AUTOMORPHISMS AND AUTOEQUIVALENCES OF GENERIC ANALYTIC K3 SURFACES

EMANUELE MACRÌ AND PAOLO STELLARI

ABSTRACT. This is a systematic exposition of recent results which completely describe the group of automorphisms and the group of autoequivalences of generic analytic K3 surfaces. These groups, hard to determine in the algebraic case, are very well described and very closely related for generic analytic K3 surfaces.

1. INTRODUCTION

In this survey paper we present some recent results about the group of automorphisms and the group of autoequivalences of generic analytic K3 surfaces. Recall that a surface of this type is a non-algebraic K3 surface with trivial Picard group. They are dense in the moduli space of K3 surfaces and their geometry is considerably simpler if compared to the one of algebraic K3 surfaces. This is mainly due to the lack of curves.

The interesting geometry of K3 surfaces has been widely studied during the last forty years with particular attention to the algebraic case. Very deep results such as the Torelli Theorem showed that many interesting geometric properties of these surfaces can be determined by lattice theory and Hodge structures. For example, the cup product and the weight-two Hodge structure defined on the second cohomology group of a K3 surface X determine it uniquely up to isomorphisms. As a consequence, we get a key tool to describe the automorphism group of K3 surfaces.

Not surprisingly, the algebraic case is extremely rich and by now we do not have a complete description of the automorphism group of any possible K3 surface. In fact, the isomorphism class of the Picard lattice can deeply influence this group. One can easily produce examples where this group is trivial, finite, infinite and cyclic or infinite but not cyclic (Section 3.2). Very often the geometry of the surface and not just the lattice and Hodge structures of its second cohomology group gives significant clues to approach a description of this group. From this intricate situation, a quite large number of interesting results emerged in the algebraic setting.

The case of generic analytic K3 surfaces is very different. In particular the description of the automorphisms group is very neat and only two possibilities are realized: either the group is trivial or the group is cyclic and infinite (see Theorem 3.6 and [81]). The latter case is quite rare and the generic analytic K3 surfaces with infinite automorphisms group were completely described by Gross and McMullen (see Section 2.4 and [39, 66]).

The interest for K3 surfaces recently increased also from the point of view of physics. Indeed, these surfaces are examples of Calabi-Yau manifolds and irreducible holomorphic symplectic manifolds at the same time and their geometry is relatively well understood. Hence they immediately became one of the favorite examples to test the homological interpretation of Mirror Symmetry due to Kontsevich ([62] but see also [26, 47, 91]). Roughly speaking this conjecture predicts that for two Calabi-Yau manifolds X and Y which are mirror the bounded derived category of coherent sheaves on one variety should be equivalent to the Fukaya category of the other one.

Morally, the bounded derived category of coherent sheaves should take into account the complex structure of a variety while the Fukaya category should encode the symplectic geometry. Indeed

2000 *Mathematics Subject Classification.* 18E30, 14J28.
Key words and phrases. K3 surfaces, derived categories, automorphisms.

the objects of the latter category are special Lagrangian submanifolds with a suitable notion of morphisms obtained by means of Floer homology.

It is known that the abelian category of coherent sheaves $\mathbf{Coh}(X)$ on a smooth projective variety X determines the variety uniquely up to isomorphism. This is no longer true if we consider the bounded derived category of $\mathbf{Coh}(X)$. More precisely, Bondal and Orlov proved in [11] that if either the canonical or the anticanonical bundle of X is ample then X is determined uniquely up to isomorphism by the bounded derived category of $\mathbf{Coh}(X)$. But as soon as the canonical bundle becomes trivial (as for K3 surfaces and, more generally, for Calabi-Yau manifolds) then a large number of phenomena can happen. Just restricting to the case of algebraic K3 surfaces, one can produce examples of K3 surfaces X and arbitrarily many non-isomorphic K3 surfaces Y with the same derived category ([80, 96]). Nevertheless the number of isomorphism classes of K3 surfaces with such a close relationship to X is always finite ([19]). Conjecturally the number of isomorphism classes of smooth algebraic varieties with the same derived categories is always expected to be finite.

One can wonder if the same results are still true when X is non-algebraic but we will see that this is not the case. As a first step, we will observe that to be sure that all functors and dualities we need are well-defined we should work with a suitable subcategory of the derived category of $\mathcal{O}_X$-modules instead of the bounded derived category of coherent sheaves (Section 4.3). The main problem in this sense is the validity of Serre duality. Notice that in our geometric setting, just involving complex non-projective surfaces, this is not a painful point since, as in the algebraic case, this special subcategory is naturally equivalent to the bounded derived category of coherent sheaves (see Proposition 4.15 and [10]) although some problems appear when we deal with the derived category of the product of two K3 surfaces. Moreover, the geometric similarities with the algebraic setting appear in many other useful instances. For example, any coherent sheaf admits a locally free resolution of finite length. Hence the Grothendieck group of these surfaces is generated by the classes of locally free sheaves, we can easily define Chern classes for coherent sheaves and the Riemann-Roch Theorem in its non-algebraic version holds true for any coherent sheaf. More generally we will recall how Grothendieck-Riemann-Roch Theorem was proved in [76] after Chern classes were introduced in [77] for coherent sheaves on analytic varieties. A sketchy definition of these Chern classes will be included in Section 4.1.

Assuming the point of view of physics one can see that many interesting dualities in string theory can be reinterpreted as equivalences (or autoequivalences) of derived categories. This motivates the growing interests for the study of the group of autoequivalences of the derived categories of K3 surfaces or, more generally, of varieties with trivial canonical bundle.

While a very nice and complete description of the group of autoequivalences is now available for abelian varieties ([85]) the same is not true for algebraic K3 surfaces. Even for very explicit examples where the geometry of the surface is very well understood, this group remains mysterious.

A conjectural description of this group was first proposed by Szendrői in [97]. Indeed, Orlov ([84]) proved using results of Mukai ([70]) that any autoequivalence of the derived category of a smooth projective K3 surface induces an isomorphism of the total cohomology groups preserving some special lattice and Hodge structures (Section 5.1). Szendrői observed that such an isomorphism should preserve the orientation of a particular four-dimensional subspace of the real cohomology (Conjecture 5.6). So we would expect a surjective morphism from the autoequivalences group onto a group of orientation preserving isometries of the total cohomology group of an algebraic K3 surface. Notice that this is in perfect analogy with the Strong Torelli Theorem where the preservation of the Kähler cone is required.

On the other hand, untill recently no clue about the kernel of this morphism was available. The real breakthrough in this direction is due to Bridgeland and to his notion of stability condition on triangulated categories ([14]). Roughly speaking a stability condition is determined by a t-structure on the triangulated category and a $\mathbb{C}$-linear function on the heart of this t-structure satisfying special properties which imitate the usual definition of μ-stability for coherent sheaves. Bridgeland's motivation was to axiomatize the notion of Π-stability emerged in Douglas' work ([29]). In particular, the manifold parametrizing stability conditions on the derived category $\mathbf{D}$ of some Calabi-Yau manifold quotiented by the natural action of the autoequivalence group of $\mathbf{D}$ contains the stringy Kähler moduli space of the Calabi-Yau manifold.

Studying stability conditions on the bounded derived category of coherent sheaves on smooth projective K3 surfaces, Bridgeland conjectured that the kernel of the morphism considered by Szendrői should be the fundamental group of some period domain associated to the variety parametrizing stability conditions on the derived category ([15]).

An attempt to verify this conjectural description of the autoequivalence group due to Bridgeland and Szendrői was made in [53] where this group was described for generic twisted K3 surfaces (not considered in this paper) and generic analytic K3 surfaces. Very surprisingly, the groups of automorphisms and autoequivalences which are quite mysterious in the algebraic case, can be explicitly studied in the generic analytic case. Moreover it turns out that these groups are very closely related in the non-algebraic case since the group of autoequivalences of Fourier-Mukai type of a generic analytic K3 surface just consists of shifts, twists by a special object called spherical and automorphisms (Theorem 5.17). Notice that the techniques used in [53] to prove this result and briefly recalled in the second part of the present paper will be applied in [54] to prove that any equivalence of the derived categories of smooth projective(!) K3 surfaces induces an orientation preserving Hodge isometry on cohomology.

The aim of this paper is to describe these two groups in full detail collecting results contained in many papers and trying to give a comprehensive, systematic and integrated approach.

The paper is organized as follows. Section 2 deals with the geometry of K3 surfaces. We briefly recall some results in lattice theory and we give some examples of algebraic and non-algebraic K3 surfaces. Particular attention is paid to the examples of non-algebraic K3 surfaces proposed by McMullen. We also discuss the lattice and Hodge structures defined on the second and total cohomology groups of K3 surfaces.

Section 3 is devoted to the complete description of the automorphism group of generic analytic K3 surfaces. In particular we present the works of Gross, McMullen and Oguiso. These results are compared through examples with the case of algebraic K3 surfaces.

In Section 4 we identify the appropriate triangulated category where all functors with geometrical meaning can be derived (as a special case we consider the Serre functor). We recall a result of Verbitsky about the abelian category of coherent sheaves on generic analytic K3 surfaces and we show that any coherent sheaf admits a locally free resolution of finite length. Chern classes for coherent sheaves on smooth complex varieties are defined according to [76].

In the last section we prove that a smooth complex variety with the same derived category as a generic analytic K3 surface is a surface of the same type and we state Szendrői's conjecture for algebraic K3 surfaces. We discuss the notion of stability condition due to Bridgeland and we use it to fully describe the group of autoequivalences of generic analytic K3 surfaces.

A few warnings are needed at this point. First of all, there are no new results in this presentation and we deliberately omitted many nice results concerning algebraic K3 surfaces. For example, the section about automorphisms of algebraic K3 surfaces is mainly based on easy examples and none of the beautiful deep results available in the literature are considered. This is justified by the fact that the algebraic setting is out of the scope of this paper. Secondly, the proofs are described in detail when the length and the complexity are reasonable. When they can be of some interests to give to the reader an impression of the techniques involved but they are too articulated we just briefly outline the main passages. The final warning concerns the fact that the reader is supposed to have some familiarity with the geometry of K3 surfaces and with the language of derived categories. Although some introductory sections are present in this paper, the less prepared reader is encouraged to have a look at [1], [4] and at the first chapters of [49].

2. The geometry of K3 surfaces

In this section we discuss a few geometric properties of algebraic and generic analytic K3 surfaces. After describing some easy examples, we will introduce the Hodge and lattice structures defined on the second and on the total cohomology groups of any K3 surface. The culminations of the theory are certainly the Torelli Theorem and the theorem stating the surjectivity of the period map.

While it is very easy to describe explicit examples of algebraic K3 surfaces, the non-algebraic ones are usually presented with a very abstract approach (e.g. invoking the surjectivity of the period map). Nevertheless McMullen constructed very beautiful examples of complex non-algebraic surface that we will discuss at the end of this section.

For sake of completeness we include a short section where we recollect well-known results in lattice theory.

2.1. **Lattices.**

Lattice theory plays a special role in the study of the geometry of K3 surfaces. As we will see in a while, very interesting aspects of the geometry of K3 surfaces are encoded by the Hodge and lattice structures defined on some cohomology groups. Due to a celebrated result of Orlov (see Theorem 5.2) the lattice theoretic properties of the cohomology of a K3 surface are also important to detect equivalences between derived categories.

Therefore, not surprisingly, in this paper we will often need a few algebraic results in lattice theory. We list them in this section for the convenience of the reader. The proofs can be found in [75] (see also [25] for a readable account).

First of all, recall that a *lattice* is a free abelian group L of finite rank with a non-degenerate symmetric bilinear form $b : L \times L \to \mathbb{Z}$. For simplicity, we always write $x \cdot y := b(x, y)$ and we denote by $\mathrm{O}(L)$ the group of isometries of a lattice L (i.e. the automorphisms of L preserving the bilinear form).

Example 2.1. (i) The *hyperbolic lattice* U is the free abelian group $\mathbb{Z} \oplus \mathbb{Z}$ endowed with the quadratic form represented by the matrix

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

with respect to the standard basis.

(ii) Consider the free abelian group $\mathbb{Z}^{\oplus 8}$ with the quadratic form

$$\begin{pmatrix} 2 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & -1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 2 & -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 2 \end{pmatrix}.$$

Such a lattice is denoted by E_8.

(iii) We denote by $\langle d \rangle$ the lattice $L = (\mathbb{Z}, b)$ where $b(h, h) = d$ for a generator h of the underlaying rank one abelian group.

(iv) Given a lattice L with bilinear form b_L and an integer $m \in \mathbb{Z}$, the lattice $L(m)$ coincides with L as a group but its bilinear form $b_{L(m)}$ is such that $b_{L(m)}(l_1, l_2) = m b_L(l_1, l_2)$, for any $l_1, l_2 \in L$.

A lattice (L, b) is *even* if, for all $x \in L$, $x^2 := b(x,x) \in 2\mathbb{Z}$, while it is *odd* if there exists $x \in L$ such that $b(x,x) \notin 2\mathbb{Z}$. Given an integral basis for L, we associate to the bilinear form b_L a symmetric $(\operatorname{rk} L \times \operatorname{rk} L)$-matrix S_L. The *discriminant* of L is the integer $\operatorname{disc}(L) := \det S_L$. A lattice is *unimodular* if $\operatorname{disc}(L) = \pm 1$ while it is *non-degenerate* if $\operatorname{disc}(L) \neq 0$. Moreover, a lattice L is positive definite if $x^2 > 0$, for any $0 \neq x \in L$ while L is negative definite if $x^2 < 0$, for any $0 \neq x \in L$. It is indefinite if it is non-degenerate but neither positive definite nor negative definite. (Notice that $L(-1)$ is negative definite if and only if L is positive definite.)

Unimodular indefinite lattices are particularly relevant for the study of K3 surfaces and they are completely classified by the following result.

Theorem 2.2. ([65]) *Let L be a unimodular indefinite lattice. If L is odd then $L \cong \langle 1\rangle^{\oplus m} \oplus \langle -1\rangle^{\oplus n}$, while if L is even $L \cong U^{\oplus m} \oplus E_8(\pm 1)^{\oplus n}$, for some positive integers m and n.*

From now on, L will be a non-degenrate lattice. We can easily associate a finite group to L. Indeed, consider the *dual lattice*

$$L^\vee := \operatorname{Hom}_{\mathbb{Z}}(L, \mathbb{Z}) \cong \{l \in L \otimes \mathbb{Q} : l \cdot p \in \mathbb{Z}, \text{ for any } p \in L\}$$

of a lattice L and the natural inclusion $L \hookrightarrow L^\vee$. The *discriminant group of* L is the quotient

$$A_L := L^\vee / L.$$

The order of A_L is $|\operatorname{disc}(L)|$ (see [4, Lemma 2.1]). Moreover, the bilinear form b of L induces a symmetric bilinear form $b_L : A_L \times A_L \to \mathbb{Q}/\mathbb{Z}$ and thus a quadratic form

$$q_L : A_L \to \mathbb{Q}/\mathbb{Z}.$$

Whenever L is even, q_L takes values in $\mathbb{Q}/2\mathbb{Z}$. By $\mathrm{O}(A_L)$ we denote the group of automorphisms of A_L preserving q_L. We denote by $\ell(L)$ the minimal number of generators of A_L. The inclusion of L into $L^\vee$ yields a homomorphism $\Psi : \mathrm{O}(L) \to \mathrm{O}(A_L)$.

The discriminant group of a lattice L and the group of its automorphisms preserving the quadratic form just defined always play a role when we want to extend isometries of sublattices to isometry of L itself. Since at some point we will have to do this, we briefly recall the main results proved by Nikulin in this direction.

Proposition 2.3. **([75], Theorem 1.14.2.)** *Let L be an even indefinite lattice such that $\operatorname{rk} L \geq \ell(L) + 2$. Then the map $\Psi : \mathrm{O}(L) \to \mathrm{O}(A_L)$ is onto.*

An embedding $i : V \hookrightarrow L$ of a lattice V into a lattice L is *primitive* if $L/i(V)$ is a free abelian group. One can show that given a primitive sublattice K of a unimodular lattice L, there exists a morphism of groups $\gamma : A_K \longrightarrow A_{K^\perp}$. Moreover,

Proposition 2.4. **([75], Corollary 1.6.2.)** *If K and L are as above, then the morphism γ induces an isometry $\gamma : (A_K, q_K) \to (A_{K^\perp}, -q_{K^\perp})$.*

Hence, γ defines an isomorphism $\psi : \mathrm{O}(A_K) \to \mathrm{O}(A_{K^\perp})$. Furthermore, given an even unimodular lattice L, there are two homomorphism $\Psi_1 : \mathrm{O}(K) \to \mathrm{O}(A_K)$ and $\Psi_2 : \mathrm{O}(K^\perp) \to \mathrm{O}(A_{K^\perp})$ naturally associated to a primitive sublattice K of L.

Proposition 2.5. **([75], Theorem 1.6.1 and Corollary 1.5.2.)** *An isometry $f \in \mathrm{O}(K)$ lifts to an isometry in $\mathrm{O}(L)$ if and only if $\psi(\Psi_1(f)) \in \operatorname{im}(\Psi_2)$.*

For later use we include here an easy example:

Example 2.6. Consider a primitive sublattice $L = \langle 2d\rangle$ of Λ, with $d > 0$. Following the definition we have that $A_L \cong \mathbb{Z}/2d\mathbb{Z}$. If there is a prime p dividing d, then it was proved in [89] that $\mathrm{O}(A_L)$ has order greater or equal to 2. In particular, keeping the previous notations, $\Psi_1(\mathrm{id}) \neq \Psi_1(-\mathrm{id})$.

2.2. K3 surfaces.

An *analytic K3 surface* (or simply a *K3 surface*) is a complex smooth surface X with trivial canonical bundle and such that $H^1(X,\mathbb{Z})=0$. Notice that Siu proved in [94] that these properties automatically imply that X carries a Kähler structure. Although in this paper we will be mainly interested in complex non-projective K3 surfaces let us start with very easy examples of algebraic K3 surfaces.

Example 2.7. (Complete intersections) Let X be a smooth surface in $\mathbb{P}^{n+2}$ which is the complete intersection of n hypersurfaces of degrees $d_1,\ldots,d_n$, with $d_i>1$ for $i\in\{1,\ldots,n\}$. The adjunction formula shows that the dualizing sheaf ω_X of X is $\mathcal{O}_X((d_1+\ldots+d_n)-(n+3))$. Hence X is a K3 surface if and only if $d_1+\ldots+d_n=n+3$. In particular the only possibilities are:

$$\begin{array}{llll} n=1 & d_1=4; & & \\ n=2 & d_1=2 & d_2=3; & \\ n=3 & d_1=2 & d_2=2 & d_3=2. \end{array}$$

A result of Kodaira proves that K3 surfaces, considered as differentiable manifolds, are all diffeomorphic. In particular, Example 2.7 shows that any K3 surface is diffeomorphic to a smooth quartic surface in $\mathbb{P}^3$. Since a surface of this type is simply connected, any K3 surface is simply connected as well.

The *Néron-Severi group* of a K3 surface X is the free abelian group

$$\mathrm{NS}(X):=\mathrm{c}_1(\mathrm{Pic}(X))\subset H^2(X,\mathbb{Z}),$$

where c_1 is the natural map

$$H^1(X,\mathbb{Z})\longrightarrow H^1(X,\mathcal{O}_X)\longrightarrow \mathrm{Pic}(X)\xrightarrow{\mathrm{c}_1} H^2(X,\mathbb{Z})$$

induced by the exponential exact sequence

$$0\longrightarrow\mathbb{Z}\longrightarrow\mathcal{O}_X\longrightarrow\mathcal{O}_X^*\longrightarrow 0.$$

Since a K3 surface X is simply connected, c_1 is injective and then $\mathrm{Pic}(X)\cong\mathrm{NS}(X)$. The rank $\rho(X)$ of the free abelian group $\mathrm{NS}(X)$ is the *Picard number of* X.

