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ŁOJASIEWICZ-SIMON GRADIENT INEQUALITIES FOR ANALYTIC AND MORSE-BOTT FUNCTIONALS ON BANACH SPACES AND APPLICATIONS TO HARMONIC MAPS

PAUL M. N. FEEHAN AND MANOUSOS MARIDAKIS

Abstract. We prove two abstract versions of the Łojasiewicz-Simon gradient inequality for an analytic functional on a Banach space (stated earlier without proof as [52, Theorem 2.4.5]) that generalize previous abstract versions of this inequality, significantly weakening their hypotheses and, in particular, the well-known infinite-dimensional version of the gradient inequality due to Łojasiewicz [66] proved by Simon as [80, Theorem 3]. We also prove that the optimal exponent of the Łojasiewicz-Simon gradient inequality is obtained when the functional is Morse-Bott and not necessarily analytic, improving on similar results due to Chill [18, Corollary 3.12], [19, Corollary 4], Haraux and Jendoubi [44, Theorem 2.1], and Simon [82, Sections 3.12 and 3.13].

We apply our abstract Łojasiewicz-Simon gradient inequality to prove a Łojasiewicz-Simon gradient inequality for the harmonic map energy functional using Sobolev spaces which impose minimal regularity requirements on maps between closed, Riemannian manifolds. Our Łojasiewicz-Simon gradient inequality for the harmonic map energy functional significantly generalizes those of Kwon [60, Theorem 4.2], Liu and Yang [64, Lemma 3.3], Simon [80, Theorem 3], [81, Equation (4.27)], and Topping [89, Lemma 1].

Contents

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1. Introduction

Since its discovery by Łojasiewicz in the context of analytic functions on Euclidean spaces [66, Proposition 1, p. 92] and subsequent generalization by Simon to a class of analytic functionals on certain Hölder spaces [80, Theorem 3], the *Łojasiewicz-Simon gradient inequality* has played a significant role in analyzing questions such as *a*) global existence, convergence, and analysis of singularities for solutions to nonlinear evolution equations that are realizable as gradient-like systems for an energy functional, *b*) uniqueness of tangent cones, and *c*) energy gaps and discreteness of energies. For applications of the Łojasiewicz-Simon gradient inequality to gradient flows arising in geometric analysis, beginning with the harmonic map energy functional, we refer to Irwin [54], Kwon [60], Liu and Yang [64], Simon [81], and Topping [88, 89]; for gradient flow for the Chern-Simons functional, see Morgan, Mrowka, and Ruberman [68]; for gradient flow for the Yamabe functional, see Brendle [13, Lemma 6.5 and Equation (100)] and Carlotto, Chodosh, and Rubinstein [16]; for Yang-Mills gradient flow, we refer to our monograph [30], Råde [74], and Yang [92]; for mean curvature flow, we refer to the survey by Colding and Minicozzi [24]; and for Ricci curvature flow, see Ache [2], Haslhofer [47], Haslhofer and Müller [48], and Kröncke [59, 58].

For applications of the Łojasiewicz-Simon gradient inequality to proofs of global existence, convergence, convergence rate, and stability of non-linear evolution equations arising in other areas of mathematical physics (including the Cahn-Hilliard, Ginzburg-Landau, Kirchoff-Carrier, porous medium, reaction-diffusion, and semi-linear heat and wave equations), we refer to the monograph by Huang [52] for a comprehensive introduction and to the articles by Chill [18, 19], Chill and Fiorenza [20], Chill, Haraux, and Jendoubi [21], Chill and Jendoubi [22, 23], Feireisl and Simondon [35], Feireisl and Takáč [36], Grasselli, Wu, and Zheng [40], Haraux [42], Haraux and Jendoubi [43, 44, 45], Haraux, Jendoubi, and Kavian [46], Huang and Takáč [53], Jendoubi [55], Rybka and Hoffmann [76, 77], Simon [80], and Takáč [86]. For applications to fluid dynamics, see the articles by Feireisl, Laurençot, and Petzeltová [34], Frigeri, Grasselli, and Krejčí [38], Grasselli and Wu [39], and Wu and Xu [91]

For applications of the Łojasiewicz-Simon gradient inequality to proofs of energy gaps and discreteness of energies for Yang-Mills connections, we refer to our articles [29, 28]. A key feature of our version of the Łojasiewicz-Simon gradient inequality for the pure Yang-Mills energy functional [30, Theorem 22.8] is that it holds for $W^{1,p}$ Sobolev norms and thus considerably weaker than the $C^{2,\alpha}$ Hölder norms originally employed by Simon in [80, Theorem 3] and this affords considerably greater flexibility in applications. For example, when (X, g) is a closed, four-dimensional, Riemannian manifold, the $W^{1,2}$ Sobolev norm on (bundle-valued) one-forms is (in a suitable sense) *quasi-conformally invariant* with respect to conformal changes in the Riemannian metric g. In particular, that observation is exploited in our proof of [28, Theorem 1], which asserts discreteness of L^2 energies of Yang-Mills connections on arbitrary G-principal bundles over X, for any compact Lie structure group G. In our companion article [32], we apply Theorem 1 to prove Łojasiewicz-Simon gradient inequalities for coupled Yang-Mills energy functionals.

There are essentially three approaches to establishing a Łojasiewicz-Simon gradient inequality for a particular energy functional arising in geometric analysis or mathematical physics: *1*) establish the inequality from first principles, *2*) adapt the argument employed by Simon in the proof of his [80, Theorem 3], or *3*) apply an abstract version of the Łojasiewicz-Simon gradient inequality for an analytic or Morse-Bott functional on a Banach space. Most famously, the first approach is exactly that employed by Simon in [80], although this is also the avenue followed by Kwon [60], Liu and Yang [64] and Topping [88, 89] for the harmonic map energy functional and by Råde for the Yang-Mills energy functional. Occasionally a development from first principles may be necessary, as discussed by Colding and Minicozzi in [24]. However, in almost all of the remaining examples cited, one can derive a Łojasiewicz-Simon gradient inequality for a specific application from an abstract version for an analytic or Morse-Bott functional on a Banach space. For this strategy to work well, one desires an abstract Łojasiewicz-Simon gradient inequality with the weakest possible hypotheses and a proof of such a gradient inequality (Theorem 1) is the one purpose of the present article. We also prove an abstract Łojasiewicz-Simon gradient inequality, with the optimal exponent, for a Morse-Bott functional on a Banach space, generalizing and unifying previous versions of the Łojasiewicz-Simon gradient inequality with optimal exponent obtained in specific examples.

Moreover, we establish versions of the Łojasiewicz-Simon gradient inequality for the harmonic map energy functional (Theorem 4), using systems of Sobolev norms in these applications that are (as best we can tell) as *weak as possible*. Our gradient inequality for the harmonic map energy functional is a significant generalization of previous inequalities due to Kwon [60, Theorem 4.2], Liu and Yang [64, Lemma 3.3], Simon [80, Theorem 3], [81, Equation (4.27)], and Topping [89, Lemma 1]. When the source manifold is a Riemann surface, our gradient inequality for the harmonic map L^2 energy functional uses the quasi-conformally invariant $W^{2,1}$ norm for harmonic maps from a Riemann surface into a Riemannian manifold of arbitrary dimension.

While our abstract versions of the Łojasiewicz-Simon gradient inequality (Theorems 1 and 3 and Corollary 2) are versatile enough to apply to many problems in geometric analysis, mathematical physics, and applied mathematics, it is worth noting that there are situations where it appears difficult to derive a Łojasiewicz-Simon gradient inequality for a specific application from an abstract version. For example, a gradient inequality due to Feireisl, Issard-Roch, and Petzeltová applies to functionals that are not C^2 [33, Proposition 4.1 and Remark 4.1]. Colding and Minicozzi describe certain gradient inequalities [24, Theorems 2.10 and 2.12] employed in their work on non-compact singularities arising in mean curvature flow that do not appear to follow from abstract Łojasiewicz-Simon gradient inequalities or even the usual arguments underlying their proofs [24, Section 1]. Nevertheless, that should not preclude consideration of abstract Łojasiewicz-Simon gradient inequalities with the broadest possible application.

In the remainder of our Introduction, we summarize the principal results of our article, beginning with a version of the abstract Łojasiewicz-Simon gradient inequality for analytic functionals on Banach spaces in Section 1.1 and Łojasiewicz-Simon gradient inequalities for the harmonic map L^2 energy functional in Section 1.2.

1.1. **Łojasiewicz-Simon gradient inequalities for analytic functionals on Banach spaces and Morse-Bott functionals on Hilbert spaces.** We begin with the following generalization of Simon's infinite-dimensional version [80, Theorem 3] of the Łojasiewicz gradient inequality [66]. Theorem 1 is stated by Huang as [52, Theorem 2.4.5] but no proof is given and it does not follow from his less general [52, Theorem 2.4.2]. Huang cites [53, Proposition 3.3] for the proof of Theorem 1 but the hypotheses of [53, Proposition 3.3] assume that $\mathscr{X}$ is a Hilbert space. That distinction is important because we shall need Theorem 1 when $\mathscr{X}$ is a Banach space that may not even be reflexive, as in our application to the harmonic map L^2 energy functional in Section 1.2. The proof of Theorem 1 that we include in Section 2 is similar to that of Feireisl and Takáč for their [36, Proposition 6.1] in the case of the Ginzburg-Landau energy functional. If $\mathscr{X}$ is a Banach space, we let $\mathscr{X}^*$ denote its continuous dual space.

Theorem 1 (Łojasiewicz-Simon gradient inequality for analytic functionals on Banach spaces)**.** *Let $\mathscr{X}$ be a Banach space that is continuously embedded in a Hilbert space $\mathscr{H}$. Let $\mathscr{U} \subset \mathscr{X}$ be an open subset, $\mathscr{E} : \mathscr{U} \to \mathbb{R}$ be an analytic function, and $x_\infty \in \mathscr{U}$ be a critical point of $\mathscr{E}$, that is, $\mathscr{E}'(x_\infty) = 0$. Assume that $\mathscr{E}''(x_\infty) : \mathscr{X} \to \mathscr{X}^*$ is a Fredholm operator with index zero. Then there are constants $Z \in [1, \infty)$, and $\sigma \in (0, 1]$, and $\theta \in [1/2, 1)$, with the following significance. If $x \in \mathscr{U}$ obeys*

$$\|x - x_\infty\|_{\mathscr{X}} < \sigma, \tag{1.1}$$

then

$$\|\mathscr{E}'(x)\|_{\mathscr{X}^*} \geq Z|\mathscr{E}(x) - \mathscr{E}(x_\infty)|^\theta. \tag{1.2}$$

Remark 1.1 (Index of a Fredholm Hessian operator on a reflexive Banach space)*.* If $\mathscr{X}$ is a reflexive Banach space in Theorem 1, then the hypothesis that $\mathscr{E}''(x_\infty) : \mathscr{X} \to \mathscr{X}^*$ has index zero can be omitted, since $\mathscr{E}''(x_\infty)$ is always a symmetric operator and thus necessarily has index zero when $\mathscr{X}$ is reflexive by Lemma 2.3.

Surprisingly, it is possible to substantially weaken the traditional Łojasiewicz-Simon neighborhood condition (1.1). Indeed, by adapting the proof of a nice observation [36, Corollary 6.2] due to Feireisl and Takáč for the Ginzburg-Landau energy functional we obtain the

Corollary 2 (Łojasiewicz-Simon gradient inequality for C^2 functionals on Banach spaces)**.** *Let $\mathscr{X}$ be a Banach space that is continuously embedded in a Hilbert space $\mathscr{H}$. Let $\mathscr{U} \subset \mathscr{X}$ be an open subset, $\mathscr{E} : \mathscr{U} \to \mathbb{R}$ be a C^2 function, and $x_\infty \in \mathscr{U}$ be a critical point of $\mathscr{E}$, that is, $\mathscr{E}'(x_\infty) = 0$. Assume that there are constants, $Z \in [1, \infty)$, and $\sigma \in (0, 1]$, and $\theta \in [1/2, 1)$, such that the Łojasiewicz-Simon gradient inequality* (1.2) *holds with constant Z for all $x \in \mathscr{U}$ obeying the Łojasiewicz-Simon neighborhood condition* (1.1) *with constant σ. If $M \in [1, \infty)$ is a constant, then there are positive constants, Z_0, σ_0, with the following significance. If $x \in \mathscr{U}$ obeys*

$$|\mathscr{E}(x) - \mathscr{E}(x_\infty)| \leq M \quad \text{and} \quad \|x - x_\infty\|_{\mathscr{H}} < \sigma_0, \tag{1.3}$$

then (1.2) *holds with Z replaced by Z_0.*

We note that unlike in the proof of [36, Corollary 6.2] in their application, we do not assume compactness of the embedding $\mathscr{X} \subset \mathscr{H}$ in the hypotheses of Corollary 2, although this property typically does hold in applications.

Remark 1.2 (Previous versions of the Łojasiewicz-Simon gradient inequality for analytic functionals on Banach spaces)*.* The [18, Theorem 3.10 and Corollary 3.11] and [19, Corollary 3] due to Chill provide versions of the Łojasiewicz-Simon gradient inequality for an analytic functional on a Banach space, but the hypotheses of Theorem 1 are considerably simpler and more general.

The [45, Theorem 4.1] due to Haraux and Jendoubi is an abstract Łojasiewicz-Simon gradient inequality which they argue is optimal based on examples that they discuss in [45, Section 3]. However, while the hypothesis in Theorem 1 is replaced by their alternative requirements that $\operatorname{Ker} \mathscr{E}''(x_\infty)$ be finite-dimensional and $\mathscr{E}''(x_\infty)$ obey a certain coercivity condition on the orthogonal complement of $\operatorname{Ker} \mathscr{E}''(x_\infty)$, they require $\mathscr{X}$ to be a Hilbert space.

In [52, Theorem 2.4.2], Huang provides a version of the Łojasiewicz-Simon gradient inequality for analytic functionals on Banach spaces in which boundedness of the gradient map $\mathscr{E}'(x) : \mathscr{X} \to \mathscr{H}$ replaces boundedness of the gradient map $\mathscr{E}'(x) : \mathscr{X} \to \mathscr{X}^*$ and the Banach space dual norm $\|\cdot\|_{\mathscr{X}^*}$ in (1.2) is replaced by the Hilbert space norm $\|\cdot\|_{\mathscr{H}}$. However, the hypotheses of Theorem 1 are again considerably simpler and more general than those of [52, Theorem 2.4.2].

Remark 1.3 (Replacement of Hilbert by Banach space dual norms in Łojasiewicz-Simon gradient inequalities). The structure of the original result of Simon [80, Theorem 3] was improved in specific applications by Feireisl and Simondon [35, Proposition 6.1], Råde [74, Proposition 7.2], Rybka and Hoffmann [76, Theorem 3.2], [77, Theorem 3.2], and Takáč [86, Proposition 8.1] by replacing the $L^2(X)$ norm used by Simon in his [80, Theorem 3] with dual Sobolev norms, such as $W^{-1,2}(X)$, and replacing the $C^{2,\alpha}$ Hölder norm used by Simon to define the neighborhood of the critical point with a Sobolev $W^{1,2}(X)$ norm.

It is of considerable interest to know when the optimal exponent $\theta = 1/2$ is achieved, since in that case one can prove (see [30, Theorem 23.19], for example) that a global solution, $u : [0, \infty) \to \mathscr{X}$, to a gradient system governed by the Łojasiewicz-Simon gradient inequality,

$$\frac{du}{dt} = -\mathscr{E}'(u(t)), \quad u(0) = u_0,$$

has *exponential* rather than mere power-law rate of convergence to the critical point, u_∞. One simple version of such an optimal Łojasiewicz-Simon gradient inequality is provided by the following result of Huang; note that the functional $\mathscr{E}$ is *not* required to be analytic.

Proposition 1.4 (Optimal Łojasiewicz-Simon gradient inequality for C^2 functionals on Hilbert spaces)**.** [52, Proposition 2.7.1] *Let $\mathscr{H}$ be a real Hilbert space, $\mathscr{U} \subset \mathscr{H}$ be an open subset, $\mathscr{E} : \mathscr{U} \to \mathbb{R}$ be a C^2 function, and $x_\infty \in \mathscr{U}$ be a critical point of $\mathscr{E}$, that is, $\mathscr{E}'(x_\infty) = 0$. Assume that the self-adjoint operator $\mathscr{E}''(x_\infty) : \mathscr{H} \to \mathscr{H}$ is injective. Then the following assertions are equivalent.*

(1) *The operator $\mathscr{E}''(x_\infty) : \mathscr{H} \to \mathscr{H}$ is surjective (and thus invertible);*
(2) *There are constants, $Z \in [1, \infty)$ and $\sigma \in (0, 1]$, such that if $x \in \mathscr{U}$ obeys*

$$\|x - x_\infty\|_{\mathscr{H}} < \sigma, \tag{1.4}$$

then

$$\|\mathscr{E}'(x)\|_{\mathscr{H}} \geq Z|\mathscr{E}(x) - \mathscr{E}(x_\infty)|^{1/2}. \tag{1.5}$$

Remark 1.5 (Related optimal abstract Łojasiewicz-Simon gradient inequalities). See Haraux and Jendoubi [44, Theorem 2.1] and Haraux, Jendoubi, and Kavian [46, Proposition 1.1] for results which are similar to Proposition 1.4.

Proposition 1.4, though of interest, has rather limited applications since it is only valid in a very restrictive setting of Hilbert spaces and when the Hessian, $\mathscr{E}''(x_\infty) : \mathscr{H} \to \mathscr{H}$, is an invertible operator.

For the harmonic map energy functional, a more interesting optimal Łojasiewicz-Simon-type gradient inequality,

$$\|\mathscr{E}'(f)\|_{L^p(S^2)} \geq Z|\mathscr{E}(f) - \mathscr{E}(f_\infty)|^{1/2},$$

has been obtained by Kwon [60, Theorem 4.2] for maps $f : S^2 \to N$, where N is a closed Riemannian manifold and f is close to a harmonic map f_∞ in the sense that

$$\|f - f_\infty\|_{W^{2,p}(S^2)} < \sigma,$$

where p is restricted to the range $1 < p \leq 2$, and f_∞ is assumed to be *integrable* in the sense of [60, Definitions 4.3 or 4.4 and Proposition 4.1]. Her [60, Proposition 4.1] quotes results of Simon [81, pp. 270–272] and Adams and Simon [3].

