28. MATHEMATISCHE ARBEITSTAGUNG

1988

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für Mathematik
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Programm der Mathematischen Arbeitstagung 1988 (I)

Donnerstag, den 16.6.1988

16.00 - 17.00 Uhr	M.F. ATIYAH: Topological quantum field theory

Freitag, den 17.6.1988

10.15 - 11.15 Uhr	A. SUSLIN: K-theory of fields
12.00 - 13.00 Uhr	G. MARGULIS: Minima of indefinite quadratic forms
17.00 - 18.00 Uhr	A. PARSHIN: Inequalities for algebraic surfaces and connections with number theory

Samstag, den 18.6.1988

10.00 - 10.15 Uhr	Festlegung der nächsten Vorträge
10.15 - 11.15 Uhr	J. BISMUT: Quillen metrics and degenerating Riemann surfaces
12.00 - 13.00 Uhr	K. RUBIN: Finiteness of the Shafarevich group (Kolyvagin's work)
17.00 - 18.00 Uhr	R. THOM: Animal organization as a stratified space

Die Vorträge finden alle im "*Großen Hörsaal*", Wegelerstraße 10, statt.

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Alle Tagungsteilnehmer mit ihren Damen oder Herren sind herzlich zum *Empfang des Rektors* eingeladen. Zeit: Donnerstag, den 16.6., 20.00 Uhr. Ort: Festsaal der Universität, Hauptgebäude; Eingang von der Straße "Am Hof" durch das Tor gegenüber Buchhandlung Röhrscheid.

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Programm der Mathematischen Arbeitstagung (II)

Sonntag, den 19.6.1988

10.15 - 11.15 Uhr	W. BALLMANN: Dirichlet problem at ∞
12.00 - 13.00 Uhr	D. KOTSCHICK: Differentiable classification of algebraic surfaces
17.00 - 18.00 Uhr	J. FRANKE: Chow categories

Montag, den 20.6.1988

10.30 - 10.45 Uhr	Festlegung der nächsten Vorträge
10.45 - 11.45 Uhr	G. WÜSTHOLZ: Tate conjecture by transcendence theory
13.00 Uhr	Schiffsausflug nach Andernach. Abfahrt um 13.00 Uhr mit Motorschiff "Carmen Sylva", Ablegestelle Alter Zoll. Rückkehr ca. 20.00 Uhr.

Dienstag, den 21.6.1988

10.15 - 11.15 Uhr	Y. NAMIKAWA: Conformal field theory on Riemann surfaces

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Erfrischungspausen mit Tee: Sonntag und Dienstag, 11.15 - 12.00 Uhr und 16.00 - 17.00 Uhr vor dem Großen Hörsaal.

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Programm der Mathematischen Arbeitstagung 1988 (III)

Dienstag, den 21.6.1988

12.00 - 13.00 Uhr J.-M. FONTAINE: Semi-stable Galois representations

17.00 - 18.00 Uhr J. LEITERER: Extension of CR cohomology. Deformation of vector bundles on pseudoconvex manifolds with 2 convex holes

Mittwoch, den 22.6.1988

10.15 - 11.15 Uhr R. HAIN: A trilogarithm

12.00 - 13.00 Uhr K. GROVE: Bounding homotopy types by geometry

15.00 - 16.00 Uhr J. HUBBARD: Iteration of cubic polynomials

Die Vorträge finden alle im "*Großen Hörsaal*", Wegelerstraße 10, statt.

Erfrischungspausen mit Tee: Dienstag, 16.00 - 17.00 Uhr vor dem Großen Hörsaal; Mittwoch, 11.15 - 12.00 Uhr vor dem Großen Hörsaal.

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Titel: TOPOLOGICAL QUANTUM FIELD THEORY

Autor: MICHAEL ATIYAH

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§ 1 Introduction

In previous years at the Arbeitstagungs there have been many lectures on the important work of Donaldson [2] concerning the structure of smooth 4-dimensional manifolds, and more recently on the related work by Floer [3] on 3-dimensional manifolds. I have in [1] tried to present these in the framework of super-symmetric quantum field theory. Very recently Witten [5] has found an appropriate model of such a theory which yields the Donaldson/Floer theory in the classical limit.

Witten's theory is quite complicated and involves a large number of auxiliary fields, both bosonic and fermionic. Their geometric significance and origin is not transparent.

In this lecture I will indicate how to view Witten's theory from the standpoint of geometry.

§ 2 The self-duality equation

I review first the essentials of Donaldson theory.
We fix a compact oriented smooth 4-dimensional
manifold X with a Riemannian metric ρ, and
a compact Lie group G which we take ~~to be~~ $SU(2)$.
Let A be an $SU(2)$-connection for a principal
bundle P over X with $c_2 = k$. Using the Hodge
$*$ operator we can decompose the curvature F_A into
$F_A^+ \oplus F_A^-$, the ± 1-eigenspaces of $*$. Donaldson studies
the anti-self-duality equation $F_A^+ = 0$. Modulo
the action of the gauge group $\mathcal{G} = \mathrm{Aut}(P)$ the
solutions are parametrized by a space $M_k(X, \rho)$
whose dimension is generically given by a formula

$$\dim M_k = 8k - 3(1 - b_1 + b_2^+)$$

where b_i are the Betti numbers of X and b_2^+ is
the number of positive eigenvalues of the intersection
form on $H_2(X)$.

Assume for simplicity that $\dim M_k = 0$, then Donaldson shows that the number of points in M_k, counted with suitable signs, is independent of ρ and is an <u>invariant</u> of X [He also defines similar invariants in the general case, for higher values of k].

Formally if $\mathcal{A}$ denotes the space of all connections A, and ignoring reducible connections, F_A^+ can be viewed as a section of an infinite-dimensional vector bundle (fibre the space of self-dual Lie algebra valued 2-forms on X) over the infinite-dimensional manifold $\mathcal{B} = \mathcal{A}/\mathcal{G}$.

If we can write down a suitable infinite-dimensional version of the Gauss-Bonnet theorem, computing the number of zeros of a section of a vector bundle in terms of an integrand, we will be led to a Witten type Lagrangian.

§3 Euler classes

The standard Gauss-Bonnet integrand, involving the Pfaffian of the curvature, can be generalized to a formula, depending also on a given section, which is due to Mathai & Quillen [4]. This has several advantages;

(i) It has Gaussian behaviour (rather than compact support)

(ii) It is expressed in terms of "fermion integration"

(iii) It explicitly uses a section and, by a rescaling argument, leads naturally to counting zeros.

This formula involves of course the curvature. When the principal bundle comes from a group action on a Riemannian manifold the curvature involves a denominator measuring the size of group orbits [and this would blow up at fixed points for a non-free action]. To eliminate such denominator one can use a double integral (expressing the Fourier inversion formula) over the Lie algebra.

The final formula for the Euler class obtained in this way can be shown to coincide with the supersymmetric Lagrangian introduced by Witten.

REFERENCES

[1] M.F. Atiyah, New Invariants of Three and Four Dimensional Manifolds, A.M.S Symposium, North Carolina (to appear)

[2] S.K. Donaldson, Proc. International Congress, Berkeley (to appear)

[3] A. Floer, An Instanton invariant for three Manifolds (to appear)

[4]. V. Mathai and D. Quillen, Superconnections, Thom classes and equivariant differential forms, Topology 25 (1986), 85-110.

[5]. E. Witten, Topological Quantum Field Theory Institute for Advanced Study, preprint 1988

Titel: *Algebraic K-theory of fields*

Autor: A. Suslin

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One of the important tools in algebraic K-theory of schemes is a spectral sequence (due to D. Quillen), relating K'-theory of a scheme X to K-theory of it's residue fields

$$E_1^{pq} = \coprod_{x \in X^p} K_{-p-q}(k(x)) \Rightarrow K'_{-p-q}(X)$$

This spectral sequence shows that to prove any sufficiently general conjecture relating K-theory to other cohomology theories it's usually sufficient to treat the case of a field.

The K-theory of any ring A may be defined in terms of Quillen's plus construction : $K_i(A) = \pi_i(BGL(A)^+)$, where $BGL(A)^+$ is the space, having the same homology as $BGL(A)$, but whose fundamental group is obtained from $\pi_1(BGL(A)) = GL(A)$ by killing the perfect normal subgroup $E(A)$ (= the subgroup generated by elementary matrices). In particular $K_1(A) = GL(A)/E(A)$ (this definition of K_1 due to H. Bass was historically the first step towards the developement of higher K-theory). Specializing to the case of fields one sees immediately that for a field F $K_0(F) = \mathbb{Z}$, $K_1(F) = F^*$.

Furthermore products in K-theory make $K_*(F)$ a graded commutative ring, so that one has a canonical map $F^* \otimes F^* \to K_2(F)$ and a well-known theorem due to C. Moore and H. Matsumoto states, that this map is surjective and it's kernel is generated by elements of the form $x \otimes (1-x)$ $(x \in F^* - 1)$.

For any field F set $K_*^M(F) = T(F^*)/I$, where T stands for the tensor algebra and I is a graded ideal, generated by tensors of the form $x \otimes (1-x) \in T_2(F^*)$ $(x \in F^* - 1)$. By what was said above there is a canonical map $K_*^M(F) \to K_*(F)$, which is bijective in dimensions ≤ 2. To produce a map in the opposite sence we need the following result

Theorem 1([6]) For any infinite field F the canonical maps $H_n(GL_n(F)) \to H_n(GL_{n+1}(F)) \to \ldots \to H_n(GL(F))$ are isomorphisms. Furthermore the map $\underbrace{F^* \otimes \ldots \otimes F^*}_{n} \to H_n(GL_n(F))/H_n(GL_{n-1}(F))$ (defined by means of homology products) induces an isomorphism $K_n^M(F) \xrightarrow{\sim} H_n(GL_n(F))/H_n(GL_{n-1}(F))$

Thus we get a map $K_n(F) = \pi_n(BGL(F)^+) \to H_n(BGL(F)^+) = H_n(GL(F)) = H_n(GL_n(F)) \to K_n^M(F)$. It's easy to check that the composition $K_n^M(F) \to K_n(F) \to K_n^M(F)$ coincides with multiplication by $(-1)^{n-1}(n-1)!$ This shows in particular that the map $K_3^M(F) \to K_3(F)$ is injective up to two-torsion

Set $K_3(F)_{ind} = K_3(F)/K_3^M(F)$. This group is closely related to the so called Block's group of F. For an infinite field F denote by $\mathfrak{p}(F)$ an abelian group with generators $[x]$ $(x \in F^* - 1)$ and relations $[x]-[y]+[\frac{y}{x}]-[\frac{1-x^{-1}}{1-y^{-1}}]+[\frac{1-x}{1-y}] = 0$ $(x \neq y \in F^* - 1)$, denote further by σ the involution of $F^* \otimes F^*$, given by the formula $(x \otimes y)^{\sigma} = -(y \otimes x)$. It's easy to see, that $[x] \mapsto x \otimes (1-x)$ gives a well defined homomorphism $\mathfrak{p}(F) \longrightarrow (F^* \otimes F^*)_{\sigma}$, whose cokernel coincides with $K_2(F)$. Block's group $B(F)$ is defined to be the kernel of the above map.

Theorem 2 ([7]). There is a natural exact sequence $0 \to \mathrm{Tor}(F^*, F^*)^{\sim} \longrightarrow K_3(F)_{ind} \longrightarrow B(F) \to 0$, where $\mathrm{Tor}(F^*, F^*)^{\sim}$ is the unique non trivial $\mathbb{Z}/2$-extension of the (ind-) cyclic group $\mathrm{Tor}(F^*, F^*)$ if char $F \neq 2$ and $\mathrm{Tor}(F^*, F^*)^{\sim} = \mathrm{Tor}(F^*, F^*)$ if char $F = 2$.

Let n be an integer prime to char F, then Kummer's theory defines an isomorphism $\chi : F^*/F^{*n} \xrightarrow{\sim} H^1(F, \mu_n)$. It's easy to check that $\chi(x) \cup \chi(1-x) = 0 \in H^2(F, \mu_n^{\otimes 2})$ and hence we get a graded ring homomorphism $K_*^M(F)/n \longrightarrow \coprod_{i=0}^{\infty} H^i(F, \mu_n^{\otimes i})$. One of the most interesting conjectures in algebraic K-theory states that the above map is an isomorphism. This conjecture is trivial in dimensions 0, 1 and was proved in dimension two by Merkurjev-Suslin [2].

In dimension three this conjecture was proved for $n=2$ by M. Rost [5] and independently by Merkurjev-Suslin [3].

Another piece of information, concerning the relations between K-theory and Galois cohomology is given by the following theorem due to M. Levin [1] and Merkurjev-Suslin [4].

Theorem 3. Suppose that n is prime to char F, then ${}_n(K_3(F)_{ind}) = H^0(F, \mu_n^{\otimes 2})$ and there is a natural exact sequence

$$0 \to K_3(F)_{ind}/n \to H^1(F, \mu_n^{\otimes 2}) \to {}_nK_2(F) \to 0$$

If $p = \text{char } F > 0$, then $K_3(F)_{ind}$ is uniquely p-divisible.

Corollary 1. Set $\mathbb{Q}/\mathbb{Z}(2) = \coprod_{\ell \neq \text{char } F} \mathbb{Q}_\ell/\mathbb{Z}_\ell(2)$, then torsion subgroup in $K_3(F)_{ind}$ coincides with $H^0(F, \mathbb{Q}/\mathbb{Z}(2))$.

Corollary 2. Let F be an algebraic number field and let r_1 (resp. r_2) denote the number of real (resp. complex) places of F, then

$$K_3(F) \cong \begin{cases} \mathbb{Z}^{r_2} \oplus H^0(F, \mathbb{Q}/\mathbb{Z}(2)) & \text{if } r_1 = 0 \\ \mathbb{Z}^{r_2} \oplus (\mathbb{Z}/2)^{r_1-1} \oplus H^0(F, \mathbb{Q}/\mathbb{Z}(2))^{\sim} & \end{cases}$$

where $\sim$ has the same meaning as above

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Titel: Minima of indefinite quadratic forms

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Let B be a real nondegenerate indefinite quadratic form in n variables. According to the classical theorem of Meyer, if $n \geq 5$ and the coefficients of B are rational, then B represents zero nontrivially, i.e. there exist integers $x_1, \ldots, x_n$ not all equal to 0 such that $B(x_1, \ldots, x_n) = 0$. Theorem 1 stated below can be considered as an analogy of this assertion in the case where B is not a multiple of a rational form. Note that in theorem 1 the condition "$n \geq 5$" is replaced by a weaker condition "$n \geq 3$".

Theorem 1 ([3], [4], [5]). Suppose that $n \geq 3$ and that B is not a multiple of a rational form, or equivalently the ratio of some two coefficients of B is irrational. Then for any $\varepsilon > 0$ there exist integers $x_1, \ldots, x_n$ not all equal to 0 such that $|B(x_1, \ldots, x_n)| < \varepsilon$.

Let us formulate a stronger version of theorem 1 (see [5]).

Theorem 1'. Let n and B are the same as in theorem 1. Then for any $\varepsilon > 0$ there exist integers $x_1, \ldots, x_n$ such that

$$0 < |B(x_1, \dots, x_n)| < \varepsilon .$$

Theorems 1 and 1′ are equivalent for forms which do not represent zero nontrivially. Theorem 1 was conjectured by Oppenheim for $n \geq 5$ (see [6], [7]). Theorem 1′ had been proved earlier for $n \geq 21$ (see [2]) and also for some special types of quadratic forms. In [8] it is shown for $n \geq 3$ that if the set $(0, \varepsilon) \cap B(\mathbb{Z}^n)$ is not empty for any $\varepsilon > 0$, then the same is true for the form $-B$. On the other hand it is clear that $B(\mathbb{Z}^n)$ is invariant under multiplication by square of integers. Thus theorem 1′ implies, under the conditions of that theorem, that $B(\mathbb{Z}^n)$ is dense in $\mathbb{R}$.

