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Abstract

It is well-known that if A is a finitely generated degree-graded algebra and there is $n \in \mathbb{N}$ such that $\dim A_n \leqslant n$, then A has linear growth. More specifically, the sequence of dimensions $(\dim A_m)_{m\in\mathbb{N}}$ is bounded. Having in mind applications to a number of classification problems, we characterize all possible sequences $(\dim A_n, \dim A_{n+1}, \dots)$ in the case $\dim A_n \leqslant 3$ and $n \geqslant 3$. It turns out that there are surprisingly few options and we list them all. Thus by characterising all Hilbert series with a coefficient at most three, we specify a class of Hilbert series which are indeed possible.

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1 Introduction

We start by recalling relevant definitions. Throughout this article $\mathbb{K}$ is fixed field. All algebras we deal with are unital associative algebras over $\mathbb{K}$ and all vector spaces are over the field $\mathbb{K}$. The only dimension we use (always denoted $\dim$) is the dimension of vector spaces over $\mathbb{K}$. If B is a $\mathbb{Z}_+$-graded vector space, B_m stands for the m^{th} component of B. We always assume that each B_m is finite dimensional, which allows to consider the ***Hilbert series*** of B:

$$H_B(t) = \sum_{m=0}^{\infty} \dim B_m \ t^m \in \mathbb{Z}[[t]].$$

If V is an n-dimensional vector space over $\mathbb{K}$, then $F = F(V)$ is the tensor algebra of V. For any choice of a basis $X = \{x_1, \dots, x_n\}$ in V,

F is naturally identified with the free algebra $\mathbb{K}\langle X\rangle$,

always assumed to be degree graded (=all elements of X are of degree 1.)

The set of all words (including the empty word 1) in the alphabet X is denoted $\langle X\rangle$.

For each n, we denote

$$\langle X\rangle_n = \{u \in \langle X\rangle : \mathtt{deg}\, u = n\}.$$

For a quotient $A = F/I$ (I is a two-sided ideal in F different from F), I is called the ***ideal of relations***. If an ideal I in F is degree-graded (=coincides with the linear span of its homogeneous elements), then A is naturally ***degree-graded***.

If R is a subspace of the n^2-dimensional space $V^2 = V \otimes V$, then the quotient of $F(V)$ by the ideal $I(V,R)$ generated by R is called a ***quadratic algebra*** (see the book [3] for a vast exposition on quadratic algebras) and denoted $A(V,R)$. For any choice of bases $x_1, \dots, x_n$ in V and $g_1, \dots, g_k$ in R, $A(V,R)$ is exactly the algebra presented by generators $x_1, \dots, x_n$ and relations $g_1, \dots, g_k$.

There is a well-known result [4] asserting that if A is a finitely generated degree-graded algebra and there is $n \in \mathbb{N}$ such that $\dim A_n \leqslant n$, then A has linear growth. More specifically, the sequence of dimensions $(\dim A_m)_{m\in\mathbb{N}}$ is bounded. As nice as this general result is, it is not sufficient for some applications. We would like to say a lot more about H_A in the case when $\dim A_n$ is particularly small (at most 3). For the sake of brevity, we shall introduce the following notation:

$$H_A^{[n]} = (\dim A_n, \dim A_{n+1}, \dim A_{n+2}, \dots).$$

That is, $H_A^{[n]}$ is the sequence of coefficients of the Hilbert series of A starting from the one with t^n. Next, we denote

$$\overline{m} := m, m, m, \dots$$

That is $\overline{m}$ is the constant m sequence: m repeats infinitely many times. We shall use it in the following way: $56\overline{3}$ stands for the sequence $5, 6, 3, 3, 3, \dots$

Theorem 1.1. *Let $n \in N$ and A be a finitely generated degree graded algebra such that $\dim A_n = 1$ and the ideal of relations of A is generated by some homogeneous elements of degree at most n. Then*

$$H_A^{[n]} \in \{\overline{1}, 1\overline{0}\}.$$

Moreover, $H_A^{[n]} = \overline{1}$ if $n = 1$.

Theorem 1.2. *Let $n \geqslant 2$ and A be a finitely generated degree graded algebra such that $\dim A_n = 2$ and the ideal of relations of A is generated by some homogeneous elements of degree at most n. Then*

$$H_A^{[n]} \in \{\overline{2}, 2\overline{1}, 21\overline{0}, 2\overline{0}\}.$$

Theorem 1.3. *Let $n \geqslant 3$ and A be a finitely generated degree graded algebra such that $\dim A_n = 3$ and the ideal of relations of A is generated by some homogeneous elements of degree at most n. Then*

$$H_A^{[n]} \in \{3\overline{4}, \overline{3}, 332\overline{1}, 3\overline{2}, 32\overline{1}, 321\overline{0}, 32\overline{0}, 3\overline{1}, 31\overline{0}, 3\overline{0}\}.$$

Moreover,

$$H_A^{[n]} \in \{3\overline{4}, \overline{3}, 3\overline{2}, 32\overline{1}, 321\overline{0}, 32\overline{0}, 3\overline{1}, 31\overline{0}, 3\overline{0}\} \quad \textit{if } n \geqslant 4.$$

The above theorems can not be improved as asserted by the following result.