The orthogonal complement $T(X):=\mathrm{NS}(X)^\perp\subset H^2(X,\mathbb{Z})$ with respect to the cup product is the *transcendental lattice* of X. Notice that if X is algebraic, the Hodge Index Theorem implies that $\mathrm{NS}(X)$ has signature $(1,\rho(X)-1)$. In the non algebraic case this is no longer true. Indeed, we can consider the following example:

Example 2.8. (Kummer surfaces) Let T be a complex torus of dimension 2 and consider the quotient $S:=T/\langle\iota\rangle$, where $\iota(a)=-a$ for any $a\in T$. The surface S has 16 singular points corresponding to the points fixed by ι. Let $\widetilde{T}$ be the blow-up of T along those 16 points and let $\tilde{\iota}$ be the natural extension of ι to $\widetilde{T}$ (i.e. the extension of ι acting trivially on the points of the exceptional curves). The quotient

$$\mathrm{Km}(T):=\widetilde{T}/\langle\tilde{\iota}\rangle$$

is a K3 surface and it is called the *Kummer surface of* T. One can also observe that $\mathrm{Km}(T)$ is the crepant resolution of the singular quotient $T/\langle\iota\rangle$ and $\mathrm{Km}(T)$ is algebraic if and only if T is an abelian surface. Moreover, if the signature of $\mathrm{NS}(T)$ is (s,r), then the signature of $\mathrm{NS}(\mathrm{Km}(T))$ is $(s,r+16)$. For example, the signature $(0,16)$ is realized considering a 2-dimensional complex torus T with $\mathrm{Pic}(T)=\{0\}$.

The second cohomology group $H^2(X,\mathbb{Z})$ endowed with the cup product $(-,-)$ is an even unimodular lattice of signature $(3,19)$ (and hence of rank 22). More precisely, due to Theorem 2.2, there exists an isometry

$$H^2(X,\mathbb{Z})\cong\Lambda:=U^{\oplus 3}\oplus E_8(-1)^{\oplus 2},$$

where U and E_8 are defined in Example 2.1. The lattice Λ is usually called the *K3 lattice*. Notice that due to this description of $H^2(X,\mathbb{Z})$, $T(X)$ has signature $(2, 20-\rho(X))$.

Given a K3 surface X, a *marking* for X is the choice of an isometry $\varphi : H^2(X,\mathbb{Z}) \xrightarrow{\sim} \Lambda$. A *marked K3 surface* is a pair (X,φ), where X is a K3 surface and φ is a marking for X. Two marked K3 surfaces (X,φ) and (X',φ') are isomorphic if there exists an isomorphism $f : X \xrightarrow{\sim} X'$ such that $\varphi' = \varphi \circ f^*$.

2.3. **Hodge structures.**

Let X be a K3 surface. The usual weight-two Hodge structure on $H^2(X,\mathbb{Z})$ is the decomposition

$$H^2(X,\mathbb{C}) = H^{2,0}(X) \oplus H^{1,1}(X) \oplus H^{0,2}(X).$$

Since X has trivial dualizing sheaf, $H^{2,0}(X) = \langle \sigma_X \rangle \cong \mathbb{C}$, where

$$(\sigma_X, \sigma_X) = 0 \quad \text{and} \quad (\sigma_X, \overline{\sigma_X}) > 0. \tag{2.1}$$

Let X and Y be K3 surfaces. An isometry (with respect to the cup product) $f : H^2(X,\mathbb{Z}) \xrightarrow{\sim} H^2(Y,\mathbb{Z})$ is a *Hodge isometry* if $f(H^{2,0}(X)) = H^{2,0}(Y)$. Since $\mathrm{NS}(X) = H^{1,1}(X) \cap H^2(X,\mathbb{Z})$ and $\mathrm{NS}(Y) = H^{1,1}(Y) \cap H^2(Y,\mathbb{Z})$, an isometry $g : T(X) \xrightarrow{\sim} T(Y)$ is a Hodge isometry if $g(H^{2,0}(X)) = H^{2,0}(Y)$.

Remark 2.9. Notice that the Lefschetz $(1,1)$-theorem implies that the transcendental lattice $T(X)$ of a K3 surface X can be described as the minimal primitive sublattice of $H^2(X,\mathbb{Z})$ such that $\sigma_X \in T(X) \otimes \mathbb{C}$.

The second cohomology group of a K3 surface contains two interesting cones. Indeed, the *positive cone* $\mathcal{C}_X \subset H^{1,1}(X) \cap H^2(X,\mathbb{R})$ is the connected component of the cone $\{x \in H^{1,1}(X) \cap H^2(X,\mathbb{R}) : x \cdot x > 0\}$ containing a Kähler class. We denote by $\mathcal{K}_X$ the *Kähler cone* of X (i.e. the cone generated by the Kähler classes).

Remark 2.10. One can prove that $\mathcal{K}_X = \{x \in \mathcal{C}_X : (x,\delta) > 0 : \delta \in \Delta(X)\}$, where $\Delta(X) = \{y \in \mathrm{NS}(X) : (y,y) = -2\}$. It it clear that if X is such that $\mathrm{NS}(X) = 0$, then $\mathcal{K}_X = \mathcal{C}_X$.

Of course, the Hodge isometry induced on cohomology by any isomorphism $f : X \xrightarrow{\sim} X$ preserves the Kähler cone. Even more is true. Indeed the Torelli Theorem shows that the converse of this statement holds. Namely the Hodge and lattice structures on the second cohomology group identify a K3 surface up to isomorphism.

Theorem 2.11. (Strong Torelli Theorem) *Let X and Y be K3 surfaces. Suppose that there exists a Hodge isometry $g : H^2(X,\mathbb{Z}) \xrightarrow{\sim} H^2(Y,\mathbb{Z})$ which maps a Kähler class on X into the Kähler cone of Y. Then there exists a unique isomorphism $f : X \xrightarrow{\sim} Y$ such that $f_* = g$.*

Many people gave different proofs of the previous result at different levels of generality. Among them we recall [21, 63, 87] (see also [1, 4]). The following is a weaker form of the previous result:

Theorem 2.12. (Torelli Theorem) *Let X and Y be K3 surfaces. Then X and Y are isomorphic if and only if there exists a Hodge isometry $H^2(X,\mathbb{Z}) \cong H^2(Y,\mathbb{Z})$.*

There is a second way to use the weight-two Hodge structure on H^2 to study the geometry of a K3 surface X. In fact, consider the quasi-projective variety

$$Q := \{\sigma \in \mathbb{P}(\Lambda \otimes \mathbb{C}) : \sigma \cdot \sigma = 0 \text{ and } \sigma \cdot \overline{\sigma} > 0\}.$$

which is called the *period domain*. If (X,φ) is a marked K3 surface, (2.1) implies $[\varphi(\mathbb{C}\sigma_X)] \in Q$ (here $[\varphi(\mathbb{C}\sigma_X)]$ is the class in $\mathbb{P}(\Lambda \otimes \mathbb{C})$ of the line $\varphi(\mathbb{C}\sigma_X)) \subset \Lambda \otimes \mathbb{C})$.

Let $\mathcal{M}$ be the moduli space of marked K3 surfaces. The map

$$\Theta : \mathcal{M} \longrightarrow Q$$

sending (X,φ) to $[\varphi(\mathbb{C}\sigma_X)]$ is the *period map* and $[\varphi(\mathbb{C}\sigma_X)]$ is usually called *period* of the K3 surface X.

Theorem 2.13. (Surjectivity of the Period Map) *The map Θ is surjective.*

The interested reader can find a readable account of the proof of this deep result in [4].

Consider a marked K3 surface (X,φ) such that $\mathrm{NS}(X)\neq\{0\}$ and let $H^{2,0}(X)=\langle\sigma_X\rangle$. In particular, $\mathrm{rk}\,T(X)<22$ and there exists a primitive sublattice $L\hookrightarrow\Lambda$ with $\mathrm{rk}\,L<22$ and such that $[\varphi(\mathbb{C}\sigma_X)]\in Q\cap\mathbb{P}(L\otimes\mathbb{C})$. Since there exist at most countably many primitive sublattices L of Λ with rank strictly smaller than 22, the periods of marked K3 surfaces (X,φ) such that $\mathrm{NS}(X)=\{0\}$ are contained in the complement of the union of countably many hyperplane sections of the period domain Q. Therefore, a K3 surface X with $\mathrm{NS}(X)=\{0\}$ is *generic*.

Passing from $H^2(X,\mathbb{Z})$ to the total cohomology group $H^*(X,\mathbb{Z})$ of a K3 surface X, we see that $H^*(X,\mathbb{Z})$ has a natural lattice structure induced by the *Mukai pairing* defined by

$$\langle\alpha,\beta\rangle:=-(\alpha_0,\beta_4)+(\alpha_2,\beta_2)-(\alpha_4,\beta_0), \tag{2.2}$$

for every $\alpha=(\alpha_0,\alpha_2,\alpha_4)$ and $\beta=(\beta_0,\beta_2,\beta_4)$ in $H^*(X,\mathbb{Z})$. The group $H^*(X,\mathbb{Z})$ endowed with the Mukai pairing is called *Mukai lattice* and we write $\widetilde{H}(X,\mathbb{Z})$ for it. Since $H^0(X,\mathbb{Z})\oplus H^4(X,\mathbb{Z})$ with the cup product is isometric to U, we have a natural isometry of lattices

$$\widetilde{H}(X,\mathbb{Z})\cong\widetilde{\Lambda}:=\Lambda\oplus U.$$

We can now define a weight-two Hodge structure on the total cohomology group $H^*(X,\mathbb{Z})$ in the following natural way (see [43, 48, 51] for a more general setting):

$$\begin{aligned}\widetilde{H}^{2,0}(X)&:=H^{2,0}(X),\\ \widetilde{H}^{0,2}(X)&:=H^{0,2}(X),\\ \widetilde{H}^{1,1}(X)&:=H^0(X,\mathbb{C})\oplus H^{1,1}(X)\oplus H^4(X,\mathbb{C}).\end{aligned} \tag{2.3}$$

Clearly, an isomorphism $f:\widetilde{H}(X,\mathbb{Z})\xrightarrow{\sim}\widetilde{H}(Y,\mathbb{Z})$ is a Hodge isometry if it is an isometry of lattices which preserves the Hodge structures just defined.

As in the case of the second cohomology group, for a K3 surface X, the lattice $\widetilde{H}(X,\mathbb{Z})$ has some very interesting substructure. Indeed, let σ be a generator of $H^{2,0}(X)$ and ω a Kähler class (e.g. an ample class if X is algebraic), then

$$P(X,\sigma,\omega):=\langle\mathrm{Re}(\sigma),\mathrm{Im}(\sigma),1-\omega^2/2,\omega\rangle, \tag{2.4}$$

is a positive four-space in $\widetilde{H}(X,\mathbb{R})$ (here $\mathrm{Re}(\sigma)$ and $\mathrm{Im}(\sigma)$ are the real and imaginary part of σ). It comes, by the choice of the basis, with a natural orientation. It is easy to see that this orientation is independent of the choice of σ and ω.

In general, let $g:\Gamma\to\Gamma'$ be an isometry of lattices with signature $(4,t)$. Suppose that positive four-spaces $V\subset\Gamma_{\mathbb{R}}:=\Gamma\otimes\mathbb{R}$ and $V'\subset\Gamma'_{\mathbb{R}}:=\Gamma'\otimes\mathbb{R}$ and orientations for both of them have been chosen. Then one says that the isometry g *preserves the given orientation of the positive directions* (or, simply, that g is *orientation preserving*) if the composition of

$$g_{\mathbb{R}}:V\longrightarrow\Gamma'_{\mathbb{R}}$$

with the orthogonal projection $\Gamma'_{\mathbb{R}}\to V'$ is compatible with the given orientations of V and V'. By $\mathrm{O}_+(\Gamma)$ one denotes the group of all orientation preserving isometries of Γ.

A Hodge isometry $f:\widetilde{H}(X,\mathbb{Z})\xrightarrow{\sim}\widetilde{H}(Y,\mathbb{Z})$ is *orientation preserving* if preserves the orientation of the four positive directions previously defined.

Example 2.14. (i) It is easy to see that, for any K3 surface X, the Hodge isometry $j:=\mathrm{id}_{H^2(X,\mathbb{Z})}\oplus(-\mathrm{id})_{H^0(X,\mathbb{Z})\oplus H^4(X,\mathbb{Z})}$ is not orientation preserving.

(ii) Consider now $\delta = (1, 0, 1) \in \widetilde{H}(X, \mathbb{Z})$ and take the Hodge isometry defined by

$$s_\delta(x) = x - 2\frac{\langle \delta, x\rangle}{\langle \delta, \delta\rangle}\delta = x + \langle \delta, x\rangle\delta,$$

for any $x \in \widetilde{H}(X, \mathbb{Z})$. Clearly s_δ is a reflection and is orientation preserving.

(iii) Let $f : X \xrightarrow{\sim} Y$ be an isomorphism of K3 surfaces and take the natural identifications $H^0(X, \mathbb{Z}) = H^0(Y, \mathbb{Z}) = \mathbb{Z}$ and $H^4(X, \mathbb{Z}) = H^4(Y, \mathbb{Z}) = \mathbb{Z}$. The Hodge isometry $f^*|_{H^2(Y,\mathbb{Z})} \oplus (\mathrm{id})_{\mathbb{Z}\oplus\mathbb{Z}} : \widetilde{H}(Y, \mathbb{Z}) \xrightarrow{\sim} \widetilde{H}(X, \mathbb{Z})$ preserves the orientation of the four positive directions (see [12, 28]).

2.4. **McMullen's examples.**

In this section we describe some generic analytic K3 surfaces following McMullen's original idea in [66] (see also [39, Thm. 1.7]). This beautiful construction requires some preliminary results about Salem numbers and Salem polynomials (mainly contained in [39]) which we briefly summarize.

Definition 2.15. (i) An irreducible monic polynomial $S(x) \in \mathbb{Z}[x]$ is a *Salem polynomial* if the set of its roots is $\{\lambda, \lambda^{-1}, a_1, \overline{a_1}, \dots, a_n, \overline{a_n}\}$, where $\lambda \in \mathbb{R}$, $\lambda > 1$ and $|a_i| = 1$, for any $i \in \{1, \dots, n\}$.

(ii) A Salem polynomial $S(x)$ is *unramified* if $|S(-1)| = |S(1)| = 1$.

(iii) A real number $\lambda > 1$ is a *Salem number* if the minimal irreducible monic polynomial $P(x)$ such that $P(\lambda) = 0$ is a Salem polynomial.

By definition a Salem polynomial has even degree. Let $n \geq 3$ be an odd integer and consider a separable polynomial $C(x) \in \mathbb{Z}[x]$ of degree $n - 3$ and whose roots are contained in the interval $[-2, 2]$ (recall that $C(x)$ is separable if it does not have multiple roots in $\mathbb{C}$). Take now the polynomial

$$R(x) = C(x)(x^2 - 4)(x - a) - 1,$$

where $a \in \mathbb{Z}$. Finally, consider the polynomial $S(x) = x^n R(x + x^{-1})$ which has degree $2n$. In [39] Gross and McMullen showed that $S(x)$ is an unramified Salem polynomial for any $a \gg 0$. Hence the following holds true:

Theorem 2.16. **([39], Theorem 1.6.)** *For any odd integer $n \geq 3$ there are infinitely many unramified Salem polynomials of degree $2n$.*

An interesting property of Salem polynomials contained in [66, Sect. 10] and which will be used in Section 3.1 is given by the following lemma.

Lemma 2.17. *Given a positive integer n and two real numbers $a_1 < a_2$, there exist finitely many Salem polynomials $S(x)$ such that $\deg S(x) = 2n$ and $a_1 < \operatorname{tr} S(x) < a_2$, where $\operatorname{tr} S(x)$ is the sum of the roots of $S(x)$.*

Proof. By definition, the set of roots of a Salem polynomial of degree $2n$ is of the form

$$\{\lambda, \lambda^{-1}, a_1, \overline{a_1}, \dots, a_{n-1}, \overline{a_{n-1}}\},$$

where $\lambda \in \mathbb{R}$, $\lambda > 1$ and $|a_i| = 1$, for any $i \in \{1, \dots, n - 1\}$. In particular,

$$1 - n \leq \sum_{i=1}^{n-1} a_i + \overline{a_i} \leq n - 1$$

and, by assumption,

$$a_1 \leq \lambda + \lambda^{-1} + \sum_{i=1}^{n-1} a_i + \overline{a_i} \leq a_2.$$

Hence λ is bounded from above and all the roots of $S(x)$ are contained in a compact subset of $\mathbb{C}$.

Let us write $S(x) = x^{2n} + \sum_{k=1}^{2n-1} b_k x^k + 1$, where b_k is an integer which is a symmetric function of the roots of $S(x)$. Therefore, the coefficients of $S(x)$ are contained in a compact subset of $\mathbb{C}$. This concludes the proof. $\square$

The main idea in McMullen's construction is to define a K3 surface X with an automorphism $f \in \mathrm{Aut}(X)$ such that the minimal polynomial of the induced isometry $f^* : H^2(X,\mathbb{Z}) \to H^2(X,\mathbb{Z})$ is a Salem polynomial.

As a first step in this direction we consider an unramified Salem polynomial $S(x)$ of degree $22 = \mathrm{rk}\,\Lambda$ whose existence is granted by Theorem 2.16. Let λ be the unique root of $S(x)$ such that $\lambda \in \mathbb{R}$ and $\lambda > 1$. Consider the ring

$$B := \mathbb{Z}[x]/(S(x))$$

and let K be its field of fractions.

The ring B contains a subring R generated by the element $y = x + x^{-1}$ (see [66, Sect. 8]). Chosen a unit $u \in R$, we define on B the following bilinear symmetric form:

$$\langle f_1, f_2\rangle_{B,u} := \mathrm{Tr}^K_{\mathbb{Q}}\left(\frac{u f_1(x) f_2(x^{-1})}{r'(y)}\right), \tag{2.5}$$

where r is the minimal polynomial of $\lambda + \lambda^{-1}$ and r' is its derivative. McMullen showed that $\langle f_1, f_2\rangle_{B,u} \in \mathbb{Z}$, for any $f_1, f_2 \in B$, and so the pair $(B, \langle -,-\rangle_{B,u})$ is a lattice.

The choice of the unit u is extremely relevant in McMullen's construction. In fact, in [66, Sect. 8] he proved the following result:

Proposition 2.18. *There exists $u \in R$ such that the lattice $(B, \langle -,-\rangle_{B,u})$ is isometric to the K3 lattice Λ.*

In particular, the lattice $(B, \langle -,-\rangle_{B,u})$ has the same lattice structure as the second cohomology group of any K3 surface.

The second key step consists in defining an appropriate weight-two Hodge structure on B. In this sense, consider now the isomorphism $F : B \to B$ such that $F(g) = xg$. By definition F preserves the bilinear form (2.5) and hence yields an isometry of the lattice $(B, \langle -,-\rangle_{B,u})$. McMullen observes that the morphism $F + F^{-1} : B \otimes \mathbb{C} \to B \otimes \mathbb{C}$ has a unique eigenspace E whose signature is $(2,0)$. Moreover E can be decomposed as the direct sum $E = H_1 \oplus H_2$ where H_1 is the eigenspace of an eigenvalue δ of F, while H_2 is the eigenspace of the eigenvalue δ^{-1}. In particular E is invariant under the action of the isometry F. Therefore, the lattice $(B, \langle -,-\rangle_{B,u})$ has a natural weight-two Hodge structure defined by

$$(B\otimes\mathbb{C})^{2,0} := H_1 \qquad (B\otimes\mathbb{C})^{0,2} := H_2 \qquad (B\otimes\mathbb{C})^{1,1} := (H_1)^{\perp_{\langle -,-\rangle_{B,u}}}. \tag{2.6}$$

Moreover, if $H_1 = \langle\sigma\rangle$, then $H_2 = \langle\overline{\sigma}\rangle$ and

$$\langle\sigma,\sigma\rangle_{B,u} = 0 \qquad \langle\sigma,\overline{\sigma}\rangle_{B,u} > 0. \tag{2.7}$$

Since by Theorem 2.13 a K3 surface is determined (up to isomorphism) by the choice of a weight-two Hodge structure as in (2.6) and (2.7), there exists a K3 surface X and a Hodge isometry $(H^2(X,\mathbb{Z}), (-,-)) \cong (B, \langle -,-\rangle_{B,u})$.

Now we can easily prove (using [66, Thm. 3.4]) that the surface X just constructed is a generic analytic K3 surface.

Lemma 2.19. *Let X be a K3 surface as above. Then* $\mathrm{NS}(X) = \{0\}$.

Proof. Due to [66, Sect. 8], the minimal polynomial of the isometry $F : B \xrightarrow{\sim} B$ is the Salem polynomial $S(x)$. Suppose $\mathrm{NS}(X) \neq \{0\}$ and let $F_{\mathbb{Q}}$ be the $\mathbb{Q}$-linear extension of F. Then $F_{\mathbb{Q}} = F_1 \times F_2$, where $F_1 \in \mathrm{O}(\mathrm{NS}(X)\otimes\mathbb{Q})$ and $F_2 \in \mathrm{O}(T(X)\otimes\mathbb{Q})$. But this is impossible, since $S(x)$ is irreducible and then F has no rational invariant subspaces. $\square$

By abuse of notation, let us denote by $F \in \mathrm{O}(H^2(X,\mathbb{Z}))$ the isometry induced by $F : B \to B$.

Lemma 2.20. *There exists $f \in \mathrm{Aut}(X)$ such that $f^* = F : H^2(X, \mathbb{Z}) \to H^2(X, \mathbb{Z})$.*

Proof. Since $\mathrm{NS}(X) = \{0\}$, due to Remark 2.10, $\mathcal{K}_X = \mathcal{C}_X$. Hence, by Theorem 2.11, it is enough to show that $F(\mathcal{C}_X) = \mathcal{C}_X$.

Let $E(\lambda)$ be the eigenspace of the eigenvalue λ (which is a Salem number) and let $\alpha \in \mathbb{C}$ be such that $F(\sigma_X) = \alpha\sigma_X$, where $H^{2,0}(X) = \langle\sigma_X\rangle$. As

$$(\sigma_X, \overline{\sigma_X}) = (F(\sigma_X), F(\overline{\sigma_X})) = |\alpha|^2(\sigma_X, \overline{\sigma_X}) \neq 0,$$

the eigenvalue α is in the unit circle and $E(\lambda) \subset H^{1,1}(X) \cap H^2(X, \mathbb{R})$. Moreover, since λ is real, there exists $0 \neq v \in E(\lambda) \cap H^2(X, \mathbb{R})$. As F is an isometry, $(v, v) = (F(v), F(v)) = \lambda^2(v, v)$ and then $(v, v) = 0$. Hence, up to changing v with $-v$ we can suppose v in the boundary of the positive cone. In particular, F preserves the closure of the positive cone and, therefore, $\mathcal{K}_X$. □

At the end we produced a K3 surface X with $\mathrm{NS}(X) = \{0\}$ and such that $\mathrm{Aut}(X)$ contains an automorphism f whose induced isometry $f^* : H^2(X, \mathbb{Z}) \xrightarrow{\sim} H^2(X, \mathbb{Z})$ has characteristic polynomial of Salem type.

3. Automorphisms

In this section we study the automorphism groups of non-algebraic K3 surfaces with Picard number 0. As it will turn out (Theorem 3.6) such a group is either trivial or isomorphic to $\mathbb{Z}$. The latter holds true if and only if the K3 surface is constructed as in Section 2.4. Section 3.2 is devoted to show a few differences with the algebraic case.

We will reconsider the automorphisms of a non-projective generic K3 surface under a different light in Section 5. There we will show the close relation between the automorphisms group and the group of autoequivalences of the derived categories.

3.1. **The non-algebraic generic case.**

Since any automorphism $f : X \xrightarrow{\sim} X$ of a K3 surface X induces a Hodge isometry $f^* : H^2(X, \mathbb{Z}) \xrightarrow{\sim} H^2(X, \mathbb{Z})$, there exists $\chi(f) \in \mathbb{C}^\times$ such that $f^*(\sigma_X) = \chi(f)\sigma_X$, where $H^{2,0}(X) = \langle\sigma_X\rangle$. Hence we get a natural map

$$\chi : \mathrm{Aut}(X) \longrightarrow \mathbb{C}^\times \tag{3.1}$$

sending f to $\chi(f)$.

Lemma 3.1. (i) *If $\rho(X) = 0$, then the morphism χ is injective.*

(ii) *For any $f \in \mathrm{Aut}(X)$ of finite order, the complex number $\chi(f)$ is a root of unity.*

Proof. Suppose that there exists $f \in \mathrm{Aut}(X)$ such that $f^*(\sigma_X) = \sigma_X$. This means that f^* has eigenvalue 1 and then the primitive sublattice $T := \{x \in H^2(X, \mathbb{Z}) : f^*(x) = x\}$ is non empty. Moreover $\sigma_X \in T \otimes \mathbb{C}$. Hence, by Remark 2.9, $T(X) = T = H^2(X, \mathbb{Z})$ and so $f^* = \mathrm{id}$. By Theorem 2.11, $f = \mathrm{id}$.

Suppose that $f \in \mathrm{Aut}(X)$ has finite order. Then $\chi(\langle f\rangle)$ is a finite cyclic subgroup of $\mathbb{C}^\times$. Hence all the eigenvalues of f^* are roots of unity (see also [104]). □

We now prove that the automorphism group of non-algebraic generic K3 surfaces does not contain non-trivial automorphisms of finite order.

Proposition 3.2. **([74], Theorem 3.1.)** *Let X be a K3 surface such that $\rho(X) = 0$ and let $f \in \mathrm{Aut}(X)$ be of finite order. Then $f = \mathrm{id}$.*

Proof. Due to Lemma 3.1, it is enough to prove that if f has finite order then $\chi(f) = 1$. Suppose $\chi(f) \neq 1$ and consider the quotient $X' := X/\langle f\rangle$. Since $\langle f\rangle$ is finite, X' has a finite number of singular points. Let $g : Y \to X'$ be a resolution of these singularities. Of course, since X is Kähler, Y is Kähler.

Let σ be a holomorphic 2-form on Y. We can lift σ to a meromorphic 2-form σ' on X via the natural quotient map $X \to X'$. The form σ' is not defined just in a finite number of points. Hence it can be extended to a holomorphic 2-form σ'' on X which, moreover, must be invariant under the action of $\langle f\rangle$. Since we are supposing $\chi(f) \neq 1$, σ'' must be trivial and σ must be trivial as well. In particular $\dim H^{2,0}(Y) = 0$ and Y is a smooth algebraic surface (indeed, $\mathrm{NS}(Y)$ must contain h such that $h^2 > 0$). This would imply that $\mathrm{NS}(X) \neq \{0\}$ which contradicts our assumption. □

Although the group $\mathrm{Aut}(X)$ cannot contain automorphisms of finite order if $\rho(X) = 0$, there may be $f \in \mathrm{Aut}(X)$ with infinite order. McMullen's construction in Section 2.4 provides examples of this type. Indeed, if X is a K3 surface as in Section 2.4, Lemma 2.20 shows that there exists $f \in \mathrm{Aut}(X)$ whose minimal polynomial $P(x) \in \mathbb{Z}[x]$ is a Salem unramified polynomial. Since $P(x)$ has a real eigenvalue $\lambda > 1$, f has infinite order (Proposition 3.2).

Proposition 3.3. **([66], Theorem 3.2.)** *Let X be a K3 surface such that $\rho(X) = 0$ and let $f \in \mathrm{Aut}(X)$ be of infinite order. Then the minimal polynomial of f^* is a Salem polynomial.*

Proof. First notice that $|\chi(f)| \neq 1$. Indeed, if $\chi(f)$ is a n-th root of unity, then $\chi(f^n) = 1$ and by item (i) of Lemma 3.1 $f^n = \mathrm{id}$, which is a contradiction.

More precisely, f^* has an eigenvalue λ such that $|\lambda| \neq 1$. Indeed, all the eigenvalues of f^* are also roots of its characteristic polynomial in $\mathbb{Z}[x]$. Since $\det(f^*) = \pm 1$, the eigenvalues of f^* with absolute value 1 would be roots of unity. This contradicts the previous remark.

Let λ be the eigenvalue of f^* such that $|\lambda| > 1$. The pull-back f^* preserves the subspace $H^{1,1}(X) \subset H^2(X,\mathbb{C})$. Hence $f^* = F_1 \times F_2 \in \mathrm{O}(1,19) \times \mathrm{O}(2,0)$. Due to [66, Lemma 3.1], $h \in \mathrm{O}(p,q)$ has at most $\min\{p,q\}$ eigenvalues outside the unit circle. Hence $f^*_{\mathbb{C}}$ has at most one eigenvalue λ with $|\lambda| > 1$. Notice that the uniqueness also implies that such an eigenvalue is real.

The fact that $\lambda > 1$ follows easily from the remark that f^*, being induced by an automorphism, must preserve the cone $\mathcal{K}_X$ (for more details on this, see the proof of Lemma 2.20). □

We are now ready to describe the automorphism group of any K3 surface X with $\rho(X) = 0$.

Theorem 3.4. **([81], Theorem 1.5.)** *Let X be a K3 surface such that $\rho(X) = 0$. Then either $\mathrm{Aut}(X) = \{\mathrm{id}\}$ or $\mathrm{Aut}(X) = \langle f\rangle \cong \mathbb{Z}$.*

Proof. First recall that due to [81, Prop. 4.3] the group $\mathrm{Aut}(X)$ is a finitely generated free abelian group. Now suppose that $\mathrm{id} \neq f, g \in \mathrm{Aut}(X)$ are two distinct generators.

To shorten the notation we put $F := f^*$ and $G := g^*$. Let $S_F(x)$, $S_G(x)$, λ_F and λ_G be the corresponding Salem polynomials and Salem numbers. We already know (Lemma 3.1) that the morphism

$$\chi : \langle f, g\rangle \longrightarrow \mathbb{C}^\times$$

is injective. Therefore, since $\mathbb{C}^\times$ is commutative, $FG = GF$.

For any pair $(m,n) \in \mathbb{Z}\times\mathbb{Z}$, we denote by $S_{(m,n)}(x)$ the Salem polynomial of the isometry F^mG^n. Without loss of generality we can suppose that the Salem number of $S_{(m,n)}(x)$ is $\lambda_F^m\lambda_G^n$ (notice that as a consequence of Proposition 3.3 and of the fact that $FG = GF$, F and G are simultaneously diagonalizable).

There exists two real numbers $1 < c_1 < c_2$ such that the set $M_1 := \{(m,n) \in \mathbb{Z}\times\mathbb{Z}\setminus\{(0,0)\} : c_1 < \lambda_F^m\lambda_G^n < c_2\}$ is infinite. Indeed, choose c_1 and c_2 such that $\lambda_G < c_2/c_1$ (i.e. $\log(\lambda_G) < \log(c_2) - \log(c_1)$). Hence for m negative there always exists at least a positive integer n such that

$$\log(c_1) - m\log(\lambda_F) < n\log(\lambda_G) < \log(c_2) - m\log(\lambda_F).$$

This is equivalent to $c_1 < \lambda_F^m\lambda_G^n < c_2$.

Due to Lemma 2.17, there exists a Salem polynomial $S(x)$ such that the set $M_2 := \{(m,n) \in M_1 : S_{(m,n)}(x) = S(x)\}$ is infinite. Since $\chi(f^m \circ g^n)$ is a root of $S(x)$ for any $(m,n) \in M_2$ and as $S(x)$ has only finitely many roots, there are $m_1, n_1 \in \mathbb{Z}$ such that $\chi(f^{m_1} \circ g^{n_1}) = \chi(f^{m_2} \circ g^{n_2})$, for any $(m_2, n_2) \in M_1$. This means that $\chi(f^{m_1-m_2} \circ g^{n_1-n_2}) = 1$ and then $f^{m_1-m_2} \circ g^{n_1-n_2} = \mathrm{id}$ (by Proposition 3.2).

As $\chi(\langle f, g\rangle)$ is a free abelian group of finite rank ([81, Prop. 4.3]) and f and g are generators, $F^{m_1-m_2} \circ G^{n_1-n_2}$ is the identity if and only if $m_1 = m_2$ and $n_1 = n_2$. In particular, $M_2 = \{(m_1, n_1)\}$ which is a contradiction. □

The following result shows that the case of K3 surfaces X with $\mathrm{NS}(X) = \{0\}$ and with an automorphism of infinite order is relatively rare.

Corollary 3.5. **([81], Proposition 6.2.)** *There are at most countably many K3 surfaces X with $\rho(X) = 0$ and with $f \in \mathrm{Aut}(X)$ of infinite order.*

Proof. Suppose that $F \in \mathrm{O}(\Lambda)$ has infinite order and that its minimal polynomial is a Salem polynomial $S(x)$ of degree 22. Since $S(x)$ is irreducible, the eigenvalues of F are distinct and the corresponding eigenspaces have dimension 1. Due to Lemma 2.17 there are at most countably many Salem polynomials of degree 22. Hence there are only countably many subspaces of dimension 1 of $\Lambda \otimes \mathbb{C}$ which are eigenspaces of the $\mathbb{C}$-linear extension of some $F \in \mathrm{O}(\Lambda)$ of infinite order with minimal polynomial of Salem type.

To conclude the proof observe that if $f \in \mathrm{Aut}(X)$ has infinite order, then the characteristic polynomial of f^* is a Salem polynomial of degree 22 and that $H^{2,0}(X)$ is an eigenspace of dimension 1 of the $\mathbb{C}$-linear extension of f^*. □

For sake of simplicity, we summarize the previous results in the following compact form:

Theorem 3.6. *Let X be a K3 surface such that $\rho(X) = 0$. Then either $\mathrm{Aut}(X) = \{\mathrm{id}\}$ or $\mathrm{Aut}(X) \cong \mathbb{Z}$. Moreover the latter happens for countably many K3 surfaces.*

3.2. **The algebraic case.**

The main difference in the descriptions of the automorphism groups of generic non-algebraic and algebraic K3 surfaces is that in the first case a complete description can be given while in the second one only partial results are available.

This is mainly motivated by the richer geometry of algebraic K3 surfaces. Although the many partial results about the automorphism group of algebraic K3 surfaces are very interesting, the techniques appearing in their proofs are far beyond the scope of this paper and so we will not consider them here.

More modestly, the aim of this section is to establish a comparison between the algebraic and the non-algebraic settings mainly through few easy examples.

Given a K3 surface X, define the group

$$\mathrm{O}_{\mathrm{Hodge}}(X) := \{f \in \mathrm{O}(T(X)) : f(H^{2,0}(X)) = H^{2,0}(X)\}.$$

The following lemma is essentially [80, Lemma 4.1].

Lemma 3.7. *Let X be an algebraic K3 surface with odd Picard number. Then $\mathrm{O}_{\mathrm{Hodge}}(X) = \{\pm\mathrm{id}\}$.*

Proof. Let $H^{2,0}(X) = \langle \sigma_X \rangle$. We can write $T(X) \otimes \mathbb{R} = P \oplus N$, where $P = T(X) \otimes \mathbb{R} \cap (\mathbb{C}\sigma_X \oplus \mathbb{C}\overline{\sigma_X})$, $N = T(X) \otimes \mathbb{R} \cap H^{1,1}(X)$ and σ_X is a generator of $H^{2,0}(X)$. By the Hodge Index Theorem, N is negative definite and any Hodge isometry $h : T(X) \to T(X)$ is such that the $\mathbb{C}$-linear extension $h_{\mathbb{C}}$ is in $\mathrm{O}(P) \times \mathrm{O}(N)$. The subspaces P and N are definite and thus h is diagonalizable over $\mathbb{C}$ with eigenvalues with absolute value 1.

Since h is defined over $\mathbb{Z}$, the eigenvalues are algebraic integers and thus they are roots of unity. In particular, there is a natural number n such that $h(\sigma_X) = \zeta_n \sigma_X$, where ζ_n is a primitive n-th root of unity. This implies that the order of h is n. Indeed, the primitive sublattice $T = \{x \in T(X) : h^n(x) = x\}$ is such that $\sigma_X \in T \otimes \mathbb{C}$. Remark 2.9 implies $T = T(X)$.

Obviously, $\dim N = \mathrm{rk}T(X) - 2$ is odd. Since h restricts to N, there exists a non-trivial $x \in T(X)$ such that either $h(x) = x$ or $h(x) = -x$. We can show that if h has eigenvalue 1 then $h = \mathrm{id}$. Indeed, if $h(x) = x$, then $(\zeta_n \sigma_X, x) = (h(\sigma_X), h(x)) = (\sigma_X, x)$. As $n \neq 1$, $(\sigma_X, x) = 0$ and $x \in \mathrm{NS}(X)$. This is a contradiction because $x \in T(X)$.

Similarly one shows that if there exists an $x \in T(X)$ such that $h(x) = -x$, then $h = -\mathrm{id}$. □

Applying the previous result we get a complete description of the group of automorphisms of some K3 surfaces with Picard number 1. This shows a nice analogy with the non-algebraic case analyzed in the previous section.

Proposition 3.8. *Let X be a projective K3 surface such that* $\mathrm{NS}(X) = \langle \ell \rangle$, *where* $(\ell, \ell) = 2d > 0$ *and there exists a prime p dividing d. Then* $\mathrm{Aut}(X) = \{\mathrm{id}\}$.

Proof. Take $f \in \mathrm{Aut}(X)$. By Lemma 3.7, $F := f^*|_{T(X)} = \pm\mathrm{id}$. On the other hand $G := f^*|_{\mathrm{NS}(X)} = \mathrm{id}$ because G must preserve $\mathcal{K}_X$. Since F extends to f^*, Proposition 2.5 and Example 2.6 imply $F = \mathrm{id}$ and $f^* = \mathrm{id}$. By the Strong Torelli Theorem (Theorem 2.11), $f = \mathrm{id}$. □

Example 3.9. An example of an algebraic K3 surface with Picard number 1 not satisfying the hypotheses in the previous proposition is obtained considering a smooth plane sextic $C \subset \mathbb{P}^2$. The double cover of $\mathbb{P}^2$ ramified along C is an algebraic K3 surface. If C is sufficiently generic, then $\rho(X) = 1$. Notice that X has always an automorphism of order two given by the involution exchanging the two leaves of the covering.

One can consider other algebraic K3 surfaces X with non-trivial automorphisms of finite order which do not exist in the generic non-projective case. For example, there exist K3 surfaces X with an involution $f \in \mathrm{Aut}(X)$ such that $f^*|_{T(X)} = \mathrm{id}$. Such an automorphism is usually called *Nikulin involution* (see [34, 35, 67, 72, 74]). More generally an automorphism $f \in \mathrm{Aut}(X)$ of finite order and such that $f^*|_{T(X)} = \mathrm{id}$ is called *symplectic*. If a K3 surface X has a symplectic automorphisms, then $\mathrm{rk}\,\mathrm{NS}(X) \geq 9$ ([74]). Just to give some precise references, we recall that the case of K3 surfaces X such that $\mathrm{Aut}(X)$ contains a finite subgroup G whose elements are symplectic automorphism was first studied in [74] for G abelian and in [72] for G non-abelian.

To produce examples of this type, take $\sigma \in (U \oplus U) \otimes \mathbb{C}$ such that

$$(\sigma, \sigma) = 0 \quad \text{and} \quad (\sigma, \overline{\sigma}) > 0.$$

Since there is a primitive embedding $U \oplus U \hookrightarrow \Lambda$, the class of the line $\mathbb{C}\sigma$ lies in Q and by the Surjectivity of the Period Map (Theorem 2.13), there exists a marked K3 surface (X, φ) such that $\varphi(\mathbb{C}\sigma_X) = \mathbb{C}\sigma$, where $H^{2,0}(X) = \langle \sigma_X \rangle$. Notice that, by the choice of σ, $T(X) \hookrightarrow U \oplus U$. In particular, there exists a primitive embedding $E_8(-1)^{\oplus 2} \hookrightarrow \mathrm{NS}(X)$. Theorem 5.7 in [67] ensures that X has a Nikulin involution and hence a non-trivial automorphism of finite order. To be precise the isometry induced on cohomology by this automorphim exchanges the two copies of $E_8(-1)$ and fixes the rest of the K3 lattice. Other examples of K3 surfaces with automorphisms group isomorphic to $\mathbb{Z}/2\mathbb{Z}$ are discussed in [32].

Nevertheless, as in the non-algebraic case, the automorphism group of a K3 surface could be infinite. Indeed, for an algebraic K3 surface X and for any $\delta \in \Delta(X)$ (see Remark 2.10 for the definition of $\Delta(X)$), define the reflection in $\mathrm{O}(H^2(X, \mathbb{Z}))$

$$r_\delta(x) := x + (x, \delta)\delta,$$

for any $x \in H^2(X, \mathbb{Z})$ (see [27] for an extensive use of reflections in the study of automorphism groups of K3 surfaces). Let $W(X)$ be the Weyl group generated by the reflections r_δ. Since any

$\delta \in \Delta(X)$ is orthogonal to the transcendental lattice $T(X)$, $W(X)$ can be also seen as a subgroup of $\mathrm{O}(\mathrm{NS}(X))$.

Theorem 3.10. ([87]) *Let X be an algebraic K3 surface. Then* $\mathrm{Aut}(X)$ *is finite if and only if*

$$\mathrm{O}(\mathrm{NS}(X))/W(X)$$

is a finite group.

Example 3.11. (i) Following [103], one can prove that the generic algebraic K3 surface X which is a hypersurface of bidegree $(1,1)$ in $\mathbb{P}^2 \times \mathbb{P}^2$ has automorphism group isomorphic to the infinite dihedral group. Other examples of projective K3 surfaces with infinite (non-cyclic) automorphisms group are provided in [33] where the case of the double cover of $\mathbb{P}^2$ is considered (see Example 3.9).

(ii) One can also produce K3 surfaces X such that $\mathrm{Aut}(X) \cong \mathbb{Z}$. Indeed, following [7], this is the case if X is a generic projective K3 surface such that the intersection form on $\mathrm{NS}(X)$ is represented by a diagonal matrix

$$\begin{pmatrix} 2nd & 0 \\ 0 & -2n \end{pmatrix},$$

where $n \geq 2$ and d is not a square. The proof of this fact involves a direct calculation of the solutions of some Pell equations.

4. Coherent sheaves and derived categories

In this section we move towards the second main topic of this paper: the description of the group of autoequivalences of the derived category of a generic analytic K3 surface.

As a first step we discuss few properties of the abelian category of coherent sheaves on surfaces of this type. Not surprisingly, this will lead to another key difference with respect to the algebraic case. Indeed a very nice result of Verbitsky (Theorem 4.2) shows that two smooth analytic K3 surfaces with trivial Néron-Severi lattices have equivalent abelian categories of coherent sheaves. Notice that in the algebraic case this does not happen due to a well-known result of Gabriel (Theorem 4.1). As a consequence of Verbitsky result, not all equivalences are of Fourier-Mukai type (Proposition 4.22). Hence the celebrated result of Orlov about the structure of equivalences is no longer true in the non-projective setting.

One of the main topics treated in Section 4.1 is the definition of Chern classes for coherent sheaves on compact complex varieties. These classes will be used in the forthcoming sections.

An extensive introduction to derived categories and derived functors with particular attention to the non-algebraic case is included for the conveninece of the reader. In Section 4.5 we study compact complex manifolds derived equivalent to a complex K3 surface.

4.1. **Coherent sheaves, Chern classes and locally free resolutions.**

In the algebraic setting the abelian category $\mathbf{Coh}(X)$ of a smooth projective variety is a very strong invariant. Indeed the following classical result of Gabriel shows that it actually encodes all the geometric information about the variety.

Theorem 4.1. ([30]) *Let X and Y be smooth projective varieties such that $\mathbf{Coh}(X) \cong \mathbf{Coh}(Y)$. Then there exists an isomorphism $X \cong Y$.*

This is not the case for generic analytic K3 surfaces. Indeed a very deep result of Verbitsky shows that coherent sheaves are not very interesting for varieties of this type. The original proof in [99] is quite involved and we plan to simplify it in a future paper.

Theorem 4.2. ([99], **Theorem 1.1.**) *Let X and Y be K3 surfaces such that $\rho(X) = \rho(Y) = 0$. Then $\mathbf{Coh}(X) \cong \mathbf{Coh}(Y)$.*

The same result was proved in [100] for generic (non-projective) complex tori following very closely the approach in [99].

Remark 4.3. Since $\mathbf{Coh}(X)$ does not encode univocally the geometry of a compact complex manifold X, one should work with larger categories of sheaves. Unfortunately quasi-coherent sheaves are not the best choice, since many different definitions of quasi-coherent sheaves are present in the literature and none of them seems to be exhaustive. Moreover the relations between these definitions are not completely explored at the moment (see the discussion at the end of Section 5 of [6] for more details). In Section 4.3 we will work with the abelian category of $\mathcal{O}_X$-modules.

Coherent sheaves on non-algebraic K3 surfaces share very useful properties with coherent sheaves on algebraic K3 surfaces. More precisely, due to [93] any coherent sheaf on a projective algebraic variety admits a finite locally free resolution. In [90] Schuster proved that this holds true for any compact complex surface. Notice that this does not hold true for any analytic variety as it was proved in [102].

Theorem 4.4. ([90], Theorem 2.) *Any coherent sheaf $\mathcal{E} \in \mathbf{Coh}(X)$ on a K3 surface X with $\rho(X) = 0$ admits a locally free resolution of finite length.*

As an easy consequence of Schuster's theorem, it makes perfect sense to define the Chern character of any coherent sheaf $\mathcal{E} \in \mathbf{Coh}(X)$ for a generic K3 surface X. Indeed, one consider a locally free resolution of finite length of $\mathcal{E}$ and takes the (well-defined) alternating sum of the Chern characters of the locally free sheaves appearing in the resolution. Unfortunately this is not sufficient for the purposes of the forthcoming sections since we will also need to calculate the Chern character of coherent sheaves (or complexes) on the product of two K3 surfaces, where Theorem 4.4 is not known to hold true. For this we need to define Chern classes in a different way.

Let X be, more generally, a compact complex manifold. Chern classes for coherent sheaves on X with values in the Hodge cohomology of X (i.e. $\oplus_p H^p(X, \Omega_X^p)$) were defined by O'Brian, Toledo, and Tong in [76], substituting locally free resolutions by what they called *twisted cochain complexes*. They also proved Hirzebruch-Riemann-Roch and Grothendieck-Riemann-Roch formulas for theses classes ([77, 78, 79]). Notice that this definition is better suited for derived category theory, due to the close relationship with Hochschild homology (see e.g. [22, 23], as pointed out in [8]).

In our work we will morally follow a different definition by Green contained in his unpublished Ph.D.-thesis [37] (for a review of it, see [98]). In Green's approach these classes are defined via *complexes of simplicial vector bundles* and they take values in the de Rham cohomology of the manifold. Moreover, due to [98, Theorem 2], they precisely coincide with Chern classes defined by Atiyah-Hirzebruch by considering locally free resolutions in the real-analytic setting.

Althoght the reader is very much encouraged to consult the original works for precise definitions and results, we try to give at least few details. In particular, let $\mathcal{E}$ be a coherent sheaf on X and consider the sheaf $\mathcal{O}_X^{\mathrm{an}}$ of germs of complex-valued real-analytic functions on X. Denote by $\mathcal{E}^{\mathrm{an}}$ the $\mathcal{O}_X^{\mathrm{an}}$-module $\mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{O}_X^{\mathrm{an}}$. Then $\mathcal{E}^{\mathrm{an}}$ is a coherent $\mathcal{O}_X^{\mathrm{an}}$-module.

Proposition 4.5. ([3], Proposition 2.6.) *Let $\mathcal{S}$ be a coherent sheaf of $\mathcal{O}_X^{\mathrm{an}}$-modules. Then there exists a resolution of $\mathcal{S}$ over $\mathcal{O}_X^{\mathrm{an}}$*

$$0 \to L_k \to \ldots \to L_0 \to \mathcal{S} \to 0,$$

where L_i is a locally free $\mathcal{O}_X^{\mathrm{an}}$-module, for all i.

According to [3], the previous proposition allows to define Chern classes of coherent sheaves on X with values in $H^*(X, \mathbb{Q})$. All usual properties of Chern classes hold with this definition. For example: commutativity with flat pull-backs, additivity under exact sequences of sheaves (which in particular implies that Chern classes are actually well-defined at the level of Grothendieck group $K(X) := K(\mathbf{Coh}(X))$ of the abelian category $\mathbf{Coh}(X)$) and Whitney's formula. Moreover, Riemann-Roch formulas (Theorem 4.20) for Green's Chern classes and a comparison between the various definitions are contained in [37] and [98]. Clearly in the case of either compact complex surfaces or projective complex manifolds all these definitions coincide with the classical one via locally free resolutions in the complex analytic category.

Definition 4.6. For $e \in K(X)$, the *Mukai vector of* e is the vector $v(e) := \mathrm{ch}(e) \cdot \sqrt{\mathrm{td}(X)} \in H^*(X, \mathbb{Q})$.

Just to consider the easiest example, suppose E to be a locally free sheaf on a K3 surface X with rank $r := \mathrm{rk}\, E$, first Chern class $\mathrm{c}_1 := \mathrm{c}_1(E)$ and second Chern class $\mathrm{c}_2 := \mathrm{c}_2(E)$. If e is the class of E in $K(X)$, explicit calculations yields

$$v(e) = (r, \mathrm{c}_1, \mathrm{c}_1^2/2 - \mathrm{c}_2 + r).$$

From this we deduce that if X is a generic analytic K3 surface (i.e. $\mathrm{NS}(X) = 0$), the Mukai vector of any $e \in K(X)$ is $v(e) = (r, 0, s)$. For the rest of this paper, given a smooth compact complex variety X and $\mathcal{E} \in \mathbf{Coh}(X)$, we denote by $[\mathcal{E}]$ the class of $\mathcal{E}$ in the Grothendieck group of X.

4.2. **Derived categories and functors.**

For the convenience of the reader we briefly recall the algebraic construction of the derived category of an abelian category. Since we tend to be very brief in this section, the reader interested in more details and in all the proofs is encouraged to consult [36] and [49]. On the other hand the expert reader can easily skip this rather formal section.

Let $\mathbf{A}$ be an abelian category. Our main examples will be the abelian category $\mathcal{O}_X$-$\mathbf{Mod}$ of $\mathcal{O}_X$-modules and its full abelian subcategory $\mathbf{Coh}(X)$ of coherent sheaves, for a complex manifold X. A complex $M^\bullet$ over $\mathbf{A}$ is a pair $(\{M^i\}_{i\in\mathbb{Z}}, \{d^i\}_{i\in\mathbb{Z}})$ such that $M^i \in \mathbf{A}$ and $d^i : M^i \to M^{i+1}$ is a morphism such that $d^j \circ d^{j-1} = 0$, for any $j \in \mathbb{Z}$. For simplicity, a complex will be sometimes denoted by $M^\bullet$ or by the pair $(M^\bullet, d^\bullet)$. It can be also useful to think of it as a diagram

$$\cdots \to M^{p-1} \xrightarrow{d^{p-1}} M^p \xrightarrow{d^p} M^{p+1} \to \cdots,$$

where $p \in \mathbb{Z}$. A morphism of complexes is a sets of vertical arrows $\{f^i\}_{i\in\mathbb{Z}}$ as in the following diagram

$$\begin{tikzcd}
\cdots \arrow[r, "d^{i-2}_{M^\bullet}"] & M^{i-1} \arrow[r, "d^{i-1}_{M^\bullet}"] \arrow[d, "f^{i-1}"] & M^i \arrow[r, "d^i_{M^\bullet}"] \arrow[d, "f^i"] & M^{i+1} \arrow[r, "d^{i+1}_{M^\bullet}"] \arrow[d, "f^{i+1}"] & \cdots \\
\cdots \arrow[r, "d^{i-2}_{L^\bullet}"] & L^{i-1} \arrow[r, "d^{i-1}_{L^\bullet}"] & L^i \arrow[r, "d^i_{L^\bullet}"] & L^{i+1} \arrow[r, "d^{i+1}_{L^\bullet}"] & \cdots
\end{tikzcd}$$

and such that, for any $i \in \mathbb{Z}$, $f^i \circ d^{i-1}_{M^\bullet} = d^{i-1}_{L^\bullet} \circ f^{i-1}$. In this way we obtain the abelian category of complexes $\mathrm{C}(\mathbf{A})$.

A morphism of complexes is a *quasi-isomorphism* if, for any $p \in \mathbb{Z}$, it induces isomorphisms between all the cohomology objects

$$H^p(M^\bullet) := \ker d^p_{M^\bullet} / \operatorname{im} d^{p-1}_{M^\bullet}$$

of the complex $M^\bullet$. We define the derived category $\mathrm{D}(\mathbf{A})$ as the localization (see [31]) of the category of complexes with respect to the class of quasi-isomorphisms. So we get a category with the same objects as $\mathrm{C}(\mathbf{A})$ and in which we formally invert all quasi-isomorphism.

This quick definition is not very explicit. To get a description of the morphisms in $\mathrm{D}(\mathbf{A})$ we should first consider another category $\mathrm{Kom}(\mathbf{A})$ whose construction goes as follows. First observe that a morphism of complexes $f : L^\bullet \to M^\bullet$ is *null-homotopic* if there are morphisms $r^p : L^p \to M^{p-1}$ such that

$$f^p = d^{p-1}_{L^\bullet} \circ r^p + r^{p+1} \circ d^p_{M^\bullet},$$

for all $p \in \mathbb{Z}$. The objects of $\mathrm{Kom}(\mathbf{A})$ are the complexes in $\mathrm{C}(\mathbf{A})$ while the morphisms are obtained from the ones in $\mathrm{C}(\mathbf{A})$ by taking the quotient modulo null-homotopic morphisms. In $\mathrm{Kom}(\mathbf{A})$ the quasi-isomorphisms form a *localizing class* (in the sense of Verdier) and then it is possible to have an explicit description of the morphisms in $\mathrm{D}(\mathbf{A})$ as "fractions", essentially in the same way as for the product in a localized ring.

The basic properties of derived categories are summarized in the following proposition.

Proposition 4.7. (i) *The category* $\mathrm{D}(\mathbf{A})$ *is additive.*

(ii) *The canonical functor* $Q : \mathrm{C}(\mathbf{A}) \to \mathrm{D}(\mathbf{A})$ *is additive and makes all quasi-isomorphisms invertible and is universal among all functors* $\mathrm{C}(\mathbf{A}) \to \mathbf{C}$ *having this property, where* $\mathbf{C}$ *is any category.*

(iii) *The canonical functor* $\mathbf{A} \to \mathrm{D}(\mathbf{A})$*, which maps an object* $A \in \mathbf{A}$ *to the complex with all zero besides* A *in degree* 0*, is fully faithful.*

For the geometric applications we have in mind, we will be interested in some full subcategories of $\mathrm{D}(\mathbf{A})$.

Example 4.8. (i) We denote by $\mathrm{D}^+(\mathbf{A})$ (respectively $\mathrm{D}^-(\mathbf{A})$ and $\mathrm{D}^{\mathrm{b}}(\mathbf{A})$), the full additive subcategory of complexes $M^\bullet$ such that $M^n = 0$, for all $n \gg 0$ (respectively $n \ll 0$ and $|n| \gg 0$). Such a category is equivalent to the category of complexes where $H^n(M^\bullet) = 0$, for all $n \gg 0$ (respectively $n \ll 0$ and $|n| \gg 0$).

(ii) $\mathrm{D}^*_{\mathbf{A}'}(\mathbf{A})$ is the full additive subcategory of complexes $M^\bullet$ such that $H^n(M^\bullet) \in \mathbf{A}'$, for all $n \in \mathbb{Z}$, where $\mathbf{A}'$ is a full *thick* subcategory of $\mathbf{A}$ (i.e. any extension in $\mathbf{A}$ of two objects of $\mathbf{A}'$ is in $\mathbf{A}'$) and $* = \emptyset, +, -, \mathrm{b}$. For example $\mathbf{Coh}(X) \subseteq \mathcal{O}_X\text{-}\mathbf{Mod}$ is a thick subcategory.

Denote by $[1] : \mathrm{D}(\mathbf{A}) \to \mathrm{D}(\mathbf{A})$ the *shift functor* which maps a complex $(M^\bullet, d^\bullet_{M^\bullet})$ to the complex $(M^\bullet[1], d_{M^\bullet[1]})$, defined by $(M^\bullet[1])^i = M^{i+1}$ and $d_{M^\bullet[1]} = -d_{M^\bullet}$. Notice that a morphism of complexes is mapped to the same morphisms with degrees shifted by 1. A *standard triangle* in $\mathrm{D}(\mathbf{A})$ is a sequence

$$X^\bullet \xrightarrow{Q(p)} Y^\bullet \xrightarrow{Q(q)} Z^\bullet \xrightarrow{\partial\varepsilon} X[1]^\bullet,$$

where $Q : \mathrm{C}(\mathbf{A}) \to \mathrm{D}(\mathbf{A})$ is the canonical functor,

$$\varepsilon \ : \ 0 \to X^\bullet \xrightarrow{p} Y^\bullet \xrightarrow{q} Z^\bullet \to 0$$

is a short exact sequence of complexes in $\mathrm{C}(\mathbf{A})$, and $\partial\varepsilon$ is a special morphism in $\mathrm{D}(\mathbf{A})$ functorial in ε and which lifts the connecting morphism $H^i(Z^\bullet) \to H^{i+1}(X^\bullet)$ in the long exact cohomology sequence associated to ε. A *triangle* in $\mathrm{D}(\mathbf{A})$ is a sequence

$$X^\bullet \to Y^\bullet \to Z^\bullet \to X^\bullet[1]$$

isomorphic to a standard triangle.

More generally, a *triangulated category* is an additive category $\mathbf{T}$ endowed with an autoequivalence $X \mapsto X[1]$ and a class of sequences (called *triangles*) of the form

$$X \to Y \to Z \to X[1]$$

briefly denoted by (X, Y, Z) which are stable under isomorphisms and satisfy a list of axioms (see, for example, [73, Ch. 1] for a nice presentation). If $\mathbf{T}$ is a triangulated category, a *full triangulated subcategory* of $\mathbf{T}$ is a full subcategory $\mathbf{U} \subseteq \mathbf{T}$ such that $\mathbf{U}[1] = \mathbf{U}$ and, whenever we have a triangle (X, Y, Z) of $\mathbf{T}$ such that X and Z belong to $\mathbf{U}$, then there is $Y' \in \mathbf{U}$ isomorphic to Y. A triangulated subcategory is again triangulated in the natural way. For example, all the subcategories of $\mathrm{D}(\mathbf{A})$ considered in Example 4.8 are full triangulated subcategories.

To a triangulated category $\mathbf{T}$, it is naturally associated an abelian group called *Grothendieck group* and denoted by $K(\mathbf{T})$. It is defined as the quotient of the free abelian group generated by the isomorphism classes $[X]$ of objects of $\mathbf{T}$ divided by the subgroup generated by the relations

$$[X] - [Y] + [Z],$$

where (X, Y, Z) runs through the triangles of $\mathbf{T}$.

We have just seen how to pass from an abelian category to a triangulated one. Later we will need to invert this process. This is strictly related to the notion of t-structure. Indeed, a *t-structure* (see [5]) on a triangulated category $\mathbf{T}$ is a strictly (i.e. closed under isomorphism) full additive subcategory $\mathbf{F} \subseteq \mathbf{T}$ such that $\mathbf{F}[1] \subseteq \mathbf{F}$ and, if

$$\mathbf{F}^{\perp} := \{G \in \mathbf{T} : \mathrm{Hom}_{\mathbf{T}}(F, G) = 0, \text{ for all } F \in \mathbf{F}\},$$

then for any non-zero $E \in \mathbf{T}$, there exists a triangle

$$F \to E \to G \to F[1]$$

with $F \in \mathbf{F}$ and $G \in \mathbf{F}^{\perp}$. The *heart* $\mathbf{A} := \mathbf{F} \cap \mathbf{F}^{\perp}[1]$ of the t-structure is a full abelian subcategory of $\mathbf{T}$, with the short exact sequences being precisely the triangles in $\mathbf{T}$ whose vertices are objects of $\mathbf{A}$. A t-structure is called *bounded* if

$$\mathbf{T} = \bigcup_{i,j} (\mathbf{F}[i] \cap \mathbf{F}^{\perp}[j]).$$

Note that for a bounded t-structure one has $K(\mathbf{T}) \cong K(\mathbf{A})$.

Let us now briefly recall how one can pretend to produce functors between derived categories starting from functors between the corresponding abelian categories. We will follows Deligne's approach as outlined in [57]. Let $\mathbf{A}$ and $\mathbf{B}$ be abelian categories and let $F : \mathbf{A} \to \mathbf{B}$ be an additive functor. Take $M^{\bullet} \in \mathrm{D}(\mathbf{A})$ and define a functor

$$(\mathbf{r}F)(-, M^{\bullet}) : (\mathrm{D}(\mathbf{B}))^{\mathrm{op}} \to \mathbf{Mod}\text{-}\mathbb{Z}$$

in the following way: for any $L^{\bullet} \in \mathrm{D}(\mathbf{A})$.

$$(\mathbf{r}F)(L^{\bullet}, M^{\bullet}) = \left\{ (f, s) : \begin{array}{ll} (1) & f \in \mathrm{Hom}_{\mathrm{D}(\mathbf{B})}(L^{\bullet}, F(N^{\bullet})) \\ (2) & s \in \mathrm{Hom}_{\mathrm{Kom}(\mathbf{A})}(M^{\bullet}, N^{\bullet}) \text{ is a quasi-isomorphism} \end{array} \right\} / \sim,$$

where $(f, s) \sim (g, t)$ if and only if there exists (h, u) making all obvious diagrams commutative. If the functor $(\mathbf{r}F)(-, M^{\bullet})$ is representable, then we denote by $\mathbf{R}F(M^{\bullet})$ the object representing it and we say that $\mathbf{R}F$ is defined at $M^{\bullet} \in \mathrm{D}(\mathbf{A})$. We call the functor $\mathbf{R}F$ constructed in this way (and defined in general only on a subcategory of $\mathrm{D}(\mathbf{A})$) the *right derived functor* of F. Of course, the above construction can be applied without changes to arbitrary functors $F : \mathrm{Kom}(\mathbf{A}) \to \mathrm{Kom}(\mathbf{B})$. Moreover, one can dually define *left derived functors* $\mathbf{L}F : \mathrm{D}(\mathbf{A}) \to \mathrm{D}(\mathbf{B})$ by inverting the direction of the arrows in the above definition. The link between the previous definition and the classical one is illustrated by the following result.

Proposition 4.9. *Suppose that $\mathbf{A}$ has enough injectives and $M^{\bullet}$ is left bounded. Then $\mathbf{R}F$ is defined at $M^{\bullet}$ and we have*

$$\mathbf{R}F(M^{\bullet}) = F(I^{\bullet}),$$

where $M^{\bullet} \to I^{\bullet}$ is a quasi-isomorphism with a left bounded complex with injective components.

Example 4.10. If $M^{\bullet} = M \in \mathbf{A}$ is concentrated in degree zero, then $I^{\bullet}$ may be chosen to be an injective resolution of M and we find that

$$H^n(\mathbf{R}F(M)) = (\mathbf{R}^n F)(M),$$

the n-th right derived functor of F in the classical sense of Cartan-Eilenberg (see [42]). For example, if $F = \mathrm{Hom}(N, -)$, with $N \in \mathbf{A}$, then, for $M \in \mathbf{A}$, we have the chain of natural isomorphisms

$$\mathrm{Ext}^n_{\mathbf{A}}(N, M) \cong H^n(\mathbf{R}\mathrm{Hom}(N, M)) \cong \mathrm{Hom}_{\mathrm{D}(\mathbf{A})}(N, M[n]), \tag{4.1}$$

for all $n \in \mathbb{Z}$.

An additive functor $F : \mathbf{T} \to \mathbf{T}'$ between two triangulated categories is called *exact* if there exists a functorial isomorphism

$$\phi_X : F(X[1]) \xrightarrow{\sim} (F(X))[1]$$

such that the sequence

$$F(X) \xrightarrow{F(u)} F(Y) \xrightarrow{F(v)} F(Z) \xrightarrow{(\phi_X)F(w)} (F(X))[1]$$

is a triangle of $\mathbf{T}'$, for any triangle of $\mathbf{T}$.

For the rest of this paper, we will write M and f instead of $M^\bullet$ and $f^\bullet$ to denote complexes and morphisms of complexes. Moreover all functors will be supposed to be exact.

4.3. **Derived categories of complex manifolds.**

In this section we study the derived and triangulated categories associated to the abelian categories of coherent sheaves on complex manifolds. In particular, X will always be a compact complex manifold.

Definition 4.11. The *(bounded) derived category* of X, denoted by $\mathrm{D^b}(X)$ is defined as the triangulated subcategory $\mathrm{D}^{\mathrm{b}}_{\mathbf{Coh}}(\mathcal{O}_X\text{-}\mathbf{Mod})$ of $\mathrm{D}(\mathcal{O}_X\text{-}\mathbf{Mod})$ whose objects are bounded complexes of $\mathcal{O}_X$-modules whose cohomologies are in $\mathbf{Coh}(X)$.

Example 4.12. For any closed point $x \in X$, the skyscraper sheaf $\mathcal{O}_x$ is in $\mathrm{D^b}(X)$. Moreover,

$$\mathrm{Hom}(\mathcal{O}_x, \mathcal{O}_x[i]) \cong \begin{cases} \mathbb{C} & \text{if } i \in \{0, 2\} \\ \mathbb{C} \oplus \mathbb{C} & \text{if } i = 1 \\ 0 & \text{otherwise} \end{cases} \tag{4.2}$$

Any object $\mathcal{E} \in \mathrm{D^b}(X)$ such that $\mathrm{Hom}(\mathcal{E}, \mathcal{E}[i])$ is as in (4.2) is called *semi-rigid*.

Now recall that given an abelian category $\mathbf{A}$, $A \in \mathrm{Ob}(\mathbf{A})$ is *minimal* if any non-trivial surjective morphism $A \to B$ in $\mathbf{A}$ is an isomorphism. It is an easy exercise to prove that the minimal objects of $\mathbf{Coh}(X)$ are the skyscraper sheaves.

Example 4.13. In [92] a special class of objects was introduced. An object $\mathcal{E}^\bullet \in \mathrm{D^b}(X)$ is a *spherical object* if $\mathcal{E}^\bullet \otimes \omega_X \cong \mathcal{E}^\bullet$, where ω_X is the dualizing sheaf of X and

$$\mathrm{Hom}(\mathcal{E}^\bullet, \mathcal{E}^\bullet[i]) \cong \begin{cases} \mathbb{C} & \text{if } i \in \{0, \dim X\} \\ 0 & \text{otherwise.} \end{cases}$$

In particular if X is a K3 surface, any line bundle (and hence also $\mathcal{O}_X$) is a spherical object of $\mathrm{D^b}(X)$.

The definition of $\mathrm{D^b}(X)$ may seem not appropriate. A better candidate to work with would be the derived category $\mathrm{D^b}(\mathbf{Coh}(X))$. As we will see in the next section, unfortunately this category is not the natural place to derive functors and to verify Serre duality. Clearly there is a natural exact functor

$$F : \mathrm{D^b}(\mathbf{Coh}(X)) \to \mathrm{D^b}(X). \tag{4.3}$$

In particular $\mathbf{Coh}(X)$ is the heart of a bounded t-structure on $\mathrm{D^b}(X)$ and there is a natural identification $K(X) \cong K(\mathrm{D^b}(X))$ (note that, via this identification, Chern classes and the Chern character are well-defined for objects in $\mathrm{D^b}(X)$). The best situation is when this functor is actually an equivalence. In the algebraic case this is a well-known result (see, for example, Corollary 2.2.2.1 of Exposé II in [55]).

Proposition 4.14. *Let X be a smooth projective complex manifold. Then the functor F in (4.3) is an equivalence.*

In the pure analytic case the situation is more complicated due to the absence of a good notion of quasi-coherent sheaves (see Remark 4.3). But in the surface case the result is still true due to Serre duality (see Section 4.5).

Proposition 4.15. **([10], Proposition 5.2.1.)** *Let X be a smooth compact complex surface. Then the functor F in* (4.3) *is an equivalence.*

Proof. We denote by Ext_1^n and Ext_2^n respectively the Ext-groups $\mathrm{Ext}^n_{\mathbf{Coh}(X)}$ and $\mathrm{Ext}^n_{\mathcal{O}_X\text{-}\mathbf{Mod}}$ in the categories $\mathbf{Coh}(X)$ and $\mathcal{O}_X$-**Mod**, that is, by (4.1), the groups $\mathrm{Hom}(-,-[n])$ in the corresponding derived categories.

The result follows using [55, Lemma 2.1.3] once we show that for any $\mathcal{E},\mathcal{F}\in\mathbf{Coh}(X)$, the natural morphism $f_n:\mathrm{Ext}_1^n(\mathcal{E},\mathcal{F})\to\mathrm{Ext}_2^n(F(\mathcal{E}),F(\mathcal{F}))$ is an isomorphism for any n. For $n=0$ this is true by the very definition of the functor Hom in both categories. For $n=1$ this property is trivially verified since the extension of coherent sheaves is still coherent.

To show that f_n is invertible for $n\neq 1$, we apply once more [55, Lemma 2.1.3] which asserts that this happens if for any $\mathcal{E},\mathcal{F}\in\mathcal{O}_X$-**Mod** and any $f\in\mathrm{Ext}_2^n(\mathcal{E},\mathcal{F})$, there exists a surjective morphism $\mathcal{E}'\twoheadrightarrow\mathcal{E}$ such that f is sent to zero by the natural morphism $\mathrm{Ext}_2^n(\mathcal{E},\mathcal{F})\to\mathrm{Ext}_2^n(\mathcal{E}',\mathcal{F})$.

If $n>2$, Serre duality discussed in Section 4.5 for analytic varieties implies that $\mathrm{Ext}_2^n(\mathcal{E},\mathcal{F})=\{0\}$ and so there is nothing to prove. Hence we just need to consider the case $n=2$.

Let us first show that for any $\mathcal{E},\mathcal{F}\in\mathbf{Coh}(X)$ and any $x\in X$, there is n such that $\mathrm{Ext}_2^2(m_x^n\mathcal{E},\mathcal{F})=\{0\}$, where m_x is the maximal ideal of $\mathcal{O}_X$ in x. Due to Serre duality, we can just show that for $n\gg 0$ one has $\mathrm{Hom}(\mathcal{G},m_x^n\mathcal{E})=\{0\}$, where $\mathcal{G}=\mathcal{F}\otimes\omega_X$ and ω_X is the dualizing sheaf of X.

Since $\mathrm{Hom}(\mathcal{G},m_x^n\mathcal{E})$ is finite dimensional over $\mathbb{C}$, it is sufficient to to show that for $a\in\mathbb{N}$ there is $b>a$ such that $\mathrm{Hom}(\mathcal{G},m_x^a\mathcal{E})\neq\mathrm{Hom}(\mathcal{G},m_x^b\mathcal{E})$. If we pick $f:\mathcal{G}\to m_x^a\mathcal{E}$ then it is an easy check using stalks that there is b such that $\mathrm{im}(f_x)\not\subset m_x^b\mathcal{E}_x$. Hence $f\notin\mathrm{Hom}(\mathcal{G},m_x^b\mathcal{E})$.

Now we take $x,y\in X$ such that $x\neq y$ and we choose $n\in\mathbb{N}$ such that $\mathrm{Ext}_2^2(m_x^n\mathcal{E},\mathcal{F})=\mathrm{Ext}_2^2(m_y^n\mathcal{E},\mathcal{F})=\{0\}$. The canonical map $m_x^n\mathcal{E}\oplus m_y^n\mathcal{E}\to\mathcal{F}$ is clearly surjective. This concludes the proof. $\square$

Going back to the case of generic analytic K3 surfaces X, we can now state the following crucial property of $\mathrm{D^b}(X)$ which completely classifies its spherical objects. This will be important when we will study stability conditions on the derived category of these K3 surfaces.

Lemma 4.16. **([53], Lemma 3.1.)** *Let X be a generic analytic K3 surface. The trivial line bundle $\mathcal{O}_X$ is up to shift the only spherical object in* $\mathrm{D^b}(X)$.

4.4. Derived functors on complex manifolds.

Let X and Y be compact complex manifolds. Consider their corresponding derived categories. As we pointed out in Section 4.2, not all functors can be derived. Nevertheless, the following result shows that all functors with a geometric meaning are subject to this procedure.

Proposition 4.17. *Let $f,g:X\to Y$ be morphisms and assume f to be proper. Then the following functors are defined:*

$$\begin{gathered}
\mathbf{R}\mathcal{H}om^\bullet:\mathrm{D^b}(X)^{\mathrm{op}}\times\mathrm{D^b}(X)\longrightarrow\mathrm{D^b}(X),\\
-\overset{\mathbf{L}}{\otimes}-:\mathrm{D^b}(X)\times\mathrm{D^b}(X)\longrightarrow\mathrm{D^b}(X),\\
\mathbf{L}g^*:\mathrm{D^b}(Y)\longrightarrow\mathrm{D^b}(X),\\
\mathbf{R}f_*:\mathrm{D^b}(X)\longrightarrow\mathrm{D^b}(Y),\\
\mathbf{R}\mathrm{Hom}^\bullet:\mathrm{D^b}(X)^{\mathrm{op}}\times\mathrm{D^b}(X)\longrightarrow\mathrm{D^b}(\mathbf{vect}),
\end{gathered}$$

where we denote by **vect** *the abelian category of finite dimensional $\mathbb{C}$-vector spaces.*

Proof. According to [95], the functors listed in the proposition are all well-defined if we consider the corresponding unbounded derived categories of modules $\mathrm{D}(\mathcal{O}_X\text{-}\mathbf{Mod})$, $\mathrm{D}(\mathcal{O}_Y\text{-}\mathbf{Mod})$ and $\mathrm{D}(\mathbf{Vect})$ (where we denote by $\mathbf{Vect}$ the abelian category of $\mathbb{C}$-vector spaces). To prove the proposition we only have to show that, by restricting the domain to the corresponding derived categories of bounded complexes with coherent cohomology, the images of these functors are still the bounded derived categories of complexes with coherent cohomology. But these are well-known results (see, for example, [38]). □

To give to the reader a first impression of the relationships between the derived functors just defined, we list here a few of their properties whose proofs are, for example, in [95]. As before, X, Y and Z are compact complex manifolds while $f : X \to Y$ and $g : Y \to Z$ are morphisms. Then there exist the following natural isomorphisms of functors:

- Assuming f and g proper, $\mathbf{R}(g_* \circ f_*) \cong \mathbf{R}g_* \circ \mathbf{R}f_*$;
- $\mathbf{L}(f^* \circ g^*) \cong \mathbf{L}f^* \circ \mathbf{L}g^*$.
- $\mathbf{R}\mathrm{Hom}^\bullet(\mathcal{F},\mathcal{G}) \cong \mathbf{R}\Gamma(X, \mathbf{R}\mathcal{H}om(\mathcal{F},\mathcal{G}))$ as bi-functors.
- $\mathbf{R}f_*\mathcal{H}om^\bullet(\mathcal{F},\mathcal{G}) \cong \mathbf{R}\mathcal{H}om(\mathbf{R}f_*(\mathcal{F}), \mathbf{R}f_*(\mathcal{G}))$.
- Projection Formula: assuming f proper, $\mathbf{R}f_*(\mathcal{F}) \overset{\mathbf{L}}{\otimes} \mathcal{G} \cong \mathbf{R}f_*(\mathcal{F} \overset{\mathbf{L}}{\otimes} \mathbf{L}f^*(\mathcal{G}))$.
- Assuming f proper, $\mathrm{Hom}^\bullet(\mathbf{L}f^*(\mathcal{F}),\mathcal{G}) \cong \mathrm{Hom}(\mathcal{F}, \mathbf{R}f_*(\mathcal{G}))$. In other words, $\mathbf{L}f^*$ is left adjoint to $\mathbf{R}f_*$.
- $\mathbf{L}f^*(\mathcal{F} \overset{\mathbf{L}}{\otimes} \mathcal{G}) \cong \mathbf{L}f^*(\mathcal{F}) \overset{\mathbf{L}}{\otimes} \mathbf{L}f^*(\mathcal{G})$.
- $\mathcal{F} \overset{\mathbf{L}}{\otimes} \mathcal{G} \cong \mathcal{G} \overset{\mathbf{L}}{\otimes} \mathcal{F}$ and $\mathcal{F} \overset{\mathbf{L}}{\otimes} (\mathcal{G} \overset{\mathbf{L}}{\otimes} \mathcal{H}) \cong (\mathcal{F} \overset{\mathbf{L}}{\otimes} \mathcal{G}) \overset{\mathbf{L}}{\otimes} \mathcal{H}$.
- $\mathbf{R}\mathcal{H}om^\bullet(\mathcal{F},\mathcal{G}) \overset{\mathbf{L}}{\otimes} \mathcal{H} \cong \mathbf{R}\mathcal{H}om^\bullet(\mathcal{F},\mathcal{G} \overset{\mathbf{L}}{\otimes} \mathcal{H})$.
- $\mathbf{R}\mathcal{H}om^\bullet(\mathcal{F}, \mathbf{R}\mathcal{H}om^\bullet(\mathcal{G},\mathcal{H})) \cong \mathbf{R}\mathcal{H}om^\bullet(\mathcal{F} \overset{\mathbf{L}}{\otimes} \mathcal{G},\mathcal{H})$.

Moreover, if $u : Y' \to Y$ is a flat morphism fitting in the following commutative diagram

$$\begin{tikzcd} X \times_Y Y' \arrow[r, "v"] \arrow[d, "g"'] & X \arrow[d, "f"] \\ Y' \arrow[r, "u"] & Y, \end{tikzcd}$$

then there exists a functorial isomorphism

$$u^*\mathbf{R}f_*(\mathcal{F}) \cong \mathbf{R}g_*v^*(\mathcal{F}).$$

Such a property is usually called Flat Base Change Theorem.

Definition 4.18. (i) Let X and Y be compact complex manifolds. A functor $F : \mathrm{D^b}(X) \to \mathrm{D^b}(Y)$ is of *Fourier-Mukai type* (or a *Fourier-Mukai functor*) if there exists $\mathcal{E} \in \mathrm{D^b}(X \times Y)$ and an isomorphism of functors $F \cong \Phi_\mathcal{E}$, where, denoting by $p : X \times Y \to Y$ and $q : X \times Y \to X$ the natural projections, $\Phi_\mathcal{E} : \mathrm{D^b}(X) \to \mathrm{D^b}(Y)$ is the exact functor defined by

$$\Phi_\mathcal{E} := \mathbf{R}p_*(\mathcal{E} \overset{\mathbf{L}}{\otimes} q^*(-)). \tag{4.4}$$

The complex $\mathcal{E}$ is a *kernel* of F.

(ii) Two compact complex manifolds X and Y are *Fourier-Mukai partners* if there exists a Fourier-Mukai transform $\Phi : \mathrm{D^b}(X) \to \mathrm{D^b}(Y)$.

It is a nice exercise to prove that the equivalences in the following example are of Fourier-Mukai type.

Example 4.19. Here X and Y are always meant to be compact complex smooth surfaces.

(i) Let $f : X \xrightarrow{\sim} Y$ be an isomorphism. Then $f^* : \mathrm{D^b}(Y) \xrightarrow{\sim} \mathrm{D^b}(X)$ is an equivalence.

(ii) Let $\mathcal{E}$ be a spherical object (see Example 4.13). Consider the *spherical twist* $T_{\mathcal{E}} : \mathrm{D^b}(X) \to \mathrm{D^b}(X)$ that sends $\mathcal{F} \in \mathrm{D^b}(X)$ to the cone of

$$\mathrm{Hom}(\mathcal{E}, \mathcal{F}) \otimes \mathcal{E} \to \mathcal{F}.$$

The kernel of $T_{\mathcal{E}}$ is given by the cone of the natural map $\mathcal{E}^{\vee} \boxtimes \mathcal{E} \to \mathcal{O}_{\Delta}$, where $\mathcal{O}_{\Delta} := \Delta_* \mathcal{O}_X$ and Δ is the diagonal embedding $X \hookrightarrow X \times X$ (notice that here we need the existence of locally free resolutions as stated in Theorem 4.4).

(iii) For any line bundle $L \in \mathrm{Pic}(X)$ we have the *line bundle twist* $(-) \otimes L : \mathrm{D^b}(X) \xrightarrow{\sim} \mathrm{D^b}(X)$.

Theorem 4.20. ([79, 98]) *Let X and Y be compact complex manifolds and let $f : X \to Y$ be a holomorphic map. Then, for all $\mathcal{E} \in \mathrm{D^b}(X)$, Grothendieck-Riemann-Roch formula holds in $H^*(Y, \mathbb{Q})$:*

$$\mathrm{ch}(\mathbf{R}f_* \mathcal{E}) \cdot \mathrm{td}(Y) = f_*(\mathrm{ch}(\mathcal{E}) \cdot \mathrm{td}(X)).$$

In particular, if Y is a point, for all $\mathcal{E} \in \mathrm{D^b}(X)$, Hirzebruch-Riemann-Roch formula holds:

$$\chi(\mathcal{E}) := \sum_i (-1)^i \dim H^i(X, \mathcal{E}) = (\mathrm{ch}(\mathcal{E}) \cdot \mathrm{td}(X))_{\dim X},$$

where $(-)_{\dim X}$ denotes the $\dim X$-component of a class in $H^*(X, \mathbb{Q})$.