Using Mahler compactness criterion and Borel density theorem, it is not difficult to deduce theorem 1 from the following theorem 2 (which is really equivalent to theorem 1). One can also deduce theorem 1′ from theorem 2 but it is more complicated (see [5], §4).

Theorem 2. Let $G = SL(3, \mathbb{Z})$ and $\Gamma = SL(3, \mathbb{Z})$. Let us denote by H the group of elements of G preserving the form $2x_1x_3 - x_2^2$ and by Ω

the space of lattices in $\mathbb{R}^3$ having determinant 1. (The quotient space G/Γ can be naturally identified with Ω. Under this identification the coset $g\Gamma$ goes to the lattice $g\mathbb{Z}^3$.) Let G_y denote the stabilizer $\{g \in G \mid gy = y\}$ of $y \in \Omega$. If $z \in \Omega = G/\Gamma$ and the orbit Hz is relatively compact in Ω, then the quotient space $H/H \cap G_z$ is compact.

The statement of theorem 2 is a very special case of the following

Conjecture 1. Let G be a connected Lie group, Γ be a lattice in G (i.e. Γ be a discrete subgroup with finite covolume) and H be a subgroup of G. Suppose that H is generated by unipotent elements (an element u of G is called unipotent if the transformation $Ad\, u$ of the Lie algebra of G is unipotent). Then for any point $x \in G/\Gamma$, there exists a closed subgroup $P \subset G$ containing H such that the closure of Hx coincides with Px.

The following conjecture, due to Raghunathan, is a special case of conjecture 1.

Conjecture 2. Let G be a connected Lie group,

Γ be a lattice in G and U be a unipotent subgroup of G. Then for any point $x \in G/\Gamma$, there exists a closed subgroup $P \subset G$ containing U such that the closure of Ux coincides with Px.

It seems that conjecture 1 should follow from conjecture 2. At least this is so if Γ is an arithmetic subgroup of G. (In this case, as it was noted by Raghunathan, the abovementioned reduction follows from the countability of the set of $\mathbb{Q}$-subgroups.)

Borel and Prasad (see [1]) gave the following generalization of theorem 1'.

Theorem 3. Let K be a finite extension of $\mathbb{Q}$, let S be a finite set of places of K containing the set S_∞ of the archimedean ones, let $n \geq 3$ and let for any $v \in S$ an isotropic nondegenerate quadratic form B_v on K_v^n be given (K_v is the completion of K at v). Let us denote by $K(S)$ the ring of S-integers in K. Let us assume that it is not possible to find a quadratic form B on K^n such that B_v is proportional to B for any

$v \in S$. Then for any $\varepsilon > 0$ there exists $x \in K(S)^n$ such that $0 < |B_v(x)|_v < \varepsilon$ for any $v \in S$.

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Titel: Inequalities for the algebraic surfaces and connections with number theory

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1. Let V be an algebraic surface over algebraically closed field k of characteristic 0, c_1 and c_2 be it's Chern classes. Then if V is of general type, $c_1^2 \leq 3c_2$ and actually this inequality is valid for all unruled surfaces. Now assume that we have a stable fibration $f: V \to B$ over a curve B with smooth general fiber of genus g. Denote by $\omega_{V/B}$ the relative cotangent bundle, by ~~the~~ δ the number of ~~over~~ all double points of the bad fibers (so $\delta = \sum_{s \in S} \delta_s$, where S is the set of the points of bad reduction) and let $\chi_0 = (2g-2)(2g(B)-2)$. The above inequality can be rewritten in these terms as follows: $\omega^2_{V/B} \leq 3\delta + \chi_0$ and if

belongs to the series of inequalities proved by different methods. All of them are of the following type

$$\omega_{V/B}^2 \le a_1 \delta + a_2 \chi_0 + a_3 . \qquad (*)$$

a_1	a_2	a_3	Conditions	Method	Author
8	6	0	V of general type	Hodge theory + Castelnuovo's lemma	Van de Ven 1966
4	2	0	— " —	Stability theory + Existence of tangent bundle + Castelnuovo's lemma	Bogomolov 1976
3	1	0	— " —	— " —	Miyaoka 1978
— " —	— " —	— " —	— " —	Reduction to char p>0	Gieseker 198?
— " —	— " —	— " —	— " —	Differential Geometry	Yau 1978
5	15	18	$g \geq 1$, $g(B) \geq 1$	Topological proof, using Hirzebruch's signature formula	Folklore (?)
6g	1	30	$g > 1$, $g(B) > 1$	"Pure" Hodge theory	Parshin 1968
2	0	0	Special coverings of $\mathbb{P}^2$	"Local" considerations	Holzapfel 1986
∞	0	0	char k>0, examples	Observation	Parshin 1972

2. Let K be a number field, $B = \operatorname{Spec} \Lambda_K$, Λ_K be the ring of integers, $\sigma : K \hookrightarrow \mathbb{C}$ be all complex embeddings of K. Take an arithmetical surface $\bar{V}$ over B. It means that we have a scheme V of dimension 2 fibrated over the "curve" B (and with the properties like the fibration $f: V \to B$ in the part above), and the Arakelov's metrics μ_σ on the Riemann surfaces $X_\sigma = V \otimes_\sigma \mathbb{C}$ (which are induced by the canonical embeddings $X_\sigma \hookrightarrow \operatorname{Jac}(X_\sigma)$ and the flat metrics on $\operatorname{Jac}(X_\sigma)$). By the general principles of diophantine geometry an analogue of (*) has the following view

$$(**) \qquad \omega^2_{\bar{V}|B} \leq a_1 \delta + a_2 (2g-2) \log |D_{K/\mathbb{Q}}| + a_3$$

where $\omega_{\bar{V}|B}$ is the relative cotangent bundle on $\bar{V}$ defined by Arakelov [Ar], $\omega^2_{\bar{V}|B}$ be it's selfintersection in the Arakelov's sense,

$\delta = \sum_{v \in S} (\#\text{ dble pts}) \log Nv$ (the sum over the points of bad reduction $S \subset B$) $+ \sum_\sigma \delta_\sigma$ is Faltings's definition of the Euler characteristic of $\tilde{V}^{[Fa]}$). As was shown by Bost [Bo] we have $\delta_\sigma = \delta(X_\sigma) = c(g) - 6 \log \frac{\operatorname{Det} \Delta}{\mu_\sigma(X)}$, where $\operatorname{Det}\Delta$ is a regularized determinant of Laplace operator Δ on X and μ_σ is the area of X_σ. At last, $D_{K/\mathbb{Q}}$ is a discriminant of the field K over $\mathbb{Q}$ and $a_{1,2,3}$ are some absolute constants.

Theorem. Assume $(**)$ for arithmetical surfaces of genus ≥ 1. Then we have

1) an effective bound for the heights of rational points on the curves of genus > 1

2) an effectively computable constant $C = C_K$ s.th. for every elliptic curve E/K

$$\# E(K)_{tor} \leq C_K$$

3) an effectively computable constant

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p_0 s. th. for all prime $p \geq p_0$ an equation $x^p + y^p = 1$ has only three rational solutions (in $\mathbb{D}_2$).

To prove that the result of G. Frey and a conjecture of L. Szpiro should be used.

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Titel: Quillen metrics and degenerating Riemann surfaces.

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This is a report on a joint work with J.B. Bost.

Let $\pi : X \to S$ be a holomorphic map with compact fiber Z of dimension 1, with ordinary singularities. π has the following two local models

$$\pi(z_0, \ldots, z_n) = (z_1, \ldots z_n)$$
$$\pi(z_0, \ldots z_n) = (z_0 z_1, z_2, \ldots, z_n)$$

Let $\Delta \subset S$ be the divisor of exceptional fibers, and let $\Sigma \subset \pi^{-1}(\Delta)$ be the singular locus.

Let ξ be a holomorphic vector bundle on X.

Let $\lambda(\xi)$ be the holomorphic line bundle on S which is the inverse of the Grothendieck-Knudsen-Mumford determinant of the direct image.

Let ω_X, ω_S be the canonical lines on X, S. Set $\omega_{X/S} = \omega_X \otimes \pi^* \omega_S^{-1}$. Then on $X - \Sigma$, $TX/S \simeq \omega_{X/S}^{-1}$.

Let $\| \|_{\omega_{X/S}}$, $\| \|_{\xi}$ be smooth Hermitian metrics on $\omega_{X/S}$, ξ.

For $s \in S-\Delta$, let $\| \|_{Q_s}$ be the Quillen metric on $\lambda(\xi)_s$ associated with the metrics on TX/S, ξ.

By [BGS], the Quillen metric $\| \|_Q$ is smooth on $\lambda(\xi)|_{S-\Delta}$ and its first Chern class in Chern-Weil theory associated with the corresponding holomorphic Hermitian connection is given by

$$c_1(\lambda(\xi), \| \|_Q)|_{S-\Delta} = -\Big[\int_{X/S} Td(\omega^{-1}_{X/S}, \| \|_{\omega^{-1}_{X/S}}) \, ch(\xi, \| \|_\xi)\Big]^{(2)}$$

Our purpose is to obtain a corresponding statement on S.

Definition: Let $\mathcal{L}$ be a holomorphic line bundle on S, let $\| \|_{\mathcal{L}}$ be a smooth Hermitian metric on $\mathcal{L}$. By definition a generalized metric $\| \|'_{\mathcal{L}}$ is given by $\| \|'_{\mathcal{L}} = e^{\varphi} \| \|_{\mathcal{L}}$, with

$\varphi \in L^1_{loc}(S)$.

We can still calculate the curvature of the holomorphic Hermitian connection on $(\mathcal{L}, \| \ \|'_{\mathcal{L}})$, which is given by $\bar{\partial}\partial \operatorname{Log} \|\sigma\|'^2_{\mathcal{L}}$ (where σ is a locally defined non zero holomorphic section of $\mathcal{L}$), which is a current on S. $c_1(\mathcal{L}, \| \ \|'_{\mathcal{L}})$ is then a current on S.

Our main result is as follows.

<u>Theorem 1</u>: $\| \ \|_Q$ is a generalized metric on the line bundle $\lambda(\xi)$. The associated current on S which represents the first Chern class on $\lambda(\xi)$ is given by

$$c_1(\lambda(\xi), \| \ \|_Q) = -\left[\int_{X/S} Td(\omega^{-1}_{X/S}, \| \ \|_{\omega^{-1}_{X/S}}) ch(\xi, \| \ \|_\xi)\right]^{(2)}$$

$$- \frac{rk(\xi)\,\delta_\Delta}{12}$$

Theorem 1 is a refinement of the Theorem of Riemann-Roch-Grothendieck, with metrics.

To prove our Theorem, we use quantitative versions of the Kodaira's vanishing theorem, and also a formula of [BGS], which describes the behavior of the Quillen metric for exact sequences of holomorphic Hermitian vector bundles on X.

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Titel: Kolyvagin's work on Mordell-Weil groups and Tate-Shafarevich groups

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Let E be an elliptic curve defined over $\mathbf{Q}$, and assume that E is modular: for some integer N there is a nonconstant map defined over $\mathbf{Q}$

$$\pi : X_0(N) \rightarrow E$$

which we may assume sends the cusp ∞ to 0. Here $X_0(N)$ is the modular curve whose points correspond to pairs (A, C) where A is an elliptic curve and C is a cyclic subgroup of A of order N. It follows that the L-function of E, $L(E, s) = \sum a_n n^{-s}$, has an analytic continuation to all of $\mathbf{C}$ and satisfies a functional equation $s \rightarrow 2-s$. We will further assume that the sign in this functional equation is +1.

Fix an imaginary quadratic field K in which all primes dividing N split, and fix an ideal $\mathfrak{n}$ of K such that $\mathcal{O}_K/\mathfrak{n} \cong \mathbf{Z}/N\mathbf{Z}$. Let x_1 be the point in $X_0(N)(\mathbf{C})$ corresponding to the pair

$$(\mathbf{C}/\mathcal{O}_K,\ \mathfrak{n}^{-1}/\mathcal{O}_K).$$

Fix an embedding of $\overline{\mathbf{Q}}$ into $\mathbf{C}$; then the theory of complex multiplication shows that $x_1 \in X_0(N)(H)$ where H denotes the Hilbert class field of K. Define $y_1 = \pi(x_1) \in E(H)$ and $y = \mathrm{Tr}_{H/K} y_1 \in E(K)$.

Theorem. (Kolyvagin) *Suppose* E *and* y *are as above. If* y *has infinite order in* E(K) *then* $E(\mathbf{Q})$ *and* $Ш_{E/\mathbf{Q}}$ *are finite.*

To relate this result to the Birch and Swinnerton-Dyer conjecture one needs the following:

Theorem. (Gross and Zagier) *With* E *and* y *as above,* y *has infinite order in* E(K) *if and only if* $L(E, 1) \neq 0$ *and* $L'(E, \chi_K, 1) \neq 0$, *where* χ_K *is the quadratic character attached to* K.

Analytic Conjecture. *For any* E *as above, there exists at least one imaginary quadratic field* K, *in which all primes dividing* N *split, such that* $L'(E, \chi_K, 1) \neq 0$.

This analytic conjecture, as yet unproved, together with the theorems of Kolyvagin and Gross and Zagier, would imply:

(*) *For any modular elliptic curve* E, *if* $L(E,1) \neq 0$ *then* $E(\mathbf{Q})$ *and* $Ш_{E/\mathbf{Q}}$ *are finite.*

Assertion (*) is known for elliptic curves with complex multiplication, by theorems of Coates and Wiles (for $E(\mathbf{Q})$) and Rubin (for $Ш_{E/\mathbf{Q}}$).

Sketch of the proof of Kolyvagin's theorem. Fix a prime number ℓ satisfying

(i) $\ell \nmid 2h_K$, where h_K is the class number of K,

(ii) for all v, $\ell \nmid \#[H^1(\mathbf{Q}_v^{unr}/\mathbf{Q}_v, E(\mathbf{Q}_v^{unr}))]$,

(iii) E has no subgroup of order ℓ rational over $\mathbf{Q}$,

(iv) $y \notin \ell E(K)$.

Note that these conditions eliminate only finitely many primes ℓ. For any completion $\mathbf{Q}_v$ of $\mathbf{Q}$ we have the diagram

$$\begin{array}{ccc} E(\mathbf{Q})/\ell E(\mathbf{Q}) & \hookrightarrow & H^1(\mathbf{Q}, E_\ell) \\ \downarrow & & \downarrow \mathrm{res}_v \\ E(\mathbf{Q}_v)/\ell E(\mathbf{Q}_v) & \hookrightarrow & H^1(\mathbf{Q}_v, E_\ell) \end{array}$$

and we define the Selmer group $S^{(\ell)}$ and the ℓ-torsion in the Tate-Shafarevich group, $Ш_\ell$, by

$$S^{(\ell)} = \bigcap_v \mathrm{res}_v^{-1}(\text{image } E(\mathbf{Q}_v)), \quad 0 \to E(\mathbf{Q})/\ell E(\mathbf{Q}) \to S^{(\ell)} \to Ш_\ell \to 0.$$

For $s \in S^{(\ell)}$ write s_v for the inverse image of $\mathrm{res}_v(s)$ in $E(\mathbf{Q}_v)/\ell E(\mathbf{Q}_v)$. We will sketch the proof that for ℓ as above, $S^{(\ell)} = 0$. To complete the proof of Kolyvagin's theorem it is then necessary to show that for the finitely many primes we have left out, $S^{(\ell^n)}$ is bounded independent of n; this is proved using the same ideas and is not significantly more difficult.

We will show $S^{(\ell)} = 0$ by showing that for $s \in S^{(\ell)}$ and for a large set of primes p, $s_p = 0$. Our main tool is the following, which is proved using the local Tate pairings.

Proposition 1. *Suppose* p *is a prime such that* $\#[E(\mathbf{Q}_p)_\ell] = \ell$, *and suppose there exists a cohomology class* $c_p \in H^1(\mathbf{Q}, E)_\ell$ *satisfying*

(i) *for all* $v \neq p$, $\mathrm{res}_v(c_p) = 0$,

(ii) $\mathrm{res}_p(c_p) \neq 0$.

Then for every $s \in S^{(\ell)}$, $s_p = 0$.

It remains now to construct such a cohomology class c_p for sufficiently many p, which Kolyvagin does using Heegner points. Write τ for the complex conjugation on $\overline{\mathbf{Q}}$ induced by our embedding of $\overline{\mathbf{Q}}$ into $\mathbf{C}$.

Lemma 2. *Suppose* p *is a prime not dividing* ℓN *and* $\mathrm{Frob}_p(K(E_\ell)/\mathbf{Q}) = [\tau]$. *Then*

(i) $\ell \mid a_p$ *and* $\ell \mid p+1$,

(ii) p *remains prime in* K,

(iii) $\#[E(\mathbf{Q}_p)_\ell] = \ell$.

Proof. The first assertion follows by comparing the characteristic polynomials of Frobenius and τ acting on E_ℓ, namely $T^2 - a_pT + p$ and $T^2 - 1$, respectively; the other two statements are clear.

Fix a prime p as in the lemma, and let $\mathcal{O}_p$ be the order of conductor p in $\mathcal{O}_K$. Let x_p be the point in $X_0(N)(\mathbf{C})$ corresponding to the pair

$$(\mathbf{C}/\mathcal{O}_p,\ (\mathfrak{n} \cap \mathcal{O}_p)^{-1}/\mathcal{O}_p).$$

The theory of complex multiplication shows that $x_p \in X_0(N)(K[p])$ where $K[p]$ denotes the ring class field of K modulo p. The field $K[p]$ is a cyclic extension of H of degree $p+1$, totally ramified at p and unramified everywhere else, and τ acts on $\mathrm{Gal}(K[p]/K)$ by -1. Define $y_p = \pi(x_p) \in E(K[p])$. The only facts about Heegner point which we will need are contained in the following proposition.

Proposition 3. i) $\mathrm{Tr}_{K[p]/H}(y_p) = a_p y_1$.

ii) *There exist* $\sigma \in \mathrm{Gal}(K[p]/K)$ *and* $w \in E(\mathbf{Q})_{\mathrm{tors}}$ *such that* $y_p^\tau = -y_p^\sigma + w$, $y^\tau = -y + w$.

iii) *For any prime* $\mathfrak{p}$ *of* $K[p]$ *above* p, $\tilde{y}_p^{\mathrm{Frob}} = \tilde{y}_1 \in \tilde{E}(\mathbf{F}_{p^2})$, *where* ~ *denotes reduction modulo* $\mathfrak{p}$.

Proof. The first two assertions come from the action of the Hecke operators T_p and w_N, respectively, on x_1 and x_p. The third statement holds because the elliptic curves corresponding to the points x_1 and x_p are related by a p-isogeny, which modulo $\mathfrak{p}$ reduces to the Frobenius map.

Since $\ell \mid p+1$ and $\ell \nmid h_K$, there is a unique extension K' of K of degree ℓ in $K[p]$, and K'/K is totally ramified at p and unramified at all other primes. Define

$$z = \mathrm{Tr}_{K[p]/K'}(y_p) - (a_p/\ell)y \in E(K').$$

By (i) of Proposition 3, $\mathrm{Tr}_{K'/K}(z) = 0$. Therefore (fixing a generator of $\mathrm{Gal}(K'/K)$) z gives rise to a cohomology class

$$c_p \in H^1(K'/K) \subset H^1(K, E)_\ell.$$

Proposition 4. i) $c_p \in H^1(K, E)_\ell^+ \cong H^1(\mathbf{Q}, E)_\ell$, *where* $+$ *denotes the fixed space under* $\mathrm{Gal}(K/\mathbf{Q})$.
ii) $\mathrm{res}_v(c_p) = 0$ *for all* $v \neq p$.
iii) $\mathrm{res}_p(c_p) = 0$ *if and only if* $y \in \ell E(K_p)$.

Proof. By (ii) of Proposition 3, $z^\tau = -z$ (modulo $(\sigma-1, \ell)$), and τ also acts by -1 on $\mathrm{Gal}(K'/K)$, so $c_p^\tau = c_p$. For $v \neq p$, since K'/K is unramified at v,

$$\mathrm{res}_v(c_p) \in H^1(\mathbf{Q}_v^{\mathrm{unr}}/\mathbf{Q}_v, E(\mathbf{Q}_v^{\mathrm{unr}}))_\ell = 0$$

by assumption (ii) on ℓ. For the third statement, one checks that $\mathrm{res}_p(c_p) = 0$ if and only if $\tilde{z} = 0$ in $\tilde{E}(\mathbf{F}_{p^2})$. Using (ii) and (iii) of Proposition 3 one sees that $\tilde{z} = 0$ in $\tilde{E}(\mathbf{F}_{p^2})$ if and only if $\tilde{y} \in \ell\tilde{E}(\mathbf{F}_{p^2})$, or equivalently if and only if $y \in \ell E(K_p)$.

Corollary 5. *Suppose* $p \nmid \ell N$, $\mathrm{Frob}_p(K(E_\ell)/\mathbf{Q}) = [\tau]$, *and* $y \notin \ell E(K_p)$. *Then for all* $s \in S^{(\ell)}$, $s_p = 0$.

It remains only to show that we have found enough primes p with $s_p = 0$ to force s to be 0. This is done using that $y \notin \ell E(K)$ and that E has no subgroup of order ℓ rational over $\mathbf{Q}$.

Titel: The animal organism as a stratified space.

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I. Stratified spaces.

Let E be a closed subset of some Euclidean space $\mathbb{R}^n$. Let τ the equivalence-relation defined as follows: two points x, y in E are τ-equivalent if they have neighborhoods U of x in $\mathbb{R}^n$, V of y, such that there exists an homeomorphism $h: (U, x) \to (V, y)$ which carries $(U \cap E, x)$ onto $(V \cap E, y)$. Then the question arises: which are the spaces E such that their τ-equivalence classes form a finite set? I do not know the answer. But there are well known cases:

1) All equivalence classes are manifolds imbedded in $\mathbb{R}^n$; any connected component of such a class is called a stratum (of dimension k). In the good cases (see below), strata have in their boundaries strata of lower dimension, and good continuity properties of tangent planes on these boundaries are satisfied (the (a, b) conditions of Whitney)

2) there is the case exemplified by the Cantor set in $\mathbb{R}$; it is a perfect 'set' with empty interior. Such situation is the one qualified as "fractal" by B. Mandelbrot; the Von Koch curve is also called a fractal although its situation seems to be more of a wild embedding of a stratum.

3) there are mixtures of the two cases, as shown by the example of Alexander's horned spheres

Good cases are those defined by some constructive procedures:

Algebraic sets extended by projection to Semi-Algebraic sets
(Tarski-Seidenberg theorem)

Analytic Sets → Semi-analytic → Subanalytic set
(proper projection)

All these sets are polyhedra - But a polynomial mapping may not be triangulated if it has some "collapse" along a stratum.

II Spaces with ~~qu~~ sensible qualities

The notion of stratification generalizes to material systems exhibiting local observable properties. We say that two local appearances (at x, y in our system E) belong to the same genus if the local quality in x may be transformed by Gedankenexperiment in the local quality in y.. For instance a red fleck may be transformed into a blue fleck by continuous variation across the violet. But one cannot imagine how to transform continuously a color into an odor, or into a sound. These ideas come from Aristotle who speaks about "incommunicability between different genera".
As a result the totality of observable properties in a material medium (U) fall into different, independent genera $g_1 \dots g_k$. Each of its genus defines a stratification of the system: the τ equivalence relation ~~is equival~~ requires that the local homeomorphism $h: U_x \to V_y$ is compatible with the local quality belonging to g_i. More precisely:
Usually a genus g splits into different subqualities called kind or species. (French: Genre → Espèce, Grec classique Γένος → Εἶδος). [Par exemple l'espace des couleurs visibles R^3 se partage en bassins des attracteurs de couleur, Si on munit ces bassins de frontières nettes, alors la considération des espèces de couleur va induire un raffinement de la stratification initiale] (Considering the species qualities of ~~the~~ color leads to a refinement of the stratification of the spacial medium,
An other important instance of "genera" appears with the classification of phase states of matter: Solid-liquid-Gas.. Although-

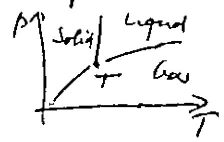
Titel: The animal organism as a stratified set

Seite 3

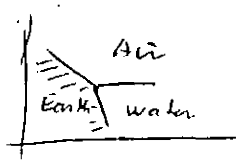
Autor: René Thom

Adresse:

It is possible to imagine how to vaporize a volume of water, the distinction between liquid and gas appears as an abrupt discontinuity in the usual conditions of ~~experience~~ every-day experience - (the critical point is not easy to reach -). But Solid - liquid - Gas appear as "species" in the global phase diagram



which realizes itself in our terrestrial world as the biosphere



III Relations with classical Logic.

A sentence as : X is α and β
is acceptable only if α and β belong to two different genera
X is α or β is acceptable only if α and β belong to the same genus.

Counterexample

(WB) : This cat is black and white

The counter example shows that the "logical atomism" prevailing both in Aristotelian ~~to~~ and in Boolean logic is not acceptable, as the (WB) sentence is quite correct.
The solution is then to divide the cat's fur into two parts, one white and one black : thus we can recover logical atomism on the level of parts.

IV. This is (probably) the origin of Aristotle's theory of biological organization. Aristotle introduces the concept of homeomeric : this is a subset in the animal's body where any two points have neighborhood having the same appearance. Blood, bone's

marrow, the "flesh" are (3 dimensional) homeomeres.
The lung is a two dimensional homeomere, a membrane separating the blood from the air in the lung's vesicles. The same for the intestinal mucosa.
Here one sees that the decomposition of the living organism into homeomeres makes the organism a stratified space — with the proviso that there are tendencies to fractality: blood for instance is limited by the walls of the vascular system, but veins and arteries end by ramifications into invisible capillaries. The lung, according to B. Mandelbrot, is also a "fractal". A nerve can be considered as a one-dimensional stratum of biological organization.
Most homeomerous strata have no names, and escape the usual description of organs. This led Aristotle to introduce the "complementary" notion of anhomeomere. He wanted to find the usual parts of the human body - head, trunk, members, ... there is a strong universality among languages in describing and denominating these parts. But it is difficult to ascertain what makes the individuality of an anhomoeomere. In particular the boundary of a well defined anhomeomere may not consist of strata of the homeomerous stratification. Example: Consider the thumb as a part of the hand. The boundary between the thumb and the palm of the hand may be seen on the skeleton, as the joint surface between Phalanx and Metacarpian. But such a discontinuity surface disappears morphologically in the surrounding flesh and skin, although it is still existing physiologically due to great angular discontinuities of the skin in the interdigital areas. Anhomeomerous are defined (according to Aristoteles) by their "works and activities", i.e. their physiological, functional vocation.

This leads to considering the relation between structure (as defined by the homeomere's stratification), and function (as defined by the anhomeomere's activities). Roughly speaking, there exists between structure and function a topological relation akin to Poincaré duality. For instance in a membranous organ as the lung is a 2dimensional stratum separating flesh from air. Its function is described by the transversal ~~transference~~ of CO^2 from inside to outside, of O^2 from outside to inside. See attached diagramm. In the same way, the joint surface between two associated bones (as humerus and radius facing the elbow) is a discontinuity locus where some rotation takes place between these bones. A more comprehensive theory describing embryological development as seen as a stratified space in $R^3 \times T$ is developed in the reference . Here one describes a ~~universal~~ dynamical scheme describing the universal physiological function of the animal Kingdom seen as a global attractor of the metabolism. This attractor implodes locally on some tissues, leading to cellular differentiations (for instance the three germ layers ectoderm, mesoderm, entoderm, at the ~~b~~ gastrulation stage). These implosions are localized as effects of morphogenetic gradients (as the cephalo-caudal gradient in Vertebrates) whose underlying physico-chemical basis is still unknown. One may interpret also with this formal Haeckel's recapitulation laws: the embryo in its development has to pass through the adult stages of its ancestors, and explain why this law is false.

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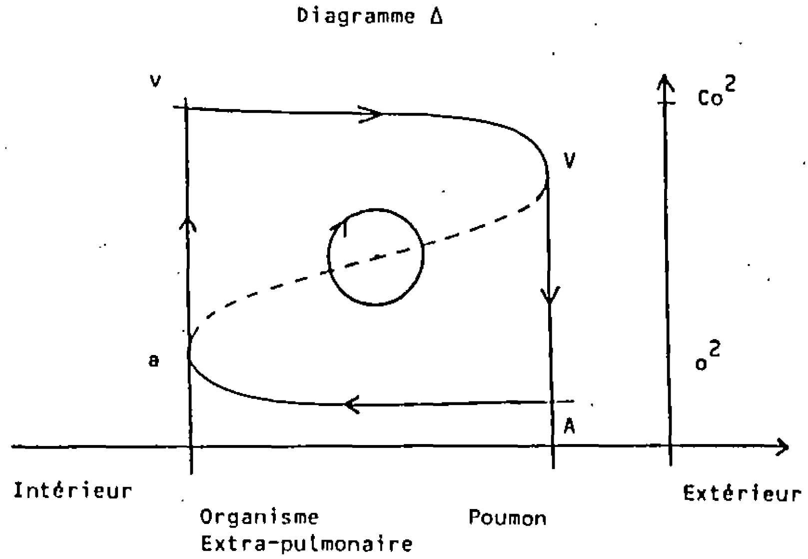
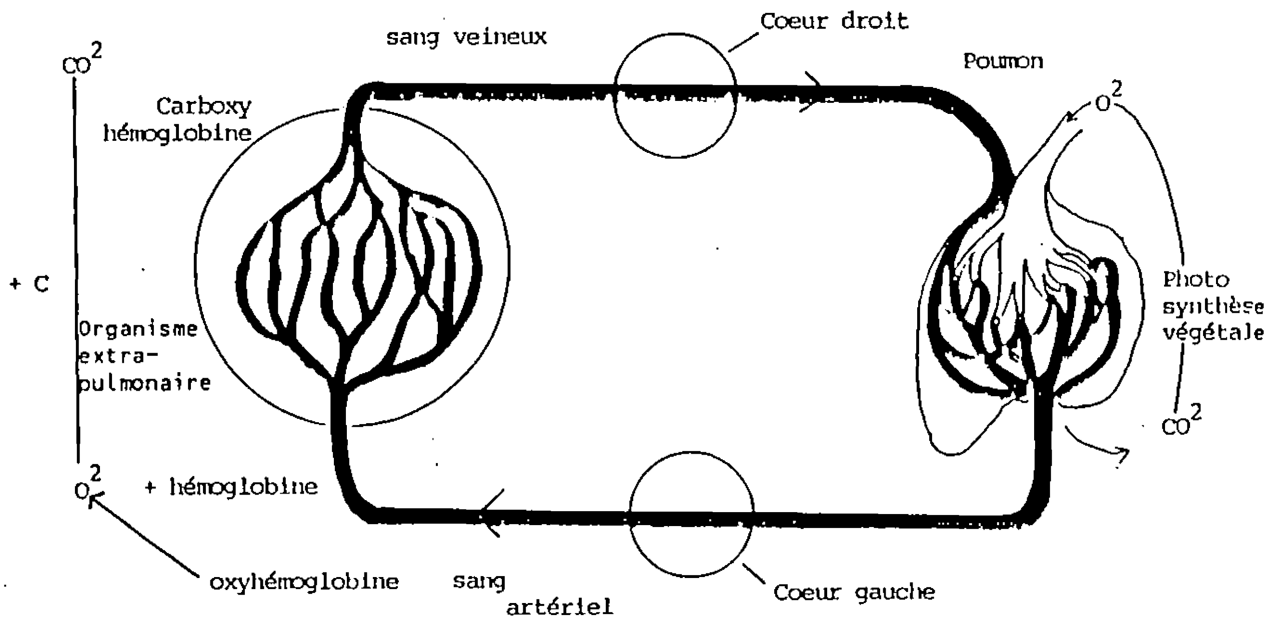


Diagramme Δ
V
V
a
A
Co^2
o^2
Intérieur
Organisme
Extra-pulmonaire
Poumon
Extérieur



sang veineux
Coeur droit
Poumon
CO^2
Carboxy
hémoglobine
+ C
Organisme
extra-
pulmonaire
O^2
+ hémoglobine
oxyhémoglobine
sang
artériel
Coeur gauche
Photo
synthèse
végétale

Fig. 9

Titel: The Dirichlet Problem at ∞

Autor: Werner Ballmann

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Let M be a complete, simply connected Riemannian manifold with sectional curvature $K \le 0$. Given two geodesics g_1 and g_2 in M, the function $dist(g_1(t), g_2(t))$ is convex in t. It follows that for any two points p, q in M there is a unique geodesic from p to q, up to parametrization. Choose $p \in M$. For $q \neq p$ let $v(q)$ be the unique unit vector at p pointing at q and $r(q) = dist(p,q)$. By what was said above, the map

$$q \longmapsto \tanh\left(\frac{r(q)}{2}\right) \cdot v(q)$$

is a diffeomorphism from M to the unit ball B^n in $T_pM = \mathbb{R}^n$. This corresponds to the Poincaré model in the case that M is the hyperbolic plane. The above model also comes with a natural compactification $\overline{M} \cong M \cup M(\infty)$, where $M(\infty) = \partial B^n = S^{n-1}$. Up to homeomorphism, this compactification is independent of the choice of p.

The Dirichlet problem at ∞ is the following:

Given a continuous function $f: M(\infty) \to \mathbb{R}$, is there a continuous function $h: \overline{M} \to \mathbb{R}$ such that $h|M$ is harmonic and such that $h|M(\infty) = f$.

<u>Theorem (Anderson / Sullivan, 1983).</u> If $-b^2 \le K \le -a^2 < 0$, then the Dirichlet problem at ∞ is solvable for any f.

In the case that M is a symmetric space of negative curvature, this result was proved earlier by Fürstenberg. Under more restrictive assumptions than the ones above, the theorem had been established by Kifer (about 75).

If f is as above, then a solution must satisfy $\min f \le h \le \max f$. Thus if M is Euclidean space, there is no solution unless f is constant. It follows from the work of Fürstenberg that the problem is not solvable for any f if M is a symmetric space of higher rank.

In the work discussed, an additional assumption on the geometry of M is made:

(*) There exists a discrete group Γ of isometries of M such that M/Γ is compact.

Then it is known that if M is not a Riemannian product, then M is either a symmetric space of higher rank or M has "rank one". By the latter is meant that Γ and M share many properties with manifolds of strictly negative curvature (cf. Ballmann et al. in Annals of Math. 85). However, in general there are no uniform estimates on the divergence of geodesics etc. The proof of the following result was discussed.

<u>Theorem.</u> If M satisfies (*) and has rank one, then the Dirichlet problem at ∞ is solvable for any f.

The proof uses ideas from the work of Fürstenberg and Lyons-Sullivan. The principal idea is to show that Brownian motion starting from $p \in M$ converges at $M(\infty)$ and that the hitting measures tend to the Dirac measure at $z \in M(\infty)$ if p tends to z. For that purpose random walks on Γ are considered and connected to Brownian motion on M as in the work of Lyons-Sullivan.

Titel: Differentiable classification of algebraic surfaces

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This talk gave a survey of the diffeomorphism classification of algebraic surfaces, using methods from gauge theory. For reasons of time only results on simply - connected surfaces were discussed. For the non - simply - connected case see (HK) and the references quoted there.

§1. Donaldson invariants for smooth 4-manifolds

Let X be a smooth, compact, oriented, connected and simply - connected Riemannian 4-manifold. Consider G-bundles $E \to X$, where G is SU(2) or SO(3). These are classified by characteristic classes c_2 and p_1, w_2 respectively. In the latter case $w_2^2 = p_1$ mod 4 must be satisfied.

Define $k=c_2$ and $k=-\frac{1}{4}p_1$ respectively. Let

$\mathcal{A}^* =$ (irreducible connections on E) and

$\mathcal{G} =$ Aut(E) be the gauge group of E. Define

$\mathcal{B}^* = \mathcal{A}^*/\mathcal{G}$.

The curvature F of a connection is decomposed into self-dual and anti-self-dual (ASD) parts under the action of the Hodge operator defined by the metric. If the self-dual part vanishes, the connection is called ASD. Let

$M_k^g =$ (gauge equivalence classes of ASD connections on E) .

The method of Donaldson defining differential topological invariants of X using the moduli spaces M_k^g proceeds as follows: There is a universal bundle $\mathbb{E} \to \mathcal{B}^* \times X$ .Define

$$\mu : H_2(X,\mathbf{Z}) \to H^2(\mathcal{B}^*,\mathbf{Z})$$
$$a \mapsto -\tfrac{1}{4}p_1(\mathbb{E})/a$$

by the usual slant product. Now try to evaluate

$$\langle \mu(a_1) \cup \ldots \cup \mu(a_d) \ , [M_k^g] \rangle .$$

This can only be nonzero if $\dim(M_k^g)$ is even. The deformation theory of Atiyah - Hitchin - Singer gives the following virtual dimension:

$$\dim(M_k^g) = 8k - 3(1+b_2^+) \ .$$

For manifolds with an indefinite intersection form and a generic metric M_k^g will be a smooth manifold of this dimension, which is even iff $b_2^+ = 1+2p$. This is always true for algebraic surfaces: $p = p_g$, the geometric genus. However, the following problems remain:

1. The moduli spaces are not compact in general, so they do not carry a fundamental class in the obvious way.
2. The moduli spaces depend on the metric g .

There are two ways to deal with the first problem:

a) Compactify M_k^g by introducing "correction terms" at infinity.

b) Donaldson's "dimension counting argument", which shows that the intersection of M_k^g with representatives of $\mu(a_i)$ is compact if $k > \frac{3}{2}(1+p)$, where p is as above.

As for the second problem, we have to analyze how $M_k^{g(t)}$ behaves for a 1-parameter family of metrics $g(t)$. This shows that for $b_2^+ \geq 3$ there is no essential dependence on the metric. For $b_2^+=1$ this is no longer true, because $g(t)$ in general contains metrics for which reducible ASD connections occur. In this case the invariants defined by the above procedure are more complicated, because they take into account the change in $M_k^{g(t)}$.

Here is a survey of the possible invariants:

	$b_2^+ = 1$	$b_2^+ \geq 3$
compactify cf. a)	Donaldson's Γ - invariant for $G=SU(2)$, $k=1$ (D2) (Applications in (D2), (FM1), (OV1) and other papers on $\pi_1 \neq 0$.)	Mong's invariant for $G=SU(2)$, $k=2$ (M)
compact intersections	Polynomial invariants depending on the metric. (K1) (Applications in (K1,2), (OV2).)	Donaldson polynomials (D4) (Applications in (D4), (FMM), (FM2).) $q_k \in Sym^*(H^2(X,\mathbb{Z}))$

The lower part of the diagram contains invariants coming from both, G=SU(2) or SO(3) . In the latter case it sometime happens that the M_k^g is a priori compact, giving simple invariants.

§2. Applications to algebraic surfaces

The Γ- invariant was used to prove the following:

Theorem1 a) (D2) The Dolgachev surface $D_{2,3}$ is not diffeomorphic to the nine-fold blow-up of CP(2).

b) (FM1),(OV1) If the Dolgachev surfaces $D_{2,q}$ and $D_{2,q'}$ are diffeomorphic, then q=q' .

All the surfaces in this theorem are of course homeomorphic by the result of Freedman. a) provided the first counterexample to the 4-dimensional h-cobordism conjecture. Part b) shows that the underlying topological manifold carries infinitely many distinct smooth structures.

The only example of a simply-connected surface with $q=p_g=0$ which is not a Dolgachev surface, is the Barlow surface, which is a minimal surface of general type homeomorphic to the 8-fold blow-up of CP(2). The Barlow surface resisted evaluation of the Γ-invariant, but using the simple invariants mentioned at the end of §1. one proves:

Theorem2 (a) (K2) The Barlow surface is not diffeomorphic to the 8-fold blow-up of CP(2).

(b) (OV2) The blown-up Barlow surface is not diffeomorphic to any of the Dolgachev surfaces.

Moreover, these SO(3) invariants are used in (OV2) to give a simple proof of Theorem1 , thus showing that the invariants in the lower left-hand box gobble up the results proved using the Γ -invariant. We will see something similar in the other half of the diagram. Mong (M) has proved some partial results about the diffeomorphism type of the homotopy K3-surfaces. More complete results are obtained using the polynomial invariants in the lower right-hand box (FM2).

Theorem A (D4) If an algebraic surface decomposes as a smooth connected sum X_1+X_2, then one of the X_i has a negative-definite intersection form.

Note that such decompositions may arise from blowing up. Donaldson conjectured that this is the only possibility. He proved Theorem A from first principles using the polynomial invariants in the lower right-hand box of the diagram on page 2 . As it seems to be very hard to get more precise results from first principles, and because the stable-bundle interpretation of ASD connections gives a natural algebraic-geometric tool, Donaldson suggests the following

Programme: Find algebraic-geometric formulae for the polynomial invariants of algebraic surfaces.

We list some results in this direction, most of them due to Friedman and Morgan.

Theorem3 (FMM) Let S be a simply-connected algebraic surface with $p_g > 0$ and with a big monodromy group. Then for all k sufficiently large the polynomial $q_k(S)$ is a polynomial in K_S and the intersection form Q_S considered as an element of $H^2(S,Z)\otimes H^2(S,Z)$.

The condition of large monodromy is verified for complete intersections, elliptic surfaces, and some abelian coverings of $CP(1)\times CP(1)$. The latter give infinitely many pairs of homeomorphic but non-diffeomorphic surfaces of general type.

Theorem4 (FMM) Let S, S' be surfaces as in Theorem3 . Suppose that $p_g(S)=p_g(S')$ is even. Let $f:S\to S'$ be an orientation preserving diffeomorphism. Then $f^*K_{S'} = aK_S$ for some $a\in\mathbb{Q}$. If S=S' or if $K_S^2\neq 0$, then a=+/-1 .

Theorem5 (FM2) Suppose that S,S' are simply-connected elliptic surfaces with $p_g > 0$. If they are diffeomorphic, then $p_g(S)=p_g(S')$ and pq=p'q', where the p,q are the multiplicities of the multiple fibres.

This result is stable under blowing up. Note that it has many consequences, for example for homotopy K3 surfaces.
Now the Enriques-Kodaira classification together with Theorems 1 and 5 give:

Theorem6 (FM2) The forgetful map

(alg. surfaces mod deformation) ⟶

(smooth 4-mfds. mod diffeomorphism)

is finite-to-one.

These Theorems are just the tip of the iceberg. There are many more, about blow-ups, selfdiffeomorphisms of surfaces, and so on.
All these Theorems are evidence for the following conjecture of Van de Ven:

Conjecture: Algebraic surfaces of different Kodaira dimension are never diffeomorphic.

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Nice surveys are in (FM2), (HK) and in Donaldson's 1986 ICM talk.

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Titel: Chow Categories

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The aim of my talk is to explain why it is sometimes interesting to consider intersection theory not as a theory concerned with operations between groups but as a theory concerned with functors between Picard categories.

Let us consider a proper smooth morphism $p: X \to S$ of relative dimension one and a vector bundle E on X. The Riemann-Roch-Grothendieck formula allows us to compute the determinant line bundle $\det Rp_* E$ up to isomorphism. In /D/, Deligne proved a much more precise functorial version of this formula, i.e., constructed a canonical isomorphism

$$(\det R p_* E)^{\otimes 12} \simeq \langle \Omega_{X/S}, \Omega_{X/S} \rangle^{\otimes \dim(E)} \otimes \langle \det E, \det E \otimes \Omega^{-1}_{X/S} \rangle^{\otimes 6} \otimes \tag{1}$$
$$\otimes I_{X/S}(c_2(E))^{\otimes -12} .$$

Then he constructs a metric on the right side of (1) and compares this metric with Quillen's metric on the left side.

For line bundles L, M on X, $\langle L, M \rangle$ is a line bundle on S corresponding to $\int_{X/S} c_1(L) \cdot c_1(M)$. It is locally on S generated by symbols $\langle \ell, m \rangle$, where ℓ and m are rational sections of L and M whose divisors don't intersect, with relations $\langle g\ell, m \rangle = g(\mathrm{div}(m)) \cdot \langle \ell, m \rangle$ and $\langle \ell, gm \rangle = g(\mathrm{div}(\ell)) \cdot \langle \ell, m \rangle$.

Deligne defines the second ingredient of (1), the functor $I_{X/S}(c_2(E))$, by a system of axioms relating it to the line bundles $\langle L, M \rangle$. This definition is strictly restricted to the case $\dim(X/S) = 1$.

In the program of proving (1) in the general situation (cf. /D, §2/), one main problem is to define line bundles

(2) $$I_{X/S}(P(c_i(E_j))),$$

where P is a polynomial in Chern classes of absolute degree dim(X/S)+1, which are functorial versions of the corresponding integrals of the corresponding integral of Chern classes.

Our strategy to define (2), and also to obtain generalizations of (1), is to give "live" to each ingredient of (2). To define the i-th Chern functor of a vector bundle, we need the i-th Chow category. Let us see how such categories should be defined.

For a Noetherian irreducible universally catenary scheme X, consider the complex

(3) $$\dots \to \coprod_{x\in X_{k-2}} K_2(k(x)) \xrightarrow{\text{Tame}} \coprod_{x\in X_{k-1}} k(x)^{\times} \xrightarrow{\text{div}} \coprod_{x\in X_k} \mathbb{Z},$$

where X_k is the set of points in codimension k and $k(x)$ is the residue field of x. It is part of the spectral sequence (cf. [Q1])

$$E_1^{p,q} = K_{-p-q}(\mathcal{M}_p/\mathcal{M}_{p+1}) = \coprod_{x\in X_p} K_{-p-q}(k(x)) \Longrightarrow K'_{-p-q}(X) = K_{-p-q}(\mathcal{M}),$$

where $\mathcal{M}$ is the category of coherent sheaves and $\mathcal{M}_p$ is the category of coherent sheaves supported in codimension p.

In (3), the group at the right end of the complex is the group of cycles in codimension k, and the image of div is the group of cycles which are rationally equivalent to zero. Therefore one expects a relation between isomorphisms in the Chow category and the homology in the middle term of (3). If we pursue this idea, we are led to the following definition of the Chow categories.

Let $X_{(k)}$ be the following topology on X: Open subsets of $X_{(k)}$ are Zariski-open subsets $U\subset X$ such that $\operatorname{cod}(X-U)\geq k$. For an open subset $U\subset X_{(k)}$ let $\mathcal{G}_k(U) = E_2^{k-1,-k}(U)$ be the homology in the middle term of (3). By (3), one checks easily that $\mathcal{G}_k$ is a sheaf on $X_{(k)}$. Let $\underline{CH}^k(X)$ be the category of $\mathcal{G}_k$-principal homogeneous sheaves on $X_{(k)}$. This is a Picard category in the sense of /D/.

Let $(\mathcal{G}_k)_r(X)$ be the cokernel of "Tame" in (3). There is $d:(\mathcal{G}_k)_r(X)\to Z^k(X)$, the group of cycles in codimension k. Every $z\in Z^k(X)$ defines an object $T(z)$ of the Chow category by

(4) $T(z)(U)=\{\varphi\in(\mathcal{G}_k)_r(X) \mid d(\varphi)|_U=-z|_U\}$,

where U is an open subset of $X_{(k)}$. $\mathcal{G}_k(U)$ acts on (4) by translations. If $\alpha\in(\mathcal{G}_k)_r(X)$ satisfies $d(\alpha)=z'-z$, it defines an isomorphism $T(\alpha): T(z)\to T(z')$ which sends φ in (4) to $\varphi-\alpha|_U$. (Let us mention that in general $T(z)$ does not belong to $\underline{CH}^k(X)$ but to a more sophisticated category $\widetilde{\underline{CH}}^k(X)$, cf. /F/. For the sake of simplicity we do not stress this distinction).

Since $X_{(1)}=X_{Zar}$ and $\mathcal{G}_1=\mathcal{O}_X^*$ for a normal scheme, the first Chow category of a normal scheme is the category of line bundles and isomorphisms.

For an object A of $\underline{CH}^k(X)$, let $A_r(X)=\bigcup_{U \text{ open in } X_{(k)}} A(U)$ be its set of rational sections. It is an affine space for $(\mathcal{G}_k)_r(X)$. Every $a\in A_r(X)$ has a cycle $c(a)\in Z^k(X)$ defined by the following condition. If $g\in(\mathcal{G}_k)_r(X)$ and U is open in $X_{(k)}$, $g+a$ belongs to $A(U)$ iff $d(g)|_U=-c(a)|_U$.

If $f: X\to Y$ is flat, then $f: X_{(k)}\to Y_{(k)}$ is continuous. We have a pull-back morphism (cf. /G/) $f^*: f^+\mathcal{G}_{k,Y}\to\mathcal{G}_{k,X}$, where f^+ is the inverse image of sheaves. The composition of the functors

$$\mathcal{G}_{k,Y}\text{-p.h. sheaves} \xrightarrow{f^+} f^+(\mathcal{G}_{k,Y})\text{-p.h.s.} \xrightarrow{f^*} \mathcal{G}_{k,X}\text{-p.h.s}$$

is $f^*: \underline{CH}^k(Y)\to\underline{CH}^k(X)$.

Our next remarks aim at defining the Chern functors. For a topological space X, a complex $K^\cdot$ of sheaves and a covering $\mathfrak{U}$ of X, let

$$C^k(\mathfrak{U},K^\cdot)=\bigoplus_{p+q=k} C^p(\mathfrak{U},K^q)$$

$$d=\check{C}\text{ech differential} + (-1)^p\cdot\text{differential in } K^\cdot$$

be the complex defining Čech hypercohomology of $K^\cdot$. Let $Z^k(\mathfrak{U},K^\cdot)$ and $B^k(\mathfrak{U},K^\cdot)$ be the closed and exact chains for d.

We need the following generalization of (4). For a covering $\mathfrak{U}$ of X_{Zar} and $z\in Z^k(\mathfrak{U}, E_1^{\cdot,-k})$, we define a sheaf on $X_{(k)}$ by

(5) $T(z)(V)=\{\varphi\in(C^{k-1}/B^{k-1})(\mathfrak{U}|_V, E_1^{\cdot,-k}) \mid d(\varphi)=-z|_V\}$.

Since $E_1^{k-1,-k}(X)\subset C^0(\mathfrak{U},E_1^{k-1,-k})$, $\mathcal{G}_k(V)$ acts on (5) by translation. ~~Similar as for (4)~~ Since the sheaves $E_1^{p,q}$ are flabby on X_{Zar}, this action is easily seen to be transitive. Similar as for (4), every $\alpha\in C^{k-1}(\mathfrak{U},E_1^{\cdot,-k})$ with $d(\alpha)=z'-z$ defines $T(\alpha): T(z)\xrightarrow{\sim}T(z')$.

Let $A \in Ob(\underline{CH}^k(X))$ and let $\mathcal{L}$ be a line bundle on X. We are now ready to define an object $\underline{c}_1(L) \cup A$ of $\underline{CH}^{k+1}(X)$ by the following generalisation of the product $E_2^{p,q}(X) \times H^k(X, \mathcal{K}_\ell) \to E_2^{p+k, q-\ell}(X)$ (cf. [G, §8]). Fix a rational section $a \in A_r(X)$, a covering $\mathcal{U}$ of X_{zar}, and trivialisations $F_i \in \mathcal{L}^*(V_i)$. The product in algebraic K-theory defines $\{\cdot,\cdot\}: C^k(\mathcal{U}, K_\ell) \times E_1^{p,q}(X) \to C^{p+k}(\mathcal{U}, E_1^{\cdot, q-\ell})$. Applying this to the cycle $\varphi_{ij} = F_j F_i^{-1} \in C^1(\mathcal{U}, K_1)$, we get $\{\varphi_{ij}, \underline{c}(a)\}$. We define

(6) $\quad (\underline{c}_1(L) \cup A)_{\mathcal{U}, F_i, a} = T(\{\varphi_{ij}, \underline{c}(a)\})$.

If we replace a by another rational section b, we have a transition isomorphism

$$T(\{\varphi_{ij}, a-b\}): (\underline{c}_1(L) \cup A)_{\mathcal{U}, F_i, a} \to (\underline{c}_1(L) \cup A)_{\mathcal{U}, F_i, b}.$$

Similar isomorphisms can be contructed if we replace F_i by G_i or refine $\mathcal{U}$. It follows that the objects (6) can be identified with one object $\underline{c}_1(L) \cup A$.

If ℓ is a rational section on X whose divisor intersects $c(a)$ properly, then $\varsigma = \{\ell F_i^{-1}, \underline{c}(a)\}$ solves (5) for $\{\varphi_{ij}, c(a)\}$ on $V = X - (|\mathrm{div}(\ell)| \cap |c(a)|)$. Hence ς defines an element of $(\underline{c}_1(L) \cup A)_{\mathcal{U}, F_i, a}(V)$. Since these elements are compatible with the transition isomorphisms for replacing F_i by G_i and refining $\mathcal{U}$, they define $\ell \cup a \in (\underline{c}_1(L) \cup A)(V)$.

Now we are ready to define chern functors. For a vector bundle E of dimension

$$\prod_{j=0}^{e-1} \underline{CH}^{k-j}(X) \longrightarrow \underline{CH}^k(\mathbb{P}(E)),$$

$$(A_j) \longrightarrow \bigoplus_{j=0}^{e-1} \underline{c}_1(\mathcal{O}(1))^j \cup \underline{p}^*(A_j),$$

where $p: \mathbb{P}(E) \to X$ is the projection, is an equivalence of categories. It follows that for every $A \in Ob(\underline{CH}^k(X))$ there are objects $\underline{c}_i(E) \cup A$ of $\underline{CH}^{k+i}(X)$ and isomorphisms

(7) $\quad \underline{c}_0(E) \cup A = A$

(8) $\quad \bigoplus_{j=0}^{e} \underline{c}_1(\mathcal{O}(1))^{e-j} \cup \underline{p}^*(\underline{c}_j(E) \cup A) \xrightarrow{\sim} \mathcal{G}_{k+e, \mathbb{P}(E)}$.

It follows also that the isomorphisms (7) and (8) determine $\underline{c}_i(E) \cup A$ up to unique isomorphism.

To finish our construction of Deligne's line bundles we need a special case of the proper push-forward-functor. Let $p: X \to S$ be proper of relative dimension $N > 0$, with S normal and locally factorial. We define a homomorphism p_* from $(\mathcal{G}_{N+1})_r(X)$ to $K(S)^*$, the rational functions on S. Let η be the generic point of S. Every $g \in (\mathcal{G}_{N+1})_r(X)$ has a representative $(a_x)_{x \in X_{N+1}}$, $a_x \in k(x)^*$, and

$$p_*(g) = \prod_{\substack{x \in X_{N+1} \\ p(x) = \eta}} N_{k(x)/k(\eta)}(a_x)$$

is independent of the choice of the representative. Similar the push-forward of a $(N+1)$-cycle on X is a divisor on S.
Every $A \in Ob(\underline{CH}^{N+1}(X))$ defines a line bundle $p_*(A)$ on S which is generated by rational sections $p_*(a)$, $a \in A_v(X)$ with divisors $div(p_*(a)) = p_*(c(a))$ and relations $p_*(ga) = p_*(g)\, p_*(a)$ for $g \in (\mathcal{G}_{N+1})_v(X)$.
Our proposal for (2) is

$$p_*(P(\underline{C}_i(E_j))),$$

where $\underline{C}_i$ are the Chern functors defined on p.4. As an example we consider the case $\dim(X/S) = 1$ and line bundles L, M on X. Then it is possible to construct an isomorphism from $p_*(\underline{C}_1(L) \cup \underline{C}_1(M))$ to $\langle L, M\rangle$ which sends $p_*(l \cup m)$, where $l \cup m$ has been defined on page 4, to the section $\langle l, m\rangle$ mentioned on page 1.
Let me finally mention that functorial intersection theory has some other applications. For instance, it can be used to give a new construction of Bloch's biextension of $CH^k(X)$ and $CH^{N+1-k}(X)$ $(\dim(X) = N)$ by G_m (cf. /B/).

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Titel: Tate conjecture via transcendence
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We fix a number field K, an abelian variety A of dimension n defined over K and a prime number ℓ. We denote by π the Galois group $\mathrm{Gal}(\bar{K}/K)$ and put

$$ {}_{\ell^m}A := \mathrm{Ker}(A \xrightarrow{\ell^m} A)(\bar{\mathbb{Q}}), $$

$$ T_\ell(A) := \varprojlim {}_{\ell^m}A . $$

A homomorphism $A^* \longrightarrow A$ induces a homomorphism of Galois-modules

$$ \mathrm{Hom}(A^*, A) \otimes_{\mathbb{Z}} \mathbb{Z}_\ell \hookrightarrow \mathrm{Hom}_\pi(T_\ell(A^*), T_\ell(A)). $$

Tate proved the injectivity of this homomorphism and conjectured its surjectivity. Furthermore he proved that his conjecture is implied by the following statement due to Lichtenbaum.

$\underline{Hyp(K, A, d, \ell)}$: Given an abelian variety A of dimension n defined over a number field K, a prime ℓ and an integer $d \geqq 1$ there exist only finitely many abelian varieties A^* over K such that

(i) there exists a polarisation φ of A^* of degree d^2 defined over K,

(ii) there exists a K-isogeny $\phi: A^* \to A$ with

$$\deg \phi = \ell^m \quad , \text{ for some } m \geqq 1,$$

This hypothesis was proved by G. Faltings in 1983. It is also implied by the following Theorem proved by D.W. Masser and the author.

Theorem. There exists an effectively computable constant $c > 0$ depending only on the height $h(A)$ of A, $\dim A$ and the degree of K over $\mathbb{Q}$ with the following property. If A^* is an abelian variety over K isogeneous to A over K then there exists an isogeny $\phi : A^* \to A$ over K with

$$\deg \phi \leq c.$$

The proof goes as follows: Let ϕ be a minimal isogeny from A^* to A. Then ϕ induces a period relation and this can be used to define a homomorphism

$$\Psi : A^{2n} \to A^* .$$

The graph $\Gamma \subset A^{2n} \times A^*$ is an analytic subgroup. Now we apply transcendence technics and an effective version of the author's analytic subgroup theorem leads to an algebraic subgroup $H \subseteq \Gamma$. One then shows that $H = \Gamma$ and that the degree of H can be bounded by a constant as described in the theorem. Thus we can bound the degree of Γ. But now it is easy to get an isogeny from A to A^* using Γ of bounded degree and to bound finally the degree of ϕ.

Remark. Of course, this theorem gives another proof of the Mordell conjecture and again a proof of Siegels theorem using diophantine inequalities.

Titel: Conformal Field Theory on Riemann Surfaces

Autor: Yukihiko NAMIKAWA

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The string theory in elementary particle theory brought us an unexpected close relation with algebraic geometry, in particular the moduli theory of Riemann surfaces.

Recently Friedan and Shenker proposed a program to understand the string theory by using the 2-dimensional conformal field theory which originated from statistical mechanics.

So far there are two main methods:

I) Path-integral method – geometric
global (arbitrary genus) but not justifiable

II) Operator formalism method – algebraic
local (difficult for $g \geq 2$) but rigorous.

Our goal is:

To construct a CFT for charged, free, chiral fermions by unifying these 2 methods.

As a mathematical system,

this CFT = a dynamical system on the determinant bundle over the moduli space.

We follow mainly [KNTY], but see [W], [ACKP], [A-G.G.-M.V.], [Kr.-N]. [B.Sh.] also.

We use the Sato's theory of KP equations in an essential way. Its main ingredients are: the universal Grassmann manifolds and τ-function.

We work in formal power series and use adic topology.

§1. Weierstrass system.

- Notations.

$$\begin{array}{rcl} \hat{K} & = & \mathbb{C}((z^{-1})) \\ \cup & & \\ \hat{\mathcal{O}} & = & \mathbb{C}[[z^{-1}]] \\ \cup & & \\ \hat{m} & = & z^{-1}\mathbb{C}[[z^{-1}]] \end{array} \quad \Bigg] \text{ with complete adic topology}$$

$V = \hat{K}$ as vector space $/\mathbb{C}$, $\{e^n := z^{-n},\ n \in \mathbb{Z}\}$

$\Omega = \hat{K}$ as scalar (Gauge algebra)

$\vartheta = \mathrm{Der}(\hat{K})$: Witt algebra

$\mathcal{D} = \mathrm{Diff}^1(\hat{K})$: (formal scalar) Atiyah algebra

$S_j : \vartheta \to \mathcal{D}$, $\quad \ell(z^{-1})\frac{d}{dz} \longmapsto \ell(z^{-1})\frac{d}{dz} + j\,\ell'(z^{-1})$

$\mathcal{M}_g$: the moduli space of Riemann surfaces of genus g,

$\mathcal{T}_g$: the Torelli space $= \{(R, \{\alpha,\beta\}$ can. basis of $H_1)\}$,

$\pi : \mathcal{C}_g \to \mathcal{T}_g$: universal family,

$\hat{\mathcal{C}}_g = \{X = (R, Q, u : \hat{\mathcal{O}}_Q \xrightarrow{\sim} \hat{\mathcal{O}})\}$
formal uniformization at Q

$\hat{\pi} : \hat{\mathcal{C}}_g \longrightarrow \mathcal{C}_g$ fibre bundle with fibre $\mathrm{Aut}(\hat{\mathcal{O}})$

$$\begin{array}{ccccc} & & \hat{\mathcal{C}}_g & \overset{\hat{\sigma}_j}{\rightleftarrows} & \mathbb{P}(\hat{\mathcal{X}}_g) \\ & & \hat{\pi}\downarrow & & \downarrow\hat{p} \\ & & \mathcal{C}_g & \longleftarrow & \mathcal{X}_g \\ & & \pi\downarrow & & \downarrow \\ \mathcal{M}_g & \underset{/Sp(g,\mathbb{Z})}{\longleftarrow} & \mathcal{T}_g & \underset{\sigma_j}{\rightleftarrows} & \mathcal{J}_g \end{array}$$

Thm [B-Sh]. ϑ acts on $T\hat{\mathcal{C}}_g$ as global vector fields and generates $T_X\hat{\mathcal{C}}_g$ at each $X \in \hat{\mathcal{C}}_g$.

$\mathcal{J}_g$: the fam. of Picard varieties,

$\mathcal{X}_g = \mathcal{C}_g \times_{\mathcal{T}_g} \mathcal{J}_g$: Cartesian product

$\hat{\mathcal{X}}_g = \{(R, Q, \mathcal{L}, u, s : \mathcal{L}_Q \xrightarrow{\sim} \hat{\mathcal{O}})\}$
local trivialization at Q

$\hat{p} : \hat{\mathcal{X}}_g \to \mathcal{X}_g$ fibre bundle with fibre $\mathrm{Aut}(\hat{\mathcal{O}}) \times \hat{\mathcal{O}}^*$

$\mathbb{P}(\hat{\mathcal{X}}_g) = \hat{\mathcal{X}}_g / \mathbb{C}^*$: the Weierstrass system.

Thm. $\mathcal{D}$ acts on $\mathbb{P}(\hat{\mathcal{X}}_g)$ similarly as above.

For $j \in \frac{1}{2}\mathbb{Z}$, $\sigma_j : \mathcal{C}_g \to \mathcal{P}_g$, $R \mapsto \Omega_R^{\otimes j}$

$\hat{\sigma}_j : \hat{\mathcal{C}}_g \to \mathbb{P}(\hat{\mathfrak{X}}_g)$, $u \mapsto (du)^{\otimes j}$

Prop. Compatible with δ_j.

§2. Universal Grassmann Manifold (UGM).

Def. UGM $= \{ U \subset V$; ① U : closed,

② $U \to V/\hat{\mathcal{O}}$, dim Ker, Coker finite$\}$

$p(U) = \dim \mathrm{Ker} - \dim \mathrm{Coker}$: charge of U

Def. Krichever map : $\Gamma : \mathbb{P}(\hat{\mathfrak{X}}_g) \to$ UGM

$((X, L)) \mapsto (\zeta H^0(R, \mathcal{L}(*R)))$

For this the Torelli theorem holds.

The UGM admits a canonical projective embedding, the Plücker embedding, where the projective space is ass. with the semi-infinite exterior product of V which is called the fermion Fock space:

$$\mathfrak{P} : \mathrm{UGM} \to \mathbb{P}(\mathcal{F}) \qquad \mathcal{F} = \oplus \mathcal{F}_p$$

$$\mathrm{UGM}^p \to \mathbb{P}(\mathcal{F}_p), \qquad \mathcal{F}_p = \bigwedge^{\infty/2+p} V.$$

This UGM is also an infinitesimally homogeneous space with the Lie algebra:

$$\mathcal{D}_z = \{ P(z, \tfrac{d}{dz}) = \sum_{n<<\infty} z^n a_n(\tfrac{d}{dz}) \; : a_n(t) \in \mathbb{C}[[t]] \}$$

The Canonical inclusion $\mathcal{D} \subset \mathcal{D}_z$ is compatible with Γ.

§3. The second quantization

The space $\mathcal{F}$ has the Fermion algebra

$$\mathcal{A} = \mathbb{C}\{ \psi_\mu, \bar{\psi}_\mu \} \quad \mu \in \mathbb{Z} + 1/2$$

as the operator algebra ($\psi_\mu = e^{\mu - 1/2} \lrcorner$, $\bar{\psi}_\mu = e^{-\mu-1/2} \wedge$).

By the second quantization, we define a map

$$\mathfrak{X} : \mathcal{D}_z \to \hat{\mathcal{A}},$$

which defines an extension of Lie algebras

$$0 \to \mathbb{C}\,Id \to \tilde{\mathcal{D}}_2 \to \mathcal{D}_2 \to 0$$
$$0 \to \mathbb{C}\,Id \to \tilde{\mathcal{D}} \to \mathcal{D} \to 0$$
$$0 \to \mathbb{C}\,Id \to \tilde{\vartheta}^{(j)} \to \vartheta \to 0$$
$$0 \to \mathbb{C}\,Id \to \tilde{\sigma} \to \sigma \to 0$$

$\tilde{\vartheta}^{(j)}$ is called the Virasoro alg. $\tilde{\sigma}$ the current alg.

This defines a connection on the tautological line bundle on $\mathbb{P}(\mathcal{F})$, and the induced one on UGM, called the determinant bundle. Its connection form is explicitly given, called Schwinger term.

§4. Bosonization and the τ-function

Def. The boson Fock space is defined to be

$$\mathcal{H} = \mathbb{C}[[t_1, t_2, \cdots]] \otimes \mathbb{C}[e^{t_0}, e^{-t_0}]$$
$$\mathcal{H}_p = \mathbb{C}[[\qquad]] \otimes e^{pt_0} \quad \text{(charge } p \text{ sector)}$$

The bosonization rule says that there is an isomorphism

$$\beta : \mathcal{F} \xrightarrow{\sim} \mathcal{H}$$

preserving charge, by which the current algebra is transformed the Heisenberg algebra on $\mathcal{H}$.

Then the τ-function is defined as a canonical section on the determinant bundle on $\hat{\mathcal{E}}$ (induced from $\hat{\sigma}^{(1/2)}$), which behaves well. By this τ-function one can obtain all correlation functions.

Moreover this τ satisfies 2 systems of linear differential equations called the gauge conditions and the equations of motions (the generalized Ward-Takahashi identity) and it is the unique solution of these systems up to a function on $\mathcal{Y}$. It has a nice modular transformation property, which characterizes τ uniquely up to constant for $g \geq 3$.

§5. Further developments.

There will be 2 main directions to generalize our CFT system:

① non-abelian gauge algebra

② arithmetic, i.e. over $\mathbb{Z}$.

The first case is treated in [B-Sh.] which develop a method to analyze the determinant bundle for general Atiyah algebra. Their result implies that for an irr. integral representation of an affine Lie algebra the gauge condition admits a finite dimensional solutions (i.e. this system of diff. eqns is a holonomic one in the sense of Sato-Kashiwara)

For the second recently [K.S.U.] generalized our system over Z by introducing a new boson Fock space which is defined to be the coordinate ring of the universal Witt scheme.

We expect further development in both (analysis and arithmetic) directions, which would yield grand unification in math.

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Titel: Semi-stable Galois representations

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The purpose of this talk is to discuss a possible characterisation of the ℓ-adic representations of $Gal(\overline{\mathbb{Q}}/\mathbb{Q})$ that we get from the ℓ-adic cohomology of proper and smooth varieties defined over $\mathbb{Q}$.

1) p-adic semi-stable representations of $G_{\mathbb{Q}_p}$.

Let $\overline{\mathbb{Q}}_p$ be a fixed algebraic closure of $\mathbb{Q}_p$ (the field of p-adic numbers) and $G_p = G_{\mathbb{Q}_p} = Gal(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$.

Let K be the maximal (unramified) extension of $\mathbb{Q}_p$ contained in $\overline{\mathbb{Q}}_p$ and σ the absolute Frobenius acting on K.

__Definition__: A (φ, N, G_p) - __module__ is a finite dimensional K vector space D equipped with

a) A «Frobenius», i.e, a σ-semi-linear, bijective map

$$\varphi : D \to D;$$

b) A «monodromy operator», i.e. a K-linear endomorphism N of D satisfying

$$N\varphi = p\varphi N \quad (\Rightarrow N \text{ is nilpotent}),$$

c) A discrete semi-linear action of G_p ($\Rightarrow$ the action of the inertia sub-group factors through a finite quotient), commuting with φ and N.

If D is such a module and if $\Delta = (\overline{\mathbb{Q}}_p \otimes_K D)^{G_p}$, then $\dim_{\mathbb{Q}_p} \Delta = \dim_K D$.

Definition: A (φ, N, G_p)-filtered module consists of a (φ, N, G_p)-module D together with a decreasing filtration $(\mathrm{Fil}^i \Delta)_{i \in \mathbb{Z}}$ of Δ by sub-$\mathbb{Q}_p$-vector spaces satisfying $\mathrm{Fil}^i \Delta = 0$ if $i \gg 0$ and $\mathrm{Fil}^i \Delta = \Delta$ if $i \ll 0$.

Fact: One can define a functor

D_p : (p-adic representations of G_p) $\to$ ((φ, N, G_p)-filtered modules):

We say that a p-adic representation V is potentially semi-stable if $\dim_K D_p(V) = \dim_{\mathbb{Q}_p} V$ ($\Leftrightarrow \dim_{\mathbb{Q}_p} \Delta_p(V) = \dim_{\mathbb{Q}_p} V$ if $\Delta_p(V) = (\overline{\mathbb{Q}}_p \otimes_K D_p(V))^{G_p}$).

The functor D_p induces an equivalence between the category of pot. semi-stable p-adic rep'ns of G_p and the category of «admissible» (φ, N, G_p) filtered modules.

Conjecture: Let X be a proper and smooth variety over $\mathbb{Q}_p$. Then $V = H^m_{et}(X \times \overline{\mathbb{Q}}_p, \mathbb{Q}_p)$ is pot. semi-stable. Moreover

(i) $\Delta_p(V)$ can be identified to $H^m_{DR}(X)$;

(ii) if L is a finite Galois extension of $\mathbb{Q}_p$, contained in $\overline{\mathbb{Q}}_p$, on which $X \times L$ has good reduction, $D_p(V)$ can be identified to the crystalline cohomology of the special fiber of a smooth model of $X \times L^{nr}$ over the integers (where L^{nr} is the maximal unramified extension of L in $\overline{\mathbb{Q}}_p$);

(iii) if L is a finite Galois extension of $\mathbb{Q}_p$, on which $X \times L$ has semi-stable reduction, $D_p(V)$ can be identified to a new cohomology, the «crystalline with log poles cohomology» of $X \times L^{ur}$.

There is a lot of partial results in this direction and this conjecture is known for a wild class of varieties, and also for some other «motivic representations» (Tate, Raynaud, Bloch, Kato, Faltings, Messing, Hyodo, Scholl, Illusie, ... and the autor). The new cohomology theory seems to work (according to a quite recent work of Kato).

The definition of D_p uses the construction of three rings

$$B_{cris} \subset B_{st} \subset B_{DR} .$$

I constructed B_{cris} and B_{DR} a few years ago; the idea that something like B_{st} should exist is due to Uwe Jannsen.

2) <u>ℓ-adic semi-stable representations of</u> $G_{\mathbb{Q}}$ (joint work with B. Mazur).

<u>Definition</u>: Let ℓ be a prime number and V be a ℓ-adic representation of $G_{\mathbb{Q}} = Gal(\bar{\mathbb{Q}}/\mathbb{Q})$. We say that V is <u>geometric</u> if

(i) V is unramified almost everywhere;

(ii) V is potentially semi-stable at $p = \ell$.

Assume we have such a V, plus an embedding of a finite extension E_λ of $\mathbb{Q}_\ell$ into $End_{G_{\mathbb{Q}}}(V)$. Let $d = \dim_{E_\lambda} V$. For each prime p, one can associate to V a d-dimensional linear representation of the Weil-Deligne group $W'_p = W'_{\mathbb{Q}_p}$ of $\mathbb{Q}_p$ (for $p \neq \ell$, this is the usual construction and for $p = \ell$, one uses prop. (ii) and $D_p(V)$).

Using also $D_p(V)$, for $p=\ell$, one can define the Hodge numbers of V, its weight, whenever V is simple, and (under a mild assumption) an isomorphism class of a d-dimensional linear representation of the Weil group $W_{\mathbb{R}}$.

Therefore, if we choose an embedding of E_λ into $\mathbb{C}$, we can define the conductor N_V of V, the L-function $L(V,s)$ (with or without the Γ-factor), the ε-factor $\varepsilon(V,s)$.

We then have a lot of natural questions (which are not unrelated), e.g:

(1) If V is a semi-simple ℓ-adic geometric rep'n, does it exist $i \in \mathbb{Z}$ and a proper and smooth variety X over $\mathbb{Q}$ s.t. $V(i)$ is isomorphic to a direct summand of $H^{*}_{et}(X \times \bar{\mathbb{Q}}, \mathbb{Q}_\ell)$?

(2) If V is a geometric representation, is it true that $L(V,s)$ has a meromorphic continuation in the whole complex plane? Does it satisfy the functional equation

$$L(V,s) = \varepsilon(V,s) \cdot L(V^*, 1-s) \qquad ?$$

(3) Are there only finitely many isomorphism classes of semi-simple geometric E_λ-representations with a given conductor and given Hodge numbers?

(4) If $\dim_{E_\lambda} V = 2$, and if V is simple and geometric, with conductor N and $h^{0,k-1} = h^{k-1,0} = 1$ for a suitable integer $k \geq 1$, is it true that V is the representation associated to a modular form of weight k and level N (of course we already know the Fourier coefficients of the modular form).

(5) If V is a simple geometric representation of weight m, does V satisfy the Weil's conjecture (i.e. is it true that, for all p such that V is unramified at p, all the absolute values of the eigenvalues of the geometric Frobenius at p are equal to $p^{m/2}$) ?

Titel: Extension of CR cohomology. Deformation of holomorphic vector bundles over pseudoconvex manifolds with 2-convex holes
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The topic of this talk is a version with uniform estimates of the Andreotti-Vesentini separation theorem (joint work with G. Henkin) and two applications: The first application is a generalization of the classical Hartogs-Bochner extension theorem to CR cohomology classes (joint work with C. Laurent-Thiébaut). The second application is a theorem on deformation of holomorphic vector bundles over pseudoconvex manifolds with 2-convex holes (joint work with G. Henkin and P. Poljakov).

1. The Andreotti-Vesentini theorem with uniform estimates

Let X be an n-dimensional complex manifold. A C^2 function $\varrho: X \to \mathbb{R}$ will be called *q-convex* in $x \in X$ ($1 \le q \le n$) if the Levi form of ϱ at x has at least q positive eigenvalues. We say X is *q-convex* ($1 \le q \le n-1$) if there exists a C^2 function $\varrho: X \to \mathbb{R}$ which is exhausting for X and $(q+1)$-convex outside some compact set. A C^2 domain $D \subset\subset X$ will be called *strictly* q-convex ($1 \le q \le n-1$) if there exists a $(q+1)$-convex function ϱ in a neighborhood U of ∂D such that $D \cap \{\varrho < 0\} = U \cap \{\varrho < 0\}$. We say X is a *q-convex extension* of a domain $D \subset\subset X$ ($1 \le q \le n-1$) if there exists a C^2 function $\varrho: X \to \mathbb{R}$ which is exhausting for X and $(q+1)$-convex in some neighborhood of $X \setminus D$ and such that $D = \{\rho < 0\}$.

Wow we suppose that X is $(n-q)$-convex, where $1 \leq q \leq n-1$ (for instance, X may be an arbitrary pseudoconvex, i.e. $(n-1)$-convex, manifold). Further, let $G \subset\subset X$ be a strictly q-convex C^2 domain, and $D := X \setminus \overline{G}$. Then it is possible that $\dim H^{p,q}(D) = \infty$, $0 \leq p \leq n$ (Andreotti-Norguet theorem, 1966); however, by the Andreotti-Vesentini theorem, 1965, then nevertheless the natural topology of $H^{p,q}(D)$ is Hausdorff, i.e. the space of continuous $\bar\partial$-closed (p,q)-forms on D is closed with respect to uniform convergence on the compact subsets of D. This facts admits the following strengthening: Denote by $E^{1/2 \to 0}_{p,q}(\overline{D})$ the space of all continuous (p,q)-forms f on $\overline{D}$ $(= X \setminus G)$ such that the equation $\bar\partial u = f$ admits a solution u which is Hölder continuous with exponent $1/2$ on $\overline{D}$.

<u>Theorem 1.</u> The space $E^{1/2 \to 0}_{p,q}(\overline{D})$ is closed with respect to uniform convergence on the compact subsets of $\overline{D}$ (and not only of D).

This theorem is proved in Sect. 19 of the book [HL 1988] by means of the method of integral representation formulas for $\bar\partial$, which was introduce in 1970 by Grauert/Lieb/Ramirez and Henkin. Actually in [HL 1988] a version of Theorem 1 for differential forms with values in holomorphic vector bundles over X is proved. For $q = n-1$ Theorem 1 was obtained already in 1977 by Henkin.

2. Extension of CR cohomology classes

Let $D \subset\subset \mathbb{C}^n$ be a C^2 domain, $V \subseteq \partial D$ an open part of ∂D, and $f \in C^0_{p,q}(V)$, $0 \le p \le n$, $0 \le q \le n-2$. f is called CR if $\int_V f \wedge \bar\partial \alpha = 0$ for any $\alpha \in C^\infty_{n-p,n-q-2}(\mathbb{C}^n)$ such that the intersection $V \cap \operatorname{supp} \alpha$ is compact. f is called complex normal if $\int_V f \wedge \alpha = 0$ for all $\alpha \in C^\infty_{n-p,n-q-1}(\mathbb{C}^n)$ such that the intersection $V \cap \operatorname{supp} \alpha$ is compact. Recall the classical

Theorem 2 (Hartogs-Bochner). If $D \subset\subset \mathbb{C}^n$ is a C^1 domain such that $\mathbb{C}^n \setminus D$ is connected, then every CR function $f \in C^0_{0,0}(\partial D)$ admits a continuous extension onto $\overline{D}$ which is holomorphic in D.

This theorem admits a generalization to differential forms:

Theorem 3. If $D \subset\subset \mathbb{C}^n$ is a strictly q-convex C^2 domain ($1 \le q \le n-2$), then for any CR form $f \in C^\alpha_{p,q}(\partial D)$, $\alpha > 0$, $0 \le p \le n$, there exist a continuous (p,q)-form f_+ on $\overline{D}$ which is $\bar\partial$-closed in D as well as a form $g_- \in C^{1/2}_{p,q-1}(\mathbb{C}^n \setminus D)$ such that $\bar\partial g_- \in C^0_{p,q}(\mathbb{C}^n \setminus D)$ and the form $f - f_+|_{\partial D} + \bar\partial g_-|_{\partial D}$ is complex normal.

Theorem 2 admits generalizations to CR functions which are given only on some open part of ∂D (cf. [LuTo 1984], [LT 1988]). One of these generalizations is the following

<u>Theorem 2'</u> [Lu 1988]. Let $D \subset\subset \mathbb{C}^n$ be a C^2 domain and $K \subseteq \overline{D}$ a closed set such that: $\mathbb{C}^n \setminus D$ is connected and there exists a neighborhood U of $\overline{D}$ which is an 1-convex extension of K. Then any CR function $f \in C^0_{0,0}(\partial D \setminus K)$ admits a continuous extension onto $\overline{D} \setminus K$ which is holomorphic in $D \setminus K$.

This theorem also admits a generalization to forms:

<u>Theorem 3'.</u> Let $D \subset\subset \mathbb{C}^n$ be a C^2 domain and $K \subseteq \overline{D}$ a closed set such that, for some $1 \leq q \leq n-2$: (i) D is strictly q-convex and there exists an $(n-q)$-convex neighborhood $X \subseteq \mathbb{C}^n$ of $\overline{D}$ which is a q-convex extension of D; (ii) there exists a neighborhood $U \subseteq X$ of $K \cup \partial D$ which is a $(q+1)$-convex extension of K (i.e. of arbitrarily small neighborhoods of K).

Then, for any CR form $f \in C^{\alpha}_{p,q}(\partial D \setminus K)$, $\alpha > 0$, $0 \leq p \leq n$, there exist a continuous (p,q)-form f_+ on $\overline{D} \setminus K$ which is $\overline{\partial}$-closed on $D \setminus K$ as well as a form $g_- \in C^{1/2}_{p,q-1}(\mathbb{C}^n \setminus (D \cup K))$ such that $\overline{\partial} g_-$ is continuous on $\mathbb{C}^n \setminus (D \cup K)$ and the form

$$f - f_+\big|_{\partial D \setminus K} + \overline{\partial} g_-\big|_{\partial D \setminus K}$$

is complex normal.

Theorem 3' was obtained in [LTL 1988] by means of Theorem 1 and a jump theorem as in [LuTo 1984] and [LT 1988]. If D is strictly $(q+1)$-convex, Theorem 3' is contained in [P 1988]. In [LTL 1988] also several generalizations of Theorem 3' are given: $\mathbb{C}^n$ can be replaced by more general manifolds, and forms with values in holomorphic vector bundles may be admitted.

3. Deformation of holomorphic vector bundles

Let X be an n-dimensional complex manifold which is pseudoconvex, and let $G \subset\subset X$ be a strictly 2-convex domain. Set $D = X \setminus \overline{G}$. Then:

Theorem 4. For any holomorphic vector bundle E over X there is a complex space S with a distinguished point $s \in S$ as well as a holomorphic vector bundle F over $S \times D$ such that the following conditions are fulfilled: (i) $F|_{s \times D} \cong E|_D$. (ii) If $\tilde{S}$ is a second complex space, $\tilde{s} \in \tilde{S}$, U is a neighborhood of $\overline{D}$, and $\tilde{F}$ is a holomorphic vector bundle over $\tilde{S} \times U$ with $\tilde{F}|_{\tilde{s} \times U} \cong E|_U$, then after shrinking $\tilde{S}$ there is a holom. map $\varphi: \tilde{S} \to S$ with $\varphi(\tilde{s}) = s$ and a holomorphic homomorphism $h: \tilde{F}|_{\tilde{S} \times D} \to F|_{S \times D}$ over $\varphi \times id$ such that h induces an isomorphism from $\tilde{F}|_{\tilde{s} \times D}$ onto $F|_{s \times D}$. (iii) If φ is as in condition (ii), then the differential of φ at $\tilde{s}$ is uniquely determined.

This theorem is obtained in a joint work with G. Henkin and P. Poljakov (to appear). The "gap" between the neighborhood U in condition (ii) and D can be removed by introducing an appropriate notion of a holomorphic vector bundle over D with boundary values at ∂D.

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Titel: A Trilogarithm

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Seattle, WA 98195, U.S.A.

This talk was a summary of joint work done 2 years ago with Bob MacPherson.

After a change of variables, the classical logarithm, $\log x$, can be written as the power series

$$L_1(x) := -\log(1-x) = \sum_{n=1}^{\infty} \frac{x^n}{n}, \quad |x|<1.$$

In 1768, Euler defined the dilogarithm

$$L_2(x) = \sum_{n=1}^{\infty} \frac{x^n}{n^2}, \quad |x|<1$$

which was soon followed by the polylogarithms

$$L_k(x) = \sum_{n=1}^{\infty} \frac{x^n}{n^k}, \qquad |x|<1.$$

(See [8].) With the aid of the integral formula

$$L_k(x) = \int_0^x L_{k-1}(z)\,\frac{dz}{z} \qquad k>1,$$

one sees that each $L_k(x)$ can be

analytically continued to a multivalued holomorphic function on $\mathbb{C}-\{0,1\}$.

Each $L_k(x)$ is independent of its predecessors in the sense that

$$L_k(x) \notin \mathbb{C}[1, \log x, L_1(x), \ldots, L_{k-1}(x)]$$

as can be seen by an elementary monodromy argument, (cf. [9]).

The well known 3 term functional equation

$$\log x - \log xy + \log y \equiv 0 \mod 2\pi i\mathbb{Z}$$

plays an essential role in the construction of the first Chern class

$$c_1 : \left\{ \begin{matrix} \text{complex line bundles} \\ \text{over a topological space } X \end{matrix} \right\} \longrightarrow H^2(X,\mathbb{Z})$$

as does the multivaluedness of $\log x$.

Spence (1809) and Abel (1827) independently discovered the 5 term functional equation

$$\mathcal{L}_2(x) - \mathcal{L}_2(y) + \mathcal{L}_2(x/y) - \mathcal{L}_2\left(\frac{1-x}{1-y}\right) + \mathcal{L}_2\left(\frac{y(1-x)}{x(1-y)}\right) = 0$$

where

$$\mathcal{L}_2(x) = L_2(x) + \begin{pmatrix} \text{suitably chosen} \\ \text{quadratic polynomial} \\ \text{in } 1, \log x, \log(1-x) \end{pmatrix}.$$

[One has to be careful when writing functional equations of multivalued functions – cf. [7].]

Part of the recent interest in the dilogarithm stems from its connection with the second Chern class, both in topological situations [5] and in algebraic K-theory [2], where Chern classes are also known as regulators.

Spence's 5 term functional equation above plays the role of a 4-cocycle condition.

One is tempted to generalize this and relate $L_k(x)$ to the k^{th} Chern class. To do this, one needs $L_k(x)$ to satisfy a natural $2k+1$ term functional equation which would correspond to a $2k$-cocycle condition. Unfortunately, no such equations are known for

$k \geq 3$, despite much work on this topic during the 19th century (cf. [8]).

The idea behind the current work [7] is to propose a radically new approach to polylogarithms. In this approach, the k^{th} <u>higher logarithm</u> $\mathcal{L}_k$, if it exists, will automatically satisfy a natural $(2k+1)$-term functional equation. The domain of $\mathcal{L}_k$, if it exists, will be the Zariski open subset G_{k-1}^{k} of the grassmannian of k-planes in $\mathbb{C}^{2k}$ which intersect the k-dimensional coordinate flats transversally.

eg: $G_1^1 = \mathbb{C}^*$, $G_1^2 = \mathbb{C} - \{0,1\} \times (\mathbb{C}^*)^3$.

Unfortunately, establishing the existence of the $\mathcal{L}_k$ seems difficult and we have only constructed $\mathcal{L}_1$, $\mathcal{L}_2$ and $\mathcal{L}_3$. — $\mathcal{L}_1$ being the classical logarithm, $\mathcal{L}_2$ the pullback of the classical dilogarithm to G_1^2 along the natural projection to $\mathbb{C}-\{0,1\}$. $\mathcal{L}_3$ is a new function which satisfies a natural 7 term functional equation.

The main obstruction to constructing $\mathcal{L}_k$ in the current approach is understanding the topology of the spaces G_g^k $0 \le g \le k$ which comprises the set of k by $g+1$ complex matrices, all of whose minors are non zero.

The main technique for constructing $\mathcal{L}_k$ is the Hodge theory of π_1 due to Morgan (cf. [6]) and the iterated integrals of Chen [3,6]

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Titel: Bounding homotopy types by curvature and volume

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It is a basic problem in geometry to find relations between topology and geometry of riemannian manifolds. As a first general question one might ask to which extend "bounded geometry" implies "bounded topology".

For a closed, connected n-manifold, M, we consider the three basic geometric invariants: (sectional) curvature, K_M, diameter, diam M, and volume, Vol M. In terms of these invariants there are now three theorems in the above spirit:

<u>Theorem A</u> (Cheeger [C], Peters [P]) For each $k, K, D > 0$, and $v > 0$, there are only finitely many diffeomorphism classes of riemannian n-manifolds satisfying: $k \le K_M \le K$, $\operatorname{diam} M \le D$, and $\operatorname{Vol} M \ge v$.

<u>Theorem B</u> (Grove, Petersen [GP]) For each $k, D > 0$, and $v > 0$, there are only finitely many homotopy types of riemannian n-manifolds satisfying: $k \le K_M$, $\operatorname{diam} M \le D$, and $\operatorname{Vol} M \ge v$.

<u>Theorem C</u> (Gromov [G]) For each $k, D > 0$ there is an a priori number of generators for the homology of riemannian n-manifolds satisfying: $k \le K_M$, and $\operatorname{diam} M \le D$.

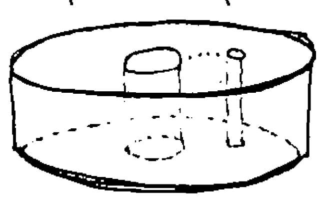
The following Corollary of Theorem B generalizes a theorem of Weinstein [W].

<u>Corollary D</u> For each $v > 0$ there are only finitely many homotopy types of riemannian n-manifolds satisfying: $K_M \geq 1$, and $\mathrm{Vol}\, M \geq v$.

<u>Theorem E</u> (Grove, Petersen [GP_2]) Let M be a riemannian n-manifold with $K_M \geq 1$, and $\mathrm{Vol}\, M \geq v$. Then if $v < \frac{1}{2} \mathrm{Vol}\, S^n(1)$ is sufficiently large, M has the homotopy type of either S^n or of $\mathbb{R}P^n$.

We will confine our discussion essentially to Theorem B. Before outlining the proof we want to indicate by examples, that this last theorem in some sense is obtimal (assumptions needed for the conclusion)

<u>Example 1</u> The family of oriented surfaces with metrics (to be "rounded off") of the form



provide examples where only $k \leq K_M$ is violated! (among all inequalities for the invariants considered)

Assume we have such a V, plus an embedding of a finite extension E_λ of $\mathbb{Q}_\ell$ into $\mathrm{End}_{G_{\mathbb{Q}}}(V)$. Let $d = \dim_{E_\lambda} V$. For each prime p, one can associate to V a d-dimensional linear representation of the Weil-Deligne group $W'_p = W'_{\mathbb{Q}_p}$ of $\mathbb{Q}_p$ (for $p \neq \ell$, this is the usual construction and for $p = \ell$, one uses prop. (ii) and $D_p(V)$).

Example 2 The 3-manifolds, $M_g = N_g^2 \times S^1$, where N_g^2 has constant negative curvature, and $\mathrm{Vol}\, S^1 = 1/\mathrm{Vol}\, N_g^2$ provide examples where only $\mathrm{diam}\, M_g \le D$ is violated!

Example 3 The Lense-spaces, $M_q = S^3(1)/\mathbb{Z}_q$, provide examples, where only $\mathrm{Vol}\, M_q \ge v$ is violated!

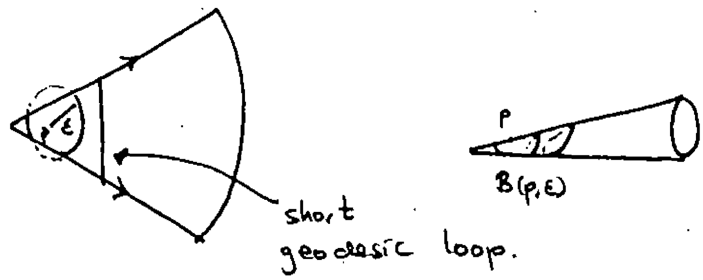
Remark: In all of the above theorems one can of course normalize (scale) the metrics, so that all manifolds considered have the same diameter (thereby only leaving bounds for curvature and volume).

The first trace of finiteness (in all cases) is easily obtained via the Bishop-Gromov volume comparison theorem. Namely for $k \le K_M$ (and $\mathrm{diam}\, M \le D$), the number of ε-balls in a minimal cover of M (i.e. the $\varepsilon/2$-balls are disjoint) is a priori bounded in terms of ε, k (and D). Moreover any point of M is in at most an apriori bounded number of ε-balls independent of ε.

Assuming $K_M \le k$ (as in Theorem A) $B(p,\varepsilon)$ is convex for sufficiently small ε and also the overlaps of balls are "controled".

Without an upper curvature bound, however, arbitrarily small balls can be topologically non-trivial:

Example



Say that minimal ε-covers $\{B(x_i,\varepsilon)\}$, $\{B(y_i,\varepsilon)\}$ of manifolds M, M' are equivalent their 1-skeleta are equal, i.e. the number of x_i's is the same as the number of y_i's and $B(x_i,\varepsilon) \cap B(x_j,\varepsilon) \neq \emptyset$ if and only if $B(y_i,\varepsilon) \cap B(y_j,\varepsilon) \neq \emptyset$.

Theorem B now follows from the following

Claim For $\varepsilon > 0$ small (fixed) manifolds M, M' with equivalent minimal ε-covers are homotopy equivalent.

The claim is proved by explicitely exhibiting maps $F: M \to M'$, $G: M' \to M$ such that $F \circ G$, $G \circ F$ are homotopic to the identity maps. – The idea in the construction of F (and G) is to "average" the constant maps $B(x_i,\varepsilon) \to y_i$ by a partition of unity subordinate to $\{B(x_i,\varepsilon)\}$. The crucial step is to be able to average two constant maps, i.e. to construct in a continuous manner a curve between any two nearby points, or, equivalently a deformation retraction of an a priory neighborhood of the diagonal Δ in $M \times M$ onto Δ. In order to achieve this, one proves, using the Toponogov distance comparison theorem as well as the Bishop-Gromov volume comparison theorem, that the continuous function dist: $M\times M \to \mathbb{R}$ has no "critical points" in a uniform r-neighborhood of the diagonal $\Delta \subset M\times M$.

Remark It should be noticed that by construction our homotopy equivalences are "small" in the sense of S. Ferry. This gives hope that it may be possible to obtain finiteness for homeo- and hence diffeomorphism types in dimensions ≥ 5

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ITERATION OF POLYNOMIALS.
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The iteration of complex analytic functions was begun by P. Fatou and G. Julia in about 1915-1920. It then went into a coma, from which it emerged in about 1980, and has been growing ever since.

The simplest analytic functions to iterate are the polynomials, and the object of this talk will be to describe the main results in this field.

There have been many contributors, listed here in random order:
A. Douady, W. Thurston, D. Sullivan, J. Milnor, B. Branner, B. Mandelbrot, M. Shishikura, H. Brolin, Ushiki, Tan Lei, R. Devaney, L. Keen, L. Goldberg, C. McMullen, F. von Haeseler, H.-O. Peitgen, J. Guckenheimer, P. Lavaurs, M. Hermann, J.-C. Yoccoz, P. Sentenac, Lyubich, Eremenko, Jakobsen, J. Kotus, M. Rees, P. Blanchard, B. Wittner, Y. Fisher, B. Bielefeld, and no doubt others which I have forgotten.

Generalities. When iterating a polynomial $P(z)$ of degree $d > 1$, the main object of interest is the filled in Julia set

$$K_P = \{ z \mid |P^{\circ n}(z)| \text{ remains bounded } \} ;$$

more precisely, we wish to know whether K_P is connected, if not what its connected components look like, do they have any interior, etc. Answers to such questions usually involve periodic points, i.e numbers z such that $P^{\circ n}(z) = z$ for some n ; the smallest number for which this occurs is the period of the point. Periodic points come in three flavors:

attractive if $|(P^{\circ n})'(z)| < 1$;
indifferent if $|(P^{\circ n})'(z)| = 1$;
repelling if $|(P^{\circ n})'(z)| > 1$;

the names should be self explanatory. An attractive periodic point has a basin, the set of points which are attracted to it, and an immediate basin, the union of the connected components containing the cycle.
A fundamental principle in this study, discovered by Fatou, is that the first thing to ask about a polynomial is:

What are the orbits of the critical points?

Fatou proved the following.

Theorem 1. ***Every attractive periodic point must have a critical point in its immediate basin.***

This shows for instance that a polynomial of degree d can have at most $d-1$ attractive cycles in all, and in particular a quadratic polynomial can have at most 1; considering that it has approximately $2^k/k$ cycles of period k, this is rather surprizing. Fatou's proof uses the hyperbolic metric; the result is hard to prove without complex analysis even for real quadratic polynomials and real periodic cycles.

Another theorem, also due to Fatou, is:

Theorem 2. ***If*** K_P ***contains all the critical points, then*** K_P ***is connected.***

If K_P ***contains no critical points, then*** K_P ***is a Cantor set, and***

$$P: K_P \to K_P$$

is conjugate to the 1-sided Bernoulli shift on d symbols.

One good reason for being interested in the dynamics of polynomials is that the **Staightening Theorem** says that this dynamical behaviour is in some sense universal; there are whole analytic classes of mappings which have the same quasi-conformal behavior.

Quadratic Polynomials. Theorem 2 above says that for the quadratic polynomial $P_c(z) = z^2 + c$, the following dichotomy holds:

either $0 \in K_c$, in which case K_c is connected,

or 0 is not in K_c , in which case K_c is a Cantor set.

The Mandelbrot set M is defined to be the set

$$M = \{c \mid K_c \text{ is connected}\} = \{c \mid 0 \in K_c\}$$

$$= \{c \mid \text{the sequence } 0, c, c^2+c, (c^2+c)^2+c, \ldots \text{ is bounded}\} .$$

This set is remarkably complicated. Douady and the author proved:

Theorem 3. *The set* M *is connected.*

In fact, the proof explicitly constructs the Riemann mapping for the complement, showing that the infinite product

$$\Phi(c) = c\,(1 + 1/c)^{1/2} \ldots (1 + c/(P_c^{\circ k}(c))^2)^{1/2^k} \ldots$$

is convergent in a neighborhood of infinity, and can be extended to an isomorphism $\Phi: \mathbb{C}-M \to \mathbb{C}-\overline{D}$, where $\overline{D}$ is the closed unit disc.

If we define the **exterior ray of angle** θ of M to be

$$R_M(\theta) = \Phi^{-1}(\{r e^{2\pi i\theta}, r > 1\}) ,$$

then the specification of precisely when rays $R_M(\theta_1)$ and $R_M(\theta_2)$ land at the same point would competely describe the topology of M as a subset of $\mathbb{C}$, at least if the following conjecture holds:

Conjecture: ***The Mandelbrot set is locally connected.***

In any case, the combinatorics of when external rays land at the same point is now well understood, and can be read in [D-H], or more easily without proofs in [P-R].

Polynomials of higher degree. Above we considered the quadratic polynomials $P_c(z) = z^2 + c$; this was justified because every quadratic polynomial is conjugate to a unique one of this form. More generally, any polynomial of degree d is conjugate to d-1 polynomials of the form $P(z) = z^d + a_{d-2}z^{d-2} + \ldots + a_0$. Let $\mathbf{P_d}$ be the space of polynomials of this form. Theorem 2 suggests several possible generalizations of the Mandelbrot set to higher degree; the best appears to be the **connectedness locus** $C_d \subset \mathbf{P_d}$:

$$C_d = \{P \in \mathbf{P_d} \mid K_P \text{ is connected}\} .$$

Theorem 4. ***For all*** d ***the space*** C_d ***is a cell-like compact subset of*** P_d.

This is of course exactly theorem 3 for quadratic polynomials. It was proved in [B-H] for cubics, and very recently by P. Lavaurs (unpublished) for all d. Lavaur's proof incorporates the ideas used for cubics, sketched below, and techniques for cell-like mappings, as well as using the Poincare conjecture in dimensions ≥ 5 and the Schonfliess theorem.

Cubic polynomials. Changing our notation slightly, we will let P_3 be the space of polynomials

$$P_{a,b}(z) = z^3 - 3\,a^2 z + b \ ,$$

so that the critical points are at $\pm a$. To show that C_3 is cell like, we need to show that the complement of the connectedness locus $P_3 - C_3$ is foliated by spheres bounding ball. If we define the potential of K_P by the formula

$$h_P(z) = \lim_{n \to \infty} \log_+ \left| P^{\circ n}(z) \right|$$

and then $H: P_3 \to \mathbb{R}$ by $H(P) = \sup(h_P(+a) \, , h_P(-a))$, then we have:

Theorem 5. ***The mapping*** $H: P_3 \to]0,\infty[$ ***is a trivial fibration, with fibers homeomorphic to spheres.***

For cubics, an explicit trivialization is given in [B-H] using quasi-conformal surgery. Lavaurs proves that the fibers of the analog of H in higher degrees are always spheres, but not by constructing a trivialization; he needs the generalized Poincare conjecture.

If we let $S_r = H^{-1}(\log r)$, then this sphere is naturally decomposed into two parts S_r^+ and S_r^- , according to whether the sup in the definition of H is realized by $+a$ or $-a$. These two parts are each solid tori, which link with linking number 3 ; a topological model is given by decomposing the standard 3-sphere S^3 into two solid tori as usual, and taking the triple cover ramified along the diagonal.

The space S_r^+ is characterized by the property that the critical point $+a$ escapes at rate r and $-a$ more slowly, if at all. It has a whole internal structure classifying how this occurs. The main result to understand this classification is the following:

Theorem 6. ***Let*** P ***be a cubic polynomial for which one critical point*** ω_1 ***escapes to*** ∞ ***and the other*** ω_2 ***does not. Then if the component of*** K_P ***containing*** ω_2 ***is periodic, this component is quasi-conformally homeomorphic to the filled in Julia set of some quadratic polynomial. If the component of*** K_P ***containing*** ω_2 ***is not periodic, then*** K_P ***is a Cantor set.***

The techiques used in the proof involve estimating the moduli of infinitely many annuli. These techniques definitely do not extend to degree 4, opening the possibility that really new phenomena occur in higher degrees.

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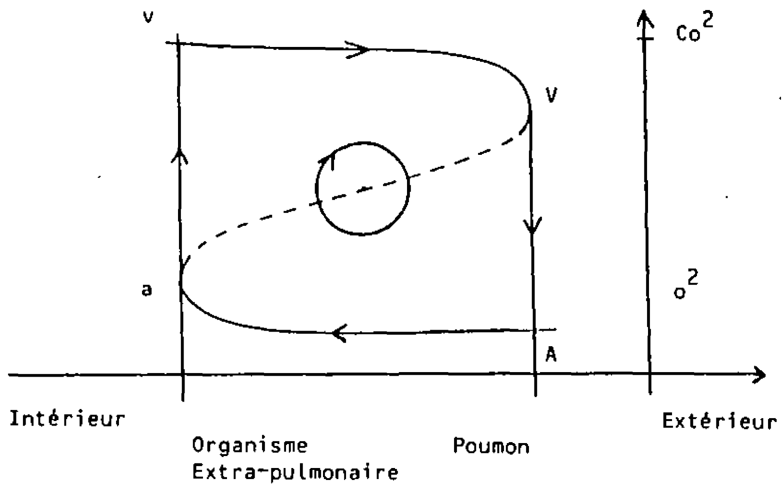
Diagramme Δ



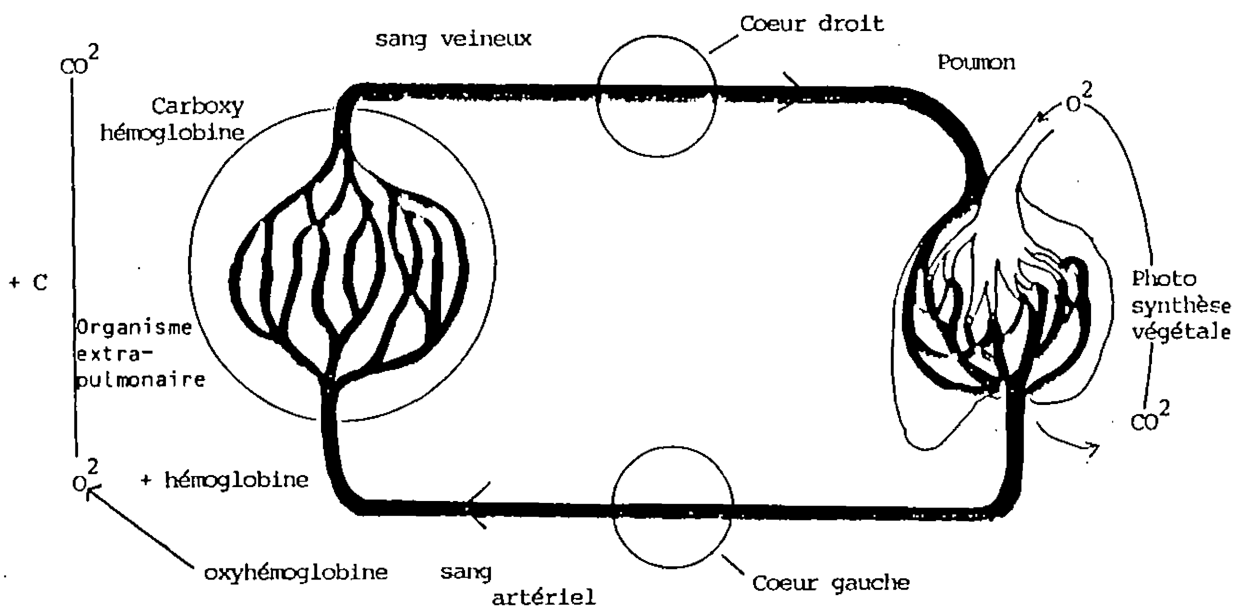


Fig. 9

semi-stable reduction, $D_p(V)$ can be identified to a new cohomology, the «crystalline with log poles cohomology» of $X \times L^{ur}$.

There is a lot of partial results in this direction and this conjecture is known for a wild class of varieties, and also for some other «motivic representations» (Tate, Raynaud, Bloch, Kato, Faltings, Messing, Hyodo, Scholl, Illusie, ... and the autor). The new cohomology theory seems to work (according to a quite recent work of Kato).

The definition of D_p uses the construction of three rings

$$B_{cris} \subset B_{st} \subset B_{DR} .$$

I constructed B_{cris} and B_{DR} a few years ago; the idea that something like B_{st} should exist is due to Uwe Jannsen.

2) ℓ-adic <u>semi-stable representations of</u> $G_{\mathbb{Q}}$ (joint work with B. Mazur).

<u>Definition</u>: Let ℓ be a prime number and V be a ℓ-adic representation of $G_{\mathbb{Q}} = Gal(\bar{\mathbb{Q}}/\mathbb{Q})$. We say that V is <u>geometric</u> if

(i) V is unramified almost everywhere;

(ii) V is potentially semi-stable at $p = \ell$.

~~If we have such a V, for each prime number p, one can ~~define~~ associate to V a linear representation of the Weil-Deligne group W'_p of~~