The [64, Lemma 3.3] due to Liu and Yang is another example of an optimal Łojasiewicz-Simon-type gradient inequality for the harmonic map energy functional, but restricted to the setting of maps $f : S^2 \to N$, where N is a Kähler manifold of complex dimension $n \geq 1$ and nonnegative bisectional curvature, and the energy $\mathscr{E}(f)$ is sufficiently small. The result of Liu and Yang generalizes that of Topping [89, Lemma 1], who assumes that $N = S^2$.

For the Yamabe functional, an optimal Łojasiewicz-Simon gradient inequality, has been obtained by Carlotto, Chodosh, and Rubinstein [16] under the hypothesis that the critical point is *integrable* in the sense of their [16, Definition 8], a condition that they observe in [16, Lemma 9] (quoting [3, Lemma 1] due to Adams and Simon) is equivalent to a function on Euclidean space given by the *Lyapunov-Schmidt reduction* of $\mathscr{E}$ being constant on an open neighborhood of the critical point.

For the Yang-Mills L^2 energy functional for connections on a principal $U(n)$-bundle over a closed Riemann surface, an optimal Łojasiewicz-Simon gradient inequality, has been obtained by Råde [74, Proposition 7.2] when the Yang-Mills connection is *irreducible*.

Given the desirability of treating an energy functional as a *Morse function* whenever possible, for example in the spirit of Atiyah and Bott [5] for the Yang-Mills equation over Riemann surfaces, it is useful to rephrase these integrability conditions in the spirit of Morse theory.

Definition 1.6 (Morse-Bott function)**.** [6, Section 3.1] Let $\mathscr{B}$ be a smooth Banach manifold, $\mathscr{E} : \mathscr{B} \to \mathbb{R}$ be a C^2 function, and $\operatorname{Crit}\mathscr{E} := \{x \in \mathscr{B} : \mathscr{E}'(x) = 0\}$. A smooth submanifold $\mathscr{C} \hookrightarrow \mathscr{B}$ is called a *nondegenerate critical submanifold of* $\mathscr{E}$ if $\mathscr{C} \subset \operatorname{Crit}\mathscr{E}$ and

$$(T\mathscr{C})_x = \operatorname{Ker}\mathscr{E}''(x), \quad \forall\, x \in \mathscr{C}, \tag{1.6}$$

where $\mathscr{E}''(x) : (T\mathscr{B})_x \to (T\mathscr{B})_x^*$ is the Hessian of $\mathscr{E}$ at the point $x \in \mathscr{C}$. One calls $\mathscr{E}$ a *Morse-Bott function* if its critical set $\operatorname{Crit}\mathscr{E}$ consists of nondegenerate critical submanifolds.

We say that a C^2 function $\mathscr{E} : \mathscr{B} \to \mathbb{R}$ is *Morse-Bott at a point* x_0 if there is an open neighborhood $U \subset \mathscr{B}$ of x_0 such that $U \cap \operatorname{Crit}\mathscr{E}$ is a (relatively open) smooth submanifold of $\mathscr{B}$ and (1.6) holds at x_0.

Definition 1.6 is a restatement of definitions of a Morse-Bott function on a finite-dimensional manifold, but we omit the condition that $\mathscr{C}$ be compact and connected as in Nicolaescu [71, Definition 2.41] or the condition that $\mathscr{C}$ be compact in Bott [11, Definition, p. 248]. Note that if $\mathscr{B}$ is a Riemannian manifold and $\mathscr{N}$ is the normal bundle of $\mathscr{C} \hookrightarrow \mathscr{B}$, so $\mathscr{N}_x = (T\mathscr{C})_x^\perp$ for all $x \in \mathscr{C}$, where $(T\mathscr{C})_x^\perp$ is the orthogonal complement of $(T\mathscr{C})_x$ in $(T\mathscr{B})_x$, then (1.6) is equivalent to the assertion that the restriction of the Hessian to the fibers of the normal bundle of $\mathscr{C}$,

$$\mathscr{E}''(x) : \mathscr{N}_x \to (T\mathscr{B})_x^*,$$

is *injective* for all $x \in \mathscr{C}$; using the Riemannian metric on $\mathscr{B}$ to identify $(T\mathscr{B})_x^* \cong (T\mathscr{B})_x$, we see that $\mathscr{E}''(x) : \mathscr{N}_x \cong \mathscr{N}_x$ is an isomorphism for all $x \in \mathscr{C}$. In other words, the condition (1.6) is equivalent to the assertion that the Hessian of $\mathscr{E}$ is an isomorphism of the normal bundle $\mathscr{N}$ when $\mathscr{B}$ has a Riemannian metric.

The Yang-Mills L^2 energy functional for connections on a principal G-bundle over X is Morse-Bott when X is a closed Riemann surface — see the article by Atiyah and Bott [5] and the discussion by Swoboda [85, p. 161]. However, it appears difficult to extend this result to the case where X is a closed four-dimensional Riemannian manifold. To gain a sense of the difficulty, see the analysis by Bourguignon and Lawson [12] and Taubes [87] of the Hessian for the Yang-Mills L^2 energy functional when $X = S^4$ with its standard round metric of radius one. For a development of Morse-Bott theory and a discussion of and references to its numerous applications, we refer to Austin and Braam [6].

However, given a Morse-Bott energy functional, we then have the

Theorem 3 (Optimal Łojasiewicz-Simon gradient inequality for Morse-Bott functionals on Banach spaces)**.** *Assume the hypotheses of Theorem 1, except for the hypothesis that $\mathscr{E} : \mathscr{U} \subset \mathscr{X} \to \mathbb{R}$ be real analytic and require instead that this functional be C^2 and Morse-Bott function at x_∞ in the sense of Definition 1.6. Then the conclusions of Theorem 1 and Corollary 2 hold with $\theta = 1/2$.*

We refer to Appendix A for a discussion of integrability and the Morse-Bott condition for the harmonic map energy functional, together with examples.

Remark 1.7 (Previous versions of the optimal Łojasiewicz-Simon gradient inequality). Theorem 3 was first proved by Chill [18, Corollary 3.12], [19, Corollary 4] and Haraux and Jendoubi [44, Theorem 2.1] but under the additional hypothesis that $\mathscr{X}$ is a Hilbert space and $\mathscr{H} = \mathscr{X}$, although in less general settings it is originally due to Simon [82, Sections 3.12 and 3.13].

Remark 1.8 (Distinguishing between Fredholm index and Morse-Bott index). It is worth pointing out that term 'index' arises here in several different, albeit traditional ways. Suppose that $\mathscr{B}$ is a closed, oriented, smooth manifold and that $\mathscr{E} : M \to \mathbb{R}$ is a Morse function. If $x_\infty \in \mathscr{B}$ is a critical point of $\mathscr{E}$, then the *(Morse) index* of $\mathscr{E}$ at x_∞ is the number of negative eigenvalues of the Hessian, $\mathscr{E}''(x_\infty) : (T\mathscr{B})_{x_\infty} \times (T\mathscr{B})_{x_\infty} \to \mathbb{R}$, of $\mathscr{E}$ at x_∞. More generally, if $\mathscr{B}$ also Riemannian, $\mathscr{E}$ is a Morse-Bott function, and $\mathscr{C} \hookrightarrow \mathscr{B}$ is a connected component of a critical set of $\mathscr{E}$, then the *(Morse-Bott) index* [6, Section 3.1] of $\mathscr{E}$ along $\mathscr{C}$ is the rank of $\mathscr{N}^-(\mathscr{C})$, where

$$\mathscr{N}(\mathscr{C}) = \mathscr{N}^+(\mathscr{C}) \oplus \mathscr{N}^-(\mathscr{C}),$$

is the normal bundle of $\mathscr{C}$ and the fibers $\mathscr{N}^\pm_{x_\infty}(\mathscr{C})$ of the subbundles $\mathscr{N}^\pm(\mathscr{C})$ are defined by the positive and negative eigenspaces of $\mathscr{E}''(x_\infty) : \mathscr{N}_{x_\infty}(\mathscr{C}) \times \mathscr{N}_{x_\infty}(\mathscr{C}) \to \mathbb{R}$ for each $x_\infty \in \mathscr{C}$.

On the other hand, in Theorem 1, we view the Hessian $\mathscr{E}''(x_\infty)$ is a linear operator $\mathscr{E}''(x_\infty) : (T\mathscr{B})_{x_\infty} \to (T\mathscr{B})^*_{x_\infty}$ and require that $\mathscr{E}''(x_\infty)$ be Fredholm with index zero, that is,

$$\operatorname{index} \mathscr{E}''(x_\infty) = \dim \operatorname{Ker} \mathscr{E}''(x_\infty) - \dim \operatorname{Coker} \mathscr{E}''(x_\infty) = 0,$$

in the usual sense of linear operators on Banach spaces.

1.2. Łojasiewicz-Simon gradient inequality for the harmonic map L^2-energy functional.

Finally, we describe a consequence of Theorem 1 for the harmonic map L^2-energy functional. For background on harmonic maps, we refer to Hélein [49], Jost [56], Simon [82], Struwe [84], and references cited therein. We begin with the

Definition 1.9 (Harmonic map energy functional)**.** Let (M, g) and (N, h) be a pair of closed, Riemannian, smooth manifolds. One defines the *harmonic map L^2-energy functional* by

$$\mathscr{E}_{g,h}(f) := \frac{1}{2} \int_M |df|^2_{g,h} \, d\operatorname{vol}_g, \tag{1.7}$$

for smooth maps, $f : M \to N$, where $df : TM \to TN$ is the differential map.

When clear from the context, we omit explicit mention of the Riemannian metrics g on M and h on N and write $\mathscr{E} = \mathscr{E}_{g,h}$. Although initially defined for smooth maps, the energy functional $\mathscr{E}$ in Definition 1.9, extends to the case of Sobolev maps of class $W^{1,2}$. To define the gradient of the energy functional $\mathscr{E}$ in (1.7) with respect to the L^2 metric on $C^\infty(M;N)$, we first choose an isometric embedding, $(N,h) \hookrightarrow \mathbb{R}^n$ for a sufficiently large n (courtesy of the isometric embedding theorem due to Nash [69]), and recall that by [56, Equations (8.1.10) and (8.1.13)] we have

$$\begin{aligned}\left(\mathscr{E}'(f), u\right)_{L^2(M,g)} &:= \left.\frac{d}{dt}\mathscr{E}(\exp_f(tu))\right|_{t=0} \\ &= (\Delta_g f, u)_{L^2(M,g)} \\ &= (\Pi_h(f)\Delta_g f, u)_{L^2(M,g)},\end{aligned}$$

for all $u \in C^\infty(M; f^*TN)$, where $\Pi_h(y) : \mathbb{R}^n \to T_yN$ is orthogonal projection and $\exp_y : T_yN \to N$ is the exponential map, so $\exp_y(0) = y \in N$, for all $y \in N$. (Note that one could alternatively define

$$\left(\mathscr{E}'(f), u\right)_{L^2(M,g)} = \left.\frac{d}{dt}\mathscr{E}(\pi(f+tu))\right|_{t=0}$$

as implied by [82, Equations (2.2)(i) and (ii)], where π is the nearest point projection onto N from a normal tubular neighborhood.) Thus, viewing the gradient as an operator and applying [49, Lemma 1.2.4],

$$\mathscr{E}'(f) = \Pi_h(f)\Delta_g = \Delta_g f - A_h(df, df), \tag{1.8}$$

as in [82, Equations (2.2)(iii) and (iv)]. Here, A_h denotes the second fundamental form of the isometric embedding, $(N,h) \subset \mathbb{R}^n$ and

$$\Delta_g := -\operatorname{div}_g \operatorname{grad}_g = d^{*,g}d = -\frac{1}{\sqrt{\det g}}\frac{\partial}{\partial x^\beta}\left(\sqrt{\det g}\,\frac{\partial f}{\partial x^\alpha}\right) \tag{1.9}$$

denotes the Laplace-Beltrami operator for (M,g) (with the opposite sign convention to that of [17, Equations (1.14) and (1.33)]) acting on the scalar components f^i of $f = (f^1, \dots, f^n)$ and $\{x^\alpha\}$ denote local coordinates on M. As usual, the gradient vector field, $\operatorname{grad}_g f^i \in C^\infty(TM)$, is defined by $\langle \operatorname{grad}_g f^i, \xi\rangle_g := df^i(\xi)$ for all $\xi \in C^\infty(TM)$ and $1 \le i \le n$ and the divergence function, $\operatorname{div}_g \xi \in C^\infty(M;\mathbb{R})$, by the pointwise trace, $\operatorname{div}_g \xi := \operatorname{tr}(\eta \mapsto \nabla^g_\xi \eta)$, for all $\eta \in C^\infty(TM)$.

One says that a smooth map $f : M \to N$ is *harmonic* if it is a critical point of the L^2 energy functional (1.7), that is

$$\mathscr{E}'(f) = \Delta_g f - A_h(df, df) = 0.$$

Given a smooth map $f : M \to N$, an isometric embedding $(N,h) \hookrightarrow \mathbb{R}^n$, a non-negative integer k, and $p \in [1,\infty)$, we define the Sobolev norms,

$$\|f\|_{W^{k,p}(M)} := \left(\sum_{i=1}^n \|f^i\|^p_{W^{k,p}(M)}\right)^{1/p},$$

with

$$\|f^i\|_{W^{k,p}(M)} := \left(\sum_{j=0}^k \int_M |(\nabla^g)^j f^i|^p \, d\operatorname{vol}_g\right)^{1/p},$$

where ∇^g denotes the Levi-Civita connection on TM and all associated bundles (that is, T^*M and their tensor products), and if $p = \infty$, we define

$$\|f\|_{W^{k,\infty}(M)} = \|f\|_{C^k(M)} := \sum_{i=1}^{n} \sum_{j=0}^{k} \operatorname*{ess\,sup}_M |(\nabla^g)^j f^i|.$$

If $k = 0$, then we denote $\|f\|_{W^{0,p}(M)} = \|f\|_{L^p(M)}$. For $p \in [1, \infty)$ and nonnegative integers k, we use [4, Theorem 3.12] (applied to $W^{k,p}(M; \mathbb{R}^n)$ and noting that M is a closed manifold) and Banach space duality to define

$$W^{-k,p'}(M; \mathbb{R}^n) := \left(W^{k,p}(M; \mathbb{R}^n)\right)^*,$$

where $p' \in (1, \infty]$ is the dual exponent defined by $1/p + 1/p' = 1$. Elements of the Banach space dual $(W^{k,p}(M; \mathbb{R}^n))^*$ may be characterized via [4, Section 3.10] as distributions in the Schwartz space $\mathscr{D}'(M; \mathbb{R}^n)$ [4, Section 1.57].

In particular, when $p = 1$ and $p' = \infty$ and k is a non-negative integer, we have

$$W^{-k,\infty}(M; \mathbb{R}^n) := \left(W^{k,1}(M; \mathbb{R}^n)\right)^*.$$

Lastly, we note that if (N, h) is real analytic, then the isometric embedding $(N, h) \hookrightarrow \mathbb{R}^n$ may also be chosen to be analytic by the analytic isometric embedding theorem due to Nash [70], with a simplified proof due to Greene and Jacobowitz [41]).

The statement of the forthcoming Theorem 4 includes the most delicate dimension for the source Riemannian manifold, (M, g), namely the case where M has dimension $d = 2$ and allows a Sobolev norm for the definition of the Łojasiewicz-Simon neighborhood of a harmonic map that appears to be optimal for that case, namely, $W^{2,1}(M; N)$, as well as the suboptimal $W^{1,p}(M; N)$ with $p > 2$. Following the landmark articles by Sacks and Uhlenbeck [79, 78], the case where the domain manifold M has dimension two is well-known to be critical.

Theorem 4 (Łojasiewicz-Simon gradient inequality for the energy functional for maps between pairs of Riemannian manifolds). *Let $d \geq 2$ and $k \geq 1$ be integers and $p \in [1, \infty)$ be such that*

$$kp > d \quad \text{or} \quad k = d \text{ and } p = 1.$$

Let (M, g) and (N, h) be closed, Riemannian, smooth manifolds, with M of dimension d. If (N, h) is real analytic (respectively, C^∞) and $f \in W^{k,p}(M; N)$, then the gradient map[1]

$$\mathscr{E}'(f) : T_f W^{k,p}(M; N) \to T_f^* W^{k,p}(M; N),$$

is a real analytic (respectively, C^∞) map of Banach spaces. If (N, h) is real analytic and $f_\infty \in W^{k,p}(M; N)$ is a harmonic map, then there are positive constants $Z \in [1, \infty)$, and $\sigma \in (0, 1]$, and $\theta \in [1/2, 1)$, depending on f_∞, g, h, k, p, with the following significance. If $f \in W^{k,p}(M; N)$ obeys the $W^{k,p}$ Łojasiewicz-Simon neighborhood *condition,*

$$\|f - f_\infty\|_{W^{k,p}(M)} < \sigma, \tag{1.10}$$

then the harmonic map energy functional (1.7) *obeys the* Łojasiewicz-Simon gradient inequality*,*

$$\|\mathscr{E}'(f)\|_{W^{-k,p'}(M)} \geq Z|\mathscr{E}(f) - \mathscr{E}(f_\infty)|^\theta. \tag{1.11}$$

Furthermore, if the hypothesis that (N, h) is analytic is replaced by the condition that $\mathscr{E}$ is Morse-Bott at f_∞, then (1.11) *holds with the optimal exponent $\theta = 1/2$.*

---

[1]Thus $T_f^* W^{k,p}(M; N)$ is the dual of the tangent space $T_f W^{k,p}(M; N)$ of the Banach manifold $W^{k,p}(M; N)$ at the point f.

Finally, if (N,h) *is real analytic or* (N,h) *is* C^∞ *and* $\mathscr{E}$ *is Morse-Bott at* f_∞*, and* $C_0 \in [1,\infty)$ *is a constant, and* $\theta \in [1/2,1)$ *is as in* (1.2)*, then there are constants,* $Z_0 \in [1,\infty)$ *and* $\sigma_0 \in (0,1]$*, depending in addition on* C_0*, with the following significance. If the hypothesis that* $f \in W^{k,p}(M;N)$ *obeys* (1.10) *is replaced by the* L^2 Łojasiewicz-Simon neighborhood *condition,*

$$|\mathscr{E}(f) - \mathscr{E}(f_\infty)| \leq C_0 \quad \text{and} \quad \|f - f_\infty\|_{L^2(M)} < \sigma_0, \tag{1.12}$$

then (1.2) *holds with constant* Z *replaced by* Z_0*.*

Remark 1.10 (Previous versions of the Łojasiewicz-Simon gradient inequality for the harmonic map energy functional). As noted earlier, and Topping [89, Lemma 1] proved a Łojasiewicz-type gradient inequality for maps, $f : S^2 \to S^2$, with small L^2 energy, with the latter criterion replacing the usual small $C^{2,\alpha}$ norm criterion of Simon for the difference between a map and a critical point [80, Theorem 3]. Simon uses a $C^2(\Sigma)$ norm to measure distance between maps, $f : \Sigma \to N$, in [81, Equation (4.27)]. Topping's result is generalized by Liu and Yang in [64, Lemma 3.3]. Kwon [60, Theorem 4.2] obtains a Łojasiewicz-type gradient inequality for maps, $f : S^2 \to N$, that are $W^{2,p}(\Sigma)$-close to a harmonic map, with $1 < p \leq 2$. However, her proof explicitly uses the fact that $p > 1$.

Our interest in Łojasiewicz-Simon gradient inequalities for harmonic map energy functionals is motivated by the wealth of potential applications. We shall survey some of those applications below.

1.3. **Applications of the Łojasiewicz-Simon gradient inequality for the harmonic map energy functional.** Simon applied his Łojasiewicz-Simon gradient inequality [80, Theorem 3] to obtain a global existence and convergence (see [60, Theorem 1.15]) for harmonic map gradient flow in [81]. In [28], we applied the Łojasiewicz-Simon gradient inequality to prove discreteness of L^2 energies of Yang-Mills connections over closed four-dimensional Riemannian smooth manifolds when $d = 4$. In a sequel [31] to the present article, we adapt our proof of that result to the case of harmonic maps from a closed Riemann surface to an analytic closed Riemannian manifold and address the following conjecture due to Fang-Hua Lin [63]:

Conjecture 1.11 (Discreteness for energies of harmonic maps from closed Riemann surfaces into analytic closed Riemannian manifolds)**.** (Lin [63, Conjecture 5.7].) Assume the hypotheses of Theorem 4. Then the subset of critical values of the L^2-energy functional, $\mathscr{E} : W^{2,1}(\Sigma;N) \to [0,\infty)$, is closed and discrete.

1.4. **Outline of the article.** In Section 2, we derive an abstract Łojasiewicz-Simon gradient inequality for an analytic functional over a Banach space, proving Theorem 1 and Corollary 2, and for a Morse-Bott energy functional over a Banach space, proving Theorem 3. In Section 3, we establish the Łojasiewicz-Simon gradient inequality for the harmonic map energy functional, proving Theorem 4. Appendix A reviews the relationship between the Morse-Bott property and the integrability in the setting of harmonic maps.

1.5. **Notation and conventions.** For the notation of function spaces, we follow Adams and Fournier [4], and for functional analysis, Brezis [14] and Rudin [75]. We let $\mathbb{N} := \{0,1,2,3,\dots\}$ denote the set of non-negative integers. We use $C = C(*,\dots,*)$ to denote a constant which depends at most on the quantities appearing on the parentheses. In a given context, a constant denoted by C may have different values depending on the same set of arguments and may increase from one inequality to the next. If $\mathscr{X},\mathscr{Y}$ is a pair of Banach spaces, then $\mathscr{L}(\mathscr{X},\mathscr{Y})$ denotes the Banach space of all continuous linear operators from $\mathscr{X}$ to $\mathscr{Y}$. We denote the continuous dual space of $\mathscr{X}$ by $\mathscr{X}^* = \mathscr{L}(\mathscr{X}, \mathbb{R})$. We write $\alpha(x) = \langle x, \alpha\rangle_{\mathscr{X}\times\mathscr{X}^*}$ for the pairing between $\mathscr{X}$ and its dual space, where $x \in \mathscr{X}$ and $\alpha \in \mathscr{X}^*$. If $T \in \mathscr{L}(\mathscr{X}, \mathscr{Y})$, then its adjoint is denoted by $T^* \in \mathscr{L}(\mathscr{Y}^*, \mathscr{X}^*)$, where $(T^*\beta)(x) := \beta(Tx)$ for all $x \in \mathscr{X}$ and $\beta \in \mathscr{Y}^*$.

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2. Łojasiewicz-Simon gradient inequalities for analytic and Morse-Bott energy functionals

Our goal in this section is to prove the abstract Łojasiewicz-Simon gradient inequalities for analytic and Morse-Bott energy functionals stated in our Introduction, namely Theorems 1 and 3 and Corollary 2. In Sections 2.1 and 2.2, respectively, we review or establish some of the results in linear and nonlinear functional analysis which we will subsequently require. As in Simon's original approach to the proof of his gradient inequality for analytic functionals, one establishes the result in infinite dimensions via a Lyapunov-Schmidt reduction to finite dimensions and an application of the finite-dimensional Łojasiewicz gradient inequality, whose statement we recall in Section 2.3. Section 2.4 contains the proofs of the corresponding gradient inequalities for infinite-dimensional applications.

2.1. **Linear functional analysis preliminaries.** In this subsection, we gather a few elementary observations from linear functional analysis that we will subsequently need.

Lemma 2.1 (Maps of dual spaces induced by an embedding of a Banach space into a Hilbert space)**.** [14, Remark 3, page 136] *Let $\mathscr{H}$ be a Hilbert space, $\mathscr{X}$ be a Banach space with a continuous embedding $\varepsilon : \mathscr{X} \subset \mathscr{H}$, and $\varepsilon^* : \mathscr{H}^* \to \mathscr{X}^*$ be the canonical map of dual spaces (adjoint of the embedding) defined by*

$$\langle x, \varepsilon^* h^*\rangle_{\mathscr{X}\times\mathscr{X}^*} = \langle \varepsilon(x), h^*\rangle_{\mathscr{H}\times\mathscr{H}^*} = (\varepsilon(x), h)_{\mathscr{H}} = \iota(h)\varepsilon(x) \quad \forall\, x \in \mathscr{X}, h \in \mathscr{H}, \tag{2.1}$$

and $h^* \in \mathscr{H}^*$ is defined by the Riesz isomorphism*,*

$$\iota : \mathscr{H} \cong \mathscr{H}^*, \quad h \mapsto h^* := (\cdot, h)_{\mathscr{H}}. \tag{2.2}$$

Then the following hold.

(1) $\|\varepsilon^*\alpha\|_{\mathscr{X}^*} \leq C\|\alpha\|_{\mathscr{H}^*}$ *for all $\alpha \in \mathscr{H}^*$, where C is the norm of the embedding $\varepsilon : \mathscr{X} \subset \mathscr{H}$, and the linear map $\varepsilon^* : \mathscr{H}^* \to \mathscr{X}^*$ is continuous;*

(2) The composition,

$$\jmath \equiv \varepsilon^* \circ \iota \circ \varepsilon : \mathscr{X} \to \mathscr{X}^*, \tag{2.3}$$

is injective and thus a continuous embedding;

(3) If $\mathscr{X}$ is dense in $\mathscr{H}$, then $\varepsilon^* : \mathscr{H}^* \to \mathscr{X}^*$ is injective and thus a continuous embedding;

(4) $\operatorname{Ran}\varepsilon^* \subset \mathscr{X}^*$ *is dense if $\mathscr{X}$ is dense in $\mathscr{H}$ and $\mathscr{X}$ is reflexive.*

Proof. We only need to prove Item (2), since the remaining Items are given by [14, Remark 3, page 136]. Observe that, for all $x, y \in \mathscr{X}$, the definition of $\jmath$ gives

$$\jmath(x)(y) = \langle y, \varepsilon^* \iota \varepsilon(x) \rangle_{\mathscr{X} \times \mathscr{X}^*} = (\varepsilon(y), \varepsilon(x))_{\mathscr{H}},$$

so $(\varepsilon(y), \varepsilon(x))_{\mathscr{H}} = 0$ for all $y \in \mathscr{X}$ if $\jmath(x)(y) = 0$ for all $y \in \mathscr{X}$. Choosing $y = x$ gives $\|\varepsilon(x)\|_{\mathscr{H}} = 0$, that is, $x = 0$ since $\varepsilon : \mathscr{X} \subset \mathscr{H}$ is a continuous embedding. Moreover, $\jmath$ is continuous by Item (1). □

Remark 2.2 (Notation for embedding of a Banach space into a Hilbert space or into its Banach dual space). Although the adjoint, $\varepsilon^* : \mathscr{H}^* \to \mathscr{X}^*$, of a continuous embedding, $\mathscr{X} \subset \mathscr{H}$, is not necessarily also an embedding unless $\mathscr{X}$ is dense in the pivot Hilbert space, $\mathscr{H}$, Lemma 2.1 shows that the composition, $\jmath = \varepsilon^* \circ \iota \circ \varepsilon : \mathscr{X} \to \mathscr{X}^*$, *is* an embedding and thus we may identify $\mathscr{X}$ with its image $\jmath(\mathscr{X}) \subset \mathscr{X}^*$, suppress further explicit mention of $\jmath$ unless otherwise noted (for example, in Lemma 2.4), and write $\mathscr{X} \subset \mathscr{X}^*$. Similarly, unless otherwise noted (for example, in Lemma 2.4), we identify $\mathscr{X}$ with its image $\varepsilon(\mathscr{X}) \subset \mathscr{H}$, suppress explicit mention of ε, and write $\mathscr{X} \subset \mathscr{H}$.

We often encounter symmetric operators in the form of the Hessian of a functional on a Banach space and thus it is convenient to have a simple criterion for when they are Fredholm with index zero.

Lemma 2.3 (Fredholm property and index of a bounded, linear, symmetric operator with closed range)**.** *Let $\mathscr{X}$ be a reflexive Banach space. If $\mathscr{A} : \mathscr{X} \to \mathscr{X}^*$ is a bounded, linear, symmetric operator with closed range, then $\mathscr{A}$ is Fredholm with index zero.*

Proof. If $M \subset X^*$ is a subspace, we recall from [75, Section 4.6] that its annihilator is

$$M^\circ := \{\phi \in \mathscr{X}^{**} : \langle \alpha, \phi \rangle_{\mathscr{X}^* \times \mathscr{X}^{**}} = 0, \ \forall\, \alpha \in M\},$$

and that by [75, Theorem 4.12],

$$(\operatorname{Ran} \mathscr{A})^\circ = \operatorname{Ker} \mathscr{A}^*,$$

where $\mathscr{A}^* : \mathscr{X}^{**} \to \mathscr{X}^*$ is the adjoint operator defined by

$$(\mathscr{A}^* \phi)(x) = \phi(\mathscr{A} x), \quad \forall\, x \in \mathscr{X} \quad \text{and} \quad \phi \in \mathscr{X}^{**}.$$

If $J : \mathscr{X} \to \mathscr{X}^{**}$ is the canonical map defined by $J(x)\alpha = \alpha(x)$ for all $x \in \mathscr{X}$ and $\alpha \in \mathscr{X}^*$, then J is an isomorphism by hypothesis that $\mathscr{X}$ is reflexive and thus

$$\langle y, \mathscr{A}^* J(x) \rangle_{\mathscr{X} \times \mathscr{X}^*} = (\mathscr{A}^* J(x))(y) = J(x)(\mathscr{A} y) = \langle x, \mathscr{A} y \rangle_{\mathscr{X} \times \mathscr{X}^*} \quad \forall\, x, y \in \mathscr{X}.$$

Hence,

$$\begin{aligned}
\operatorname{Ker} \mathscr{A}^* &= \{\phi \in \mathscr{X}^{**} : \langle y, \mathscr{A}^* \phi \rangle_{\mathscr{X} \times \mathscr{X}^*} = 0, \ \forall\, y \in \mathscr{X}\} \\
&\cong \{x \in \mathscr{X} : \langle y, \mathscr{A}^* J(x) \rangle_{\mathscr{X} \times \mathscr{X}^*} = 0, \ \forall\, y \in \mathscr{X}\} \\
&= \{x \in \mathscr{X} : \langle x, \mathscr{A} y \rangle_{\mathscr{X} \times \mathscr{X}^*} = 0, \ \forall\, y \in \mathscr{X}\} \\
&= \{x \in \mathscr{X} : \langle y, \mathscr{A} x \rangle_{\mathscr{X} \times \mathscr{X}^*} = 0, \ \forall\, y \in \mathscr{X}\} \quad \text{(by symmetry of } \mathscr{A}) \\
&= \operatorname{Ker} \mathscr{A}.
\end{aligned}$$

On the other hand, using the quotient map $\pi : \mathscr{X}^* \to \mathscr{X}^* / \operatorname{Ran} \mathscr{A} = \operatorname{Coker} \mathscr{A}$ and employing [14, Proposition 11.9] with $M = \operatorname{Ran} \mathscr{A} \subset E = \mathscr{X}^*$ in that proposition, which is closed by our hypothesis, the adjoint map,

$$\pi^* : (\mathscr{X}^* / \operatorname{Ran} \mathscr{A})^* \to (\operatorname{Ran} \mathscr{A})^\circ,$$

is a well-defined isometric isomorphism. Therefore $\operatorname{Ker}\mathscr{A} \cong (\operatorname{Ran}\mathscr{A})^\circ$ has the same dimension as $(\operatorname{Coker}\mathscr{A})^*$ and hence the same dimension as $\operatorname{Coker}\mathscr{A}$. Thus, $\operatorname{Coker}\mathscr{A}$ is finite-dimensional and $\mathscr{A}$ is Fredholm with index zero. □

When $\mathscr{X}$ is a Banach space that is continuously embedded in a Hilbert space $\mathscr{H}$, we next observe that the orthogonal projection of $\mathscr{H}$ onto a finite-dimensional subspace $K \subset \mathscr{X}$ extends to a continuous linear projection on the dual space $\mathscr{X}^*$.

Lemma 2.4 (Extension of orthogonal projection onto a finite-dimensional subspace to a continuous linear projection on a Banach dual space)**.** *Let $\mathscr{X}$ be a Banach space with a continuous embedding, $\varepsilon : \mathscr{X} \subset \mathscr{H}$, into a Hilbert space $\mathscr{H}$, let $\iota : \mathscr{H} \cong \mathscr{H}^*$ denote the Riesz isomorphism* (2.2)*, let $\jmath = \varepsilon^* \circ \iota \circ \varepsilon : \mathscr{X} \to \mathscr{X}^*$ denote the embedding* (2.3)*, and let $K \subset \mathscr{X}$ be a finite-dimensional subspace. If $\Pi : \mathscr{H} \to K$ is the $\mathscr{H}$-orthogonal projection, then $\widetilde{\Pi} \equiv (\Pi \circ \varepsilon)^* : \mathscr{X}^* \to \mathscr{X}^*$, has the following properties:*

(1) $\widetilde{\Pi}$ is continuous with operator norm $\|\widetilde{\Pi}\|_{\mathscr{L}(\mathscr{X}^*)} = \|\Pi \circ \varepsilon\|_{\mathscr{L}(\mathscr{X})}$;
(2) $\operatorname{Ran}\widetilde{\Pi} = \jmath(K) \subset \mathscr{X}^*$;
(3) $\widetilde{\Pi}\varepsilon^*\iota(h) = \jmath(\Pi h)$ for all $h \in \mathscr{H}$;
(4) $\widetilde{\Pi}\jmath(x) = \jmath(\Pi\varepsilon(x))$ for all $x \in \mathscr{X}$;
(5) $\widetilde{\Pi}\jmath(k) = \jmath(k)$ for all $k \in K \subset \mathscr{X}$.

Remark 2.5 (Notation for projection on dual Banach space). Henceforth, when this can cause no confusion, we shall suppress explicit mention of the embedding ε and identify $\varepsilon(\mathscr{X})$ with its image in $\mathscr{H}$. Similarly, we shall suppress explicit mention of the embedding $\jmath$ and identify $K, \mathscr{X}$ with their images $\jmath(K), \jmath(\mathscr{X}) \subset \mathscr{X}^*$ and write $K, \mathscr{X} \subset \mathscr{X}^*$, together with $\widetilde{\Pi} = \Pi$ on $\mathscr{X}$ or $\mathscr{H}$, and $\widetilde{\Pi} = \mathrm{id}_K$ on K, and $\operatorname{Ran}\widetilde{\Pi} = K \subset \mathscr{X}^*$.

Remark 2.6 (On the role of the Hilbert space). Suppose that $\mathscr{X}$ is a Banach space. By [14, Example 1, page 38] or [75, Lemma 4.21(a)] (an application of the Hahn-Banach [75, Theorem 3.3]), a finite-dimensional subspace $K \subset \mathscr{X}$ admits a topological complement, a closed subspace $\mathscr{L} \subset \mathscr{X}$ such that $\mathscr{X} = K \oplus \mathscr{L}$, that is, $\mathscr{X} = K + \mathscr{L}$ and $K \cap \mathscr{L} = \{0\}$. The linear projection $\Pi : \mathscr{X} \to K$ onto the factor K is continuous by [14, Theorem 2.10]. If one assumed the existence of a continuous embedding $\jmath : \mathscr{X} \to \mathscr{X}^*$, with suitable conditions, it may be possible to reproduce the conclusions of Lemma 2.4 without the introduction of an auxiliary Hilbert space, $\mathscr{H}$, but any such slight increase in generality appears to have limited practical application. See Capraro and Rossi [15] for a discussion of related issues.

Proof of Lemma 2.4. The assertions in Item (1) follow from [75, Theorem 4.10].

Let $\{k_i\}_{i=1}^\kappa$ be an $\mathscr{H}$-orthonormal basis for K and define $\{\alpha_i\}_{i=1}^\kappa \subset \mathscr{X}^*$ by setting $\alpha_i := \jmath(k_i) \in \mathscr{X}^*$ for $i = 1, \dots, \kappa$. Hence, noting that $K \subset \mathscr{X}$ and identifying X with its image $\varepsilon(X) \subset \mathscr{H}$,

$$\alpha_i(k_j) = \jmath(k_i)(k_j) = \varepsilon^*\iota\varepsilon(k_i)(k_j) = (\varepsilon(k_j), \varepsilon(k_i))_{\mathscr{H}} = \delta_{ij}, \quad \forall\, i, j,$$

where δ_{ij} is the Kronecker delta.

For Item (2), we first note that the projection operator Π is given by

$$\Pi h = \sum_{i=1}^{\kappa} (h, k_i)_{\mathscr{H}}\, k_i, \quad \forall\, h \in \mathscr{H},$$

and, for all $x \in \mathscr{X}$, since $\alpha_i(x) = \jmath(k_i)(x) = (\varepsilon(x), \varepsilon(k_i))_{\mathscr{H}}$,

$$\Pi\varepsilon(x) = \sum_{i=1}^{\kappa} (\varepsilon(x), \varepsilon(k_i))_{\mathscr{H}}\, k_i = \sum_{i=1}^{\kappa} \alpha_i(x) k_i.$$

Consequently, for all $x \in \mathscr{X}$ and $\alpha \in \mathscr{X}^*$,

$$\left(\widetilde{\Pi}\alpha\right)(x) = ((\Pi \circ \varepsilon)^*\alpha)(x) = \alpha(\Pi\varepsilon(x)) = \sum_{i=1}^{\kappa} \alpha_i(x)\alpha(k_i) = \left(\sum_{i=1}^{\kappa} \alpha(k_i)\alpha_i\right)(x),$$

and therefore,

$$\widetilde{\Pi}\alpha = \sum_{i=1}^{\kappa} \alpha(k_i)\alpha_i.$$

Hence, $\operatorname{Ran}\widetilde{\Pi} = \jmath(K)$, which is Item (2).

For Item (3), we observe that

$$\left(\widetilde{\Pi}\varepsilon^*\iota(h)\right)(y) = \left(\sum_{i=1}^{\kappa} \varepsilon^*\iota(h)(k_i)\,\alpha_i\right)(y) = \sum_{i=1}^{\kappa} \varepsilon^*\iota(h)(k_i)\,\alpha_i(y)$$

while

$$\jmath(\Pi h)(y) = \jmath\left(\sum_{i=1}^{\kappa}(h, \varepsilon(k_i))_{\mathscr{H}}\, k_i\right)(y) = \left(\sum_{i=1}^{\kappa}(h, \varepsilon(k_i))_{\mathscr{H}}\, \jmath(k_i)\right)(y) = \sum_{i=1}^{\kappa}(h, \varepsilon(k_i))_{\mathscr{H}}\, \jmath(k_i)(y).$$

We have $\varepsilon^*\iota(h)(k_i) = (\varepsilon(k_i), h)_{\mathscr{H}} = (h, \varepsilon(k_i))_{\mathscr{H}}$ for all $h \in \mathscr{H}$, while $\jmath(k_i)(y) = \alpha_i(y)$ for all $y \in \mathscr{X}$ by definition of the α_i. Hence,

$$\jmath(\Pi h)(y) = \sum_{i=1}^{\kappa} \varepsilon^*\iota(h)(k_i)\,\alpha_i(y) = \left(\widetilde{\Pi}\varepsilon^*\iota(h)\right)(y), \quad \forall\, h \in \mathscr{H}, y \in \mathscr{X},$$

which is Item (3). Then Item (4) is an immediate consequence of Item (3) by taking $h = \varepsilon(x)$ and recalling that $\jmath(x) = \varepsilon^*\iota\varepsilon(x)$ for all $x \in \mathscr{X}$. Item (5) follows from the fact that $\Pi = \mathrm{id}_K$ on K. □

2.2. **Nonlinear functional analysis preliminaries.** In this subsection, we gather a few elementary observations from nonlinear functional analysis that we will subsequently need.

2.2.1. *Smooth and analytic inverse and implicit function theorems for maps on Banach spaces.* Statements and proofs of the Inverse Function Theorem for C^k maps of Banach spaces are provided by Abraham Marsden, and Ratiu [1, Theorem 2.5.2], Deimling [25, Theorem 4.15.2], Zeidler [93, Theorem 4.F]; statements and proofs of the Inverse Function Theorem for *analytic* maps of Banach spaces are provided by Berger [7, Corollary 3.3.2] (complex), Deimling [25, Theorem 4.15.3] (real or complex), and Zeidler [93, Corollary 4.37] (real or complex). The corresponding C^k or Analytic Implicit Function Theorems are proved in the standard way as corollaries, for example [1, Theorem 2.5.7] and [93, Theorem 4.H].

2.2.2. *Differentiable and analytic maps on Banach spaces.* We refer to [52, Section 2.1A]; see also [7, Section 2.3]. Let $\mathscr{X}, \mathscr{Y}$ be a pair of Banach spaces, let $\mathscr{U} \subset \mathscr{X}$ be an open subset, and $\mathscr{F} : \mathscr{U} \to \mathscr{Y}$ be a map. Recall that $\mathscr{F}$ is *Gâteaux differentiable* at a point $u \in \mathscr{U}$ with a Gâteaux derivative, $\mathscr{F}'(x) \in \mathscr{L}(\mathscr{X}, \mathscr{Y})$, if

$$\lim_{t\to 0} \frac{1}{t}\|\mathscr{F}(x + ty) - \mathscr{F}(x) - \mathscr{F}'(x)ty\|_{\mathscr{Y}} = 0, \quad \forall\, y \in \mathscr{X}.$$

Furthermore, if $\mathscr{F}$ is Gâteaux differentiable at $u \in \mathscr{U}$ and

$$\lim_{y\to 0} \frac{1}{\|y\|_{\mathscr{X}}}\|\mathscr{F}(x + y) - \mathscr{F}(x) - \mathscr{F}'(x)y\|_{\mathscr{Y}} = 0,$$

then $\mathscr{F}$ is said to be *Fréchet differentiable* at $x \in \mathscr{U}$. If $\mathscr{F}$ is Gâteaux differentiable near x and the Gâteaux derivative is continuous at x, then $\mathscr{F}$ is *Fréchet differentiable* at u [25, Proposition 2.7.5].

Recall from [7, Definition 2.3.1], [25, Definition 15.1], [93, Definition 8.8] that $\mathscr{F}$ is (real) *analytic* at $x \in \mathscr{U}$ if there exists a constant $r > 0$ and a sequence of continuous symmetric n-linear forms, $L_n : \mathscr{X} \times \cdots \times \mathscr{X} \to \mathscr{Y}$, such that $\sum_{n\geq 1} \|L_n\| r^n < \infty$ and there is a positive constant $\delta = \delta(x)$ such that

$$\mathscr{F}(x+y) = \mathscr{F}(x) + \sum_{n\geq 1} L_n(y^n), \quad \|y\|_{\mathscr{X}} < \delta, \tag{2.4}$$

where $y^n \equiv (y, \dots, y) \in \mathscr{X} \times \cdots \times \mathscr{X}$ (n-fold product). If $\mathscr{F}$ is differentiable (respectively, analytic) at every point $x \in \mathscr{U}$, then $\mathscr{F}$ is differentiable (respectively, analytic) on $\mathscr{U}$. It is a useful observation that if $\mathscr{F}$ is analytic at $x \in \mathscr{X}$, then it is analytic on a ball $B_x(\varepsilon)$ [90, p. 1078].

2.2.3. *Gradient maps.* We refer to [52, Section 2.1B]; see also [7, Section 2.5].

Definition 2.7 (Gradient map)**.** [52, Definition 2.1.1] Let $\mathscr{U}$ be an open subset of a Banach space, $\mathscr{X}$. A continuous map, $\mathscr{M} : \mathscr{U} \to \mathscr{X}^*$, is called a *gradient map* if there exists a C^1 functional, $\mathscr{E} : \mathscr{U} \to \mathbb{R}$, such that $\mathscr{M}(x) = \mathscr{E}'(x)$ for all $x \in \mathscr{U}$ in the sense that,

$$\mathscr{E}'(x)y = \langle h, \mathscr{M}(x)\rangle_{\mathscr{X}\times\mathscr{X}^*}, \quad \forall\, x \in \mathscr{U}, \quad y \in \mathscr{X},$$

where $\langle\cdot,\cdot\rangle_{\mathscr{X}\times\mathscr{X}^*}$ is the canonical bilinear form on $\mathscr{X} \times \mathscr{X}^*$. The real-valued function, $\mathscr{E}$, is called a *potential* for the map $\mathscr{M}$.

We recall the following basic facts concerning gradient maps.

Proposition 2.8 (Properties of gradient maps)**.** [52, Proposition 2.1.2] *Let $\mathscr{U} \subset \mathscr{X}$ be an open subset of a Banach space, $\mathscr{X}$, and let $\mathscr{M} : \mathscr{U} \to \mathscr{X}^*$ be a continuous map. Then the following hold.*

(1) If $\mathscr{M}$ is of class C^1, then $\mathscr{M}$ is a gradient map if and only if all of its Fréchet derivatives, $\mathscr{M}'(x)$ for $x \in \mathscr{U}$, are symmetric in the sense that

$$\langle w, \mathscr{M}'(x)y\rangle_{\mathscr{X}\times\mathscr{X}^*} = \langle y, \mathscr{M}'(x)w\rangle_{\mathscr{X}\times\mathscr{X}^*}, \quad \forall\, x, y, w \in \mathscr{U}.$$

(2) A bounded linear operator $\mathscr{A} : \mathscr{X} \to \mathscr{X}^*$ is a gradient operator if and only if $\mathscr{A}$ is symmetric, in which case a potential for $\mathscr{A}$ is given by $\mathscr{E}(x) = \frac{1}{2}\langle x, \mathscr{A}x\rangle_{\mathscr{X}\times\mathscr{X}^*}$, for all $x \in \mathscr{X}$.

(3) If $\mathscr{M}$ is an analytic gradient map, then any potential $\mathscr{E} : \mathscr{U} \to \mathbb{R}$ such that $\mathscr{M} = \mathscr{E}'$ is analytic as well.

2.3. **Finite dimensional Łojasiewicz and Simon gradient inequalities.** We recall the finite-dimensional versions of the Łojasiewicz-Simon gradient inequality.

Theorem 2.9 (Finite-dimensional Łojasiewicz and Simon gradient inequalities)**.** [52, Theorem 2.3.1] [2] *Let $U \subset \mathbb{R}^n$ be an open subset, $z \in U$, and let $\mathscr{E} : U \to \mathbb{R}$ be a real-valued function.*

(1) If $\mathscr{E}$ is real analytic on a neighborhood of z and $\mathscr{E}'(z) = 0$, then there exist constants $\theta \in (0, 1)$ and $\sigma > 0$ such that

$$|\mathscr{E}'(x)| \geq |\mathscr{E}(x) - \mathscr{E}(z)|^\theta, \quad \forall\, x \in \mathbb{R}^n,\ |x - z| < \sigma. \tag{2.5}$$

[2] There is a typographical error in the statement of [52, Theorem 2.3.1 (i)], as Huang omits the hypothesis that $\mathscr{E}'(z) = 0$; also our statement differs slightly from that of [52, Theorem 2.3.1 (i)], but is based on original sources.

(2) *Assume that $\mathscr{E}$ is a C^2 function and $\mathscr{E}'(z) = 0$. If the connected component, C, of the critical point set, $\{x \in U : \mathscr{E}'(x) = 0\}$, that contains z has the same dimension as the kernel of the Hessian matrix $\operatorname{Hess}_{\mathscr{E}}(z)$ of $\mathscr{E}$ at z locally near z, and z lies in the interior of the component, C, then there are positive constants, c and σ, such that*

$$|\mathscr{E}'(x)| \geq c|\mathscr{E}(x) - \mathscr{E}(z)|^{1/2}, \quad \forall\, x \in \mathbb{R}^n,\ |x - z| < \sigma. \tag{2.6}$$

Theorem 2.9 (1) is well known and was stated by Łojasiewicz in [65] and proved by him as [66, Proposition 1, p. 92] and Bierstone and Milman as [10, Proposition 6.8]; see also the statements by Chill and Jendoubi [22, Proposition 5.1 (i)] and by Łojasiewicz [67, p. 1592].

Theorem 2.9 (2) was proved by Simon as [82, Lemma 1, p. 80] and Haraux and Jendoubi as [44, Theorem 2.1]; see also the statement by Chill and Jendoubi [22, Proposition 5.1 (ii)].

Łojasiewicz used methods of *semi-analytic sets* [66] to prove Theorem 2.9 (1). For the inequality (2.5), unlike (2.6), the constant, c, is equal to one while $\theta \in (0, 1)$. In general, so long as c is positive, its actual value is irrelevant to applications; the value of θ in the infinite-dimensional setting [52, Theorem 2.4.2 (i)], at least, is restricted to the range $[1/2, 1)$ and $\theta = 1/2$ is optimal [52, Theorem 2.7.1].

2.4. **Łojasiewicz-Simon gradient inequalities for analytic or Morse-Bott functionals on Banach spaces.** We note that if $\mathscr{E} : U \to \mathbb{R}$ is a C^2 functional on an open subset U of a Banach space $\mathscr{X}$, then its Hessian operator at a point $x_0 \in U$ is symmetric, that is

$$\langle x, \mathscr{E}''(x_0)y\rangle_{\mathscr{X}\times\mathscr{X}^*} = \langle y, \mathscr{E}''(x_0)x\rangle_{\mathscr{X}\times\mathscr{X}^*}, \tag{2.7}$$

for all $x, y \in \mathscr{X}$; compare Proposition 2.8, Item (1). Moreover, Lemma 2.3 shows that if $\mathscr{E}''(x_0) : \mathscr{X} \to \mathscr{X}^*$ is Fredholm, then it necessarily has index zero if $\mathscr{X}$ is reflexive, though we emphasize that we do not assume that $\mathscr{X}$ is reflexive except when that property is explicitly stated. Throughout this section we employ the conventions of Remarks 2.2 and 2.5.

Lemma 2.10 (Properties of C^2 functionals with Fredholm Hessian maps of index zero)**.** *Let $\mathscr{X}$ be a Banach space that is continuously embedded in a Hilbert space $\mathscr{H}$ and let $U \subset \mathscr{X}$ be an open subset. Let $\mathscr{E} : U \to \mathbb{R}$ be a C^2 functional with gradient map, $\mathscr{E}' : U \to \mathscr{X}^*$, let $x_\infty \in U$ be a critical point of $\mathscr{E}$, and assume that the Hessian $\mathscr{E}''(x_\infty) : \mathscr{X} \to \mathscr{X}^*$ of $\mathscr{E}$ at x_∞ is a Fredholm operator with index zero. Let $\Pi : \mathscr{H} \to K$ be the $\mathscr{H}$-orthogonal projection onto $K := \operatorname{Ker} \mathscr{E}''(x_\infty) \subset \mathscr{X} \subset \mathscr{H}$ and let $\Pi : \mathscr{X}^* \to \mathscr{X}^*$ denote its bounded linear extension provided by Lemma 2.4. Then there exist an open neighborhood $U_0 \subset U$ of x_∞ and an open neighborhood $V_0 \subset \mathscr{X}^*$ of the origin such that the C^1 map,*

$$\Phi : U \to \mathscr{X}^*, \quad x \mapsto \mathscr{E}'(x) + \Pi(x - x_\infty), \tag{2.8}$$

when restricted to U_0, has a C^1 inverse, $\Psi : V_0 \to U_0$. Moreover, there is a constant $C = C(\mathscr{E}, U_0, V_0) \in [1, \infty)$ such that

$$\|\Psi(\Pi\alpha) - \Psi(\alpha)\|_{\mathscr{X}} \leq C\|\mathscr{E}'(\Psi(\alpha))\|_{\mathscr{X}^*}, \quad \forall\, \alpha \in V_0. \tag{2.9}$$

Proof. The derivative of Φ at x_∞ is given by $D\Phi(x_\infty) = \mathscr{E}''(x_\infty) + \Pi : \mathscr{X} \to \mathscr{X}^*$. Thus if $D\Phi(x_\infty)(x) = 0$ for some $x \in \mathscr{X}$, then $\mathscr{E}''(x_\infty)(x) = -\Pi x \in K \subset \mathscr{H}$ and if $y \in K$,

$$\begin{aligned}
(y, \Pi x)_{\mathscr{H}} &= -(y, \mathscr{E}''(x_\infty)(x))_{\mathscr{H}} \\
&= -\langle y, \mathscr{E}''(x_\infty)(x)\rangle_{\mathscr{X}\times\mathscr{X}^*} \qquad \text{(by (2.3))} \\
&= -\langle x, \mathscr{E}''(x_\infty)(y)\rangle_{\mathscr{X}\times\mathscr{X}^*} \qquad \text{(by (2.7))} \\
&= 0 \qquad (\text{since } y \in \operatorname{Ker} \mathscr{E}''(x_\infty)),
\end{aligned}$$

where in the application of (2.3) we use the fact that $\jmath : \mathscr{X} \subset \mathscr{X}^*$ is an embedding and identify $\mathscr{X}$ with $\jmath(\mathscr{X}) \subset \mathscr{X}^*$. In particular, for $y = \Pi x \in K$ we deduce that $\mathscr{E}''(x_\infty)(x) = -\Pi x = 0$, so that $x \in K$ and $x = \Pi x = 0$. Therefore $D\Phi(x_\infty)$ has trivial kernel.

