \{l \in V^{\vee} \mid \mathrm{Im}\,(l(\Gamma)) \subset \mathbb{Z}\}.$$

The structure of a complex vector space on $V^{\vee}$ is defined by the operator $J^{\vee} \in \mathrm{End}_{\mathbb{R}}(V^{\vee})$ such that $J^{\vee}(l) = il$, i.e. $J^{\vee}(l)(v) = il(v)$. By construction,

$$J^{\vee}(l) = -l \circ J \ \forall l \in V^{\vee} \tag{14}$$

(recall that l is *antilinear*).

Let $f : T \to T^\vee$ be a morphism of complex tori (viewed as complex Lie group). Then (see [BL, Ch. 1, Sect. 1, p. 4] and [OSVZ, Sect. 3.3]) there exists (a lifting of f, i.e.,) a $\mathbb{C}$-linear map $F : V \to V^\vee$ such that $F(\Gamma) \subset \Gamma^\vee$ and

$$f(v+\Gamma) = F(v) + \Gamma^\vee \in V^\vee/\Gamma^\vee = T^\vee \ \forall v+\Gamma \in V/\Gamma = T.$$

Let us consider the sesquilinear form

$$H : V \times V \to \mathbb{C}, v_1, v_2 \mapsto F(v_1)(v_2)$$

and its imaginary part (which is a $\mathbb{R}$-bilinear form)

$$B = \mathrm{Im}(H) : V \times V \to \mathbb{R}, \ v_1, v_2 \mapsto \mathrm{Im}\left((F(v_1)(v_2)\right).$$

Clearly,

$$B(\Gamma,\Gamma) \subset \mathbb{Z}, \ H(Jv_1, Jv_2) = H(v_1, v_2) \ \forall v_1, v_2 \in V = \Gamma_{\mathbb{R}}.$$

This implies that

$$B(Jv_1, Jv_2) = B(v_1, v_2) \ \forall v_1, v_2 \in V = \Gamma_{\mathbb{R}}.$$

By Lemma 3.6, $B \equiv 0$. This implies that $H \equiv 0$ and therefore $F = 0$. It follows that $f = 0$, which ends the proof. □

Proof of Proposition 1.9. Clearly, $\mathrm{End}(T)$ is an order in the purely imaginary number field $E = \mathrm{End}(T) \otimes \mathbb{Q}$ of degree $2s$; its group of invertible elements (units) $\mathrm{End}(T)^*$ coincides with $\mathrm{Aut}(T)$. It is also clear that the roots of unity in $\mathrm{End}(T)$ are precisely 1 and -1. Now the desired result follows from Dirichlet's theorem about units [BS, Ch. II, Sect. 4, Th. 5]. □

Proof of Theorem 1.3. We keep the notation of Theorem 1.3. Let us put $T := T_{E,\Psi,\Lambda}$ and consider

$$O := \{u \in E \mid u \cdot \Lambda \subset \Lambda\} \subset E. \tag{15}$$

Then O is an *order* in E [BS, Ch. VII, Sect. 2, Th. 3]. Multiplications by elements of O in $E_{\mathbb{R}}$ give rise to the ring embedding

$$O \hookrightarrow \mathrm{End}(T), \tag{16}$$

which extends by $\mathbb{Q}$-linearity to the $\mathbb{Q}$-algebra embedding

$$E = O \otimes \mathbb{Q} \hookrightarrow \mathrm{End}(T) \otimes \mathbb{Q} = D(T). \tag{17}$$

This allows us to view E as a certain $\mathbb{Q}$-subalgebra of $D(T)$. Note that $1 \in E$ is mapped to $1 \in D(T)$. Recall that

$$[E : \mathbb{Q}] = 2g = 2\dim(T).$$

Appying Lemma 3.3, we conclude that T is simple and $D(T) = E$.

Recall that $\dim(T) \geq 2$. Applying Lemma 3.4 to T and taking into account Remark 3.5, we obtain that the algebraic dimension of T is 0. It follows from already proven Propositions 1.7 and 1.8 that $\mathrm{NS}(T) = \{0\}$ and $\mathrm{Hom}(T, T^\vee) = \{0\}$.

