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RAMANUJAN-STYLE CONGRUENCES FOR PRIME LEVEL

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Abstract. We establish Ramanujan-style congruences modulo certain primes ℓ between an Eisenstein series of weight k, prime level p and a cuspidal newform in the ε-eigenspace of the Atkin-Lehner operator inside the space of cusp forms of weight k for $\Gamma_0(p)$. Under a mild assumption, this refines a result of Gaba-Popa. We use these congruences and recent work of Ciolan, Languasco and the third author on Euler-Kronecker constants, to quantify the non-divisibility of the Fourier coefficients involved by ℓ. The degree of the number field generated by these coefficients we investigate using recent results on prime factors of shifted prime numbers.

1. Introduction

Let E_k be the Eisenstein series of even weight $k \geq 2$ for the group $SL_2(\mathbb{Z})$, normalized so that its Fourier series expansion is

$$E_k(z) = -\frac{B_k}{2k} + \sum_{n=1}^{\infty} \sigma_{k-1}(n) e^{2\pi i n z},$$

where B_k is the kth Bernoulli number and $\sigma_r(n) = \sum_{d|n} d^r$ is the r-th sum of divisors function. The prototype of a Ramanujan congruence goes back to 1916 and asserts that

$$\tau(n) \equiv \sigma_{11}(n) \pmod{691}, \tag{1.1}$$

for every positive integer n. This can be viewed as a (coefficient-wise) congruence between the unique cusp form $\Delta(z) = \sum_{n=1}^{\infty} \tau(n) e^{2\pi i n z}$ of weight 12 and the Eisenstein series $E_{12}(z)$, namely $\Delta \equiv E_{12} \pmod{691}$. There are several well-known ways to prove, interpret, and generalize this. For example, to higher weights eigenforms of level 1 by Datskovsky-Guerzhoy [DG96], to newforms of weight k and prime level p by Billerey-Menares [BM16], and to Fourier coefficients of index coprime to p by Dummigan-Fretwell [DF14]. The latter two authors were primarily motivated by an interesting relation of these congruences with the Bloch-Kato conjecture for the partial Riemann zeta function. Gaba-Popa [GP18] refined these results, by determining, under some technical conditions, also the Atkin-Lehner eigenvalue of the involved newform, and thus obtained congruences for all coefficients.

To make our statements more concrete, we first define for $\varepsilon \in \{\pm 1\}$ an Eisenstein series of even weight $k \geq 2$ and prime level p, namely

$$E_{k,p}^{\varepsilon}(z) := E_k(z) + \varepsilon E_k|W_p(z) = E_k(z) + \varepsilon p^{k/2} E_k(pz), \tag{1.2}$$

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where W_p is the Atkin-Lehner operator. By $M_k^{\varepsilon}(p)$ (resp. $S_k^{\varepsilon}(p)$) we denote the ε-eigenspace of the Atkin-Lehner operator W_p inside $M_k(p)$ (resp. $S_k(p)$), the space of modular forms (resp. cusp forms) of weight k and for the group $\Gamma_0(p)$. It is known that $E_{2,p}^{-1} \in M_2^{-1}(p)$ and $E_{k,p}^{\varepsilon} \in M_k^{\varepsilon}(p)$ for $k \geq 4$. Using the Fourier series expansion of E_k, we obtain

$$E_{k,p}^{\varepsilon}(z) = -\frac{B_k}{2k}\varepsilon(\varepsilon + p^{k/2}) + \sum_{n\geq 1}\left(\sigma_{k-1}(n) + \varepsilon p^{k/2}\sigma_{k-1}\left(\frac{n}{p}\right)\right)e^{2\pi i n z},$$

where $\sigma_{k-1}\left(\frac{n}{p}\right) = 0$ if $p \nmid n$. We now recall the main result of Gaba-Popa, the proof of which relies on the theory of period polynomials for congruence subgroups developed by Paşol and Popa [PP13].

Theorem 1.1. [GP18, Theorem 1] *Let $k \geq 4$ be an even integer, p a prime and $\varepsilon \in \{\pm 1\}$. Let $\ell \geq k+2$ be a prime such that*

$$\ell \mid \tfrac{B_k}{2k}(\varepsilon + p^{k/2}) \text{ and } \ell \mid (\varepsilon + p^{k/2})(\varepsilon + p^{k/2-1}).$$

In case $\ell \nmid (\varepsilon + p^{k/2})$ we assume in addition that there exists an even integer n with $0 < n < k$ such that $\ell \nmid B_n B_{k-n}(p^{n-1} - 1)$. Then, there exists a newform $f \in S_k^{\varepsilon}(p)$ and a prime ideal λ lying above ℓ in the coefficient field of f such that

$$f \equiv E_{k,p}^{\varepsilon} \pmod{\lambda}.$$

We remark that from the latter congruence it follows that if $\overline{\rho}_{f,\lambda}$ denotes the mod λ Galois representation, then (up to semisimplification) $\overline{\rho}_{f,\lambda} \simeq 1 \oplus \chi_\ell^{k-1}$, where χ_ℓ is the mod ℓ cyclotomic character. Therefore, Theorem 1.1 gives a sufficient conditon on the prime ℓ such that the representation $1 \oplus \chi_\ell^{k-1}$ arises from a newform in $S_k^{\varepsilon}(p)$.

The purpose of this paper is to strengthen Theorem 1.1 and, on a somewhat different note, to quantify the non-divisibility of the Fourier coefficients of $E_{k,p}^{\varepsilon}$. The corresponding results are presented in the next section, respectively in Section 1.2.

1.1. **Strengthening of Theorem 1.1.** We sharpen Theorem 1.1 in the next two theorems.

Theorem 1.2. *Let $k \geq 2$ be an even integer, p a prime and $\varepsilon \in \{\pm 1\}$. If $k = 2$, we also assume that $\varepsilon = -1$. Let $\ell \geq \max\{5, k-1\}$ be a prime such that $p \not\equiv -1 \pmod{\ell}$. Then the following are equivalent:*

1. *$\ell \mid \frac{B_k}{2k}(\varepsilon + p^{k/2})$ and $\ell \mid (\varepsilon + p^{k/2})(\varepsilon + p^{k/2-1})$;*
2. *the existence of a newform $f \in S_k^{\varepsilon}(p)$ and a prime ideal λ lying above ℓ in the coefficient field of f such that*

$$f \equiv E_{k,p}^{\varepsilon} \pmod{\lambda}.$$

This result improves on Theorem 1.1 in three different aspects.

(a) Instead of $\ell > k+1$, now also $\ell = k \pm 1$ is allowed. Gaba-Popa [GP18] pointed out that, based on several numerical examples, they expect that their result should hold even for $\ell = k \pm 1$, although their method breaks down for these values of ℓ. Therefore, it is reasonable to expect that [BM16, Conjecture 3.2] and [GP18, Conjecture on p. 53] should also hold for $\ell = k \pm 1$.

(b) Taking $k = 2$ is allowed and hence this recovers an earlier result of Mazur [Maz77, Proposition 5.12 (ii)].

(c) There is no condition on $B_n B_{k-n}(p^{n-1} - 1)$ anymore in case $\ell \nmid (\varepsilon + p^{k/2})$.

Comparing Theorem 1.2 with Theorem 1.1, we see that there is now the extra condition $p \not\equiv -1 \pmod{\ell}$ (redundant for $k = 2$). In some special cases we remove this condition, together with the assumption $\ell \geq k - 1$, by proving the following variant of Theorem 1.2.

Theorem 1.3. *Let $k \geq 2$ be even integer, $\ell \geq 5$ and p be primes and $\varepsilon \in \{\pm 1\}$. If $k = 2$, we also assume that $\varepsilon = -1$. Suppose that $\ell \mid \frac{B_k}{2k}(\varepsilon + p^{k/2})$. We further assume that $k \not\equiv 0 \pmod{\ell - 1}$ and $\ell \nmid \frac{B_k}{2k}$. Then there exists a newform $f \in S_k^{\varepsilon}(p)$ and a prime ideal λ over ℓ in the coefficient field of f such that*

$$f \equiv E_{k,p}^{\varepsilon} \pmod{\lambda}.$$

Our proof of the above theorems uses some classical results from the theory of mod ℓ modular forms and Deligne's theorem on Galois representations attached to eigenforms. Further, it is based on the ideas used in [DF14], is quite classical in nature, and completely avoids the use of period polynomials. More precisely, we first prove that the assumptions on ℓ ensure that the reduction of $E_{k,p}^{\varepsilon}$ modulo ℓ is a cuspidal eigenform in characteristic ℓ, and then using the Deligne-Serre lifting lemma we lift it to an eigenform in characteristic zero. In the final step we apply the Diamond-Ribet level raising theorem and a result of Langlands to obtain the desired newform.

Next using some elementary ideas, we establish the following result in which the resulting cusp form may not be an eigenform as before, but it will always have rational Fourier coefficients.

Theorem 1.4. *Let $k \geq 2$ be an even integer, p a prime and $\varepsilon \in \{\pm 1\}$. If $k = 2$, we also assume that $\varepsilon = -1$. Let $N_{k,p}^{\varepsilon}$ be the reduced numerator of $\frac{B_k}{2k}(\varepsilon + p^{k/2})$. Suppose that at least one of the following conditions hold:*

(a) $p \in \{2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 41, 47, 59, 71\}$.

(b) $k \geq 8$, $k \equiv 1 - \varepsilon \pmod{4}$ *and* $N_{k,p}^{-1}$ *is coprime to* $p - 1$.

(c) $k \geq 10$ *and* $k \equiv 1 - \varepsilon \pmod{10}$, *and* $N_{k,p}^{\varepsilon}$ *is coprime to* $(p + \varepsilon)p(p + 1)$.