Because $\mathscr{E}''(x_\infty)$ is Fredholm and $\Pi : \mathscr{X}^* \to \mathscr{X}^*$ is finite-rank by Lemma 2.4 (2), it follows that

$$D\Phi(x_\infty) = \mathscr{E}''(x_\infty) + \Pi : \mathscr{X} \to \mathscr{X}^*$$

is Fredholm. Now $D\Phi(x_\infty) : \mathscr{X} \to \mathscr{X}^*$ is an injective Fredholm operator with index zero and therefore surjective too. By the Open Mapping Theorem, $D\Phi(x_\infty)$ has a bounded inverse. Applying the Inverse Function Theorem for Φ near x_∞, there exist an open neighborhood $U_1 \subset U$ of x_∞ and a convex open neighborhood $V_1 \subset \mathscr{X}^*$ of the origin in $\mathscr{X}^*$ so that the C^1 inverse $\Psi : V_1 \to U_1$ of Φ is well-defined. Since $\Pi : \mathscr{X}^* \to \mathscr{X}^*$ is bounded by Lemma 2.4, we may choose $V_0 \subset V_1$, a smaller open neighborhood of the origin in $\mathscr{X}^*$, with $\Pi(V_0) \subset V_1$ and set $U_0 := \Psi(V_0)$. From (2.8), we have

$$\Phi(x) = \mathscr{E}'(x) + \Pi(x - x_\infty), \quad \forall\, x \in U_0,$$

and the inverse function property and writing $\alpha = \Phi(x) \in V_0$ and $x = \Psi(\alpha)$ for $x \in U_0$, we obtain

$$\alpha = \mathscr{E}'(\Psi(\alpha)) + \Pi(\Psi(\alpha) - x_\infty)), \quad \forall\, \alpha \in V_0. \tag{2.10}$$

The Fundamental Theorem of Calculus then yields

$$\begin{aligned}\Psi(\Pi\alpha) - \Psi(\alpha) &= \int_0^1 \left(\frac{d}{dt}\Psi(\alpha + t(\Pi\alpha - \alpha))\right) dt \\ &= \left(\int_0^1 D\Psi(\alpha + t(\Pi\alpha - \alpha))\, dt\right)(\Pi\alpha - \alpha), \quad \forall\, \alpha \in V_0,\end{aligned}$$

where we use the fact that for $\alpha \in V_0$, we have $\alpha, \Pi\alpha \in V_1$ and, by convexity of V_1, the map Ψ is well defined on the line segment joining α to $\Pi\alpha$. (Note also that in the preceding identity, we implicitly make use of the embedding [75, Section 4.5] $\mathscr{X} \subset \mathscr{X}^{**}$ defined by $x(\alpha) = \alpha(x)$, for all $x \in \mathscr{X}$ and $\alpha \in \mathscr{X}^*$.) Therefore,

$$\|\Psi(\Pi\alpha) - \Psi(\alpha)\|_{\mathscr{X}} \le M\|\Pi\alpha - \alpha\|_{\mathscr{X}^*}, \quad \forall\, \alpha \in V_0,$$

where, since $D\Psi(\alpha_1) \in \mathscr{L}(\mathscr{X}^*, \mathscr{X})$ is a continuous function of $\alpha_1 \in V_1$ (as $\Psi : V_1 \to U_1$ is C^1 by construction), we have

$$M := \sup_{\alpha_1 \in V_1} \|D\Psi(\alpha_1)\|_{\mathscr{L}(\mathscr{X}^*, \mathscr{X})} < \infty,$$

because we may assume without loss of generality that $V_1 \supset V_0$ is a sufficiently small and bounded (convex) open neighborhood of the origin. Also, for all $\alpha \in V_0$,

$$\begin{aligned}\Pi\alpha - \alpha &= \Pi\alpha - \mathscr{E}'(\Psi(\alpha)) - \Pi(\Psi(\alpha) - x_\infty)) \qquad \text{(by (2.10))} \\ &= \Pi(\alpha - \Pi(\Psi(\alpha) - x_\infty)) - \mathscr{E}'(\Psi(\alpha)) \qquad \text{(since } \Pi^2 = \Pi \text{ by Lemma 2.4 (5))},\end{aligned}$$

and

$$\begin{aligned}\|\Pi(\alpha - \Pi(\Psi(\alpha) - x_\infty))\|_{\mathscr{X}^*} &\le C_1\|\alpha - \Pi(\Psi(\alpha) - x_\infty)\|_{\mathscr{X}^*} \qquad \text{(by Lemma 2.4 (1))} \\ &= C_1\|\mathscr{E}'(\Psi(\alpha))\|_{\mathscr{X}^*} \qquad \text{(by (2.10))}.\end{aligned}$$

Taking norms, we conclude that

$$\|\Pi\alpha - \alpha\|_{\mathscr{X}^*} \le (C_1 + 1)\|\mathscr{E}'(\Psi(\alpha))\|_{\mathscr{X}^*}, \quad \forall\, \alpha \in V_0.$$

Therefore, by combining the preceding inequalities, we obtain

$$\|\Psi(\Pi\alpha) - \Psi(\alpha)\|_{\mathscr{X}} \le M(C_1 + 1)\|\mathscr{E}'(\Psi(\alpha))\|_{\mathscr{X}^*}, \quad \forall\, \alpha \in V_0,$$

and this concludes the proof of the lemma. ☐

Recall the Definition 1.6 of a Morse-Bott functional $\mathscr{E}$ and its set $\operatorname{Crit}\mathscr{E}$ of critical values.

Definition 2.11 (Lyapunov-Schmidt reduction of a C^2 functional with a Fredholm Hessian map)**.** Assume the hypotheses of Lemma 2.10 and let $\Psi : V_0 \cong U_0$ be the C^1 diffeomorphism of open neighborhoods, $V_0 \subset \mathscr{X}^*$ of the origin and $U_0 \subset \mathscr{X}$ of x_∞, provided by that lemma. We define the *Lyapunov-Schmidt reduction of* $\mathscr{E} : U_0 \to \mathbb{R}$ *at* x_∞ by

$$\Gamma : K \cap V_0 \to \mathbb{R}, \quad \alpha \mapsto \mathscr{E}(\Psi(\alpha)),$$

where $K = \operatorname{Ker}\mathscr{E}''(x_\infty) \subset \mathscr{X}$ as in Lemma 2.10 and we are implicitly applying the embedding $\jmath : \mathscr{X} \subset \mathscr{X}^*$ defined by (2.3) in Lemma 2.1.

Note that the origin in $\mathscr{X}^*$ is a critical point of Γ since $\Psi(0) = x_\infty$, the critical point of $\mathscr{E} : U \to \mathbb{R}$ in Lemma 2.10 and

$$\Gamma'(0)(x) = \mathscr{E}'(\Psi(0))D\Psi(0)(x) = \mathscr{E}'(x_\infty)D\Psi(0)(x) = 0, \quad \forall\, x \in \mathscr{X}.$$

The following lemma plays a crucial role in the proofs of Theorems 1 and 3.

Lemma 2.12 (Properties of the Lyapunov-Schmidt reduction of a C^2 functional with a Fredholm Hessian map)**.** *Assume the hypotheses of Lemma 2.10 and the notation of Definition 2.11.*

(1) If $\mathscr{E}$ is Morse-Bott at x_∞, then there is an open neighborhood $\mathscr{V}$ of the origin in $K \cap V_0$ where the Lyapunov-Schmidt reduction of $\mathscr{E}$ is a constant function, that is,

$$\Gamma \equiv \mathscr{E}(x_\infty) \quad \text{on } \mathscr{V}.$$

.

(2) If $\mathscr{E}$ is real analytic on U, then Γ is real analytic on $K \cap V_0$.

Proof. If $\mathscr{E}$ is Morse-Bott at x_∞ then, by shrinking U_0 if necessary, we may assume that the set $\operatorname{Crit}\mathscr{E} \cap U_0$ is a submanifold of U_0 with tangent space $T_{x_\infty}\operatorname{Crit}\mathscr{E} = K$. Then the restriction of the map $\Phi : U_0 \to V_0$ in (2.8),

$$\Phi : \operatorname{Crit}\mathscr{E} \cap U_0 \to K \cap V_0, \tag{2.11}$$

has differential at x_∞ given by

$$D\Phi(x_\infty) = \mathscr{E}'(x_\infty) + \Pi = \Pi : K \to K.$$

The preceding operator comprises the embedding $\jmath$ provided by Lemma 2.1 (2) and in particular is an isomorphism. An application of the Inverse Function Theorem shows that the inverse of the map (2.11) is defined in a neighborhood $\mathscr{V}$ of the origin in $K \cap V_0$ and is the restriction of the map $\Psi : V_0 \to U_0$ to $K \cap V_0$. Therefore, $\Psi(\mathscr{V}) \subset \operatorname{Crit}\mathscr{E} \cap U_0$ and we compute

$$\Gamma'(\alpha) = \mathscr{E}'(\Psi(\alpha))D\Psi(\alpha) = 0, \quad \forall\, \alpha \in \mathscr{V}.$$

Therefore, $\Gamma(\alpha) = \Gamma(0) = \mathscr{E}(x_\infty)$, for every $\alpha \in \mathscr{V}$. This proves Item (1).

To prove Item (2), we recall from Lemma 2.10 that $\Phi : U_0 \to V_0$ is a diffeomorphism. Moreover, Φ is real analytic since $\mathscr{E}'$ is real analytic. By the Analytic Inverse Function Theorem (see Section 2.2.1) the inverse map, $\Psi : V_0 \to U_0$, is also real analytic and therefore its restriction to the intersection $K \cap V_0$ of a finite-dimensional linear subspace $K \subset \mathscr{X}^*$ with the open set $V_0 \subset \mathscr{X}^*$ is still real analytic. Since $\mathscr{E} : U \to \mathbb{R}$ is real analytic by hypothesis, the composition $\Gamma = \mathscr{E} \circ \Psi$ is real analytic. ☐

We then have the

Proposition 2.13 (Łojasiewicz-Simon gradient inequalities for analytic and Morse-Bott functionals on Banach spaces)**.** *Assume the hypotheses of Lemma 2.10. Then the following hold.*

(1) *If $\mathscr{E}$ is Morse-Bott at x_∞, then there exist an open neighborhood $W_0 \subset U$ of x_∞ and a constant $C = C(\mathscr{E}, W_0) \in [1,\infty)$ such that*
$$|\mathscr{E}(x) - \mathscr{E}(x_\infty)| \leq C\|\mathscr{E}'(x)\|_{\mathscr{X}^*}^2, \quad \forall\, x \in W_0.$$
(2) *If $\mathscr{E}$ is analytic on U, then there exist an open neighborhood $W_0 \subset U$ of x_∞ and constants $C = C(\mathscr{E}, W_0) \in [1,\infty)$ and $\beta \in (1,2]$ such that*
$$|\mathscr{E}(x) - \mathscr{E}(x_\infty)| \leq C\|\mathscr{E}'(x)\|_{\mathscr{X}^*}^\beta, \quad \forall\, x \in W_0.$$

Proof. Denote $x = \Psi(\alpha) \in U_0$ for $\alpha \in V_0$ and recall the definitions of the open neighborhoods U_1 and V_1 from the proof of Lemma 2.10. By shrinking U_1 if necessary, we may assume that U_1 is contained in a bounded convex open subset $U_2 \subset U$. For $\alpha \in V_0$ we have $\alpha, \Pi\alpha \in V_1$ (as in the proof of Lemma 2.10) and therefore $\Psi(\alpha), \Psi(\Pi\alpha) \in U_0$ and the line segment joining $\Psi(\alpha)$ to $\Psi(\Pi\alpha)$ lies in U_2. The Definition 2.11 of Γ, the fact that
$$\Pi\alpha \in K \cap V_0, \quad \forall\, \alpha \in V_0 \quad \text{by Lemma 2.4 (2)},$$
and the Fundamental Theorem of Calculus then give
$$\begin{aligned}\mathscr{E}(\Psi(\alpha)) - \Gamma(\Pi\alpha) &= \mathscr{E}(\Psi(\alpha)) - \mathscr{E}(\Psi(\Pi\alpha))\\ &= -\int_0^1 \frac{d}{dt}(\mathscr{E}(\Psi(\alpha) + t(\Psi(\Pi\alpha) - \Psi(\alpha))))\,dt, \quad \forall\, \alpha \in V_0,\end{aligned}$$
and thus
$$\begin{aligned}&\mathscr{E}(\Psi(\alpha)) - \Gamma(\Pi\alpha)\\ &\quad = \left(-\int_0^1 \mathscr{E}'(\Psi(\alpha) + t(\Psi(\Pi\alpha) - \Psi(\alpha)))\,dt\right)(\Psi(\Pi\alpha) - \Psi(\alpha)), \quad \forall\, \alpha \in V_0.\end{aligned} \tag{2.12}$$
Note that
$$\begin{aligned}&\|\mathscr{E}'(\Psi(\alpha) + t(\Psi(\Pi\alpha) - \Psi(\alpha)))\|_{\mathscr{X}^*}\\ &\quad \leq \|\mathscr{E}'(\Psi(\alpha) + t(\Psi(\Pi\alpha) - \Psi(\alpha))) - \mathscr{E}'(\Psi(\alpha))\|_{\mathscr{X}^*} + \|\mathscr{E}'(\Psi(\alpha))\|_{\mathscr{X}^*} \quad \forall\, \alpha \in V_0.\end{aligned} \tag{2.13}$$
Similarly, the Fundamental Theorem of Calculus yields
$$\begin{aligned}&\mathscr{E}'(\Psi(\alpha) + t(\Psi(\Pi\alpha) - \Psi(\alpha))) - \mathscr{E}'(\Psi(\alpha))\\ &= \int_0^1 \frac{d}{ds}\left(\mathscr{E}'(\Psi(\alpha) + st(\Psi(\Pi\alpha) - \Psi(\alpha)))\right)\,ds\\ &= t\left(\int_0^1 \mathscr{E}''(\Psi(\alpha) + st(\Psi(\Pi\alpha) - \Psi(\alpha)))\,ds\right)(\Psi(\Pi\alpha) - \Psi(\alpha)), \quad \forall\, \alpha \in V_0.\end{aligned}$$
Thus, by taking norms of the preceding equality we obtain
$$\|\mathscr{E}'(\Psi(\alpha) + t(\Psi(\Pi\alpha) - \Psi(\alpha))) - \mathscr{E}'(\Psi(\alpha))\|_{\mathscr{X}^*} \leq M_1\|\Psi(\Pi\alpha) - \Psi(\alpha)\|_{\mathscr{X}}, \quad \forall\, \alpha \in V_0, \tag{2.14}$$
where, since $\mathscr{E} : U \to \mathbb{R}$ is C^2 by hypothesis, we have
$$M_1 := \sup_{x \in U_2} \|\mathscr{E}''(x)\|_{\mathscr{L}(\mathscr{X}, \mathscr{X}^*)} < \infty,$$
because we may assume (by further shrinking U_1 if necessary) that $U_2 \subset U$ is a sufficiently small and bounded (convex) open neighborhood of x_∞.

Combining the inequalities (2.13) and (2.14) with the equality (2.12) yields

$$|\mathscr{E}(\Psi(\alpha)) - \Gamma(\Pi\alpha)| \leq \left(M_1\|\Psi(\Pi\alpha) - \Psi(\alpha)\|_{\mathscr{X}} + \|\mathscr{E}'(\Psi(\alpha))\|_{\mathscr{X}^*}\right)\|\Psi(\Pi\alpha) - \Psi(\alpha)\|_{\mathscr{X}},$$

and so combining the preceding inequality with (2.9) gives

$$|\mathscr{E}(\Psi(\alpha)) - \Gamma(\Pi\alpha)| \leq C\|\mathscr{E}'(\Psi(\alpha))\|^2_{\mathscr{X}^*}, \quad \forall\, \alpha \in V_0. \tag{2.15}$$

We now invoke the hypotheses that $\mathscr{E}$ is Morse-Bott at x_∞ or analytic near x_∞.

When $\mathscr{E}$ is Morse-Bott at x_∞, Lemma 2.12 (1) provides an open neighborhood $\mathscr{V}$ of the origin in $K \cap V_0$ such that $\Gamma \equiv \mathscr{E}(x_\infty)$ on $\mathscr{V}$. Choosing $W_0 = \Psi(V_0 \cap \Pi^{-1}(\mathscr{V}))$, noting that $\Pi : \mathscr{X}^* \to \mathscr{X}^*$ is a continuous (linear) map by Lemma 2.4, we obtain from (2.15) that

$$|\mathscr{E}(x) - \mathscr{E}(x_\infty)| \leq C\|\mathscr{E}'(x)\|^2_{\mathscr{X}^*}, \quad \forall\, x = \Psi(\alpha) \in W_0,$$

which proves Item (1).

Finally, when $\mathscr{E}$ is analytic on U then Lemma 2.12 (2) implies that Γ is analytic on $K \cap V_0$. The finite-dimensional Łojasiewicz gradient inequality (2.5) in Theorem 2.9 (1) applies to give, for a possibly smaller neighborhood $V_2 \subset V_0$ of the origin, constants $C \in [1, \infty)$ and $\alpha \in (1, 2]$, such that

$$|\Gamma(\Pi\alpha) - \mathscr{E}(x_\infty)| \leq C\|\Gamma'(\alpha)\|^\beta, \quad \forall\, \alpha \in V_2. \tag{2.16}$$

But $\Gamma'(\Pi\alpha) = \mathscr{E}'(\Psi(\Pi\alpha))D\Psi(\Pi\alpha)$ by Definition 2.11 of Γ and thus

$$\|\Gamma'(\Pi\alpha)\| \leq M_2\|\mathscr{E}'(\Psi(\Pi\alpha))\|_{\mathscr{X}^*}, \quad \forall\, \alpha \in V_2, \tag{2.17}$$

where, since $D\Psi(\alpha_1) \in \mathscr{L}(\mathscr{X}^*, \mathscr{X})$ is a continuous function of $\alpha_1 \in V_1$ (as $\Psi : V_1 \to U_1$ is C^1 by construction), we have

$$M_2 := \sup_{\alpha_1 \in V_1} \|D\Psi(\alpha_1)\|_{\mathscr{L}(\mathscr{X}^*, \mathscr{X})} < \infty,$$

because we may assume without loss of generality that $V_1 \supset V_2$ is a sufficiently small and bounded (convex) open neighborhood of the origin. Hence, for every $\alpha \in V_2$,

$$\begin{aligned}
|\Gamma(\Pi\alpha) - \mathscr{E}(x_\infty)| &\leq C\|\mathscr{E}'(\Psi(\Pi\alpha))\|^\beta_{\mathscr{X}^*} \qquad \text{(by (2.16) and (2.17))} \\
&\leq C\left(\|\mathscr{E}'(\Psi(\Pi\alpha)) - \mathscr{E}'(\Psi(\alpha))\|_{\mathscr{X}^*} + \|\mathscr{E}'(\Psi(\alpha))\|_{\mathscr{X}^*}\right)^\beta \\
&\leq C\left(\|\Psi(\Pi\alpha) - \Psi(\alpha)\|_{\mathscr{X}} + \|\mathscr{E}'(\Psi(\alpha))\|_{\mathscr{X}^*}\right)^\beta \qquad \text{(by (2.14) for } t = 1),
\end{aligned}$$

and thus, by combining the preceding inequality with (2.9),

$$|\Gamma(\Pi\alpha) - \mathscr{E}(x_\infty)| \leq C\|\mathscr{E}'(\Psi(\alpha))\|^\beta_{\mathscr{X}^*}. \tag{2.18}$$

Consequently, for every $\alpha \in V_2$,

$$\begin{aligned}
|\mathscr{E}(\Psi(\alpha)) - \mathscr{E}(x_\infty)| &\leq |\mathscr{E}(\Psi(\alpha)) - \Gamma(\Pi\alpha)| + |\Gamma(\Pi\alpha) - \mathscr{E}(x_\infty)| \\
&\leq C\left(\|\mathscr{E}'(\Psi(\alpha))\|^2_{\mathscr{X}^*} + \|\mathscr{E}'(\Psi(\alpha))\|^\beta_{\mathscr{X}^*}\right) \qquad \text{(by (2.15) and (2.18))} \\
&\leq C\|\mathscr{E}'(\Psi(\alpha))\|^\beta_{\mathscr{X}^*}\left(1 + \|\mathscr{E}'(\Psi(\alpha))\|^{2-\beta}_{\mathscr{X}^*}\right) \\
&\leq CM_3\|\mathscr{E}'(\Psi(\alpha))\|^\beta_{\mathscr{X}^*},
\end{aligned}$$

where, for small enough V_2 and noting that $\mathscr{E}'(\Psi(\alpha)) \in \mathscr{X}^*$ is a continuous function of $\alpha \in V_2$ (since $\mathscr{E} : U \to \mathbb{R}$ is analytic and $\Psi : V_1 \to U_1$ is C^1 by construction), we have

$$M_3 := 1 + \sup_{\alpha \in V_2} \|\mathscr{E}'(\Psi(\alpha))\|^{2-\beta}_{\mathscr{X}^*} < \infty.$$

Setting $x = \Psi(\alpha)$ for $\alpha \in V_2$ yields

$$|\mathscr{E}(x) - \mathscr{E}(x_\infty)| \leq CM_3 \|\mathscr{E}'(x)\|_{\mathscr{X}^*}^{\beta}, \quad \forall\, x \in \Psi(V_2).$$

We now choose $W_0 = \Psi(V_2)$ to complete the proof of Item (2) and hence the proposition. □

We can now complete the

Proofs of Theorems 1 and 3. The conclusions follow immediately from Proposition 2.13. □

Proof of Corollary 2. According to the hypotheses, there exist $Z \in [1, \infty)$, and $\sigma \in (0, 1]$, and $\theta \in [1/2, 1)$, depending on the critical point $x_\infty \in \mathscr{U}$, so that for every $x \in \mathscr{U}$ with $\|x - x_\infty\|_{\mathscr{X}} < \sigma$ we have

$$|\mathscr{E}(x) - \mathscr{E}(x_\infty)|^\theta \leq Z \|\mathscr{E}'(x)\|_{\mathscr{X}^*}. \tag{2.19}$$

We claim that for each $M \in [1, \infty)$ there exist constants $\sigma_0 \in (0, 1]$ and $Z_0 \in [1, \infty)$ such that if $x \in \mathscr{U}$ obeys

$$|\mathscr{E}(x) - \mathscr{E}(x_\infty)| \leq M \quad \text{and} \quad \|x - x_\infty\|_{\mathscr{H}} < \sigma_0,$$

then (2.19) holds with constant Z_0 in place of Z. To prove this claim we argue by contradiction and suppose it is false. Hence, there is a constant $M \in [1, \infty)$ such that, for each positive $n \in \mathbb{N}$ and $\sigma_0 = 1/n$ and $Z_0 = n$, there exists $x_n \in \mathscr{U}$ obeying

$$|\mathscr{E}(x_n) - \mathscr{E}(x_\infty)| \leq M \quad \text{and} \quad \|x_n - x_\infty\|_{\mathscr{H}} < \frac{1}{n},$$

with

$$|\mathscr{E}(x_n) - \mathscr{E}(x_\infty)|^\theta > n \|\mathscr{E}'(x)\|_{\mathscr{X}^*}. \tag{2.20}$$

Consequently, we have a sequence $\{x_n\}_{n \in \mathbb{N}} \subset \mathscr{U}$ with $x_n \to x_\infty$ in $\mathscr{H}$ as $n \to \infty$ and

$$n \|\mathscr{E}'(x_n)\|_{\mathscr{X}^*} < |\mathscr{E}(x_n) - \mathscr{E}(x_\infty)|^\theta \leq M^\theta, \quad \forall\, n \in \mathbb{N}.$$

Therefore, we must have $\mathscr{E}'(x_n) \to 0$ in $\mathscr{X}^*$ as $n \to \infty$. Since $\jmath \circ \Pi = \widetilde{\Pi} \circ \varepsilon^* \circ \iota : \mathscr{H} \to \mathscr{X}^*$ and is a continuous operator by Lemma 2.4 (1) and (3), we obtain, as $n \to \infty$,

$$\jmath \Pi x_n \to \jmath \Pi x_\infty \quad \text{in } \mathscr{X}^*.$$

For $\Phi : U \to \mathscr{X}^*$ defined as in Lemma 2.10 we see that, as $n \to \infty$,

$$\Phi(x_n) = \mathscr{E}'(x_n) + \jmath \Pi(x_n - x_\infty) \to 0 \quad \text{in } \mathscr{X}^*.$$

Since Φ restricts to a C^1 diffeomorphism from an open neighborhood $U_0 \subset U$ of x_∞ onto an open neighborhood $V_0 \subset \mathscr{X}^*$ of the origin in $\mathscr{X}^*$ with a C^1 inverse Ψ, we see that $x_n \to x_\infty$ strongly in $\mathscr{X}$ as $n \to \infty$. Thus, for large enough n, we have $\|x_n - x_\infty\|_{\mathscr{X}} < \sigma$ and so by (2.19) and (2.20),

$$n \|\mathscr{E}'(x_n)\|_{\mathscr{X}^*} < |\mathscr{E}(x_n) - \mathscr{E}(x_\infty)|^\theta \leq Z \|\mathscr{E}'(x_n)\|_{\mathscr{X}^*}.$$

This leads to contradiction by choosing $n \geq Z$. This proves the claim and hence Corollary 2. □

3. Łojasiewicz-Simon gradient inequality for the harmonic map energy functional

Our overall goal in this section is to prove Theorem 4, the Łojasiewicz-Simon gradient inequality for the harmonic map L^2 energy functional $\mathscr{E}$ in the cases where (N,h) is a closed, real analytic, Riemannian target manifold or $\mathscr{E}$ is Morse-Bott at a critical point f_∞, under the hypotheses that f belongs to a traditional $W^{k,p}$ or an L^2 Łojasiewicz-Simon neighborhood of f_∞. By way of preparation we prove in Section 3.1 that $W^{k,p}(M;N)$ is a real analytic (respectively, C^∞) Banach manifold when (N,h) is real analytic (respectively, C^∞). In Section 3.2, we prove that $\mathscr{E}$ is real analytic (respectively, C^∞) when (N,h) is real analytic (respectively, C^∞). Finally, in Section 3.3 we take up the proof of Theorem 4 proper.

3.1. **Real analytic manifold structure on Sobolev spaces of maps.** The [72, Theorems 13.5 and 13.6] due to Palais imply that the space $W^{k,p}(M;N)$ of $W^{k,p}$ maps (with $kp > d$) from a closed, C^∞ manifold M of dimension d into a closed, C^∞ manifold N can be endowed with the structure of a C^∞ manifold by choosing the fiber bundle, $E \to M$, considered by Palais to be the product $E = M \times N$ and viewing maps $f : M \to N$ as sections of $E \to M$. In particular, [72, Theorem 13.5] establishes the C^∞ structure while [72, Theorem 13.6] identifies the tangent spaces.

While other authors have also considered the smooth manifold structure of spaces of maps between smooth manifolds (see Eichhorn [27], Krikorian [57], or Piccione and Tausk [73]) or approximation properties (see Bethuel [8]), none appear to have considered the specific question of interest to us here, namely, the real analytic manifold structure of the space of Sobolev maps from a closed, Riemannian, C^∞ manifold into a closed, real analytic, Riemannian manifold. Moreover, the question does not appear to be considered directly in standard references for harmonic maps (such as Hélein [49], Jost [56], or Struwe [83, 84], or references cited therein). Those consideration aside, it will be useful to establish this property directly and, in so doing, develop the framework we shall need to prove the Łojasiewicz-Simon gradient inequality for the harmonic map energy functional (Theorem 4).

We shall assume the notation and conventions of Section 1.2, so (M,g) is a closed, Riemannian, smooth manifold of dimension d and (N,h) is a closed, real analytic (or C^∞), Riemannian, manifold that is embedded analytically (or smoothly) and isometrically in $\mathbb{R}^n$. We shall view N as a subset of $\mathbb{R}^n$ with Riemannian metric h given by the restriction of the Euclidean metric. Therefore, a map $f : M \to N$ will be viewed as a map $f : M \to \mathbb{R}^n$ such that $f(x) \in N$ for every $x \in M$ and similarly a section $Y : N \to TN$ will be viewed as a map $Y : N \to \mathbb{R}^{2n}$ such that $Y(y) \in T_yN$ for every $y \in N$.

The space of maps,

$$W^{k,p}(M;N) := \{f \in W^{k,p}(M;\mathbb{R}^n) : f(x) \in N, \text{ for a.e. } x \in M\},$$

inherits the Sobolev norm from $W^{k,p}(M;\mathbb{R}^n)$ and by [4, Theorem 4.12] embeds continuously into the Banach space of continuous maps, $C(M;\mathbb{R}^n)$, when $kp > d$ or $p = 1$ and $k = d$. Furthermore, for this range of exponents, $W^{k,p}(M;N)$ can be given the structure of a real analytic Banach manifold, as we prove in Proposition 3.2. A definition of coordinate charts on $W^{k,p}(M;N)$ is given [60, Section 4.3], which we now recall.

Let $\mathscr{O}$ denote a normal tubular neighborhood [50, p. 11] of radius δ_0 of N in $\mathbb{R}^n$, so $\delta_0 \in (0,1]$ is sufficiently small that there is a well-defined projection map, $\pi : \mathscr{O} \to N \subset \mathbb{R}^n$, from $\mathscr{O}$ to the nearest point of N. When $y \in N$, the value $\pi(y+\eta)$ is well defined for $\eta \in \mathbb{R}^n$ with $|\eta| < \delta_0$ and the differential,

$$d\pi(y+\eta) : T_{y+\eta}\mathbb{R}^n \cong \mathbb{R}^n \to T_{\pi(y+\eta)}N, \tag{3.1}$$

is given by orthogonal projection.

Lemma 3.1 (Analytic diffeomorphism of a neighborhood of the zero-section of the tangent bundle onto an open neighborhood of the diagonal)**.** *Let (N,h) be a closed, real analytic, Riemannian manifold that is analytically and isometrically embedded in $\mathbb{R}^n$ and let $(\pi,\mathscr{O})$ be a normal tubular neighborhood of radius δ_0 of $N \subset \mathbb{R}^n$, where $\pi : \mathscr{O} \to N \subset \mathbb{R}^n$ is the projection to the nearest point of N. Then there is a constant $\delta_1 \in (0,\delta_0]$ such that the map,*

$$\Phi : \{(y,\eta) \in TN : |\eta| < \delta_1\} \to N \times N \subset \mathbb{R}^{2n}, \quad (y,\eta) \mapsto (y, \pi(y+\eta)), \tag{3.2}$$

is an analytic diffeomorphism onto an open neighborhood of the diagonal of $N \times N \subset \mathbb{R}^{2n}$.

Proof. For each $y \in N$, we have $\Phi(y,0) = (y,y) \in \operatorname{diag}(N\times N)$, where $\operatorname{diag}(N\times N)$ denotes the diagonal of $N \times N$. Moreover, $T_{(y,0)}(TN) = T_yN \times T_yN$ and the differential $d\Phi(y,0) : (TN)_{(y,0)} \to T_yN \times T_yN$ is given by $(\zeta,\eta) \mapsto (\zeta, d\pi(y)(\eta)) = (\zeta,\eta)$, that is, the identity. By [50, Theorem 5.1 and remark following proof, p. 110], the projection π is C^∞ and by replacing the role of the C^∞ Inverse Function Theorem in its proof by the real analytic counterpart, one can show that π is real analytic; see [82, Section 2.12.3, Theorem 1] due to Simon for a proof. Thus Φ is real analytic and the Analytic Inverse Function Theorem now yields the conclusion of the lemma. □

For a map $f \in W^{k,p}(M;N)$, we note that, because of the Sobolev embedding $W^{k,p}(M;N) \subset C(M;N)$, we can regard f as a continuous map $f : M \to \mathbb{R}^n$ such that $f(M) \subset N$. Consider the vector bundle,

$$V_f := f^*TN \to M,$$

that is, $(V_f)_x = T_{f(x)}N \subset \mathbb{R}^n$ for all $x \in M$. Let $B_f(\delta)$ denote the ball of center zero and radius $\delta > 0$ in the Banach space of sections,

$$W^{k,p}(V_f) := \left\{u \in W^{k,p}(M;\mathbb{R}^n) : u(x) \in T_{f(x)}N, \ \forall\, x \in M\right\}, \tag{3.3}$$

and denote

$$\mathscr{U}_f := B_f(\kappa(f)^{-1}\delta) \subset W^{k,p}(V_f), \tag{3.4}$$

where $\kappa(f)$ denotes the norm of the Sobolev embedding $W^{k,p}(V_f) \subset C(V_f)$.

Proposition 3.2 (Banach manifold structure on the Sobolev space of maps between Riemannian manifolds)**.** *Let $d \geq 1$ and $k \geq 1$ be integers and $p \in [1,\infty)$ be such that*

$$kp > d \quad \text{or} \quad k = d \text{ and } p = 1.$$

Let (M,g) be a closed, Riemannian, C^∞ manifold of dimension d and (N,h) be a closed, real analytic, Riemannian, manifold that is isometrically and analytically embedded in $\mathbb{R}^n$ and identified with its image. Then the space of maps, $W^{k,p}(M;N)$, has the structure of a real analytic Banach manifold and for each $f \in W^{k,p}(M;N)$, there is a constant $\delta = \delta(N,h) \in (0,1]$ such that, with the definition of $\mathscr{U}_f$ from (3.4), the map,

$$\Phi_f : \mathscr{U}_f \to W^{k,p}(M;N), \quad u \mapsto \pi(f+u), \tag{3.5}$$

defines an inverse coordinate chart on an open neighborhood of $f \in W^{k,p}(M;N)$ and a real analytic manifold structure on $W^{k,p}(M;N)$. Finally, if the hypothesis that (N,h) is real analytic is relaxed to the hypothesis that it is C^∞, then $W^{k,p}(M;N)$ inherits the structure of a C^∞ manifold.

Proof. Because $N \subset \mathbb{R}^n$ is a real analytic submanifold, it follows from arguments of Palais [72, Chapter 13] that $W^{k,p}(M;N)$ is a real analytic submanifold of the Banach space $W^{k,p}(M;\mathbb{R}^{2n})$. Because Palais treats the C^∞ but not explicitly the real analytic case, we provide details.

Let $f \in W^{k,p}(M;N)$ and define an open ball with center f and radius $\varepsilon \in (0,1]$,

$$\mathbb{B}_f(\varepsilon) := \{v \in W^{k,p}(M;\mathbb{R}^{2n}) : \|v - f\|_{W^{k,p}(M)} < \varepsilon\},$$

Recall from Lemma 3.1, that the assignment $\Phi(y,\eta) = (y, \pi(y+\eta))$ defines an analytic diffeomorphism from an open neighborhood of the zero section $N \subset TN$ onto an open neighborhood of the diagonal $N \subset N \times N \subset \mathbb{R}^{2n}$. In particular, the assignment $\Phi_f(u) = \pi(f+u)$, for u belonging to a small enough open ball, $B_f(\delta_2)$, centered at the origin in $W^{k,p}(V_f)$, defines a real analytic embedding of $B_f(\delta_2)$ into $W^{k,p}(M;\mathbb{R}^{2n})$ and onto a relatively open subset, $\Phi_f(B_f(\delta_2)) \subset W^{k,p}(M;N)$. Thus, for small enough ε,

$$\mathbb{B}_f(\varepsilon) \cap W^{k,p}(M;N) \subset \Phi_f(B_f(\delta_2)).$$

The assignment $\Phi_f(u) = \pi(f+u) \in W^{k,p}(M;N)$, for $u \in B_f(\delta_2)$, may be regarded as the restriction of the real analytic map,

$$W^{k,p}(M;\mathbb{R}^{2n}) \ni u \mapsto \pi(f+u) \in W^{k,p}(M;\mathbb{R}^{2n}).$$

Therefore, the collection of inverse maps, defined by each $f \in W^{k,p}(M;N)$,

$$\Phi_f^{-1} : \mathbb{B}_f(\varepsilon) \cap W^{k,p}(M;N) \to W^{k,p}(V_f),$$

defines an atlas for a real analytic manifold structure on $W^{k,p}(M;N)$ as a real analytic submanifold of $W^{k,p}(M;\mathbb{R}^{2n})$.

Lastly, we relax the assumption of real analyticity and require only that (N,h) be a C^∞ closed, Riemannian manifold and isometrically and smoothly embedded in $\mathbb{R}^n$ and identified with its image. The conclusion that $W^{k,p}(M;N)$ is a C^∞ manifold is immediate from the proof in the real analytic case by just replacing real analytic with C^∞ diffeomorphisms. □

Remark 3.3 (Identification of the tangent spaces). The existence of a C^∞ Banach manifold structure for $W^{k,p}(M;N)$ in the case of a smooth isometric embedding $(N,h) \subset \mathbb{R}^n$ is also provided in [72, Theorem 13.5]. In [72, Theorem 13.6] the Banach space $W^{k,p}(V_f)$ is identified as the tangent space of the Banach manifold $W^{k,p}(M;N)$ at the point f. Note that for $f \in W^{k,p}(M;N)$, the differential $(d\Phi_f)(0) : W^{k,p}(V_f) \to T_f W^{k,p}(M;N)$ is the identity map.

Remark 3.4 (Properties of coordinate charts). For the inverse coordinate chart $(\Phi_f, \mathscr{U}_f)$ and $u \in \mathscr{U}_f$ with $f_1 := \pi(f+u) \in W^{k,p}(M;N)$, the differential

$$(d\Phi_f)(u) : W^{k,p}(V_f) \to W^{k,p}(V_{f_1}) \subset W^{k,p}(M;\mathbb{R}^n),$$

is an isomorphism of Banach spaces. By choosing $\delta \in (0,1]$ in Proposition 3.2 sufficiently small we find that the norm of the operator

$$(d\Phi_f)(u) - (d\Phi_f)(0) : W^{k,p}(V_f) \to W^{k,p}(M;\mathbb{R}^n)$$

obeys

$$\|(d\Phi_f)(u) - (d\Phi_f)(0)\| \leq 1, \quad \forall\, u \in \mathscr{U}_f,$$

and therefore $C_3 := \sup_{u \in \mathscr{U}_f} \|(d\Phi_f)(u)\| \leq 2$. By applying the Mean Value Theorem to Φ_f and its inverse, we obtain

$$C_4^{-1}\|f - f_1\|_{W^{k,p}(M)} \leq \|u\|_{W^{k,p}(V_f)} \leq C_4\|f - f_1\|_{W^{k,p}(M)} \tag{3.6}$$

for every $f \in W^{k,p}(M;N)$ and every $u \in W^{k,p}(V_f)$ with $f_1 = \pi(f+u)$, where $C_4 \geq C_3$ depends on (N,h) and f. (Compare [60, Inequality (4.7)].)

3.2. **Smoothness and analyticity of the harmonic map energy functional.** We shall assume the notation and conventions of Section 3.1. Recall Definition 1.9 of the harmonic map L^2-energy functional,

$$\mathscr{E} : W^{k,p}(M;N) \to \mathbb{R}, \quad f \mapsto \frac{1}{2}\int_M |df|^2 \, d\operatorname{vol}_g,$$

and define

$$\mathscr{E}_f \equiv \mathscr{E} \circ \Phi_f : \mathscr{U}_f \subset W^{k,p}(V_f) \to \mathbb{R}, \quad u \mapsto \frac{1}{2}\int_M |d(\pi(f+u))|^2 \, d\operatorname{vol}_g . \tag{3.7}$$

We now establish the following proposition.

Proposition 3.5 (Smoothness and analyticity of the harmonic map L^2-energy functional)**.** *Let $d \geq 2$ and $k \geq 1$ be integers and $p \in [1,\infty)$ be such that*

$$kp > d \quad \text{or} \quad k = d \text{ and } p = 1.$$

Let (M,g) and (N,h) be closed, Riemannian, smooth manifolds with (N,h) real analytic and analytically and isometrically embedded in $\mathbb{R}^n$ and identified with its image. If $f \in W^{k,p}(M;N)$, then $\mathscr{E}_f : \mathscr{U}_f \to \mathbb{R}$ in (3.7) is a real analytic map, where $\mathscr{U}_f \subset W^{k,p}(V_f)$ is as in (3.4) and the image of a coordinate neighborhood in $W^{k,p}(M;N)$. In particular, the functional

$$\mathscr{E} : W^{k,p}(M;N) \to \mathbb{R}$$

is real analytic. Finally, if the hypothesis that (N,h) is real analytic is relaxed to the hypothesis that it is C^∞, then the functional $\mathscr{E} : W^{k,p}(M;N) \to \mathbb{R}$ is C^∞.

Proof. Our hypotheses on d,k,p ensure that there is a continuous Sobolev embedding, $W^{k,p}(M;N) \subset C(M;N)$ by [4, Theorem 4.12] and that $W^{k,p}(M;\mathbb{R})$ is a Banach algebra by [4, Theorem 4.39]. By hypothesis, $f \in W^{k,p}(M;N)$, so $f \in C(M;N)$. We view $N \subset \mathbb{R}^n$ as isometrically and real analytically embedded as the zero section of its tangent bundle, TN, and which is in turn isometrically and real analytically embedded in $\mathbb{R}^{2n}$ and identified with its image. Moreover, if $u \in W^{k,p}(V_f) = W^{k,p}(M; f^*TN)$, then $u \in C(V_f) = C(M; f^*TN)$.

As in Lemma 3.1, let $(\pi, \mathscr{O})$ be a normal tubular neighborhood of $N \subset \mathbb{R}^n$ of radius $\delta_0 \in (0,1]$. Because the nearest-point projection map, $\pi : \mathscr{O} \subset \mathbb{R}^n \to N$, is real analytic, its differential, $(d\pi)(y) \in \operatorname{Hom}_\mathbb{R}(\mathbb{R}^n, T_yN) \subset \operatorname{End}_\mathbb{R}(\mathbb{R}^n)$, is a real analytic function of $y \in \mathscr{O}$ and $d\pi(y) : \mathbb{R}^n \to \mathbb{R}^n$ is orthogonal projection. We choose $\varepsilon \in (0,1]$ small enough that $d\pi(y+z)$ has a power series expansion centered at each point $y \in \mathscr{O}$ with radius of convergence ε,

$$d\pi(y+z) = \sum_{m=0}^\infty a_m(y) z^m, \quad \forall\, y, z \in \mathbb{R}^n \text{ with } |z| < \varepsilon,$$

where (see, for example, Whittlesey [90] in the case of analytic maps of Banach spaces), for each $y \in \mathscr{O}$, the coefficients $a_m(y; z_1, \ldots, z_m)$ are continuous, multilinear, symmetric maps of $(\mathbb{R}^n)^m$ into $\operatorname{End}_\mathbb{R}(\mathbb{R}^n)$ and we abbreviate $a_m(y; z, \ldots, z) = a_m(y) z^m$. The coefficient maps, $a_m(y)$, are (analytic) functions of $y \in \mathscr{O}$, intrinsically defined as derivatives of $d\pi$ at $y \in \mathscr{O}$. We shall use the convergent power series for $d\pi(y+z)$, in terms of z with $|z| < \varepsilon$, to determine a convergent power series for $\mathscr{E}_f(u)$ in (3.7), namely

$$\mathscr{E}_f(u) = \frac{1}{2}\int_M |d(\pi(f+u))|^2 \, d\operatorname{vol}_g = \frac{1}{2}\int_M |d\pi(f+u)(df+du)|^2 \, d\operatorname{vol}_g,$$

in terms of $u \in W^{k,p}(V_f)$ with $\|u\|_{W^{k,p}(V_f)} < \delta$, where $\delta = \varepsilon/\kappa$ and $\kappa = \kappa(f,g,h)$ is the norm of the Sobolev embedding, $W^{k,p}(V_f) \subset C(V_f)$. Recall that

$$d\pi(f+u)(df+du)|_x = d\pi(f(x)+u(x))(df(x)+du(x)), \quad \forall\, x \in M,$$

where $f(x)+u(x) \in \mathscr{O}$ and $f(x)+du(x) \in T_{f(x)}N$. We have the pointwise identity,

$$|d\pi(f+u)(df+du)|^2 = \left|\left(\sum_{m=0}^{\infty} a_m(f)u^m\right)(df+du)\right|^2 \quad \text{on } M,$$

and thus,

$$|d(\pi(f+u))|^2 = \sum_{m=0}^{\infty} |(a_m(f)u^m)(df+du)|^2 \\ + 2\sum_{l=1}^{\infty}\sum_{m=0}^{\infty} \langle |(a_m(f)u^m)(df+du), (a_{m+l}(f)u^{m+l})(df+du)\rangle \quad \text{on } M.$$

After substituting the preceding expression and noting that M is compact and that all integrands are continuous functions on M, the Lebesgue Dominated Convergence Theorem yields a convergent power series as a function of $u \in W^{k,p}(V_f)$ with $\|u\|_{W^{k,p}(V_f)} < \delta$,

$$\mathscr{E}_f(u) = \frac{1}{2}\int_M \left|\left(\sum_{m=0}^{\infty} a_m(f)u^m\right)(df+du)\right|^2 d\operatorname{vol}_g,$$

and thus $\mathscr{E}_f(u)$ is an analytic function of $u \in W^{k,p}(V_f)$ with $\|u\|_{W^{k,p}(V_f)} < \delta$.

We now relax the assumption of real analyticity of (N,h) and require only that (N,h) be a C^∞ closed, Riemannian manifold and isometrically and smoothly embedded in $\mathbb{R}^n$ and identified with its image. The conclusion that the map $\mathscr{E}_f : W^{k,p}(V_f) \to \mathbb{R}$ is C^∞ is immediate from the fact that $W^{k,p}(V_f) \subset C(V_f)$ because the pointwise expressions for $|d\pi(f(x)+u(x))(df(x)+du(x))|^2$, for $x \in M$, and all higher-order derivatives with respect to $z = u(x) \in \mathscr{O} \subset \mathbb{R}^n$ will be continuous functions on the compact manifold, M. □

3.3. **Application to the Łojasiewicz-Simon gradient inequality for the harmonic map energy functional.** We continue to assume the notation and conventions of Section 3.1. The covariant derivative, with respect to the Levi-Civita connection for the Riemannian metric h on N, of a vector field $Y \in C^\infty(TN)$ is given by

$$(\nabla^h Y)_y = d\pi(y)(dY), \tag{3.8}$$

where $\pi = \pi_h$ is as discussed around (3.1) and the second fundamental form [56, Definition 4.7.2] of the embedding $N \subset \mathbb{R}^n$ is given by

$$A_h(X,Y) := \left(\nabla^h_X Y\right)^\perp = dY(X) - d\pi(dY(X)), \quad \forall\, X, Y \in C^\infty(TN), \tag{3.9}$$

where dY is the differential of the map $Y : N \to \mathbb{R}^{2n}$ and we recall from (3.1) that $d\pi(y) : \mathbb{R}^n \to T_yN$ is orthogonal projection, so $\operatorname{id} - d\pi(y) : \mathbb{R}^n \to (T_yN)^\perp$ is orthogonal projection onto the normal plane. By [56, Lemma 4.7.2] we know that $A_h(y) : T_yN \times T_yN \to (T_yN)^\perp$ is a symmetric bilinear form with values in the normal space, for all $y \in N$.

We assume that d, k, p obey the conditions of Proposition 3.5 and recall from (1.8) that the gradient of $\mathscr{E}$ at $f \in W^{k,p}(M;N)$ in the direction of $v \in T_f W^{k,p}(M;N) = W^{k,p}(V_f)$ is given by

$$\mathscr{E}'(f)(v) = (d\pi(f)(\Delta_g f), v)_{L^2(M;\mathbb{R}^n)},$$

noting again that $d\pi(y) : \mathbb{R}^n \to T_y N$ is orthogonal projection (3.1), for all $y \in N$. Thus [49, Lemma 1.2.4] gives

$$d\pi(f)(\Delta_g f) = \Delta_g f - A_h(f)(df, df),$$

and as an operator, we have

$$\mathscr{E}'(f) : T_f W^{k,p}(M;N) \to T_f^* W^{k,p}(M;N),$$

where $p' \in (1, \infty]$ is the dual exponent defined by $1/p + 1/p' = 1$, and

$$T_f^* W^{k,p}(M;N) = \left(W^{k,p}(V_f)\right)^* = W^{-k,p'}(V_f).$$

The Hessian of $\mathscr{E}$ at $f \in W^{k,p}(M;N)$ is defined by [56, p. 427],

$$\mathscr{E}''(f)(v,w) := \frac{\partial^2}{\partial s \partial t} \mathscr{E}(\exp_f(sv + tw))\Big|_{s=t=0} = \frac{d}{dt} \mathscr{E}'(\exp_f(tw))(v)\Big|_{t=0}.$$

Just as in the expression (1.8) for the gradient, one can replace the variation $\exp_f(sv + tw)$ by $\pi(f + sv + tw)$. When f is harmonic, that is, $\mathscr{E}'(f) = 0$, one finds that [56, Theorem 8.2.1]

$$\mathscr{E}''(f)(v,w) = \int_M \langle \Delta_g v, w \rangle_{V_f} \, d\operatorname{vol}_g - \int_M \operatorname{tr}_g \langle R^h(df, v)w, df \rangle_{V_f} \, d\operatorname{vol}_g, \quad \forall\, v, w \in W^{k,p}(V_f),$$

as a bilinear symmetric form on the tangent space, $T_f W^{k,p}(M;N)$. Viewing $\mathscr{E}''(f)$ as a linear operator from the tangent to cotangent space,

$$\mathscr{E}''(f) : T_f W^{k,p}(M;N) \to T_f^* W^{k,p}(M;N),$$

one also finds that [60, Equation (4.3)]

$$\mathscr{E}''(f)(v) = \Delta_g v - 2A_h(f)(df, dv) - (dA_h)(v)(df, df). \tag{3.10}$$

We now have the

Proposition 3.6 (Fredholm and index zero properties for the Hessian of the harmonic map L^2-energy functional)**.** *Assume the hypotheses of Proposition 3.5, but exclude the case $d = k = 2$ and $p = 1$. If $f \in W^{k,p}(M;N)$ is a critical point of $\mathscr{E}$, then the Hessian,*

$$\mathscr{E}''(f) : W^{k,p}(V_f) \to W^{-k,p'}(V_f),$$

is a Fredholm operator with index zero, where $p' \in (1, \infty]$ is defined by $1/p + 1/p' = 1$.

Proof. We first consider the case $d \geq 2$ and $k \geq 1$ and $1 < p < \infty$ and $kp > d$. We need to show that the operator $\mathscr{E}''(f) - \Delta_g : W^{k,p}(M;V_f) \to W^{-k,p'}(M;V_f)$ is compact, where $p' \in (1, \infty)$ is the dual exponent defined by $1/p + 1/p' = 1$. The Sobolev embedding $W^{k,p}(M) \subset C(M)$ is continuous by [4, Theorem 4.12] and the embedding $W^{k,p}(M) \Subset L^{q'}(M)$ is compact by [4, Theorem 6.3], for $1 \leq q' < \infty$. Hence, the dual embedding $L^q(M) \Subset W^{-k,p'}(M)$ is compact for the dual exponent, $1 < q \leq \infty$, defined by $1/q + 1/q' = 1$, using [14, Theorem 6.4]. Therefore, we aim to show that the operator,

$$\mathscr{E}''(f) - \Delta_g : W^{k,p}(M;V_f) \to L^q(M;V_f),$$

is bounded for some $q \in (1, \infty]$ and compose with the compact embedding,

$$L^q(M;V_f) \Subset W^{-k,p'}(M;V_f),$$

to obtain the desired compactness result by [14, Proposition 6.3].

Note that $W^{k,p}(M) \subset W^{1,r}(M)$ is a continuous Sobolev embedding by [4, Theorem 4.12] for

(1) $k = 1$ and $r = p$ (and $p > d \geq 2$), or
(2) $k \geq 2$ and $(k-1)p > d$ and $1 \leq r \leq \infty$, or
(3) $k \geq 2$ and $(k-1)p = d$ and $1 \leq r < \infty$, or
(4) $k \geq 2$ and $(k-1)p < d$ and $1 \leq r \leq 1^* = d/(d-(k-1)p)$. To ensure the possibility of a choice $r \in (2, 1^*]$, we require in this case that $1^* > 2$, that is, $d/(d-(k-1)p) > 2$ or $d/2 > d-(k-1)p$ or $(k-1)p > d/2$, that is, $p > d/(2k-2)$. But $2k-2 \geq k \iff k \geq 2$ and thus, for $k \geq 2$, the hypothesis $p > d/k \implies p > d/(2k-2)$.

Hence, for $d \geq 2$ and each of the preceding possibilities for k and p, we may choose $r = 2q$ with $1 < q < \infty$. Indeed, the choice $q = r/2$ suffices to give both a continuous embedding, $W^{k,p}(M) \subset W^{1,r}(M) = W^{1,2q}(M)$, and a compact embedding, $L^q(M) \Subset W^{-k,p'}(M)$.

Using the expression (3.10) for the Hessian of $\mathscr{E}$ at a harmonic map f, we estimate

$$\begin{aligned}
\|\mathscr{E}''(f)(v) - \Delta_g v\|_{L^q(M;\mathbb{R}^n)} &\leq 2\|A_h(f)\|_{C(M;\mathbb{R}^n)}\||df||dv|\|_{L^q(M;\mathbb{R})} \\
&\quad + \|dA_h\|_{C(M;\mathbb{R}^{2n})}\||v||df|^2\|_{L^q(M;\mathbb{R})} \\
&\leq 2\|A_h\|_{C(M,\mathbb{R}^n)}\|df\|_{L^{2q}(M;\mathbb{R}^n)}\|dv\|_{L^{2q}(M;\mathbb{R}^{2n})} \\
&\quad + \|dA_h\|_{C(M;\mathbb{R}^{2n})}\|v\|_{C(M;\mathbb{R}^n)}\|df\|^2_{L^{2q}(M;\mathbb{R}^n)} \\
&\leq 2\|A_h\|_{C(M,\mathbb{R}^n)}\|f\|_{W^{1,2q}(M;\mathbb{R}^n)}\|v\|_{W^{1,2q}(M;\mathbb{R}^n)} \\
&\quad + \|dA_h\|_{C(M;\mathbb{R}^{2n})}\|v\|_{C(M;\mathbb{R}^n)}\|f\|^2_{W^{1,2q}(M;\mathbb{R}^n)} \\
&\leq C\|A_h\|_{C^1(M;\mathbb{R}^n)}\left(\|f\|_{W^{k,p}(M;N)} + \|f\|^2_{W^{k,p}(M;N)}\right)\|v\|_{W^{k,p}(V_f)},
\end{aligned}$$

where the last inequality follows from the Sobolev embeddings just described, and thus

$$\mathscr{E}''(f) - \Delta_g : W^{k,p}(V_f) \to L^q(V_f)$$

is a bounded linear operator. Since the embedding $L^q(V_f) \Subset W^{-k,p'}(V_f)$ is compact and composition of a bounded linear operator and a compact operator is compact by [14, Proposition 6.3], the operator, $\mathscr{E}''(f) - \Delta_g : W^{k,p}(V_f) \to W^{-k,p'}(V_f)$, is compact. Thus, $\mathscr{E}''(f) : W^{k,p}(V_f) \to W^{-k,p'}(V_f)$ is a Fredholm operator with index zero by [51, Corollary 19.1.8], since the same is true of Δ_g.

We now turn to the case $d \geq 3$ and $k = d$ and $p = 1$. We need to show that the operator $\mathscr{E}''(f) - \Delta_g : W^{d,1}(M; V_f) \to W^{-d,\infty}(M; V_f)$ is compact. The Sobolev embedding $W^{d,1}(M) \subset C(M)$ is continuous by [4, Theorem 4.12] and the embedding $W^{d,1}(M) \Subset L^{q'}(M)$ is compact by [4, Theorem 6.3], for $1 \leq q' < \infty$. Hence, the dual embedding $L^q(M) \Subset W^{-d,\infty}(M)$ is compact for $1 < q \leq \infty$ defined by $1/q + 1/q' = 1$ and the dual exponent $1 \leq q' < \infty$, using [14, Theorem 6.4]. Therefore, we aim to show that the operator,

$$\mathscr{E}''(f) - \Delta_g : W^{d,1}(M; V_f) \to L^q(M; V_f),$$

is bounded for some $q \in (1, \infty]$ and compose with the compact embedding,

$$L^q(M; V_f) \Subset W^{-d,\infty}(M; V_f),$$

to obtain the desired compactness result by [14, Proposition 6.3].

Note that $W^{d,1}(M) \subset W^{1,r}(M)$ is a continuous Sobolev embedding by [4, Theorem 4.12] for $1 \leq r \leq 1^* = d/(d-(d-1)) = d$, that is, $1 \leq r \leq d$. Hence, for $d \geq 3$, we may choose $r = 2q$

with $1 < q \leq d/2$. Indeed, the choice $q = d/2$ suffices to give both a continuous embedding, $W^{d,1}(M) \subset W^{1,d}(M) = W^{1,2q}(M)$, and a compact embedding, $L^q(M) = L^{\frac{d}{2}}(M) \Subset W^{-d,\infty}(M)$.