Theorem 1.4. *For every $n \geqslant 2$, each of the sequences $1\overline{0}$, $\overline{1}$, $2\overline{0}$, $2\overline{1}$, $21\overline{0}$, $\overline{2}$, $3\overline{0}$, $31\overline{0}$, $3\overline{1}$, $32\overline{0}$, $32\overline{1}$, $3\overline{2}$ and $\overline{3}$ occurs as $H_A^{[n]}$ for a monomial quadratic algebra A. Moreover, for every $n \geqslant 3$, the sequence $321\overline{0}$ is also among the sequences $H_A^{[n]}$ for monomial quadratic algebras A, while the sequence $3\overline{4}$ is among $H_A^{[n]}$ for monomial algebras A, whose (monomial) relations are of degree at most n. Finally, there is an algebra A given by homogeneous relations of degree at most three for which $H_A^{[3]} = 332\overline{1}$.*

1.1 Gröbner bases

The concept of a Gröbner basis plays a key role in our proofs. We recall its main features. Let $A = F(V)/I$ and X be a linear basis in V. A well-ordering $\leqslant$ on $\langle X \rangle$ is said to be ***compatible with mutiplication*** if

$$1 \leqslant u \text{ for all } u \in \langle X \rangle \text{ and } u \leqslant v \implies uw \leqslant vw,\ wu \leqslant wv \text{ for all } u, v, w \in \langle X \rangle.$$

If we fix a well-ordering $\leqslant$ on $\langle X \rangle$ compatible with multiplication, we can talk of the leading monomial $\overline{f}$ of a non-zero $f \in \mathbb{K}\langle X \rangle$ (=the biggest with respect to $\leqslant$ monomial, which features in f with non-zero coefficient). A subset G of an ideal I in $\mathbb{K}\langle X \rangle$ is called a ***Gröbner basis of*** I if $0 \notin G$, G generates I as an ideal and for each non-zero $f \in I$, there is $g \in G$ such that $\overline{g}$ is a subword of $\overline{f}$. That is the two sets $\{\overline{g} : g \in G\}$ and $\{\overline{f} : f \in I \setminus \{0\}\}$ generate the same ideal. Such a G is by no means unique: for one, $G = I \setminus \{0\}$ fits the bill. However, a couple of extra conditions pinpoint G. Namely, if we additionally assume that a Gröbner basis G satisfies

- for every two distinct $f, g \in G$, $\overline{f}$ is not a subword of any monomial featuring in g;
- every $f \in G$ is monic: the $\overline{f}$-coefficient in f equals 1,

then G becomes unique. Such a basis is called the ***reduced Gröbner basis of*** I. Note that I possesses a finite Gröbner basis if and only if its reduced Gröbner basis is finite.

The non-commutative Buchberger algorithm [1] applied to a set of defining relations (any collection of elements of I generating I as a two-sided ideal) yields the reduced Gröbner basis for I. In general, one of the problems though is that (unlike for the commutative case) the procedure does not have to terminate in finitely many steps. What is even worse, there is no *a-priory* way to say if it does (the problem of recognizing finiteness of the reduced Gröbner basis is algorithmically unsolvable). Furthermore, everything is highly sensitive to the choice of the generators and the ordering.

The words $u \in \langle X \rangle$, which have no leading monomials of elements of the ideal I of relations of A as subwords, are called ***normal words*** for A. It is easy to see that normal words form a linear basis in A. Clearly, if G is a Gröbner basis for I, then a word $u \in \langle X \rangle$ is normal if and only if it has no leading monomials of elements of G as subwords.

For the sake of brevity, we shall introduce the following concept. An order $\leqslant$ on $\langle X \rangle$ is called ***admissible*** if it is a well-ordering compatible with multiplication respecting the degree in the following sense: $u < v$ if $\mathtt{deg}\, u < \mathtt{deg}\, v$. In most cases, when choosing an admissible order we opt for the **left-to-right** or **right-to-left degree lexicographical orders** corresponding to some total order on X. Since this is a veritable mouthful, we shall abbreviate these orders as

the LR order and ***the RL order***,

respectively.

1.2 Examples: Proof of Theorem 1.4

As Theorem 1.4 states the existence of a bunch of examples and as it is frankly easy but entertaining, we deal with it right away. We start with very simple examples, used as building blocks later. Assume that A and B are finitely presented degree graded $\mathbb{K}$-algebras given by generating sets X_A and X_B and defining relation sets $R_A \subset \mathbb{K}\langle X_A \rangle$ and $R_B \subset \mathbb{K}\langle X_B \rangle$, where X_A and X_B are assumed disjoint. Then we can consider the algebra C given by the generating set X_C and the relation set R_C, where

$$X_C = X_A \cup X_B \quad \text{and} \quad R_C = R_A \cup R_B \cup \{ab : a \in X_A,\ b \in X_B\} \cup \{ba : a \in X_A,\ b \in X_B\}.$$

We call C the ***direct sum*** of A and B, which constitutes a slight abuse of notation because the degree 0 component of C. However, C_n is the direct sum of the vector spaces A_n and B_n for every $n \geqslant 1$. In particular, the Hilbert series of these three algebras satisfy $H_C = H_A + H_B - 1$. Note also that C is quadratic if A and B are quadratic and that C is a monomial algebra if A and B are monomial algebras.

Example 1.5. *The free algebra $S = \mathbb{K}\langle x \rangle = \mathbb{K}[x]$ on one generator satisfies* $\dim S_n = 1$ *for each $n \in \mathbb{N}$.*

Example 1.6. *For each $j \in \mathbb{N}$ and each $n \geqslant \max\{j, 2\}$, let $A = A^{(n,j)}$ be the monomial quadratic algebra given by $n + j - 1$ generators $x_1, \dots, x_{n+j-1}$ and relations $x_p x_q = 0$ if $p \geqslant q$ or $q - p \geqslant 2$. Then the sequence $H_A^{[n]}$ coincides with the sequence $(j, j-1, \dots, 1, 0, 0, \dots)$.*

Proof. For each $r \in \mathbb{N}$ satisfying $r \leqslant n + j - 1$, A_r is easily seen to be spanned by the linearly independent monomials $x_m x_{m+1} \dots x_{m+r-1}$ for $1 \leqslant m \leqslant n + j - r$, while $A_r = \{0\}$ for $r \geqslant n + j$. The result follows. $\square$