Then there exists a non-zero cusp form $f \in S_k^{\varepsilon}(p)$ with rational Fourier coefficients such that

$$f \equiv E_{k,p}^{\varepsilon} \pmod{N_{k,p}^{\varepsilon}}.$$

Remark 1.5. The primes p in (a) are exactly those for which the genus of the Fricke group of level p is zero. They are also precisely the prime factors of $2^{46} \cdot 3^{20} \cdot 5^9 \cdot 7^6 \cdot 11^2 \cdot 13^3 \cdot 17 \cdot 19 \cdot 23 \cdot 29 \cdot 31 \cdot 41 \cdot 47 \cdot 59 \cdot 71$, the order of the Monster group. That is not a coincidence! For more details, see, e.g., Gannon [GaA06, GaB06].

Using Theorem 1.4 one can prove Theorem 1.6, which gives congruences even for small primes, namely for $\ell = 2$ and 3. The proof is completely patterned after the proof of [DG96, Lemma 2.1, Theorem 2] and so we omit it here. It rests on the fact that the set

$$\mathcal{B} := \{f + \varepsilon f|W_p : f \in S_k(1) \text{ is a normalized eigenform}\} \cup \{g : g \in S_k^{\varepsilon}(p) \text{ newform}\}$$

forms a basis of $S_k^{\varepsilon}(p)$ consisting of normalized Hecke eigenfunctions for all T_q for $q \neq p$.

Theorem 1.6. *Suppose at least one of the conditions $(a), (b), (c)$ of Theorem 1.4 holds. Let $\mathcal{B}$ be a basis of $S_k^{\varepsilon}(p)$ of normalized Hecke eigenfunctions for all T_q for $q \neq p$. Suppose some prime ideal λ divides $N_{k,p}^{\varepsilon}$ in the coefficient field $\mathbb{Q}(a_f(q) : f \in \mathcal{B},\, q \neq p)$. Then there exists a cusp form $f = \sum_{n \geq 1} a_f(n) e^{2\pi i n z} \in \mathcal{B}$, such that for all integers n coprime to p, we have*

$$a_f(n) \equiv \sigma_{k-1}(n) \pmod{\lambda}.$$

As an application of our results (especially of Theorem 4.1), we give a non-trivial lower bound of the degree of the number field generated by all normalized eigenforms (and newforms) in the space $S_k(p)$, see Section 8. These bounds improve a similar result of [BM16] and are valid in a subset of the primes with natural density close to one.

1.2. **Quantification of Fourier coefficient non-divisibility.** The second goal of this article is to quantify how often $\ell \nmid a(n)$ for certain prime numbers ℓ and Fourier coefficients $a(n)$. This problem was first considered by Ramanujan for his tau function (in [BO01] that remained unpublished for many years). He made various claims of the form

$$\sum_{n \leq x,\, \ell \nmid \tau(n)} 1 = C_\ell \int_2^x \frac{dt}{(\log t)^{\delta_\ell}} + O\left(\frac{x}{(\log x)^r}\right), \tag{1.3}$$

that he thought to be valid for arbitrary r. For example, he claimed (1.3) for $\ell = 691$ and $\delta_\ell = 1/690$. Partial integration gives

$$C_\ell \int_2^x \frac{dt}{(\log t)^{1/\delta_\ell}} = \frac{C_\ell\, x}{(\log x)^{1/\delta_\ell}} \left(1 + \frac{1}{\delta_\ell \log x} + O_\ell\left(\frac{1}{(\log x)^2}\right)\right). \tag{1.4}$$

The functions

$$\frac{C_\ell\, x}{(\log x)^{1/\delta_\ell}} \quad \text{and} \quad C_\ell \int_2^x \frac{dt}{(\log t)^{1/\delta_\ell}},$$

are now called the *Landau approximation*, respectively *Ramanujan approximation* of the counting function in (1.3), the true behavior of which is, see Serre [Ser76],

$$\sum_{n \leq x,\, \ell \nmid \tau(n)} 1 = \frac{C_\ell\, x}{(\log x)^{1/\delta_\ell}} \left(1 + \frac{1 - \gamma_{\tau;\ell}}{\delta_\ell \log x} + O_\ell\left(\frac{1}{(\log x)^2}\right)\right), \tag{1.5}$$

with $\gamma_{\tau;\ell}$ a constant sometimes called *Euler-Kronecker constant.* Note that if $\gamma_{\tau;\ell} > 1/2$ the Landau approximation asymptotically gives a better approximation to $T_\ell(x)$ than the Ramanujan approximation, and that if $\gamma_{\tau;\ell} < 1/2$ it is the other way around. Comparing (1.4) and (1.5) we see that Ramanujan's claim (1.3) entails $\gamma_{\tau;\ell} = 0$. For $\ell = 3, 5, 7, 23$ and 691 this was disproved by Moree [Mor04]. For the true value of these numerical constants see Table 1 (data taken from [CLM21]).

Table 1: Euler-Kronecker constants $\gamma_{\tau;\ell}$

ℓ	$\gamma_{\tau;\ell}$	winner
3	$+0.534921\ldots$	Landau
5	$+0.399547\ldots$	Ramanujan
7	$+0.231640\ldots$	Ramanujan
23	$+0.216691\ldots$	Ramanujan
691	$+0.571714\ldots$	Landau

Let ℓ be an odd prime. An arithmetic function $\mathfrak{f}$ assuming integer values has a *refined ℓ-non-divisibility asymptotic* with Euler-Kronecker constant $\gamma_{\mathfrak{f};\ell}$ if there exist positive constants C_ℓ and h_1 such that

$$\sum_{n\le x,\, \ell\nmid \mathfrak{f}(n)} 1 = \frac{C_\ell\, x}{(\log x)^{1/h_1}}\left(1+\frac{1-\gamma_{\mathfrak{f};\ell}}{h_1\log x}+o_{\mathfrak{f}}\left(\frac{1}{\log x}\right)\right), \tag{1.6}$$

where the implicit constant in the error term may depend on $\mathfrak{f}$.

Theorem 1.7. *Let $\mathfrak{f}$ be an integer valued multiplicative function. If*

$$\#\{p_1\le x : p_1 \text{ prime and } \ell \mid \mathfrak{f}(p_1)\} = \delta \sum_{p_1\le x} 1 + O_{\mathfrak{f}}\Big(\frac{x}{(\log x)^{2+\rho}}\Big), \tag{1.7}$$

for some real numbers $\rho>0$ and $0<\delta<1$, then (1.6) holds with $h_1 = 1/\delta$ for some positive constant C_ℓ.

Corollary 1.8. *Let $m\ge 1$ be an integer and ℓ an odd prime such that $h_2 := (\ell-1)/(\ell-1,m)$ is even. Then the m-th sum of divisors function σ_m has a refined ℓ-non-divisibility asymptotic with $h_1 = h_2$ for some positive constant C_ℓ.*

The proof of the corollary is left as an exercise, cf. [CLM21]. We remark that in case h_2 is odd, (1.6) takes a more trivial form.

Theorem 1.9. *Let $k\ge 4$ be an even integer, p a prime and $\varepsilon\in\{\pm1\}$. Let $\ell\ge 5$ be a prime such that $\varepsilon p^{k/2}\equiv -1 \pmod{\ell}$. Set $r = gcd(\ell-1,k-1)$. Let g_1 be the multiplicative order of p^r modulo ℓ. Put $\mu_p=\ell$ if $g_1=1$ and $\mu_p = g_1$ otherwise. Then $\mathfrak{f}(n)=\sigma_{k-1}(n)+\varepsilon p^{k/2}\sigma_{k-1}(n/p)$ has a refined ℓ-non-divisibility asymptotic (1.6) with $h_1=(\ell-1)/r$, and Euler-Kronecker constant*

$$\gamma_{\mathfrak{f};\ell} = \gamma_{\sigma_{k-1};\ell} + \left(\frac{\mu_p}{p^{\mu_p}-1} - \frac{(\mu_p-1)}{p^{\mu_p-1}-1}\right)\log p, \tag{1.8}$$

provided that h_1 is even.

This result reduces the study of $\gamma_{\mathfrak{f};\ell}$ to that of $\gamma_{\sigma_{k-1};\ell}$, which was studied in extenso by Ciolan et al. [CLM21]. They gave a (long and involved) formula for this Euler-Kronecker constant that allows one to evaluate it with a certified accuracy of several decimals. The relevant computer programs are made available at www.math.unipd.it/~languasc/CLM.html.

1.3. **Plan for the remainder of the article.** In Section 2 we revisit some basic preliminaries needed to prove Theorem 1.2, such as Hecke operators, Artin-Lehner newforms theory, mod ℓ modular forms and ℓ-adic Galois representations associated to modular forms. In Section 3 we give a proof of Theorem 1.2 by assuming Theorem 4.1 (proven in Section 4 along with a variant of it). In Section 5, we prove Theorem 1.4 followed by a discussion of several interesting numerical examples in Section 6. In Section 8, as an application of our results, we give a non-trivial lower bound for the degree of the field of coefficients of any normalized eigenform of fixed weight and level, with Section 7 recalling some relevant results on large prime factors of shifted primes. Finally, in Section 9, we prove Theorem 1.9.