Using the expression (3.10) for the Hessian of $\mathscr{E}$ at a harmonic map f and $q = d/2$, we again find that

$$\|\mathscr{E}''(f)(v) - \Delta_g v\|_{L^q(M;\mathbb{R}^n)} \leq C\|A_h\|_{C^1(M;\mathbb{R}^n)} \left(\|f\|_{W^{d,1}(M;N)} + \|f\|^2_{W^{d,1}(M;N)}\right) \|v\|_{W^{d,1}(M;V_f)},$$

and thus

$$\mathscr{E}''(f) - \Delta_g : W^{d,1}(M; V_f) \to L^q(M; V_f)$$

is a bounded linear operator. The remainder of the argument for the first case ($d \geq 2$ and $k \geq 1$ and $1 < p < \infty$ and $kp > d$) again shows that $\mathscr{E}''(f) : W^{d,1}(V_f) \to W^{-d,\infty}(V_f)$ is a Fredholm operator with index zero. ☐

The proof that the Hessian operator is Fredholm with index zero in the borderline case $k = d = 2$ and $p = 1$ relies on a regularity theorem for weakly harmonic maps from surfaces due to Heléin [49, Theorem 4.1.1].

Proposition 3.7 (Fredholm and index zero properties for the Hessian of the harmonic map L^2-energy functional in the borderline case for $d = 2$)**.** *Let (M, g) be a closed, smooth Riemann surface and (N, h) be a closed, Riemannian, smooth manifold that is isometrically embedded in $\mathbb{R}^n$ and identified with its image. If $f \in W^{2,1}(M; N)$ is a critical point of $\mathscr{E}$, then the Hessian,*

$$\mathscr{E}''(f) : W^{2,1}(V_f) \to W^{-2,\infty}(V_f),$$

is a Fredholm operator with index zero.

Proof. We need to show that the operator $\mathscr{E}''(f) - \Delta_g : W^{2,1}(V_f) \to W^{-2,\infty}(V_f)$ is compact. The Sobolev embedding $W^{2,1}(M; V_f) \Subset L^r(M; V_f) \subset L^r(M; \mathbb{R}^n)$ is compact by [4, Theorem 6.3] for $1 \leq r < \infty$. We choose $r = 2$ and observe that the dual embedding $L^2(M; \mathbb{R}^n) \Subset W^{-2,\infty}(M; V_f)$ is compact by [14, Theorem 6.4]. Since the embedding $W^{1,1}(M) \subset L^s(M)$ is continuous by [4, Theorem 4.12] for $1 \leq s \leq 1^* = 2/(2-1) = 2$, we can choose $s = 2$ and observe the composition

$$W^{1,1}(M; \mathbb{R}^n) \subset L^2(M; \mathbb{R}^n) \Subset W^{-2,\infty}(M; V_f).$$

is compact by [14, Proposition 6.3].

Using the expression (3.10) for the Hessian of $\mathscr{E}$ at a harmonic map f, we estimate

$$\begin{aligned}\|\mathscr{E}''(f)(v) - \Delta_g v\|_{W^{1,1}(M;\mathbb{R}^n)} &\leq 2\|A_h(f)(df, dv)\|_{W^{1,1}(M;\mathbb{R}^n)} + \|(dA_h)(v)(df, df)\|_{W^{1,1}(M;\mathbb{R}^{2n})} \\ &\leq 2\|A_h(f)(df)\|_{C^1(M,\mathbb{R}^n)}\|v\|_{W^{2,1}(M;\mathbb{R}^n)} \\ &\quad + \|dA_h(df, df)\|_{C^1(M;\mathbb{R}^{2n})}\|v\|_{W^{1,1}(M;\mathbb{R}^n)} \\ &\leq 2\left(\|A_h(f)(df)\|_{C^1(M;\mathbb{R}^n)} + \|dA_h(df, df)\|_{C^1(M,\mathbb{R}^{2n})}\right)\|v\|_{W^{2,1}(V_f)}.\end{aligned}$$

Because f is a critical point of $\mathscr{E}$, it is a weakly harmonic map in the sense of [49, Definition 1.4.9]. By a regularity theorem due to Hélein [49, Theorem 4.1.1], the map f is C^∞ and the quantity

$$\|A_h(f)(df)\|_{C^1(M,\mathbb{R}^{2n})} + \|dA_h(df, df)\|_{C^1(M,\mathbb{R}^{2n})},$$

is finite. Hence,

$$\mathscr{E}''(f) - \Delta_g : W^{2,1}(V_f) \to W^{1,1}(V_f)$$

is a bounded linear operator. Since the embedding $W^{1,1}(V_f) \Subset W^{-2,\infty}(V_f)$ is compact and composition of a bounded linear operator and a compact operator is compact by [14, Proposition 6.3], the operator, $\mathscr{E}''(f) - \Delta_g : W^{2,1}(V_f) \to W^{-2,\infty}(V_f)$, is compact. Thus, $\mathscr{E}''(f) : W^{2,1}(V_f) \to W^{-2,\infty}(V_f)$ is a Fredholm operator with index zero, since the same is true of Δ_g. ☐

Remark 3.8 (Regularity for weakly harmonic maps from higher-dimensional Riemannian manifolds). We note that the regularity theorem for weakly harmonic maps, $f : M \to N$, from Riemann surfaces, M, due to Hélein [49, Theorem 4.1.1] has been partly generalized by Bethuel [9, Theorem 1.1], [49, Theorem 4.3.1] to the case of weakly harmonic maps from Riemannian manifolds M of dimension $d \geq 3$, to show that a weakly harmonic map, $f : M \to N$, is $C^{1,\alpha}_{\mathrm{loc}}$ on $M \setminus S$, where $S \subset M$ is a closed subset with $(d-2)$-dimensional Hausdorff measure zero.

We are now ready to complete the

Proof of Theorem 4. By Remark 3.4, there is a constant $C_4 = C_4(f, g, h, k, p) \in [1, \infty)$ such that for every $u \in \mathscr{U}_{f_\infty} \subset W^{k,p}(V_{f_\infty})$ and $f = \Phi_{f_\infty}(u) = \pi(f_\infty + u) \in W^{k,p}(M; N)$, we have

$$C_4^{-1}\|f - f_\infty\|_{W^{k,p}(M)} \leq \|u\|_{W^{k,p}(V_{f_\infty})} \leq C_4\|f - f_\infty\|_{W^{k,p}(M)}, \tag{3.11}$$

and

$$(d\Phi_{f_\infty})(u) = d\pi(f_\infty + u) : W^{k,p}(V_{f_\infty}) \to T_f W^{k,p}(M, N) = W^{k,p}(V_f),$$

is a Banach space isomorphism with norm $C_3 := \sup_{u \in \mathscr{U}_{f_\infty}} \|(d\Phi_{f_\infty})(u)\| \in [1, \infty)$.

We shall first derive the Łojasiewicz-Simon gradient inequalities for the map

$$\mathscr{E}_{f_\infty} = \mathscr{E} \circ \Phi_{f_\infty} : \mathscr{U}_{f_\infty} \subset W^{k,p}(V_{f_\infty}) \to \mathbb{R}.$$

Consider first the case where (N, h) is real analytic. Propositions 3.5, 3.6, and 3.7 ensure that the hypotheses of Theorem 1 are fulfilled with

$$x_\infty = 0 \in \mathscr{X} = W^{k,p}(V_{f_\infty}) \quad \text{and} \quad \mathscr{H} = L^2(V_{f_\infty}),$$

noting that $\Phi_{f_\infty}(0) = f_\infty$, so $\mathscr{E}_{f_\infty}$ has a critical point at the origin. Hence, there exist constants $\theta \in [1/2, 1)$, and $\sigma_0 \in (0, \delta]$, and $Z_0 \in [1, \infty)$ (where $\delta \in (0, 1]$ is the constant in (3.4) that defines the open neighborhood $\mathscr{U}_{f_\infty}$ of the origin) such that for every $v \in W^{k,p}(V_{f_\infty})$ obeying $\|v\|_{W^{k,p}(V_{f_\infty})} < \sigma_0$ we have

$$|\mathscr{E}_{f_\infty}(v) - \mathscr{E}_{f_\infty}(0)|^\theta \leq Z_0\|\mathscr{E}'_{f_\infty}(v)\|_{W^{-k,p'}(V_{f_\infty})}.$$

Thus, if $f = \Phi_{f_\infty}(u) \in W^{k,p}(M; N)$ obeys $\|f_\infty - f\|_{W^{k,p}(M)} < C_4^{-1}\sigma_0$, then (3.11) implies that $\|u\|_{W^{k,p}(V_{f_\infty})} < \sigma_0$. Also

$$\mathscr{E}'_{f_\infty}(u) = \mathscr{E}'(f) \circ (d\Phi_{f_\infty})(u) = \mathscr{E}'_f(0) \circ (d\Phi_{f_\infty})(u),$$

and therefore

$$\|\mathscr{E}'_{f_\infty}(u)\|_{W^{-k,p'}(V_{f_\infty})} \leq \|\mathscr{E}'_f(0)\|_{W^{-k,p'}(V_f)}\|(d\Phi_{f_\infty})(u)\| \leq C_3\|\mathscr{E}'(f)\|_{W^{-k,p'}(V_f)}.$$

We conclude that if $\|f - f_\infty\|_{W^{k,p}(M)} < C_4^{-1}\sigma_0$, then

$$|\mathscr{E}(f) - \mathscr{E}(f_\infty)|^\theta \leq C_3 Z_0\|\mathscr{E}'(f)\|_{W^{-k,p'}(V_f)},$$

where $W^{-k,p'}(V_f)$ is the dual of the tangent space $W^{k,p}(V_f) = T_f W^{k,p}(M; N)$ of the analytic Banach manifold $W^{k,p}(M; N)$ at f. This yields inequality (1.11) for constants $Z = C_3 Z_0$ and $\sigma = C_4^{-1}\sigma_0$.

We now allow (N, h) to be any C^∞ closed, Riemannian manifold and consider the case where $\mathscr{E}$ is a C^2 functional that is Morse-Bott at a critical point $f_\infty \in W^{k,p}(M; N)$. Though no longer real analytic, $W^{k,p}(M; N)$ is still a C^∞ Banach manifold by Proposition 3.2 and the functional

$$\mathscr{E}_{f_\infty} = \mathscr{E} \circ \Phi_{f_\infty} : \mathscr{U}_{f_\infty} \subset W^{k,p}(V_{f_\infty}) \to \mathbb{R}$$

is C^2 with critical point at the origin, where it is also Morse-Bott. Propositions 3.6 and 3.7 ensure that the hypotheses of Theorem 3 are fulfilled with

$$x_\infty = 0 \in \mathscr{X} = W^{k,p}(V_{f_\infty}) \quad \text{and} \quad \mathscr{H} = L^2(V_{f_\infty}),$$

so there exist constants $\sigma_0 \in (0, \delta]$ and $Z_0 \in [1, \infty)$ such that for every $v \in W^{k,p}(V_{f_\infty})$ obeying $\|v\|_{W^{k,p}(V_{f_\infty})} < \sigma_0$ we have

$$|\mathscr{E}_{f_\infty}(v) - \mathscr{E}_{f_\infty}(0)| \leq Z_0 \|\mathscr{E}'_{f_\infty}(v)\|^2_{W^{-k,p'}(V_{f_\infty})}.$$

The proof that optimal the Łojasiewicz-Simon gradient inequality (1.11) holds with $\theta = 1/2$ under the condition (1.10) now follows *mutatis mutandis* the proof of the inequality with $\theta \in [1/2, 1)$ in the real analytic case.

It remains to prove that the Łojasiewicz-Simon gradient inequality (1.11) holds under the L^2 Łojasiewicz-Simon neighborhood condition (1.12). We have seen thus far that Theorems 1 and 3 apply to $\mathscr{E}_{f_\infty} : \mathscr{U}_f \to \mathbb{R}$ with $x_\infty = 0 \in \mathscr{X} = W^{k,p}(V_{f_\infty})$ and $\mathscr{H} = L^2(V_{f_\infty})$, for $\mathscr{E}$ analytic or Morse-Bott, respectively. Thus, Corollary 2 also applies to $\mathscr{E}_{f_\infty}$ with the same value of $\theta \in [1/2, 1)$: for every constant $C_0 \in [1, \infty)$, there exist constants $\sigma_1 \in (0, 1]$ and $Z_1 \in [1, \infty)$ such that for every $v \in W^{k,p}(V_{f_\infty})$ obeying $\|v\|_{L^2(V_{f_\infty})} < \sigma_1$ and $|\mathscr{E}_{f_\infty}(v) - \mathscr{E}_{f_\infty}(0)| \leq C_0$, we have

$$|\mathscr{E}_{f_\infty}(v) - \mathscr{E}_{f_\infty}(0)|^\theta \leq Z_1 \|\mathscr{E}'_{f_\infty}(v)\|_{W^{-k,p'}(V_{f_\infty})}.$$

However, recalling the notation of Lemma 3.1, for every $y \in N$ the map

$$\Phi_y \equiv \Phi(y, \cdot) : \{\eta \in T_yN : |\eta| < \delta\} \to N, \quad \eta \mapsto \pi(y + \eta),$$

is a diffeomorphism onto its image, a normal tubular neighborhood of $N \subset \mathbb{R}^n$. Therefore, by applying the Mean Value Theorem to Φ_y^{-1}, we obtain

$$|\eta| \leq C_5 |\pi(y + \eta) - y|, \quad \forall\, \eta \in T_yN \text{ with } |\eta| < \delta,$$

where $C_5 := \sup\{|d(\Phi_y^{-1})(z)| : (y, \eta) \in TN \text{ with } |\eta| < \delta \text{ and } z = \pi(y + u)\}$. Using the preceding inequality with $y = f_\infty(x) \in N$ and $\eta = v(x) \in T_{f_\infty(x)}N$ for $x \in M$ and taking L^2 norms, we obtain

$$\|v\|_{L^2(V_f)} \leq C_5 \|f - f_\infty\|_{L^2(M)}.$$

Therefore, given $C_0 \in [1, \infty)$, we choose $\sigma = C_5^{-1}\sigma_1$ and observe that for every $f = \pi(f_\infty + v) \in W^{k,p}(M; N)$ with $v \in W^{k,p}(V_{f_\infty})$ obeying $\|f - f_\infty\|_{L^2(M)} < \sigma$ and $|\mathscr{E}(f) - \mathscr{E}(f_\infty)| \leq C_0$, the preceding inequality yields $\|v\|_{L^2(V_{f_\infty})} < \sigma_1$. Since $|\mathscr{E}_{f_\infty}(v) - \mathscr{E}_{f_\infty}(0)| \leq C_0$, we thus have

$$|\mathscr{E}(f) - \mathscr{E}(f_\infty)|^\theta \leq Z_1 \|\mathscr{E}'_{f_\infty}(v)\|_{W^{-k,p'}(V_{f_\infty})} \leq C_3 Z_1 \|\mathscr{E}'(f)\|_{W^{-k,p'}(V_f)},$$

which is just (1.11) with constant $Z = C_3 Z_1$. This concludes the proof of Theorem 4. □

Appendix A. Integrability and the Morse-Bott condition for the harmonic map energy functional

Following Lemaire and Wood [61, Section 1], we review the concept of *integrability* of a *Jacobi field* along a harmonic map, describe the relation between integrability and the Morse-Bott condition for the harmonic map energy functional at a harmonic map. We then indicate some of the few examples where integrability is known for harmonic maps.

We begin by recalling the *second variation of the energy*. For a smooth two-parameter variation, $f_{t,s} : M \to N$, of a map $f : M \to N$ with $\partial f_{t,s}/\partial t|_{(0,0)} = v$ and $\partial^2 f_{t,s}/\partial s|_{(0,0)} = w$, the *Hessian* of f is defined by

$$\operatorname{Hess}_f(v, w) := \left.\frac{\partial^2 \mathscr{E}(f_{t,s})}{\partial t \partial s}\right|_{(0,0)}.$$

One has

$$\operatorname{Hess}_f(v, w) = (J_f(v), w)_{L^2(M,g)},$$

where

$$J_f(v) := \Delta v - \operatorname{tr} R^N(df, v)df$$

is called the *Jacobi operator*, a self-adjoint linear elliptic differential operator. Here, Δ denotes the Laplacian induced on $f^{-1}TN$ and the sign conventions on Δ and the curvature R^N are those of Eells and Lemaire [26].

Let v be a *vector field along* f, that is, a smooth section of $f^{-1}TN$, where $f : M \to N$ is a smooth map. Then v is called a *Jacobi field* (for the energy) if $J_f(v) = 0$. The space of Jacobi fields, $\operatorname{Ker} J_f$, is finite-dimensional and its dimension is called the ($\mathscr{E}$)-*nullity* of f.

Definition A.1 (Integrability of a Jacobi field along a harmonic map)**.** [61, Definition 1.2] A Jacobi field v along a harmonic map, $f : M \to N$, is said to be *integrable* if there is a smooth family of harmonic maps, $f_t : M \to N$ for $t \in (-\varepsilon, \varepsilon)$, such that $f_0 = f$ and $v = \partial f_t/\partial t|_{t=0}$.

Adams and Simon proved the following alternative characterization of the integrability condition in Definition A.1.

Lemma A.2. [3, Lemma 1] *Let $\varphi_0 : (M, g) \to (N, h)$ be a harmonic map between real-analytic Riemannian manifolds. Then all Jacobi fields along φ_0 are integrable if and only if the space of harmonic maps ($C^{2,\alpha}$-) close to φ_0 is a smooth manifold, whose tangent space at φ_0 is* $\operatorname{Ker} \mathscr{E}''(\varphi_0)$.

It follows that for two real-analytic manifolds, all Jacobi fields along all harmonic maps are integrable if and only if the space of harmonic maps is a manifold whose tangent bundle is given by the Jacobi fields [61, p. 470]. By Definition 1.6, the conclusion of Lemma A.2 is equivalent to the assertion that all Jacobi fields along φ_0 are integrable if and only if the harmonic map energy functional $\mathscr{E}$ is Morse-Bott at φ_0.

For a further discussion of integrability and additional references, see [3, Section 1], Kwon [60, Section 4.1], and Simon [81, pp. 270–272].

According to [61, Theorem 1.3] any Jacobi field along a harmonic map from S^2 to $\mathbb{CP}^2$ is integrable, where the two-sphere S^2 has its unique conformal structure and the complex projective space $\mathbb{CP}^2$ has its standard Fubini-Study metric of holomorphic sectional curvature 1.

From the list of examples provided by Lemaire and Wood [61, p. 471], there are few other examples of families of harmonic maps that are guaranteed to be integrable, with the list including harmonic maps from S^2 to S^2 but excluding harmonic maps from S^2 to S^3 or S^4 [62].

We note that Fernández [37] has proved a dimension formula for the space of degree-d harmonic maps from S^2 into S^{2n}. However, thus far, integrability for such maps is known only when $n = 1$.

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