In order to prove assertion (d), notice that E does not contain any roots of unity except $\{1, -1\}$. Indeed, if this is not the case then either E contains a primitive fourth root of unity $\sqrt{-1}$ or a primitive pth root of unity ζ where p is an odd prime. In all these cases E contains a quadratic subfield that is either $\mathbb{Q}(\sqrt{-1})$ or $\mathbb{Q}(\sqrt{-p})$ (if p is congruent to 3 mod 4) or $\mathbb{Q}(\sqrt{p})$ (if p is congruent to 1 mod 4). Since E does not contain a quadratic subfield, it does not contain any roots of unity except $\{1, -1\}$. Now the assertion (d) follows readily from Proposition 1.9.

□

Example 3.7. Let $T = V/\Gamma$ be a complex torus of dimension $g \geq 2$ where V is a g-dimensional complex vector space and Γ is a discrete lattice of rank $2g$ in V. Let ϕ_T be a holomorphic endomorphism of the complex Lie group T that enjoys the following properties.

If ϕ_Γ is the endomorphism of Γ induced by ϕ_T and $f(x) \in \mathbb{Z}[x]$ the characteristic polynomial of ϕ_Γ (which is monic of degree $2g$) then it is separable, has no real roots and its Galois group $\mathrm{Gal}(f)$ over $\mathbb{Q}$ is a *transitive primitive* subgroup of the full symmetric group $\mathbf{S}_{2g}$.

Let E be the $\mathbb{Q}$-subalgebra of $\mathrm{End}^0(T)$ generated by ϕ_T. The conditions on $f(x)$ and $\mathrm{Gal}(f)$ imply that $f(x)$ is irreducible and $E \cong \mathbb{Q}[x]/f(x)\mathbb{Q}[x]$ is a purely imaginary number field of degree $2g$. In light of Proposition 2.1, the condition on $\mathrm{Gal}(f)$ implies that E has no proper subfields except $\mathbb{Q}$. Applying Theorem 1.3, we conclude that T is a *special torus* and $\mathrm{End}^0(T) = E$.

4. Poor tori

Definition 4.1 (See [BZ20])**.** We say that a compact connected complex manifold Y of positive dimension is *poor* if it enjoys the following properties.

- The algebraic dimension $a(Y)$ of Y is 0.
- Y does not contain analytic subspaces of codimension 1.
- Y contains no rational curve, i.e., an image of a non-constant holomorphic map $\mathbb{P}^1 \to X$. (In other words, every holomorphic map $\mathbb{P}^1 \to Y$ is constant.)

Let Y be a poor manifold. Obviously, $\dim(Y) \geq 2$. For a surface, *poor* means the absence of any curve $C \subset Y$. An explicit example of a $K3$ surface having this property may be found in [McM] and in [BHPV, Proposition 3.6, Chapter VIII]). Explicit examples of complex 2-dimensional tori Y with $a(Y) = 0$ are given in [BL, Example 7.4]. It is proven in [CDV, Theorem 1.2] that if a compact Kähler 3-dimensional manifold has no nontrivial subvariety then it is a complex torus.

On the other hand, a complex torus T with $\dim(T) \geq 2$ and $a(T) = 0$ is a poor Kähler manifold. Indeed, a complex torus T is a Kähler manifold that does not contain rational curves. If $a(T) = 0$, it contains no analytic subsets of codimension 1 [BL, Corollary 6.4, Chapter 2]. Thus a complex torus T is poor if and only if $a(T) = 0$.

We will use the following properties of poor manifolds.

Lemma 4.2. *Let X, Y be two complex compact connected manifolds and let $f : X \to Y$ be a surjective holomorphic map. Assume that Y is poor. Then*

(1) *if $F_y := f^{-1}(y)$ is finite for every $y \in Y$ then X is poor;*
(2) *if $F_y := f^{-1}(y)$ is a poor manifold with dim$(F_y) = \dim(X) - \dim(Y)$ for every $y \in Y$ then X is poor.*

In particular, the direct product of poor manifolds is a poor manifold.

Proof. For proving (1), let us note that f is an unramified cover of Y. Indeed, the image R under f of the ramification locus is either empty or has pure codimension 1 in Y ([DG, Section 1, 9], [Pe, Theorem1.6], [Re]). Since Y is poor, R is empty. Now statement (1) follows from [BZ20, Lemma 3.1].