2. Required Preliminaries

2.1. **Notation.** The letters p, p_1, q and ℓ will denote prime numbers throughout, except in Section 6, where $q = e^{2\pi i z}$. For a rational number $\frac{m}{n}$, by $\ell \mid \frac{m}{n}$, we mean that ℓ divides the reduced numerator of $\frac{m}{n}$. Given a newform f we denote its n-th Fourier coefficient by $a_f(n)$ and its *coefficient field* $\mathbb{Q}(a_f(n) : n \geq 1)$ by K_f. We say two forms f and g are *congruent* mod ℓ or mod λ if $a_f(n)$ is congruent to $a_g(n)$ for every integer n. For notational convenience, we also abbreviate $M_k^{\pm 1}(p)$, $S_k^{\pm 1}(p)$ and $E_{k,p}^{\pm 1}$ by $M_k^{\pm}(p)$, $S_k^{\pm}(p)$ and $E_{k,p}^{\pm}$, respectively.

2.2. **Atkin-Lehner operators and newform theory.** Let $M_k(N)$ and $S_k(N)$ be the $\mathbb{C}$-vector space of modular forms and cusp forms, respectively of even weight $k \geq 2$ and level N with respect to $\Gamma_0(N)$ of trivial nebentypus. These spaces have actions of the Hecke operators $T_1, T_2, \ldots$ which satisfy the following relations: $T_1 = 1$, $T_{mn} = T_m T_n$ if $(m, n) = 1$ and for prime powers q^r with $q \nmid N$ we have the recurrence

$$T_{q^r} = T_q T_{q^{r-1}} - q^{k-1} T_{q^{r-2}}.$$

For a prime $p \mid N$, the action of T_p (we will denote it by U_p) on $f \in M_k(N)$ is given by $a_{T_p f}(n) = a_f(np)$ and such an operator T_p is generally called an U_p *operator*. Next, we recall some newform theory from [AL70]. A modular form $f \in M_k(N)$ is called a *Hecke eigenform* if it is an eigenfunction for all the Hecke operators T_q for $(q, N) = 1$ and U_p for all $p \mid N$. It is a well-known result that if f is a Hecke cusp eigenform, then $a_f(1) \neq 0$. We say such f is *normalized* if $a_f(1) = 1$.

Suppose that $p \mid N$, but $p^2 \nmid N$. Then there are two ways to embed $S_k(N/p)$ inside $S_k(N)$; one by the identity and the other $f(z) \mapsto f(pz)$ which give rise to a map

$$S_k(N/p) \oplus S_k(N/p) \to S_k(N) \text{ defined by } (f, g) \mapsto f(z) + g(pz).$$

The image of this map is called the *space of p-oldforms* in $S_k(N)$, and is denoted by $S_k(N)^{p-\mathrm{old}}$. The orthogonal complement of $S_k(N)^{p-\mathrm{old}}$ in $S_k(N)$ with respect to the Petersson inner product is called the *space of p-newforms*, and denoted by $S_k(N)^{p-\mathrm{new}}$. Finally, in case N is squarefree, we define the *space of newforms* $S_k(N)^{\mathrm{new}}$ as the intersection $\bigcap_{p \mid N} S_k(N)^{p-\mathrm{new}}$. A normalized Hecke eigenform f in $S_k(N)^{\mathrm{new}}$ is called a *newform*.

Let W_p be the *Atkin-Lehner operator* on $M_k(p)$ defined by

$$f|W_p(z) = p^{-k/2} z^{-k} f\Big(\frac{-1}{pz}\Big).$$

It preserves the space $M_k(p)$ and $S_k(p)$ and also since it is an involution its eigenvalue ε is in $\{\pm 1\}$. Next we state some standard facts about the operators T_q $(q \neq p)$, U_p, and W_p and newforms for the space $S_k(p)$ that can be, for example, found in [AL70, Lemma 17, Theorem 5].

Lemma 2.1. *We have the following.*

(1) *Both $\{T_q, U_p : q \neq p\}$ and $\{T_q, W_p : q \neq p\}$ are commuting families of operators.*
(2) *$f \in S_k(p)$ is a newform if and only if it is an eigenfunction for all T_q $(q \neq p)$, U_p and W_p.*
(3) *If $f \in S_k(p)^{\mathrm{new}}$ is a newform with Atkin-Lehner eigenvalue ε, then $a_f(p) = -\varepsilon p^{k/2-1}$.*

2.3. **Modular forms with coefficients in a ring A.** Let $M_k(N, \mathbb{Z}) \subset \mathbb{Z}[\![q]\!]$ denote the set of elements of $M_k(N)$ having integer Fourier coefficients at the cusp infinity. For a commutative ring A, we define

$$M_k(N, A) = M_k(N, \mathbb{Z}) \otimes_{\mathbb{Z}} A.$$

By the q-expansion principle, the map $M_k(N, A) \to A[\![q]\!]$ is injective, so we may view $M_k(N, A)$ as a submodule of $A[\![q]\!]$. Note that $S_k(N, \mathbb{Z}) = M_k(N, \mathbb{Z}) \cap S_k(N)$. Hence we can define $S_k(N, A)$ similarly, and we identify it with an A-submodule of $M_k(N, A)$.

The Hecke operators T_n defined earlier also act on the space $M_k(N, A)$ with the small modification that the action of T_ℓ on $M_k(N, A)$ coincides with the action of U_ℓ if A is a domain of positive characteristic r and $\ell \mid r$.

2.4. **Mod ℓ modular forms.** For a prime ℓ, let $\overline{\mathbb{F}}_\ell$ denote an algebraic closure of the finite field $\mathbb{F}_\ell$, with ℓ elements. In this section we recall the notion of modular forms with coefficients in $\overline{\mathbb{F}}_\ell$ (see [Ser87, Section 3.1]).

Fix an embedding $\iota_\ell : \overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_\ell$ in particular we have $\overline{\mathbb{Z}} \hookrightarrow \overline{\mathbb{Z}}_\ell$. Therefore the ring $\overline{\mathbb{Z}}_\ell$ has a natural reduction map to its residue field $\overline{\mathbb{F}}_\ell$ and we obtain a homomorphism

$$\overline{\mathbb{Z}}_\ell \to \overline{\mathbb{F}}_\ell \text{ defined by } a \mapsto \overline{a}.$$

For $k \geq 2$ and an integer N, coprime to ℓ, we define the space of modular forms of type (N, k) with coefficients in $\overline{\mathbb{F}}_\ell$, denoted by $M_k(N, \overline{\mathbb{F}}_\ell)$, consisting of formal power series

$$F(z) = \sum_{n \geq 1} A_n e^{2\pi i n z}, \ A_n \in \overline{\mathbb{F}}_\ell,$$

for which there exists a modular form $f(z) = \sum_{n\geq 1} a_n e^{2\pi i n z} \in M_k(N), \ a_n \in \overline{\mathbb{Z}}$, such that $\overline{a}_n = A_n$ for all $n \geq 1$. The space $S_k(N, \overline{\mathbb{F}}_\ell)$ is defined analogously.

As mentioned in Section 2.3, we have the action of the Hecke algebra generated by the operators T_q, $q \nmid \ell N$ and U_p, $p \mid \ell N$ on the space $M_k(N, \overline{\mathbb{F}}_\ell)$, they also preserve $S_k(N, \overline{\mathbb{F}}_\ell)$. Observe that by the Deligne-Serre lifting lemma if $F \in S_k(N, \overline{\mathbb{F}}_\ell)$ is a non-zero normalized Hecke eigenform then F is the reduction mod ℓ of some normalized Hecke eigenform $f \in S_k(N, \overline{\mathbb{Z}}_\ell)$.

We say $F \in S_k(N, \overline{\mathbb{F}}_\ell)$ is an *eigenfunction for the Atkin-Lehner operator* W_N with eigenvalue ε if it is a reduction of $f \in S_k(N, \overline{\mathbb{Z}}_\ell)$ which is an eigenfunction for W_N with eigenvalue ε. Notice that this is a well-defined operator on $M_k(N, \overline{\mathbb{F}}_\ell)$ as we know that if f and f' are characteristic zero modular forms of the same weight and level N that are congruent modulo ℓ then $W_N f$ and $W_N f'$ are congruent modulo ℓ as well.

2.5. **Galois representations attached to modular forms.** In this section, we briefly recall some standard facts about 2-dimensional Galois representations of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ associated to Hecke eigenforms.

Let $f(z) = \sum_{n \geq 1} a_f(n) e^{2\pi i n z}$ be a normalized Hecke eigenform of weight k and level N. It is well-known that the Fourier coefficients $a_f(n)$ belong to the ring of integers $\mathcal{O}_{K_f}$ of a finite extension field K_f of $\mathbb{Q}$. For a given prime ℓ, due to a theorem of Deligne, corresponding to such an eigenform f and a prime ideal λ above ℓ in K_f, there is a continuous ℓ-adic Galois representation

$$\rho_{f,\lambda} : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to GL(2, K_{f,\lambda}),$$

where $K_{f,\lambda}$ is the completion of K_f at the place λ. The representation $\rho_{f,\lambda}$ is irreducible, unique up to isomorphism and it is unramified outside the primes dividing N and the norm of λ, and has the following properties:

$$\mathrm{tr}(\rho_{f,\lambda}(\mathrm{Frob}_q)) = a_f(q) \text{ and } \det(\rho_{f,\lambda}(\mathrm{Frob}_q)) = q^{k-1},$$

where $\mathrm{Frob}_q \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ is the Frobenius element at the prime q. Conjugating by a matrix in $GL(2, K_{f,\lambda})$, one can assume that the image of $\rho_{f,\lambda}$ lands inside $GL(2, \mathcal{O}_{K_{f,\lambda}})$. Reducing this representation with values in $GL(2, \mathcal{O}_{K_{f,\lambda}})$ modulo λ, we get a mod-ℓ representation of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$

$$\overline{\rho}_{f,\lambda} : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to GL(2, \mathcal{O}_{K_f}/\lambda).$$

The representation $\overline{\rho}_{f,\lambda}$ is well defined up to semi-simplification and depends only on the cusp form f modulo λ.