As in the projective case, the Chern class of a coherent sheaf on a K3 surface is integral. The previous theorem can be used to proved that $\mathrm{ch}(\mathcal{E})$ (and then $v([\mathcal{E}])$) is integral for any $\mathcal{E} \in \mathrm{D^b}(X \times X)$, where X is a K3 surface following line by line the original proof of Mukai in [49, Lemma 10.6] of the same result in the projective case. Moreover Hirzebruch-Riemann-Roch formula can be rewritten for K3 surfaces as

$$\chi(\mathcal{E}, \mathcal{F}) := \sum_i (-1)^i \dim \mathrm{Hom}_{\mathrm{D^b}(X)}(\mathcal{E}, \mathcal{F}[i]) = -\langle v([\mathcal{E}]), v([\mathcal{F}]) \rangle, \tag{4.5}$$

for any $\mathcal{E}, \mathcal{F} \in \mathrm{D^b}(X)$ (here $\langle -, - \rangle$ is the Mukai pairing introduced in (2.2)). The pairing $\chi(-,-)$ will be sometimes called *Euler pairing.*

Given two compact complex manifolds X and Y, let $\mathrm{Eq}(\mathrm{D^b}(X), \mathrm{D^b}(Y))$ be the set of equivalences between the derived categories $\mathrm{D^b}(X)$ and $\mathrm{D^b}(Y)$. Moreover we denote by $\mathrm{Eq^{FM}}(\mathrm{D^b}(X), \mathrm{D^b}(Y))$ the subset of $\mathrm{Eq}(\mathrm{D^b}(X), \mathrm{D^b}(Y))$ containing all equivalences of Fourier-Mukai type.

The projective case is very special, as stated in the following deep result due to Orlov (see the main result in [84] and have a look to [24] for a more general statement).

Theorem 4.21. ([84]) *Let X and Y be smooth projective varieties. Then any equivalence $\Phi : \mathrm{D^b}(X) \xrightarrow{\sim} \mathrm{D^b}(Y)$ is of Fourier-Mukai type and the kernel is uniquely determined up to isomorphism.*

Going back to the case of non-algebraic K3 surfaces, the situation immediately becomes more complicated. Indeed, reasoning as in the proof of [49, Cor. 5.23, Cor. 5.24] and using Theorem 4.2, we can prove the following result.

Proposition 4.22. *Let X and Y be non-isomorphic K3 surfaces such that $\rho(X) = \rho(Y) = 0$. Then $\mathrm{Eq}(\mathrm{D^b}(X), \mathrm{D^b}(Y)) \neq \emptyset$ and $\mathrm{Eq^{FM}}(\mathrm{D^b}(X), \mathrm{D^b}(Y)) \neq \mathrm{Eq}(\mathrm{D^b}(X), \mathrm{D^b}(Y))$.*

Proof. By Theorem 4.2, there exists an equivalence $\Psi : \mathbf{Coh}(X) \xrightarrow{\sim} \mathbf{Coh}(Y)$ which naturally extends by Proposition 4.15 to an equivalence $\Upsilon : \mathrm{D^b}(X) \xrightarrow{\sim} \mathrm{D^b}(Y)$. If $\mathrm{Eq^{FM}}(\mathrm{D^b}(X), \mathrm{D^b}(Y)) = \mathrm{Eq}(\mathrm{D^b}(X), \mathrm{D^b}(Y))$, then Υ is of Fourier-Mukai type and there exists $\mathcal{E} \in \mathrm{D^b}(X \times Y)$ and an isomorphism of functors

$$\Upsilon \cong \Phi_{\mathcal{E}} := \mathbf{R}p_*(\mathcal{E} \overset{\mathbf{L}}{\otimes} q^*(-)).$$

Since the minimal objects of $\mathbf{Coh}(X)$ are the skyscraper sheaves $\mathcal{O}_x$, where x is a closed point of X (Example 4.12), Υ maps skyscraper sheaves to skyscraper sheaves. Therefore $\mathcal{E}|_{\{x\} \times Y}$ is isomorphic to a skyscraper sheaf and we naturally get a morphism $f : X \to Y$ and $L \in \mathrm{Pic}(X)$ such that $\Upsilon \cong L \otimes f_*(-)$. The morphism f is actually an isomorphism since Ψ is an equivalence ([49, Cor. 5.22]). In particular X and Y would be isomorphic. $\square$

Actually in the last section we will show that if $X \not\cong Y$ then $\mathrm{Eq}^{\mathrm{FM}}(\mathrm{D^b}(X), \mathrm{D^b}(Y)) = \emptyset$.

As before, if X is a K3 surface such that $\rho(X) = 0$, we denote by $\mathrm{Aut}(\mathrm{D^b}(X))$ the group of autoequivalences of $\mathrm{D^b}(X)$ and by $\mathrm{Aut}^{\mathrm{FM}}(\mathrm{D^b}(X))$ the subgroup of $\mathrm{Aut}(\mathrm{D^b}(X))$ containing all Fourier-Mukai autoequivalences.

In the forthcoming pages, to shorten the notations we will often write a functor and its derived version in the same way (e.g. we write f_* instead of $\mathbf{R}f_*$).

4.5. **Serre functor.**

For a compact complex manifold X, the functor

$$\begin{array}{rcl} S_X : \mathrm{D^b}(X) & \longrightarrow & \mathrm{D^b}(X) \\ \mathcal{E} & \longmapsto & \mathcal{F} \otimes \omega_X[\dim(X)], \end{array} \tag{4.6}$$

where ω_X is the canonical bundle of X, is called *Serre functor*. The main property of this functor is the existence of the isomorphism of $\mathbb{C}$-vector spaces

$$\mathrm{Hom}_{\mathrm{D^b}(X)}(\mathcal{E}, \mathcal{F}) \cong \mathrm{Hom}_{\mathrm{D^b}(X)}(\mathcal{F}, S_X(\mathcal{E}))^\vee$$

which is functorial in $\mathcal{E}$ and $\mathcal{F}$ (see [49, Thm. 3.12] for the projective case and [10, Prop. 5.1.1] for the general case).

Since in $\mathrm{D^b}(X)$ and $\mathrm{D^b}(Y)$, the Hom's are finite dimensional complex vector spaces, any equivalence $F : \mathrm{D^b}(X) \to \mathrm{D^b}(Y)$ (not necessarily of Fourier-Mukai type) commutes with S_X and S_Y, i.e. there exists an isomorphism $S_Y \circ F \cong F \circ S_X$. Indeed, since F is fully faithful, $\mathrm{Hom}(\mathcal{F}, S_X(\mathcal{G})) = \mathrm{Hom}(F(\mathcal{F}), F(S_X(\mathcal{G})))$ and

$$\mathrm{Hom}(\mathcal{G}, \mathcal{F}) = \mathrm{Hom}(F(\mathcal{G}), F(\mathcal{F})),$$

for any $\mathcal{F}, \mathcal{G} \in \mathrm{D^b}(X)$. Moreover, $\mathrm{Hom}(\mathcal{F}, S_X(\mathcal{G})) = \mathrm{Hom}(\mathcal{G}, \mathcal{F})^\vee$ and

$$\mathrm{Hom}(F(\mathcal{G}), F(\mathcal{F})) = \mathrm{Hom}(F(\mathcal{F}), S_Y(F(\mathcal{G})))^\vee.$$

Hence, we get a functorial (in $\mathcal{F}$ and $\mathcal{G}$) isomorphism

$$\mathrm{Hom}(F(\mathcal{F}), S_Y(F(\mathcal{G})) \cong \mathrm{Hom}(F(\mathcal{F}), F(S_X(\mathcal{G}))).$$

Using the fact that F is an equivalence, we get an isomorphism between the representable functors $\mathrm{Hom}(-, S_Y(F(\mathcal{G})))$ and $\mathrm{Hom}(-, F(S_X(\mathcal{G})))$ and hence an isomorphism $S_Y(F(\mathcal{G}))) \cong F(S_X(\mathcal{G}))$ (see [20]).

Lemma 4.23. *Let X be a K3 surface and let Y be a compact complex manifold with an equivalence*

$$F : \mathrm{D^b}(X) \xrightarrow{\sim} \mathrm{D^b}(Y).$$

Then Y is a surface with trivial canonical bundle.

Proof. The *order* of the dualizing sheaf ω_X of a compact complex manifold X is the minimal $k \in \mathbb{Z} \cup \{\pm\infty\}$ such that $\omega_X^{\otimes k} \cong \mathcal{O}_X$. Following the proof of [49, Prop. 4.1] and using the property of the Serre functor previously described, we can show that $\dim X = \dim Y$ and that the orders of ω_X and ω_Y are the same. $\square$

Hence, by Kodaira classification (see [4]), Y is either a K3 surface or a 2-dimensional complex torus or a Kodaira surface.

Proposition 4.24. *If Y is also assumed to be Kähler, then Y is a K3 surface with the same Picard number of X.*

Proof. By Example 4.13 we know that $\mathcal{O}_X$ is a spherical object of $\mathrm{D^b}(X)$. It is enough to prove that a 2-dimensional complex torus T does not contain spherical objects.

Suppose that $\mathbf{Coh}(T)$ contains a spherical sheaf $\mathcal{E}$ of positive rank. Since $\mathcal{E}^\vee \otimes \mathcal{E} = \mathcal{O}_T \oplus \mathcal{E}nd_0(\mathcal{E})$ (here $\mathcal{E}nd_0(\mathcal{E})$ is the sheaf of endomorphisms with trace 0), $\mathrm{Hom}(\mathcal{E},\mathcal{E}[1]) = \mathrm{Ext}^1(\mathcal{E},\mathcal{E}) = H^1(T,\mathcal{O}_T) \oplus H^1(T,\mathcal{E}nd_0(\mathcal{E})) \neq \{0\}$, because T is a torus and $H^1(T,\mathcal{O}_T) \cong \mathbb{C}$. Moreover, any torsion sheaf $\mathcal{T}$ can be deformed. Thus $\mathrm{Ext}^1(\mathcal{T},\mathcal{T}) \neq \{0\}$ and $\mathcal{T}$ is not spherical.

Now let $\mathcal{E}$ be a spherical object in $\mathrm{D^b}(T)$ and let r be the maximal integer such that $H^r(\mathcal{E}) \neq 0$. There exists a spectral sequence

$$E_2^{p,q} = \bigoplus_{i-j=p} \mathrm{Ext}^q_{\mathbf{Coh}(Y)}(H^i(\mathcal{E}), H^j(\mathcal{E})) \Longrightarrow \mathrm{Hom}_{\mathrm{D^b}(Y)}(\mathcal{E},\mathcal{E}[p+q]),$$

where $H^j(\mathcal{E})$ is the j-th cohomology sheaf of $\mathcal{E}$. For $q=1$, $p=0$ and $i=j=1$ there is some element surviving in $\mathrm{Hom}(\mathcal{E},\mathcal{E}[1])$ and so $\mathcal{E}$ cannot be spherical.

Hence Y is a K3 surface. Let $\mathcal{N}(X) := K(X)/K(X)^\perp$ and $\mathcal{N}(Y) := K(Y)/K(Y)^\perp$, where the orthogonal is taken with respect to the Euler pairing (4.5). The existence of the Chern character and the Hirzebruch-Riemann-Roch Theorem give the identifications $\mathcal{N}(X) = H^0(X,\mathbb{Z}) \oplus \mathrm{NS}(X) \oplus H^4(X,\mathbb{Z})$ and $\mathcal{N}(Y) = H^0(Y,\mathbb{Z}) \oplus \mathrm{NS}(Y) \oplus H^4(Y,\mathbb{Z})$. Clearly an exact equivalence $\mathrm{D^b}(X) \cong \mathrm{D^b}(Y)$ induces an isometry (with respect to the Euler pairing) $\mathcal{N}(X) \cong \mathcal{N}(Y)$. $\square$

We will see in the next section (Corollary 5.3) that the previous proposition is indeed general: all Fourier-Mukai partners of a K3 surface X are K3 surfaces, with the same Picard number of X.

5. Autoequivalences

We are now ready to move to the second main problem that we want to discuss in this paper: the description of the group of autoequivalences of Fourier-Mukai type of the derived category of a generic analytic K3 surface.

The first step will be to study the action induced by any equivalence of Fourier-Mukai type either on the Grothendieck group or on the Mukai lattice. Secondly we will use the machinery of stability conditions introduced by Bridgeland to produce a complete description.

5.1. **Action on K-theory and cohomology.**

In a first step, let X and Y be compact complex manifolds. Let $\Phi_{\mathcal{E}} : \mathrm{D^b}(X) \to \mathrm{D^b}(Y)$ be a Fourier-Mukai equivalence with kernel $\mathcal{E} \in \mathrm{D^b}(X \times Y)$. Then the induced isomorphism at level of Grothendieck groups is given by the morphism $\Phi^K_{[\mathcal{E}]} : K(X) \to K(Y)$ defined by

$$\Phi^K_{[\mathcal{E}]}(e) := q_*([\mathcal{E}] \cdot p^*(e)),$$

where $p : X \times Y \to X$ and $q : X \times Y \to Y$ are the natural projections. In other words, the following diagram commutes

$$\begin{tikzcd} \mathrm{D^b}(X) \arrow[r, "\Phi_{\mathcal{E}}"] \arrow[d, "{[-]}"'] & \mathrm{D^b}(Y) \arrow[d, "{[-]}"] \\ K(X) \arrow[r, "\Phi^K_{[\mathcal{E}]}"] & K(Y). \end{tikzcd} \tag{5.1}$$

Using the definition of Mukai vector, our analysis of Fourier-Mukai equivalences of the bounded derived categories can be further developed. Indeed, for $\mathcal{E} \in \mathrm{D^b}(X)$, one can easily consider the Mukai vector $v([\mathcal{E}])$ of $\mathcal{E}$ just by means of Definition 4.6. Now the morphism $\Phi^K_{[\mathcal{E}]} : K(X) \to K(Y)$ gives rise to a map $\Phi^H_{v([\mathcal{E}])} : H^*(X,\mathbb{Q}) \to H^*(Y,\mathbb{Q})$ such that

$$\Phi^H_{v([\mathcal{E}])} : b \longmapsto q_*(v([\mathcal{E}]) \cdot p^*(b)).$$

The Grothendieck-Riemann-Roch Theorem shows that the following diagram commutes:

$$\begin{tikzcd} K(X) \arrow[r, "\Phi^K_{[\mathcal{E}]}"] \arrow[d, "v(-)"'] & K(Y) \arrow[d, "v(-)"] \\ H^*(X,\mathbb{Q}) \arrow[r, "\Phi^H_{v([\mathcal{E}])}"] & H^*(Y,\mathbb{Q}). \end{tikzcd} \tag{5.2}$$

From now on, given a Fourier-Mukai equivalence $\Phi_{\mathcal{E}} : \mathrm{D^b}(X) \to \mathrm{D^b}(Y)$, we denote by

$$\Phi^H_{\mathcal{E}} : H^*(X,\mathbb{Q}) \longrightarrow H^*(Y,\mathbb{Q})$$

the morphism induced on cohomology via the diagrams (5.1) and (5.2) and, to simplify the notation, we write $v(\mathcal{E}) := v([\mathcal{E}])$, for any $\mathcal{E} \in \mathrm{D^b}(X)$.

Lemma 5.1. *Let X and Y be compact complex manifolds and let $\Phi_{\mathcal{F}} : \mathrm{D^b}(X) \to \mathrm{D^b}(Y)$ be a Fourier-Mukai equivalence whose kernel is $\mathcal{F} \in \mathrm{D^b}(X \times Y)$. Then*

$$\Phi^H_{\mathcal{F}} : H^*(X,\mathbb{Q}) \longrightarrow H^*(Y,\mathbb{Q})$$

is an isomorphim of $\mathbb{Q}$-vector spaces.

Proof. The proof in the non-algebraic case follows line by line [49, Prop. 5.33]. Since it is quite easy we reproduce it here for the convenience of the reader. For simplicity, we put $\Phi := \Phi_{\mathcal{F}}$ and $\Phi^H := \Phi^H_{\mathcal{F}}$.

The $\mathbb{Q}$-linearity is clear form the definition of Φ^H. So it is enough to prove that Φ^H is bijective. Since Φ is an equivalence, its right and left adjoints coincide with the inverses. Moreover, $\Phi \circ \Phi^{-1} = \mathrm{id} = \Phi_{\mathcal{O}_\Delta}$, where Δ is the image of the diagonal embedding $i : \Delta \hookrightarrow X \times X$. Indeed, for any $\mathcal{E} \in \mathrm{D^b}(X)$,

$$\begin{aligned} \Phi_{\mathcal{O}_\Delta}(\mathcal{E}) &= p_*(\mathcal{O}_\Delta \otimes q^*(\mathcal{E})) \\ &= p_*(i_*\mathcal{O}_X \otimes q^*(\mathcal{E})) \\ &= p_*(i_*(\mathcal{O}_X \otimes i^*q^*(\mathcal{E})) \\ &= (p \circ i)_*(q \circ i)^*\mathcal{E} \\ &= \mathcal{E}, \end{aligned}$$

where p and q are the natural projections, all the functors are supposed to be derived and the second equality is due to the Projection Formula (see Section 4.4).

Hence it suffices to prove that $\Phi^H_{\mathcal{O}_\Delta} = \mathrm{id}$. Since $\mathrm{ch}(\mathcal{O}_X) = (1,0,0)$ and the diagonal is isomorphic to X, using the Grothendieck-Riemann-Roch Theorem we get

$$\mathrm{ch}(\mathcal{O}_\Delta) \cdot \mathrm{td}(X \times X) = i_*(\mathrm{td}(X)).$$

Dividing by $\sqrt{\mathrm{td}(X \times X)}$ and using the Projection Formula, we get

$$\mathrm{ch}(\mathcal{O}_\Delta) \cdot \sqrt{\mathrm{td}(X \times X)} = i_*(\mathrm{td}(X) \cdot i^*((\sqrt{\mathrm{td}(X \times X)})^{-1})).$$

Since $i^*\mathrm{td}(X \times X) = \mathrm{td}(X)^2$, $i_*(\mathrm{td}(X) \cdot i^*((\sqrt{\mathrm{td}(X \times X)})^{-1})) = i_*(1,0,0)$ and thus

$$\begin{aligned} \Phi^H_{\mathcal{O}_\Delta}(b) &= p_*(\mathrm{ch}(\mathcal{O}_\Delta) \cdot \sqrt{\mathrm{td}(X \times X)} \cdot q^*(b)) \\ &= p_*(i_*(1) \cdot q^*(b)) \\ &= p_*(i_*(i^*(\cdot q^*(b)))) \\ &= b. \end{aligned}$$

The penultimate equality is given by the Projection Formula, while the last one is trivially due to the fact that $p \circ i = q \circ i = \mathrm{id}$. □

Lemma 5.1 can be made more precise for K3 surfaces.

Theorem 5.2. *Let X and Y be K3 surfaces and let $\Phi_{\mathcal{F}} : \mathrm{D}^{\mathrm{b}}(X) \to \mathrm{D}^{\mathrm{b}}(Y)$ be a Fourier-Mukai equivalence whose kernel is $\mathcal{F} \in \mathrm{D}^{\mathrm{b}}(X \times Y)$. Then $\Phi^H_{\mathcal{F}}$ restricts to a Hodge isometry*

$$\Phi^H_{\mathcal{F}} : \widetilde{H}(X, \mathbb{Z}) \longrightarrow \widetilde{H}(Y, \mathbb{Z}).$$

Proof. The complete proof of this result (originally due to Orlov, [84]) is in [49]. Hence we will simply sketch the main steps. As before we put $\Phi := \Phi_{\mathcal{F}}$ and $\Phi^H := \Phi^H_{\mathcal{F}}$.

First notice that, the isomorphism Φ^H defined over $\mathbb{Q}$ (Lemma 5.1) restricts to integral coeffients by the integrality of Chern character as discussed in Section 4.4. This is an easy exercise which we leave to the reader.

To prove that $\Phi^H_{\mathcal{F}}$ preserves the Hodge structure defined in Section 2.3, it is enough to show that $\Phi^H_{\mathcal{F}}(H^{p,q}(X)) \subset \bigoplus_{r-s=p-q} H^{r,s}(X)$. The Künneth decomposition of $c := \mathrm{ch}(\mathcal{F}) \cdot \sqrt{\mathrm{td}(X \times Y)}$ is $\sum \alpha_{u,v} \boxtimes \beta_{r,s}$, where $\alpha_{u,v} \in H^{u,v}(X)$ and $\beta_{r,s} \in H^{r,s}(Y)$. Since c is algebraic, the non-trivial terms are the ones such that $u + r = v + s$. If $\alpha \in H^{p,q}(X)$, then

$$\Phi^H_{\mathcal{F}}(\alpha) = \sum \left(\int_X (\alpha \wedge \alpha_{u,v}) \right) \beta_{r,s} \in \bigoplus H^{r,s}(Y).$$

In particular the non-zero terms satisfy the condition $u + p = v + q = \dim X$. Thus $p - q = v - u = r - s$. □

This is enough to generalize Proposition 4.24.

Corollary 5.3. *If X is a K3 surface and Y is a smooth compact complex variety which is a Fourier-Mukai partner of X, then Y is a K3 surface and $\rho(X) = \rho(Y)$.*

Proof. As a consequence of Lemma 4.23, we have already proved that Y is either a K3 surface or a 2-dimesional complex torus or a Kodaira surface. To exclude the last two possibilities observe that if Y is a torus or a Kodaira surface, then $\dim H^*(X, \mathbb{Q}) \neq \dim H^*(Y, \mathbb{Q})$. But this would contradict Lemma 5.1.

The Picard numbers of X and Y are the same because, by Theorem 5.2, any equivalence $\Phi : \mathrm{D}^{\mathrm{b}}(X) \xrightarrow{\sim} \mathrm{D}^{\mathrm{b}}(Y)$ is such that $\Phi^H(H^{2,0}(X)) = \Phi^H(\widetilde{H}^{2,0}(X)) = \widetilde{H}^{2,0}(Y) = H^{2,0}(Y)$. As usual, this yields and isometry $T(X) \cong T(Y)$. □

In the generic non-algebraic case the following holds:

Corollary 5.4. *Let X and Y be generic analytic K3 surfaces which are Fourier-Mukai partners. Then $X \cong Y$.*

Proof. By the previous result an equivalence $\Phi_{\mathcal{E}} : \mathrm{D}^{\mathrm{b}}(X) \xrightarrow{\sim} \mathrm{D}^{\mathrm{b}}(Y)$ of Fourier-Mukai type induces a Hodge isometry $\widetilde{H}(X, \mathbb{Z}) \cong \widetilde{H}(Y, \mathbb{Z})$. The facts that the weight-two Hodge structures are preserved and that $T(X) = H^2(X, \mathbb{Z})$ and $T(Y) = H^2(Y, \mathbb{Z})$ imply that there is a Hodge isometry $H^2(X, \mathbb{Z}) \cong H^2(Y, \mathbb{Z})$. Theorem 2.12 yields now the desired isomorphism $X \cong Y$. □

Remark 5.5. As a consequence a generic analytic K3 surface does not have non-isomorphic Fourier-Mukai partners. On the other hand, the algebraic case is very much different. In fact, in [80] and [96] it is proved that for any positive integer N, there are N non-isomorphic algebraic K3 surfaces $X_1, \ldots, X_N$ such that $\mathrm{D}^{\mathrm{b}}(X_i) \cong \mathrm{D}^{\mathrm{b}}(X_j)$, for $i, j \in \{1, \ldots, N\}$.

Let us develop further the consequences of Theorem 5.2. Indeed, consider a (non necessarily projective) K3 surface X. If $\mathrm{O}_{\mathrm{Hodge}}(\widetilde{H}(X, \mathbb{Z}))$ denotes the set of Hodge isometries of the Mukai lattice $\widetilde{H}(X, \mathbb{Z})$, then Theorem 5.2 yields a group homorphism

$$\Pi : \mathrm{Aut}^{\mathrm{FM}}(\mathrm{D}^{\mathrm{b}}(X)) \longrightarrow \mathrm{O}_{\mathrm{Hodge}}(\widetilde{H}(X, \mathbb{Z})).$$

Of course, the information about the image of Π would give interesting properties of $\mathrm{Aut}^{\mathrm{FM}}(\mathrm{D^b}(X))$. Notice that a complete description of the group of autoequivalences in terms of a group homomorphism analogous to Π is available for the closest relatives of K3 surfaces: abelian variaties ([85]).

Nevertheless, the situation for K3 surfaces is widely unknown and the expectation is summarized by the following conjecture.

Conjecture 5.6. For a K3 surface X, the image of Π coincides with the subgroup

$$\mathrm{O}_+(\widetilde{H}(X,\mathbb{Z})) \subseteq \mathrm{O}_{\mathrm{Hodge}}(\widetilde{H}(X,\mathbb{Z}))$$

of orientation preserving Hodge isometries.

This conjecture was first stated by Szendrői in [97] and then improved by Bridgeland ([15]) who proposed a very nice description of the kernel of Π in terms of the fundamental group of a domain naturally associated to a variety parametrizing stability conditions on the triangulated category $\mathrm{D^b}(X)$ (see Section 5.2). In the algebraic case, evidences for the truth of Conjecture 5.6 were found in [51]. The generic twisted case is proved in [53]. In Section 5.3 the conjecture will be proved for generic analytic K3 surface following the proof in [53] while in the forthcoming paper [54] the conjecture is solved for smooth projective K3 surfaces.

Example 5.7. (i) The Hodge isometry induced on cohomolgy by the shift functor $[1] : \mathrm{D^b}(X) \xrightarrow{\sim} \mathrm{D^b}(X)$ is $-\mathrm{id} : \widetilde{H}(X,\mathbb{Z}) \xrightarrow{\sim} \widetilde{H}(X,\mathbb{Z})$ which is orientation preserving.

(ii) Let $f : X \xrightarrow{\sim} Y$ be an isomorphism of K3 surfaces and consider the Fourier-Mukai equivalence $f^* : \mathrm{D^b}(Y) \xrightarrow{\sim} \mathrm{D^b}(X)$. Obviously, $(f^*)^H = f^*|_{H^2(X,\mathbb{Z})} \oplus \mathrm{id}_{H^0(X,\mathbb{Z})\oplus H^4(X,\mathbb{Z})}$ which preserves the natural orientation of the four positive directions (Example 2.14).

(iii) The sheaf $\mathcal{O}_X$ is a spherical object (see Example 4.13) and, if X is a K3 surface, the isometry induced on cohomology by the spherical twist $T_{\mathcal{O}_X}$ (Example 4.19) is the reflection with respect to the vector $(1,0,1) \in H^*(X,\mathbb{Z})$ described in Example 2.14 which preserves the orientation.

Using the previous examples we can prove the following:

Proposition 5.8. *Let X be a generic analytic K3 surface and let $g \in \mathrm{O}_+(\widetilde{H}(X,\mathbb{Z}))$. Then there exists $\Phi \in \mathrm{Aut}^{\mathrm{FM}}(\mathrm{D^b}(X))$ such that $g = \Pi(\Phi)$.*

Proof. From the very definition of orientation preserving Hodge isometries and using the fact that $\rho(X) = \rho(Y) = 0$, it follows easily that $g = h_1 \circ h_2$, where $h_1 = k_1 \oplus \mathrm{id}_{H^2(X,\mathbb{Z})}$ and $h_2 = \mathrm{id}_{H^0(X,\mathbb{Z})\oplus H^4(X,\mathbb{Z})} \oplus k_2$, for $k_1 \in \mathrm{O}(H^0(X,\mathbb{Z}) \oplus H^4(X,\mathbb{Z}))$ and $k_2 \in \mathrm{O}(H^2(X,\mathbb{Z}))$. Up to composing g with $-\mathrm{id}_{\widetilde{H}(X,\mathbb{Z})}$ (which, by Example 5.7 (i), is induced by the shift functor) we can also suppose that h_1 and h_2 are orientation preserving.

In this situation the Torelli Theorem immediately yields an automorphism $f : X \xrightarrow{\sim} X$ such that $h_2 = f^*$. On the other hand, h_1 is either the identity or the reflection with respect to the vector $(1,0,1)$. The second possibility is induced by the spherical twist (Example 5.7 (iii)). $\square$

5.2. **Stability conditions.**

Stability conditions were introduced by Bridgeland in [14] to deal with Douglas' notion of Π-stability ([2, 29]). Nevertheless, the interest for stability conditions and for the topology of the varieties parametrizing them became immediately apparent in the geometric context as well. In particular, stability conditions defined on the bounded derived categories of coherent sheaves on complex varieties were studied in many interesting geometric cases (see, for example, [15, 16, 64, 82, 83]). Very nice surveys for this are [17, 18].

As we anticipated in the previous section, Bridgeland suggested in [15] that a possible strategy to prove Conjecture 5.6 would involve the topology of the manifold parametrizing stability conditions on the derived categories of K3 surfaces. In this paper, we will not deal with this topological problems since in Section 5.3 we will see that the study of the equivalences of the derived categories of generic analytic K3 surface just relies on the notion of stability conditions and does not involve any topological property.

Since due to Proposition 4.15 there exists a natural equivalence $\mathrm{D^b}(X) \cong \mathrm{D^b}(\mathbf{Coh}(X))$, through out this section we will not distinguish between these two categories. Moreover X will be a generic analytic K3 surface and $\mathcal{N}(X)$ will denote the lattice $H^0(X,\mathbb{Z}) \oplus H^4(X,\mathbb{Z}) \cong U$. We will restrict our attention to stability conditions of special type.

Definition 5.9. A *(numerical) stability condition* on $\mathrm{D^b}(X)$ is a pair $\sigma = (Z, \mathcal{P})$ where

$$Z : \mathcal{N}(X) \longrightarrow \mathbb{C}$$

is a linear map (the *central charge*) while $\mathcal{P}(\phi) \subset \mathrm{D^b}(X)$ are full additive subcategories for each $\phi \in \mathbb{R}$ satisfying the following conditions:

(a) If $0 \neq \mathcal{E} \in \mathcal{P}(\phi)$, then $Z(\mathcal{E}) = m(\mathcal{E})\exp(i\pi\phi)$ for some $m(\mathcal{E}) \in \mathbb{R}_{>0}$.

(b) $\mathcal{P}(\phi+1) = \mathcal{P}(\phi)[1]$ for all ϕ.

(c) $\mathrm{Hom}_{\mathrm{D^b}(X)}(\mathcal{E}_1, \mathcal{E}_2) = 0$ for all $\mathcal{E}_i \in \mathcal{P}(\phi_i)$ with $\phi_1 > \phi_2$.

(d) Any $0 \neq \mathcal{E} \in \mathrm{D^b}(X)$ admits a *Harder-Narasimhan filtration* given by a collection of distinguished triangles $\mathcal{E}_{i-1} \to \mathcal{E}_i \to \mathcal{A}_i$ with $\mathcal{E}_0 = 0$ and $\mathcal{E}_n = \mathcal{E}$ such that $\mathcal{A}_i \in \mathcal{P}(\phi_i)$ with $\phi_1 > \ldots > \phi_n$.

Following Bridgeland's terminology, the objects in the category $\mathcal{P}(\phi)$ are called *semistable* of phase ϕ while the objects $\mathcal{A}_i$ in (d) are called the *semistable factors* of $\mathcal{E}$. It is an easy exercise to show that they are unique up to isomorphism. The minimal objects of $\mathcal{P}(\phi)$ are called *stable* of phase ϕ.

Given a stability condition $(Z, \mathcal{P})$ and $\mathcal{E} \in \mathrm{D^b}(X)$, we put $\phi_-(\mathcal{E}) := \phi_1$ and $\phi_+(\mathcal{E}) := \phi_n$, where ϕ_1 and ϕ_n are the phases of the first and the last semistable factor in the Harder-Narasimhan filtration of $\mathcal{E}$ (see item (d) of the previous definition). One defines $\mathcal{P}((0,1])$ to be the subcategory of $\mathrm{D^b}(X)$ containing the zero object and all $\mathcal{E} \in \mathrm{D^b}(X)$ such that $0 < \phi_-(\mathcal{E}) \leq \phi_+(\mathcal{E}) \leq 1$.

Remark 5.10. (i) All stability conditions studied in this section are tacitly assumed to be *locally finite* (see [15] for the precise definition). For our purposes it is enough to know that this additional condition implies that any semistable object $\mathcal{E} \in \mathcal{P}(\phi)$ admits a finite Jordan-Hölder filtration, i.e. a finite filtration $\mathcal{E}_0 \subset \ldots \subset \mathcal{E}_n = \mathcal{E}$ with stable quotients $\mathcal{E}_{i+1}/\mathcal{E}_i \in \mathcal{P}(\phi)$.

(ii) There is an alternative way to give a stability condition on $\mathrm{D^b}(X)$. Indeed [14, Prop. 5.3] shows that a stability condition can equivalently be described by a bounded t-structure on $\mathrm{D^b}(X)$ with heart $\mathbf{A}$ and a stability function $Z : K(\mathbf{A}) \to \mathbb{C}$ which has the Harder-Narasimhan property (recall that a stability function is a $\mathbb{C}$-linear function which takes values in $\mathbb{H} \cup \mathbb{R}_{\leq 0}$, where $\mathbb{H}$ is the complex upper half plane). Indeed, given a stability condition $\sigma = (Z, \mathcal{P})$ one can prove that $\mathbf{A} := \mathcal{P}((0,1])$, is the heart (usually called the *heart of* σ) of a bounded t-structure.

Definition 5.11. The space of all locally finite numerical stability conditions on $\mathrm{D^b}(X)$ is denoted by $\mathrm{Stab}(\mathrm{D^b}(X))$.

In general, it is a highly non-trivial task to show that the space parametrizing stability conditions on a triangulated category is non-empty but, once this is achieved, [14, Thm. 1.2] shows that such a space is a manifold. Moreover, if the triangulated category is the bounded derived category of coherent sheaves on some complex manifold, then the space parametrizing stability conditions is actually a finite-dimensional complex manifold ([14, Cor. 1.3]).

Hence, let us first show that, in our specific case, $\mathrm{Stab}(\mathrm{D^b}(X))$ is non-empty. Consider the open subset

$$R := \mathbb{C} \setminus \mathbb{R}_{\geq -1} = R_+ \cup R_- \cup R_0, \tag{5.3}$$

where $R_+ := R\cap\mathbb{H}$, $R_- := R\cap(-\mathbb{H})$ and $R_0 := R\cap\mathbb{R}$ with $\mathbb{H}$ denoting the upper half-plane. Given $z = u + iv \in R$, following [15], the subcategories $\mathcal{F}(z), \mathcal{T}(z) \subset \mathbf{Coh}(X)$ are defined as follows:

- $\mathcal{F}(z)$ is the full subcategory of all torsion free sheaves of degree $\leq v$;
- $\mathcal{T}(z)$ is the full subcategory that contains all torsion sheaves and all torsion free sheaves of degree $> v$.

Hence, if $z \in R_+ \cup R_0$ then $\mathcal{F}(z)$ and $\mathcal{T}(z)$ are respectively the full subcategories of all torsion free sheaves respectively torsion sheaves while if $z \in R_-$ then $\mathcal{F}(z)$ is trivial and $\mathcal{T}(z) = \mathbf{Coh}(X)$.

Consider the subcategories defined by means of $\mathcal{F}(z)$ and $\mathcal{T}(z)$ as follows. If $z \in R_+ \cup R_0$, we put

$$\mathcal{A}(z) := \left\{ \mathcal{E} \in \mathrm{D^b}(X) : \begin{array}{l} \bullet\ H^i(\mathcal{E}) = 0 \text{ if } i \notin \{-1, 0\} \\ \bullet\ H^{-1}(\mathcal{E}) \in \mathcal{F}(z) \\ \bullet\ H^0(\mathcal{E}) \in \mathcal{T}(z) \end{array} \right\}.$$

On the other hand, let $\mathcal{A}(z) = \mathbf{Coh}(X)$ when $z \in R_-$. Due to [15, Lemma 3.3], $\mathcal{A}(z)$ is the heart of a bounded t-structure for any $z \in R$.

Now, for any $z = u + iv \in R$ we define the function

$$\begin{array}{rcl} Z : \mathcal{A}(z) & \longrightarrow & \mathbb{C} \\ \mathcal{E} & \longmapsto & \langle v(\mathcal{E}), (1, 0, z)\rangle = -u\cdot r - s - i(r\cdot v), \end{array}$$

where $v(\mathcal{E}) = (r, 0, s) = (r, 0, r - \mathrm{c}_2(\mathcal{E}))$ is the Mukai vector of $\mathcal{E}$.

The main properties of the pair $(\mathcal{A}(z), Z)$ just defined are summarized by the following proposition.

Proposition 5.12. **([53], Lemma 3.2 and Proposition 3.4)** *For any $z \in R$ the function Z defines a stability function on $\mathcal{A}(z)$ which has the Harder-Narasimhan property.*

Using item (ii) of Remark 5.10, we conclude that for any $z \in R$ we have a stability condition σ_z obtained from the t-structure associated to the pair $(\mathcal{F}(z), \mathcal{T}(z))$ and the stability function Z on $\mathcal{A}(z)$. In [53, Prop. 3.4] it is also proved that such a stability condition is locally finite. Hence $\sigma_z \in \mathrm{Stab}(\mathrm{D^b}(X))$ and $\mathrm{Stab}(\mathrm{D^b}(X))$ is non-empty. Some additional interesting topological properties of $\mathrm{Stab}(\mathrm{D^b}(X))$ (not used in the rest of this paper) are determined by the following result.

Theorem 5.13. **([53], Theorem 3.)** *The space* $\mathrm{Stab}(X)$ *is connected and simply-connected.*

Remark 5.14. As pointed out in [14], there is a right action of the group $\mathrm{Aut^{FM}}(X)$ onto the manifold $\mathrm{Stab}(\mathrm{D^b}(X))$. Indeed, given $\sigma = (Z, \mathcal{P}) \in \mathrm{Stab}(\mathrm{D^b}(X))$ and $\Phi \in \mathrm{Aut^{FM}}(X)$ we define $\Phi(\sigma)$ to be the stability condition $(Z', \mathcal{P}') \in \mathrm{Stab}(\mathrm{D^b}(X))$ where $Z' = Z \circ (\Phi^H)^{-1}$ and $\mathcal{P}'(\phi) = \Phi(\mathcal{P}(\phi))$.

Using Lemma 4.16 one can show the following remarkable property of $\mathrm{Stab}(\mathrm{D^b}(X))$ which will be used in Section 5.3.

Proposition 5.15. **([53], Proposition 1.18 and Corollary 1.19.)** *For any $\sigma \in \mathrm{Stab}(\mathrm{D^b}(X))$, there is $n \in \mathbb{Z}$ such that $T^n_{\mathcal{O}_X}(\mathcal{O}_x)$ is stable in σ, for any closed point $x \in X$.*

5.3. **Equivalences.**

For generic analytic K3 surfaces X, we obtain a complete description of the group of autoequivalences of $\mathrm{D^b}(X)$ using the notion of stability condition. The proof immediately shows that the topology of $\mathrm{Stab}(X)$ is irrelevant in this case.

Proposition 5.16. **([53], Lemma 3.9.)** *Let X and Y be complex K3 surfaces such that $\rho(X) = \rho(Y) = 0$. If $\Phi_{\mathcal{E}} : \mathrm{D}^{\mathrm{b}}(X) \xrightarrow{\sim} \mathrm{D}^{\mathrm{b}}(Y)$ is an equivalence of Fourier-Mukai type, then up to shift*

$$\Phi_{\mathcal{E}} \cong T^n_{\mathcal{O}_Y} \circ f_*$$

for some $n \in \mathbb{Z}$. Here, $T_{\mathcal{O}_Y}$ is the spherical twist with respect to $\mathcal{O}_Y$ and $f : X \xrightarrow{\sim} Y$ is an isomorphism.

Proof. Suppose $\sigma = \sigma_{(u,v=0)}$ is one of the distinguished stability conditions constructed in Section 5.2 with $(u, v) \in R$ and $v = 0$. Let $\tilde{\sigma} := \Phi_{\mathcal{E}}(\sigma)$.

Denote by T the spherical twist $T_{\mathcal{O}_Y}$. By Proposition 5.15, there exists an integer n such that all skyscraper sheaves $\mathcal{O}_x$ are stable of the same phase in the stability condition $T^n(\tilde{\sigma})$.

The composition $\Psi := T^n \circ \Phi_{\mathcal{E}}$, is again an equivalence of Fourier-Mukai type, i.e. $\Psi := \Phi_{\mathcal{F}}$, for some kernel $\mathcal{F} \in \mathrm{D}^{\mathrm{b}}(X \times Y)$. Moreover it sends the stability condition σ to a stability condition σ' for which all skyscraper sheaves are stable of the same phase.

Up to shifting the kernel $\mathcal{F}$ sufficiently, we can assume that $\phi_{\sigma'}(\mathcal{O}_y) \in (0, 1]$ for all closed points $y \in Y$. Thus, the heart $\mathcal{P}((0, 1])$ of the t-structure associated to σ' (see Remark 5.10), which under $\Phi_{\mathcal{F}}$ is identified with $\mathcal{A}(u)$, contains as stable objects the images $\Psi(\mathcal{O}_x)$ of all closed points $x \in X$ and all point sheaves $\mathcal{O}_y$.

In [53, Rmk. 3.3] it was observed that the only semi-rigid stable objects in $\mathcal{A}(u)$ are the skyscraper sheaves. Hence, for all $x \in X$ there exists a point $y \in Y$ such that $\Psi(\mathcal{O}_x) \cong \mathcal{O}_y$.

This suffices to conclude that the Fourier-Mukai equivalence $\Psi_{\mathcal{F}}$ is a composition of f_* for some isomorphism $f : X \xrightarrow{\sim} Y$ and a line bundle twist (see [49, Cor. 5.23]). But there are no non-trivial line bundles on Y. $\square$

This immediately yield the following complete description of all Fourier-Mukai autoequivalences in the generic case.

Theorem 5.17. **([53], Theorem 4.)** *If X is a K3 surface with $\mathrm{Pic}(X) = \{0\}$, then*

$$\mathrm{Aut}^{\mathrm{FM}}(\mathrm{D}^{\mathrm{b}}(X)) \cong \mathbb{Z} \oplus \mathbb{Z} \oplus \mathrm{Aut}(X).$$

The first two factors are generated respectively by the shift functor and the spherical twist $T_{\mathcal{O}_X}$.

This nice description shows that the interesting (and possibly non-trivial) part of $\mathrm{Aut}^{\mathrm{FM}}(\mathrm{D}^{\mathrm{b}}(X))$ is given by the group of automorphisms of X. Due to Theorem 3.6, such a group is usually trivial while it is isomorphic to $\mathbb{Z}$ for countably many generic K3 surfaces characterized by the fact that $\mathrm{Aut}(X)$ is generated by an automorphism f such that the induced isometry f^* on the second cohomology group has minimal polynomial of Salem type. Hence, in the complex analytic setting, $\mathrm{Aut}^{\mathrm{FM}}(\mathrm{D}^{\mathrm{b}}(X))$ is either isomorphic to $\mathbb{Z}^{\oplus 2}$ (for K3 surfaces not of McMullen's type) or isomorphic to $\mathbb{Z}^{\oplus 3}$ (for the countably many K3 surfaces of McMullen type).

We conclude observing that, together with Proposition 5.8, Theorem 5.17 proves Conjecture 5.6 for analytic generic K3 surfaces. More precisely, there exists a short exact sequence of groups

$$0 \longrightarrow \mathbb{Z} \oplus \mathbb{Z} \longrightarrow \mathrm{Aut}^{\mathrm{FM}}(\mathrm{D}^{\mathrm{b}}(X)) \longrightarrow \mathrm{O}_+(\widetilde{H}(X, \mathbb{Z})) \longrightarrow 0,$$

where the two copies of $\mathbb{Z}$ are generated by the shift by 2 and by $T^2_{\mathcal{O}_Y}$ respectively.

Acknowledgements. The project was completed when the second named author was visiting the Institute for Mathematical Sciences of the Imperial College (London), whose hospitality is gratefully acknowledged. The stay at the Institute was funded by a grant of the Istituto Nazionale di Alta Matematica (Italy). The results about the equivalences of the derived categories of generic analytic K3 surfaces presented in this paper were obtained by the authors in collaboration with Daniel Huybrechts. It is a great pleasure to thank him for the fruitful collaboration. Thanks also to Alice Garbagnati and Bert van Geemen who read a preliminary version of this paper.

References

[1] *Géometrie des surfaces K3: modules et périodes*, Astérisque **126** (1985).
[2] P. Aspinwall, *D-Branes, Π-Stability and Theta-Stability*, hep-th/0407123.
[3] M.F. Atiyah, F. Hirzebruch, *Analytic cycles on complex manifolds*, Topology **1** 1962, 25–45.
[4] W. Barth, K. Hulek, C. Peters, A. Van de Ven, *Compact complex surfaces*, Springer (2004).
[5] A. Beilinson, J. Bernstein, P. Deligne, *Faisceaux perverses*, Astérisque **100** (1982).
[6] O. Ben-Bassat, J. Block, T. Pantev, *Non-commutative tori and Fourier-Mukai duality*, to appear in: Compositio Math., math.AG/0509161.
[7] G. Bini, *On the automorphism group of some K3 surfaces with Picard number two*, MCFA Annals **1** (2006), 1–4.
[8] J. Block, *Duality and equivalence of module categories in noncommutative geometry I*, math.QA/0509284.
[9] A. Bondal, M. Kapranov, *Representable functors, Serre functors and mutations*, USSR Izvestiya **35** (1990), 519–541.
[10] A. Bondal, M. Van den Bergh, *Generators and representability of functors in commutative and noncommutative geometry*, Moscow Math. J. **3** (2003), 1–36.
[11] A. Bondal, D. Orlov, *Reconstruction of a variety from the derived category and groups of autoequivalences*, Compositio Math. **125** (2001), 327–344.
[12] C. Borcea, *Diffeomorphisms of a K3 surface*, Math. Ann. **275** (1986), 1–4.
[13] T. Bridgeland, *Equivalences of triangulated categories and Fourier-Mukai transforms*, Bull. London Math. Soc. **31** (1999), 25–34.
[14] T. Bridgeland, *Stability conditions on triangulated categories*, to appear in: Ann. Math., math.AG/0212237.
[15] T. Bridgeland, *Stability conditions on K3 surfaces*, math.AG/0307164.
[16] T. Bridgeland, *Stability conditions on a non-compact Calabi-Yau threefold*, Com. Math. Phys. **266** (2006) 715–733.
[17] T. Bridgeland, *Derived categories of coherent sheaves*, to appear in: Proceedings of ICM 2006, math.AG/0602129.
[18] T. Bridgeland, *Spaces of stability conditions*, math.AG/0611510.
[19] T. Bridgeland, A. Maciocia, *Complex surfaces with equivalent derived categories*, Math. Z. **236** (2001), 677–697.
[20] E.H. Brown, *Cohomologies theories*, Ann. Math. **75** (1962), 497–484.
[21] D. Burns, M. Rapoport, *On the Torelli problem for khlerian K-3 surfaces*, Ann. Sci. cole Norm. Sup. **8** (1975), 235–273.
[22] A. Căldăraru, *The Mukai pairing I: the Hochschild pairing*, math.AG/0308079.
[23] A. Căldăraru, *The Mukai pairing. II. The Hochschild-Kostant-Rosenberg isomorphism*, Adv. Math. **194** (2005), 34–66.
[24] A. Canonaco, P. Stellari, *Twisted Fourier-Mukai functors*, to appear in: Adv. Math., math.AG/0605229.
[25] I.V. Dolgachev, *Integral quadratic forms: applications to algebraic geometry (after V. Nikulin)*, Seminaire Bourbaki **611** 1982/83, Astérisque **105-106** (1983), 251–278.
[26] I.V. Dolgachev, *Mirror symmetry for lattice polarized K3 surfaces*, Algebraic geometry 4, J. Math. Sci. **81** (1996), 2599–2630.
[27] I.V. Dolgachev, *Reflection groups in algebraic geometry*, math.AG/0610938.
[28] S. Donaldson, *Polynomial invariants for smooth four-manifolds*, Topology **29** (1990), 257–315.
[29] M.R. Douglas, *Dirichlet branes, homological mirror symmetry, and stability*, Proceedings of the International Congress of Mathematicians, Vol. III, Higher Ed. Press (2002), 395–408.
[30] P. Gabriel, *Des catégories abéliennes*, Bull. Soc. Math. France **90** (1962), 323–348.
[31] P. Gabriel, M. Zisman, *Calculus of fractions and homotopy theory*, Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 35, Springer-Verlag New York, Inc., New York (1967).
[32] F. Galluzzi, G. Lombrado, *On the automorphisms of some $K3$ surface double cover of the plane*, math.AG/0512603.
[33] F. Galluzzi, G. Lombrado, *On automorphisms group of some K3 surfaces*, math.AG/0610972.
[34] A. Garbagnati, A. Sarti, *Symplectic automorphisms of prime order on K3 surfaces*, math.AG/0603742.
[35] B. van Geemen, A. Sarti, *Nikulin involutions on K3 surfaces*, to appear in: Math. Zeit., math.AG/0602015.
[36] S.I. Gelfand, Yu.I. Manin, *Methods of homological algebra*, Springer-Verlag, Berlin (1997).
[37] H.I. Green, *Chern classes for coherent sheaves*, Ph.D. Thesis, University of Warwick (1980).
[38] P. Griffiths, J. Harris, *Principles of algebraic geometry*, Pure and Applied Mathematics, Wiley-Interscience, New York (1978).
[39] B. Gross, C.T. McMullen, *Automorphisms of even unimodular lattices and unramified Salem polynomials*, J. Algebra **257** (2002), 265–290.
[40] A. Grothendieck, *Sur quelques points d'algèbre homologique*, Thoku Math. J. **9** (1957), 119–221.
[41] R. Hartshorne, *Residues and duality*, Lecture Notes in Mathematics Vol. 20, Springer-Verlag (1966).

[42] R. Hartshorne, *Algebraic geometry*, Graduated Texts in Mathematics Vol. 52, Springer-Verlag (1977).
[43] N. Hitchin, *Generalized Calabi-Yau manifolds*, Q. J. Math. **54** (2003), 281–308.
[44] S. Hosono, B.H. Lian, K. Oguiso, S.-T. Yau, *Fourier-Mukai partners of a K3 surface of Picard number one*, Vector bundles and representation theory (Columbia, 2002), Contemp. Math **322**, AMS, Providence (2003), 43–55.
[45] S. Hosono, B.H. Lian, K. Oguiso, S.-T. Yau, *Autoequivalences of derived category of a K3 surface and monodromy transformations*, J. Alg. Geom. **13** (2004), 513–545.
[46] S. Hosono, B.H. Lian, K. Oguiso, S.-T. Yau, *Fourier-Mukai number of a K3 surface*, CRM Proc. and Lect. Notes **38** (2004), 177–192.
[47] D. Huybrechts, *Moduli spaces of hyperkhler manifolds and mirror symmetry*, in: Intersection theory and moduli, ICTP Lect. Notes XIX, Trieste (2004), 185–247.
[48] D. Huybrechts, *Generalized Calabi-Yau structures, K3 surfaces, and B-fields*, Int. J. Math. **16** (2005), 13–36.
[49] D. Huybrechts, *Fourier-Mukai transforms in algebraic geometry*, Oxford Mathematical Monographs, Oxford Science Publications (2006).
[50] D. Huybrechts, M. Lehn, *The geometry of the moduli spaces of sheaves*, Vieweg (1997).
[51] D. Huybrechts, P. Stellari, *Equivalences of twisted K3 surfaces*, Math. Ann. **332** (2005), 901–936.
[52] D. Huybrechts, P. Stellari, *Proof of Căldăraru's conjecture*, The 13th MSJ Inter. Research Inst. - Moduli Spaces and Arithmetic Geometry, Adv. Stud. Pure Math. **45** (2006), 31–42.
[53] D. Huybrechts, E. Macrì, P. Stellari, *Stability conditions of generic K3 categories*, math.AG/0608430.
[54] D. Huybrechts, E. Macrì, P. Stellari, *Derived equivalences of K3 surfaces and orientation*, in preparation.
[55] L. Illusie, *Existence de résolutions globales*, SGA6, Lecture Notes in Mathematics, vol. 20, Springer (1971).
[56] Y. Kawamata, *Equivalences of derived categories of sheaves on smooth stacks*, Am. J. Math. **126** (2004), 1057–1083.
[57] B. Keller, *Derived categories and tilting*, preprint 2003.
[58] J. Keum, S. Kondō, *The automorphism groups of Kummer surfaces associated with the product of two elliptic curves*, Trans. Amer. Math. Soc. **353** (2001), 1469–1487.
[59] S. Kondō, *Maximal subgroups of the Mathieu group M_{23} and symplectic automorphisms of supersingular $K3$ surfaces*, Int. Math. Res. Not. (2006), 9 pp.
[60] S. Kondō, *The automorphism group of a generic Jacobian Kummer surface*, J. Algebraic Geom. **7** (1998), 589–609.
[61] S. Kondō, *Niemeier lattices, Mathieu groups, and finite groups of symplectic automorphisms of $K3$ surfaces*, Duke Math. J. **92** (1998), 593–603.
[62] M. Kontsevich, *Homological algebra of Mirror Symmetry*, in: Proceedings of the International Congress of Mathematicians (Zurich, 1994, ed. S.D. Chatterji), Birkhauser, Basel (1995), 120-139.
[63] E. Looijenga, P.Ch. Peters, *Torelli theorems for Kähler K3 surfaces*, Compositio Math. **42** (1980/81), 145–186.
[64] E. Macrì, *Some examples of spaces of stability conditions on derived categories*, math.AG/0411613.
[65] J.S. Milnor, *On symply connected 4-manifolds*, Symposium Intenational de Geometria Algebraica, La Universidad Atonoma de Mèxico e UNESCO (1958), 122–128.
[66] C.T. McMullen, *Dynamics on K3 surfaces: Salem numbers and Siegel disks*, J. Reine Angew. Math. **545** (2002), 201–233.
[67] D.R. Morrison, *On K3 surfaces with large Picard number*, Invent. Math. **75** (1984), 105–121.
[68] S. Mukai, *Duality between $D(X)$ and $D(\widehat{X})$ with its applications to Picard sheaves*, Nagoya Math. J. **81** (1981), 153–175.
[69] S. Mukai, *Symplectic structure of the moduli space of sheaves on an abelian or K3 surface*, Invent. Math. **77** (1984), 101–116.
[70] S. Mukai, *On the moduli space of bundles on K3 surfaces, I*, In: Vector Bundles on Algebraic Varieties, Oxford University Press, Bombay and London (1987), 341–413.
[71] S. Mukai, *Moduli of vector bundles and symplectic manifolds*, Sugako Expositions **1** (1988), 139–174.
[72] S. Mukai, *Finite groups of automorphisms of K3 surfaces and the Mathieu group*, Invent. Math. **94** (1988), 183–221.
[73] A. Neeman, *Triangulated categories*, Annals of Mathematics Studies, **148**. Princeton University Press, Princeton, NJ (2001).
[74] V.V. Nikulin, *Finite automorphism groups of Kähler K3 surfaces*, Trans. Moscow Math. Soc. **38** (1980), 71–135.
[75] V.V. Nikulin, *Integral symmetric bilinear forms and some of their applications*, Math. USSR Izvestija **14** (1980), 103–167.
[76] N. O'Brian, D. Toledo, Y.L.L. Tong, *The trace map and characteristic classes for coherent sheaves*, Amer. J. Math. **103** (1981), 225–252.
[77] N. O'Brian, D. Toledo, Y.L.L. Tong, *Hirzebruch-Riemann-Roch for coherent sheaves*, Amer. J. Math. **103** (1981), 253–271.

[78] N. O'Brian, D. Toledo, Y.L.L. Tong, *Grothendieck-Riemann-Roch for complex manifolds*, Bull. Amer. Math. Soc. **5** (1981), 182–184.
[79] N. O'Brian, D. Toledo, Y.L.L. Tong, *A Grothendieck-Riemann-Roch formula for maps of complex manifolds*, Math. Ann. **271** (1985), 493–526.
[80] K. Oguiso, *K3 surfaces via almost-primes*, Math. Research Letters **9** (2002), 47–63.
[81] K. Oguiso, *Bimeromorphic automorphism groups of non-projective hyperkähler manifolds - a note inspired by C. T. McMullen*, math.AG/0312515.
[82] S. Okada, *Stability manifold of* $\mathbb{P}^1$, J. Alg. Geom. **15** (2006), 487–505.
[83] S. Okada, *On stability manifolds of Calabi–Yau surfaces*, Int. Math. Res. Not. (2006), ID 58743.
[84] D. Orlov, *Equivalences of derived categories and K3 surfaces*, J. Math. Sci. **84** (1997), 1361–1381.
[85] D. Orlov, *Derived categories of coherent sheaves on abelian varieties and equivalences between them*, Izv. Math. **66** (2002), 569–594.
[86] D. Orlov, *Derived category of coherent sheaves and equivalences between them*, Russ. Math. Surveys **58:3** (2003), 511–591.
[87] I. Pijateckiĭ-Šapiro, I. Šafarevič, *A Torelli theorem for algebraic surfaces of type K3*, Math. USSR Izvestija **5** (1971), 547–588.
[88] J.-P. Ramis, G. Ruget, *Résidues et dualité*, Invent. Math. **26** (1974), 89–131.
[89] F. Scattone, *On the compactification of moduli spaces for algebraic K3 surfaces*, Mem. Amer. Math. Soc **70** (1987), no. 374.
[90] H.-W. Schuster, *Locally free resolutions of coherent sheaves on surfaces*, J. Reine Angew. Math. **337** (1982), 159–165.
[91] P. Seidel, *Homological mirror symmetry for the quartic surface*, math.SG/0310414.
[92] P. Seidel, R. Thomas, *Braid group actions on derived categories of coherent sheaves*, Duke Math. J. **108** (2001), 37–108.
[93] J.P. Serre, *Géométrie algébrique et géométrie analytique*, Ann. Inst. Fourier **6** (1956), 1–42.
[94] Y.T. Siu, *Every* $K3$ *surface is Kähler*, Invent. Math. **73** (1983), 139–150.
[95] N. Spaltenstein, *Resolutions of unbounded complexes*, Compositio Math. **65** (1988), 121–154.
[96] P. Stellari, *Some remarks about the FM-partners of K3 surfaces with Picard numbers 1 and 2*, Geom. Dedicata **108** (2004), 1–13.
[97] B. Szendrői, *Diffeomorphisms and families of Fourier-Mukai transforms in mirror symmetry*, Applications of Alg. Geom. to Coding Theory, Phys. and Comp. NATO Science Series, Kluwer (2001), 317–337.
[98] D. Toledo, Y.L.L. Tong, *Green's theory of Chern classes and the Riemann-Roch formula*, The Lefschetz centennial conference, Part I (Mexico City, 1984), Contemp. Math. **58** (1986), 261–275.
[99] M. Verbitsky, *Coherent sheaves on general K3 surfaces and tori*, math.AG/0205210.
[100] M. Verbitsky, *Coherent sheaves on generic compact tori. Algebraic structures and moduli spaces*, CRM Proc. Lecture Notes **38** (2004), AMS, 229–247.
[101] J.-L. Verdier, *Des catégories dérivées des catégories abéliennes*, Astérisque **239** (1986).
[102] C. Voisin, *A counterexample to the Hodge conjecture extended to Kähler varieties*, Int. Math. Res. Not. **20** (2002), 1057–1075.
[103] J. Wehler, $K3$*-surfaces with Picard number* 2, Arch. Math. **50** (1988), 73–82.
[104] Yu.G. Zarhin, *Hodge groups of K3 surfaces*, J. Reine Angew. Math. **341** (1983), 193–220.

E.M.: Max-Planck-Institut für Mathematik, Vivatsgasse 7, 53111 Bonn, Germany
E-mail address: `macri@mpim-bonn.mpg.de`

P.S.: Dipartimento di Matematica "F. Enriques", Università degli Studi di Milano, Via Cesare Saldini 50, 20133 Milano, Italy
E-mail address: `stellari@mat.unimi.it`