Let us prove (2). Assume that $C \subset Y$ is a rational curve. If $C \subset F_y$ for some $y \in Y$ then F_y is not poor, which is not the case. Thus $f(C)$ is a rational curve in Y which is also impossible, since Y is poor. Assume that $D \subset Y$ is an analytic irreducible subspace of codimension 1 and $y \in Y$. If $D \cap F_y \neq \emptyset$, then $D \supset F_y$, since otherwise $D \cap F_y$ would have codimension one in F_y. Thus $f(D)$ is an analytic subspace of Y ([Re], [Nar, Theorem 2, Chapter VII]) and

$$\dim(f(D)) = \dim(D) - \dim(F_y) = \dim(Y) - 1,$$

which is impossible since Y is poor. Thus, X is poor: it contains neither rational curves nor analytic subspaces of codimension 1. □

Remark 4.3. Let X be as in Lemma 4.2. The fact that $a(X) = 0$ follows also from [Ue, Theorem 3.8]).

Proof of Theorem 1.10. Notice that T is a torus with smCM by E. Thanks to Proposition 3.2, T is isogenous to S^r where S is a simple torus such that its endomorphism algebra $D(S)$ is isomorphic to a subfield of E. Hence, $D(S)$ is not a CM field. Applying Lemma 3.4, we conclude that the algebraic dimension of S is 0. According to Lemma 4.2, this implies that $a(T) = 0$ as well.

□

5. Explicit examples

Our goal is to describe an explicit construction of simple complex tori T with algebraic dimension 0 in all complex dimensions $g \geq 2$. In order to apply Theorem 1.3 and Proposition 2.1, let us find a degree $2g$ monic irreducible polynomial $f(x) \in \mathbb{Q}[x]$ such that

- $f(x)$ has no real roots;
- Gal(f) is *primitive*.

Suppose that we are given such a $f(x)$ (see this section below). Then the quotient $E = \mathbb{Q}[x]/f(x)\mathbb{Q}[x]$ is a degree $2g$ purely imaginary field that does not contain proper subfields except $\mathbb{Q}$. We write $\tilde{x}$ for the image of x in E.

Then $\{1, \tilde{x}, \tilde{x}^2, \dots, \tilde{x}^{2g-1}\}$ is a basis of the $\mathbb{Q}$-vector space E of dimension $2g$. It follows that

$$\Lambda = \mathbb{Z}[\tilde{x}] = \mathbb{Z} \cdot 1 + \mathbb{Z} \cdot \tilde{x} + \cdots + \mathbb{Z} \cdot \tilde{x}^{2g-1} \subset E$$

is a free $\mathbb{Z}$-module of rank $2g$ with the basis

$$\{1, \tilde{x}, \tilde{x}^2, \dots, \tilde{x}^{2g-1}\}.$$

Let $\alpha_1, \dots, \alpha_g \in \mathbb{C}$ be all the roots of $f(x)$ with positive imaginary part. Then

$$\{\alpha_1, \bar{\alpha}_1, \dots, \alpha_g, \bar{\alpha}_g\}$$

is the set of all complex roots of $f(x)$. Let

$$\tau_j : E = \mathbb{Q}[x]/f(x)\mathbb{Q}[x] \hookrightarrow \mathbb{C}, \ u(x) + f(x)\mathbb{Q}[x] \mapsto u(\alpha_j)$$

be the $\mathbb{Q}$-algebra homomorphism that sends $\tilde{x} \in E$ to α_j $(1 \le j \le g)$. As in the beginning of Section 2, the direct sum of all τ_i defines an injective $\mathbb{Q}$-algebra homomorphism

$$\Phi : E \hookrightarrow \mathbb{C}^g, \beta \mapsto (\tau_1(\beta), \dots \tau_g(\beta)),$$

which extends to the isomorphism $\Phi : E_{\mathbb{R}} \cong \mathbb{C}^g$ of $\mathbb{R}$-algebras. If $\beta \in E \subset E_{\mathbb{R}}$ then

$$\Phi(\beta) = (\tau_1(\beta), \dots \tau_g(\beta)) \in \mathbb{C}^g.$$

In particular,

$$\Phi(1) = (1, \dots, 1), \Phi(\tilde{x}) = (\alpha_1, \dots, \alpha_g)$$

and therefore

$$\Phi(\tilde{x}^k) = (\alpha_1^k, \dots, \alpha_g^k)$$

for all nonnnegative integers k. This implies that the $2g$-element set

$$(1, \dots, 1), (\alpha_1, \dots, \alpha_g), (\alpha_1^2, \dots, \alpha_g^2), \dots, (\alpha_1^{2g-1}, \dots, \alpha_g^{2g-1})$$

is a basis of the lattice $\Phi(\Lambda) \subset \mathbb{C}^g$.

Let us consider the g-dimensional complex torus

$$T(f) := \mathbb{C}^g/\Phi(\Lambda).$$

It follows from Theorem 1.3 combined with and Proposition 2.1 that $T(f)$ is a *special torus* and $\mathrm{End}^0(T(f)) \cong \mathbb{Q}[x]/f(x)\mathbb{Q}[x]$.

Below we present such polynomials for every even degree $2g \ge 2$. (See also an explicit example for $g = 4$ in [GS, Sect. 3A, pp. 271–272].)

6. Truncated exponents

Let $n \ge 1$ be an integer. Let us consider the *truncated exponent*

$$\exp_n(x) = \sum_{j=0}^{n} \frac{x^j}{j!} \in \mathbb{Q}[x] \subset \mathbb{R}[x].$$

Notice that its derivative

$$\exp_n'(x) = \exp_{n-1}(x) = \exp_n(x) - \frac{x^n}{n!} \quad \forall n \ge 2.$$

Lemma 6.1. *If $n \ge 2$ is an even integer then $\exp_n(x)$ has no real roots.*

Proof. Since $\exp_n(x)$ is an even degree polynomial with positive leading coefficient, it takes on the smallest possible value on $\mathbb{R}$ at a certain $x_0 \in \mathbb{R}$. Then

$$0 = \exp_n'(x_0) = \exp_n(x_0) - \frac{x_0^n}{n!}$$

and therefore

$$\exp_n(x_0) = \frac{x_0^n}{n!}. \tag{18}$$

If $x_0 = 0$ then $1 = \exp_n(0) = 0$, which is not the case. This implies that $x_0 \ne 0$. Taking into account that n is even, we obtain from (18) that $\exp_n(x_0) > 0$. Since $\exp_n(x_0)$ is the smallest value of the function $\exp_n$ on the whole $\mathbb{R}$, the polynomial $\exp_n$ takes on only positive values on $\mathbb{R}$ and therefore has *no* real roots. □

By a theorem of Schur [Col], $\mathrm{Gal}(\exp_n(x)) = \mathbf{S}_n$ or $\mathbf{A}_n$. It follows from Example 2.3 combined with Lemma 6.1 that if $n = 2g$ is even then

$$E = K_g := \mathbb{Q}[x]/\exp_{2g}(x)\mathbb{Q}[x]$$

is a degree $2g$ purely imaginary field that has no proper subfields except $\mathbb{Q}$.

Now the construction of Section 5 applied to $f(x) = \exp_{2g}(x)$ gives us for all $g \ge 2$ a *special* g-dimensional complex torus $T(\exp_{2g})$ with endomorphism algebra $K_g = \mathbb{Q}[x]/\exp_{2g}(x)\mathbb{Q}[x]$.

7. Selmer polynomials

Another series of examples is provided by polynomials

$$\mathrm{selm}_{2g}(x) = x^{2g} + x + 1 \in \mathbb{Z}[x] \subset \mathbb{Q}[x] \subset \mathbb{R}[x]. \tag{19}$$

Notice that $\mathrm{selm}_{2g}(x)$ takes on only *positive values* on the real line $\mathbb{R}$, hence, it does *not* have real roots. Indeed, if $a \in \mathbb{R}, |a| \ge 1$ then $a^{2g} + a \ge 0$ and therefore

$$\mathrm{selm}_{2g}(a) = (a^{2g} + a) + 1 \ge 1 > 0.$$

If $a \in \mathbb{R}, |a| < 1$ then $a + 1 > 0$ and therefore

$$\mathrm{selm}_{2g}(a) = a^{2g} + (a + 1) \ge a + 1 > 0.$$

Let us assume that g is *not* congruent to 1 mod 3. Then $2g$ is *not* congruent to 2 mod 3 and therefore, by a theorem of Selmer [Sel56, Th. 1], $\mathrm{selm}_{2g}(x)$ is irreducible over $\mathbb{Q}$. Notice that the coefficient of the trinomial $\mathrm{selm}_{2g}(x)$ at x and its constant term are relatively prime, square free and coprime to both $2g$ and $2g - 1$. It follows from [NV] (see also [Os, Cor. 2 on p. 233]) applied to $a_0 = b_0 = c = 1, n = 2g$) that $\mathrm{Gal}(\mathrm{selm}_{2g}(x)) = \mathbf{S}_{2g}$.

It follows from Example 2.3 that if a positive integer g is *not* congruent to 1 mod 3 then

$$E = M_g := K_{\mathrm{selm}_{2g}} = \mathbb{Q}[x]/\mathrm{selm}_{2g}(x)\mathbb{Q}[x]$$

is a degree $2g$ purely imaginary field that has no proper subfields except $\mathbb{Q}$.

Now the construction of Subsection 5 gives us for all $g \geq 2$ that are *not* congruent to 1 mod 3, a g-dimensional *special* complex torus $T(\mathrm{selm}_{2g})$ with endomorphism algebra M_g.

Notice that if $g \geq 5$ is *not* congruent to $1 \bmod 3$ then g-dimensional special complex tori $T(\exp_{2g})$ and $T(\mathrm{selm}_{2g})$ are *not* isogenous. Indeed, suppose that they are isogenous. Then their endomorphism algebras (which are actually number fields) K_g and M_g are isomorphic. It follows from [Os, the last assertion of Cor. 2 on p. 233] (applied to $a_0 = b_0 = c = 1, n = 2g$) that all the ramification indices in the field extension $M_g/\mathbb{Q}$ do not exceed 2. On the other hand, it is proven in [Zar03, Sect. 5] that there is a prime p that enjoys the following properties.

- $g + 1 \leq p \leq 2g + 1 := n - 1$.
- One of ramification indices over p in the field extension $K_g/\mathbb{Q}$ is divisible by p. In particular, this index

$$\geq p \geq g + 1 \geq 5 + 1 = 6 > 2.$$

This implies that number fields K_g and M_g are not isomorphic. The obtained contradiction proves that the tori $T(\tilde{\exp}_{2g})$ and $T(\mathrm{selm}_{2g})$ are *not* isogenous.

8. Polynomials with doubly transitive Galois group.

The following construction was inspired by so called *Mori polynomials* [Mori, Zar16]. As above, $g \geq 2$ is an integer, hence $2g - 1 \geq 3$. Let us fix

- a prime divisor l of $2g - 1$;
- a prime p that is congruent to 1 modulo $2g - 1$;
- an integer b that is *not* divisible by l and that is a primitive root mod p;
- an integer c that is *not* divisible by l.

We call such a (l, p, b, c) a *g-admissible quadruple*.

Remark 8.1. Let $g \geq 2$ and l be any prime divisor of $2g - 1$. In light of Dirichlet's Theorem about primes in arithmetic progressions (which allows us to choose p) and Chinese Remainder Theorem (which allows us to choose b), there are infinitely many g-admissible quadruples (l, p, b, c).

Now let us consider a monic degree $2g$ polynomial

$$f_g(x) = f_{g,l,p,b,c}(x) := x^{2g} - bx - \frac{pc}{l^l} \in \mathbb{Z}[1/l][x] \subset \mathbb{Q}[x]. \qquad (20)$$

Lemma 8.2. (i) *The polynomial $f_g(x) = f_{g,l,p,b,c}(x)$ is irreducible over the field $\mathbb{Q}_l$ of l-adic numbers and therefore over $\mathbb{Q}$.*

(ii) *The polynomial $(f_g(x) \bmod p) \in \mathbb{F}_p[x]$ is a product $x\left(x^{2g-1} - b \bmod p\right)$ of a linear factor x and an irreducible (over $\mathbb{F}_p$) degree $2g - 1$ polynomial $x^{2g-1} - (b \bmod p)$.*

(iii) *Let $\mathrm{Gal}(f_g)$ be the Galois group of $f_g(x)$ over $\mathbb{Q}$ viewed as a transitive subgroup of $\mathrm{Perm}(\mathscr{R}_{f_g})$.*

Then transitive $\mathrm{Gal}(f_g)$ *contains a permutation* σ *that is a cycle of length* $2g-1$*. In particular,* $\mathrm{Gal}(f_g)$ *is a doubly transitive permutation subgroup of* $\mathrm{Perm}(\mathscr{R}_{f_g})$*.*

(iv) *The polynomial* $f_g(x)$ *has no real roots if and only if*

$$c < \frac{l^l \left(\frac{b}{2g}\right)^{1/(2g-1)} \left(\frac{b}{2g} - 1\right)}{p}. \tag{21}$$

(v) *Let* ℓ *be a prime that divides* b*, does not divide* $2glp$*, and such that* c *is congruent to* ℓ *modulo* ℓ^2*. Then the discriminant of the number field* $\mathbb{Q}[x]/f_g(x)\mathbb{Q}[x]$ *is divisible by* ℓ*.*

(vi) *Let* ℓ *be a prime that divides* c *and does not divide* $(2g-1)pb$*. Then the discriminant of the number field* $\mathbb{Q}[x]/f_g(x)\mathbb{Q}[x]$ *is not divisible by* ℓ*.*

Remark 8.3. Let (l, p, b, c) be a g-admissible quadruple. Let N be a positive integer such that

$$N > \frac{l^l \left(\frac{b}{2g}\right)^{1/(2g-1)} \left(\frac{b}{2g} - 1\right)}{p} - c.$$

(1) Replacing c by $c_1 = c - Nlp$, we get a g-admissible quadruple (l, p, b, c_1) such that the corresponding polynomial $f_{l,g,p,b,c_1}(x)$ has no real roots, in light of Lemma 8.2(iii) and (21).

(2) Let ℓ be a prime that satisfies conditions (v) (respectively (vi)) of Lemma 8.2 with respect to (l, p, b, c). Let $c_2 = c - Nlp\ell^2$. Then (l, p, b, c_2) is also a g-admissible quadruple and $f_{l,g,p,b,c_1}(x)$ has no real roots, in light of the previous remark (applied to $N\ell^2$ instead of N). In addition, c_2 is congruent to ℓ modulo ℓ^2 (respectively, is not divisible by ℓ). In other words, ℓ also satisfies the congruence properties similar to conditions (v) (respectively to (vi)) of Lemma 8.2 where c is replaced by c_2. It follows from Lemma 8.2(v) (respectively (vi)) that the discriminant of $\mathbb{Q}[x]/f_{l,g,p,b,c_1}(x)\mathbb{Q}[x]$ is divisible by ℓ (respectively, not divisible by ℓ).

Proof of Lemma 8.2. (i) The l-adic Newton polygon of $f_g(x)$ consists of one segment that connects its endpoints $(0, -l)$ and $(2g, 0)$, which are its only integer points, since prime l does *not* divide $2g$. Now the irreducibility of $f_g(x)$ follows from Eisenstein–Dumas Criterion ([Mott, Corollary 3.6, p. 316], [Gao, p. 502]).

(ii) The conditions on b and p imply that for each divisor $d > 1$ of $2g-1$ the residue $b \bmod p$ is *not* a dth power m^d for any $m \in \mathbb{F}_p$. It follows from theorem 9.1 of [Lang, Ch. VI, Sect. 9] that the polynomial $x^{2g-1} - (b \bmod p)$ is *irreducible* over $\mathbb{F}_p$ and therefore its Galois group over $\mathbb{F}_p$ is a cyclic group of order $2g-1$.

(iii) Let us consider the reduction

$$\bar{f}_g(x) = (f_g(x) \bmod\ p) \in \mathbb{F}_p[x]$$

of $f_g(x)$ modulo p. Clearly, $\bar{f}_g(x) = x(x^{2g-1} - (b \bmod\ p))$ is a product in $\mathbb{F}_p[x]$ of relatively prime linear x and irreducible $x^{2g-1} - (b \bmod p)$. This implies that $\mathbb{Q}\left(\mathscr{R}_{f_g}\right)/\mathbb{Q}$ is unramified at p and a corresponding Frobenius element in

$$\mathrm{Gal}\left(\mathbb{Q}(\mathscr{R}_{f_g}\right)/\mathbb{Q}) = \mathrm{Gal}(f_g) \subset \mathrm{Perm}\left(\mathscr{R}_{f_g}\right)$$

is a cycle of length $2g-1$. This proves (iii).

(iv). Since $f_g(x)$ has even degree and positive leading coefficient, it reaches its smallest value on $\mathbb{R}$ at a certain real point that is zero of its derivative $f_g'(x) = 2gx^{2g-1} - b$. The only real zero of $f_g'(x)$ is $\beta = (b/2g)^{\frac{1}{2g-1}}$. Hence, $f_g(x)$ has no real roots if and only if $f_g(\beta) > 0$. We have

$$f_g(\beta) = \beta^{2g} - b\beta - \frac{pc}{l^l} = \left(\frac{b}{2g}\right)^{2g/2g-1} - b\left(\frac{b}{2g}\right)^{1/(2g-1)} - \frac{pc}{l^l} =$$

$$\left(\frac{b}{2g}\right)^{1/(2g-1)}\left(\frac{b}{2g} - 1\right) - \frac{pc}{l^l}.$$

This implies that $f_g(\beta) > 0$ if and only if

$$\left(\frac{b}{2g}\right)^{1/(2g-1)}\left(\frac{b}{2g} - 1\right) > \frac{pc}{l^l},$$

i.e.,

$$c < \frac{l^l}{p}\left(\frac{b}{2g}\right)^{1/(2g-1)}\left(\frac{b}{2g} - 1\right).$$

This proves (iv).

(v)-(vi). Let us consider the degree $2g$ number field $E := \mathbb{Q}[x]/f_g(x)\mathbb{Q}[x]$ and its discriminant $\mathbf{\Delta}_E \in \mathbb{Z}$. The formula for the discriminant of a trinomial [FS, Example 834] tells us that the discriminant $\mathrm{Discr}(f_g)$ of $f_g(x)$ is

$$\mathrm{Discr}(f_g) = (-1)^{g(2g-1)}(2g)^{2g}\left(\frac{pc}{l^l}\right)^{2g-1} + (-1)^{(2g-1)(g-1)}(2g-1)^{2g-1}b^{2g} = \tag{22}$$

$$\pm\left(\frac{p}{l^l}\right)^{2g-1}(2g)^{2g}c^{2g-1} \mp (2g-1)^{2g-1}b^{2g} \in \mathbb{Z}[1/l].$$

Notice that there is a *nonzero* rational number r such that

$$r^2 \cdot \mathbf{\Delta}_E = \mathrm{Discr}(f_g) \tag{23}$$

(see, e.g., [BS, Algebraic Extensions, Sect. 2.3, especially, formula 2.12] applied to $k = \mathbb{Q}$ and $K = E$).

In the case of (v), there are integers $c_1, b_1 \in \mathbb{Z}$ such that

$$c = \ell(1 + c_1\ell),\ b = \ell b_1.$$

It follows from (22) that

$$\mathrm{Discr}(f_g) = \ell^{2g-1} \cdot u_1 + \ell^{2g}u_2$$

where $u_1 \in \mathbb{Z}[1/l]$ is an ℓ-adic unit and $u_2 \in \mathbb{Z}$ is an integer. This implies that $\mathrm{Discr}(f_g) = \ell^{2g-1}u$ where $u \in \mathbb{Q}$ is an ℓ-adic unit. Since $2g-1$ is odd, it follows from (23) that $\mathbf{\Delta}_E$ is divisible by ℓ, which proves (v).

In the case of (vi), it follows from (22) that

$$\mathrm{Discr}(f_g) = \ell^{2g-1}v_1 + v_2$$

where $v_1 \in \mathbb{Z}[1/l]$ is an ℓ-adic unit and $v_2 \in \mathbb{Z}$ is an integer *not* divisible by ℓ. This implies that $\mathrm{Discr}(f_g) \in \mathbb{Z}[1/l]$ is an ℓ-adic unit. Taking into account that $\ell \neq l$, we obtain that the reduction modulo ℓ

$$f_g(x) \bmod \ \ell \in (\mathbb{Z}[1/l]/\ell\mathbb{Z}[1/l])[x] = \mathbb{F}_\ell[x]$$

of $f_g(x)$ is a degree $2g$ monic polynomial with coefficients in $\mathbb{F}_\ell$ and without repeated roots. It follows from [FS, Ch. III, Sect 2, Th. 23 on p. 129] (applied to $\mathfrak{o} = \mathbb{Z}[1/l]$ and $\mathfrak{p} = \ell\mathbb{Z}[1/l]$) that the prime ideal $\ell\mathbb{Z}[1/l]$ of the Dedekind ring $\mathbb{Z}[1/l]$ is unramified in E. This means that the *discriminant ideal* $\mathbf{\Delta}_E \cdot \mathbb{Z}[1/l]$ of $\mathbb{Z}[1/l]$ is *not* contained in $\ell\mathbb{Z}[1/l]$. It follows that $\mathbf{\Delta}_E$ is *not* divisible by ℓ, which proves (vi). □

Now assume that we have chosen c in such a way that inequality (21) holds. It can be done, in light of Remark 8.3. Then we have:

- $f_g(x)$ is irreducible over $\mathbb{Q}$ and has no real roots (Lemma 8.2(i));
- the group $\mathrm{Gal}(f_g)$ is doubly transitive (Lemma 8.2(iii));
- the group $\mathrm{Gal}(f_g)$ is primitive (Remark 2.2).

It follows from Lemma 8.2 that

$$E = L_g = L_{g,l,p,b,c} := \mathbb{Q}[x]/f_{g,l,p,b,c}(x)\mathbb{Q}[x]$$

is a degree $2g$ purely imaginary field that has no proper subfields except $\mathbb{Q}$.

Now the construction of Section 5 gives us for all $g \geq 2$ a *special* g-dimensional complex torus $T_{g,l,p,b,c} := T(f_{g,l,p,b,c})$ with endomorphism algebra $L_{g,l,p,b,c}$.

9. Isogeny classes.

Let $g \geq 2$ be an integer. The aim of this section is to construct infinitely many mutually non-isogenous special g-dimensional complex tori.

Let us choose a g-admissible quaduple (l, p, b, c) that satisfies (21). The construction of Section 8 gives us a special complex torus $T^{(1)} := T_{g,l,p,b,c}$ of dimension g. Suppose that n is a positive integer and we have already constructed n mutually non-isogenous g-dimensional special complex tori

$$T^{(k)} = T_{g,l,p,b_k,c_k}, \ 1 \leq k \leq n$$

where each (l, p, b_k, c_k) is a g-admissible quaduple such that $f_{g,l,p,b_k,c_k}(x)$ has no real roots. In particular, the endomorphism algebra of $T^{(k)}$ is isomorphic to the purely imaginary number field L_{g,l,p,b_k,c_k}.

Let us choose

- an odd prime $\ell \neq l, p$ that does *not* divide g, and is *unramified* in all number fields L_{g,l,p,b_k,c_k} $(1 \leq k \leq n)$, i.e., does not divide the discriminant of any L_{g,l,p,b_k,c_k};
- an integer b_{n+1} that is *not* divisible by l and is a primitive root mod p.

Assume additionally, that b_{n+1} is divisible by ℓ. Since all three primes l, p, ℓ are distinct, such a b_{n+1} does exist, thanks to Chinese Remainder Theorem. Now let us choose an integer c_{n+1} that is not divisible by l and congruent to ℓ modulo ℓ^2. Then (l, p, b_{n+1}, c_{n+1}) is a g-admissible quadruple such that the discriminant of the number field $L_{g,l,p,b_{n+1},c_{n+1}}$ is *divisible* by ℓ, thanks to Lemma 8.2(v). According to Remark 8.3, one may also choose c_{n+1} in such a way that $f_{g,l,p,b_{n+1},c_{n+1}}(x)$ has no real roots., i.e., the field $L_{g,l,p,b_{n+1},c_{n+1}}$ is purely imaginary. This gives us a special g-dimensional complex torus $T^{(n+1)} = T_{g,l,p,b_{n+1},c_{n+1}}$, whose endomorphism algebra $\mathrm{End}^0(T^{(n+1)})$ is isomorphic to the field $L_{g,l,p,b_{n+1},c_{n+1}}$, which is *ramified* at ℓ.

Our choice of ℓ implies that $L_{g,l,p,b_{n+1},c_{n+1}}$ is not isomorphic to any of L_{g,l,p,b_k,c_k} with $k \leq n$. It follows that $T^{(n+1)}$ is *not* isogenous to any of $T^{(k)}$ with $k \leq n$. In light of results of Section 8, all $T^{(1)}, \dots, T^{(n)}, T^{(n+1)} \dots$ are *special* g-dimensional *mutually non-isogenous* complex tori.

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