2.6. **Diamond-Ribet level raising theorem and a refinement.** We now recall the following celebrated result of Ribet [Rib90] (for weight two and trivial character) and Diamond [Dia91] (for higher weight and non-trivial characters), called *level raising theorem*, which gives a criterion for the existence of a congruence between two newforms of the same weight, but of different level. This plays an important role in the proof of our results.

Theorem 2.2 (Diamond-Ribet level raising theorem)**.** *Let $g \in S_k(N)$ be a newform of weight $k \geq 2$ and let p and ℓ be distinct primes not dividing N with $\ell \nmid \frac{1}{2}\varphi(N)Np(k-2)!$. Let λ be a prime ideal above ℓ in the field generated by the eigenvalues of all eigenforms in $S_k(Np)$ and $S_k(N)$. Then the following are equivalent:*

(1) $a_g(p)^2 \equiv p^{k-2}(1+p)^2 \pmod{\lambda}$.

(2) *There exists a p-newform $f \in S_k(Np)$ such that for every prime q coprime to pN,*

$$a_f(q) \equiv a_g(q) \pmod{\lambda}.$$

In [GP18, Theorem 2], Gaba-Popa obtained a refinement of Diamond's level raising theorem and their proof is (loosely speaking) a part of the proof of their main result. In the same vein, we record the following refinement of the above theorem which also strengthens [GP18, Theorem 2] under a mild assumption.

Theorem 2.3. *Let $k \geq 2$ be an even integer, p a prime, N a positive integer coprime to p and $\varepsilon \in \{\pm 1\}$. If $k = 2$ we assume furthermore that $\varepsilon = -1$. Suppose $g(z) = \sum_{n\geq 1} a_g(n)e^{2\pi i n z} \in S_k(N)$ is a newform. Let $\ell \geq k-1$ be a prime such that $p \not\equiv -1 \pmod{\ell}$ and $\ell \nmid N\phi(N)$ and let λ be a prime ideal above ℓ in the field generated by the eigenvalues of all eigenforms in $S_k(Np)$ and $S_k(N)$. Then the following are equivalent:*

(1) $a_g(p) \equiv -\varepsilon p^{k/2-1}(1+p) \pmod{\lambda}$.

(2) *There exists an eigenform $f \in S_k^{\varepsilon}(Np)$ which is new at p such that*

$$f(z) \equiv g(z) + \varepsilon p^{k/2} g(pz) \pmod{\lambda}.$$

Proof. We omit the proof because for $N = 1$ its proof is the content of the second half of the proof of Theorem 1.2 (c.f. (3.4)), and for general N the same arguments apply. ∎

3. Proof of Theorem 1.2

We give a proof of Theorem 1.2 using Theorem 4.1 (the proof of which is given in the next section). We first prove that (1) implies (2). The assumptions on ℓ, in view of Theorem 4.1, ensure that there is a normalized eigenform $h(z) = \sum_{n\geq 1} a_h(n)e^{2\pi i n z} \in S_k(p)$ and a prime ideal λ above ℓ in K_h such that

$$h \equiv E^{\varepsilon}_{k,p} \pmod{\lambda}. \tag{3.1}$$

We now prove that this eigenform h can be replaced by a newform under our assumptions on ℓ and k. We distinguish between two cases:

Case (i) *The eigenform h is a newform.* We will show that the W_p-eigenvalue of h is ε. Writing $h|W_p = \delta h$ with $\delta \in \{\pm 1\}$, we obtain $a_h(p) = -\delta p^{k/2-1}$ by Lemma 2.1. Now by considering the p-th Fourier coefficients of both functions appearing in (3.1), and using the fact that $\ell \mid (\varepsilon + p^{k/2})(\varepsilon + p^{k/2-1})$, we obtain

$$-\delta p^{k/2-1} \equiv 1 + p^{k-1} + \varepsilon p^{k/2} \equiv -\varepsilon p^{k/2-1} \pmod{\ell}.$$

Since the prime ℓ is odd and different from p, this proves that $h \in S_k^{\varepsilon}(p)$.

Case (ii) *The eigenform h is not a newform.* Then there is a level 1 eigenform g such that the corresponding ℓ-adic Galois representations $\rho_{h,\Lambda}$ and $\rho_{g,\Lambda}$ are the same, where Λ is a prime ideal above ℓ in the compositum of coefficient fields of all normalized eigenforms in $S_k(p)$ and $S_k(1)$. By

(3.1), we have $a_h(q) \equiv 1 + q^{k-1} \pmod{\Lambda}$ for all $q \neq p$. A standard application of the Chebotarev density theorem then shows that $\bar{\rho}_{h,\Lambda}$ is isomorphic to $1 \oplus \chi_\ell^{k-1}$, where χ_ℓ denotes the mod ℓ cyclotomic character. Thus, we conclude that

$$\bar{\rho}_{g,\Lambda} \simeq 1 \oplus \chi_\ell^{k-1}. \tag{3.2}$$

Because g is of level 1, the representation $\bar{\rho}_{g,\Lambda}$ is unramified outside ℓ. In particular, $\bar{\rho}_{g,\Lambda}$ and χ_ℓ are unramified at the prime p. Taking the trace of the image of Frob_p on both sides of (3.2) yields

$$a_g(p) \equiv 1 + p^{k-1} \pmod{\Lambda}. \tag{3.3}$$

Now using that ℓ divides $1 + p^{k-1} + \varepsilon p^{k/2} + \varepsilon p^{k/2-1}$, we infer that

$$a_g(p) \equiv -\varepsilon p^{k/2-1}(1+p) \pmod{\Lambda}, \tag{3.4}$$

which gives

$$a_g(p)^2 \equiv p^{k-2}(1+p)^2 \pmod{\Lambda}.$$

Hence, by apply Theorem 2.2 and using the hypothesis $\ell > k-2$, we obtain a newform $f \in S_k(p)$ for which

$$a_f(q) \equiv a_g(q) \pmod{\Lambda}, \text{ for all } q \neq p.$$

Taken together with (3.3) this results in

$$a_f(q) \equiv 1 + q^{k-1} \pmod{\Lambda}, \text{ for all } q \neq p. \tag{3.5}$$

Let δ be the W_p-eigenvalue of f and so $a_f(p) = -\delta p^{k/2-1}$, from Lemma 2.1. In order to complete the proof of Theorem 1.2, we only need to show that $\delta = \varepsilon$. The reason is that if $\delta = \varepsilon$, then $a_f(p) = -\varepsilon p^{k/2-1} \equiv 1 + p^{k-1} + \varepsilon p^{k/2} \pmod{\Lambda}$. Combining this with (3.5) and using that $E_{k,p}^{\varepsilon}$ mod ℓ is an eigenform, which has been established in the course of the proof of Theorem 4.1, gives that $f \equiv E_{k,p}^{\varepsilon} \pmod{\Lambda}$, thus completing the proof (note that we can restrict Λ to K_f to get the required prime ideal above ℓ in K_f).

Let G_p denote the decomposition group of the absolute Galois group $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ at a place over p. Denote, for any algebraic integer α, the unique unramified character $G_p \to \overline{\mathbb{F}}_\ell^\times$ sending the arithmetic Frobenius Frob_p to α mod ℓ by μ_α. It follows from the work of Langlands [Lan73] (see also [LW12, Proposition 2.8 (2)]) that the restriction of $\bar{\rho}_{f,\Lambda}$ to G_p is given by

$$\bar{\rho}_{f,\Lambda}|_{G_p} \simeq \begin{pmatrix} \chi_\ell^{k/2} & * \\ & \chi_\ell^{k/2-1} \end{pmatrix} \otimes \mu_{a_f(p)/p^{k/2-1}}.$$

Since μ_α and χ_ℓ are unramified at p, one can consider the trace of Frob_p on the right hand side and hence $\mathrm{tr}(\bar{\rho}_{f,\Lambda}(\mathrm{Frob}_p))$ is well-defined. Since $\bar{\rho}_{f,\Lambda} \simeq \bar{\rho}_{g,\Lambda}$ (up to semisimplification), we have $\bar{\rho}_{f,\Lambda}|_{G_p} \simeq \bar{\rho}_{g,\Lambda}|_{G_p}$. Taking the trace of the image of Frob_p yields

$$\left(p^{k/2} + p^{k/2-1}\right) \frac{a_f(p)}{p^{k/2-1}} \equiv a_g(p) \pmod{\Lambda}.$$

The congruence (3.3) together with $a_f(p) = -\delta p^{k/2-1}$ gives

$$-\delta(p^{k/2} + p^{k/2-1}) \equiv 1 + p^{k-1} \equiv -\varepsilon(p^{k/2} + p^{k/2-1}) \pmod{\Lambda},$$

which yields $\ell \mid (\delta - \varepsilon)p^{k/2-1}(p+1)$. Since by assumption $(\ell, p(p+1)) = 1$, we infer from this that $\delta = \varepsilon$, and so (1) implies (2).

It remains to show that (2) implies (1). Let $\ell \geq 5$ be a prime for which there exists a newform $f(z) = \sum_{n\geq 1} a_f(n)e^{2\pi i n z} \in S_k^{\varepsilon}(p)$ such that

$$f \equiv E_{k,p}^{\varepsilon} \pmod{\lambda}, \tag{3.6}$$

for some prime ideal λ above ℓ in K_f. Since the constant term of f is zero and the norm of λ is a power of ℓ, we get $\ell \mid \frac{B_k}{2k}(\varepsilon + p^{k/2})$. The fact that f is a newform with W_p-eigenvalue ε, along with Lemma 2.1 yields $a_f(p) = -\varepsilon p^{k/2-1}$. Taken together with the congruence (3.6) at the prime p, this leads to $-\varepsilon p^{k/2-1} \equiv 1 + p^{k-1} + \varepsilon p^{k/2} \pmod{\lambda}$. Since λ is a prime ideal above ℓ, this completes the proof. $\square$

4. Variants of Theorem 1.2 and proof of Theorem 1.3

As promised in the previous section, we now give a proof of Theorem 4.1. This result may be of independent interest because of the limited assumptions on ℓ when compared with Theorem 1.2. Recall that the proof of Theorem 1.3 is an immediate consequence of Theorem 4.1. We also remind the reader that by "eigenform" we mean an eigenfunction of all the Hecke operators T_n, $n \geq 1$.

Theorem 4.1. *Let $k \geq 2$ be even integer, $\ell \geq 5$ and p be primes and $\varepsilon \in \{\pm 1\}$. If $k = 2$, we also assume that $\varepsilon = -1$. Suppose that ℓ divides both $\frac{B_k}{2k}(\varepsilon + p^{k/2})$ and $(\varepsilon + p^{k/2})(\varepsilon + p^{k/2-1})$. Then there exists a normalized eigenform $h \in S_k(p)$ and a prime ideal λ over ℓ in the coefficient field of h such that*

$$h \equiv E_{k,p}^{\varepsilon} \pmod{\lambda}.$$

Moreover, if $k = 2$, then $h \in S_2^-(p)$ is a newform.

Proof. We observe that $E_{k,p}^{\varepsilon}|W_p = \varepsilon E_{k,p}^{\varepsilon}$ and that W_p interchanges both the cusps of $\Gamma_0(p)$, which implies that the constant term of $E_{k,p}^{\varepsilon}$ at both the cusps is $-\frac{B_k}{2k}\varepsilon(\varepsilon + p^{k/2})$, up to a sign and powers of p. Since $\ell \mid \frac{B_k}{2k}(\varepsilon + p^{k/2})$, it follows from the q-expansion principle (here $q = e^{2\pi i z}$) that the reduction of $E_{k,p}^{\varepsilon}$ modulo ℓ gives rise to an element $\overline{E}_{k,p}^{\varepsilon} \in S_k^{\varepsilon}(p, \mathbb{F}_\ell) \subset S_k(p, \overline{\mathbb{F}}_\ell)$. As E_k is an eigenfunction for all the Hecke operators T_q $(q \neq p)$, all of which commute with W_p, we see that $E_{k,p}^{\varepsilon}$, hence $\overline{E}_{k,p}^{\varepsilon}$ is a common eigenfunction of all T_q $(q \neq p)$. We next claim that the assumptions on the prime ℓ ensure that $\overline{E}_{k,p}^{\varepsilon}$ is also an eigenfunction of the operators U_p. For $k = 2$, it is easy to see that $U_p E_{2,p}^- = E_{2,p}^-$ which shows that $E_{2,p}^-$, and so in particular $\overline{E}_{2,p}^-$, is an eigenfunction for U_p with eigenvalue 1. For $k \geq 4$, a simple computation gives that if $a(n)$ and $b(n)$ are the nth Fourier coefficients of $E^{\varepsilon}_{k,p}(z)$, respectively $U_pE^{\varepsilon}_{k,p}(z)$, then

$$b(n)=\begin{cases}a(0) & \text{if } n=0;\\ (1+p^{k-1}+\varepsilon p^{k/2})a(n) & \text{if } p\nmid n;\\ (1+p^{k-1}+\varepsilon p^{k/2})a(n)-\varepsilon\sigma_{k-1}(n/p)p^{k/2}(\varepsilon+p^{k/2})(\varepsilon+p^{k/2-1}) & \text{otherwise.}\end{cases}$$

It shows that $E^{\varepsilon}_{k,p}$ is not an eigenfunction of U_p, but since by assumption $\ell \mid \frac{B_k}{2k}(\varepsilon+p^{k/2})$ and $\ell\mid(\varepsilon+p^{k/2})(\varepsilon+p^{k/2-1})$, it follows that

$$U_pE^{\varepsilon}_{k,p}(z)\equiv(1+p^{k-1}+\varepsilon p^{k/2})E^{\varepsilon}_{k,p}(z) \pmod{\ell}. \tag{4.1}$$

In other words, $\overline{E}^{\varepsilon}_{k,p}$ is an eigenfunction of U_p with eigenvalue $1+p^{k-1}+\varepsilon p^{k/2}$ and this proves our claim.

The reduction map from $S_k(p,\bar{\mathbb{Z}}_\ell)$ to $S_k(p,\overline{\mathbb{F}}_\ell)$ is surjective by Carayol's lemma [Edi97, Proposition 1.10]. Hence, there exists an element in $S_k(p,\mathcal{O}_K)$ having $\overline{E}^{\varepsilon}_{k,p}$ as mod ℓ reduction, where $\mathcal{O}_K$ is the ring of integers of some finite extension K of $\mathbb{Q}_\ell$. In other words, $\overline{E}^{\varepsilon}_{k,p}$ is the reduction of a characteristic 0 cusp form, which may not be an eigenfunction for the Hecke operators. Now we use the Deligne-Serre lifting lemma [DS74, Lemma 6.11] guaranteeing the existence of an $h'\in S_k(p,\mathcal{O}_L)$ that is a normalized common eigenfunction for every element of $\{T_q,U_p : q\neq p\}$, such that

$$h'\equiv E^{\varepsilon}_{k,p} \pmod{\lambda'},$$

for some prime ideal λ' lying above ℓ in $\mathcal{O}_L$. Here $L\supseteq K$ is a finite extension of $\mathbb{Q}_\ell$ which is a completion of some number field at a prime over ℓ. Moreover, such an h' arises from some $h(z)=\sum_{n\geq1}a_h(n)e^{2\pi inz}\in S_k(p)$ via the embedding of K_h into L, and hence there exists a prime ideal λ above ℓ in K_h such that

$$h\equiv E^{\varepsilon}_{k,p} \pmod{\lambda}.$$

This proves the first part of Theorem 4.1. Next, if $k=2$, then we know that $S_2(1)=0$, and therefore there are no oldforms in $S_2(p)$, and hence h must be a newform. Let δ be the W_p-eigenvalue of h. Then by Lemma 2.1 we have $a_h(p)=-\delta$, whereas the p-th Fourier coefficient of $E^{-1}_{2,p}$ is 1. This gives rise to $-\delta\equiv1 \pmod{\lambda}$, and hence $h\in S_2^-(p)$, as ℓ (the characteristic of λ) is odd. □

The next result refines [DF14, Theorem 1] and it is a direct consequence of Theorem 4.1 and [DG96, Theorem 1].

Corollary 4.2. *Let $k\geq2$ be an even integer, p a prime and $\varepsilon\in\{\pm1\}$. If $k=2$, we also assume that $\varepsilon=-1$. Let $\ell\geq5$ be a prime divisor of $\frac{B_k}{2k}(\varepsilon+p^{k/2})$. Then there exists a normalized eigenfunction $f\in S^{\varepsilon}_k(p)$ for all T_q with $q\neq p$, and a prime ideal λ over ℓ in the coefficient field of f such that*

$$f\equiv E^{\varepsilon}_{k,p} \pmod{\lambda}.$$

Proof. If $\ell\mid\frac{B_k}{2k}$, applying [DG96, Theorem 1] gives a normalized eigenform $g\in S_k(1)$ and a prime ideal λ above ℓ such that $g\equiv E_k \pmod{\lambda}$. In this case, the form $f:=g+\varepsilon g|W_p$ is a desired eigenfunction. We now assume that $\ell\nmid\frac{B_k}{2k}$, which implies $\ell\mid(\varepsilon+p^{k/2})$. Then, by Theorem 4.1, we obtain a normalized eigenform $h \in S_k(p)$ and a prime ideal λ above ℓ in the coefficient field of h such that $h \equiv E^{\varepsilon}_{k,p} \pmod{\lambda}$. If h is a newform, then Case (i) in the proof of Theorem 1.2 gives $h \in S^{\varepsilon}_k(p)$ and hence we are done. Whereas, if h is not a newform, then following the arguments in Case (ii) and by (3.3), we have a normalized eigenform $g \in S_k(1)$ of level 1 such that $g \equiv E_k \pmod{\lambda}$, where λ is a prime ideal above ℓ in the coefficient field of g. As before, $f := g + \varepsilon g|W_p$ serves our purpose. □

Proof of Theorem 1.3. The proof uses Theorem 4.1 and some ideas used in the proof of Theorem 1.2. So we only give a sketch here. We first notice that $\ell \mid (\varepsilon + p^{k/2})$ and hence from Theorem 4.1 we have an eigenform $h \in S_k(p)$ such that $h \equiv E^{\varepsilon}_{k,p} \pmod{\lambda}$. We now claim that because of the conditions $\ell \nmid \frac{B_k}{2k}$ and $k \not\equiv 0 \pmod{\ell - 1}$, the eigenform h has to be a newform with W_p-eigenvalue ε, i.e., the Case (ii) in the proof of Theorem 1.2 does not occur at all. Suppose it does occur, then by (3.3) we have an eigenform g of level 1 such that $g \equiv E_k \pmod{\Lambda}$. Since g has integral Fourier coefficients in its coefficient field and $k \not\equiv 0 \pmod{\ell-1}$, applying [Lan76, Chapter X, Theorem 8.4] gives $\ell \mid \frac{B_k}{2k}$, which is a contradiction.

5. Proof of Theorem 1.4

The main idea of the proof is to construct, in each of the three cases, a modular form $g \in M^{\varepsilon}_k(p)$ with integral Fourier coefficients and having non-zero constant term $a_g(0)$, coprime to $N^{\varepsilon}_{k,p}$, such that if one would define

$$f := E^{\varepsilon}_{k,p} + \frac{B_k}{2k}\varepsilon(\varepsilon + p^{k/2})\frac{g}{a_g(0)}, \tag{5.1}$$

then such f should be a non-zero cusp form. The existence of such g then ensures that the corresponding f has rational Fourier coefficients and, moreover,

$$f \equiv E^{\varepsilon}_{k,p} \pmod{N^{\varepsilon}_{k,p}}.$$

5.1. **Case (a).** Let $\dim M^{\varepsilon}_k(p) = d + 1$. From [CK13] for $\varepsilon = +1$ and [CKL19] for $\varepsilon = -1$, we know that for such choices of primes p there exists a basis $\{f_0, f_1, \cdots, f_d\}$ of $M^{\varepsilon}_k(p)$, known as Victor-Miller basis, such that all the Fourier coefficients of the f_j are integers and have Fourier series expansion of the form

$$f_j(z) = e^{2\pi ijz} + O(e^{2\pi i(d+1)z}) \text{ for } 0 \leq j \leq d.$$

So, the obvious choice for g in this case is f_0, completing the proof.

5.2. **Case (b).** For any even integer $\alpha \geq 4$ observe that $G_\alpha(z)G_\alpha(pz) \in M^{+}_{2\alpha}(p)$, where G_α is the Eisenstein series of weight α defined by

$$G_\alpha(z) = 1 - \frac{2\alpha}{B_\alpha}\sum_{n\geq1}\sigma_{\alpha-1}(n)e^{2\pi inz}.$$

We first consider the situation when $\varepsilon = +1$ and $k \geq 8$ with $k \equiv 0 \pmod 4$. Such k can be written as $k = 8a + 12b = 4\beta$ for some non-negative integers a, b and $\beta \geq 2$, and so

$$g(z) := (G_4(z)G_4(pz))^a (G_6(z)G_6(pz))^b \in M_{4\beta}^{+}(p)$$

has integer Fourier coefficients (as G_4 and G_6 have integer Fourier coefficients) with constant term 1. Next we claim that the corresponding function f defined by (5.1) is non-zero by showing that its $e^{2\pi iz}$ coefficient is non-zero, i.e.,

$$1 + (30a - 63b)\frac{B_{4\beta}(1 + p^{2\beta})}{\beta} \neq 0.$$

For that, we write $\frac{B_{4\beta}}{8\beta} = \frac{m}{n}$, where $m \in \mathbb{N}, n \in \mathbb{Z}$ and $(m, n) = 1$ and hence $1 + (30a - 63b)\frac{B_{4\beta}(1+p^{2\beta})}{\beta}$ is zero if and only if $n = 8(-30a + 60b)(1 + p^{2\beta})$ and $m = 1$. But one can easily check that $n \neq -8(30a - 60b)(1 + p^{2\beta})$ for $\beta = 2$ and $m > 1$ for $\beta \geq 3$.

For $\varepsilon = -1$ and $k \geq 8$ with $k \equiv 2 \pmod 4$, we write $k = 2 + 8a + 12b = 2 + 4\beta$. Define

$$g(z) = G_{2,p}^{-}(z)(G_4(z)G_4(pz))^a (G_6(z)G_6(pz))^b \in M_{2+4\beta}^{-}(p),$$

where $G_{2,p}^{-}(z) = G_2(z) - pG_2(pz) \in M_2^{-}(p)$. This g has integer Fourier coefficients and constant term $1 - p$. The first Fourier coefficient of the corresponding f is

$$1 + (240a - 504b - 24)\frac{B_{2+4\beta}(p^{1+2\beta} - 1)}{2(2 + 4\beta)},$$

and as before one can verify that it is non-zero.

5.3. **Case (c).** For $\varepsilon = +1$, write $k = 10\alpha$ with $\alpha > 0$. Define

$$g(z) := (p^2 G_4(pz)G_6(z) + p^3 G_4(z)G_6(pz))^\alpha.$$

Observe that $p^2G_4(pz)G_6(z) + p^3G_4(z)G_6(pz) \in M_{10}^{+}(p)$ and hence $g \in M_k^{+}(p)$. Further, g has integer Fourier coefficients and constant term $(p^2 + p^3)^\alpha$. Since by assumption, $p(p + 1)$ does not divide $N_{k,p}^{+}$, it follows that the corresponding f defined by (5.1) satisfies

$$f \equiv E_{k,p}^{\varepsilon} \pmod{N_{k,p}^{\varepsilon}}.$$

To show that f is non zero we prove that its $e^{2\pi iz}$ coefficient is non-zero, i.e.,

$$1 + \frac{(240p^2 - 504p^3)\alpha(p^2 + p^3)B_{10\alpha}(1 + p^{5\alpha})}{20\alpha(p^2 + p^3)^\alpha} \neq 0.$$

First suppose that $\alpha > 1$. Write

$$\frac{B_{10\alpha}(1 + p^{5\alpha})}{20\alpha(p^2 + p^3)^{\alpha-1}} = \frac{m}{n} \text{ with } (m, n) = 1.$$

Then one checks that m is always greater than 1, and therefore the coefficient can not be equal to zero. If $\alpha = 1$ we write $B_{10}(1 + p^5)/20 = m/n$ with $(m, n) = 1$. Again the coefficient is zero if and only if $n = -(240p^2 - 504p^3)m$. Since $(m, n) = 1$, we must have $m = 1$. But for $\alpha = 1$, we have $B_{10}(1+p^5)/20 = (1+p^5)/264$, i.e., if $p \geq 5$ then it can be easily seen that $m > 1$. For the remaining two primes we have $n \neq -(240p^2 - 504p^3)$.

Finally we assume that $\varepsilon = -1$ and $k = 2 + 10\alpha$, for some α, integer. Define

$$g(z) := G^{-}_{2,p}(z)(p^2 G_4(pz)G_6(z) + p^3 G_4(z)G_6(pz))^{\alpha}.$$

This g has all the required properties and the corresponding f, defined by (5.1), has $e^{2\pi iz}$ coefficient

$$1 + \frac{(\alpha(240p^2 - 504p^3)(p^2 + p^3) - 24)B_{2+10\alpha}(1 + p^{1+5\alpha})}{2(1 + 10\alpha)(p^2 + p^3)^{\alpha}}.$$

As before, one can easily check that this coefficient is non-zero and hence we are done. □

6. Numerical Examples

In this section, we give several numerical examples of Ramanujan-style congruences and will write q for $e^{2\pi iz}$. We first recall some basic facts that will be used. To prove that two normalized eigenforms of weight k and level p are congruent, it is enough to check that their first $k(p+1)/12$ Fourier coefficients at prime indices are congruent (this is due to the Sturm bound). Moreover, if $f(z) = \sum_{n\geq 1} a_f(n)q^n$ is a newform and $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$, then its Galois conjugate $f^{\sigma}(z) := \sum_{n\geq 1} \sigma(a_f(n))q^n$ is a newform. In fact, it is easy to see that if two modular forms f and g are congruent modulo some prime ideal λ above ℓ and $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$, then $\sigma(\lambda)$ is a prime ideal above ℓ and

$$f^{\sigma} \equiv g^{\sigma} \pmod{\sigma(\lambda)}. \tag{6.1}$$

To simplify the notation in this section, we put

$$N^{\varepsilon}_{k,p} := \text{the reduced numerator of } \tfrac{B_k}{2k}(\varepsilon + p^{k/2}) \quad \text{and} \quad M^{\varepsilon}_{k,p} := (\varepsilon + p^{k/2})(\varepsilon + p^{k/2-1}).$$

Example 6.1. Take $p = 11$ and $k = 6$. For $\varepsilon = -1$, we easily see that $N^{-}_{6,11} = 5 \cdot 19$ and both the primes 5 and 19 divide $M^{-}_{6,11}$, so $\ell = 5$ or 19. We see that $S^{-}_6(11)$ is 3-dimensional, spanned by the newforms

$$\begin{aligned} g(z) = q + aq^2 &- (1/6a^2 + 5/3a - 64/3)q^3 + (a^2 - 32)q^4 - (3/2a^2 + 7a - 98)q^5 \\ &- (5/3a^2 - 19/3a - 94/3)q^6 + O(q^7), \end{aligned}$$

where a is any root of the polynomial $x^3 - 90x + 188$. Because any two newforms in $S^{-}_6(11)$ are Galois conjugates, in view of (6.1), Theorem 1.2 ensures the existence of a congruence between the newform g and $E^{-}_{6,11}$ modulo some prime in $\mathbb{Q}(a)$ above 5 (resp. for 19) which we verify now. Factoring the ideals (5) and (19) in the ring of integers of $\mathbb{Q}(a)$ gives $5 = \lambda\lambda'$ and $19 = \beta\beta'^2$, where $\lambda = (5, -1/6a^2 + 1/3a + 28/3)$, $\lambda' = (5, 1/6a^2 + 2/3a - 28/3)$, $\beta = (19, 1/6a^2 + 2/3a - 31/3)$ and $\beta' = (19, 1/6a^2 + 2/3a - 58/3)$. We then check that

$$g \equiv E^{-}_{6,11} \pmod{\lambda'} \quad \text{and} \quad g \equiv E^{-}_{6,11} \pmod{\beta} \tag{6.2}$$

We emphasize the fact that the congruence (6.2) for the prime above $\ell = 5 = 6 - 1$ is new and corresponds to the non-covered case $\ell = k - 1$ in [GP18].

For $\varepsilon = +1$, we have $N^+_{6,11} = 37$ and $37 \mid M^+_{6,11}$. As $S^+_6(11)$ is 1-dimensional and spanned by the newform

$$f(z) = q - 4q^2 - 15q^3 - 16q^4 - 19q^5 + 60q^6 + O(q^7),$$

Theorem 1.2 guarantees the congruence

$$f \equiv E^+_{6,11} \pmod{37}.$$

As $11^3 \equiv -1 \pmod{37}$, the conditions of Theorem 1.9 are satisfied with $\ell = 37, p = 11, k = 6$ and $\varepsilon = 1$. On using that $\gamma_{1,37} = 0.47464\ldots$, we conclude that f has a refined ℓ-non-divisibility asymptotic (1.6) with $h_1 = 36$, and Euler-Kronecker constant

$$\gamma_{f;37} = \gamma_{1,37} + \frac{6}{11^6 - 1} - \frac{5}{11^5 - 1} = 0.47464\ldots - 0.000027\ldots = 0.47461\ldots$$

Hence, in this case the Ramanujan approximation is better than the Landau one.

Example 6.2. Although Theorem 1.2 holds for $\ell > k - 2$, we have checked several numerical examples for the case $\ell \leq k - 2$ and in all those cases, we find that the assertion of Theorem 1.2 is true. We give one such example here. Take $k = 12$ and $p = 7$. Then the space of newforms in $S^+_{12}(7)$ is 3-dimensional and spanned by the newforms

$$\begin{aligned} f(z) = q + aq^2 &- (11/21a^2 - 103/7a - 33758/21)q^3 + (a^2 - 2048)q^4 + (59/7a^2 - 517/7a - 203864/7)q^5 \\ &+ (-538/21a^2 + 788/7a + 2476144/21)q^6 + O(q^7), \end{aligned}$$

where a is any root of the polynomial $x^3 - 77x^2 - 2854x + 225104$. Here $N^+_{12,7} = 5 \cdot 181 \cdot 691$ and $5 \cdot 181 \mid M^+_{12,7}$. Therefore, by using the same reasoning used in the previous example, Theorem 1.2 guarantees the existence of the congruence of f with $E^+_{12,7}$ modulo a prime ideal above 181 in $\mathbb{Q}(a)$, something we have verified by direct computation. Note that Theorem 1.2 is not applicable for the prime $\ell = 5$, as $\ell < k - 1 = 11$. But factoring the ideal (5) in the ring of integers of $\mathbb{Q}(a)$ gives $5 = \lambda\lambda'$, where $\lambda = (5, -1/14a^2 + 23/14a + 1628/7)$ and $\lambda' = (5, 1/42a^2 - 3/14a - 1775/21)$ and we check that

$$f \equiv E^+_{12,7} \pmod{\lambda'}.$$

Example 6.3. The dimension of $S^+_k(p)$ turns out to be 1 for $p = 2, 3, 5, 7, 11$ and for certain finite values of k, and the unique newforms have integer Fourier coefficients. In these cases, Theorem 1.4 gives a suitable congruence modulo $N^+_{k,p}$. For example if $p = 2$ and $k = 8$, then one easily check that $N^+_{8,2} = 17$. Then $S^+_8(2)$ is 1-dimensional and spanned by the newform

$$\Delta_{8,2}(z) := \eta^8(z)\eta^8(2z) \in S^+_8(2).$$

Theorem 1.4 gives that a constant multiple of f and $E^+_{8,2}$ are same modulo 17. By comparing the Fourier coefficients, we see that the constant must be 1 and hence $\Delta_{8,2} \equiv E^+_{8,2} \pmod{17}$.

Example 6.4. This example, which falls outside the scope of Theorem 1.2, gives a congruence modulo 6 using Theorem 1.4. Take $k = 4$ and $p = 17$, and hence $N^-_{4,17} = 6$. The space $S^-_4(17)$ is 1-dimensional, spanned by the newform

$$f(z) = q - 3q^2 - 8q^3 + q^4 + 6q^5 + 24q^6 - 28q^7 + 21q^8 + 37q^9 + O(q^{10}).$$

Theorem 1.4 guarantees a congruence between a constant multiple of f and $E^-_{4,17}$ modulo 6. An easy computation shows that the first 6 Fourier coefficients of f and $E^-_{4,17}$ are the same modulo 6. Hence, invoking the Sturm bound,

$$f \equiv E^-_{4,17} \pmod 6.$$

We end this section by making some comments on Examples 5.6 and 5.7 considered in [DF14]. These involve congruences between the coefficients of a newform in $S_k(p)$ and E_k away from the level p. More precisely, in those examples, it is shown that if $\ell \mid \frac{B_k}{2k}(1-p^k)$ and $\ell \geq 5$, then there exists a newform $f \in S_k(p)$ such that, for all primes $q \neq p$, modulo a prime ideal λ above ℓ we have $a_f(q) \equiv 1 + q^{k-1} \pmod{\lambda}$. On using Theorem 1.2 more can be said. For both examples the prime ℓ divides $\frac{B_k}{2k}(\varepsilon + p^{k/2})$ and $(\varepsilon + p^{k/2})(\varepsilon + p^{k/2-1})$, for some $\varepsilon \in \{\pm 1\}$ and also satisfies the further requirements of Theorem 1.2, which then yields that the W_p-eigenvalue of f is ε and that $a_f(p) \equiv 1 + p^{k-1} + \varepsilon p^{k/2} \pmod{\lambda}$.

7. Intermezzo: Anatomy of integers

This section is a preamble for the next one.

Let $P^+(n)$ denote the largest prime divisor of an integer $n \geqslant 2$. Put $P^+(1) = 1$. A number n is said to be y-*friable*[1] if $P^+(n) \leqslant y$. The number of integers $1 \leqslant n \leqslant x$ such that $P^+(n) \leqslant y$ is denoted by $\Psi(x, y)$. The study of the smoothness of integers was dubbed psixyology by the third author in his PhD thesis, but is currently called the anatomy of integers (thus the focus shifted from the mind of numbers to their body...).

In 1930, Dickman [Dic30] proved that

$$\lim_{x\to\infty} \frac{\Psi(x, x^{1/u})}{x} = \rho(u), \tag{7.1}$$

where the *Dickman function* $\rho(u)$ is defined by

$$\rho(u) = \begin{cases} 1 & \text{for } 0 \leqslant u \leqslant 1; \\ \frac{1}{u}\int_0^1 \rho(u-t)dt & \text{for } u > 1. \end{cases} \tag{7.2}$$

We have $0 < \rho(u) < 1/\Gamma(u+1)$, where Γ is the Gamma function. Thus ρ is rapidly decreasing. Dickman's result also remains true if we ask for the proportion of integers $n \leq x$ such that $P^+(n) \leq n^{1/u}$. Thus if p is a prime number and $p-1$ would behave like a typical integer, then one would expect that $P^+(p-1) \leq p^{1/u}$ with probability $\rho(u)$. The following known result partially confirms this.

[1] Some authors use y-smooth. Friable is an adjective meaning easily crumbled or broken.

Theorem 7.1. *Let s be any fixed non-zero integer. The set $\{p : P^+(p+s) \geq p^{1/u}\}$ has density $1-\rho(u)$ under the Elliott-Halberstam conjecture and unconditionally a lower density at least $1-4\rho(u)$ for $u > u_1$, where $u_1 \in (2.677, 2.678)$ is the unique solution of the equation $4u_1\rho(u_1) = 1$.*

Proof. A detailed proof of the first assertion was given by Lamzouri [Lam07] and, independently, by Wang [Wan18]. The second assertion is due to Feng-Wu [FW18] and Liu-Wu-Xi [LWX20]. □

Unconditionally the set considered in Theorem 7.1 is not known to have a density, and therefore we work with the notion of lower and upper density.

Interestingly, (7.1) was already known to Ramanujan, for details and more about the behaviour of the Dickman function, see, e.g., Moree [Mor13].

8. Application to the degree of the coefficient field

Let K_f be the coefficient field of a normalized eigenform $f \in S_k(p)$. Put

$$d_k^{\text{new}}(p) := \max\{[K_f : \mathbb{Q}] : f \in S_k(p),\ f \text{ newform}\}$$

and

$$d_k(p) := \max\{[K_f : \mathbb{Q}] : f \in S_k(p),\ f \text{ normalized eigenform}\}.$$

Billerey-Menares [BM16] showed that for every even integer $k \geq 2$ and every prime $p \geq (k+1)^4$ with $P^+(p-1) \geq 5$ one has

$$d_k^{\text{new}}(p) \geq \frac{5\log(P^+(p-1))}{2k}. \tag{8.1}$$

(Actually the original result has the factor $\log(1+2^{(k-1)/2})$ in the denominator, which we prefer to replace by the upper bound $k/5$.) In 2015, Luca et al. [LMP15] showed that there exists a set of primes of natural density at least $3/4$ such that $P^+(p-1) \geq p^{1/4}$. This result in combination with inequality (8.1), then yields that

$$d_k^{\text{new}}(p) \geq \frac{5\log p}{8k} \tag{8.2}$$

for a set of primes of density at least $3/4$. If one wants a lower bound valid for all large enough primes p, we still cannot do better than Bettin et al. [BPR21] who showed that

$$d_k^{\text{new}}(p) \gg_k \log\log p,\ p \to \infty.$$

On combining Theorem 7.1 and inequality (8.1), we obtain the following improvement of (8.2).

Theorem 8.1. *Let $k \geq 2$ be an even integer and $u > 1$ any real number. Under the Elliott-Halberstam conjecture the set of primes p for which*

$$d_k^{\text{new}}(p) \geq \frac{5\log p}{2uk}$$

has lower density at least $1-\rho(u)$. Unconditionally this set has at least lower density $1-4\rho(u)$ for $u > 2.678$.

Corollary 8.2. *The set of primes p for which (8.2) holds has lower density at least $1-4\rho(4) \geq 0.98$.*

In the same spirit and as an application of Proposition 4.1, we establish analogues for $d_k(p)$ of the latter theorem and corollary, by following the ideas used in the proof of [BM16, Theorem 2].

Theorem 8.3. *If $k \geq 4$ is an even integer and p any prime, then*

$$d_k(p) \geq \frac{5\log(P^+(p^2-1))}{2k}.$$

Proof. Define $\ell := P^+(p^2-1)$. First assume that $\ell \geq 5$. Then we can choose $\varepsilon \in \{\pm 1\}$ such that $\ell \mid (\varepsilon + p^{k/2})$. Applying Proposition 4.1 yields a normalized eigenform $f = \sum_{n\geq 1} a_f(n)q^n \in S_k^\varepsilon(p)$ and a prime ideal $\lambda \subset \mathcal{O}_{K_f}$ above ℓ such that

$$f \equiv E_{k,p}^\varepsilon \pmod{\lambda}.$$

The coefficients of the eigenform f and those of any Galois conjugate of f all satisfy Deligne's estimate and hence the non-zero algebraic integer $b := a_f(2) - 1 - 2^{k-1}$ and all its Galois conjugates have absolute value bounded above by $(1+2^{(k-1)/2})^2$. Because of the above congruence, it is clear that $b \in \lambda$ and hence ℓ divides the absolute value of the norm of b, which is at most $(1+2^{(k-1)/2})^{2d}$, where $d = [K_f : \mathbb{Q}]$. Therefore we conclude that

$$d_k(p) \geq d \geq \frac{\log \ell}{2\log\left(1+2^{(k-1)/2}\right)} \geq \frac{5\log \ell}{2k}.$$

In the remaining case $\ell \leq 3$ we have $5(\log \ell)/2k \leq 5(\log 3)/8 < 1 \leq d_k(p)$, and there is nothing to prove. □

Combining this with Theorem 7.1 we obtain the following corollary.

Corollary 8.4. *Let $k \geq 4$ be even integer and $u > 1$ any real number. The set of primes p for which*

$$d_k(p) \geq \frac{5\log p}{2uk}$$

has lower density at least $1 - 4\rho(u)$ for $u > 2.678$.

This can be sharpened if one solves the following problem.

Open problem 8.5. *Show that the set of primes p for which $P^+(p^2-1) < p^{1/u}$ has an upper density not exceeding $4\rho(u)$ for all u large enough.*

By Theorem 7.1 under the Elliott-Halberstam conjecture each of the inequalities $P^+(p-1) < p^{1/u}$ and $P^+(p+1) < p^{1/u}$ is satisfied with probability $\rho(u)$. Assuming independence of the two events leads to the following conjecture on invoking Theorem 8.3.

Conjecture 8.6. *Let $k \geq 4$ be even integer and $u > 1$ any real number. The set of primes p for which*

$$d_k(p) \geq \frac{5\log p}{2uk},$$

has lower density at least $1 - \rho(u)^2$.

The independence assumption was already made by Erdős and Pomerance, see the two articles by Wang [Wan18, Wan21], who also made partial progress in proving it.

Using elementary number theory it is easy to show that $P^+(p^2-1) \leq 3$ if and only if $p \in \{2, 3, 5, 7, 17\}$. A much deeper result is the following trivial consequence of a celebrated result of Evertse (where as usual we denote by $\pi(x)$ the number of primes $p \leq x$).

Proposition 8.7. *Let x be a real number. The number of primes p for which $P^+(p^2-1) \leq x$ is at most $3 \cdot 7^{1+2\pi(x)}$.*

Proof. Put $S := \{p_1, \ldots, p_s\}$. Integers of the form $\pm p_1^{e_1} \cdot p_2^{e_1} \cdots p_s^{e_s}$ are called S-units. Evertse [Ev84, Theorem 1] proved that the equation $a + b = 1$ has at most $3 \cdot 7^{1+2s}$ solutions (a, b) with a and b both S-units. If $P^+(p^2-1) \leq x$, then both $p-1$ and $p+1$ are S-units, with S the set of consecutive primes not excedding x, which has cardinality $\pi(x)$. Taking the difference and dividing by two leads to the S-unit equation $a + b = 1$. □

Combining this result with Theorem 8.3, we obtain the following result.

Theorem 8.8. *Let $m \geq 1$ be an integer. Then, with the exception of at most*

$$O(e^{e^{2km/5}})$$

primes p, we have $d_k(p) \geq m$.

Proving a similar result for $d_k^{\text{new}}(p)$ is an open problem.

9. Proof of Theorem 1.9

A set of natural numbers S is said to be *multiplicative* if its characteristic function is a multiplicative function. The set B of natural numbers that can be written as a sum of two squares provides an example (as already Fermat knew). One can wonder about the asymptotic behavior of $S(x)$, the number of positive integers $n \leq x$ that are in S. An important role in understanding $S(x)$ is played by the Dirichlet series

$$L_S(s) := \sum_{n \in S} n^{-s}, \tag{9.1}$$

which converges for $\Re(s) > 1$. If the limit

$$\gamma_S := \lim_{s \to 1^+} \left(\frac{L'_S(s)}{L_S(s)} + \frac{\alpha}{s-1} \right) \tag{9.2}$$

exists for some $\alpha \neq 0$, we say that the set S admits an *Euler-Kronecker constant* γ_S. A leisurely account of the theory of Euler-Kronecker constants with plenty of examples is given in Moree [Mor11].

Proof of Theorem 1.9. We first consider the non-divisibility of $\sigma_{k-1}(n)$ by ℓ. Note that $\ell \nmid \sigma_{k-1}(n)$ if and only if $\ell \nmid \sigma_r(n)$, where $r = (k-1, \ell-1)$. Let g_{p_1} be the multiplicative order of p_1^r modulo ℓ, with $p_1 \neq \ell$ an arbitrary prime number. We let

$$\mu_{p_1} = \begin{cases} \ell & \text{if } g_{p_1} = 1, \\ g_{p_1} & \text{if } g_{p_1} > 1. \end{cases} \tag{9.3}$$

Put $S_1 := \{n : \ell \nmid \sigma_r(n)\}$. Rankin [Ran61] showed that

$$L_{S_1}(s) = \frac{1}{1-\ell^{-s}} \prod_{p_1 \neq \ell} \frac{1 - p_1^{-(\mu_{p_1}-1)s}}{(1-p_1^{-s})(1-p_1^{-\mu_{p_1}s})}. \tag{9.4}$$

The proof of this is short and can also be found in [CLM21]. Set $f(n) = \sigma_{k-1}(n) + \varepsilon p^{k/2}\sigma_{k-1}(n/p)$. Write $n = mp^e$, with m coprime to p. Note that $f(m) = \sigma_{k-1}(m)$. Writing momentarily σ instead of σ_{k-1}, we have

$$f(mp^e) = \sigma(mp^e) + \varepsilon p^{k/2}\sigma(mp^{e-1}) = \sigma(m)\sigma(p^e) + \varepsilon p^{k/2}\sigma(m)\sigma(p^{e-1}) = f(m)f(p^e),$$

and so f is a multiplicative function. Put $S_2(p) := \{n : \ell \nmid f(n)\}$. The multiplicativity of f implies that

$$L_{S_2(p)}(s) = \prod_{p_1} \sum_{\substack{a \geq 0 \\ \ell \nmid f(p_1^a)}} \frac{1}{p_1^{as}}$$

has an Euler product, the sums being the Euler product factors. For the primes $p_1 \neq p$, we have $\ell \nmid f(p_1^a)$ if and only if $\ell \nmid \sigma_r(p_1^a)$ and we get the same Euler product factors as in (9.4). Since $\varepsilon p^{k/2} = -1 \pmod{\ell}$ by assumption, we have

$$f(p^a) \equiv \sigma_{k-1}(p^a) - \sigma_{k-1}(p^{a-1}) \equiv p^{a(k-1)} \not\equiv 0 \pmod{\ell},$$

and we conclude that the Euler product factor at $p_1 = p$ is $(1-p^{-s})^{-1}$. We infer that

$$L_{S_2(p)}(s) = \frac{1}{(1-\ell^{-s})} \frac{1}{(1-p^{-s})} \prod_{\substack{p_1 \neq \ell \\ p_1 \neq p}} \frac{1 - p_1^{-(\mu_{p_1}-1)s}}{(1-p_1^{-s})(1-p_1^{-\mu_{p_1}s})} = L_{S_1}(s) \frac{(1-p^{-\mu_p s})}{1-p^{-(\mu_p-1)s}}. \tag{9.5}$$

Comparing the logarithmic derivative of $L_{S_2(p)}(s)$ with that of $L_{S_1}(s)$, we obtain

$$\frac{L'_{S_2(p)}(s)}{L_{S_2(p)}(s)} = \frac{L'_{S_1}(s)}{L_{S_1}(s)} + \log p \left(\frac{\mu_p}{p^{\mu_p s}-1} - \frac{(\mu_p - 1)}{p^{(\mu_p-1)s}-1} \right).$$

The proof of (1.8) is then completed on invoking (9.2).

The prime counting functions $\#\{p_1 \leq x :\ \ell \mid \sigma_r(p_1)\}$ and $\#\{p_1 \leq x :\ \ell \mid f(p_1)\}$ differ by at most one for every x. As the first counting function satisfies (1.7) with $\delta = r/(\ell-1)$, so does the second, and the proof is completed on account of Theorem 1.7. □

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