Example 1.7. *Let $n \geqslant 2$ and B be the quadratic algebras given by $n+2$ generators $z, x_0, \dots x_n$ and and monomial quadratic relations $x_j x_k = 0$ if $k \leqslant j$ or $k - j \geqslant 2$, $x_j z = 0$ for $0 \leqslant j \leqslant n$, $z^2 = 0$ and $zx_j = 0$ for $j \neq 1$. Then $H_B^{[n]} = 32\overline{0}$.*

Proof. It is easy to see Now B_n is spanned by $x_0 \dots x_{n-1}$, $zx_1 \dots, x_{n-1}$ and $x_1 \dots x_n$, B_{n+1} is spanned by $x_0 \dots x_n$ and $zx_1 \dots x_n$, while $B_{n+2} = \{0\}$. □

Remark 1.8. Taking direct sums of (sometimes more than one) copies of S of Example 1.5 and of copies of $A^{(n,j)}$ of Example 1.6 with $j \in \{1, 2\}$ and throwing in the algebra B of Example 1.7, we see that the sequences $1\overline{0}$, $\overline{1}$, $2\overline{0}$, $2\overline{1}$, $21\overline{0}$, $\overline{2}$, $3\overline{0}$, $31\overline{0}$, $3\overline{1}$, $32\overline{0}$, $32\overline{1}$, $3\overline{2}$ and $\overline{3}$ occur as $H_A^{[n]}$ for a monomial quadratic algebras A. The algebra $A^{(n,3)}$ of Example 1.6 shows that the sequence $321\overline{0}$ is also among the sequences $H_A^{[n]}$ for monomial quadratic algebras A provided $n \geqslant 3$.

It remains to deal with two sequences $3\overline{4}$ and $332\overline{1}$.

Example 1.9. *Let $n \geqslant 2$ and C be the algebra given by 3 generators x, y, z and monomial relations $zx = yz = x^2 = y^2 = xy = yx = 0$ and $xz^{n-2}y = 0$. Then $H_C^{[n]} = 3\overline{4}$.*

Proof. It is easy to see that C_n is spanned by three linearly independent monomials z^n, xz^{n-1} and $z^{n-1}y$, while all other degree n monomials vanish in C. It follows that for each $m > n$, C_m is spanned by four linearly independent monomials z^{n+1}, xz^n, $z^n y$ and $xz^{n-1}y$. The result follows. □

Example 1.10. *Let D be the algebra given by 2 generators z and s and 5 cubic relations $zs^2 = s^2 z = s^3 = sz^2 = 0$ and $szs = z^2 s$. Then $H_D^{[n]} = 332\overline{1}$.*

Proof. We set $z < s$ and equip $\langle z, s \rangle$ with the LR order. Computing the reduced Gröbner basis of the ideal of relations of D, we easily see that it consists of the defining relations zs^2, $s^2 z$, s^3, sz^2 and $szs - z^2 s$ together with just one extra element $z^4 s$. This easily yields $H_D(t) = 1 + 2t + 4t^2 + 3t^3 + 3t^4 + 2t^5 + t^6 + t^7 + t^8 + \dots$ and therefore $H_D^{[n]} = 332\overline{1}$. □

As Theorem 1.4 follows immediately from Remark 1.8 and Examples 1.9 and 1.10, we declare it proven and forget about it from now on.

1.3 Our tools

We would like to mention the following fact, which simplifies our life considerably.

Remark 1.11. Let A be a finitely generated degree graded algebra and $n \in N$ be such that the ideal of relations of A is generated by its elements of degree $\leqslant n$. Let also $V = A_1$ and $I \subset F(V)$ be the ideal of relations of A. Consider the algebra $B = F/J$, where J is the ideal generated by I_n. It is easy to see that $I_m = J_m$ for all $m \geqslant n$. Hence $\dim A_m = \dim B_m$ for all $m \geqslant n$. Thus in Theorems 1.1, 1.2 and 1.3, one can, without loss of generality, assume that the ideal of relations of A is generated by homogeneous elements of degree **exactly** n.

The proof of Theorems 1.1, 1.2 and 1.3 requires considering a vast number of cases. We split considerations in different ways. One of the ways we sort possibilities for A is by the following pair of integer isomorphism invariants. Let A be a finitely generated degree graded algebra and $n \in N$. We denote

$$\lambda(A, n) = \max_{x \in A_1} \dim xA_{n-1} \quad \text{and} \quad \rho(A, n) = \max_{x \in A_1} \dim A_{n-1}x, \tag{1.1}$$

where xA_{n-1} and $A_{n-1}x$ are considered as subspaces of A. Clearly,

$$1 \leqslant \lambda(A, n), \rho(A, n) \leqslant \dim A_n, \quad \text{provided } A_n \neq \{0\}.$$

One way to whittle down options is to use the **general position type arguments**. Since for a finitely presented $\mathbb{K}$-algebra A replacing $\mathbb{K}$ by a field extension $\mathbb{F}$ does not change the Hilbert series (it swaps A with the $\mathbb{F}$-algebra $\mathbb{F}\otimes_{\mathbb{K}} A$), we can and will **always** assume that

the field $\mathbb{K}$ ***is infinite.***

This allows us, on a number of occasions, to talk about ***generic*** elements of a finite dimensional vector space W over $\mathbb{K}$. Namely, we say that generic $x\in W$ have a property P if the set of $x\in W$ for which P fails is contained in a proper algebraic variety. Equivalently P holds for generic x if it holds for all elements of a non-empty Zariski open set. Since $\mathbb{K}$ is infinite, if each of finitely many properties holds for generic $x\in W$, then all of them hold simultaneously for generic $x\in W$. Note also that if it is clear that the set of x for which a certain property is satisfied is Zariski open, then either this property is satisfied for no x at all or it is satisfied for generic x. We shall use these observations repeatedly and without extra comments.

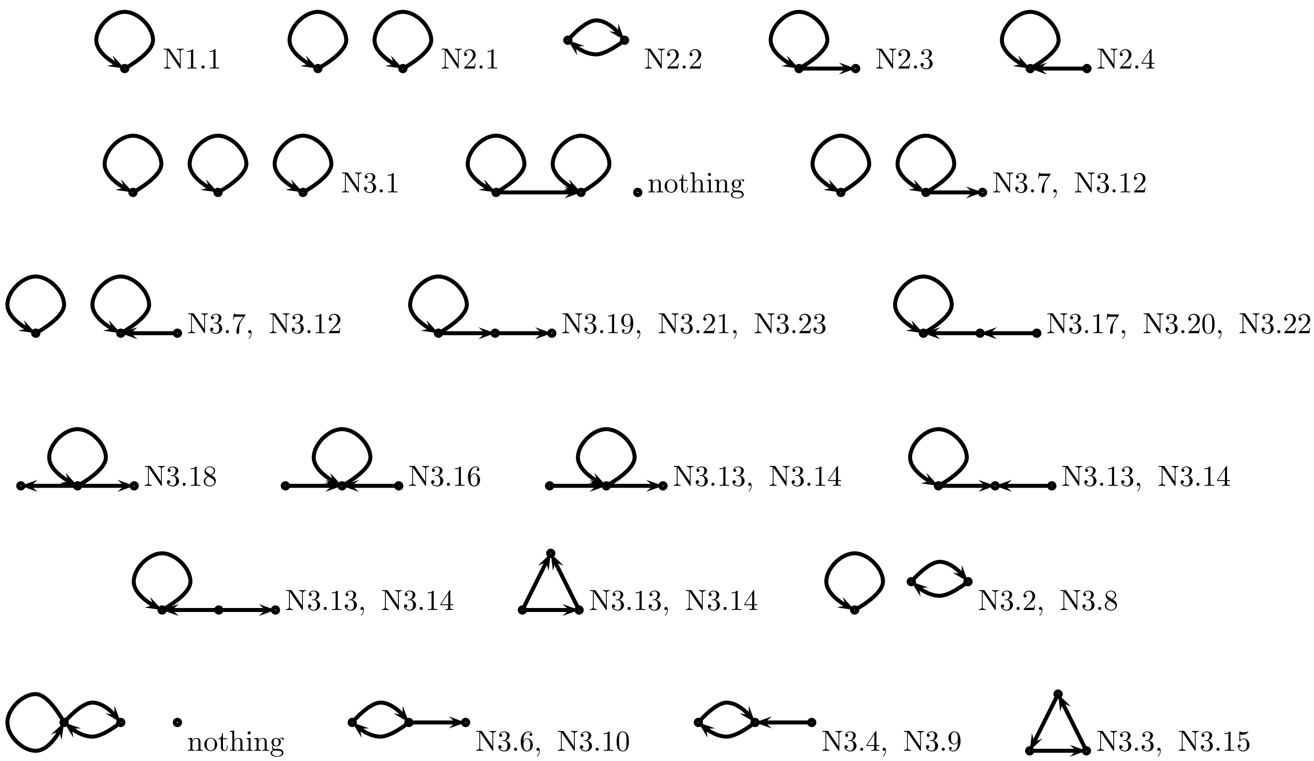
2 Possible normal words

We start by listing all possibilities for the set of degree n normal words in the relevant cases. In this section we shall also list degree n normal words occurring for generic choice of generators.

Lemma 2.1. *Let A be a finitely generated degree graded algebra, $k,n\in N$, $n\geqslant k$ and $1\leqslant k\leqslant 3$. Assume also that $\dim A_n=k$, $\dim A_{n+1}\geqslant k$, X is a linear basis in $V=A_1$ and $\langle X\rangle$ is equipped with an admissible order. Then the set of normal words for A of degree n (there are exactly k of them) have to be of one of the forms presented in the following table, where a,b,c are pairwise distinct elements of X :*

Label	k	degree n normal words	Label	k	degree n normal words
N1.1	1	a^n	N3.10	3	$\dots ababab, \dots bababa, \dots ababaa$
N2.1	2	a^n, b^n	N3.11	3	a^n, b^n, ba^{n-1}
N2.2	2	$abab\dots, baba\dots$	N3.12	3	$a^n, b^n, a^{n-1}b$
N2.3	2	$a^n, a^{n-1}b$	N3.13	3	$a^n, ca^{n-1}, a^{n-1}b$
N2.4	2	a^n, ba^{n-1}	N3.14	3	$a^n, ba^{n-1}, a^{n-1}b$
N3.1	3	a^n, b^n, c^n	N3.15	3	$aabaab\dots, abaaba\dots, baabaa\dots$
N3.2	3	$a^n, bcbc\dots, cbcb\dots$	N3.16	3	a^n, ba^{n-1}, ca^{n-1}
N3.3	3	$abcabc\dots, bcabca\dots, cabcab\dots$	N3.17	3	a^n, ba^{n-1}, cba^{n-2}
N3.4	3	$ababab\dots, bababa\dots, cabab\dots$	N3.18	3	$a^n, a^{n-1}b, a^{n-1}c$
N3.5	3	a^n, b^n, ca^{n-1}	N3.19	3	$a^n, a^{n-1}b, a^{n-2}bc$
N3.6	3	$\dots ababab, \dots bababa, \dots ababac$	N3.20	3	a^n, ba^{n-1}, aba^{n-2}
N3.7	3	$a^n, b^n, a^{n-1}c$	N3.21	3	$a^n, a^{n-1}b, a^{n-2}ba$
N3.8	3	$a^n, abab\dots, baba\dots$	N3.22	3	$a^n, ba^{n-1}, b^2a^{n-2}$
N3.9	3	$ababab\dots, bababa\dots, aabab\dots$	N3.23	3	$a^n, a^{n-1}b, a^{n-2}b^2$

Proof. Let $u^{(1)},\dots,u^{(k)}$ be all degree n normal words. Clearly, $u^{(j)}$ are pairwise distinct. Since $\dim A_{n+1}\geqslant k$, there are at least k normal words of degree $n+1$. Since every subword of a normal word is normal, there exists k-element set $M\subset\{1,\dots,k\}^2$ such that for every $(j,m)\in M$, $v=v^{(j,m)}\in\langle X\rangle_{n+1}$ is a normal word for A and satisfies $v_1\dots v_n=u^{(j)}$ and $v_2\dots v_{n+1}=u^{(m)}$. Now M can be interpreted as a directed graph on k vertices $1,\dots,k$ with exactly k edges (loops are allowed, multiple edges are forbidden). As $k\leqslant 3$, one can easily run through all possible (isomorphism classes of) graphs like that and solve the corresponding systems $u^{(j)}_2\dots u^{(j)}_n=u^{(m)}_1\dots u^{(m)}_{n-1}$ of symbolic equations, which yields only the options from the above table. The following picture shows the complete list of these graphs. To the right of each graph, labels of corresponding solutions from the above table are written (the word 'nothing' indicates that no solutions correspond to the graph).



Note that one has to use the condition $n \geqslant k$. If we drop it, much more solutions occur including solutions corresponding to graphs with 'nothing' to the right of them. □

2.1 Generic normal words and the structure of the proofs

Let A be a finitely generated degree graded algebra and let $q = \dim A_1$. For a totally ordered tuple Y (possible empty) of linearly independent elements of A_1, we denote by the symbol

$$\Omega(A, Y)$$

the set of all totally ordered linear bases X in A_1 containing Y as an initial segment. The latter means that the order on Y induced from X coincides with the original order on Y and that $y < x$ whenever $y \in Y$, $x \in X \setminus Y$. Clearly, the set

$$\Omega(A) := \Omega(A, \varnothing)$$

is just the set of all ordered linear bases in A_1. If $Y = \{y_1, \dots, y_m\}$ with the order $y_1 < \dots < y_m$, we write

$$\Omega(A, y_1, \dots, y_m)$$

when referring to $\Omega(A, Y)$. This notation is convenient because it contains the description of the order on Y.

Note that if $q = \dim A_1$, then a generic element $(x_1, \dots, x_q)$ of A_1^q provides a linear basis in A_1. This allows us to speak of generic bases in A_1. One of the ways to reduce the number of options to consider is to figure out what happens under the assumptions of Lemma 2.1 if the set of generators X is a generic ordered basis in A_1. More specifically, we want to know which of the normal word patterns listed in Lemma 2.1 are the patterns occurring for generic basis with respect to, say, the left-to-right degree-lexicographical order. As described above, we say that a pattern (N3.23, for example) occurs for a generic basis X if the set of X, for which it happens contains a non-empty Zariski open subset of A_1^q.

The following lemma collects information of this type. For the sake of brevity, from now on we shall use the following notation. Once a degree graded $A = F(V)/I$, a linear basis X in $V = A_1$ and an admissible order on $\langle X \rangle$ are fixed, we denote

$$\mathrm{NW}_n = \text{the set of all degree } n \text{ normal words.}$$

Lemma 2.2. *Let $1 \leqslant k \leqslant 3$, $n \geqslant \max\{2, k\}$ and let A be a finitely generated degree-graded algebra such that $\dim A_n = k$ and $\dim A_{n+1} \geqslant k$. For an ordered basis X in A_1, we denote the corresponding LR and RL orders on $\langle X \rangle$ by $<$ and $\prec$, respectively. Then for generic $X = (x_1, x_2, \dots) \in \Omega(A)$ the following statements hold*:

(G1) *If $\lambda(A, n) = \rho(A, n) = 1$, then with respect to both $<$ and $\prec$,* $\mathrm{NW}_n = \{x_1^n, \dots, x_k^n\}$ (patterns N1.1, N2.1 and N3.1);

(G2) *If $k = 2$ and $\lambda(A, n) = 2$, then with respect to $<$,* $\mathrm{NW}_n = \{x_1^n, x_1^{n-1}x_2\}$ (N2.3);

(G3) *If $k = 2$ and $\rho(A, n) = 2$, then with respect to $\prec$,* $\mathrm{NW}_n = \{x_1^n, x_2x_1^{n-1}\}$ (N2.4);

(G4) *If $k = 3$ and $\max\{\lambda(A, n), \rho(A, n)\} = 2$, then with respect to both $<$ and $\prec$,* $\mathrm{NW}_n = \{x_1^n, x_1^{n-1}x_2, x_2x_1^{n-1}\}$ (N3.14);

(G5) *If $k = 3$ and $\lambda(A, n) = 3$, then with respect to $<$,* $\mathrm{NW}_n = \{x_1^n, x_1^{n-1}x_2, w\}$*, where* $w \in \{x_1^{n-1}x_3, x_1^{n-2}x_2x_1, x_1^{n-2}x_2^2\}$ (N3.18, N3.21 and N3.23);

(G6) *If $k = 3$ and $\rho(A, n) = 3$, then with respect to $\prec$,* $\mathrm{NW}_n = \{x_1^n, x_2x_1^{n-1}, w\}$*, where* $w \in \{x_3x_1^{n-1}, x_1x_2x_1^{n-2}, x_2^2x_1^{n-2}\}$ (N3.16, N3.20 and N3.22).

We shall prove the above lemma in the following section. The punch-line though is that it allows to significantly reduce the number of options to consider. Item (G4) is a pleasant surprise: only one pattern of normal words. We have multiple options in (G5) and (G6). A priori, this could mean we could reduce the spectrum of options further by showing that some of the options do not occur generically. Unfortunately, this is a no go: each of the three options in (G5) and in (G6) occurs as the normal word pattern for generic basis in A_1 for some A.

The proof of Theorems 1.1, 1.2 and 1.3 is structured in the following way:

- Prove Lemma 2.2;
- Consider the normal word patterns emerging from Lemma 2.2 one at a time.

2.2 Proof of Lemma 2.2

Lemma 2.3. *Let $1 \leqslant k \leqslant 3$, $n \geqslant \max\{2, k\}$ and let A be a finitely generated degree-graded algebra such that $\dim A_n = k$ and $\dim A_{n+1} \geqslant k$. Then $x^n \neq 0$ in A for generic $x \in A_1$.*

Proof. Assume the contrary: $x^n = 0$ for all $x \in A$. First, we show that this assumption implies

$$x^3 A_{n-3} = A_{n-3} x^3 = \{0\} \text{ for every } x \in A_1 \text{ provided } n \geqslant 3. \tag{2.1}$$

Assume the contrary. Then there is $a \in A_1$ such that either $a^3 A_{n-3}$ or $A_{n-3} a^3$ is non-zero. The two options are obviously equivalent (they swap when we pass to the opposite multiplication). Thus we can assume that $a^3 A_{n-3} \neq \{0\}$. Now for $X \in \Omega(A, a)$ with the corresponding LR order on $\langle X \rangle$, the condition $a^3 A_{n-3} \neq \{0\}$ implies that at least one degree n normal word must start with a^3.

Looking at the possibilities for degree n normal words provided by Lemma 2.1, we see that then a^n must be a normal word. Hence $a^n \neq 0$ in A, which is incompatible with the assumption $x^n = 0$ for all $x \in A$. Hence (2.1) must be satisfied.

Next, we show that

$$x^2 A_{n-2} \neq \{0\} \text{ and } A_{n-2} x^2 \neq \{0\} \text{ in } A \text{ for generic } x \in A_1. \tag{2.2}$$

Indeed, assume the contrary. Then either $x^2 A_{n-2} = \{0\}$ for all $x \in A_1$ or $A_{n-2}x^2 = \{0\}$ for all $x \in A_1$. Again the two statements reduce to one another by passing to the opposite multiplication. Thus we can assume that $x^2 A_{n-2} = \{0\}$ for all $x \in A_1$. Let $X \in \Omega(A)$ and $<$ be the corresponding LR order on $\langle X \rangle$. Lemma 2.1 provides all possible forms of the degree n normal words. Since $x^2 A_{n-2} = \{0\}$ for all $x \in A_1$, none of the normal words can start with x^2 for $x \in X$. This excludes all options except for N2.2, N3.3, N3.4, N3.6 and N3.10. For each of these options, we either have two degree n normal words starting with ab and ba respectively for distinct $a, b \in X$ or (for N3.3) the three normal words start with ab, bc and ca respectively for pairwise distinct $a, b, c \in X$. It follows that there is a normal word w of degree n for which the first letter is greater than the second one: $w = xyw'$ with $x > y$. Since $xy + yx = (x + y)^2 - x^2 - y^2$ and $(x + y)^2 w' = x^2 w' = y^2 w' = 0$, we have $(xy + yx)w' = 0$ in A. Hence $w = -yxw'$ in A. Since w is a normal word and $yxw' < w$, we arrive to a contradiction. This contradiction proves (2.2).

Now let $z \in A_1$ be such that $z^2 A_{n-2} \neq \{0\}$. Then for every $X \in \Omega(A, z)$ with $\langle X \rangle$ carrying the corresponding LR order, at least one degree n normal word must start with z^2. Since (as we have assumed) $x^n = 0$ in A for all $x \in A_1$, none of degree n normal words is the n^{th} power of an element of X. Of all the possibilities offered by Lemma 2.1 only N3.9 and N3.15 have the desired properties. First, assume the degree n normal words are of the form N3.15. Then they are $w = yzzyzz \cdots = yzzw'$, $zyzzyz \dots$ and $zzyzzy \dots$. Note that $(z + \alpha y)^3 - z^3 - \alpha^3 y^3 = \alpha(z^2 y + zyz + yz^2) + \alpha^2(y^2 z + yzy + zy^2)$ for all $\alpha \in \mathbb{K}$. Since our $\mathbb{K}$ is infinite, it follows that $z^2 y + zyz + yz^2$ is a linear combination of cubes. According to (2.1), we then have $(z^2 y + zyz + yz^2)w' = 0$ and therefore $w = -zyzw' - z^2 yw'$. Since $zyzw' < w$, $z^2 yw' < w$ and w is a normal word, we arrive to a contradiction. This contradiction eliminates normal words shape N3.15 from our considerations leaving N3.9 only. Hence the degree n normal words are $yzyz \dots$, $zyzy \dots$ and $z^2 yzyz \dots$ for some $y \in X \setminus \{z\}$. In particular, $yzyz \dots$, $zyzy \dots$ and $z^2 yzyz \dots$ are linearly independent in A (and therefore form a basis in A_n) for generic $y, z \in A_1$. Hence

$$\begin{array}{l}\text{for generic } y, z \in A_1, \text{ for every } X \in \Omega(A, z, y) \text{ with } \langle X \rangle \text{ carrying the} \\ \text{corresponding LR order, the degree } n \text{ normal words are } yzyz \dots, zyzy \dots \text{ and } z^2 yzyz \dots\end{array} \tag{2.3}$$

Applying the same argument to the RL order, or alternatively applying (2.3) to A with the opposite multiplication, we get

$$\begin{array}{l}\text{for generic } y, z \in A_1, \text{ for every } X \in \Omega(A, z, y) \text{ with } \langle X \rangle \text{ carrying the} \\ \text{corresponding RL-order, the degree } n \text{ normal words are } yzyz \dots, zyzy \dots \text{ and } \dots zyzyz^2.\end{array} \tag{2.4}$$

By (2.3) and (2.4), for generic $y, z \in A_1$, $w_1 = yzyz \dots$ and $w_2 = zyzy \dots$ are linearly independent in A and each of the four triples (w_1, w_2, v_1), (w_1, w_2, v_2), (w_1, w_2, v_3) and (w_1, w_2, v_4) forms a linear basis in A_n, where $v_1 = z^2 yzyz \dots$, $v_2 = y_2 zyzy \dots$, $v_3 = \dots zyzyz^2$ and $v_4 = \dots yzyzy^2$. Pick $y, z \in A_1$ having all these properties and choose a linear basis X in A_1 containing y and z. Consider two total orders on X such that z is the smallest element and y is second smallest for the first one, while y is the smallest element and z is second smallest for the second one. Now we have four admissible orders on $\langle X \rangle$: the LR and RL orders corresponding to the two orders on X. We denote the LR one satisfying $z < y$ by $<_1$, the LR one satisfying $y < z$ by $<_2$, the RL one satisfying $z < y$ by $<_3$, the RL one satisfying $y < z$ by $<_4$. By the above observations, the degree n normal words with respect to $<_j$ are w_1, w_2 and v_j for $1 \leqslant j \leqslant 4$. Since the only words of degree $n+1$ with both degree n subwords being normal are words of the same form, the inequality $\dim A_{n+1} \geqslant 3$ yields that $\dim A_{n+1} = 3$ and that the degree $n+1$ normal words exactly as described above only one letter longer. On our way to a contradiction we consider the following two cases.

Case 1: n is even. Then $n = 2k+2$ for some $k \in \mathbb{N}$, $w_1 = (yz)^{k+1}$ and $w_2 = (zy)^{k+1}$. We take the third normal word $v_4 = (yz)^k y^2$ with respect to $<_4$ and write it as a linear combination of normal words with respect to $<_1$:

$$(yz)^k y^2 = p(yz)^{k+1} + q(zy)^{k+1} + rz^2(yz)^k \text{ in } A, \text{ where } p, q, r \in \mathbb{K}.$$

By (2.1), $A_{n-3}y^3 = \{0\}$ and $z^3 A_{n-3} = \{0\}$. Multiplying the equality in the above display by y on the right and these facts, we get

$$\begin{aligned} 0 = p(yz)^{k+1}y + qz(yz)^k y^2 + rz^2(yz)^k y &= p(yz)^{k+1}y + qz(p(yz)^{k+1} + q(zy)^{k+1} + rz^2(yz)^k) + rz^2(yz)^k y \\ &= p(yz)^{k+1}y + (q^2+r)z(yz)^{k+1} + pq(zy)^{k+1}z. \end{aligned}$$

Since three monomials in the last line are the three degree $n+1$ normal words with respect to $<_1$, we have $p = pq = q^2 + r = 0$. Hence $p = 0$ and $r = -q^2$ and therefore

$$(yz)^k y^2 = q(zy)^{k+1} - q^2 z^2 (yz)^k \text{ in } A, \text{ where } q \in \mathbb{K}.$$

Now we multiply this equality by z on the right:

$$(yz)^k y^2 z = q(zy)^{k+1} z - q^2 z^2 (yz)^k z.$$

Since $y^2 z + yzy + zyy$ is a linear combination of cubes and $A_{n-3}x^3 = 0$ for all $x \in A_1$, we have that $(yz)^k y^2 z + (yz)^k yzy + (yz)^k zy^2 = 0$. Plugging this into the above display, we get

$$(yz)^{k+1}y + (yz)^k zy^2 + q(zy)^{k+1}z - q^2 z^2 (yz)^k z = 0.$$

Since $(yz)^{k+1}y$ is a degree $n+1$ normal word with respect to $<_1$ and the remaining three monomials featuring in the above display are smaller that $(yz)^{k+1}y$ with respect to $<_1$, we arrive to a contradiction.

Case 2: n is odd. Then $n = 2k+1$ for some $k \in \mathbb{N}$, $w_1 = (yz)^k y$ and $w_2 = (zy)^k z$. First, we write $y(yz)^k$ as a linear combination of normal words with respect to $<_1$:

$$y(yz)^k = p(yz)^k y + q(zy)^k z + rz(zy)^k \text{ in } A, \text{ where } p, q, r \in \mathbb{K}.$$

Since of the four monomials featuring in the above display, the biggest one with respect to $<_3$ is $(yz)^k y$ and it also happens to be a normal word for the same order, we must have $p = 0$. Since of the four monomials featuring in the above display, the biggest one with respect to $<_4$ is $(zy)^k z$ and it also happens to be a normal word for the same order, we must have $q = 0$. Hence

$$y(yz)^k = rz(zy)^k \text{ in } A, \text{ where } r \in \mathbb{K}. \tag{2.5}$$

Now we do the same with $(yz)^k z$:

$$(yz)^k z = a(yz)^k y + b(zy)^k z + cz(zy)^k \text{ in } A, \text{ where } a, b, c \in \mathbb{K}.$$

Since the $<_1$-normal word $(yz)^k y$ is the biggest with respect to $<_1$ monomial present in the above display, $a = 0$. Note that, according to (2.5), $y[(yz)^k z] = [y(yz)^k]z = rz^2(yz)^k$. Thus, multiplying the above display by y on the left, we obtain

$$rz^2(yz)^k = b(yz)^{k+1} + cyz(zy)^k.$$

Since the $<_1$-normal word $(yz)^{k+1}$ is the biggest with respect to $<_1$ monomial present in the above display, $b = 0$. Coming back to the expression for $(yz)^k z$, we see that $(yz)^k z = a(yz)^k y$. Since both $(yz)^k z$ and $(yz)^k y$ are degree $n+1$ normal words with respect to $<_3$, we arrive to a contradiction, which completes the proof. $\square$

2.3 Proof of Part (G1) of Lemma 2.2

Let $1 \leqslant k \leqslant 3$, $n \geqslant \max\{2, k\}$ and let A be a finitely generated degree-graded algebra such that $\dim A_n = k$, $\dim A_{n+1} \geqslant k$ and $\lambda(A, n) = \rho(A, n) = 1$. For an ordered basis X in A_1, we denote the corresponding LR and RL-orders on $\langle X \rangle$ by $<$ and $\prec$, respectively. We have to show that for generic $X = (x_1, x_2, \dots) \in \Omega(A)$ with respect to both $<$ and $\prec$, the degree n normal words are $x_1^n, \dots, x_k^n$.

Note that according to the assumption $\lambda(A, n) = \rho(A, n) = 1$, we have that for every linear basis X in A_1 and every admissible order on $\langle X \rangle$, degree n normal words must start with pairwise distinct letters and they must end with pairwise distinct letters. Hence according to Lemma 2.1,

for every linear basis X in A_1 and every admissible order on $\langle X \rangle$,
degree n normal words must be of one of the following shapes: N1.1, N2.1, N2.2, N3.1, N3.2 or N3.3. (2.6)

We start by showing that

$$\text{there exist } x_1, \dots, x_k \in A_1 \text{ for which } x_1^n, \dots, x_k^n \text{ are linearly independent.} \tag{2.7}$$

By Lemma 2.3, $z^n \neq 0$ in A for generic $z \in A_1$, which proves (2.7) for $k = 1$. For $k \geqslant 2$, we pick $z \in A_1$ for which $z^n \neq 0$ in A and pick $X \in \Omega(A, z)$. We equip $\langle X \rangle$ with the LR order $<$. Since z^n is the smallest degree n word and $z^n \neq 0$ in A, z^n is one of the degree n normal words with respect to $<$. If additionally $k = 2$, then by (2.6), the second degree n normal word is also a power of some $x \in X$, which proves (2.7) in the case $k = 2$. It remains to consider the case $k = 3$. Since we also know that z^n is a normal word, (2.6) implies that the degree n normal words must be of one of the forms N3.1 or N3.2. If they are of the form N3.1, (2.7) is satisfied. It remains to consider the case N3.2. Then the degree n normal words are z^n, $w = xyxy \cdots = xyw'$ and $yxyx \dots$ for distinct $x, y \in X \setminus \{z\}$. Without loss of generality, $x > y$. Obviously $w = -yxw' + (x+y)^2 w' - x^2 w' - y^2 w'$. Since w is a normal word and $z^n < w$, $yxw' < w$ at least one of the elements $(x+y)^2 w'$, $x^2 w'$ or $y^2 w'$ has to be non-proportional to z^n. Hence there is $a \in A_1$ and $v \in A_{n-2}$ such that a and z are linearly independent and $a^2 v$ and z^n are linearly independent. Now take (new) $X \in \Omega(A, z, a)$, equipping $\langle X \rangle$ with the LR order. As above, z^n is the smallest degree n normal word. Since $\lambda(A, n) = 1$ no other normal word starts with z. Since $a^2 v$ and z^n are linearly independent, the second smallest degree n normal word must start with either az or a^2. Hence by (2.6), only the form N3.1 of normal words is viable: all three degree n normal words are n^{th} powers, which proves (2.7) in the final case $k = 3$.

By (2.7), $x_1^n, \dots, x_k^n$ are linearly independent in A for generic $x_1, \dots, x_k \in A_1$. Hence for generic $X = (x_1, x_2, \dots) \in \Omega(A)$, $x_1^n, \dots, x_k^n$ are linearly independent. Now assume that X is such a basis. Since x_1^n is the smallest degree n word with respect to both $<$ and $\prec$ and $x_1^n \neq 0$ in A, x_1^n is the smallest degree n normal word with respect to both $<$ and $\prec$. This completes the proof in the case $k = 1$. Assume now that $k = 2$. Since x_1^n is a normal word (2.6) implies that the second normal word with respect to each of the orders $<$ and $\prec$ is an n^{th} power. Since x_1^n and x_2^n are linearly independent in A and are the two smallest n^{th} powers for both $<$ and $\prec$, x_1^n and x_2^n are degree n normal words with respect to both $<$ and $\prec$, which completes the proof in the case $k = 2$. Finally, assume that $k = 3$. Since x_1^n is the smallest degree n normal word for both $<$ and $\prec$, (2.6) implies that the second smallest degree n normal words is either an n^{th} power of a letter or has the form $abab \dots$ for some distinct $a, b \in X \setminus \{x_1\}$. Since x_1^n and x_2^n are linearly independent and $x_2^n < abab \dots$, $x_2^n \prec abab \dots$, the latter is impossible. Hence the second smallest degree n normal words is an n^{th} power for both $<$ and $\prec$. Now by (2.6) all three normal words are n^{th} powers for both $<$ and $\prec$. Since x_1^n, x_2^n and x_3^n are linearly independent in A and are the smallest three n^{th} powers for both $<$ and $\prec$, we have that the degree n normal words are x_1^n, x_2^n and x_3^n with respect to both $<$ and $\prec$. The proof is complete.

2.4 Proof of Parts (G2) and (G3) of Lemma 2.2

Parts (G2) and (G3) are clearly equivalent. Indeed, they transform to one another when we pass to the opposite multiplication. Thus it suffices to verify (G2). Let $n \geqslant 2$ and let A be a finitely generated degree-graded algebra such that $\dim A_n = 2$, $\dim A_{n+1} \geqslant 2$ and $\lambda(A,n) = 2$. For an ordered basis X in A_1 we always equip $\langle X\rangle$ with the LR order $<$. We have to show that for generic $X = (x_1, x_2, \dots) \in \Omega(A)$, the degree n normal words are x_1^n and $x_1^{n-1}x_2$.

By Lemma 2.3, $z^n \neq 0$ for generic $z \in A_1$. Since $\lambda(A,n) = 2$, $\dim zA_{n-1} = 2$ for generic $z \in A_1$. Hence for generic $z \in A_1$, we have both $z^n \neq 0$ and $\dim zA_{n-1} = 2$. For such a z and for every $X \in \Omega(A,z)$, z^n is a degree n normal word and both degree n normal words must start with z. