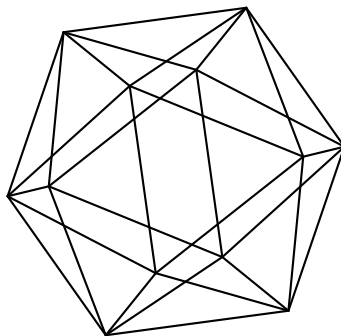


Max-Planck-Institut für Mathematik Bonn

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Abstract

The deepest arithmetic invariants attached to an algebraic variety defined over a number field F are conjecturally captured by the *integral part* of its *motivic cohomology*. There are essentially two ways of defining it when X is a smooth projective variety: one is via the K -theory of a regular model, the other is through its ℓ -adic realization. Both approaches are conjectured to coincide.

This paper initiates the study of motivic cohomology for global fields of positive characteristic, hereafter named *A -motivic cohomology*, where classical mixed motives are replaced by mixed Anderson A -motives. Our main objective is to set the definitions of the *model version* and the *ℓ -adic version* of the integral part of A -motivic cohomology, using Gardeyn's notion of *maximal models* of A -motives as the analogue of regular models of varieties. Our main result states that the *model version* is contained in the *ℓ -adic version*. As opposed to what is expected in the number field setting, we show that the two approaches do not match in general.

We conclude this work by introducing the submodule of *regulated extensions* of mixed Anderson A -motives, for which we expect the two approaches to match, and solve some particular cases of this expectation.

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1 Introduction

1.1 The number field picture

The idea of *mixed motives* and *motivic cohomology* has been gradually formulated by Deligne, Beilinson and Lichtenbaum and aims to extend Grothendieck's philosophy of pure motives. Before discussing the function fields side, subject of the paper, let us first present the classical setting to derive some motivations.

The theory, mostly conjectural, starts with a number field F . The hypothetical landscape portrays a \mathbb{Q} -linear Tannakian category \mathcal{MM}_F of *mixed motives* over F , equipped with several *realization functors* having \mathcal{MM}_F as source (see [Del, §1]). Among them, the ℓ -adic realization functor V_ℓ , for a prime number ℓ , takes values in the category of continuous ℓ -adic representations of the absolute Galois group G_F of F .

It is expected that *reasonable* cohomology theories factor through the category \mathcal{MM}_F . For instance, the ℓ -adic realization should recover the étale cohomology of algebraic varieties with coefficients in \mathbb{Q}_ℓ in the following way: for all integer i , one foresees the existence of a functor h^i , from the category of algebraic varieties over F to \mathcal{MM}_F , making the following diagram of categories commute:

$$\begin{array}{ccc} \{\text{Varieties}/F\} & \xrightarrow{h^i} & \mathcal{MM}_F \\ & \searrow & \downarrow V_\ell \\ X \mapsto H_{\text{ét}}^i(X \times_F F^s, \mathbb{Q}_\ell) & & \text{Rep}_{\mathbb{Q}_\ell}(G_F) \end{array}$$

According to Deligne [Del, §1.3], the category \mathcal{MM}_F should admit a *weight filtration* in the sense of Jannsen [Jan, Def 6.3], which would coincide with the classical weight filtration of varieties. The *weights* of a mixed motive M would then be defined as the breaks of its weight filtration.

From the Tannakian formalism, \mathcal{MM}_F admits a tensor operation, extending the fiber product on varieties, and we fix $\mathbb{1}$ a neutral object. Let M be a mixed motive over F . According to Beilinson [Bei1, §0.3] (see also [Andr, Def 17.2.11]), the *motivic cohomology* of M is defined as the complex

$$\text{RHom}_{\mathcal{MM}_F}(\mathbb{1}, M)$$

in the derived category of \mathbb{Q} -vector spaces. Its i th cohomology is the \mathbb{Q} -vector space $\text{Ext}_{\mathcal{MM}_F}^i(\mathbb{1}, M)$, the i th Yoneda extension space of $\mathbb{1}$ by M in \mathcal{MM}_F . We quote from [Sch, §2] and [Del, §1.3] respectively:

Conjecture. We expect that:

(C1) for $i \notin \{0, 1\}$, $\text{Ext}_{\mathcal{MM}_F}^i(\mathbb{1}, M) = 0$,

(C2) if the weights of M are non-negative, $\text{Ext}_{\mathcal{MM}_F}^1(\mathbb{1}, M) = 0$.

We focus on the first Yoneda extension \mathbb{Q} -vector space $\text{Ext}_{\mathcal{MM}_F}^1(\mathbb{1}, M)$. A subspace thereof of fundamental importance is the space of *extensions having everywhere good reduction*. Let us first describe its local version.

Let $F_{\mathfrak{p}}$ be the local field of F at a finite place \mathfrak{p} , and let $M_{\mathfrak{p}}$ be a mixed motive over $F_{\mathfrak{p}}$. To define the \mathfrak{p} -integral part of the motivic cohomology of $M_{\mathfrak{p}}$, we encounter two approaches, which are believed to meet.

Via the ℓ -adic realization: Let $G_{\mathfrak{p}}$ be the absolute Galois group of $F_{\mathfrak{p}}$, and let $I_{\mathfrak{p}}$ be its inertia subgroup. Given a prime number ℓ from which \mathfrak{p} is not above, one predicts that the ℓ -adic realization V_{ℓ} is an exact functor. This allows to construct a \mathbb{Q} -linear morphism, called *the ℓ -adic realization map of $M_{\mathfrak{p}}$* ,

$$r_{M,\ell,\mathfrak{p}} : \text{Ext}_{\mathcal{M}\mathcal{M}_{F_{\mathfrak{p}}}}^1(\mathbb{1}_{\mathfrak{p}}, M_{\mathfrak{p}}) \longrightarrow H^1(G_{\mathfrak{p}}, V_{\ell}M_{\mathfrak{p}})$$

which maps the class of an exact sequence $[E_{\mathfrak{p}}] : 0 \rightarrow M_{\mathfrak{p}} \rightarrow E_{\mathfrak{p}} \rightarrow \mathbb{1}_{\mathfrak{p}} \rightarrow 0$ in $\mathcal{M}\mathcal{M}_{F_{\mathfrak{p}}}$ to the class of the continuous cocycle $c : G_{\mathfrak{p}} \rightarrow V_{\ell}M_{\mathfrak{p}}$ associated to the class of the exact sequence $[V_{\ell}E_{\mathfrak{p}}] : 0 \rightarrow V_{\ell}M_{\mathfrak{p}} \rightarrow V_{\ell}E_{\mathfrak{p}} \rightarrow V_{\ell}\mathbb{1}_{\mathfrak{p}} \rightarrow 0$ in $\text{Rep}_{\mathbb{Q}_{\ell}}(G_{\mathfrak{p}})$.

Following Scholl [Sch], we define the \mathfrak{p} -integral part of the motivic cohomology as follows. It is said that $[E_{\mathfrak{p}}] \in \text{Ext}_{\mathcal{M}\mathcal{M}_{F_{\mathfrak{p}}}}^1(\mathbb{1}, M)$ has *good reduction* if $r_{M,\ell,\mathfrak{p}}([E_{\mathfrak{p}}])$ splits as a representation of $I_{\mathfrak{p}}$ (that is, $[V_{\ell}E_{\mathfrak{p}}]$ is zero in $H^1(I_{\mathfrak{p}}, V_{\ell}M_{\mathfrak{p}})$). We define $\text{Ext}_{\text{good}}^1(\mathbb{1}, M_{\mathfrak{p}})_{\ell}$ as the subspace of $\text{Ext}_{\mathcal{M}\mathcal{M}_{F_{\mathfrak{p}}}}^1(\mathbb{1}, M_{\mathfrak{p}})$ consisting of extensions having good reduction. In [Sch, §2 Rmk], Scholl conjectures:

Conjecture. We expect that:

(C3) The property that $[E_{\mathfrak{p}}]$ has good reduction is independent of the prime ℓ .

Via the K -theory of regular models: In the case where $M_{\mathfrak{p}} = h^{i-1}(X)(n)$ for a smooth projective variety X over $F_{\mathfrak{p}}$ and two integers $n, i \geq 1$, there is another conjectural way of defining the \mathfrak{p} -integral part of the motivic cohomology of $M_{\mathfrak{p}}$. Following Beilinson [Bei1], there should be an isomorphism (see *loc. cit.* for details):

$$\text{Ext}_{\mathcal{M}\mathcal{M}_{F_{\mathfrak{p}}}}^1(\mathbb{1}, M_{\mathfrak{p}}) \xrightarrow{\sim} (K_{2n-i}(X) \otimes_{\mathbb{Z}} \mathbb{Q})^{(n)}. \quad (1.1)$$

Assume that X has a regular model \mathcal{X} over $\mathcal{O}_{\mathfrak{p}}$ (i.e. \mathcal{X} is regular over $\text{Spec } \mathcal{O}_{\mathfrak{p}}$ and $\mathcal{X} \times_{\text{Spec } \mathcal{O}_{\mathfrak{p}}} \text{Spec } F_{\mathfrak{p}} = X$). Then, we define $\text{Ext}_{\mathcal{O}_{\mathfrak{p}}}^1(\mathbb{1}, M_{\mathfrak{p}})$ to be the inverse image of

$$\text{image} \left((K_{2n-i}(\mathcal{X}) \otimes_{\mathbb{Z}} \mathbb{Q})^{(n)} \longrightarrow (K_{2n-i}(X) \otimes_{\mathbb{Z}} \mathbb{Q})^{(n)} \right),$$

which does not depend on \mathcal{X} [Bei1, Lem. 8.3.1], through (1.1). The next conjecture supersedes (C3):

Conjecture. We expect that

(C4) For any prime ℓ from which \mathfrak{p} is not above, $\text{Ext}_{\mathcal{O}_{\mathfrak{p}}}^1(\mathbb{1}, M_{\mathfrak{p}}) = \text{Ext}_{\text{good}}^1(\mathbb{1}, M_{\mathfrak{p}})_{\ell}$.

The global version of the integral part of the motivic cohomology of a mixed motive M over F is recovered as follows. The motive M induces a motive $M_{\mathfrak{p}}$ over $F_{\mathfrak{p}}$ by localization. Admitting (C3), we say that an extension $[E] \in \text{Ext}_{\mathcal{M}\mathcal{M}_F}^1(\mathbb{1}, M)$ has *everywhere good reduction* if, for all \mathfrak{p} , $[E_{\mathfrak{p}}]$ belongs to $\text{Ext}_{\mathcal{M}\mathcal{M}_{F_{\mathfrak{p}}}}^1(\mathbb{1}_{\mathfrak{p}}, M_{\mathfrak{p}})_{\ell}$ for some prime ℓ . We denote by $\text{Ext}_{\text{good}}^1(\mathbb{1}, M)$ the subspace of $\text{Ext}_{\mathcal{M}\mathcal{M}_F}^1(\mathbb{1}, M)$ consisting of extensions having everywhere good reduction.

Similarly, in the case where $M = h^{i-1}(X)(n)$ for a smooth projective variety X over F , we let $\text{Ext}_{\mathcal{O}_F}^1(\mathbb{1}, M)$ be the subspace of extensions $[E]$ such that $[E_{\mathfrak{p}}]$ belongs to $\text{Ext}_{\mathcal{O}_{\mathfrak{p}}}^1(\mathbb{1}_{\mathfrak{p}}, M_{\mathfrak{p}})$ for all finite places \mathfrak{p} of F . In virtue of the previous conjectures, we should have:

$$\text{Ext}_{\text{good}}^1(\mathbb{1}, M) = \text{Ext}_{\mathcal{O}_F}^1(\mathbb{1}, M) \cong (K_{2n-i}(\mathcal{X}) \otimes_{\mathbb{Z}} \mathbb{Q})^{(n)},$$

where \mathcal{X} is a regular model of X over \mathcal{O}_F . The above space is at the heart of Beilinson's conjectures, which begin by the next expectation:

Conjecture. We expect that

(C5) The space $\text{Ext}_{\mathcal{O}_F}^1(\mathbb{1}, M)$ has finite dimension over \mathbb{Q} .

1.2 The function field picture

Despite its intrinsic obscurities, Motivic cohomology remains a difficult subject also because its definition sits on a completely conjectural framework. The present paper grew out as an attempt to understand the analogous picture in function fields arithmetic. There, the theory looks more promising using *Anderson A -motives*, instead of classical motives, whose definition is well-established. This parallele has been drawn by many authors and led to celebrated achievements. The analogue of the Tate conjecture [Tag] [Tam], of Grothendieck's periods conjecture [Pap] and of the Hodge conjecture [HarJu] are now theorems on the function fields side. The recent volume [tMo] records some of these feats. Counterparts of Motivic cohomology in function fields arithmetic have not been studied yet, although recent works of Taelman [Tae2] [Tae4] and Mornev [Mo1] strongly suggest the pertinence of such a project.

The setting

Let \mathbb{F} be a finite field, q its number of elements, and let (C, \mathcal{O}_C) be a geometrically irreducible smooth projective curve over \mathbb{F} . The curve C is determined up to \mathbb{F} -isomorphism by its function field $K := \mathbb{F}(C)$. Let ∞ be a closed point on C with associated valuation v_∞ and consider the ring

$$A := \Gamma(C \setminus \{\infty\}, \mathcal{O}_C)$$

whose fraction field is K . We let K_∞ be its completion with respect to v_∞ , and let K_∞^s be the separable closure of K_∞ . We let $G_\infty = \text{Gal}(K_\infty^s|K_\infty)$ be the absolute Galois group of K_∞ . The analogy with number fields that should guide us in this text is:

| | | | | | | | | |
|------------------|--------------|-----------|--------------|-----------|--------------|-----------|--------------|-------------------------------------|
| Number fields: | \mathbb{Z} | \subset | \mathbb{Q} | \subset | \mathbb{R} | \subset | \mathbb{C} | $\text{Gal}(\mathbb{C} \mathbb{R})$ |
| | \wr | | \wr | | \wr | | \wr | \wr |
| Function fields: | A | \subset | K | \subset | K_∞ | \subset | K_∞^s | G_∞ |

Let F be a finite extension of K (that is, a global function field over K) and let \mathcal{O}_F denote the integral closure of A in F .

The analogy with number fields disappears when one considers the tensor product $A \otimes F$, which is at the heart of the definition of Anderson A -motives (unlabeled fiber and tensor products are over \mathbb{F}). We consider the ring endomorphism τ of $A \otimes F$ which acts as the identity on A and as the q -Frobenius on F . $A \otimes F$ is a Dedekind domain, and we let \mathfrak{j} be its maximal ideal generated by the set $\{a \otimes 1 - 1 \otimes a | a \in A\}$.

Following [And], an *Anderson A -motive \underline{M} over F* is a pair (M, τ_M) where M designates a finite locally free $A \otimes F$ -module of constant rank, and where $\tau_M : (\tau^*M)[\mathfrak{j}^{-1}] \rightarrow M[\mathfrak{j}^{-1}]$ is an $(A \otimes F)[\mathfrak{j}^{-1}]$ -linear isomorphism (see Definition 2.1). We let \mathcal{M}_F denote the category of Anderson A -motives with obvious morphisms. \mathcal{M}_F is known to be A -linear, rigid monoidal, and is exact in the sense of Quillen but not abelian ([HarJu, §2.3] or Section 2.1). Let $\mathbb{1}$ in \mathcal{M}_F be a neutral object for the tensor operation.

Extensions of A -motives

The category \mathcal{M}_F , or rather full subcategories of it, will play the role of the category of Grothendieck's motives. Guided by this, the next theorem already describes the analogue of motivic cohomology in an explicit manner, and is the starting point of our research (see Theorem 2.13). Let \underline{M} be an A -motive over F .

Theorem A. *The complex $\left[M \xrightarrow{\text{id} - \tau_M} M[j^{-1}] \right]$ of A -modules placed in degree 0 and 1 represents the complex $\text{RHom}_{\mathcal{M}_F}(\mathbb{1}, \underline{M})$.*

We immediately deduce that $\text{Ext}_{\mathcal{M}_F}^i(\mathbb{1}, \underline{M})$ is zero for $i > 1$. For $i = 1$, one obtains a surjective morphism

$$\iota : M[j^{-1}] \longrightarrow \text{Ext}_{\mathcal{M}_F}^1(\mathbb{1}, \underline{M}) \quad (1.2)$$

whose kernel is the A -module $(\text{id} - \tau_M)(M)$. (1.2) is explicitly described by mapping a representative $m \in M[j^{-1}]$ to the class of the extension of $\mathbb{1}$ by \underline{M} given by $[M \oplus (A \otimes F), \begin{pmatrix} \tau_M & m \\ 0 & 1 \end{pmatrix}]$ (Section 2.2).

Remark. Extension groups in the full subcategory of \mathcal{M}_F consisting of *effective A -motives* (see Definition 2.3) were already determined in the existing literature (see e.g. [Tae3], [Tae4], [PapRa]). The novelty of Theorem A is to consider the whole category \mathcal{M}_F .

To pursue the analogy with number fields, we now present the notion of *mixedness* and *weights* for Anderson A -motives. In the case $A = \mathbb{F}[t]$ or $\deg(\infty) = 1$ over a complete algebraically closed base field, the corresponding definitions were carried out respectively by Taelman [Tae1] and Hartl-Juschka [HarJu]. We completed this picture in the most general way (over any A -field and without any restriction on $\deg(\infty)$).

To an Anderson A -motive \underline{M} over F , we attach an *isocrystal* $\mathcal{I}_\infty(\underline{M})$ over F at ∞ (in the sense of [Mor2]). The term *isocrystal* is borrowed from p -adic Hodge theory, where the function field setting allows to apply the non-archimedean theory at the infinite point ∞ of C as well. In Section 3, we prove that the isocrystal $\mathcal{I}_\infty(\underline{M})$ carries a uniquely determined *slope filtration* (see Definition 3.8):

$$0 = \mathcal{I}_\infty(\underline{M})_{\mu_0} \subsetneq \mathcal{I}_\infty(\underline{M})_{\mu_1} \subsetneq \mathcal{I}_\infty(\underline{M})_{\mu_2} \subsetneq \cdots \subsetneq \mathcal{I}_\infty(\underline{M})_{\mu_s} = \mathcal{I}_\infty(\underline{M}) \quad (1.3)$$

for uniquely determined rational numbers $\mu_1 < \dots < \mu_s$ called the *weights* of \underline{M} . We say that \underline{M} is *mixed* if there exists an increasing filtration $(W_{\mu_i} \underline{M})_{1 \leq i \leq s}$ of \underline{M} by subobjects in \mathcal{M}_F whose associated filtration $(\mathcal{I}_\infty(W_{\mu_i} \underline{M}))_{1 \leq i \leq s}$ of $\mathcal{I}_\infty(\underline{M})$ by subsocystals matches with (1.3) (Definition 3.20). We let \mathcal{MM}_F be the full subcategory of \mathcal{M}_F whose objects are mixed Anderson A -motives over F (Section 3). The main results of Section 3 are condensed in:

Theorem B. *Let \underline{M} be an object of \mathcal{MM}_F . If all the weights of \underline{M} are non-positive, then every extension of $\mathbb{1}$ by \underline{M} is mixed, that is:*

$$\text{Ext}_{\mathcal{MM}_F}^1(\mathbb{1}, \underline{M}) = \text{Ext}_{\mathcal{M}_F}^1(\mathbb{1}, \underline{M}).$$

If all the weights of \underline{M} are positive, then an extension of $\mathbb{1}$ by \underline{M} is mixed if and only if its class is torsion, that is:

$$\text{Ext}_{\mathcal{MM}_F}^1(\mathbb{1}, \underline{M}) = \text{Ext}_{\mathcal{M}_F}^1(\mathbb{1}, \underline{M})^{\text{tors}}.$$

Furthermore, for $i > 1$, $\text{Ext}_{\mathcal{MM}_F}^i(\mathbb{1}, \underline{M})$ is a torsion module for all \underline{M} .

Remark. Although the category of classical mixed motives is expected to be \mathbb{Q} -linear, the category \mathcal{MM}_F of mixed Anderson A -motives over F is only A -linear. To obtain a K -linear category, it might be convenient to introduce $\mathcal{MM}_F^{\text{iso}}$ whose objects are the ones of \mathcal{MM}_F and whose Hom-spaces are given by $\text{Hom}_{\mathcal{MM}_F}(-, -) \otimes_A K$. In the literature, $\mathcal{MM}_F^{\text{iso}}$ is called the category of mixed A -motives over F up to isogenies [Har3], [HarJu]. Theorem B implies that $\text{Ext}_{\mathcal{MM}_F^{\text{iso}}}^i(\mathbb{1}, \underline{M}) = 0$ for $i > 1$ and $\text{Ext}_{\mathcal{MM}_F^{\text{iso}}}^1(\mathbb{1}, \underline{M}) = 0$ if the weights of \underline{M} are positive. This reveals that the analogue of the number fields conjecture

(C1) and (C2) are true for function fields. Note, however, that contrary to what is expected for number fields, the full subcategory of pure A -motives is not semi-simple. Hence, we cannot expect any 1-fold Yoneda extension of two pure A -motives to split, even if they have the same weight.

Integral part: the ℓ -adic version

Let \mathfrak{p} be a finite place of F (i.e. not above ∞), $F_{\mathfrak{p}}$ the associated local field, $F_{\mathfrak{p}}^s$ a separable closure and $G_{\mathfrak{p}} = \text{Gal}(F_{\mathfrak{p}}^s|F_{\mathfrak{p}})$ the absolute Galois group of $F_{\mathfrak{p}}$ equipped with the profinite topology. Given a maximal ideal ℓ in A from which \mathfrak{p} is not above, there is an ℓ -adic realization functor from $\mathcal{M}_{F_{\mathfrak{p}}}$ to the category of continuous \mathcal{O}_{ℓ} -linear representations of $G_{\mathfrak{p}}$. For $\underline{M}_{\mathfrak{p}} = (M_{\mathfrak{p}}, \tau_M)$ an object of $\mathcal{M}_{F_{\mathfrak{p}}}$, it is given by the \mathcal{O}_{ℓ} -module

$$T_{\ell}\underline{M}_{\mathfrak{p}} = \varprojlim_n \{m \in (M_{\mathfrak{p}} \otimes_{F_{\mathfrak{p}}} F_{\mathfrak{p}}^s) / \ell^n (M_{\mathfrak{p}} \otimes_{F_{\mathfrak{p}}} F_{\mathfrak{p}}^s) \mid m = \tau_M(\tau^* m)\}$$

where $G_{\mathfrak{p}}$ acts compatibly on the right of the tensor $M_{\mathfrak{p}} \otimes_{F_{\mathfrak{p}}} F_{\mathfrak{p}}^s$ (Definition 2.16). We prove in Corollary 2.20 that T_{ℓ} is exact.

This paves the way for introducing *extensions with good reduction*, as Scholl did in the number fields setting. Let $I_{\mathfrak{p}} \subset G_{\mathfrak{p}}$ be the inertia subgroup. We consider the ℓ -adic realization map restricted to $I_{\mathfrak{p}}$:

$$r_{\underline{M}, \ell, \mathfrak{p}} : \text{Ext}_{\mathcal{M}_{F_{\mathfrak{p}}}}^1(\mathbb{1}_{\mathfrak{p}}, \underline{M}_{\mathfrak{p}}) \longrightarrow H^1(I_{\mathfrak{p}}, T_{\ell}\underline{M}_{\mathfrak{p}}) \tag{1.4}$$

(we refer to Subsection 2.3). Mimicking Scholl’s approach, we say that an extension $[\underline{E}_{\mathfrak{p}}]$ of $\mathbb{1}_{\mathfrak{p}}$ by $\underline{M}_{\mathfrak{p}}$ has *good reduction* if $[\underline{E}_{\mathfrak{p}}]$ lies in the kernel of (1.4), and we let $\text{Ext}_{\text{good}}^1(\mathbb{1}_{\mathfrak{p}}, \underline{M}_{\mathfrak{p}})_{\ell}$ denote the kernel of $r_{\underline{M}, \ell, \mathfrak{p}}$ (Definition 2.22). As in the number field setting, we expect this definition to be independent of ℓ .

Integral part: the model version

Gardeyn in [Gar2] has introduced a notion of *maximal models* for τ -sheaves. Inspired by Gardeyn’s work, we developed the notion of *maximal integral models of A -motives* (Section 4). They form the function field analogue of Néron models of abelian varieties, or more generally, of regular models of varieties. Let $\mathcal{O}_{\mathfrak{p}}$ be the valuation ring of $F_{\mathfrak{p}}$.

Definition (Definition 4.16). Let $\underline{M}_{\mathfrak{p}}$ and \underline{M} be A -motives over $F_{\mathfrak{p}}$ and F respectively.

1. An $\mathcal{O}_{\mathfrak{p}}$ -model for $\underline{M}_{\mathfrak{p}}$ is a finite sub- $A \otimes \mathcal{O}_{\mathfrak{p}}$ -module L of $M_{\mathfrak{p}}$ which generates $M_{\mathfrak{p}}$ over $F_{\mathfrak{p}}$, and such that $\tau_M(\tau^* L) \subset L[j^{-1}]$.
2. An \mathcal{O}_F -model for \underline{M} is a finite sub- $A \otimes \mathcal{O}_F$ -module L of M which generates M over F , and such that $\tau_M(\tau^* L) \subset L[j^{-1}]$.

We say that L is *maximal* if L is not strictly contained in any other models.

As opposed to [Gar2, Def 2.1 & 2.3], we do not ask for an $\mathcal{O}_{\mathfrak{p}}$ -model (resp. \mathcal{O}_F -model) to be locally free. We show that this is implicit for maximal ones using Bourbaki’s flatness criterion (Proposition 4.43). Compared to Gardeyn, our exposition is therefore simplified and avoids the use of a technical lemma due to L. Lafforgue [Gar2, §2.2]. Our next result should be compared with [Gar2, Prop 2.13] (see Propositions 4.41, 4.43 in the text).

Proposition. *A maximal $\mathcal{O}_{\mathfrak{p}}$ -model $M_{\mathcal{O}_{\mathfrak{p}}}$ for $\underline{M}_{\mathfrak{p}}$ (resp. \mathcal{O}_F -model $M_{\mathcal{O}}$ for \underline{M}) exists and is unique. It is locally free over $A \otimes \mathcal{O}_{\mathfrak{p}}$ (resp. $A \otimes \mathcal{O}_F$).*

We let $\text{Ext}_{\mathcal{O}_{\mathfrak{p}}}^1(\mathbb{1}_{\mathfrak{p}}, \underline{M}_{\mathfrak{p}})$ be the image of $M_{\mathcal{O}_{\mathfrak{p}}}[j^{-1}]$ through ι (1.2) (Definition 4.46). Our main result (repeated from Theorem 4.48) is the next:

Theorem C. *Let $\underline{M}_{\mathfrak{p}}$ be an A -motive over $F_{\mathfrak{p}}$ and let ℓ be a maximal ideal of A from which \mathfrak{p} is not above. Then, $\text{Ext}_{\mathcal{O}_{\mathfrak{p}}}^1(\mathbb{1}_{\mathfrak{p}}, \underline{M}_{\mathfrak{p}})$ is a sub- A -module of $\text{Ext}_{\text{good}}^1(\mathbb{1}_{\mathfrak{p}}, \underline{M}_{\mathfrak{p}})_{\ell}$.*

Surprisingly enough, we cannot claim equality in general. In Subsection 5.1, in the simplest case of the neutral A -motive, we construct for some ℓ and \mathfrak{p} an explicit extension in $\text{Ext}_{\text{good}}^1(\mathbb{1}_{\mathfrak{p}}, \mathbb{1}_{\mathfrak{p}})_{\ell}$ which does not belong to $\text{Ext}_{\mathcal{O}_{\mathfrak{p}}}^1(\mathbb{1}_{\mathfrak{p}}, \mathbb{1}_{\mathfrak{p}})$.

In Subsection 4.4, we define the *global* version of the above. Namely, let \underline{M} be an A -motive over F . The A -motive \underline{M} defines an A -motive $\underline{M}_{\mathfrak{p}}$ over $F_{\mathfrak{p}}$ by extending the base field. We let:

$$\text{Ext}_{\mathcal{O}_F}^1(\mathbb{1}, \underline{M}) \stackrel{\text{def}}{=} \bigcap_{\mathfrak{p}} \left\{ [E] \in \text{Ext}_{\mathcal{M}_F}^1(\mathbb{1}, \underline{M}) \mid [E_{\mathfrak{p}}] \in \text{Ext}_{\mathcal{O}_{\mathfrak{p}}}^1(\mathbb{1}_{\mathfrak{p}}, \underline{M}_{\mathfrak{p}}) \right\}$$

where the intersection is indexed over all the finite places of F . Our second main result (repeated from Theorem 4.49) is the following:

Theorem D. *The A -module $\text{Ext}_{\mathcal{O}_F}^1(\mathbb{1}, \underline{M})$ equals the image of $M_{\mathcal{O}_F}[j^{-1}]$ through ι . In addition, ι induces a natural isomorphism of A -modules:*

$$\frac{M_{\mathcal{O}}[j^{-1}]}{(\text{id} - \tau_M)(M_{\mathcal{O}})} \xrightarrow{\sim} \text{Ext}_{\mathcal{O}}^1(\mathbb{1}, \underline{M}).$$

Regulated A -motives

We are facing two main issues to pursue our analogy: the counterpart of Conjecture (C4) does not hold true, and more seriously, neither is the counterpart of (C5): the A -module $\text{Ext}_{\mathcal{O}_F}^1(\mathbb{1}, \underline{M})$ is typically not finitely generated. Those facts suggest that the category \mathcal{M}_F – even $\mathcal{M}\mathcal{M}_F$ – is *too huge* to hold a convincing motivic cohomology theory. We end this text by presenting a conjectural picture aiming to answer the analogue of (C4) and (C5).

Definition (c.f. 5.2 for details). Let $0 \rightarrow \underline{M} \rightarrow \underline{E} \rightarrow \underline{N} \rightarrow 0$ be an exact sequence of A -motives over $F_{\mathfrak{p}}$. We say that $[E]$ is regulated if the Hodge polygon of the Hodge-Pink structure attached to $\underline{M} \oplus \underline{N}$ matches the one of \underline{E} (see [Pin, §6]). We denote by $\text{Ext}^{1, \text{reg}}$ the submodule of regulated extensions.

We took inspiration for this definition from the work of Pink, more precisely from his notion of *Hodge additivity* [Pin, §. 6]. The above definition is motivated by the observation that $\text{Ext}_{\mathcal{O}_F}^1(\mathbb{1}, \underline{M})$ is non finitely generated whenever \underline{M} is non zero. One reason, consequence of Theorem D, stems from the fact that elements of $M_{\mathcal{O}_F}[j^{-1}]$ – hence the resulting extensions of $\mathbb{1}$ by \underline{M} obtained from ι – might have an arbitrary large pole at j . The notion of regulated extensions exactly prevents this to happen: in Corollary 5.10, we prove that ι induces an isomorphism of A -modules:

$$\frac{M + \tau_M(\tau^* M)}{(\text{id} - \tau_M)(M)} \xrightarrow{\sim} \text{Ext}_{\mathcal{M}_{F_{\mathfrak{p}}}}^{1, \text{reg}}(\mathbb{1}, \underline{M}).$$

We strongly expect the following to hold for a large class of A -motives $\underline{M}_{\mathfrak{p}}$:

Hope. *Let ℓ be a maximal ideal of A from which \mathfrak{p} is not above. Then,*

$$\text{Ext}_{\mathcal{O}_{\mathfrak{p}}}^{1, \text{reg}}(\mathbb{1}_{\mathfrak{p}}, \underline{M}_{\mathfrak{p}}) = \text{Ext}_{\text{good}}^{1, \text{reg}}(\mathbb{1}_{\mathfrak{p}}, \underline{M}_{\mathfrak{p}})_{\ell}.$$

In particular, the module $\text{Ext}_{\text{good}}^{1, \text{reg}}(\mathbb{1}_{\mathfrak{p}}, \underline{M}_{\mathfrak{p}})_{\ell}$ would not depend on ℓ .

We conclude this text by solving some particular cases of the above Hope (Subsection 5.3). The general case, however, remains open.

While $\text{Ext}_{\mathcal{O}_F}^{1,\text{reg}}(\mathbb{1}, \underline{M})$ is still not finitely generated over A in general, a version of Conjecture (C5) involving the infinite places holds. This will be the subject of another work.

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1.3 Plan of the paper

The paper is organized as follow.

In the beginning of Section 2, Subsection 2.1, we review the usual set up (notations, definitions, basic properties) of A -motives over an arbitrary A -algebra or field. We follow [HarJu] and [Har3] as a guideline, though the former reference is concerned with the particular choice of a closed point ∞ of degree one and over a complete algebraically closed field. Most of the results on A -motives extend without changes to our larger setting. In Subsection 2.2, A -Motivic cohomology in \mathcal{M}_F is introduced. We describe the extension modules in \mathcal{M}_F and obtain Theorem A as Theorem 2.13 in the text. In Subsection 2.3, we recall the definition and main properties of the ℓ -adic realization functor for A -motives, and introduce *extensions having good reduction with respect to ℓ* in Definition 2.22.

Section 3 is concerned with mixed A -motives. In the beginning of Subsection 3.1 we recall, and add some new material, to the theory of function fields isocrystals in the steps of [Mor2]. The main ingredient, used later on in Subsection 3.1 to define the category of mixed A -motives, is the existence and uniqueness of the slope filtration (extending [Har2, Prop. 1.5.10] to general coefficient rings A). We focus on extension modules in the category \mathcal{MM}_F in Subsection 3.2 where we deduce Theorem B from Propositions 3.31, 3.33 and Theorem 3.37.

In Section 4, we develop the notion of *maximal integral models* of A -motives over local and global function fields. It splits into four subsections. In Subsection 4.1, we present integral models of *Frobenius spaces* over local function fields. The theory is much easier than the one for A -motives, introduced over a local function field in Subsection 4.2 and over a global function field in Subsection 4.3. Although our definition of integral model is inspired by Gardeyn's work in the context of τ -sheaves [Gar2], our presentation is simpler as we removed the *locally free* assumption. That maximal integral models are locally free is automatic, as we show in Propositions 4.26 and 4.43. The chief aim of this section, however, is Subsection 4.4 where we use the results of the previous ones to prove Theorems C and D (respectively Theorems 4.48 and 4.49 in the text).

In our last Section 5, we introduce the notion of *regulated extensions of A -motives* with an eye toward understanding the lack of equality in Theorem C, highlighted in Subsection 5.1. We recall the definitions of Hodge-Pink structures and Hodge polygon, as introduced in [Pin], in Subsection 5.2. Those are used to define regulated extensions in Definition 5.7. We conclude this text by Subsection 5.3, where we present a general hope that $\text{Ext}_{\mathcal{O}_p}^{1,\text{reg}}(\mathbb{1}_p, \underline{M}_p)$ and $\text{Ext}_{\text{good}}^{1,\text{reg}}(\mathbb{1}_p, \underline{M}_p)_\ell$ match. We then prove some particular cases of this expectation, namely when

\underline{M} is effective, pure of weight 0 and has good reduction (Theorem 5.13), or when \underline{M} is a p -tensor power of Carlitz's twist $\underline{A}(1)$ (Theorem 5.17).

2 Anderson A -motives and their extension modules

Let \mathbb{F} be a finite field of cardinality q . By convention, throughout this text unlabeled tensor products and fiber products are over \mathbb{F} . Let (C, \mathcal{O}_C) be a geometrically irreducible smooth projective curve over \mathbb{F} , and fix a closed point ∞ on C . Let $A := \Gamma(C \setminus \{\infty\}, \mathcal{O}_C)$ be the ring of regular functions on $C \setminus \{\infty\}$ and let K be the function field of C .

2.1 Definition of A -motives

This subsection is devoted to define and recall the main properties of Anderson A -motives. We begin with a paragraph of notations.

Let R be a commutative \mathbb{F} -algebra and let $\kappa : A \rightarrow R$ be an \mathbb{F} -algebra morphism. R will be referred to as *the base algebra* and κ as the *characteristic morphism*. The kernel of κ is called *the characteristic of (R, κ)* . We consider the ideal $\mathfrak{j} = \mathfrak{j}_\kappa$ of $A \otimes R$ generated by the set $\{a \otimes 1 - 1 \otimes \kappa(a) \mid a \in A\}$; \mathfrak{j} is equivalently defined as the kernel of $A \otimes R \rightarrow R$, $a \otimes f \mapsto \kappa(a)f$. The ideal \mathfrak{j} is maximal if and only if R is a field, and is a prime ideal if and only if R is a domain.

Let $\text{Quot}(A \otimes R)$ be the localization of $A \otimes R$ at its non-zero-divisors (if $A \otimes R$ is an integral domain, $\text{Quot}(A \otimes R)$ is the field of fractions of $A \otimes R$).

Let M be an $A \otimes R$ -module. For $n \in \mathbb{Z}$, we denote $\mathfrak{j}^{-n}M$ the submodule of $M \otimes_{A \otimes R} \text{Quot}(A \otimes R)$ consisting of elements m for which $(a \otimes 1 - 1 \otimes \kappa(a))^n m \in M$ for all $a \in A \setminus \mathbb{F}$. We then set

$$M[\mathfrak{j}^{-1}] := \bigcup_{n \geq 0} \mathfrak{j}^{-n}M.$$

Let $\tau : A \otimes R \rightarrow A \otimes R$ be the A -linear morphism given by $a \otimes r \mapsto a \otimes r^q$ on elementary tensors. Let τ^*M denotes the pull-back of M by τ ([Bou, A.II.§5]). That is, τ^*M is the $A \otimes R$ -module

$$(A \otimes R) \otimes_{\tau, A \otimes R} M$$

where the subscript τ signifies that the relation $(a \otimes_\tau bm) = (a\tau(b) \otimes_\tau m)$ holds for $a, b \in A \otimes R$, and where the $A \otimes R$ -module structure on τ^*M corresponds to $b \cdot (a \otimes_\tau m) := (ba \otimes_\tau m)$. We let $\mathbf{1} : \tau^*(A \otimes R) \rightarrow A \otimes R$ be the $A \otimes R$ -linear morphism which maps $(a \otimes r) \otimes_\tau (b \otimes s) \in \tau^*(A \otimes R) := (A \otimes R)_{\tau, A \otimes R}(A \otimes R)$ to $ab \otimes rs^q \in A \otimes R$.

The next definition takes its roots in the work of Anderson [And], though this version is borrowed from [Har3, Def. 2.1]:

Definition 2.1. An *Anderson A -motive \underline{M} (over R)* is a pair (M, τ_M) where M is a locally free $A \otimes R$ -module of finite constant rank and where $\tau_M : (\tau^*M)[\mathfrak{j}^{-1}] \rightarrow M[\mathfrak{j}^{-1}]$ is an isomorphism of $(A \otimes R)[\mathfrak{j}^{-1}]$ -modules.

In all the following, we shall more simply write *A -motive* instead of *Anderson A -motive*. The *rank of \underline{M}* is the (constant) rank of M over $A \otimes R$.

A morphism $(M, \tau_M) \rightarrow (N, \tau_N)$ of A -motives (over R) is an $A \otimes R$ -linear morphism $f : M \rightarrow N$ such that $f \circ \tau_M = \tau_N \circ \tau^*f$. We let \mathcal{M}_R be the A -linear category of A -motives over R .

Remark 2.2. A -motives as in Definition 2.1 are called *abelian A -motives* by several authors (see e.g. [BroPa]). The word *abelian* refers to the assumption that the underlying $A \otimes R$ -module is finite locally free. Dropping this assumption is not a good strategy in our work, as too many analogies with number fields motives would fail to hold.

Definition 2.3. An A -motive $\underline{M} = (M, \tau_M)$ (over R) is called *effective* if $\tau_M(\tau^*M) \subset M$. We let $\mathcal{M}_R^{\text{eff}}$ be the full subcategory of \mathcal{M}_R whose objects are effective A -motives.

Let $\mathbb{1}$ be the *unit A -motive over R* defined as $(A \otimes R, \mathbf{1})$. The biproduct of two A -motives \underline{M} and \underline{N} , denoted $\underline{M} \oplus \underline{N}$, is defined to be the A -motive whose underlying $A \otimes R$ -module is $M \oplus N$ and whose τ -linear morphism is $\tau_M \oplus \tau_N$. Their tensor product, denoted $\underline{M} \otimes \underline{N}$, is defined to be $(M \otimes_{A \otimes R} N, \tau_M \otimes \tau_N)$. The tensor operation admits $\mathbb{1}$ as a neutral object. The dual of \underline{M} is defined to be the A -motive whose underlying $A \otimes R$ -module is $M^\vee := \text{Hom}_{A \otimes R}(M, A \otimes R)$ and where τ_{M^\vee} is defined as

$$\tau_{M^\vee} : (\tau^*M^\vee)[j^{-1}] = (\tau^*M)^\vee[j^{-1}] \xrightarrow{\sim} M^\vee[j^{-1}], \quad h \longmapsto h \circ \tau_M^{-1}$$

(we refer to [HarJu, §.2.3] for more details). Given S an R -algebra, there is a *base-change* functor $\mathcal{M}_R \rightarrow \mathcal{M}_S$ mapping $\underline{M} = (M, \tau_M)$ to $\underline{M}_S := (M \otimes_R S, \tau_M \otimes_R \text{id}_S)$. The *restriction functor* $\text{Res}_{S/R} : \mathcal{M}_S \rightarrow \mathcal{M}_R$ maps an A -motive \underline{M} over S to \underline{M} seen as an A -motive over R . Given two A -motives \underline{M} and \underline{N} over R and S respectively, we have

$$\text{Hom}_{\mathcal{M}_R}(\underline{M}, \text{Res}_{S/R} \underline{N}) = \text{Hom}_{\mathcal{M}_S}(\underline{M}_S, \underline{N}).$$

In other words, the base-change functor is left-adjoint to the restriction functor.

Example 2.4 (Carlitz's motive). Let $C = \mathbb{P}_{\mathbb{F}}^1$ be the projective line over \mathbb{F} and let ∞ be the closed point of coordinates $[0 : 1]$. If t is any element in $\Gamma(\mathbb{P}_{\mathbb{F}}^1 \setminus \{\infty\}, \mathcal{O}_{\mathbb{P}^1})$ whose order of vanishing at ∞ is 1, we have an identification $A = \mathbb{F}[t]$. For an \mathbb{F} -algebra R , the tensor product $A \otimes R$ is identified with $R[t]$. The morphism τ acts on $p(t) \in R[t]$ by raising its coefficients to the q th-power. It is rather common to denote by $p(t)^{(1)}$ the polynomial $\tau(p(t))$. Let $\kappa : A \rightarrow R$ be an injective \mathbb{F} -algebra morphism and let $\theta = \kappa(t)$. The ideal $\mathfrak{j} \subset R[t]$ is principal, generated by $(t - \theta)$.

The Carlitz $\mathbb{F}[t]$ -motive \underline{C} over R is defined by the couple $(R[t], \tau_C)$ where τ_C maps $\tau^*p(t)$ to $(t - \theta)p(t)^{(1)}$. Its n th tensor power $\underline{C}^n := \underline{C}^{\otimes n}$ is isomorphic to the $\mathbb{F}[t]$ -motive whose underlying module is $R[t]$ and where τ_{C^n} maps $\tau^*p(t)$ to $(t - \theta)^n p(t)^{(1)}$. We let $\underline{A}(n) := \underline{C}^{-n} = (\underline{C}^n)^\vee$.

For $A = \mathbb{F}[t]$, $\underline{A}(1)$ plays the role of the number fields' Tate motive $\mathbb{Z}(1)$ and, more generally, $\underline{A}(n)$ plays the role of $\mathbb{Z}(n)$.

The category \mathcal{M}_R of A -motives over R is generally *not* abelian, even if $R = F$ is a field. This comes from the fact that a morphism in \mathcal{M}_F might not admit a cokernel. However, there is a notion of exact sequences in the category \mathcal{M}_R which we borrow from [HarJu, Rmk. 2.3.5(b)]:

Definition 2.5. We say that a sequence $0 \rightarrow \underline{M}' \rightarrow \underline{M} \rightarrow \underline{M}'' \rightarrow 0$ in \mathcal{M}_R is *exact* if its underlying sequence of $A \otimes R$ -modules is exact.

The next proposition appears and is discussed in [HarJu, Rmk. 2.3.5(b)] and will allow us to consider extension modules (Subsection 3.2). Although stated in the case where R is a particular A -algebra and $\deg(\infty) = 1$, it extends without changes to our setting:

Proposition 2.6. *The category \mathcal{M}_R together with the notion of exact sequences as in Definition 2.5 is exact in the sense of Quillen [Qui1, §2].*

To remedy to the non-abelian feature, we shall introduce next the category $\mathcal{M}_R^{\text{iso}}$ of A -motives up to isogeny (over R) (see Definition 2.9), which is abelian when $R = F$ is a field. We first discuss the notion of *saturation*.

Definition 2.7. Let $\underline{M} = (M, \tau_M)$ be an Anderson A -motive over R . A *submotive* of \underline{M} is an A -motive $\underline{N} = (N, \tau_N)$ such that $N \subset M$ and $\tau_N = \tau_M|_{\tau^*N[j^{-1}]}$.

We set $\underline{N}^{\text{sat}}$ to be the submotive of \underline{M} whose underlying $A \otimes R$ -module is

$$N^{\text{sat}} := \{n \in M \mid \exists a \in A \otimes R, an \in N\}$$

and call it the *saturation of \underline{N} in \underline{M}* . We say that \underline{N} is *saturated in \underline{M}* if $\underline{N} = \underline{N}^{\text{sat}}$.

Following [Har3, Def. 5.5, Thm. 5.12], we have the next:

Definition 2.8. A morphism $f : \underline{M} \rightarrow \underline{N}$ in \mathcal{M}_R is an *isogeny* if one of the following equivalent conditions is satisfied.

- (a) f is injective and $\text{coker}(f : M \rightarrow N)$ is a finite locally free R -module,
- (b) M and N have the same rank and $\text{coker } f$ is finite locally free over R ,
- (c) M and N have the same rank and f is injective,
- (d) there exists $0 \neq a \in A$ such that f induces an isomorphism of $(A \otimes R)[a^{-1}]$ -modules $M[a^{-1}] \xrightarrow{\sim} N[a^{-1}]$,
- (e) there exists $0 \neq a \in A$ and $g : \underline{N} \rightarrow \underline{M}$ in \mathcal{M}_R such that $f \circ g = a \text{id}_N$ and $g \circ f = a \text{id}_M$.

If an isogeny between \underline{M} and \underline{N} exists, \underline{M} and \underline{N} are said to be *beisogenous*.

As a consequence of those equivalent definitions, a submotive of an A -motive \underline{M} is isogenous to its saturation in \underline{M} . This motivates the definition of the category of A -motives up to isogeny (see [HarJu, Def. 2.3.1]).

Definition 2.9. Let $\mathcal{M}_R^{\text{iso}}$ be the K -linear category whose objects are those of \mathcal{M}_R and where the hom-sets of two objects \underline{M} and \underline{N} is given by the K -vector space

$$\text{Hom}_{\mathcal{M}_R^{\text{iso}}}(\underline{M}, \underline{N}) := \text{Hom}_{\mathcal{M}_R}(\underline{M}, \underline{N}) \otimes_A K.$$

We call the objects of $\mathcal{M}_R^{\text{iso}}$ the A -motives over R up to isogeny.

An isogeny in \mathcal{M}_R then becomes an isomorphism in $\mathcal{M}_R^{\text{iso}}$. According to [HarJu, Prop. 2.3.4], the category $\mathcal{M}_F^{\text{iso}}$ is abelian.

2.2 Extension modules in \mathcal{M}_R

In this subsection, we are concerned with the computation of extension modules in the category \mathcal{M}_F . Theorem A of the introduction is proved below (Theorem 2.13).

Let R be an \mathbb{F} -algebra and let \underline{M} and \underline{N} be two A -motives over R . The morphisms from \underline{N} to \underline{M} in \mathcal{M}_R are precisely the $A \otimes R$ -linear map of the underlying modules $f : N \rightarrow M$ such that $\tau_M \circ \tau^* f = f \circ \tau_N$. The module of 0-fold extensions is described by the homomorphisms:

$$\text{Ext}_{\mathcal{M}_R}^0(\underline{N}, \underline{M}) = \text{Hom}_{\mathcal{M}_R}(\underline{N}, \underline{M}) = \{f \in \text{Hom}_{A \otimes R}(N, M) \mid \tau_M \circ \tau^* f = f \circ \tau_N\}.$$

As we saw in Proposition 2.6, \mathcal{M}_R possesses exact sequences in the sense of Quillen which turns it into an A -linear exact category. It allows us to consider higher Yoneda extension A -modules $\text{Ext}_{\mathcal{M}_R}^i(\underline{N}, \underline{M})$ (for $i \geq 0$) of two A -motives \underline{M} and \underline{N} . The next proposition computes the first extension group.

Proposition 2.10. *Let \underline{M} and \underline{N} be A -motives over R . There is a surjective morphism of A -modules, functorial in both \underline{N} and \underline{M} ,*

$$\text{Hom}_{A \otimes R}(\tau^* N, M)[j^{-1}] \twoheadrightarrow \text{Ext}_{\mathcal{M}_R}^1(\underline{N}, \underline{M})$$

whose kernel is $\{f \circ \tau_N - \tau_M \circ \tau^* f \mid f \in \text{Hom}_{A \otimes R}(N, M)\}$. It is given by mapping a morphism $u \in \text{Hom}_{A \otimes R}(\tau^* N, M)[j^{-1}]$ to the class of the extension $[M \oplus N, \begin{pmatrix} \tau_M & u \\ 0 & \tau_N \end{pmatrix}]$ in $\text{Ext}_{\mathcal{M}_R}^1(\underline{N}, \underline{M})$.

Proof. Let $[\underline{E}] : 0 \rightarrow \underline{M} \xrightarrow{\iota} \underline{E} \xrightarrow{\pi} \underline{N} \rightarrow 0$ be an exact sequence in \mathcal{M}_R , that is an exact sequence of the underlying $A \otimes R$ -modules with commuting τ -action. Because N is a projective module, there exists $s : N \rightarrow E$ a section of the underlying short exact sequence of $A \otimes R$ -modules. We let $\xi := \iota \oplus s : M \oplus N \rightarrow E$. We have an equivalence:

$$\begin{array}{ccccccc} 0 & \longrightarrow & \underline{M} & \xrightarrow{\iota} & \underline{E} & \xrightarrow{\pi} & \underline{N} \longrightarrow 0 \\ & & \text{id} \uparrow & & \xi \uparrow & & \text{id} \uparrow \\ 0 & \longrightarrow & \underline{M} & \longrightarrow & (M \oplus N, \xi^{-1} \circ \tau_E \circ \xi) & \longrightarrow & \underline{N} \longrightarrow 0 \end{array}$$

Because $\xi^{-1} \circ \tau_M \circ \xi$ is an isomorphism from $\tau^* M[j^{-1}] \oplus \tau^* N[j^{-1}]$ to $M[j^{-1}] \oplus N[j^{-1}]$ which restricts to τ_M on the left and to τ_N on the right, there exists $u \in \text{Hom}_{A \otimes R}(\tau^* N[j^{-1}], M[j^{-1}]) = \text{Hom}_{A \otimes R}(\tau^* N, M)[j^{-1}]$ such that $\xi^{-1} \circ \tau_E \circ \xi = \begin{pmatrix} \tau_M & u \\ 0 & \tau_N \end{pmatrix}$. We have just shown that the map

$$\iota : \text{Hom}_{A \otimes R}(\tau^* N, M)[j^{-1}] \rightarrow \text{Ext}_{\mathcal{M}_R}^1(\underline{N}, \underline{M}), \quad u \mapsto [M \oplus N, \begin{pmatrix} \tau_M & u \\ 0 & \tau_N \end{pmatrix}]$$

is onto. Note that $\iota(0)$ corresponds to the class of the split extension. Further, $\iota(u + v)$ corresponds to the Baer sum of $\iota(u)$ and $\iota(v)$. In addition, given the exact sequence $[\underline{E}]$ and $a \in A$, the pullback of multiplication by a on N and π gives another extension which defines $a \cdot [\underline{E}]$. If $[\underline{E}] = \iota(u)$, it is formal to check that $a \cdot [\underline{E}] = \iota(au)$. As such, ι is a surjective A -module morphism. To find its kernel, it suffices to determine whenever $\iota(u)$ is equivalent to the split extension. This happens if and only if there is a commutative diagram in \mathcal{M}_R of the form

$$\begin{array}{ccccccc} 0 & \longrightarrow & \underline{M} & \longrightarrow & \underline{M} \oplus \underline{N} & \longrightarrow & \underline{N} \longrightarrow 0 \\ & & \downarrow \text{id}_M & & \downarrow h & & \downarrow \text{id}_N \\ 0 & \longrightarrow & \underline{M} & \longrightarrow & [M \oplus N, \begin{pmatrix} \tau_M & u \\ 0 & \tau_N \end{pmatrix}] & \longrightarrow & \underline{N} \longrightarrow 0 \end{array}$$

where h is a morphism in \mathcal{M}_R . Since the diagram commutes in the category of $A \otimes R$ -modules, it follows that h is of the form $\begin{pmatrix} \text{id}_M & f \\ 0 & \text{id}_N \end{pmatrix}$ for an $A \otimes R$ -linear map $f : N \rightarrow M$. Because it is a diagram in \mathcal{M}_R , it further requires commuting τ -action, that is:

$$\begin{pmatrix} \tau_M & u \\ 0 & \tau_N \end{pmatrix} \tau^* \begin{pmatrix} \text{id}_M & f \\ 0 & \text{id}_N \end{pmatrix} = \begin{pmatrix} \text{id}_M & f \\ 0 & \text{id}_N \end{pmatrix} \begin{pmatrix} \tau_M & 0 \\ 0 & \tau_N \end{pmatrix}.$$

The above equation amounts to $u = f \circ \tau_N - \tau_M \circ \tau^* f$, and hence

$$\ker(\iota) = \{f \circ \tau_N - \tau_M \circ \tau^* f \mid f \in \text{Hom}_{A \otimes R}(N, M)\}.$$

This concludes. \square

Corollary 2.11. *Suppose that R is Noetherian. Let \underline{N} be an A -motive over R and let $f : \underline{M} \rightarrow \underline{M}''$ be an epimorphism morphism in \mathcal{M}_R . Then, the induced map $\mathrm{Ext}_{\mathcal{M}_R}^1(\underline{N}, \underline{M}) \rightarrow \mathrm{Ext}_{\mathcal{M}_R}^1(\underline{N}, \underline{M}'')$ is onto.*

Proof. As R is Noetherian, so is $A \otimes R$. Because τ^*N is finite locally-free over $A \otimes R$, it is projective. That f is an epimorphism means that f is a surjective morphism of the underlying modules. The induced morphism

$$\mathrm{Hom}_{A \otimes R}(\tau^*N, M)[j^{-1}] \rightarrow \mathrm{Hom}_{A \otimes R}(\tau^*N, M'')[j^{-1}]$$

is therefore surjective, and we conclude by Proposition 2.10. \square

Let \underline{N} be an A -motive over R . The functor $\mathrm{Hom}_{\mathcal{M}_R}(\underline{N}, -)$ from the category \mathcal{M}_R to the category Mod_A of A -modules is left-exact and therefore right-derivable. Because \mathcal{M}_R is an exact category, the higher extensions modules $\mathrm{Ext}_{\mathcal{M}_R}^i(\underline{N}, \underline{M})$ are computed by the cohomology of $\mathrm{RHom}_{\mathcal{M}_R}(\underline{N}, \underline{M})$. This implies that, given a short exact sequence in \mathcal{M}_R

$$0 \rightarrow \underline{M}' \rightarrow \underline{M} \rightarrow \underline{M}'' \rightarrow 0,$$

we derive a long-exact sequence of A -modules:

$$\mathrm{Hom}_{\mathcal{M}_R}(\underline{N}, \underline{M}') \hookrightarrow \mathrm{Hom}_{\mathcal{M}_R}(\underline{N}, \underline{M}) \rightarrow \mathrm{Hom}_{\mathcal{M}_R}(\underline{N}, \underline{M}'') \rightarrow \mathrm{Ext}_{\mathcal{M}_R}^1(\underline{N}, \underline{M}') \rightarrow \dots$$

We deduce the following from Corollary 2.11.

Proposition 2.12. *Suppose that R is Noetherian. The modules $\mathrm{Ext}_{\mathcal{M}_R}^i(\underline{N}, \underline{M})$ vanish for $i > 1$. In particular, the cohomology of $\mathrm{RHom}_{\mathcal{M}_R}(\underline{N}, \underline{M})$ is represented by the complex of A -modules*

$$\left[\mathrm{Hom}_{A \otimes R}(N, M) \xrightarrow{\tau_N^\vee - \tau_M} \mathrm{Hom}_{A \otimes R}(\tau^*N, M)[j^{-1}] \right] \quad (2.1)$$

placed in degree 0 and 1.

Proof. Let C be the complex (2.1). We have

$$\begin{aligned} \mathrm{Ext}_{\mathcal{M}_R}^0(\underline{N}, \underline{M}) &= \mathrm{Hom}_{\mathcal{M}_R}(\underline{N}, \underline{M}) = \{f \in \mathrm{End}_{A \otimes R}(M, N) \mid f \circ \tau_N = \tau_M \circ \tau^*f\} \\ &= \ker(\tau_N^\vee - \tau_M) = H^0(C). \end{aligned}$$

By Proposition 2.10, $\mathrm{Ext}_{\mathcal{M}_R}^1(\underline{N}, \underline{M}) \cong H^1(C)$ and, by Corollary 2.11, the functor $\mathrm{Ext}_{\mathcal{M}_R}^1(\underline{N}, -)$ is right-exact. It follows that $\mathrm{Ext}_{\mathcal{M}_R}^i(\underline{N}, \underline{M}) \cong H^i(C) = 0$ for $i \geq 2$ ([PetSt][Lem. A.33]). \square

Let \underline{N} be a nonzero A -motive over R . The canonical morphism of A -motives

$$\mathbf{1}_N : \mathbb{1} \rightarrow \underline{N} \otimes \underline{N}^\vee = \mathrm{Hom}(\underline{N}, \underline{N}), \quad a \mapsto a \cdot \mathrm{id}_N$$

induces functorial isomorphisms for all $i \geq 0$:

$$\mathrm{Ext}_{\mathcal{M}_R}^i(\underline{N}, \underline{M}) \xrightarrow{\sim} \mathrm{Ext}_{\mathcal{M}_R}^i(\mathbb{1}, \underline{M} \otimes \underline{N}^\vee). \quad (2.2)$$

In particular, there is no loss of generality in considering extension modules of the form $\mathrm{Ext}_{\mathcal{M}_R}^i(\mathbb{1}, \underline{M})$. From now on, we will be interested mainly in extension modules of the latter form. We shall restate the main results of this section in this case (repeated from Theorem A of the introduction in the case of a global function field).

Theorem 2.13. *Suppose that R is Noetherian, and let \underline{M} be an A -motive over R . The cohomology of $\mathrm{RHom}_{\mathcal{M}_R}(\mathbb{1}, \underline{M})$ is computed by the cohomology of the complex of A -modules*

$$\left[M \xrightarrow{\mathrm{id} - \tau_M} M[j^{-1}] \right]$$

placed in degree 0 and 1. Further, there is an A -linear surjective natural morphism

$$\iota : M[j^{-1}] \rightarrow \mathrm{Ext}_{\mathcal{M}_R}^1(\mathbb{1}, \underline{M})$$

whose kernel is $(\mathrm{id} - \tau_M)(M)$, and which is given explicitly by mapping $m \in M[j^{-1}]$ to the class of the extension

$$0 \rightarrow \underline{M} \rightarrow [M \oplus (A \otimes R), (\tau_M \ m \cdot \mathbf{1})] \rightarrow \mathbb{1} \rightarrow 0.$$

Remark 2.14. From $\mathrm{Hom}_{\mathcal{M}_R^{\mathrm{iso}}}(-, -) = \mathrm{Hom}_{\mathcal{M}_R}(-, -) \otimes_A K$ (c.f. Definition 2.9), the extension spaces of $\mathbb{1}$ by \underline{M} in the K -linear category $\mathcal{M}_R^{\mathrm{iso}}$ are computed by the complex

$$\left[M \otimes_A K \xrightarrow{\mathrm{id} - \tau_M} M[j^{-1}] \otimes_A K \right].$$

2.3 Extensions having good reduction

We now introduce the function field analogue of the ℓ -adic realization functor (Definition 2.16), and show that it is exact (Proposition 2.19). It will allow us to define *extensions with good reduction* next (Definition 2.22).

For the rest of this section, let ℓ be a maximal ideal of A , and denote by \mathcal{O}_ℓ the completed local ring of A at ℓ . We let F be a field containing K and let $\kappa : A \rightarrow F$ be the inclusion. Let F^s be a separable closure of F and denote by $G_F = \mathrm{Gal}(F^s|F)$ the absolute Galois group of F equipped with the profinite topology. The group G_F acts A -linearly on the left-hand side of the tensor $A \otimes F^s$, and this action extends by continuity to an \mathcal{O}_ℓ -linear action of the algebra

$$\mathcal{A}_\ell(F^s) := (A \hat{\otimes} F^s)_\ell = \varprojlim_n (A \otimes F^s) / \ell^n (A \otimes F^s)$$

leaving $\mathcal{A}_\ell(F) := (A \hat{\otimes} F)_\ell$ invariant. If \mathbb{F}_ℓ denotes the residue field of \mathcal{O}_ℓ and π a uniformizer, we have an identification $\mathcal{A}_\ell(F) = (\mathbb{F}_\ell \otimes F)[[\pi]]$.

Remark 2.15. In the function field/number field dictionary, the assignment $R \mapsto \mathcal{A}(R)$ is akin to the Witt-vector construction $R \mapsto W(R)$ (e.g. [Har1, §1.1]). In this analogy, $R \mapsto \mathcal{B}(R)$ would be akin to $R \mapsto W(R)[p^{-1}]$.

Let $\underline{M} = (M, \tau_M)$ be an A -motive over F of rank r . Let $\underline{M}_{F^s} = (M_{F^s}, \tau_M)$ be the A -motive over F^s obtained from \underline{M} by base-change. G_F acts \mathcal{O}_ℓ -linearly on:

$$\widehat{(M_{F^s})}_\ell := \varprojlim_n (M \otimes_F F^s) / \ell^n (M \otimes_F F^s) = M \otimes_{A \otimes F} \mathcal{A}_\ell(F^s)$$

and leaves the submodule $\widehat{M}_\ell = M \otimes_{A \otimes F} \mathcal{A}_\ell(F)$ invariant. Following [HarJu, §2.3.5], we define:

Definition 2.16. The ℓ -adic realization $T_\ell \underline{M}$ of \underline{M} consists of the \mathcal{O}_ℓ -module

$$T_\ell \underline{M} := \left\{ m \in \widehat{(M_{F^s})}_\ell \mid m = \tau_M(\tau^* m) \right\}$$

together with the compatible action of G_F it inherits as a submodule of $\widehat{(M_{F^s})}_\ell$.

Remark 2.17. In [Mor2], Mornev extended this construction to the situation where ℓ is the closed point ∞ .

The next lemma is well-known in the case of τ -sheaves (e.g. [TagWa, Prop.6.1]).

Lemma 2.18. *The map $T_\ell \underline{M} \otimes_{\mathcal{O}_\ell} \mathcal{A}_\ell(F^s) \rightarrow \widehat{(M_{F^s})}_\ell$, $\omega \otimes f \mapsto \omega \cdot f$ is an isomorphism of $\mathcal{A}_\ell(F^s)$ -modules. In particular, the \mathcal{O}_ℓ -module $T_\ell \underline{M}$ is free of rank r and the action of G_F on $T_\ell \underline{M}$ is continuous.*

Proof. Let $n \geq 1$. In the ring $A \otimes F$, the ideals ℓ^n and \mathfrak{j} are coprime. Hence, the following composition of $A \otimes F$ -linear map is a well-defined isomorphism:

$$\varphi_n : \tau^*(M/\ell^n M) \cong \frac{(\tau^* M)}{\ell^n(\tau^* M)} \cong \frac{(\tau^* M)[\mathfrak{j}^{-1}]}{\ell^n(\tau^* M)[\mathfrak{j}^{-1}]} \xrightarrow{\tau_M} \frac{M[\mathfrak{j}^{-1}]}{\ell^n M[\mathfrak{j}^{-1}]} \cong M/\ell^n M.$$

The data of φ_n induces a semi-simple q -linear map¹ (in the sense of [Kat1, §1]) on the finite dimensional F^s -vector space:

$$(M \otimes_F F^s)/\ell^n(M \otimes_F F^s) = \widehat{(M_{F^s})}_\ell / \ell^n \widehat{(M_{F^s})}_\ell.$$

By [Kat1, Prop. 1.1], the multiplication map

$$\{m \in \widehat{(M_{F^s})}_\ell / \ell^n \widehat{(M_{F^s})}_\ell \mid \tau_M(\tau^* m) = m\} \otimes_{\mathbb{F}} F^s \rightarrow \widehat{(M_{F^s})}_\ell / \ell^n \widehat{(M_{F^s})}_\ell \quad (2.3)$$

is an isomorphism. Taking the inverse limit of (2.3) over all n yields the desired isomorphism.

As $\widehat{(M_{F^s})}_\ell$ is free of rank r over $\mathcal{A}_\ell(F^s)$, the same is true for $\widehat{(M_{F^s})}_\ell / \ell^n \widehat{(M_{F^s})}_\ell$ over $(A/\ell^n) \otimes F^s$. The isomorphism (2.3) implies that the A/ℓ^n -module

$$[\widehat{(M_{F^s})}_\ell / \ell^n \widehat{(M_{F^s})}_\ell]^{\tau_M=1} := \{m \in \widehat{(M_{F^s})}_\ell \mid \tau_M(\tau^* m) = m\}$$

is free of rank r over A/ℓ^n . Their projective limit $T_\ell \underline{M}$ is thus a free \mathcal{O}_ℓ -module of rank r .

By definition, the action of G_F on $T_\ell \underline{M}$ is continuous if, and only if, the induced action of G_F on $T_\ell \underline{M}/\ell^n T_\ell \underline{M}$ factors through a finite quotient for all n . Let $\mathbf{t} = \{t_1, \dots, t_s\}$ be a basis of the finite dimensional F -vector space $\widehat{M}_\ell / \ell^n \widehat{M}_\ell$. Let F_M be the matrix of τ_M written in the basis $\tau^* \mathbf{t}$ and \mathbf{t} . Let $\boldsymbol{\omega} = \{\omega_1, \dots, \omega_s\}$ be a basis of $T_\ell \underline{M}/\ell^n T_\ell \underline{M}$ over \mathbb{F} . By (2.3), $\boldsymbol{\omega}$ is a basis of $\widehat{(M_{F^s})}_\ell / \ell^n \widehat{(M_{F^s})}_\ell$ over F^s , and we let $w_{ij} \in F^s$ be the coefficients of $\boldsymbol{\omega}$ expressed in \mathbf{t} , that is, for $i \in \{1, \dots, s\}$, $\omega_i = \sum w_{ij} t_j$. We let E_n denote the Galois closure of the finite separable extension $F(w_{ij} \mid (i, j) \in \{1, \dots, s\}^2)$ of F in F^s . Then,

$$T_\ell \underline{M}/\ell^n T_\ell \underline{M} = \{m \in (M \otimes_F E_n)/\ell^n(M \otimes_F E_n) \mid \tau_M(\tau^* m) = m\}.$$

That is, the action of G_F factors through $\text{Gal}(E_n|F) = G_F/\text{Gal}(F^s|E_n)$, as desired. \square

Proposition 2.19. *The following sequence of $\mathcal{O}_\ell[G_F]$ -modules is exact*

$$0 \rightarrow T_\ell \underline{M} \longrightarrow \widehat{(M_{F^s})}_\ell \xrightarrow{\text{id} - \tau_M} \widehat{(M_{F^s})}_\ell \longrightarrow 0.$$

Proof. Everything is clear but the surjectivity of $\text{id} - \tau_M$. Let π be a uniformizer of \mathcal{O}_ℓ and let \mathbb{F}_ℓ be its residue field. Let $f = \sum_{n \geq 0} a_n \pi^n$ be a series in $\mathcal{A}_\ell(F^s) = (\mathbb{F}_\ell \otimes F^s)[[\pi]]$. Let $b_n \in \mathbb{F}_\ell \otimes F^s$ be such that $[\text{id}_{\mathbb{F}_\ell} \otimes (\text{id} - \text{Frob}_q)](b_n) = a_n$ (which exists as F^s is separably closed), and let g be the series $\sum_{n \geq 0} b_n \pi^n$ in $\mathcal{A}_\ell(F^s)$. For $\omega \in T_\ell \underline{M}$, we have

$$(\text{id} - \tau_M)(\omega \cdot g) = \omega \cdot f.$$

It follows that any element in $\widehat{(M_{F^s})}_\ell = M \otimes_{A \otimes F} \mathcal{A}_\ell(F^s)$ of the form $\omega \cdot f$ is in the image of $\text{id} - \tau_M$. By the first part of Lemma 2.18, those elements generates $\widehat{(M_{F^s})}_\ell$. We conclude that $\text{id} - \tau_M$ is surjective. \square

¹For k a field containing \mathbb{F} and V a k -vector space, an \mathbb{F} -linear endomorphism f of V is q -linear if $f(rv) = r^q f(v)$ for all $r \in k$ and $v \in V$.

We obtain the following:

Corollary 2.20. *The functor $\underline{M} \mapsto T_\ell \underline{M}$, from \mathcal{M}_F to the category of continuous \mathcal{O}_ℓ -linear G_F -representations, is exact.*

Proof. Let $S : 0 \rightarrow \underline{M}' \rightarrow \underline{M} \rightarrow \underline{M}'' \rightarrow 0$ be an exact sequence in \mathcal{M}_F . The underlying sequence of $A \otimes F$ -modules is exact, and because $\mathcal{A}_\ell(F^s)$ is flat over $A \otimes F$, the sequence of $\mathcal{A}_\ell(F^s)$ -modules $(\widehat{S_{F^s}})_\ell$ is exact. In particular, the next commutative diagram of \mathcal{O}_ℓ -modules has exact rows:

$$\begin{array}{ccccccc} 0 & \longrightarrow & (\widehat{M'_{F^s}})_\ell & \longrightarrow & (\widehat{M_{F^s}})_\ell & \longrightarrow & (\widehat{M''_{F^s}})_\ell \longrightarrow 0 \\ & & \downarrow \text{id} - \tau_{M'} & & \downarrow \text{id} - \tau_M & & \downarrow \text{id} - \tau_{M''} \\ 0 & \longrightarrow & (\widehat{M'_{F^s}})_\ell & \longrightarrow & (\widehat{M_{F^s}})_\ell & \longrightarrow & (\widehat{M''_{F^s}})_\ell \longrightarrow 0 \end{array}$$

and the Snake Lemma together with Proposition 2.19 yields that $T_\ell S$ is exact. \square

Let \underline{M} be an A -motive over F . From Corollary 2.20, the functor T_ℓ induces an A -linear morphism:

$$\text{Ext}_{\mathcal{M}_F}^1(\mathbb{1}, \underline{M}) \longrightarrow H^1(G_F, T_\ell \underline{M}) \quad (2.4)$$

into the first continuous cohomology group of G_F with values in $T_\ell \underline{M}$. The next paragraph is devoted to the explicit determination of (2.4).

Let $[\underline{E}] : 0 \rightarrow \underline{M} \rightarrow \underline{E} \rightarrow \mathbb{1} \rightarrow 0$ be a class in $\text{Ext}_{\mathcal{M}_F}^1(\mathbb{1}, \underline{M})$ of the form $\iota(m)$ for some $m \in M[j^{-1}]$ (Theorem 2.13). The ℓ -adic realization $T_\ell \underline{E}$ of \underline{E} is the $\mathcal{O}_\ell[G_F]$ -module consisting of solutions $\xi \oplus a \in (\widehat{M_{F^s}})_\ell \oplus \mathcal{A}_\ell(F^s)$ of the equation

$$\begin{pmatrix} \tau_M & m \cdot \mathbf{1} \\ 0 & \mathbf{1} \end{pmatrix} \begin{pmatrix} \tau^* \xi \\ \tau^* a \end{pmatrix} = \begin{pmatrix} \xi \\ a \end{pmatrix}$$

(see Definition 2.16). The above equality amounts to $a \in \mathcal{O}_\ell$ and $\xi - \tau_M(\tau^* \xi) = am$. A splitting of $[T_\ell \underline{E}]$ as a sequence of \mathcal{O}_ℓ -modules corresponds to the choice of a particular solution $\xi_m \in (\widehat{M_{F^s}})_\ell$ of $\xi - \tau_M(\tau^* \xi) = m$ (whose existence is provided by Proposition 2.19). We then have

$$T_\ell \underline{M} \oplus \mathcal{O}_\ell \xrightarrow{\sim} T_\ell \underline{E}, \quad (\omega, a) \mapsto (\omega + a\xi_m, a).$$

It follows that the morphism (2.4) maps $[\underline{E}]$ to the class of the cocycle $(\sigma \mapsto \xi_m^\sigma - \xi_m)$, where ξ_m is any solution in $(\widehat{M_{F^s}})_\ell$ of the equation $\xi - \tau_M(\tau^* \xi) = m$. In other words, we have almost proved:

Proposition 2.21. *There is a commutative diagram of A -modules:*

$$\begin{array}{ccc} \text{Ext}_{\mathcal{M}_F}^1(\mathbb{1}, \underline{M}) & \xrightarrow{(2.4)} & H^1(G_F, T_\ell \underline{M}) \\ \uparrow \iota & & \uparrow \iota \\ M[j^{-1}] & \longrightarrow & \widehat{M}_\ell \\ \text{(id} - \tau_M)(M) & & \text{(id} - \tau_M)(\widehat{M}_\ell) \end{array}$$

where the right vertical morphism maps the class of $f \in \widehat{M}_\ell$ to the class of the cocycle $\sigma \mapsto \xi - \sigma \xi$, ξ being any solution in $(\widehat{M_{F^s}})_\ell$ of $f = \xi - \tau_M(\tau^* \xi)$.

Proof. The only remaining fact to check is that the right vertical morphism is an isomorphism. Applying the functor of G_F -invariants to the short exact sequence of Proposition 2.19, we obtain a long exact sequence of cohomology:

$$\widehat{M}_\ell \xrightarrow{\text{id} - \tau_M} \widehat{M}_\ell \longrightarrow H^1(G_F, T_\ell \underline{M}) \longrightarrow H^1(G_F, \widehat{(M_{F^s})}_\ell).$$

Therefore, it is sufficient to prove that $H^1(G_F, \widehat{(M_{F^s})}_\ell)$ vanishes. We have $\widehat{(M_{F^s})}_\ell = M \otimes_{A \otimes F} \mathcal{A}_\ell(F^s)$, G_F acting on the right-hand side of the tensor. Hence, it is enough to prove that $H^1(G_F, \mathcal{A}_\ell(F^s))$ vanishes. The latter stems easily from the identification $\mathcal{A}_\ell(F^s) = (\mathbb{F}_\ell \otimes F^s)[[\pi]]$. \square

For the remaining of this section, let us assume that $F = F_{\mathfrak{p}}$ is a local function field with valuation ring $\mathcal{O}_{\mathfrak{p}}$ and maximal ideal \mathfrak{p} . Let $F_{\mathfrak{p}}^{\text{ur}}$ be the maximal unramified extension of $F_{\mathfrak{p}}$ in F^s . Let $I_{\mathfrak{p}}$ be the inertia subgroup of $G_{\mathfrak{p}} = G_{F_{\mathfrak{p}}}$.

Definition 2.22. Let \underline{M} be an A -motive over $F_{\mathfrak{p}}$, and let ℓ be a maximal ideal in A . We say that an extension $[E]$ of $\mathbb{1}$ by \underline{M} has *good reduction with respect to ℓ* if $[E]$ stands in the kernel of $\text{Ext}_{\mathcal{M}_F}^1(\mathbb{1}, \underline{M}) \rightarrow H^1(I_{\mathfrak{p}}, T_\ell \underline{M})$.

From Proposition 2.21 together with the fact that $I_{\mathfrak{p}} = \text{Gal}(F_{\mathfrak{p}}^s | F_{\mathfrak{p}}^{\text{ur}})$, we easily derive:

Proposition 2.23. *Let $m \in M[j^{-1}]$. The following are equivalent:*

- (a) *The extension $\iota(m)$ has good reduction with respect to ℓ ,*
- (b) *The equation $\xi - \tau_M(\tau^* \xi) = m$ admits a solution ξ in $\widehat{(M_{F_{\mathfrak{p}}^{\text{ur}}})}_\ell$.*

Remark 2.24. If $\kappa(A) \subset \mathcal{O}_{\mathfrak{p}}$, then Definition 2.22 should presumably be independent of ℓ , as long as $\kappa(\ell)\mathcal{O}_{\mathfrak{p}} = \mathcal{O}_{\mathfrak{p}}$. The analogous statement is a conjecture in the number field setting (c.f. [Sch, §2 Rmk]). So far, I have no clue on how to prove this statement.

3 Mixed A -motives and their extension modules

We discuss here the notion of mixedness for Anderson A -motives in our setting. In the $A = \mathbb{F}[t]$ -case, the definition of *pure t -motives* is traced back to the work of Anderson [And, 1.9], but the definition of *mixed t -motives* appeared only some decades later in the work of Taelman [Tae1]. It was extended to more general coefficients ring A by Hartl and Juschka in [HarJu, §3], under the assumption that $\deg(\infty) = 1$ and that the base field is algebraically closed and complete. Our presentation deals with the case of general A (that is, without assumption on $\deg(\infty)$) and $R = F$ a field.

3.1 Mixed Anderson A -motives

Isocrystals over a field

In this subsection, we introduce *function fields isocrystals* following [Mor2]. Our objective is to prove existence and uniqueness of the slope filtration with pure subquotients. The general theory has been developed in [Andr], and the results of interest for us appear in [Har2]. The new account of this subsection is the adaptation of [Har2, Prop 1.5.10] to our slightly more general setting (see Theorem 3.9). This result will allow us to define mixedness and weights in Subsection 3.1.

We begin with some general notations. Let R be a Noetherian \mathbb{F} -algebra, and let k be a finite field extension of \mathbb{F} . Let E be the field of Laurent series over k in the formal variable π , \mathcal{O} the subring of E consisting of power series over k and \mathfrak{m} the maximal ideal of \mathcal{O} . Explicitly $E = k((\pi))$, $\mathcal{O} = k[[\pi]]$ and $\mathfrak{m} = \pi\mathcal{O}$. In the sequel, E will correspond to the local field of (C, \mathcal{O}_C) at a closed point of C .

Extending the notation introduced in Subsection 2.3 in the context of the ℓ -adic realization functor, we denote by $\mathcal{A}(R)$ the completion of the ring $\mathcal{O} \otimes R$ at its ideal $\mathfrak{m} \otimes R$, that is

$$\mathcal{A}(R) = \varprojlim_n (\mathcal{O} \otimes R) / (\mathfrak{m}^n \otimes R)$$

and we let $\mathcal{B}(R)$ be the tensor product $E \otimes_{\mathcal{O}} \mathcal{A}(R)$. Throughout the previous identifications, we readily check that $\mathcal{A}(R) = (k \otimes R)[[\pi]]$ and $\mathcal{B}(R) = (k \otimes R)((\pi))$. Since $\mathcal{O} \otimes R$ is Noetherian, $\mathcal{A}(R)$ is flat over $\mathcal{O} \otimes R$ ([Bou, AC.III.§4, Thm 3(iii)]). It follows that $\mathcal{B}(R)$ is flat over $K \otimes R$.

Let $\tau : \mathcal{O} \otimes R \rightarrow \mathcal{O} \otimes R$, be the \mathcal{O} -linear map induced by $a \otimes r \mapsto a \otimes r^q$. We shall denote by τ also its continuous extension to $\mathcal{A}(R)$ or $\mathcal{B}(R)$. Similarly, we denote by $\mathbf{1}$ the canonical $A \otimes R$ -linear morphisms $\tau^* \mathcal{A}(R) \rightarrow \mathcal{A}(R)$ and $\tau^* \mathcal{B}(R) \rightarrow \mathcal{B}(R)$.

For the remaining of this subsection, we assume that $R = F$ is a field.

Definition 3.1. An *isocrystal* \underline{D} over F is a pair (D, φ_D) where D is a free $\mathcal{B}(F)$ -module of finite rank and $\varphi_D : \tau^* D \rightarrow D$ is a $\mathcal{B}(F)$ -linear isomorphism. A morphism $(D, \varphi_D) \rightarrow (C, \varphi_C)$ of isocrystals is a $\mathcal{B}(F)$ -linear morphism of the underlying modules $f : D \rightarrow C$ such that $f \circ \varphi_D = \varphi_C \circ \tau^* f$. We let \mathcal{I}_F be the category of isocrystals over F .

Remark 3.2. Pursuing the analogy of Remark 2.15, isocrystals are the analogue of the eponymous object in p -adic Hodge theory (we refer to [Har1, §3.5]). In both settings, such objects carry a slope filtration (see Theorem 3.9 for the function fields one). For number fields, isocrystals are only defined at finite places, whereas function fields isocrystals are defined regardless of the finiteness of the place. In the next subsection, we use the slope filtration at ∞ in order to define mixedness and weight filtrations.

We define the *rank* $\text{rk } \underline{D}$ of \underline{D} to be the rank of D over $\mathcal{B}(F)$. If \underline{D} is nonzero, let \mathbf{b} be a basis of D and let U denote the matrix of φ expressed in $\tau^* \mathbf{b}$ and \mathbf{b} . A different choice of basis \mathbf{b}' leads to a matrix U' such that $U = \tau(P)U'P^{-1}$ for a certain invertible matrix P with coefficients in $\mathcal{B}(F)$. As such, the valuation of $\det U$ in π is independent of \mathbf{b} . We denote it by $\deg \underline{D}$ and we name it the *degree of \underline{D}* . We define the *slope of \underline{D}* to be the rational number $\mu(\underline{D}) = \delta \deg \underline{D} / \text{rk } \underline{D}$, where δ is the degree of k over \mathbb{F} .

From [Mor2, Prop. 4.1.1], the category \mathcal{I}_F is abelian. We can therefore consider exact sequences in \mathcal{I}_F which are simply exact sequences of the underlying modules with commuting φ -action. The degree and rank are thus additive in short exact sequences over the abelian category of isocrystals, and the association $\underline{D} \mapsto \mu(\underline{D})$ defines a slope function for \mathcal{I}_F in the sense of [Andr, Def. 1.3.1]. The second point of the next definition should be compared with [Andr, Def. 1.3.6]:

Definition 3.3. Let $\underline{D} = (D, \varphi)$ be an isocrystal over F .

1. A *subisocrystal* of \underline{D} is an isocrystal $\underline{G} = (G, \varphi_G)$ for which $G \subset D$, $\varphi_G = \varphi_D|_{\tau^* G}$. The *quotient* of \underline{D} by \underline{G} is the pair $(D/G, \varphi_D)$ (this is indeed an isocrystal by [Mor2, Prop. 4.1.1]).

2. The isocrystal \underline{D} is *semistable* (resp. *isoclinic*) if, for any nonzero subisocrystal \underline{D}' of \underline{D} , $\mu(\underline{D}') \leq \mu(\underline{D})$ (resp. $\mu(\underline{D}') = \mu(\underline{D})$).

Semistability and isoclinicity are related to the notion of *purity*, borrowed from [Mor2, Def. 3.4.6], that we next recall. We first define $\mathcal{A}(F)$ -lattices:

Definition 3.4. Let D be a free $\mathcal{B}(F)$ -module of finite rank. An $\mathcal{A}(F)$ -lattice in D is a sub- $\mathcal{A}(F)$ -module of finite type of D which generates D over E .

Note that any $\mathcal{A}(F)$ -lattice L in D is free, and its rank is the rank of D over $\mathcal{B}(F)$. We denote by $\langle \varphi_D L \rangle$ the sub- $\mathcal{A}(F)$ -module $\varphi_D(\tau^* L)$ in D : it is again an $\mathcal{A}(F)$ -lattice in D since φ_D is an isomorphism. We define $\langle \varphi_D^n L \rangle$ inductively to be the $\mathcal{A}(F)$ -lattice $\langle \varphi_D \langle \varphi_D^{n-1} L \rangle \rangle$. To include the $n = 0$ -case, we set $\langle \varphi_D^0 L \rangle = L$.

Definition 3.5. A nonzero isocrystal (D, φ_D) over F is said to be *pure of slope* μ if there exist an $\mathcal{A}(F)$ -lattice L in D and integers s and $r > 0$ such that $\langle \varphi_D^{r\delta} L \rangle = \pi^s L$ and $\mu = s/r$. By convention, the zero isocrystal is pure with *no slope*.

Example 3.6. Let D be the free $\mathcal{B}(F)$ -module of rank $s \geq 1$ with basis $\{e_0, \dots, e_{s-1}\}$ and let $\varphi_D : \tau^* D \rightarrow D$ be the unique linear map such that $\varphi_D(\tau^* e_{i-1}) = e_i$ for $1 \leq i < s$ and $\varphi_D(\tau^* e_{s-1}) = \pi^r e_0$. Then (D, φ_D) is a pure isocrystal of slope $r\delta/s$ with $\mathcal{A}(F)e_0 \oplus \dots \oplus \mathcal{A}(F)e_{s-1}$ for $\mathcal{A}(F)$ -lattice.

The following lemma relates the definition of slopes from purity and from slope functions:

Lemma 3.7. *If \underline{D} is a pure isocrystal of slope μ , then $\mu(\underline{D}') = \mu$ for any nonzero sub-isocrystal \underline{D}' of \underline{D} . In particular, \underline{D} is isoclinic (hence semistable).*

Proof. Assume there exists an $\mathcal{A}(F)$ -lattice L in D such that $\langle \varphi^{r\delta} L \rangle = \mathfrak{m}^s L$ for integers $r > 0$ and d such that $\mu = s/r$. If $\underline{D}' = (D', \varphi)$ is a nonzero subisocrystal of \underline{D} , then $L' = L \cap D'$ is an $\mathcal{A}(F)$ -lattice in D' such that $\langle \varphi^{r\delta} L' \rangle = \mathfrak{m}^s L'$. As L' is nonzero, let $\{t_1, \dots, t_\ell\}$ be a basis of L' over $\mathcal{A}(F)$. We have

$$(\det \varphi)^{r\delta} (t_1 \wedge \dots \wedge t_\ell) = \mathfrak{m}^{s\ell} (t_1 \wedge \dots \wedge t_\ell) \quad \text{in } \bigwedge^\ell L'.$$

Hence $r\delta \deg \underline{D}' = s \operatorname{rk} \underline{D}'$, which yields $\mu(\underline{D}') = \mu$. □

Definition 3.8. A *slope filtration* for \underline{D} is an increasing sequence of sub-isocrystals of \underline{D}

$$0 = \underline{D}_0 \subsetneq \underline{D}_1 \subsetneq \dots \subsetneq \underline{D}_s = \underline{D},$$

satisfying:

- (i) $\forall i \in \{1, \dots, s\}$, $\underline{D}_i/\underline{D}_{i-1}$ is semi-stable,
- (ii) we have $\mu(\underline{D}_1) > \mu(\underline{D}_2/\underline{D}_1) > \dots > \mu(\underline{D}_s/\underline{D}_{s-1})$.

It follows from [Andr, Thm 1.4.7] applied to the slope function $\underline{D} \mapsto \mu(\underline{D})$ on the abelian category \mathcal{I}_F , that a slope filtration for \underline{D} exists and is unique (up to unique isomorphism). A much stronger result holds: the quotients in (i) are pure. This is the next theorem.

Theorem 3.9. *Let \underline{D} be an isocrystal over F . In the slope filtration for \underline{D}*

$$0 = \underline{D}_0 \subsetneq \underline{D}_1 \subsetneq \dots \subsetneq \underline{D}_s = \underline{D}, \tag{3.1}$$

for all $i \in \{1, \dots, s\}$, the quotients $\underline{D}_i/\underline{D}_{i-1}$ are pure isocrystals.

Remark 3.10. It would be relevant to have a lemma stating the equivalence between semi-stable and isoclinic, so that Theorem 3.9 would follow from André's theory. Yet, the only proof I know already uses [Har2, Prop. 1.5.10] and follows from Theorem 3.9.

Proof of Theorem 3.9. If $\delta = 1$, then $\mathcal{A}(F)$ is identified with $F[[\pi]]$ and Theorem 3.9 is proved in [Har2, Prop. 1.5.10]. We now explain how the general case follows from the above. Let \mathbb{G} be the finite field extension of \mathbb{F} corresponding to

$$\mathbb{G} := \{f \in \bar{\mathbb{F}} \cap F \mid f^{q^\delta} = f\}.$$

Let $\phi : \mathbb{G} \rightarrow F$ denote the inclusion. This defines an embedding of \mathbb{G} in k , the residue field of E . Let $\mathcal{A}_\phi(F)$ be the completion of $\mathcal{O} \otimes_{\mathbb{G}} F$ at the ideal $\mathfrak{m} \otimes_{\mathbb{G}} F$. In the theory of isocrystals over F with \mathbb{G} in place of \mathbb{F} , $\mathcal{A}_\phi(F)$ appears in place of $\mathcal{A}(F)$ and $\delta = 1$. In [Mor2, §4.2], Mornev defines a functor

$$[\phi]^* : (\mathcal{A}(F) - \text{isocrystals}) \longrightarrow (\mathcal{A}_\phi(F) - \text{isocrystals})$$

By [Mor2, Prop 4.2.2] (see also [BorHa, Prop 8.5]), the functor $[\phi]^*$ defines an equivalence of categories such that $[\phi]^*(\underline{D})$ is a pure isocrystal of slope μ if \underline{D} is. Let

$$[\phi]_* : (\mathcal{A}_\phi(F) - \text{isocrystals}) \longrightarrow (\mathcal{A}(F) - \text{isocrystals})$$

be a quasi-inverse of $[\phi]^*$ and let $\ell : [\phi]_*[\phi]^* \xrightarrow{\sim} \text{id}$ be a natural transformation.

Let \underline{D} be an $\mathcal{A}(F)$ -isocrystal. We only need to prove existence of (3.1) with pure subquotients since uniqueness follows from [Andr, Thm 1.4.7]. By [Har2, Prop. 1.5.10], there exists an increasing sequence of sub- $\mathcal{A}_\phi(F)$ -isocrystals of $[\phi]^*\underline{D}$:

$$0 = \underline{G}_0 \subsetneq \underline{G}_1 \subsetneq \underline{G}_2 \subsetneq \cdots \subsetneq \underline{G}_s = [\phi]^*\underline{D}$$

the subquotients $\underline{G}_i/\underline{G}_{i-1}$ being pure of slopes μ_i with $\mu_1 > \cdots > \mu_s$. Applying $[\phi]_*$ and then ℓ , we obtain

$$0 = \underline{D}_0 \subsetneq \underline{D}_1 \subsetneq \underline{D}_2 \subsetneq \cdots \subsetneq \underline{D}_s = \underline{D} \tag{3.2}$$

with $\underline{D}_i := \ell([\phi]_*[\phi]^*\underline{D}_i)$ for all $i \in \{0, 1, \dots, s\}$. We claim that the isocrystals $\underline{D}_i/\underline{D}_{i-1}$ are pure of slope μ_i . Indeed, we have

$$\underline{D}_i/\underline{D}_{i-1} \cong [\phi]_*\underline{G}_i/[\phi]_*\underline{G}_{i-1} \cong [\phi]_*(\underline{G}_i/\underline{G}_{i-1})$$

where the last isomorphism comes from the fact that $[\phi]_*$ is an exact functor (any equivalence of categories is exact). Because $\underline{G}_i/\underline{G}_{i-1}$ is pure of slope μ_i , $\underline{D}_i/\underline{D}_{i-1}$ is also pure of slope μ_i . We conclude that (3.2) is the slope filtration for \underline{D} and satisfies the assumption of the theorem. \square

Theorem 3.9 allows us to define *weights* of isocrystals:

Definition 3.11. Let \underline{D} be an isocrystal over F and let $(\underline{D}_i)_{i \in \{0, \dots, s\}}$ be its slope filtration. The elements of the set $\{-\mu(\underline{D}_i/\underline{D}_{i-1}) \mid 1 \leq i \leq s\}$ are called the *weights* of \underline{D} . We call the *weight filtration of \underline{D}* the increasing filtration $(\underline{D}_\lambda)_{\lambda \in \mathbb{Q}}$ of \underline{D} defined by

$$\underline{D}_\lambda := \bigcup_{\mu_j \geq -\lambda} \underline{D}_j.$$

For $\lambda \in \mathbb{Q}$, we let $\text{Gr}_\lambda \underline{D} := \underline{D}_\lambda / \bigcup_{\lambda' < \lambda} \underline{D}_{\lambda'}$.

Remark 3.12. The *breaks* of the weight filtration of \underline{D} are the rational numbers λ such that $\text{Gr}_\lambda \underline{D} \neq 0$. By definition, the set of breaks equals the set of weights.

It follows from Theorem 3.9 that any semi-stable isocrystal is pure, and using Lemma 3.7, that any semi-stable isocrystal is isoclinic. Restating [Andr, Thm 1.5.9] in our setting, we obtain:

Corollary 3.13. *For all $\lambda \in \mathbb{Q}$, the assignment $\mathcal{I}_F \rightarrow \mathcal{I}_F, \underline{D} \mapsto \underline{D}_\lambda$ defines an exact functor. Equivalently, any morphism $f : \underline{D} \rightarrow \underline{C}$ of isocrystals over F is strict with respect to the weight filtration, that is:*

$$\forall \lambda \in \mathbb{Q}, \quad f(D_\lambda) = f(D) \cap C_\lambda.$$

The weight filtration is not split in general. However it splits when the ground field F is perfect.

Theorem 3.14. *If F is perfect, the weight filtration of \underline{D} splits, i.e. \underline{D} decomposes along a direct sum*

$$\underline{D} \cong \bigoplus_{\lambda \in \mathbb{Q}} \text{Gr}_\lambda(\underline{D}).$$

Remark 3.15. The proof is similar to the argument given for Theorem 3.9: the corresponding result for $\delta = 1$ is proven in [Har2, Prop 1.5.10] and the general δ -case is easily deduced from [Mor2, Prop 4.2.2].

Remark 3.16. The above theorem is the *Dieudonné-Manin decomposition for isocrystals*. When F is algebraically closed, given $\mu \in \mathbb{Q}$ there exists a unique (up to isomorphisms) simple and pure isocrystal \underline{S}_μ of slope μ (see [Mor2, Prop 4.3.4]). Any pure isocrystal of slope μ decomposes as a direct sum of \underline{S}_μ (see [Mor2, Prop 4.3.7]) and together with Theorem 3.14 yields the *Dieudonné-Manin classification* (see [Lau]). It does not hold for any F , even separably closed, as noticed by Mornev in [Mor2, Rmk 4.3.5].

Isocrystals attached to A -motives

We now explain how to attach isocrystals to A -motives over fields. This construction (Definition 3.18) is required next in the definition of mixed A -motives (Definition 3.20). In the rest of this subsection, we deal with the classical properties of the category of mixed A -motives and its objects, in order to be used in the sequel.

Let R be a Noetherian \mathbb{F} -algebra and let $\kappa : A \rightarrow R$ be an \mathbb{F} -algebra morphism.

We chose the rings $\mathcal{A}(R)$ and $\mathcal{B}(R)$ of subsection 3.1 in the following way. Given a closed point λ on C , we let $O_\lambda \subset K$ be the associated discrete valuation ring of maximal ideal \mathfrak{m}_λ . We denote \hat{O}_λ the completion of O_λ and \hat{K}_λ the completion of K . We let \mathbb{F}_λ denote the residue field of λ (of finite dimension over \mathbb{F} , its dimension being the degree of λ). We let $\mathcal{A}_\lambda(R)$ and $\mathcal{B}_\lambda(R)$ be the completions of $\mathcal{O}_\lambda \otimes R$ and $K_\lambda \otimes R$ for the \mathfrak{m}_λ -adic topology.

Recall that \mathfrak{j}_κ is the ideal of $A \otimes R$ generated by $\{a \otimes 1 - 1 \otimes \kappa(a) \mid a \in A\}$. A weak version of the next lemma was already used in the proof of Lemma 2.18.

Lemma 3.17. *We have $\mathfrak{j}_\kappa \mathcal{B}_\infty(R) = \mathcal{B}_\infty(R)$. For λ a closed point of C distinct from ∞ such that $\kappa(\mathfrak{m}_\lambda)R = R$, then $\mathfrak{j}_\kappa \mathcal{A}_\lambda(R) = \mathcal{A}_\lambda(R)$.*

Proof. We prove the first assertion. let a be a non constant element of A so that $a^{-1} \in \mathfrak{m}_\infty$. Then $a \otimes 1 - 1 \otimes \kappa(a) \in \mathfrak{j}$ is invertible with $-\sum_{n \geq 0} a^{-(n+1)} \otimes \kappa(a)^n$ as inverse, where the infinite sum converges in $\mathcal{A}_\infty(R) \subset \mathcal{B}_\infty(R)$.

To prove the second assertion, let $\ell \in \mathfrak{m}_\lambda$ be such that $\kappa(\ell)$ is invertible in R . Then $\ell \otimes 1 - 1 \otimes \kappa(\ell) \in \mathfrak{j}$ is invertible with $-\sum_{n \geq 0} \ell^n \otimes \kappa(\ell)^{-(n+1)}$ as inverse, where the infinite sum converges in $\mathcal{A}_\lambda(R)$. \square

In order to use the results of the previous subsection, we now assume that $R = F$ is a field.

Definition 3.18. Let $\underline{M} = (M, \tau_M)$ be an A -motive over F and let λ be a closed point of C . We let $\mathcal{I}_\lambda(M)$ be the $\mathcal{B}_\lambda(F)$ -module $M \otimes_{A \otimes R} \mathcal{B}_\lambda(F)$. Let $\mathcal{I}_\lambda(\underline{M})$ be the pair $(\mathcal{I}_\lambda(M), \tau_M \otimes \mathbf{1})$.

Proposition 3.19. Let $\underline{M} = (M, \tau_M)$ be a nonzero A -motive over F . Let λ be a closed point of C distinct from $\ker \kappa$.

- (i) $\mathcal{I}_\lambda(\underline{M})$ is an isocrystal over F .
- (ii) If $\lambda \neq \infty$, $\mathcal{I}_\lambda(\underline{M})$ is pure of slope 0.

Proof. Because M is locally free of constant rank and $\mathcal{B}_\lambda(F)$ is a finite product of fields, $\mathcal{I}_\lambda(M)$ is a free $\mathcal{B}_\lambda(F)$ -module. Thus, point (i) follows from Lemma 3.17. To prove (ii), it suffices to note that $L = M \otimes_{A \otimes F} \mathcal{A}_\lambda(F)$ is an $\mathcal{A}_\lambda(F)$ -lattice in $M \otimes_{A \otimes F} \mathcal{B}_\lambda(F)$ such that $\langle \tau_M L \rangle = L$. \square

We now choose $\lambda = \infty$. Let \underline{M} be an A -motive over F . The isocrystal $\mathcal{I}_\infty(\underline{M})$ admits a weight filtration (Definition 3.11) denoted $(\mathcal{I}_\infty(\underline{M})_\mu)_{\mu \in \mathbb{Q}}$:

$$0 = \mathcal{I}_\infty(\underline{M})_{\mu_0} \subsetneq \mathcal{I}_\infty(\underline{M})_{\mu_1} \subsetneq \mathcal{I}_\infty(\underline{M})_{\mu_2} \subsetneq \cdots \subsetneq \mathcal{I}_\infty(\underline{M})_{\mu_s} = \mathcal{I}_\infty(\underline{M})$$

where $\mu_1 < \mu_2 < \dots < \mu_s$ are the rational numbers such that $\mathcal{I}_\infty(\underline{M})_{\mu_i} / \mathcal{I}_\infty(\underline{M})_{\mu_{i-1}}$ is a pure isocrystal of slope $-\mu_i$. For $\mu \in \mathbb{Q}$, we write $\mathcal{I}_\infty(\underline{M})_\mu = (\mathcal{I}_\infty(M)_\mu, \tau_M)$.

Definition 3.20. Let \underline{M} be a nonzero A -motive over F .

- (a) The elements of the set $w(\underline{M}) := \{\mu_1, \dots, \mu_s\}$ are called the *weights* of \underline{M} . We agree that $w(\underline{0})$ is the empty set, and call \underline{M} pure of weight w if $\{\mu_1, \dots, \mu_s\} = \{w\}$.
- (b) We call \underline{M} *mixed* if there exists an increasing filtration $(\underline{M}_{\mu_i})_{i \in \mathbb{Q}}$ of \underline{M} by sub- A -motives such that $(\mathcal{I}_\infty(\underline{M}_{\mu_i}))_{i \in \mathbb{Q}}$ coincides with the weight filtration of $\mathcal{I}_\infty(\underline{M})$. In particular, a pure A -motive is mixed.

Remark 3.21. If \underline{M} is an A -motive over F of weights $\{\mu_1, \dots, \mu_s\}$ and F' is a field extension of F , then $\underline{M}_{F'}$ also has weights $\{\mu_1, \dots, \mu_s\}$. This follows from the uniqueness of the slope filtration (Theorem 3.9). If \underline{M} is mixed, then so is $\underline{M}_{F'}$.

Remark 3.22. In [HarJu, Ex. 2.3.13], the authors constructed an A -motive which is not mixed. The latter is constructed starting from an extension of a pure Anderson A -motive of weight 2 by another pure of weight 1.

If a filtration as in (b) exists, it might not be unique. However, if we impose that the filtration is composed with saturated sub- A -motives of \underline{M} , then it is unique. This follows from the next lemma.

Lemma 3.23. Let \underline{M} be an A -motive over F and let \underline{P} be a sub- A -motive of \underline{M} . Then $\mathcal{I}_\infty(\underline{P}) = \mathcal{I}_\infty(\underline{P}^{\text{sat}})$. If \underline{Q} is a sub- A -motive of \underline{M} such that $\mathcal{I}_\infty(\underline{P}) = \mathcal{I}_\infty(\underline{Q})$ inside $\mathcal{I}_\infty(\underline{M})$, then $\underline{P}^{\text{sat}} = \underline{Q}^{\text{sat}}$.

Proof. The inclusion $\underline{P} \subset \underline{P}^{\text{sat}}$ is an isogeny and therefore its cokernel is A -torsion (see [Har3, Thm. 5.12]). Consequently, $\mathcal{I}_\infty(\underline{P}) = \mathcal{I}_\infty(\underline{P}^{\text{sat}})$.

We prove the second part. The $A \otimes F$ -modules $P, P^{\text{sat}}, Q, Q^{\text{sat}}$ and $(P \cap Q)^{\text{sat}} = P^{\text{sat}} \cap Q^{\text{sat}}$ are locally-free of the same rank, as they become equal once \mathcal{I}_∞ is applied. Note that $(P \cap Q)^{\text{sat}} = P^{\text{sat}} \cap Q^{\text{sat}}$ is again endowed with an A -motive structure and hence so is the quotient $A \otimes F$ -module $P^{\text{sat}} / (P \cap Q)^{\text{sat}}$. The underlying $A \otimes F$ -module is locally-free and has rank 0; hence it is zero. The inclusion $P^{\text{sat}} \cap Q^{\text{sat}} \rightarrow P^{\text{sat}}$ is therefore an isomorphism which yields $P^{\text{sat}} \subset Q^{\text{sat}}$. We conclude by exchanging the roles of P and Q in the above argument to obtain the converse inclusion. \square

We deduce at once:

Proposition-Definition 3.24. Any mixed A -motive \underline{M} over F admits a unique increasing filtration by saturated sub- A -motives $(W_{\mu_i}\underline{M})_{i \in \{1, \dots, s\}}$ such that the family of isocrystals $(\mathcal{I}_\infty(W_{\mu_i}\underline{M}))_i$ coincides with the weight filtration of $\mathcal{I}_\infty(\underline{M})$ (Definition 3.11).

- (i) We call $(W_{\mu_i}\underline{M})_i$ the *weight filtration of \underline{M}* .
- (ii) For all $i \in \{1, \dots, s\}$, we let $W_{\mu_i}M$ be the underlying module of $W_{\mu_i}\underline{M}$.
- (iii) For all $\mu \in \mathbb{Q}$, we set

$$W_\mu M := \bigcup_{\mu_i \leq \mu} W_{\mu_i} M, \quad W_\mu \underline{M} := (W_\mu M, \tau_M),$$

$$W_{<\mu} M := \bigcup_{\mu_i < \mu} W_{\mu_i} M, \quad W_{<\mu} \underline{M} := (W_{<\mu} M, \tau_M),$$

and $\text{Gr}_\mu \underline{M} := W_\mu \underline{M} / W_{<\mu} \underline{M}$. Both $\text{Gr}_\mu \underline{M}$ and $W_\mu \underline{M}$, as well as $W_{<\mu} \underline{M}$, define mixed A -motives over F for all $\mu \in \mathbb{Q}$.

- (iv) We let \mathcal{MM}_F (resp. $\mathcal{MM}_F^{\text{iso}}$) be the full subcategory of \mathcal{M}_F (resp. $\mathcal{M}_F^{\text{iso}}$) whose objects are mixed.

The next lemma shows how to recover the weight filtration of a mixed A -motive from its isocrystal:

Proposition 3.25. Let \underline{M} be a mixed A -motive over F . For $\mu \in \mathbb{Q}$, we have $W_\mu M = \mathcal{I}_\infty(M)_\mu \cap M$, where the intersection is taken inside $M \otimes_{A \otimes F} \mathcal{B}_\infty(F)$.

Proof. By definition, the following sequence of inclusions hold:

$$(W_\mu M) \otimes_A K \subset \mathcal{I}_\infty(M)_\mu \cap (M \otimes_A K) \subset \mathcal{I}_\infty(M)_\mu.$$

Note that the left-hand side forms a dense subset of the right-hand side for the ∞ -adic topology. Hence, the first inclusion is an inclusion of a dense subset. Taking the completion, we obtain $\mathcal{I}_\infty(W_\mu M) = \mathcal{I}_\infty(\mathcal{I}_\infty(M)_\mu \cap M)$. As both $W_\mu M$ and $\mathcal{I}_\infty(M)_\mu \cap M$ are saturated submodules of M , the lemma follows from Lemma 3.23. \square

When \underline{M} is non mixed, $\mathcal{I}_\infty(M)_\mu \cap (M \otimes_A K)$ is generally not a dense subset of $\mathcal{I}_\infty(M)_\mu$, preventing $W_\mu M$ from being defined by the mean of $\mathcal{I}_\infty(M)_\mu \cap M$. In general, we have the following:

Proposition 3.26. Let \underline{M} be an A -motive over F . For all $\mu \in \mathbb{Q}$, we have

$$\text{rank}_{A \otimes F}(\mathcal{I}_\infty(M)_\mu \cap M) \leq \text{rank } \mathcal{I}_\infty(\underline{M})_\mu$$

with equality for all $\mu \in \mathbb{Q}$ if and only if \underline{M} is mixed.

Proof. First note that, for all μ , the couple $\underline{M}_\mu := (\mathcal{I}_\infty(M)_\mu \cap M, \tau_M)$ defines a saturated sub- A -motive of \underline{M} . Furthermore, since the underlying module of $\mathcal{I}_\infty(\underline{M}_\mu)$ corresponds to the completion of $\mathcal{I}_\infty(M)_\mu \cap (M \otimes_A K)$ for the ∞ -adic topology, $\mathcal{I}_\infty(\underline{M}_\mu)$ defines a subisocrystal of $\mathcal{I}_\infty(\underline{M})_\mu$. The rank being additive in short exact sequences in \mathcal{I}_F , we get:

$$\text{rank}_{A \otimes F}(\mathcal{I}_\infty(M)_\mu \cap M) = \text{rank } \underline{M}_\mu = \text{rank } \mathcal{I}_\infty(\underline{M}_\mu) \leq \text{rank } \mathcal{I}_\infty(\underline{M})_\mu.$$

If \underline{M} is mixed then, by Proposition 3.25, the above is an equality for all μ . Conversely, if this is an equality for all μ , then $\mathcal{I}_\infty(\underline{M}_\mu) \subseteq \mathcal{I}_\infty(\underline{M})_\mu$ is an inclusion of isocrystals of the same rank. Hence they are equal. \square

Remark 3.27. We end this subsection by describing how weights behave under linear algebra type operations. Proofs are presented in [HarJu, Prop. 2.3.11] and extend without change to our larger setting. First note that $\mathbb{1}$ is a pure A -motive over F of weight 0. Given two mixed A -motives \underline{M} and \underline{N} , their biproduct $\underline{M} \oplus \underline{N}$ is again mixed with weight filtration $W_\mu(\underline{M} \oplus \underline{N}) = W_\mu \underline{M} \oplus W_\mu \underline{N}$ ($\mu \in \mathbb{Q}$). Their tensor product $\underline{M} \otimes \underline{N}$ is also mixed, with λ -part of its weight filtration being:

$$W_\lambda(\underline{M} \otimes \underline{N}) = \left(\sum_{\mu+\nu=\lambda} W_\mu \underline{M} \otimes W_\nu \underline{N} \right)^{\text{sat}}.$$

We took the saturation A -motive to ensure that the above is a saturated sub- A -motive of $\underline{M} \otimes \underline{N}$. The dual \underline{M}^\vee is mixed, and the μ -part of its weight filtration $W_\mu \underline{M}^\vee$ has for underlying module $W_\mu M^\vee = \{m \in M^\vee \mid \forall \lambda < -\mu : m(W_\lambda M) = 0\}^{\text{sat}}$. In general, given \underline{M} and \underline{N} two A -motives over F (without regarding whether \underline{M} or \underline{N} are mixed) and an exact sequence $0 \rightarrow \underline{M}' \rightarrow \underline{M} \rightarrow \underline{M}'' \rightarrow 0$ in \mathcal{M}_F , we have

$$\begin{aligned} w(\underline{0}) &= \emptyset \\ w(\underline{M}^\vee) &= -w(\underline{M}) \\ w(\underline{M} \oplus \underline{N}) &= w(\underline{M}) \cup w(\underline{N}) \\ w(\underline{M}) &= w(\underline{M}') \cup w(\underline{M}'') \\ w(\underline{M} \otimes \underline{N}) &= \{w + v \mid w \in w(\underline{M}), v \in w(\underline{N})\} \end{aligned}$$

3.2 Extension modules of mixed A -motives

As before, let F be a field containing \mathbb{F} and consider an \mathbb{F} -algebra morphism $\kappa : A \rightarrow F$. In this subsection, we are concerned with extension modules in the category $\mathcal{M}\mathcal{M}_F$. The next proposition shows that they are well-defined.

Proposition 3.28. *The category $\mathcal{M}\mathcal{M}_F$ is an exact subcategory of \mathcal{M}_F .*

Proposition 3.28 follows from the next Lemma 3.29, which follows closely [HarJu, Prop 2.3.11(c)], and [HarJu, Rmk 2.3.12].

Lemma 3.29. *Any sub- A -motive $\underline{M}' \hookrightarrow \underline{M}$ and any quotient A -motive $\underline{M} \twoheadrightarrow \underline{M}''$ of mixed A -motives \underline{M} is itself mixed.*

Proof. Let f denotes the morphism $\underline{M} \rightarrow \underline{M}''$ in \mathcal{M}_F . For $\mu \in \mathbb{Q}$, let $W_\mu M := W_\mu M \cap M'$ and $W_\mu M'' := f(W_\mu M)^{\text{sat}} \cap M''$. Both are saturated modules in M' and M'' respectively. They are also canonically endowed with an A -motive structure. Since $\mathcal{B}_\infty(F)$ is flat over $A \otimes F$, [Bou, §I.2, Prop. 6] implies that $- \otimes_{A \otimes F} \mathcal{B}_\infty(F)$ commutes with finite intersections:

$$\begin{aligned} \mathcal{I}_\infty(W_\mu M') &= (W_\mu M') \otimes_{A \otimes F} \mathcal{B}_\infty(F) \\ &= (W_\mu M \cap M') \otimes_{A \otimes F} \mathcal{B}_\infty(F) \\ &= \mathcal{I}_\infty(M)_\mu \cap \mathcal{I}_\infty(M') \\ &= \mathcal{I}_\infty(M')_\mu \end{aligned}$$

where the last equality follows from Theorem 3.9. Similarly,

$$\begin{aligned} \mathcal{I}_\infty(W_\mu M'') &= (W_\mu M'') \otimes_{A \otimes F} \mathcal{B}_\infty(F) \\ &= (f(W_\mu M)^{\text{sat}} \cap M'') \otimes_{A \otimes F} \mathcal{B}_\infty(F) \\ &= f(\mathcal{I}_\infty(M)_\mu) \cap \mathcal{I}_\infty(M'') \\ &= \mathcal{I}_\infty(M'')_\mu. \end{aligned}$$

This shows that \underline{M}' and \underline{M}'' are both mixed with respective weight filtrations $(W_\mu \underline{M}')_{\mu \in \mathbb{Q}}$ and $(W_\mu \underline{M}'')_{\mu \in \mathbb{Q}}$. \square

As a consequence, we record:

Proposition 3.30. *Any morphism of mixed A -motives preserves the weight filtration, that is, given a morphism $f : \underline{M} \rightarrow \underline{N}$ in \mathcal{M}_F ,*

$$\forall \mu \in \mathbb{Q}, \quad f(W_\mu M) \subset W_\mu N.$$

In particular, for all $\mu \in \mathbb{Q}$, the assignation $\underline{M} \mapsto W_\mu \underline{M}$ is functorial over $\mathcal{M}\mathcal{M}_F$. Over $\mathcal{M}\mathcal{M}_F^{\text{iso}}$, this assignment defines an exact functor.

Proof. Let $f : \underline{M} \rightarrow \underline{N}$ be a morphism of mixed Anderson A -motives over F . In the category of isocrystals, f induces a morphism from $\mathcal{I}_\infty(\underline{M})$ to $\mathcal{I}_\infty(\underline{N})$. We have

$$\mathcal{I}_\infty(f(W_\mu \underline{M})) = f(\mathcal{I}_\infty(W_\mu \underline{M})) = f(\mathcal{I}_\infty(\underline{M})_\mu)$$

and, by flatness of $\mathcal{B}_\infty(F)$ over $A \otimes F$,

$$\begin{aligned} \mathcal{I}_\infty(f(M) \cap W_\mu N) &= \mathcal{I}_\infty(f(M)) \cap \mathcal{I}_\infty(W_\mu N) = f(\mathcal{I}_\infty(M)) \cap \mathcal{I}_\infty(N)_\mu \\ &= f(\mathcal{I}_\infty(M)_\mu) \end{aligned} \quad (3.3)$$

where the last equality follows from Theorem 3.9. By Lemma 3.23 applied to (3.3),

$$f(W_\mu M) \subset f(W_\mu M)^{\text{sat}} = f(\tau^* M)^{\text{sat}} \cap W_\mu N \subset W_\mu N. \quad (3.4)$$

To conclude that W_μ is exact over $\mathcal{M}\mathcal{M}_F^{\text{iso}}$, it suffices to note that all inclusions in (3.4) defines isogenies of A -motives over F . \square

Thanks to Proposition 3.28, we can consider Yoneda's extension modules of two mixed A -motives in $\mathcal{M}\mathcal{M}_F$. By Proposition 3.30, we have an equality

$$\text{Ext}_{\mathcal{M}\mathcal{M}_F}^0(\underline{M}, \underline{N}) = \text{Ext}_{\mathcal{M}\mathcal{M}_F}^0(\underline{M}, \underline{N}),$$

but this is not true for higher extension modules. $\text{Ext}_{\mathcal{M}\mathcal{M}_F}^1(\underline{M}, \underline{N})$ can be interpreted as a submodule of $\text{Ext}_{\mathcal{M}\mathcal{M}_F}^i(\underline{M}, \underline{N})$, but in general $\text{Ext}_{\mathcal{M}\mathcal{M}_F}^i$ is not even a submodule of $\text{Ext}_{\mathcal{M}\mathcal{M}_F}^i$ ($i > 1$). To that regard, we show that $\text{Ext}_{\mathcal{M}\mathcal{M}_F}^2(\mathbb{1}, \underline{M})$ does not vanish in general.

Proposition 3.31. *Let $0 \rightarrow \underline{M}' \rightarrow \underline{M} \xrightarrow{p} \underline{M}'' \rightarrow 0$ be an exact sequence of A -motives in \mathcal{M}_F .*

1. *If \underline{M} is mixed, so are \underline{M}' and \underline{M}'' .*
2. *If \underline{M}' and \underline{M}'' are mixed, and if the smallest weight of \underline{M}'' is bigger than the biggest weight of \underline{M}' , then \underline{M} is mixed.*

Proof. Point 1 is a reformulation of Lemma 3.29.

We move to point 2. By Proposition 2.10, there exists $u \in \text{Hom}_{A \otimes F}(\tau^* M'', M')[j^{-1}]$ and a diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & \underline{M}' & \xrightarrow{\iota} & \underline{M} & \xrightarrow{p} & \underline{M}'' \longrightarrow 0 \\ & & \text{id} \uparrow & & \xi \uparrow & & \text{id} \uparrow \\ 0 & \longrightarrow & \underline{M}' & \longrightarrow & (M' \oplus M'', \begin{pmatrix} \tau_{M'} & u \\ 0 & \tau_{M''} \end{pmatrix}) & \longrightarrow & \underline{M}'' \longrightarrow 0 \end{array}$$

which commutes in \mathcal{M}_F . From the the weight assumption on \underline{M}' and \underline{M}'' , u automatically respects the weight filtration. That is, for all μ ,

$$\underline{M}_\mu := \xi \left(W_\mu M' \oplus W_\mu M'', \begin{pmatrix} \tau_{M'} & u \\ 0 & \tau_{M''} \end{pmatrix} \right)$$

defines a sub- A -motive of \underline{M} inserting in a short exact sequence in \mathcal{M}_F :

$$0 \longrightarrow W_\mu \underline{M}' \longrightarrow \underline{M}_\mu \longrightarrow W_\mu \underline{M}'' \longrightarrow 0. \quad (3.5)$$

We obtain a commutative diagram of isocrystals:

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathcal{I}_\infty(\underline{M}')_\mu & \longrightarrow & \mathcal{I}_\infty(\underline{M})_\mu & \longrightarrow & \mathcal{I}_\infty(\underline{M}'')_\mu \longrightarrow 0 \\ & & \uparrow = & & \uparrow & & \uparrow = \\ 0 & \longrightarrow & \mathcal{I}_\infty(W_\mu \underline{M}') & \longrightarrow & \mathcal{I}_\infty(\underline{M}_\mu) & \longrightarrow & \mathcal{I}_\infty(W_\mu \underline{M}'') \longrightarrow 0 \end{array}$$

where the first row is exact by Corollary 3.13, and the second row is exact as (3.5) is exact. The extremal vertical maps are equalities, and hence so is the middle one. To conclude, we have found a family of sub- A -motives $(\underline{M}_\mu)_\mu$ of \underline{M} which specializes to $(\mathcal{I}_\infty(\underline{M})_\mu)_\mu$ when applying the functor \mathcal{I}_∞ . Therefore \underline{M} is mixed. \square

Remark 3.32. Contrary to the number fields situation, the full subcategory of $\mathcal{M}\mathcal{M}_F$ consisting of pure A -motives over F is not semi-simple. This follows easily from the equality $\text{Ext}_{\mathcal{M}\mathcal{M}_F}^1(\underline{N}, \underline{M}) = \text{Ext}_{\mathcal{M}_F}^1(\underline{N}, \underline{M})$ for two pure motives of the same weight.

Proposition 3.31 implies that $\text{Ext}_{\mathcal{M}_F}^1(\underline{M}'', \underline{M}') = \text{Ext}_{\mathcal{M}\mathcal{M}_F}^1(\underline{M}'', \underline{M}')$ when the weights of \underline{M}'' are bigger than the biggest weight of \underline{M}' . In general this is not true. In this direction we record:

Proposition 3.33. *Let $0 \rightarrow \underline{M}' \rightarrow \underline{M} \rightarrow \underline{M}'' \rightarrow 0$ be an exact sequence of A -motives where \underline{M}' and \underline{M}'' are mixed. We assume that all the weights of \underline{M}'' are strictly smaller than the smallest weight of \underline{M}' . Then, the sequence is torsion in $\text{Ext}_{\mathcal{M}_F}^1(\underline{M}'', \underline{M}')$ if, and only if, \underline{M} is mixed.*

Proof of Proposition 3.33. Taking $\underline{N} := \underline{M}' \otimes (\underline{M}'')^\vee$, we can assume that the exact sequence is of the form $(S) : 0 \rightarrow \underline{N} \rightarrow \underline{E} \rightarrow \mathbb{1} \rightarrow 0$, \underline{N} having positive weights. In view of Theorem 2.13, we may assume that \underline{E} is of the form $\iota(u)$ for some $u \in M[j^{-1}]$. Note that 0 is a weight of \underline{E} , the smallest.

If \underline{E} is mixed, then \underline{E} contains a sub- A -motive $\underline{L} = (L, \tau_M)$ of weight 0 which is isomorphic to $\mathbb{1}$. Let $(m \oplus a) \in M \oplus (A \otimes F)$ be a generator of L over $A \otimes F$. We have

$$\begin{pmatrix} \tau_M & u \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \tau^* m \\ \tau^* a \end{pmatrix} = \begin{pmatrix} m \\ a \end{pmatrix}.$$

This amounts to $a \in A$ and $au \in \text{im}(\text{id} - \tau_M)$, and then that $a[\underline{E}] = 0$ in $\text{Ext}_{\mathcal{M}_F}^1(\mathbb{1}, \underline{N})$. Conversely, if there exists a nonzero $a \in A$ such that $a[\underline{E}]$ is split, Theorem 2.13 implies that there exists $m \in N$ such that $au = m - \tau_M(\tau^* m)$. The nonzero $A \otimes F$ -module L generated by $m \oplus a$ together with τ_M defines a sub- A -module of \underline{E} isomorphic to $\mathbb{1}$. For all $\mu \in \mathbb{Q}$, we define the $A \otimes F$ -module

$$W_\mu \underline{E} := W_\mu M + \mathbf{1}_{\mu \geq 0} L$$

where $(W_\mu \underline{M})_{\mu \in \mathbb{Q}}$ is the weight filtration of \underline{M} . It is easy to see that $W_\mu \underline{E} := (W_\mu \underline{E}, \tau_M)$ defines a sub- A -motive of \underline{E} and that $(\mathcal{I}_\infty(W_\mu \underline{M}))_{\mu \in \mathbb{Q}}$ coincides with the slope filtration of $\mathcal{I}_\infty(\underline{E})$. Hence, \underline{E} is mixed. \square

Remark 3.34. Under the same hypothesis, Proposition 3.33 can be rephrased into

$$\text{Ext}_{\mathcal{M}_F}^1(\underline{M}'', \underline{M}')^{\text{tors}} = \text{Ext}_{\mathcal{M}\mathcal{M}_F}^1(\underline{M}'', \underline{M}').$$

In particular, the K -vector space $\text{Ext}_{\mathcal{M}\mathcal{M}_F^{\text{iso}}}^1(\underline{M}'', \underline{M}')$ vanishes. The latter is only conjectured to be true in the number fields setting ([Del, §1.3]).

Remark 3.35. Nevertheless, $\text{Ext}_{\mathcal{M}\mathcal{M}_F}^1(\mathbb{1}, \underline{M})$ is generally non zero for \underline{M} having positive weights. In the notations of Example 2.4, let n be a positive integer that is a power of the characteristic p , and consider the A -motive $\underline{A}(-n)$ over a field extension F of K that contains a $(q-1)$ -st root η of $(-1/\theta)^n$. We claim that $\text{Ext}_{\mathcal{M}\mathcal{M}_F}^1(\mathbb{1}, \underline{A}(-n))$ is non zero, i.e. $\text{Ext}_{\mathcal{M}_F}^1(\mathbb{1}, \underline{A}(-n))$ has non zero torsion ($\underline{A}(-n)$ is pure of weight n). Indeed, let $[\underline{E}] := \iota(\eta^q)$. Then,

$$-t^n \cdot [\underline{E}] = \iota(-t^n \eta^q) = \iota(\eta - (t - \theta)^n \eta^q) = 0.$$

On the other-hand, $[\underline{E}]$ is non zero: for degree reasons, there does not exist $p(t) \in F[t]$ such that $\eta^q = p(t) - (t - \theta)^n p(t)^{(1)}$.

Remark 3.36. A variation of the above argument shows that $\text{Ext}_{\mathcal{M}\mathcal{M}_F}^2(\mathbb{1}, \underline{A}(-1))$ is non zero for certain fields F . Assume that:

1. F contains a $(q-1)$ -st root η of $(-1/\theta)^p$,
2. F does not contain of $(q-1)$ -st root of $-1/\theta$
3. there exists $\beta \in F$ for which the polynomial $\theta X^q + X + \beta$ does not split over F .

For $\eta, \beta \in F$ as above, let $\alpha := \beta/\eta^q \in F$. Consider the extension \underline{M} of $\underline{A}(-p)$ by $\underline{A}(-1)$ given by

$$\underline{M} = \left[F[t]^2, \begin{pmatrix} (t-\theta)\mathbf{1} & \alpha\mathbf{1} \\ 0 & (t-\theta)^p\mathbf{1} \end{pmatrix} \right].$$

As $\underline{A}(-p)$ and $\underline{A}(-1)$ are pure of weight p and 1 respectively, \underline{M} is mixed by Proposition 3.31, with weights $\{1, p\}$. We claim that the A -linear morphism

$$\text{Ext}_{\mathcal{M}\mathcal{M}_F}^1(\mathbb{1}, f) : \text{Ext}_{\mathcal{M}\mathcal{M}_F}^1(\mathbb{1}, \underline{M}) \longrightarrow \text{Ext}_{\mathcal{M}\mathcal{M}_F}^1(\mathbb{1}, \underline{A}(-p)), \quad (3.6)$$

induced by the epimorphism $f : \underline{M} \rightarrow \underline{A}(-p)$, is not surjective. By the long-exact sequence of Ext-modules, it implies that $\text{Ext}_{\mathcal{M}\mathcal{M}_F}^2(\mathbb{1}, \ker f) = \text{Ext}_{\mathcal{M}\mathcal{M}_F}^2(\mathbb{1}, \underline{A}(-1))$ is non zero.

Because \underline{M} has positive weights, by Proposition 3.33, (3.6) can be rewritten as:

$$\left(\frac{F[t, (t-\theta)^{-1}]^{\oplus 2}}{(\text{id} - \tau_M)(F[t]^{\oplus 2})} \right)^{\text{tors}} \longrightarrow \left(\frac{F[t, (t-\theta)^{-1}]}{(\text{id} - (t-\theta)^p \tau)(F[t])} \right)^{\text{tors}}. \quad (3.7)$$

In Remark 3.34, we showed that the class of η^q defines a nonzero element in the right-hand side of (3.7). If the latter was surjective, there would exist $x \in F[t, (t-\theta)^{-1}]$ such that the class of (x, η^q) belongs to the right-hand side of (3.7). That is, there exist $a \in A = \mathbb{F}[t]$ nonzero and $(f, g) \in F[t]^{\oplus 2}$ such that

$$a \cdot \begin{pmatrix} x \\ \eta^q \end{pmatrix} = \begin{pmatrix} f \\ g \end{pmatrix} - \begin{pmatrix} (t-\theta) & \alpha \\ 0 & (t-\theta)^p \end{pmatrix} \begin{pmatrix} f^{(1)} \\ g^{(1)} \end{pmatrix}. \quad (3.8)$$

As $-t^p \eta^q = \eta - (t-\theta)^p \eta^q$, we obtain from the bottom row that t^p divides a and that $g = -(a/t^p) \eta^q$. Evaluating the top row at $t = 0$, we get:

$$0 = f(0) + \theta f(0)^q + \alpha \cdot s \eta^q \quad (3.9)$$

where $s \in \mathbb{F}$ is the evaluation of a/t^p at $t = 0$. If $s = 0$, then $g(0) = 0$ and $f(0) = 0$ by 2. Hence, dividing (3.8) by the correct power of t , we can assume without loss that $s \neq 0$. Yet, dividing (3.9) by s and using that $\alpha \eta^q = \beta$, we obtain a root of $\theta X^q + X + \beta$ in F . This contradicts our assumption 3.

Although $\text{Ext}_{\mathcal{M}\mathcal{M}_F}^2(\mathbb{1}, \underline{M})$ might be non zero, we prove next that it is always torsion:

Theorem 3.37. *For $i > 1$, the A -module $\text{Ext}_{\mathcal{M}\mathcal{M}_F}^i(\mathbb{1}, \underline{M})$ is torsion.*

Proof. We begin with a very general remark. Given three objects A, B and C in an abelian category \mathcal{A} , Yoneda's cup product:

$$\cup : \text{Ext}_{\mathcal{A}}^1(A, B) \times \text{Ext}_{\mathcal{A}}^1(B, C) \longrightarrow \text{Ext}_{\mathcal{A}}^2(A, C) \quad (3.10)$$

admits the following two descriptions:

1. Given $e = [0 \rightarrow B \xrightarrow{i} E \rightarrow A \rightarrow 0] \in \text{Ext}_{\mathcal{A}}^1(A, B)$ and $f = [0 \rightarrow C \rightarrow F \xrightarrow{j} B \rightarrow 0] \in \text{Ext}_{\mathcal{A}}^1(B, C)$, we obtain $e \cup f \in \text{Ext}_{\mathcal{A}}^2(A, C)$ as the class of the long-exact sequence:

$$0 \rightarrow C \rightarrow F \xrightarrow{ioj} E \rightarrow A \rightarrow 0.$$

2. Applying the functor $\text{Ext}_{\mathcal{A}}^1(A, -)$ to a short exact sequence associated to e , we obtain a connecting homomorphism:

$$\delta_e : \text{Ext}_{\mathcal{A}}^1(A, B) \longrightarrow \text{Ext}_{\mathcal{A}}^2(A, C).$$

Then, $e \cup f = \delta_e(f)$.

It follows from 1 that for any element g in $\text{Ext}_{\mathcal{A}}^2(A, C)$, there exists an object B in \mathcal{A} such that $g = e \cup f$ for $e \in \text{Ext}_{\mathcal{A}}^1(A, B)$ and $f \in \text{Ext}_{\mathcal{A}}^1(B, C)$. It follows from 2 that if the functor $\text{Ext}_{\mathcal{A}}^1(A, -)$ is right-exact, the image of (3.10) is zero. Combining both description, we obtain: if the functor $\text{Ext}_{\mathcal{A}}^1(A, -)$ is right-exact, then $\text{Ext}_{\mathcal{A}}^2(A, X)$ is zero for any object X in \mathcal{A} .

Back to the situation of the Theorem, let us first treat the case where \underline{M} only has non positive weights. Let \mathcal{A} be the K -linear abelian category $\mathcal{M}\mathcal{M}_F^{\text{iso}}$ and let \mathcal{A}_0 be the full subcategory of \mathcal{A} consisting of objects whose weights are all non positive. For $\underline{M} \in \mathcal{A}_0$, note that

$$\text{Ext}_{\mathcal{A}_0}^1(\mathbb{1}, \underline{M}) = \text{Ext}_{\mathcal{A}}^1(\mathbb{1}, \underline{M}) = \text{Ext}_{\mathcal{M}\mathcal{M}_F^{\text{iso}}}^1(\mathbb{1}, \underline{M})$$

where the last equality follows from Proposition 3.31. From Corollary 2.11, we deduce that the functor $\text{Ext}_{\mathcal{A}_0}^1(\mathbb{1}, -)$ is right-exact on \mathcal{A}_0 . From the above observation, $\text{Ext}_{\mathcal{A}_0}^2(\mathbb{1}, \underline{M}) = (0)$. Now, from the exactness of W_0 over \mathcal{A} (Proposition 3.30), given any $e \in \text{Ext}_{\mathcal{A}}^2(\mathbb{1}, \underline{M})$ we have a commutative diagram in \mathcal{A} :

$$\begin{array}{ccccccccccc} e : & 0 & \longrightarrow & \underline{M} & \longrightarrow & \underline{E}_1 & \longrightarrow & \underline{E}_2 & \longrightarrow & \mathbb{1} & \longrightarrow & 0 \\ & & & \text{id} \uparrow & & \uparrow & & \uparrow & & \text{id} \uparrow & & \\ W_0 e : & 0 & \longrightarrow & \underline{M} & \longrightarrow & W_0 \underline{E}_1 & \longrightarrow & W_0 \underline{E}_2 & \longrightarrow & \mathbb{1} & \longrightarrow & 0 \end{array}$$

from which we deduce $\text{Ext}_{\mathcal{A}_0}^2(\mathbb{1}, \underline{M}) = \text{Ext}_{\mathcal{A}}^2(\mathbb{1}, \underline{M})$. This amounts to:

$$\text{Ext}_{\mathcal{M}\mathcal{M}_F}^2(\mathbb{1}, \underline{M}) \otimes_A K = \text{Ext}_{\mathcal{A}}^2(\mathbb{1}, \underline{M}) = (0).$$

as desired. Now, let \underline{M} have arbitrary weights. Applying $\text{Ext}_{\mathcal{A}}^1(\mathbb{1}, -)$ to the exact sequence:

$$0 \longrightarrow W_0 \underline{M} \longrightarrow \underline{M} \longrightarrow \underline{M}/W_0 \underline{M} \longrightarrow 0,$$

we obtain from Proposition 3.33 that the natural map:

$$\text{Ext}_{\mathcal{A}}^1(\mathbb{1}, W_0 \underline{M}) \longrightarrow \text{Ext}_{\mathcal{A}}^1(\mathbb{1}, \underline{M})$$

is surjective. Given an epimorphism $f : \underline{M} \rightarrow \underline{N}$ in \mathcal{A} , we obtain a commutative square:

$$\begin{array}{ccc} \mathrm{Ext}_{\mathcal{A}}^1(\mathbb{1}, W_0\underline{M}) & \twoheadrightarrow & \mathrm{Ext}_{\mathcal{A}}^1(\mathbb{1}, \underline{M}) \\ \downarrow & & \downarrow \\ \mathrm{Ext}_{\mathcal{A}}^1(\mathbb{1}, W_0\underline{N}) & \twoheadrightarrow & \mathrm{Ext}_{\mathcal{A}}^1(\mathbb{1}, \underline{N}). \end{array}$$

The left vertical arrow is surjective: indeed, $W_0f : W_0\underline{M} \rightarrow W_0\underline{N}$ is an epimorphism by Proposition 3.30, and we already proved that over \mathcal{A}_0 the functor $\mathrm{Ext}_{\mathcal{A}_0}^1(\mathbb{1}, -) = \mathrm{Ext}_{\mathcal{A}}^1(\mathbb{1}, -)$ is right-exact. Hence, the right vertical arrow is surjective and the functor $\mathrm{Ext}_{\mathcal{A}}^1(\mathbb{1}, -)$ is right-exact on \mathcal{A} . By the above observation, this implies:

$$\mathrm{Ext}_{\mathcal{M}, \mathcal{M}_F}^2(\mathbb{1}, \underline{M}) \otimes_A K = \mathrm{Ext}_{\mathcal{A}}^2(\mathbb{1}, \underline{M}) = (0).$$

That $\mathrm{Ext}_{\mathcal{M}, \mathcal{M}_F}^i(\mathbb{1}, \underline{M})$ is torsion for all $i > 1$ follows from $\mathrm{Ext}_{\mathcal{A}}^i(\mathbb{1}, \underline{M}) = (0)$. \square

4 Models and the integral part of A -motivic cohomology

In this section we illustrate the notion of *maximal integral models*. For A -motives, maximal integral models are understood as an analogue of Néron models of abelian varieties. The notion dates back to Gardeyn's work on *models of τ -sheaves* [Gar2] and their reduction [Gar1], where he proved a Néron-Ogg-Shafarevič type criterion (see Proposition 4.39 in our context). However our setting differs by the fact that, in opposition to τ -sheaves, A -motives might not be *effective*. We also removed Gardeyn's assumption for an *integral model* to be locally free. We will show in Propositions 4.26 and 4.43 that this is implicit for maximal ones over local and global function fields. Our presentation thus allows to avoid the use of a technical lemma due to Lafforgue in Gardeyn's exposition [Gar2, §2]. In that sense, the content of this chapter is original.

In practice, to make maximal integral models of A -motives explicit is a difficult task. In section 4.1, we consider the easier problem of finding maximal integral models of *Frobenius spaces*. Those are pairs (V, φ) where V is a finite dimensional vector space over a local field E containing \mathbb{F} and φ is a q -linear endomorphism of V . We show in Proposition 4.2 that there exists a unique \mathcal{O}_E -lattice in V stable by φ and which is maximal for this property. We end this section by the review of Katz's equivalence of categories, and its application to study maximal integral models.

In Sections 4.2 and 4.3, we shall be concerned with integral models of A -motives. Given $R \subset S$ an inclusion of \mathbb{F} -algebras and an A -motive $\underline{M} = (M, \tau_M)$ over S , an R -model for \underline{M} is a finite sub- $A \otimes R$ -module of M stable by τ_M (Definition 4.16).

We study the case where \underline{M} is an A -motive over a local function field $S = E$ and where $R = \mathcal{O}_E$ is its valuation ring in Section 4.2. In Proposition 4.24, we prove existence and uniqueness of integral \mathcal{O}_E -models which are maximal for the inclusion, and we prove that they are locally free in Proposition 4.26. We show that, given a well-chosen maximal ideal $\ell \subset A$ and a positive integer n , the data of $(M/\ell^n M, \tau_M)$ defines a Frobenius space over E . Theorem 4.31, our main result of this section, describes how to recover the maximal integral model of \underline{M} in terms of the data of the maximal integral model of $(M/\ell^n M, \tau_M)$ for all n . The latter is of fundamental importance in the proof of Theorem C, and permits

to obtain the Néron-Ogg-Shafarevič type criterion for A -motives (Proposition 4.39).

In Section 4.3, we treat the case where \underline{M} is an A -motive over a global function field $S = F$ where R is a Dedekind domain whose fraction field is F . If \mathfrak{p} is a non zero prime ideal of R , we obtain an A -motive $\underline{M}_{F_{\mathfrak{p}}}$ by base field extension from F to its completion $F_{\mathfrak{p}}$. Our Proposition 4.41 explain how to recover the maximal integral model of \underline{M} from the data of the maximal integral models of $\underline{M}_{F_{\mathfrak{p}}}$ for all \mathfrak{p} .

The full force of this section is used in Subsection 4.4 to prove Theorems C and D of the introduction (respectively Theorems 4.48 and 4.49 in the text).

4.1 Models of Frobenius spaces

In this subsection we work with notations that are slightly more general to what we need in the sequel. We let k be a perfect field containing \mathbb{F} and let $E = k((\varpi))$ be the field of Laurent series over \mathbb{F} in the variable ϖ . We let $\sigma : E \rightarrow E$ denote the q -Frobenius Frob_q on E (it fixes \mathbb{F}), v_E be the valuation of E , $\mathcal{O}_E = k[[\varpi]]$ be its valuation ring with maximal ideal $\mathfrak{p} = (\varpi)$. We fix a separable closure E^s of E and denote by G_E the absolute Galois group $\text{Gal}(E^s|E)$ of E . Let also $I_E \subset G_E$ be the inertia subgroup.

Our object of study are pairs (V, φ) where V is a finite dimensional E -vector space and $\varphi : \sigma^*V \rightarrow V$ is an E -linear isomorphism. In the existing literature, there are generally referred to as *étale finite \mathbb{F} -shtukas over E* (e.g. [Har3, §4]). We prefer here the shorter name *Frobenius spaces*. By an \mathcal{O}_E -lattice in V we mean a finitely generated sub- \mathcal{O}_E -module L of V which generates V over E . A sub- \mathcal{O}_E -module L is stable by φ if $\varphi(\sigma^*L) \subset L$.

Definition 4.1. We say that L is an *integral model* for (V, φ) if L is an \mathcal{O}_E -lattice in V stable by φ . We say that L is *maximal* if it is not strictly included in another integral model for (V, φ) .

Proposition 4.2. *A maximal integral model for (V, φ) exists and is unique.*

Proof. Our proof follows closely [Gar2, Prop. 2.2, 2.13]. First note that there exists an integral model. Indeed, let T' be an arbitrary \mathcal{O}_E -lattice in V . There exists a positive integer k such that $\varphi(\sigma^*T') \subset \varpi^{-k}T'$. We let $T := \varpi^k T'$ so that

$$\varphi(\sigma^*T) = \varpi^{qk} \varphi(\sigma^*T') \subset \varpi^{(q-1)k} T' = \varpi^{(q-2)k} T \subset T.$$

Hence, the \mathcal{O}_E -module T is an \mathcal{O}_E -lattice in V stable by φ .

We turn to the existence and uniqueness of the maximal integral model. If $L' \subset L$ is an inclusion of integral models. We have:

$$\text{length}_{\mathcal{O}_E}(\varphi(\sigma^*L)/\varphi(\sigma^*L')) = q \cdot \text{length}_{\mathcal{O}_E}(L/L').$$

We define the *discriminant of L* to be the non-negative integer

$$\Delta(L) := \text{length}_{\mathcal{O}_E}(L/\varphi(\sigma^*L)).$$

Since we have:

$$\begin{aligned} \Delta(L') - \Delta(L) &= \text{length}_{\mathcal{O}_E}(L'/\varphi(\sigma^*L')) - \text{length}_{\mathcal{O}_E}(L/\varphi(\sigma^*L)) \\ &= \text{length}_{\mathcal{O}_E}(\varphi(\sigma^*L)/\varphi(\sigma^*L')) - \text{length}_{\mathcal{O}_E}(L/L') \\ &= (q-1) \cdot \text{length}_{\mathcal{O}_E}(L'/L), \end{aligned} \tag{4.1}$$

then $\Delta(L') > \Delta(L)$ whenever the inclusion $L' \subset L$ is strict.

Now let L be an integral model with minimal discriminant. We claim that L equals the union of all integral models of (V, φ) , which proves both existence and

uniqueness of the maximal integral model. Indeed, if L' is another integral model for (V, φ) not contained in L , then the inclusion $L \subset L + L'$ is strict. But this contradicts the minimality assumption as we would have $\Delta(L) > \Delta(L + L')$. \square

Example 4.3. Suppose $V := E$, $f \in \mathcal{O}_E$ a nonzero element and φ is the morphism corresponding to $x \mapsto fx^q$. Write $f = u\varpi^k h^{q-1}$ where $u \in \mathcal{O}_E^\times$, $0 \leq k < q - 1$ is an integer and $h \in \mathcal{O}_E$. Then, the maximal integral model of (V, φ) is given by $h^{-1}\mathcal{O}_E$. This is because $\Delta(h^{-1}\mathcal{O}_E) = k$ together with (4.1).

Let T be an integral model for (V, φ) and let r be its rank as a free \mathcal{O}_E -module. The cokernel of the inclusion $\varphi(\sigma^*T) \subset T$ is a torsion \mathcal{O}_E -module of finite type and there exists elements g_1, \dots, g_r in \mathcal{O}_E with $v_E(g_i) \leq v_E(g_{i+1})$ such that

$$T/\varphi(\sigma^*T) \cong \mathcal{O}_E/(g_1) \oplus \mathcal{O}_E/(g_2) \oplus \cdots \oplus \mathcal{O}_E/(g_r).$$

Equivalently, there exists a basis (v_1, \dots, v_r) of T over \mathcal{O}_E such that

$$\varphi(\sigma^*T) = (g_1)v_1 \oplus (g_2)v_2 \oplus \cdots \oplus (g_r)v_r.$$

The elements g_1, \dots, g_r are unique up to multiplication by units and are called *the elementary divisors relative to the inclusion of \mathcal{O}_E -lattices $\varphi(\sigma^*T) \subset T$* .

Lemma 4.4. *Let \mathbf{t} be a basis of T over \mathcal{O}_E and let F be the matrix of φ written in the bases $\sigma^*\mathbf{t}$ and \mathbf{t} . The elementary divisors relative to the inclusion $\varphi(\sigma^*T) \subset T$ are the elementary divisors of the matrix F , up to units in \mathcal{O}_E .*

Proof. If (f_1, \dots, f_r) denotes the elementary divisors of F , the Smith's normal form Theorem implies that there exists $U, V \in \mathrm{GL}_r(\mathcal{O}_E)$ such that $UF = \mathrm{diag}(f_1, \dots, f_r)V$. If we let $\mathbf{v} = (v_1, \dots, v_r)$ be the basis of T corresponding to $V \cdot \mathbf{t}$, this relation reads

$$\varphi(\sigma^*T) = (f_1)v_1 \oplus (f_2)v_2 \oplus \cdots \oplus (f_r)v_r.$$

By uniqueness of the ideals $(g_1), \dots, (g_r)$, we conclude that $(f_i) = (g_i)$ for all $i \in \{1, \dots, r\}$. \square

Definition 4.5. We let the *type of T* be the sequence (e_1, \dots, e_r) of the valuations of the elementary divisors relative to the inclusion $\varphi(\sigma^*T) \subset T$ ordered such that $e_1 \leq e_2 \leq \dots \leq e_r$. We define *the range r_T of T* to be the integer e_r .

Remark 4.6. We have $\Delta(T) = e_1 + \dots + e_r$ so that $r \leq \Delta(T) \leq r \cdot r_T$ where $\Delta(T)$ denotes the discriminant of T . It follows that the range of T is a *finer* invariant than its discriminant.

We should denote by $V_{\mathcal{O}}$ the maximal integral model of (V, φ) . The following proposition enables us to say *how far* an integral lattice is from being maximal given its range.

Proposition 4.7. *Let T be an \mathcal{O}_E -lattice in V stable by φ . Let s be a non-negative integer. If the range of T satisfies $r_T \leq s(q - 1)$, then $V_{\mathcal{O}} \subset \varpi^{-s}T$.*

We start by a lemma:

Lemma 4.8. *Let U be an \mathcal{O}_E -lattice in V such that $U \subset \varphi(\sigma^*U)$. Then $V_{\mathcal{O}} \subset U$.*

Proof. For $n \geq 0$, we let $\sigma^{n*} := (\sigma^n)^*$ and denote by $\varphi^n : \sigma^{n*}V \rightarrow V$ the E -linear morphism given by the composition

$$\sigma^{n*}V \xrightarrow{\sigma^{(n-1)*}\varphi} \sigma^{(n-1)*}V \longrightarrow \cdots \longrightarrow \sigma^*V \xrightarrow{\varphi} V.$$

We consider the following sub- \mathcal{O}_E -module of V :

$$V_{\mathcal{O}} \cap \left(\bigcup_{n=0}^{\infty} \varphi^n(\sigma^{n*}U) \right). \quad (4.2)$$

It is stable by φ , finitely generated because contained in $V_{\mathcal{O}}$, and generates V over E because contains the \mathcal{O}_E -lattice $V_{\mathcal{O}} \cap U$. By maximality, (4.2) equals $V_{\mathcal{O}}$ and we deduce that there exists a non-negative integer m such that $V_{\mathcal{O}} \subset \varphi^m(\sigma^{m*}U)$. Because $\varphi(\sigma^*V_{\mathcal{O}}) \subset V_{\mathcal{O}}$, we have $\sigma^*V_{\mathcal{O}} \subset \varphi^{-1}(V_{\mathcal{O}})$ and by immediate recursion one gets $\sigma^{m*}V_{\mathcal{O}} \subset \varphi^{-m}(V_{\mathcal{O}}) \subset \sigma^{m*}U$. We conclude that $V_{\mathcal{O}} \subset U$ because $\sigma : \mathcal{O}_E \rightarrow \mathcal{O}_E$ is faithfully flat. \square

Proof of Proposition 4.7. Let (e_1, \dots, e_r) be the type of T . Recall that $\mathfrak{m} = \mathfrak{m}_E$ denotes the maximal ideal of \mathcal{O}_E . There exists a basis (t_1, \dots, t_r) of T such that $\varphi(\sigma^*T) = \mathfrak{m}^{e_1}t_1 \oplus \mathfrak{m}^{e_2}t_2 \oplus \dots \oplus \mathfrak{m}^{e_r}t_r$. By assumption, $e_1, \dots, e_r \leq s(q-1)$ and thus

$$\begin{aligned} \varpi^{-s}T &\subset \varpi^{-s} \left(\mathfrak{m}^{e_1-s(q-1)}t_1 \oplus \mathfrak{m}^{e_2-s(q-1)}t_2 \oplus \dots \oplus \mathfrak{m}^{e_r-s(q-1)}t_r \right) \\ &= \mathfrak{m}^{e_1-sq}t_1 \oplus \mathfrak{m}^{e_2-sq}t_2 \oplus \dots \oplus \mathfrak{m}^{e_r-sq}t_r \\ &= \varpi^{-qs}\varphi(\sigma^*T) \\ &= \varphi(\sigma^*(\varpi^{-s}T)). \end{aligned}$$

Hence, $U := \varpi^{-s}T$ satisfies $U \subset \varphi(\sigma^*U)$ and we deduce that $V_{\mathcal{O}} \subset U$ by Lemma 4.8. \square

Akin to integral models, there is also a notion of *good models*.

Definition 4.9. Let L be a finitely generated \mathcal{O}_E -submodule of V . We say that L is a *good model* for (V, φ) if $\varphi(\sigma^*L) = L$. We say that L is *maximal* if it is not strictly included in another good model of (V, φ) .

Proposition 4.10. *A maximal good model for (V, φ) exists and is unique.*

Proof. First note that any good model L for (V, φ) is contained in $V_{\mathcal{O}}$: indeed, $L + V_{\mathcal{O}}$ is again an integral model, hence included in $V_{\mathcal{O}}$. The union U of all good model for (V, φ) exists (it is non-empty as the zero module is a good model) and therefore included in $V_{\mathcal{O}}$. Because \mathcal{O}_E is Noetherian, U is a finitely generated \mathcal{O}_E -module. We also have $\varphi(\sigma^*U) = U$. We deduce that U is maximal and unique. \square

We should denote by V_{good} the maximal good model of (V, φ) .

Definition 4.11. We say that (V, φ) has *good reduction* if $V_{\text{good}} = V_{\mathcal{O}}$. The rank of V_{good} is called the *non-degenerate rank* of (V, φ) .

Maximal good models have an interpretation in terms of *Frobenius sheaves* that we now recall. Let X be a smooth connected scheme over \mathbb{F} , and let $\pi(X)$ be its étale fundamental group. We still denote by σ the Frobenius on X . Let $\mathcal{F}(X)$ be the category whose objects are pairs (\mathcal{V}, φ) where \mathcal{V} is a locally-free \mathcal{O}_X -module of finite rank and $\varphi : \sigma^*\mathcal{V} \rightarrow \mathcal{V}$ is an isomorphism of \mathcal{O}_X -modules. Morphisms in this category are morphisms of the underlying \mathcal{O}_X -modules with commuting φ -action.

Example 4.12. Objects of $\mathcal{F}(\text{Spec } E)$ are Frobenius spaces over E , and objects of $\mathcal{F}(\text{Spec } \mathcal{O}_E)$ are pairs (V, φ) where V is a finite free \mathcal{O}_E -module, and where $\varphi : \sigma^*V \rightarrow V$ is an \mathcal{O}_E -linear isomorphism.

The following result is due to Katz in [Kat2, Prop. 4.1.1].

Theorem 4.13. *There is a rank-preserving equivalence of categories from $\mathcal{F}(X)$ to the category of \mathbb{F} -linear continuous representation of $\pi(X)$, which commutes with base change. For $X = \text{Spec } E$, it is explicitly given by*

$$\underline{V} = (V, \varphi) \longmapsto T\underline{V} = \{x \in V \otimes_E E^s \mid x = \varphi(\sigma^* x)\}$$

where $\pi(\text{Spec } E)$ is identified with $\text{Gal}(E^s|E)$, and acts on the right-hand side of $V \otimes_E E^s$.

The following proposition is almost immediate from Katz's equivalence:

Proposition 4.14. *Let \underline{V} be a Frobenius space over E . The non-degenerate rank of (V, φ) equals the rank of $(T\underline{V})^{I_E}$. In particular, \underline{V} has good reduction if and only if $T\underline{V}$ is unramified.*

Proof. As $\pi(\text{Spec } \mathcal{O}_E) \cong G_E/I_E$, the representation $(T\underline{V})^{I_E}$ is the maximal subobject of $T\underline{V}$ which comes from an object in $\mathcal{F}(\text{Spec } \mathcal{O}_E)$ by Katz's equivalence. Yet, elements in $\mathcal{F}(\text{Spec } \mathcal{O}_E)$ which specialize to subobjects of \underline{V} by base change to E are exactly the good models of \underline{V} . \square

We end this subsection by the next result, which will be the main ingredient to obtain Theorem C of the introduction.

Proposition 4.15. *Let $\underline{V} = (V, \varphi)$ be a Frobenius space over E , and let $x \in V$. The following are equivalent:*

- (i) *There exists $y \in V \otimes_E E^{ur}$ such that $x = y - \varphi(\tau^* y)$,*
- (ii) $x \in V_{\text{good}} + (\text{id}_V - \varphi)(V)$,
- (iii) $x \in V_{\mathcal{O}} + (\text{id}_V - \varphi)(V)$.

Proof. We first prove the equivalence between (i) and (ii). Let $\mathbf{1} : \sigma^* E \rightarrow E$ be the canonical E -linear isomorphism and let $\mathbb{1}$ be the *neutral* Frobenius space $(E, \mathbf{1})$ over E . Let also $\text{Ext}^1(\mathbb{1}, \underline{V})$ be the \mathbb{F} -vector space of Yoneda extensions of $\mathbb{1}$ by \underline{V} in the category $\mathcal{F}(\text{Spec } E)$. We have a isomorphism of \mathbb{F} -vector spaces, natural in \underline{V} ,

$$\frac{V}{(\text{id}_V - \varphi)(V)} \xrightarrow{\sim} \text{Ext}^1(\mathbb{1}, \underline{V}) \quad (4.3)$$

mapping a representative $v \in V$ to the class of the extension of $\mathbb{1}$ by \underline{V} whose underlying module is $V \oplus E$ and whose Frobenius action is given by $\begin{pmatrix} \varphi & v \cdot \mathbf{1} \\ 0 & \mathbf{1} \end{pmatrix}$. Katz's equivalence leads to a commutative square

$$\begin{array}{ccc} \frac{V_{\text{good}}}{(\text{id}_V - \varphi)(V_{\text{good}})} & \xrightarrow{\sim} & H^1(\pi(\text{Spec } \mathcal{O}_E), (T\underline{V})^{I_E}) \\ \downarrow & & \downarrow \\ \frac{V}{(\text{id}_V - \varphi)(V)} & \xrightarrow{\sim} & H^1(G_E, T\underline{V}) \end{array}$$

where, by diagram chasing, the bottom row is given as follows: for $v \in V$, let $w \in V \otimes_E E^s$ be such that $v = w - \varphi(\tau^* w)$, then $c_v : \rho \mapsto w - \rho w$ defines a cocycle $c_v : G_E \rightarrow T\underline{V}$ whose class does not depend on the choice of w . The bottom row maps v to c_v . Hence, (i) holds if and only if c_x comes from a cocycle in $H^1(\pi(\text{Spec } \mathcal{O}_E), (T\underline{V})^{I_E})$, that is, if and only if (ii) holds.

It remains to prove that (ii) and (iii) are equivalent. Let \mathfrak{p} be the maximal ideal of \mathcal{O}_E . For $n \geq 1$, $V_{\mathcal{O}}/\mathfrak{p}^n V_{\mathcal{O}}$ defines a finite dimensional k -vector space

equipped with a q -linear action induced by φ (as $\sigma(\mathfrak{p}^n) \subset \mathfrak{p}^{qn} \subset \mathfrak{p}^n$). We denote it (V_n, φ_n) . As k is perfect, there is a unique decomposition:

$$V_n = A_n \oplus B_n$$

by k -subspaces such that φ_n is an automorphism on A_n and nilpotent on B_n . By uniqueness and since φ_n is q -linear, those decompositions are compatible for all n and with the $\mathcal{O}_E/\mathfrak{p}^n$ -module structure. Taking projective limits, we obtain a decomposition of \mathcal{O}_E -modules:

$$V_{\mathcal{O}} = A \oplus B$$

such that $\varphi(\sigma^*A) = A$ and $(\text{id}_V - \varphi)(B) = B$. Hence $B \subset (\text{id}_V - \varphi)(V)$ and $A \subset V_{\text{good}}$. It follows that $V_{\mathcal{O}} \subset V_{\text{good}} + (\text{id} - \varphi)(V)$, as desired. \square

4.2 Models of A -motives over a local function field

General theory

Let R be a commutative \mathbb{F} -algebra given together with an \mathbb{F} -algebra morphism $\kappa : A \rightarrow R$. Let S be a commutative \mathbb{F} -algebra containing R . Let $\underline{M} = (M, \tau_M)$ be an A -motive over S (with characteristic morphism $\kappa : A \rightarrow S$).

Definition 4.16. We define an R -integral model L for \underline{M} to be a sub- $A \otimes R$ -module of M of finite type such that

- (i) L generates M over $A \otimes S$,
- (ii) $\tau_M(\tau^*L) \subset L[j^{-1}]$.

We say that L is maximal if it is not strictly contained in any other R -integral model of \underline{M} .

Lemma 4.17. *A maximal R -integral model for \underline{M} contains all the R -integral models for \underline{M} . In particular, if it exists it is unique.*

Proof. Given L_1 and L_2 two R -integral models, their sum $L_1 + L_2$ again defines an R -integral model. Hence, if L_1 is maximal, the inclusion $L_1 \subsetneq L_1 + L_2$ is not strict: we deduce $L_1 + L_2 = L_1$, then $L_2 \subset L_1$. \square

The next proposition is inspired by [Gar2, Prop. 2.2]:

Proposition 4.18. *If S is obtained from R by localization, an R -model for \underline{M} exists.*

Proof. Let (m_1, \dots, m_s) be generators of M as an $A \otimes S$ -module, and let L_0 be the sub- $A \otimes R$ -module of M generated by (m_1, \dots, m_s) . Let $d \in R$ be such that $\tau_M(\tau^*L_0) \subset d^{-1}L_0[j^{-1}]$, and set $L := dL_0$. We have

$$\tau_M(\tau^*L) = d^q \tau_M(\tau^*L_0) \subset d^{q-1}L_0[j^{-1}] = d^{q-2}L[j^{-1}] \subset L[j^{-1}].$$

Thus L is an R -model. \square

Similarly:

Definition 4.19. We define an R -good model L for \underline{M} to be a sub- $A \otimes R$ -module of M of finite type such that $\tau_M(\tau^*L)[j^{-1}] = L[j^{-1}]$. We say that L is maximal if it is not strictly contained in any other R -good model of \underline{M} .

From the argument given in the proof of Proposition 4.10, we easily deduce the next lemma.

Lemma 4.20. *Assume that there exists a maximal R -integral model for \underline{M} . Then, a maximal R -good model for \underline{M} exists and is unique.*

We continue this section by recording additional properties of maximal R -models. Those will eventually be useful in Subsection 4.4 for the proof of Theorem D (Theorem 4.49 in the text).

Let \underline{M} be an A -motive over S which admits a maximal integral R -integral models denoted M_R .

Proposition 4.21. *Let N be a finitely generated sub- $A \otimes R$ -module of M such that $\tau_M(\tau^*N) \subset N[j^{-1}]$. Then, $N \subset M_R$. In particular, any element $m \in M$ such that $\tau_M(\tau^*m) = m$ belongs to M_R .*

Proof. It suffices to notice that the module L generated by M_R and N over $A \otimes R$ is an R -model for \underline{M} , and hence $N \subset L \subset M_R$. \square

Corollary 4.22. *We have $(\text{id} - \tau_M)(M_R) = (\text{id} - \tau_M)(M) \cap M_R[j^{-1}]$.*

Proof. The inclusion $(\text{id} - \tau_M)(M_R) \subset (\text{id} - \tau_M)(M) \cap M_R[j^{-1}]$ is clear. Conversely, let $m \in M_R[j^{-1}]$ and let $n \in M$ be such that $m = n - \tau_M(\tau^*n)$. The sub- $A \otimes R$ -module $\langle M_R, n \rangle$ of M generated by elements of M_R together with n over $A \otimes R$ is an R -model for \underline{M} . In particular, $\langle M_R, n \rangle \subset M_R$ and $n \in M_R$. \square

We end this chapter with a remark on the assignment $\underline{M} \mapsto M_R$. Assume that S is such that every object in \mathcal{M}_S admits a maximal R -integral model (this is the case when $R \subset S$ is the inclusion of a Dedekind domain into its field of fractions, as shown in Proposition 4.41 below).

Corollary 4.23. *Let $f : \underline{M} \rightarrow \underline{N}$ be a morphism in \mathcal{M}_S . Then $f(M_R) \subset N_R$. In particular, the assignment $\underline{M} \mapsto M_R$ is functorial.*

Existence and first properties

Let E be a local field containing \mathbb{F} , let $\mathcal{O} = \mathcal{O}_E$ be its ring of integers and let $k = k_E$ be its residue field. In this subsection, we shall be concerned with the case where $S = E$ and $R = \mathcal{O}_E$, where the characteristic morphism $\kappa : A \rightarrow \mathcal{O}_E$ is an \mathbb{F} -linear morphism. Let \underline{M} be an A -motive over E of characteristic κ .

Proposition 4.24. *A maximal \mathcal{O}_E -model for \underline{M} exists and is unique. In particular, a maximal \mathcal{O}_E -good model for \underline{M} exists and is unique.*

Proof. It is enough to show existence of a maximal integral model. Let U be the $A \otimes \mathcal{O}_E$ -module given by the union of all the \mathcal{O}_E -models for \underline{M} . We claim that U is the maximal \mathcal{O}_E -model of \underline{M} . As U is non-empty by Proposition 4.18, it generates M over E . We also have $\tau_M(\tau^*U) \subset U[j^{-1}]$. So our task is to show that U is finitely generated.

Let T be an \mathcal{O}_E -model for \underline{M} and let $\mathbf{t} = \{t_1, \dots, t_s\}$ be a set of generators of T over $A \otimes \mathcal{O}_E$. Let \mathbf{m} be a basis of $M \otimes_{A \otimes E} \text{Quot}(A \otimes E)$ as a vector space over $\text{Quot}(A \otimes E)$, and let $F_M \in \text{GL}_r(\text{Quot}(A \otimes E))$ be the matrix of τ_M written in the bases $\tau^*\mathbf{m}$ and \mathbf{m} . Let $P \in \mathcal{M}_{s,r}(A \otimes E)$ be the matrix expressing \mathbf{t} in \mathbf{m} . Because of points (ii) in Definition 4.16, there exists $N \in \mathcal{M}_s(A \otimes \mathcal{O}_E[j^{-1}])$ such that $P^{(1)}F_M = NP$. If v denotes the valuation in $\text{Quot}(A \otimes E)$ at the special fiber $C \times \text{Spec } k_E$ of $C \times \text{Spec } \mathcal{O}_E$, and extend it to matrices over $\text{Quot}(A \otimes E)$ by taking the minimal valuation of its coefficients. Then $v(N) \geq 0$, and

$$qv(P) = v(P^{(1)}) = v(NPF_M^{-1}) \geq v(N) + v(P) + v(F_M^{-1}) \geq v(P) + v(F_M^{-1}).$$

Hence, $v(P) \geq v(F_M^{-1})/(q-1)$. We conclude that T is contained in the $A \otimes \mathcal{O}_E$ -module

$$U_0 := \left\{ a_1 m_1 \oplus \dots \oplus a_r m_r \mid a_i \in A \otimes E, v(a_i) \geq \frac{v(F_M^{-1})}{q-1} \right\}. \quad (4.4)$$

In particular, U is contained in U_0 . The latter being a finitely generated module over the Noetherian ring $A \otimes \mathcal{O}_E$, the former is finitely generated. \square

Definition 4.25. We denote by $M_{\mathcal{O}}$ the unique maximal \mathcal{O}_E -integral model of \underline{M} , and by M_{good} the maximal \mathcal{O}_E -good model of \underline{M} .

We have the next:

Proposition 4.26. *Both $M_{\mathcal{O}}$ and M_{good} are locally free over $A \otimes \mathcal{O}_E$.*

The proofs being similar, we solely explicit it in the case of $M_{\mathcal{O}}$. We start with a useful lemma.

Lemma 4.27. *Let $\mathfrak{a} \subset A$ be an ideal. Then $M_{\mathcal{O}} \cap \mathfrak{a}M = \mathfrak{a}M_{\mathcal{O}}$.*

Proof. The inclusion \supset is clear. We assume $\mathfrak{a} \neq 0$ and consider the sub- A -motive $(\mathfrak{a}M, \tau_M)$ of \underline{M} . If T is an \mathcal{O}_E -model for $(\mathfrak{a}M, \tau_M)$, then $\mathfrak{a}^{-1}T$ is an \mathcal{O}_E -model for \underline{M} and we have $\mathfrak{a}^{-1}T \subset M_{\mathcal{O}}$. This implies that $\mathfrak{a}M_{\mathcal{O}}$ is the maximal \mathcal{O}_E -model of $(\mathfrak{a}M, \tau_M)$ so that $(\mathfrak{a}M)_{\mathcal{O}} = \mathfrak{a}(M_{\mathcal{O}})$. Therefore, the inclusion $M_{\mathcal{O}} \cap \mathfrak{a}M \subset \mathfrak{a}M_{\mathcal{O}}$ follows from the fact that $M_{\mathcal{O}} \cap \mathfrak{a}M$ is an \mathcal{O}_E -model for $(\mathfrak{a}M, \tau_M)$. \square

Proof of Proposition 4.26. Because $A \otimes \mathcal{O}_E$ is a Noetherian domain and $M_{\mathcal{O}}$ is finitely generated, it is enough to show that $M_{\mathcal{O}}$ is flat. We use Bourbaki's local criterion of flatness. Let $\mathfrak{m} \subset A$ be a maximal ideal and let $\mathbb{F}_{\mathfrak{m}}$ be its residue field. Note that

$$\text{Tor}_1^{A \otimes \mathcal{O}_E}(A/\mathfrak{m} \otimes \mathcal{O}_E, M_{\mathcal{O}}) = \{m \in M_{\mathcal{O}} \mid \forall r \in \mathfrak{m}, (r \otimes 1)m = 0\} = 0.$$

Hence, by [Bou, AC §III.5.2 Thm. 1], the flatness of $M_{\mathcal{O}}$ over $A \otimes \mathcal{O}_E$ is equivalent to that of $M_{\mathcal{O}}/\mathfrak{m}M_{\mathcal{O}}$ over $\mathbb{F}_{\mathfrak{m}} \otimes \mathcal{O}_E$. The ring $\mathbb{F}_{\mathfrak{m}} \otimes \mathcal{O}_E$ is a product of discrete valuation rings and thus $M_{\mathcal{O}}/\mathfrak{m}M_{\mathcal{O}}$ is flat (and then locally free) if and only if it is \mathcal{O}_E -torsion free. The latter condition is easily seen to be equivalent to the equality:

$$\mathfrak{m}M_{\mathcal{O}} = M_{\mathcal{O}} \cap \mathfrak{m}M$$

which follows from Lemma 4.27. \square

Remark 4.28. Let \underline{M} and \underline{N} be two A -motives over E , and let $M_{\mathcal{O}}$ and $N_{\mathcal{O}}$ be their respective integral models. While the maximal integral model of $\underline{M} \oplus \underline{N}$ is easily shown to be $M_{\mathcal{O}} \oplus N_{\mathcal{O}}$, it is not true in general that the maximal integral model of $\underline{M} \otimes \underline{N}$ is the image of $M_{\mathcal{O}} \otimes_{A \otimes \mathcal{O}_E} N_{\mathcal{O}}$ in $M \otimes_{A \otimes E} N$. To find a counterexample, we assume $q > 2$ and consider $\varpi \in \mathcal{O}_E$ a uniformizer. We consider the A -motive \underline{M} over E where $M = A \otimes E$ and where $\tau_M = \varpi \cdot \mathbf{1}$. The maximal integral model of \underline{M} is $M_{\mathcal{O}} = A \otimes \mathcal{O}_E$. However, $\underline{M}^{\otimes(q-1)}$ has $\varpi^{-1}M_{\mathcal{O}}^{\otimes(q-1)}$ for maximal integral model.

Comparison with Frobenius spaces

As in Section 4.1, let $E = k((\varpi))$ for a perfect field k containing \mathbb{F} , let $\mathcal{O}_E = k[[\varpi]]$ be its valuation ring and let $\mathfrak{p} = (\varpi)$ be the maximal ideal of \mathcal{O}_E . Let ℓ be a maximal ideal of A . Note that $\mathfrak{j}(A/\ell^n \otimes E) = A/\ell^n \otimes E$ for all positive integers n .

Let \underline{M} be an A -motive over E . We have canonical isomorphisms

$$\forall n \geq 1 : \quad M/\ell^n M \cong M[j^{-1}]/\ell^n M[j^{-1}]. \quad (4.5)$$

In particular, for all $n \geq 1$, τ_M defines an $A \otimes E$ -linear morphism $\tau^*(M/\ell^n M) \rightarrow M/\ell^n M$ through the composition

$$\tau^*(M/\ell^n M) \xrightarrow{\tau_M} M[j^{-1}]/\ell^n M[j^{-1}] \xrightarrow{(4.5)} M/\ell^n M$$

which we still denote by τ_M . The pair $(M/\ell^n M, \tau_M)$ defines a Frobenius space over E in the sense of Section 4.1. Let $L_n \subset M/\ell^n M$ be its maximal integral model.

Remark 4.29. In general, we cannot claim equality between $(M_{\mathcal{O}} + \ell^n M)/\ell^n M$ and L_n . Here is a counter-example.

Suppose that $A = \mathbb{F}[t]$, so that $A \otimes \mathcal{O}_E$ is identified with $\mathcal{O}_E[t]$, and let $\ell = (t)$. Let $\kappa : A \rightarrow \mathcal{O}_E$ be the \mathbb{F} -algebra morphism which maps t to ϖ . In this setting, \mathfrak{j} is the principal ideal of $\mathcal{O}_E[t]$ generated by $(t - \varpi)$. Consider the A -motive $\underline{M} := (E[t], f \cdot \mathbf{1})$ over E where $f = \varpi^{q-1} - \varpi^{q-2}t$. We claim that the maximal integral model of \underline{M} is $\mathcal{O}_E[t]$. Clearly, $\mathcal{O}_E[t]$ is an integral model for \underline{M} so that $\mathcal{O}_E[t] \subset M_{\mathcal{O}}$. Conversely, by [Qui2, Thm. 4], $M_{\mathcal{O}}$ is free of rank one over $\mathcal{O}_E[t]$. If h generates $M_{\mathcal{O}}$, there exists $b \in \mathcal{O}_E[t]$ such that $fh^{(1)} = bh$. For $p \in E[t]$, let $v(p)$ be the infimum of the valuations of the coefficients of p . We have

$$v(h) \geq -\frac{v(f)}{q-1} = -\frac{q-2}{q-1} > -1$$

and $h \in \mathcal{O}_E[t]$. We get $M_{\mathcal{O}} \subset \mathcal{O}_E[t]$.

On the other-hand, the Frobenius space $(M/\ell M, \tau_M)$ is isomorphic to the pair $(\mathcal{O}_E, \varpi^{q-1}\mathbf{1})$, whose maximal integral model is $\varpi^{-1}\mathcal{O}_E$, not \mathcal{O}_E .

If one wants to compare $M_{\mathcal{O}}$ with $(L_n)_{n \geq 1}$, then one wishes that $(M_{\mathcal{O}} + \ell^n M)/\ell^n M$ defines an integral model for $(M/\ell^n M, \tau_M)$ for all $n \geq 1$. This is the case in Remark 4.29, although it is not maximal, because the considered A -motive \underline{M} is effective. In general, this is not true². From now on, we assume

(C_{ℓ}) The ideal $\ell \subset A$ is such that $\kappa(\ell)$ contains a unit in \mathcal{O}_E , that is,

$$\kappa(\ell)\mathcal{O}_E = \mathcal{O}_E.$$

The above assumption ensures that $\mathfrak{j}(A/\ell^n \otimes \mathcal{O}_E) = A/\ell^n \otimes \mathcal{O}_E$ for all $n \geq 1$ (e.g. the proof Proposition 3.19), and thus that $(M_{\mathcal{O}} + \ell^n M)/\ell^n M$ is an integral model for $(M/\ell^n M, \tau_M)$.

Remark 4.30. Note that there always exists a maximal ideal ℓ in A satisfying (C_{ℓ}) : it suffices to take a maximal ideal ℓ in A coprime to $\kappa^{-1}(\mathfrak{p})$.

Even though we cannot claim always equality between $(M_{\mathcal{O}} + \ell^n M)/\ell^n M$ and L_n , the data of L_n for all $n \geq 1$ is enough to recover $M_{\mathcal{O}}$ as we show in the next theorem.

Theorem 4.31. *Let L_n be the maximal integral model of the Frobenius space $(M/\ell^n M, \tau_M)$. Let $m \in M$. Then $m \in M_{\mathcal{O}}$ if and only if $m + \ell^n M \in L_n$ for all large enough positive integers n .*

We start with some lemmas:

Lemma 4.32. *The \mathcal{O}_E -module L_n is an $A/\ell^n \otimes \mathcal{O}_E$ -module.*

²For instance, consider the t -motive $(E[t], (t - \varpi)^{-1}\mathbf{1})$ over E , whose maximal \mathcal{O}_E -model is $\mathcal{O}_E[t]$, together with $\ell = (t)$.

Proof. For an elementary tensor $r \otimes f$ in $A/\ell^n \otimes \mathcal{O}_E$, the \mathcal{O}_E -module $(r \otimes f)L_n$ is stable by τ_M . Indeed, we have $\tau_M(\tau^*(r \otimes f)L_n) = (r \otimes f^q)\tau_M(\tau^*L_n) \subset (r \otimes f)L_n$. By maximality of L_n , we have $(r \otimes f)L_n \subset L_n$. \square

Lemma 4.33. *Let r_n be the range of the \mathcal{O}_E -lattice $(M_{\mathcal{O}} + \ell^n M)/\ell^n M$ in $M/\ell^n M$. Then $(r_n)_{n \geq 1}$ is bounded.*

Proof. Note that $M_{\mathcal{O}}$ is a finite projective $A \otimes \mathcal{O}_E$ -module by Proposition 4.26. Let P be a finitely generated $A \otimes \mathcal{O}_E$ -module such that $N := M_{\mathcal{O}} \oplus P$ is free of finite rank. Let r' be the rank of N and let \mathbf{n} be a basis of N . Let also $\tau_N : \tau^*N[j^{-1}] \rightarrow N[j^{-1}]$ be the morphism $\tau_M \oplus 0$, and denote by

$$F_N = (b_{ij})_{ij} \in \mathcal{M}_{r'}(A \otimes \mathcal{O}_E[j^{-1}])$$

the matrix of τ_N written in the bases $\tau^*\mathbf{n}$ and \mathbf{n} .

For $n \geq 1$, let \mathbf{t}_n be a basis of A/ℓ^n over \mathbb{F} . For $i, j \in \{1, \dots, r'\}$, let B_{ij}^n be the matrix with coefficients in \mathcal{O}_E representing the multiplication by b_{ij} on $A/\ell^n \otimes \mathcal{O}_E$ in the basis $\mathbf{t}_n \otimes 1$. Then, the matrix of $\tau_N : \tau^*(N/\ell^n N) \rightarrow N/\ell^n N$, seen as an \mathcal{O}_E -linear map, and written in the bases $\tau^*(\mathbf{t}_n \otimes \mathbf{n})$ and $\mathbf{t}_n \otimes \mathbf{n}$, takes the form of the block matrix:

$$F_N^n := (B_{ij}^n)_{ij} \in \mathcal{M}_{r'd_n}(\mathcal{O}_E)$$

where d_n is the dimension of A/ℓ^n over \mathbb{F} . One verifies that $v(B_{ij}^n)$ equals the infimum of the valuation of the coefficients of $b_{ij} \pmod{\ell^n}$ in \mathcal{O}_E written in \mathbf{t}_n . Thus, for large values of n , we have

$$\forall i, j \in \{1, \dots, r'\} : v(B_{ij}^n) = v(b_{ij}) \quad (n \text{ large enough}). \quad (4.6)$$

For all $n \geq 1$, note that $(M_{\mathcal{O}} + \ell^n M)/\ell^n M \cong M_{\mathcal{O}}/\ell^n M_{\mathcal{O}}$ by Lemma 4.27. Because $M_{\mathcal{O}}/\ell^n M_{\mathcal{O}}$ is a direct factor in $N/\ell^n N$, the range of $(M_{\mathcal{O}} + \ell^n M)/\ell^n M$ equals the maximal valuation of the (nonzero) elementary divisors relative to the inclusion of \mathcal{O}_E -modules

$$\tau_N(\tau^*(N/\ell^n N)) \subset N/\ell^n N. \quad (4.7)$$

The elementary divisors relative to (4.7) coincide, up to units of \mathcal{O}_E , to those appearing in the Smith normal form of the matrix $F_N^n \in \mathcal{M}_{r'd_n}(\mathcal{O}_E)$.

By (4.6), the valuations of the coefficients of F_N^n are stationary. The range of $(M_{\mathcal{O}} + \ell^n M)/\ell^n M$ in $M/\ell^n M$ is thus stationary and hence bounded. \square

For $n \geq 1$, let \tilde{L}_n be the inverse image in M of $L_n \subset M/\ell^n M$.

Proof of Theorem 4.31. The statement is equivalent to the equality

$$M_{\mathcal{O}} = \bigcap_{n=D}^{\infty} (\tilde{L}_n + \ell^n M)$$

for all positive integer $D \geq 1$. The sequence of subsets $(\tilde{L}_n + \ell^n M)_{n \geq 1}$ decreases for the inclusion: for $n \geq 1$, we have $\tilde{L}_{n+1} + \ell^{n+1} M \subset \tilde{L}_{n+1} + \ell^n M$ and, because $(\tilde{L}_{n+1} + \ell^n M)/\ell^n M$ defines an integral model for $(M/\ell^n M, \tau_M)$, we also have $\tilde{L}_{n+1} + \ell^n M \subset \tilde{L}_n + \ell^n M$. Consequently, it suffices to treat the case $D = 1$.

Consider

$$L := \bigcap_{n=1}^{\infty} (\tilde{L}_n + \ell^n M).$$

By Lemma 4.32, L is an $A \otimes \mathcal{O}_E$ -module. The inclusion $M_{\mathcal{O}} \subset L$ follows from the fact that, for all n , $(M_{\mathcal{O}} + \ell^n M)/\ell^n M$ is an integral model for $(M/\ell^n M, \tau_M)$. To

prove the converse inclusion, we show that L is an integral model for \underline{M} . From $M_{\mathcal{O}} \subset L$, one deduces that L generates M over E . Because $\tau_M(\tau^*(\tilde{L}_n + \ell^n M)) \subset \tilde{L}_n + \ell^n M[j^{-1}]$, we also have $\tau_M(\tau^*L) \subset L[j^{-1}]$. The theorem follows once we have proved that L is finitely generated.

Assume that L is not finitely generated. From the Noetherianity of $A \otimes \mathcal{O}_E$, for all $s \geq 0$, it follows that $L \not\subset \varpi^{-s} M_{\mathcal{O}}$. Equivalently, there exists an unbounded increasing sequence $(s_n)_{n \geq 0}$ of non-negative integers such that $\varpi^{s_n} L_n \not\subset (M_{\mathcal{O}} + \ell^n M)/\ell^n M$. By Proposition 4.7, the range of $(M_{\mathcal{O}} + \ell^n M)/\ell^n M$ is $> s_n(q-1)$. But this contradicts Lemma 4.33. \square

We record two useful Corollaries from Theorem 4.31.

Corollary 4.34. *Let T_n be the maximal good model of the Frobenius space $(M/\ell^n M, \tau_M)$. Let $m \in M$. Then $m \in M_{\text{good}}$ if and only if $m + \ell^n M \in T_n$ for all large enough positive integers n .*

Proof. For $n \geq 1$, let \tilde{T}_n be the inverse image in M of $T_n \subset M/\ell^n M$. Let D be a positive integer. By Theorem 4.31, the sub- $A \otimes \mathcal{O}_E$ -module of M :

$$T := \bigcap_{n \geq D} (\tilde{T}_n + \ell^n M)$$

is a submodule of $M_{\mathcal{O}}$, hence is finitely generated. For all $n \geq 1$, we have $\tau_M(\tau^*(\tilde{L}_n + \ell^n M)[j^{-1}]) = (\tilde{L}_n + \ell^n M)[j^{-1}]$ so that T satisfies $\tau_M(\tau^*T)[j^{-1}] = T[j^{-1}]$. In particular, T is a good model for \underline{M} . It is further maximal: as $(M_{\text{good}} + \ell^n M)/\ell^n M \subset T_n$ for all n , we have $M_{\text{good}} \subset T$. \square

Corollary 4.35. *Let $\underline{N} = (N, \tau_N)$ be an A -motive over \mathcal{O}_E and \underline{N}_E its base change to E . Then, $N = (N_E)_{\mathcal{O}} = (N_E)_{\text{good}}$.*

For the next section, we shall not only be interested in how to recover $M_{\mathcal{O}}$ from L_n , but also in how to recover $M_{\mathcal{O}} + (\text{id} - \tau_M)(M)$. While we do not give a complete answer, we at least show next how to recover its ℓ -adic closure in $M[j^{-1}]$. We continue with some finer technicalities.

Even if we do not have equality between $\tilde{L}_n + \ell^n M$ and $M_{\mathcal{O}} + \ell^n M$, the former is a good approximation of the latter as we show next.

Lemma 4.36. *Let $n \geq 1$. The sequence $(\tilde{L}_m + \ell^m M)_{m \geq n}$ is decreasing for the inclusion, stationary and converges to $M_{\mathcal{O}} + \ell^n M$.*

Proof. Let $m \geq 1$. $(\tilde{L}_{m+1} + \ell^m M)/\ell^m M$ is an \mathcal{O}_E -lattice stable by τ_M in $M/\ell^m M$ so that $\tilde{L}_{m+1} + \ell^m M \subset \tilde{L}_m + \ell^m M$. If $m \geq n$, we have $\tilde{L}_{m+1} + \ell^m M \subset \tilde{L}_m + \ell^n M$ which shows that $(\tilde{L}_m + \ell^m M)_{m \geq n}$ decreases. Similarly, $M_{\mathcal{O}} + \ell^n M \subset \tilde{L}_m + \ell^n M$ for all $m \geq n$. Because the set of \mathcal{O}_E -lattices Λ such that $M_{\mathcal{O}} + \ell^n M \subseteq \Lambda \subseteq \tilde{L}_n + \ell^n M$ is finite, the sequence $(\tilde{L}_m + \ell^m M)_{m \geq n}$ is stationary. We denote by \mathcal{L}_n its limit. By Theorem 4.31, we have

$$\mathcal{L}_n = \bigcap_{m=n}^{\infty} (\tilde{L}_m + \ell^m M) = \bigcap_{m=n}^{\infty} (\tilde{L}_m + \ell^m M) + \ell^n M = M_{\mathcal{O}} + \ell^n M.$$

This concludes the proof. \square

Lemma 4.37. *There exists an unbounded and increasing sequence $(k_n)_{n \geq 1}$ of non-negative integers such that, $\tilde{L}_n + \ell^n M \subset M_{\mathcal{O}} + \ell^{k_n} M$ (typically, $k_n \leq n$ for all n).*

Proof. For $m \geq 1$, let I_m be the set of non-negative integers k such that $\tilde{L}_m + \ell^m M \subset M_{\mathcal{O}} + \ell^k M$. I_m is nonempty as it contains 0. I_m is further bounded: otherwise we would have

$$\tilde{L}_m + \ell^m M \subset \bigcap_k (M_{\mathcal{O}} + \ell^k M) = M_{\mathcal{O}} \quad (4.8)$$

which is impossible ($\tilde{L}_m + \ell^m M$ is an $A \otimes \mathcal{O}_E$ -module which is not of finite type). Hence I_m has a maximal element, which we denote by k_m . Because $\tilde{L}_{m+1} + \ell^{m+1} M \subset \tilde{L}_m + \ell^m M$, we have $k_{m+1} \geq k_m$. This shows that $(k_m)_{m \geq 1}$ increases. We show that it is unbounded. Let $n \geq 1$. By Lemma 4.36, there exists $m \geq n$ such that $M_{\mathcal{O}} + \ell^n = \tilde{L}_m + \ell^n M$. Thus $\tilde{L}_m + \ell^m M \subset M_{\mathcal{O}} + \ell^n M$. In particular, there exists $m \geq n$ such that $k_m \geq n$. \square

Proposition 4.38. *Assume that \underline{M} is effective, and let $m \in M$. The following are equivalent:*

- (i) m belongs to the ℓ -adic closure of $M_{\mathcal{O}} + (\text{id} - \tau_M)(M)$ in M ,
- (ii) for all $n \geq 1$, $m \in \tilde{L}_n + (\text{id} - \tau_M)(M) + \ell^n M$.

Proof. By Theorem 4.31, the inclusion

$$M_{\mathcal{O}} + (\text{id} - \tau_M)(M) \subset \bigcap_{n=1}^{\infty} \left[\tilde{L}_n + (\text{id} - \tau_M)(M) + \ell^n M \right]$$

holds as subsets of M , and the right-hand side is ℓ -adically complete. Hence (i) implies (ii). The converse follows from Lemma 4.37:

$$\bigcap_{n=1}^{\infty} \left[\tilde{L}_n + (\text{id} - \tau_M)(M) + \ell^n M \right] \subset \bigcap_{n=1}^{\infty} \left[M_{\mathcal{O}} + (\text{id} - \tau_M)(M) + \ell^{k_n} M \right]$$

where the right-hand side is identified with the ℓ -adic completion of $M_{\mathcal{O}} + (\text{id} - \tau_M)(M)$. \square

Néron-Ogg-Shafarevič-type criterion

This paragraph is an *aparté* offering a good reduction criterion, very much in the spirit of Gardeyn's Néron-Ogg-Shafarevič-type criterion [Gar1, Thm. 1.1]. The results of this subsection are not needed in the sequel, although they are useful in examples to compute maximal models.

Proposition 4.39. *Let \underline{M} be an A -motive over E , and let ℓ be a maximal ideal of A such that $\kappa(\ell)\mathcal{O}_E = \mathcal{O}_E$. The following statements are equivalent:*

- (i) There exists an A -motive \underline{N} over \mathcal{O}_E such that \underline{N}_E is isomorphic to \underline{M} .
- (ii) The inclusion $M_{\text{good}} \subset M_{\mathcal{O}}$ is an equality.
- (iii) The representation $T_{\ell}\underline{M}$ is unramified.

Proof. The equivalence between (i) and (ii) follows from Theorem 4.31 and Corollary 4.34. Let \underline{M}_n denote the Frobenius space $(M/\ell^n M, \tau_M)$ of the previous section, and let L_n and T_n be its maximal integral and good model respectively. The equivalence between (ii) and (iii) follows from the following sequence of equivalent statements:

$$\begin{aligned} T_{\ell}\underline{M} \text{ is unramified} &\iff \forall n \geq 1, T\underline{M}_n \text{ is unramified} \quad (\text{notations of Thm. 4.13}) \\ &\iff \forall n \geq 1, T_n = L_n \quad (\text{by Prop. 4.14}) \\ &\iff M_{\text{good}} = M_{\mathcal{O}} \quad (\text{by Thm. 4.31 and Cor. 4.34}). \end{aligned}$$

\square

Definition 4.40. We say that \underline{M} has *good reduction* if one of the equivalent points of Proposition 4.39 is satisfied.

4.3 Models of A -motives over a global function field

We go back to Definition 4.16, where now $S = F$ is a global function field that contains \mathbb{F} , and R is a sub- \mathbb{F} -algebra of F which, as a ring, is a Dedekind domain whose fraction field is F . Given a maximal ideal \mathfrak{p} in R , we denote by $R_{\mathfrak{p}}$ the completion of R at \mathfrak{p} and we let $F_{\mathfrak{p}}$ be the fraction field of $R_{\mathfrak{p}}$. Let $\kappa : A \rightarrow F$ be an \mathbb{F} -algebra morphism.

Let $\underline{M} = (M, \tau_M)$ be an A -motive over F of rank r , and let $\underline{M}_{\mathfrak{p}} = \underline{M}_{F_{\mathfrak{p}}}$ be the A -motive over $F_{\mathfrak{p}}$ of rank r obtained from \underline{M} by base change from F to $F_{\mathfrak{p}}$. We let $M_{R_{\mathfrak{p}}}$ denote the maximal $R_{\mathfrak{p}}$ -model of $\underline{M}_{\mathfrak{p}}$.

Proposition 4.41. *There exists a unique maximal R -model for \underline{M} . It equals the intersection $\bigcap_{\mathfrak{p}} (M \cap M_{R_{\mathfrak{p}}})$ for \mathfrak{p} running over the maximal ideals of R . We denote it M_R .*

Proof. From Lemma 4.17, uniqueness is automatic. Let N be an R -model for \underline{M} (whose existence is ensured by Proposition 4.18). For any maximal ideal \mathfrak{p} of R , we have $N \subset N \otimes_R R_{\mathfrak{p}} \subset M_{R_{\mathfrak{p}}}$ by maximality of $M_{R_{\mathfrak{p}}}$. Therefore $N \subset \bigcap_{\mathfrak{p}} (M \cap M_{R_{\mathfrak{p}}})$. Hence, it is sufficient to show that $\bigcap_{\mathfrak{p}} (M \cap M_{R_{\mathfrak{p}}})$ is an R -model. First note that it is a sub- $A \otimes R$ -module of M which, as it contains N , generates M over F . To show stability by τ_M , let e be a large enough integer such that $\tau_M(\tau^* M) \subset \mathfrak{j}^{-e} M$. One easily checks that:

$$\tau_M \left(\tau^* \bigcap_{\mathfrak{p}} (M \cap M_{R_{\mathfrak{p}}}) \right) \subset \bigcap_{\mathfrak{p}} \mathfrak{j}^{-e} (M \cap M_{R_{\mathfrak{p}}}) \subset \left(\bigcap_{\mathfrak{p}} M \cap M_{R_{\mathfrak{p}}} \right) [\mathfrak{j}^{-1}].$$

It remains to show that $\bigcap_{\mathfrak{p}} (M \cap M_{R_{\mathfrak{p}}})$ is finitely generated over $A \otimes R$. Let $\mathbf{m} := (m_1, \dots, m_r)$ and \mathfrak{a} be respectively a family of elements in $M \otimes_{A \otimes F} \text{Quot}(A \otimes F)$ and a nonzero ideal of $A \otimes F$ such that $M = (A \otimes F)m_1 \oplus \dots \oplus (A \otimes F)m_{r-1} \oplus \mathfrak{a}m_r$. Let $F_M \in \text{GL}_r(\text{Quot}(A \otimes F))$ be the matrix of τ_M written in $\tau^* \mathbf{m}$ and \mathbf{m} . By Proposition 4.24 and its proof, the $A \otimes R$ -module:

$$\left\{ a_1 m_1 + \dots + a_r m_r \mid \forall i, \forall \mathfrak{p} \in \text{Spm } R : a_i \in A \otimes F, v_{\mathfrak{p}}(a_i) \geq \frac{v_{\mathfrak{p}}(F_M^{-1})}{q-1} \right\}$$

contains $\bigcap_{\mathfrak{p}} (M \cap M_{R_{\mathfrak{p}}})$ and is finitely generated (compare with (4.4)). Because $A \otimes R$ is Noetherian, $\bigcap_{\mathfrak{p}} (M \cap M_{R_{\mathfrak{p}}})$ is finitely generated, and hence is the maximal model of \underline{M} . \square

From Lemma 4.20, we obtain:

Corollary 4.42. *The maximal good model M_{good} of \underline{M} exists and is unique.*

We now state the global version of Lemma 4.27 and Proposition 4.26 (with R in place of $R_{\mathfrak{p}}$). The argument is similar, so we omit proofs.

Proposition 4.43. *Both M_R and M_{good} are locally-free over $A \otimes R$.*

Remark 4.44. Note, however, that an integral model for \underline{M} , when not maximal, is not necessarily locally-free. For instance, the $\mathbb{F}[t]$ -motive $\mathbb{1} = (\mathbb{F}[t](\theta), \mathbf{1})$ over $\mathbb{F}(\theta)$ admits $L := t\mathbb{F}[t, \theta] + \theta\mathbb{F}[t, \theta]$ as $\mathbb{F}[\theta]$ -model. But it is well-known that L is not a flat $\mathbb{F}[t, \theta]$ -module. A short way to see this consists in considering the element $\Delta := (t \otimes \theta - \theta \otimes t) \in L \otimes_{\mathbb{F}[t, \theta]} L$. Δ is nonzero in $L \otimes_{\mathbb{F}[t, \theta]} L$, but

$$\theta \cdot \Delta = (\theta t) \otimes \theta - \theta \otimes (\theta t) = (\theta t) \otimes \theta - (\theta t) \otimes \theta = 0.$$

Then L is not flat because $L \otimes_{\mathbb{F}[t, \theta]} L$ has non trivial torsion.

Definition 4.45. We say that \underline{M} has *good reduction at \mathfrak{p}* if $\underline{M}_{\mathfrak{p}}$ has good reduction. We say that \underline{M} has *everywhere good reduction* if \underline{M} has good reduction at \mathfrak{p} for all maximal ideals \mathfrak{p} of R .

4.4 The integral part of A -motivic cohomology

Over local function fields

Let $F_{\mathfrak{p}}$ be a local function field with valuation ring $\mathcal{O}_{\mathfrak{p}}$ and maximal ideal \mathfrak{p} . Let $F_{\mathfrak{p}}^{\text{ur}}$ be the maximal unramified extension of $F_{\mathfrak{p}}$ in $F_{\mathfrak{p}}^s$. Let $I_{\mathfrak{p}}$ be the inertia subgroup of $G_{\mathfrak{p}} = G_{F_{\mathfrak{p}}}$. Let $\kappa : A \rightarrow \mathcal{O}_{\mathfrak{p}}$ be the characteristic morphism.

Let \underline{M} be an A -motive over $F_{\mathfrak{p}}$ and let $M_{\mathcal{O}_{\mathfrak{p}}}$ be its integral model.

Definition 4.46. We define $\text{Ext}_{\mathcal{O}_{\mathfrak{p}}}^1(\mathbb{1}, \underline{M})$ as the sub- A -module of $\text{Ext}_{\mathcal{M}_{F_{\mathfrak{p}}}}^1(\mathbb{1}, \underline{M})$ given by the image of $M_{\mathcal{O}_{\mathfrak{p}}}[j^{-1}]$ through ι (Theorem 2.13).

From Corollary 4.22, ι induces an isomorphism of A -modules:

$$\frac{M_{\mathcal{O}_{\mathfrak{p}}}[j^{-1}]}{(\text{id} - \tau_M)(M_{\mathcal{O}_{\mathfrak{p}}})} \xrightarrow{\sim} \text{Ext}_{\mathcal{O}_{\mathfrak{p}}}^1(\mathbb{1}, \underline{M}).$$

Remark 4.47. An important remark is that the assignment $\underline{M} \mapsto \text{Ext}_{\mathcal{O}_{\mathfrak{p}}}^1(\mathbb{1}, \underline{M})$ is functorial, thanks to Corollary 4.23.

Our main result is the following:

Theorem 4.48. *Let ℓ be a maximal ideal in A such that $\kappa(\ell)\mathcal{O}_{\mathfrak{p}} = \mathcal{O}_{\mathfrak{p}}$. Then, $\text{Ext}_{\mathcal{O}_{\mathfrak{p}}}^1(\mathbb{1}, \underline{M}) \subset \text{Ext}_{\text{good}}^1(\mathbb{1}, \underline{M})_{\ell}$.*

Proof. Let $[E] \in \text{Ext}_{\mathcal{O}_{\mathfrak{p}}}^1(\mathbb{1}, \underline{M})$. By definition, there exists $m \in M_{\mathcal{O}_{\mathfrak{p}}}[j^{-1}]$ such that $[E] = \iota(m)$. If \tilde{L}_n denotes a lift in M of the maximal integral model of the Frobenius space $(M/\ell^n M, \tau_M)$, we obtain $m \in \tilde{L}_n + \ell^n M$ for all n . By Proposition 4.15, there exists $y_n \in M \otimes_{F_{\mathfrak{p}}} F_{\mathfrak{p}}^{\text{ur}}$ such that

$$m \equiv y_n - \tau_M(\tau^* y_n) \pmod{\ell^n}. \quad (4.9)$$

Note that, for each n , there are only finitely many such $y_n \pmod{\ell^n}$. We next show that we can choose *compatibly* y_n for all n (that is $y_{n+1} \equiv y_n \pmod{\ell^n}$). Let us define a tree T indexed by $n \geq 1$ whose nodes at the height n are the solutions y_n of (4.9) in $(M \otimes_{F_{\mathfrak{p}}} F_{\mathfrak{p}}^{\text{ur}})/\ell^n(M \otimes_{F_{\mathfrak{p}}} F_{\mathfrak{p}}^{\text{ur}})$. There is an edge between z_n and z_{n+1} if and only if z_{n+1} coincides with z_n modulo ℓ^n . The tree has finitely many nodes at each height and it is infinite from the fact that a solution of (4.9) exists for all n . By König's Lemma, there exists an infinite branch on T . This branch corresponds to a converging sequence $(y_n)_{n \geq 1}$ whose limit y in $(M \hat{\otimes}_{F_{\mathfrak{p}}} F_{\mathfrak{p}}^{\text{ur}})_{\ell}$ satisfies $m = y - \tau_M(\tau^* y)$. Therefore, we conclude that $[E] \in \text{Ext}_{\text{good}}^1(\mathbb{1}, \underline{M})_{\ell}$ thanks to Proposition 2.23. \square

Over global function fields

Let F be a finite field extension of K and let \mathcal{O}_F be the integral closure of A in F . We let $\kappa : A \rightarrow \mathcal{O}_F$ denote the inclusion. We fix S to be a set of nonzero prime ideals of \mathcal{O}_F and consider the subring $R := \mathcal{O}_F[S^{-1}]$ of F . The ring R is a Dedekind domain whose fraction field is F .

Let $\underline{M} = (M, \tau_M)$ be an Anderson A -motive over F . Given a maximal ideal $\mathfrak{p} \subset R$, we let $\underline{M}_{\mathfrak{p}}$ be the A -motive over $F_{\mathfrak{p}}$ obtained from \underline{M} by base-change from F to $F_{\mathfrak{p}}$. Given an extension $[E] \in \text{Ext}_{\mathcal{M}_F}^1(\mathbb{1}, \underline{M})$, the exactness of the base

change functor defines an extension $[\underline{E}_{\mathfrak{p}}] \in \text{Ext}_{\mathcal{M}_{F_{\mathfrak{p}}}}^1(\mathbb{1}_{\mathfrak{p}}, \underline{M}_{\mathfrak{p}})$. This allows us to define the following submodule of $\text{Ext}_{\mathcal{M}_F}^1(\mathbb{1}, \underline{M})$:

$$\text{Ext}_R^1(\mathbb{1}, \underline{M}) = \bigcap_{\substack{\mathfrak{p} \subset R \\ \text{maximal}}} \left\{ [\underline{E}] \in \text{Ext}_{\mathcal{M}_F}^1(\mathbb{1}, \underline{M}) \mid [\underline{E}_{\mathfrak{p}}] \in \text{Ext}_{R_{\mathfrak{p}}}^1(\mathbb{1}_{\mathfrak{p}}, \underline{M}_{\mathfrak{p}}) \right\}.$$

Our second main result consists of the next theorem.

Theorem 4.49. *Let M_R denote the maximal integral R -model of \underline{M} . The A -module $\text{Ext}_R^1(\mathbb{1}, \underline{M})$ equals the image of $M_R[j^{-1}]$ through ι . In addition, ι induces a natural isomorphism of A -modules:*

$$\frac{M_R[j^{-1}]}{(\text{id} - \tau_M)(M_R)} \xrightarrow{\sim} \text{Ext}_R^1(\mathbb{1}, \underline{M}).$$

The proof of the above theorem will result after a sequence of lemmas.

Lemma 4.50. *Let $M_{R_{\mathfrak{p}}}$ be the maximal integral model of $\underline{M}_{\mathfrak{p}} = (M_{\mathfrak{p}}, \tau_M)$. Inside $M[j^{-1}]$, we have:*

$$M[j^{-1}] \cap (M_{R_{\mathfrak{p}}}[j^{-1}] + (\text{id} - \tau_M)(M_{\mathfrak{p}})) = M[j^{-1}] \cap M_{R_{\mathfrak{p}}}[j^{-1}] + (\text{id} - \tau_M)(M).$$

Proof. The inclusion \supset is clear. Since M is generated over F by elements in $M \cap M_{R_{\mathfrak{p}}}$ and as $F_{\mathfrak{p}} = F + R_{\mathfrak{p}}$, we have $M_{\mathfrak{p}} = M + M_{R_{\mathfrak{p}}}$. Let m be an element in the left-hand side. We can write m as $m_{\mathfrak{p}} + n_{\mathfrak{p}} - \tau_M(\tau^* n_{\mathfrak{p}}) + n - \tau_M(\tau^* n)$ where $m_{\mathfrak{p}} \in M_{R_{\mathfrak{p}}}[j^{-1}]$, $n_{\mathfrak{p}} \in M_{R_{\mathfrak{p}}}$ and $n \in M$. In particular, $m_{\mathfrak{p}} + n_{\mathfrak{p}} - \tau_M(\tau^* n_{\mathfrak{p}})$ belongs to $M[j^{-1}] \cap M_{R_{\mathfrak{p}}}[j^{-1}]$ which implies that $m \in M[j^{-1}] \cap M_{R_{\mathfrak{p}}}[j^{-1}] + (\text{id} - \tau_M)(M)$. \square

Lemma 4.51. *Let $m \in M$. Then $m \in M_{R_{\mathfrak{p}}}$ for almost all maximal ideals \mathfrak{p} of R .*

Proof. There exists a nonzero element $d \in R$ such that $dm \in M_R$. Let $\{\mathfrak{q}_1, \dots, \mathfrak{q}_s\}$ be the finite set of maximal ideals in R that contain (d) . By Proposition 4.41, $m \in M_{R_{\mathfrak{p}}}$ for all \mathfrak{p} not in $\{\mathfrak{q}_1, \dots, \mathfrak{q}_s\}$. \square

Let N be a finite dimensional vector space over F (resp. $F_{\mathfrak{p}}$). By a *lattice* in N we mean a finitely generated module over R (resp. $R_{\mathfrak{p}}$) in N that contains a basis of N .

Lemma 4.52 (Strong approximation). *Let N be a finite dimensional F -vector space and, for all maximal ideals \mathfrak{p} of R , let $N_{R_{\mathfrak{p}}}$ be an $R_{\mathfrak{p}}$ -lattice in $N_{\mathfrak{p}} := N \otimes_R F_{\mathfrak{p}}$ such that the intersection $\bigcap_{\mathfrak{p}} (N \cap N_{R_{\mathfrak{p}}})$, over all maximal ideals \mathfrak{p} of R , is an R -lattice in N . Let T be a finite set of maximal ideals in R and, for $\mathfrak{q} \in T$, let $n_{\mathfrak{q}} \in N_{\mathfrak{q}}$. Then, there exists $n \in N$ such that $n - n_{\mathfrak{q}} \in N_{R_{\mathfrak{q}}}$ for all $\mathfrak{q} \in T$ and $n \in N_{R_{\mathfrak{p}}}$ for all \mathfrak{p} not in T .*

Proof. Let N_R denote the intersection $\bigcap_{\mathfrak{p}} (N \cap N_{R_{\mathfrak{p}}})$ over all maximal ideals \mathfrak{p} of R . By the structure Theorem for finitely generated modules over the Dedekind domain R , there exists a nonzero ideal $\mathfrak{a} \subset R$ and elements $\{b_1, \dots, b_r\} \subset M$ such that

$$N_R = Rb_1 \oplus \dots \oplus Rb_{r-1} \oplus \mathfrak{a}b_r.$$

Because $N_R \otimes_R R_{\mathfrak{p}} \subset N_{R_{\mathfrak{p}}}$ for $\mathfrak{p} \subset R$, we have $R_{\mathfrak{p}}b_1 \oplus \dots \oplus \mathfrak{p}^{v_{\mathfrak{p}}(\mathfrak{a})} R_{\mathfrak{p}}b_r \subset N_{R_{\mathfrak{p}}}$. For $\mathfrak{q} \in T$, let us write $n_{\mathfrak{q}} = \sum_i f_{\mathfrak{q},i} b_i$ with $f_{\mathfrak{q},i} \in F_{\mathfrak{q}}$. By the strong approximation Theorem [Ros, Thm. 6.13], for all $i \in \{1, \dots, r\}$, there exists $f_i \in F$ such that

1. for $\mathfrak{q} \in T$ and $i \in \{1, \dots, r-1\}$, $v_{\mathfrak{q}}(f_i - f_{\mathfrak{q},i}) \geq 0$,

2. for $\mathfrak{q} \in T$, $v_{\mathfrak{q}}(f_r - f_{\mathfrak{q},r}) \geq v_{\mathfrak{q}}(\mathfrak{a})$,
3. for $\mathfrak{p} \notin T$ and $i \in \{1, \dots, r-1\}$, $v_{\mathfrak{p}}(f_i) \geq 0$,
4. for $\mathfrak{p} \notin T$, $v_{\mathfrak{p}}(f_r) \geq v_{\mathfrak{p}}(\mathfrak{a})$.

The element $n = \sum_i f_i b_i \in N$ satisfies the assumption of the lemma. \square

Lemma 4.53. *We have*

$$\bigcap_{\mathfrak{p} \subset R} (M[j^{-1}] \cap M_{R_{\mathfrak{p}}}[j^{-1}] + (\text{id} - \tau_M)(M)) = M_R[j^{-1}] + (\text{id} - \tau_M)(M)$$

where the intersection is indexed over the maximal ideals of R .

Proof. The inclusion \supset follows from Proposition 4.41. Conversely, let m be an element of $\bigcap_{\mathfrak{p} \subset R} (M[j^{-1}] \cap M_{R_{\mathfrak{p}}}[j^{-1}] + (\text{id} - \tau_M)(M))$. By Lemma 4.51, there exists a finite subset T of maximal ideals of R such that $m \in M_{R_{\mathfrak{p}}}[j^{-1}]$ for $\mathfrak{p} \notin T$. For $\mathfrak{q} \in T$, there exists $n_{\mathfrak{q}} \in M$ and $m_{\mathfrak{q}} \in M[j^{-1}] \cap M_{R_{\mathfrak{q}}}[j^{-1}]$ such that $m = m_{\mathfrak{q}} + n_{\mathfrak{q}} - \tau_M(\tau^* n_{\mathfrak{q}})$.

Let N be a finite dimensional sub- F -vector space of M that contains m and $n_{\mathfrak{q}}$ for all $\mathfrak{q} \in T$. For a maximal ideal \mathfrak{p} of R , let $N_{R_{\mathfrak{p}}} := M_{R_{\mathfrak{p}}} \cap (N \otimes_F F_{\mathfrak{p}})$. We have $N_R := \bigcap_{\mathfrak{p}} (N \cap N_{R_{\mathfrak{p}}}) = N \cap M_R$. The latter is an R -lattice in N and hence we are in the situation of Lemma 4.52: there exists $n \in N$ such that $n - n_{\mathfrak{q}} \in N_{R_{\mathfrak{q}}}$ for all $\mathfrak{q} \in T$ and $n \in N_{R_{\mathfrak{p}}}$ for all \mathfrak{p} not in T . Then $m + n - \tau_M(\tau^* n) \in N_R \subset M_R$, which ends the proof. \square

Proof of Theorem 4.49. Let $[\underline{E}] \in \text{Ext}_{\mathcal{M}_F}^1(\mathbb{1}, \underline{M})$ and let $m \in M[j^{-1}]$ be such that $[\underline{E}] = \iota(m)$. The proof of Theorem 4.49 is achieved via the sequence of equivalences:

$$\begin{aligned} [\underline{E}] \in \text{Ext}_R^1(\mathbb{1}, \underline{M}) &\iff \forall \mathfrak{p} \in \text{Spm } R : [\underline{E}_{\mathfrak{p}}] \in \text{Ext}_{R_{\mathfrak{p}}}^1(\mathbb{1}_{\mathfrak{p}}, \underline{M}_{\mathfrak{p}}) \\ &\iff \forall \mathfrak{p} \in \text{Spm } R : m \in M[j^{-1}] \cap [M_{R_{\mathfrak{p}}}[j^{-1}] + (\text{id} - \tau_M)(M_{\mathfrak{p}})] \\ &\iff \forall \mathfrak{p} \in \text{Spm } R : m \in M[j^{-1}] \cap M_{R_{\mathfrak{p}}}[j^{-1}] + (\text{id} - \tau_M)(M) \\ &\iff m \in M_R[j^{-1}] + (\text{id} - \tau_M)(M) \\ &\iff [\underline{E}] \in \iota(M_R[j^{-1}]) \end{aligned}$$

where the second equivalence stems from Definition 4.16, the third from Lemma 4.50 and the fourth from Lemma 4.53. The second assertion follows from Corollary 4.22. \square

5 Regulated extensions

Let $F_{\mathfrak{p}}$ be a local function field with valuation ring $\mathcal{O}_{\mathfrak{p}}$ and maximal ideal \mathfrak{p} . Let $\kappa : A \rightarrow \mathcal{O}_{\mathfrak{p}}$ be the characteristic morphism, and consider an A -motive \underline{M} over $\mathcal{O}_{\mathfrak{p}}$. In the previous section, we proved that for a maximal ideal ℓ in A satisfying $\kappa(\ell)\mathcal{O}_{\mathfrak{p}} = \mathcal{O}_{\mathfrak{p}}$, there is an inclusion:

$$\text{Ext}_{\mathcal{O}_{\mathfrak{p}}}^1(\mathbb{1}, \underline{M}) \subset \text{Ext}_{\text{good}}^1(\mathbb{1}, \underline{M}) \quad (5.1)$$

as sub- A -modules of $\text{Ext}_{\mathcal{M}_{F_{\mathfrak{p}}}}^1(\mathbb{1}, \underline{M})$. Surprisingly, this is almost never an equality. In Subsection 5.1, we construct explicitly a class in the right-hand side of (5.1) which does not belong to the left-hand side. In the remaining part of this text, we offer a conjectural framework which we expect to solve the default of (5.1) to be an equality.

5.1 A particular extension of $\mathbb{1}$ by itself

We consider the case where $A = \mathbb{F}[t]$ and consider the maximal ideal $\ell = (t)$ in A . Let E be the local function field $\mathbb{F}((\pi))$, $\mathcal{O} = \mathbb{F}[[\pi]]$, with structure morphism $\kappa : A \rightarrow \mathcal{O}$ defined by $\kappa(t) = 1 + \pi$ (that is, $\pi = \theta - 1$ where $\theta = \kappa(t)$). We have $\kappa(\ell)\mathcal{O} = (1 + \pi)\mathbb{F}[[\pi]] = \mathcal{O}$ so $\kappa(\ell)\mathcal{O} = \mathcal{O}$. Let $\underline{M} = \mathbb{1}$ over E .

By Proposition 2.21 enriched with Proposition 4.15, there is a commutative square of A -modules:

$$\begin{array}{ccc} \text{Ext}_{\mathcal{M}_E}^1(\mathbb{1}, \mathbb{1}) & \longrightarrow & H^1(I_E, T_{(t)}\mathbb{1}) \\ \uparrow \iota & & \uparrow \iota \\ \frac{E[t, \frac{1}{t-\theta}]}{(\text{id} - \tau)(E[t])} & \longrightarrow & \frac{E^{\text{ur}}[[t]]}{\mathcal{O}[[t]] + (\text{id} - \tau)(E[[t]])} \end{array}$$

where the bottom arrow is induced by the inclusion of $E[t, (t - \theta)^{-1}]$ in $E[[t]]$.

Hereafter, we construct an element m in $E[t, (t - \theta)^{-1}]$ of the form

$$m = \frac{m_k}{(t - \theta)^k} + \cdots + \frac{m_1}{(t - \theta)}$$

for some m_1, \dots, m_k in E , not all in \mathcal{O} , such that m belongs to $(\text{id} - \tau)(E[[t]])$. Then, $\iota(m)$ has good reduction with respect to (t) (in the sense of Definition 2.22) but does not belong to $\mathcal{O}[t, (t - \theta)^{-1}] + (\text{id} - \tau)(E[t])$.

Let $k = q^2$ where q is the number of elements of \mathbb{F} , and for $i \in \{0, \dots, k - 1\}$, define n'_i as $\theta^i(\pi^{-1} - \pi^{-q})$ in E . For all $c \geq 0$, let f_{ck} be a root in E of the polynomial:

$$X^q - X + \theta^{-ck} n'_0.$$

Such a root exists in E as $\theta^{-k} \equiv 1 \pmod{\pi^{q^2}}$. For $l \geq 0$, we define $f_l \in E$ by $f_l := f_{ck}$ where $l = ck + r$ for $c \geq 0$ and $0 \leq r < k$. Then, we have

$$\forall l \geq 0: \quad \theta^{-l} n'_l = f_l - f_l^q \tag{5.2}$$

where $\bar{l} \in \{0, \dots, k - 1\}$ denotes the rest of the euclidean division of l by k .

For $l \geq 0$, let $S_k(l)$ be the *Pascal matrix* whose i th row- j th column entry is the binomial coefficient $\binom{i+j+l}{i+l}$ ($0 \leq i, j < k$). The following claims are easily proven:

- (i) The determinant of $S_k(0)$ is 1.
- (ii) Let p be the characteristic of \mathbb{F} . For $l \geq 0$, we have the formula

$$S_k(l+1) \equiv \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \\ 1 & & & & \end{pmatrix} S_k(l) \pmod{p}$$

- (iii) The application $l \mapsto S_k(l)$ is k -periodic modulo p .

We now define $m'_i \in E$ for $i \in \{0, \dots, k-1\}$ by mean of the formula:

$$S_k(0) \begin{pmatrix} m'_0 \\ m'_1 \\ \vdots \\ m'_{k-1} \end{pmatrix} = \begin{pmatrix} n'_0 \\ n'_1 \\ \vdots \\ n'_{k-1} \end{pmatrix}$$

Since the n_i 's have negative valuation, at least one of the m_i 's has negative valuation (by (i)). From (ii), we have

$$\forall l \geq 0 : S_k(l) \begin{pmatrix} m'_0 \\ m'_1 \\ \vdots \\ m'_{k-1} \end{pmatrix} = \begin{pmatrix} n'_l \\ n'_{l+1} \\ \vdots \\ n'_{l+k-1} \end{pmatrix}$$

From (5.2), we obtain:

$$\forall l \geq 0 : \theta^{-l} \left(\sum_{i=0}^{k-1} m'_i \binom{i+l}{l} \right) = f_l - f_l^q. \quad (5.3)$$

Finally, for $i \in \{1, \dots, k\}$, let $m_i := (-\theta)^i m'_{i-1}$. Formula (5.3) amounts to:

$$m := \frac{m_k}{(t-\theta)^k} + \dots + \frac{m_1}{(t-\theta)} = f - f^{(1)}$$

where $f := \sum_{l \geq 0} f_l t^l$. Therefore $\iota(m)$ has good reduction, although m does not belong to $\mathcal{O}[t, (t-\theta)^{-1}] + (\text{id} - \tau)(E[t])$.

5.2 Hodge polygons of A -motives

We recognize in the extension $\iota(m)$, constructed in the previous subsection, that m has a *large pole* at $j = (t-\theta)$. We now introduce the notion of *regulated extensions* which naturally prevent the *poles of extensions* of being *too large*.

We first recall some materials on Hodge-Pink structures. Let $F_{\mathfrak{p}}$ be a local function field with valuation ring $\mathcal{O}_{\mathfrak{p}}$ and maximal ideal \mathfrak{p} . Let $\kappa : A \rightarrow F_{\mathfrak{p}}$ be the characteristic morphism (we do not require it to have values in $\mathcal{O}_{\mathfrak{p}}$). We denote by $F_{\mathfrak{p}}[[j]]$ the discrete valuation ring obtained by taking the completion of $A \otimes F_{\mathfrak{p}}$ along powers of j . We denote by $F_{\mathfrak{p}}((j))$ its field of fractions.

Let \mathfrak{m} be the maximal ideal of A given by $\kappa^{-1}(\mathfrak{p})$, and denote by $K_{\mathfrak{m}}$ the local field of (C, \mathcal{O}_C) at \mathfrak{m} . We have the following:

Lemma 5.1. *The ring morphism $\nu : A \mapsto A \otimes F_{\mathfrak{p}}$, $a \mapsto a \otimes 1$ extends uniquely to a ring morphism $\nu : K_{\mathfrak{m}} \rightarrow F_{\mathfrak{p}}[[j]]$ such that the composition of ν followed by reduction mod j coincide with the canonical inclusion $K_{\mathfrak{m}} \hookrightarrow F_{\mathfrak{p}}[[j]]/j \cong F_{\mathfrak{p}}$.*

Proof. Uniqueness is clear. We proceed in three steps for the existence. The first step is to extend ν to K . Let $a \in A$. We have $a \otimes 1 \cong 1 \otimes a \pmod{j}$. Additionally, $F_{\mathfrak{p}}[[j]]$ is a discrete valuation ring with maximal ideal j and residue field $F_{\mathfrak{p}}$. Hence, if a is nonzero, $a \otimes 1$ is invertible because a is invertible in $F_{\mathfrak{p}}$. This extends ν to K .

Let $\pi_{\mathfrak{m}} \in K$ be a uniformizing parameter for $K_{\mathfrak{m}}$ and let $a, b \neq 0$ be elements of A such that $\pi_{\mathfrak{m}} = a/b$. We have the identification $K_{\mathfrak{m}} = \mathbb{F}_{\mathfrak{m}}((\pi_{\mathfrak{m}}))$ where $\mathbb{F}_{\mathfrak{m}}$ is the residue field of $K_{\mathfrak{m}}$. Let L be the subfield $L := \mathbb{F}((\pi_{\mathfrak{m}}))$ of $K_{\mathfrak{m}}$. Our second

step is to extend ν to L . Following Pink's observation [Pin, Prop. 3.1], we unfold the formal computation

$$\begin{aligned} L \ni \sum_k (f_k \pi_m^k \otimes 1) &= \sum_k f_k (1 \otimes \pi_m + \pi_m \otimes 1 - 1 \otimes \pi_m)^k \\ &= \sum_k f_k \left(1 \otimes \pi_m + \frac{a \otimes b - b \otimes a}{b \otimes b} \right)^k \\ &= \sum_k f_k \sum_{\ell \geq 0} \binom{k}{\ell} \left(\frac{a \otimes b - b \otimes a}{b \otimes b} \right)^\ell (1 \otimes \pi_m)^{k-\ell} \\ &= \sum_{\ell \geq 0} \left(1 \otimes \sum_k f_k \binom{k}{\ell} \pi_m^{k-\ell} \right) \left(\frac{a \otimes b - b \otimes a}{b \otimes b} \right)^\ell \end{aligned}$$

where the inner sum converges in K_m . We then set

$$\nu \left(\sum_k f_k \pi_m^k \right) := \sum_{\ell \geq 0} \left(1 \otimes \sum_k f_k \binom{k}{\ell} \pi_m^{k-\ell} \right) \left(\frac{a \otimes b - b \otimes a}{b \otimes b} \right)^\ell \in F_p[[j]].$$

It is formal to check that this defines a ring homomorphism $L \rightarrow F_p[[j]]$ which extends ν .

Finally, we extend ν to K_m . Let $\alpha \in K_m$ and let $p_\alpha(X)$ be the minimal polynomial of α over L . As K_m/L is a separable extension, $p_\alpha(X) \in L[X]$ is separable. We consider $p_\alpha(X)$ as a polynomial in $F_p[[j]][X]$ via $\nu : L \rightarrow F_p[[j]]$. It admits the image of α through $K_m \hookrightarrow F_p \cong F_p[[j]]/j$ as a root modulo j . By Hensel's Lemma, $p_\alpha(X)$ admits a unique root $\tilde{\alpha}$ in $F_p[[j]]$ which lifts α . Setting $\nu(\alpha) = \tilde{\alpha}$ extends ν to $K_m \rightarrow F_p[[j]]$ in a morphism which verifies the assumption of the lemma. \square

Definition 5.2. A \mathfrak{p} -adic Hodge-Pink structure \hat{H} is a pair (H, \mathfrak{q}_H) where H is a K_m -vector space and \mathfrak{q}_H is an $F_p[[j]]$ -lattice inside $H \otimes_{K_m, \nu} F_p((j))$. We call $\mathfrak{p}_H := H \otimes_{K_m, \nu} F_p[[j]]$ the *tautological lattice* and \mathfrak{q}_H the Hodge-Pink lattice.

A \mathfrak{p} -adic Hodge-Pink structure induces a (separated and exhaustive) decreasing filtration Fil on the F_p -vector space $H_{F_p} := H \otimes_{K_m} F_p$ as follows. For $p \in \mathbb{Z}$, $\text{Fil}^p H_{F_p}$ is defined as the image of $\mathfrak{p}_H \cap j^p \mathfrak{q}_H$ through

$$\mathfrak{p}_H = H \otimes_{K_m, \nu} F_p[[j]] \xrightarrow{(\text{mod } j)} H \otimes_{K_m} F_p = H_{F_p}.$$

The *Hodge polygon* of \hat{H} is defined as the Hodge polygon of $\text{Fil} = (\text{Fil}^p H_{F_p})_{p \in \mathbb{Z}}$.

Let \underline{M} be an A -motive over K_m . We attach to \underline{M} a \mathfrak{p} -adic Hodge-Pink structure $\hat{H} := \hat{H}_{\mathfrak{p}}(\underline{M})$ as follows. Its underlying K_m -vector space is $H := (\tau^* M)/j(\tau^* M)$ and its Hodge-Pink lattice is $\mathfrak{q}_H := \tau_M^{-1}(M) \otimes_{A \otimes K_m, \nu \otimes \text{id}} F_p[[j]]$.

Lemma 5.3. *Let \underline{M} be an A -motive over K and let F be a finite extension of K . For a finite place \mathfrak{p} of F , let $\mathfrak{m} = \mathfrak{p} \cap A$, and let $\underline{M}_{\mathfrak{m}}$ be the A -motive over K_m obtained from \underline{M} by base change. Then, the Hodge polygon of the \mathfrak{p} -adic Hodge-Pink structure $\hat{H}_{\mathfrak{p}}(\underline{M}_{\mathfrak{m}})$ does not depend on \mathfrak{p} .*

Proof. Given e a large enough integer for which $j^e \tau_M(\tau^* M) \subset M$, we have

$$M/j^e \tau_M(\tau^* M) \cong \bigoplus_{i=1}^r (A \otimes K)/j^{e+w_i}$$

for some integers $w_1 \leq \dots \leq w_r$ independent of e nor of the isomorphism. If $(\text{Fil}^p H_{F_p})_p$ is the induced Hodge filtration of $\hat{H}_p(M_m)$, we have

$$\forall p \in \mathbb{Z}, \quad \dim_{F_p}(\text{Fil}^p H_{F_p}) = \#\{i \in \{1, \dots, r\} \mid w_i \geq p\}.$$

In particular, the associated Hodge polygon only depends on the numbers $w_1 \leq \dots \leq w_r$. \square

Remark 5.4. The fact that the Hodge polygon is independent on the place also have strong connections with expectations in p -adic Hodge theory. Following [Har1, §2.9], Fontaine's \mathbb{Q}_p -algebra \mathbf{B}_{dR}^+ would be our analogue of $F_p[[j]]$ seen as a K_m -algebra via ν . The independence of p in the number fields setting would follow from p -adic comparison theorems: for X a variety over \mathbb{Q} , the independence of the Hodge-Tate polygon would be a consequence of Fontaine's conjecture, stating that there is an isomorphism of \mathbf{B}_{dR} -vector spaces:

$$\mathbf{B}_{\text{dR}} \otimes_{\mathbb{Q}_p} H_{\text{ét}}^i(X_{\mathbb{Q}_p}, \mathbb{Q}_p) \xrightarrow{\sim} \mathbf{B}_{\text{dR}} \otimes_{\mathbb{Q}} H_{\text{dR}}^i(X)$$

given naturally in X , and compatible with the filtrations on both sides.

Therefore, the next notion is well-defined.

Definition 5.5. Let L be one of the fields K_m, K, F a finite extension of K or F_p for p above m , and let \underline{M} be an A -motive over L . Depending on L , we define successively the *Hodge polygon* of \underline{M} to be:

- (a) if $L = K_m$, the Hodge polygon of $\hat{H}_p(\underline{M})$,
- (b) if $L = F_p$, the Hodge polygon of $\hat{H}_p(\text{Res}_{F_p/K_m} \underline{M})$,
- (c) if $L = K$, the Hodge polygon of $\hat{H}_p(\underline{M}_p)$ for some finite place p of F ,
- (d) if $L = F$, the Hodge polygon of $\text{Res}_{F/K}(\underline{M})$.

Remark 5.6. In the classical situation, it is expected that given an exact sequence $0 \rightarrow N \rightarrow M \rightarrow P \rightarrow 0$ of mixed motives over \mathbb{Q} , the Hodge polygon of $N \oplus P$ coincides with that of M . It is not true in our situation and this motivates what comes next.

Definition 5.7. We call an exact sequence $0 \rightarrow \underline{N} \rightarrow \underline{M} \rightarrow \underline{P} \rightarrow 0$ in \mathcal{M}_L *regulated* if the Hodge polygon of $\underline{N} \oplus \underline{P}$ coincide with that of \underline{M} .

Remark 5.8. We have chosen the naming *regulated* to allude to the notion of *regulators of A -motives*, to be introduced in an upcoming work.

The next proposition allows to compute regulated extensions in the category \mathcal{M}_L .

Proposition 5.9. *Let \underline{M} and \underline{N} be two objects in \mathcal{M}_L . Denote by $[\underline{E}]$ the extension of \underline{M} by \underline{N} given by $\iota_{\underline{N}, \underline{M}}(u)$ where $u \in \text{Hom}_{A \otimes L}(\tau^* \underline{N}, \underline{M})[j^{-1}]$. Then, $[\underline{E}]$ is regulated if and only if there exists $f : \text{Hom}_{A \otimes L}(\tau^* \underline{N}, \tau^* \underline{M})$ and $g \in \text{Hom}_{A \otimes L}(\underline{N}, \underline{M})$ such that $u = g \circ \tau_N - \tau_M \circ f$.*

Proof. The A -motives \underline{E} and $\underline{M} \oplus \underline{N}$ have the same Hodge-Polygons if and only if for all large enough integer e , the j -torsion $A \otimes L$ -modules

$$M \oplus N \Big/_{j^e} \begin{pmatrix} \tau_M & 0 \\ 0 & \tau_N \end{pmatrix} (\tau^* M) \oplus (\tau^* N), \quad M \oplus N \Big/_{j^e} \begin{pmatrix} \tau_M & u \\ 0 & \tau_N \end{pmatrix} (\tau^* M) \oplus (\tau^* N)$$

are isomorphic. This is the case if and only if there exists $F \in \text{Aut}_{A \otimes L}(\tau^* M \oplus \tau^* N)$ and $G \in \text{Aut}_{A \otimes L}(M \oplus N)$ such that $G \begin{pmatrix} \tau_M & 0 \\ 0 & \tau_N \end{pmatrix} = \begin{pmatrix} \tau_M & u \\ 0 & \tau_N \end{pmatrix} F$. By identifying G with $\begin{pmatrix} a_0 & g' \\ 0 & b_0 \end{pmatrix}$ and F with $\begin{pmatrix} a_1 & f' \\ 0 & b_1 \end{pmatrix}$ for some automorphisms a_0, b_0, a_1, b_1

of M , N , τ^*M and τ^*N respectively, and $A \otimes L$ -linear morphisms $g' : N \rightarrow M$ and $f' : \tau^*N \rightarrow \tau^*M$, the proposition follows by setting $g := g' \circ b_0^{-1}$ and $f := f' \circ b_1^{-1}$. \square

It follows from the above proposition that if $[\underline{E}]$ and $[\underline{E}']$ are equivalent extensions, then $[\underline{E}]$ is regulated if and only if so is $[\underline{E}']$. In particular, the subset $\text{Ext}_{\mathcal{M}_L}^{1,\text{reg}}(\underline{N}, \underline{M})$ of regulated extensions of \underline{N} by \underline{M} is well-defined. It is also a sub- A -module of $\text{Ext}_{\mathcal{M}_L}^1(\underline{N}, \underline{M})$. We have:

Corollary 5.10. *Let \underline{M} be an A -motive over L . Then, ι induces an isomorphism of A -modules:*

$$\frac{M + \tau_M(\tau^*M)}{(\text{id} - \tau_M)(M)} \xrightarrow{\sim} \text{Ext}_{\mathcal{M}_L}^{1,\text{reg}}(\mathbb{1}, \underline{M}).$$

Remark 5.11. In particular, the extension of $\mathbb{1}$ by itself constructed in Subsection 5.1 is not regulated.

5.3 Regulated extensions having good reduction

Let F be a finite extension of K , let \mathfrak{p} be a finite place of F . Hereafter, κ is the inclusion of A into F , and \mathcal{O}_F is the integral closure of A in F . By the field L (resp. the ring \mathcal{O}_L) we shall mean either F or $F_{\mathfrak{p}}$ (resp. \mathcal{O}_F or $\mathcal{O}_{\mathfrak{p}}$). Let $\mathfrak{m} = \mathfrak{p} \cap A$ and let ℓ be a maximal ideal in A distinct from \mathfrak{m} .

Definition 5.12. We let $\text{Ext}_{\mathcal{O}_L}^{1,\text{reg}}(\mathbb{1}, \underline{M})$ be the submodule of $\text{Ext}_{\mathcal{O}_L}^1(\mathbb{1}, \underline{M})$ consisting of regulated extensions. Similarly, by $\text{Ext}_{\text{good}}^{1,\text{reg}}(\mathbb{1}, \underline{M})_{\ell}$ we designate the submodule of $\text{Ext}_{\text{good}}^1(\mathbb{1}, \underline{M})_{\ell}$ consisting of regulated extensions in the category \mathcal{M}_L .

Let \underline{M} be an A -motive over $F_{\mathfrak{p}}$. By Theorem C (Theorem 4.49), there is an inclusion of A -modules:

$$\text{Ext}_{\mathcal{O}_{\mathfrak{p}}}^{1,\text{reg}}(\mathbb{1}, \underline{M}) \subseteq \text{Ext}_{\text{good}}^{1,\text{reg}}(\mathbb{1}, \underline{M})_{\ell}. \quad (5.4)$$

We strongly expect this to be an equality for a large class of A -motives. For the remaining of this section, we present some evidences for this expectation.

Theorem 5.13. *Assume that \underline{N} is effective, pure of weight 0 and has good reduction. Then, (5.4) is an equality. In particular, $\text{Ext}_{\text{good}}^{1,\text{reg}}(\mathbb{1}, \underline{N})_{\ell}$ does not depend on ℓ .*

The above will result as a sequence of lemmas:

Lemma 5.14. *Let \underline{N} be a pure of weight 0 effective A -motive of rank $r \geq 1$ over a separably closed field H . Then, there exists a non zero ideal $\mathfrak{a} \subset A$ such that $\underline{N} \cong \mathbb{1}^{\oplus(r-1)} \oplus \underline{\mathfrak{a}}$, where $\underline{\mathfrak{a}}$ is the A -motive $(\mathfrak{a} \otimes H, \mathbb{1})$.*

Proof. We first claim that $\tau_N(\tau^*N) = N$. By purity, let Λ be an $\mathcal{A}_{\infty}(H)$ -lattice in $N \otimes_{A \otimes H} \mathcal{B}_{\infty}(H)$ such that $\tau_N^n(\tau^{n*}\Lambda) = \Lambda$ for some $n \geq 1$. Up to replacing Λ by the lattice $\Lambda \cap \tau_N(\tau^*\Lambda) \cap \dots \cap \tau_N^{n-1}(\tau^{(n-1)*}\Lambda)$, we can assume that $n = 1$.

By the Beauville-Laszlo Theorem [BeaLa], we glue Λ and N to obtain a locally-free sheaf \mathcal{N} on $C \times H$ together with a homomorphism $\tau_{\mathcal{N}} : \tau^*\mathcal{N} \rightarrow \mathcal{N}$ extending τ_N . Let D be the divisor on $C \times H$ associated to the invertible $\mathcal{O}_{C \times H}$ -module $\det \mathcal{N}$. Taking the determinant of $\tau_{\mathcal{N}}$ yields:

$$\mathcal{O}(\tau^*D + (\det \tau_{\mathcal{N}})) = \mathcal{O}(-d \cdot V(j) + D)$$

where d is the dimension of $N/\tau_N(\tau^*N)$ over H . Comparing degrees, we obtain $d = 0$. Therefore, $N = \tau_N(\tau^*N)$.

For $n \in \mathbb{Z}$, let $N_n := \pi_\infty^{-n} \Lambda \cap N$. The sequence $(N_n)_{n \in \mathbb{Z}}$ is an increasing family of finite dimensional H -vector spaces whose union is N , and we have $\tau_N(\tau^* N_n) = N_n$. Since H is separably closed, it follows from [Kat1] that $N_n^{\tau_N=1} \otimes H \cong N_n$. Hence, we have

$$N^{\tau_N=1} \otimes H \xrightarrow{\sim} N. \quad (5.5)$$

Now, $N^{\tau_N=1}$ is torsion-free and finitely generated over the Dedekind domain A , hence it is projective of rank r . Thus, there exists a non zero ideal $\mathfrak{a} \subset A$ such that $N^{\tau_N=1} \cong A^{\oplus(r-1)} \oplus \mathfrak{a}$. We conclude by (5.5). \square

Lemma 5.15. *Assume $A = \mathbb{F}[t]$, and let \underline{N} be a pure of weight 0 effective A -motive over a field L . There exists an $\mathcal{A}_\infty(L)$ -lattice Λ in $N \otimes_{A \otimes L} \mathcal{B}_\infty(L)$ such that:*

1. $\tau_N(\tau^* \Lambda) = \Lambda$,
2. As L -vector spaces, $N \otimes_{A \otimes L} \mathcal{B}_\infty(L) = N \oplus \Lambda$.

Proof. From Lemma 5.14, and because $A = \mathbb{F}[t]$, the result is easily proven for \underline{N}_{L^s} where L^s is a separable closure of L . As such, there exists an $\mathcal{A}_\infty(L^s)$ -lattice in $N \otimes_{A \otimes L} \mathcal{B}_\infty(L^s)$ such that $\tau_N(\tau^* \Lambda_s) = \Lambda_s$ and $N \otimes_{A \otimes L} \mathcal{B}_\infty(L^s) = (N \otimes_L L^s) \oplus \Lambda_s$.

The Galois group $G_L = \text{Gal}(L^s|L)$ acts on $N \otimes_{A \otimes L} \mathcal{B}_\infty(L^s)$ through the right-hand side, leaving $N \otimes_{A \otimes L} \mathcal{B}_\infty(L)$ invariant. We claim that Λ_s is stable through the action of G_L . Indeed, let Λ_0 be an $\mathcal{A}_\infty(L)$ -lattice inside $N \otimes_{A \otimes L} \mathcal{B}_\infty(L)$, and assume without loss that

$$\Lambda_s \subset \Lambda_0 \otimes_{\mathcal{A}_\infty(L)} \mathcal{A}_\infty(L^s).$$

Then, by the elementary divisor Theorem for the discrete valuation ring $\mathcal{A}_\infty(L^s)$, there exists a basis $(\lambda_1, \dots, \lambda_r)$ of $\Lambda_0 \otimes_{\mathcal{A}_\infty(L)} \mathcal{A}_\infty(L^s)$ and non-negative integers (s_1, \dots, s_r) such that

$$\Lambda_s = \pi_\infty^{s_1} \mathcal{A}_\infty(L^s) \cdot \lambda_1 + \dots + \pi_\infty^{s_r} \mathcal{A}_\infty(L^s) \cdot \lambda_r.$$

It follows that Λ_s is stable by G_L . We let:

$$\Lambda := \Lambda_s^{G_L} = \{\lambda \in \Lambda_s \mid \forall \sigma \in G_L, \sigma(\lambda) = \lambda\},$$

so that both $\tau_N(\tau^* \Lambda) = \Lambda$ and $N \otimes_{A \otimes L} \mathcal{B}_\infty(L) = N \oplus \Lambda$. From this last equality, it easily follows that Λ is an $\mathcal{A}_\infty(L)$ -lattice in $N \otimes_{A \otimes L} \mathcal{B}_\infty(L)$. \square

Lemma 5.16. *Let ℓ be a maximal ideal of A and let \underline{N} be a pure of weight 0 effective A -motive over F_p having good reduction. Then $N_{\mathcal{O}_p} + (\text{id} - \tau_N)(N)$ is ℓ -adically closed in N .*

Proof. Without loss of generality, we may assume that $A = \mathbb{F}[t]$. Let Λ be as in Lemma 5.15. For $n \geq 0$, let N_n be the finite dimensional F_p -vector space $\ell^n \Lambda \cap N$. $(N_n)_n$ defines an increasing sequence of subspaces of N and we both have $\bigcup_{n \geq 0} N_n = N$ and $N = N_n \oplus \ell^n N$. We also claim that:

$$N_{\mathcal{O}_p} \cap N = (N_{\mathcal{O}_p} \cap N_n) \oplus \ell^n (N_{\mathcal{O}_p} \cap N).$$

To see this, note that since \underline{N} has good reduction, the image of $N_{\mathcal{O}_p} \cap N_n$ through $N \rightarrow N/\ell^n N$ equals the maximal integral model of $(N/\ell^n N, \tau_M)$. Hence $N_{\mathcal{O}_p} \cap N \subset (N_{\mathcal{O}_p} \cap N_n) \oplus \ell^n N$ and the claim follows.

Let $m \in N$ be such that there exists a sequence $(m_n)_{n \geq 0}$ in $N_{\mathcal{O}_p} + (\text{id} - \tau_N)(N)$ which converges ℓ -adically to m . We have $m \in N_d$ for a large enough integer d . If p_d denotes the projection onto N_d orthogonally to $\ell^d N$, we obtain $m = p_d(m) = p_d(m_d) \in (N_{\mathcal{O}_p} \cap N_d) + (\text{id} - \tau_N)(N_d)$ as desired. \square

Proof of Theorem 5.13. Let $[E] = \iota(m)$ be an extension in $\text{Ext}_{\text{good}}^{1,\text{reg}}(\mathbb{1}, \underline{N})_\ell$. By definition, $m \in N$, and by Proposition 2.3, there exists $\xi \in (N \hat{\otimes}_{F_p} F_p^{\text{ur}})_\ell$ such that

$$m = \xi - \tau_N(\tau^*\xi). \quad (5.6)$$

Reducing (5.6) modulo ℓ^n for all $n \geq 1$, we obtain from Proposition 4.15, applied to the Frobenius space $(N/\ell^n N, \tau_N)$, that

$$\forall n \geq 1: \quad m \in \tilde{L}_n + (\text{id} - \tau_N)(N) + \ell^n N$$

where \tilde{L}_n is a lift in N of a maximal integral model for $(N/\ell^n N, \tau_N)$. It follows from Proposition 4.38 that m belongs to the ℓ -adic closure of $N_{\mathcal{O}_p} + (\text{id} - \tau_N)(N)$. But the later is already closed by Lemma 5.16. It follows that $m \in N_{\mathcal{O}_p} + (\text{id} - \tau_N)(N)$, which amounts to $[E] \in \text{Ext}_{\mathcal{O}_p}^{1,\text{reg}}(\mathbb{1}, \underline{N})$ as desired. \square

Let now A be $\mathbb{F}[t]$, and let $\underline{A}(n)$ be the n th twist of the Carlitz motive over F_p (see the notations of Example 2.4). Let $\ell \neq \mathfrak{m}$ be a maximal ideal of A .

Theorem 5.17. *The inclusion (5.4) is an equality for $\underline{M} = \underline{A}(p^k)$, where p is the characteristic of \mathbb{F} and k is a non negative integer. In particular, $\text{Ext}_{\text{good}}^{1,\text{reg}}(\mathbb{1}, \underline{A}(p^k))_\ell$ is independent of ℓ .*

The rest of the text is devoted to the proof of Theorem 5.17. We start by stating lemma which holds for general $\underline{A}(n)$, $n > 0$. The proof is an easy computation which is left to the reader.

Lemma 5.18. *Let $m(t) \in F_p[t]$. There exists $\varepsilon(t) \in F_p[t]$ of degree $< n$ and $p(t) \in F_p[t]$ such that*

$$\frac{m(t)}{(t - \theta)^n} = \frac{\varepsilon(t)}{(t - \theta)^n} + p(t) - \frac{p(t)^{(1)}}{(t - \theta)^n}.$$

Proof of Theorem 5.17. Write n for p^k , and let $\underline{M} = \underline{A}(n)$. Let $m \in M + \tau_M(\tau^*M)$ be such that there exists $f \in (M \hat{\otimes}_{F_p} F_p^{\text{ur}})_\ell$ for which $f - \tau_M(\tau^*f) = m$. That is, $m \in F_p[t]/(t - \theta)^n$ and $f \in \widehat{F_p^{\text{ur}}[t]}_\ell$. By Lemma 5.18, we can assume that m is of the form $\varepsilon(t)/(t - \theta)^n$ for some $\varepsilon(t) \in F_p[t]$ of degree $< n$. Because n is a power of the characteristic, we may rewrite $f - \tau_M(\tau^*f) = m$ as:

$$t^n f - \varepsilon(t) = \theta^n f + f^{(1)}. \quad (5.7)$$

Let $v : F_p \rightarrow \mathbb{Z} \cup \{\infty\}$ be the valuation on F_p , which we canonically extends to $F_p^s \rightarrow \mathbb{Q} \cup \{\infty\}$. The assumption that $\mathfrak{m} \neq \ell$ implies that $v(\theta) = 0$.

We should derive a contradiction by assuming that $\varepsilon(t) = \varepsilon_0 + \varepsilon_1 t + \dots + \varepsilon_{n-1} t^{n-1}$ does not belong to $\mathcal{O}_p[t]$. To wit, for some $0 \leq i < n$, we have $v(\varepsilon_i) < 0$. We let $c \geq 0$ be the maximal integer such that there exists $j \in \{0, \dots, n-1\}$ for which both $v(\varepsilon_j) < 0$ and $q^c | v(\varepsilon_j)$.

Let N be a multiple of n greater than $n(c+1)/d$, where d is the degree of ℓ . We aim to rewrite (5.7) coefficient-wise in the basis $(1, t, \dots, t^{dN-1})$ of $\widehat{F_p^{\text{ur}}[t]}_\ell / \ell^N \widehat{F_p^{\text{ur}}[t]}_\ell = F_p^{\text{ur}}[t]/\ell^N F_p^{\text{ur}}[t]$ seen as a F_p^{ur} -vector space. To proceed, for $i \in \{0, \dots, Nd-1\}$, we denote by $x_i \in F_p^{\text{ur}}$ be the t^i -coefficient of $f \pmod{\ell^n}$ so that

$$f \equiv x_0 + x_1 t + \dots + x_{Nd-1} t^{dN-1} \pmod{\ell^N}.$$

For $1 \leq j \leq n$, we also write $t^n \cdot t^{dN-j} \equiv \sum_{i=0}^{dN-1} a_{ij} t^i \pmod{\ell^N}$ for coefficients

$a_{ij} \in \mathbb{F}$. It yields:

$$\begin{aligned} t^n f &\equiv (a_{01}x_{dN-1} + \dots + a_{0n}x_{dN-n}) \\ &\quad + (a_{11}x_{dN-1} + \dots + a_{1n}x_{dN-n})t \\ &\quad + \dots \\ &\quad + [x_0 + (a_{n1}x_{dN-1} + \dots + a_{nn}x_{dN-n})]t^n \\ &\quad + \dots \\ &\quad + [x_{dN-n-1} + (a_{dN-1,1}x_{dN-1} + \dots + a_{dN-1,n}x_{dN-n})]t^{dN-1} \end{aligned}$$

Equations (5.7) modulo ℓ^N is then reformulated by the following systems of equations, numbered (E_1) , (E_2) , ..., $(E_{dN/n})$:

System (E_1) (coefficients of (5.7) in $(1, \dots, t^{n-1})$):

$$\begin{aligned} (a_{01}x_{dN-1} + \dots + a_{0n}x_{dN-n}) - \varepsilon_0 &= \theta^n x_0 + x_0^q \\ (a_{11}x_{dN-1} + \dots + a_{1n}x_{dN-n}) - \varepsilon_1 &= \theta^n x_1 + x_1^q \\ &\vdots \\ (a_{n-1,1}x_{dN-1} + \dots + a_{n-1,n}x_{dN-n}) - \varepsilon_{n-1} &= \theta^n x_{n-1} + x_{n-1}^q \end{aligned}$$

System (E_2) (coefficients of (5.7) in (t^n, \dots, t^{2n-1})):

$$\begin{aligned} x_0 + (a_{n1}x_{dN-1} + \dots + a_{nn}x_{dN-n}) &= \theta^n x_n + x_n^q \\ x_1 + (a_{n+1,1}x_{dN-1} + \dots + a_{n+1,n}x_{dN-n}) &= \theta^n x_{n+1} + x_{n+1}^q \\ &\vdots \\ x_{n-1} + (a_{2n-1,1}x_{dN-1} + \dots + a_{2n-1,n}x_{dN-n}) &= \theta^n x_{2n-1} + x_{2n-1}^q \end{aligned}$$

and so on, until the two last systems:

System $(E_{dN/n-1})$ (coefficients of (5.7) in $(t^{dN-2n}, \dots, t^{dN-n-1})$):

$$\begin{aligned} x_{dN-3n} + (a_{dN-2n,1}x_{dN-1} + \dots + a_{dN-2n,n}x_{dN-n}) &= \theta^n x_{dN-2n} + x_{dN-2n}^q \\ x_{dN-3n+1} + (a_{dN-2n+1,1}x_{dN-1} + \dots + a_{dN-2n+1,n}x_{dN-n}) &= \theta^n x_{dN-2n+1} + x_{dN-2n+1}^q \\ &\vdots \\ x_{dN-2n-1} + (a_{dN-n-1,1}x_{dN-1} + \dots + a_{dN-n-1,n}x_{dN-n}) &= \theta^n x_{dN-n-1} + x_{dN-n-1}^q \end{aligned}$$

System $(E_{dN/n})$ (coefficients of (5.7) in $(t^{dN-n}, \dots, t^{dN-1})$):

$$\begin{aligned} x_{dN-2n} + (a_{dN-n,1}x_{dN-1} + \dots + a_{dN-n,n}x_{dN-n}) &= \theta^n x_{dN-n} + x_{dN-n}^q \\ x_{dN-2n+1} + (a_{dN-n+1,1}x_{dN-1} + \dots + a_{dN-n+1,n}x_{dN-n}) &= \theta^n x_{dN-n+1} + x_{dN-n+1}^q \\ &\vdots \\ x_{dN-n-1} + (a_{dN-1,1}x_{dN-1} + \dots + a_{dN-1,n}x_{dN-n}) &= \theta^n x_{dN-1} + x_{dN-1}^q \end{aligned}$$

There are two situations:

- (A) $x_{dN-1}, \dots, x_{dN-n}$ all have positive valuations. Then, it follows from (E_1) that $v(x_i) < 0$ and that $qv(x_i) = v(\varepsilon_i)$. From (E_2) , $v(x_{i+n}) < 0$ and $qv(x_{i+n}) = v(x_i)$. By immediate recursion from (E_k) , $1 \leq k \leq dN/n$, we obtain $qv(x_{i+kn}) = v(x_{i+(k-1)n})$. Hence, $v(\varepsilon_i) = q^{dN/n}v(x_{dN-n+i})$. As $x_{dN-n+i} \in F_{\mathfrak{p}}^{\text{ur}}$, its valuation is an integer and $q^{dN/n}$ divides $v(\varepsilon_i)$. Yet, this contradicts the maximality of c .

(B) Therefore, at least one of $x_{dN-1}, \dots, x_{dN-n}$ has negative valuation. Let x_{dN-j} , for some $j \in \{1, \dots, n\}$, be the one with the smallest (negative) valuation. From $(E_{dN/n})$, we obtain $v(x_{dN-n-j}) < 0$ and $v(x_{dN-n-j}) = qv(x_{dN-j})$. From $(E_{dN/n-1})$, we have $v(x_{dN-2n-j}) < 0$ and $v(x_{dN-2n-j}) = qv(x_{dN-n-j})$. Going backward in recursion, we obtain $v(x_{dN-kn-j}) = qv(x_{dN-(k-1)n-j})$ so that $v(\varepsilon_j) = q^{dN/n}v(x_{dN-j})$ which, once again, contradicts the maximality of c as $v(x_{dN-j})$ is an integer.

We conclude that $\varepsilon(t) \in \mathcal{O}_p[t]$, as desired. Then, the theorem follows from Proposition 2.21. \square

Remark 5.19. We believe the general case of $\underline{A}(n)$ for $n > 0$ would follow from a similar argument, but the computations are too involved to be gently written.

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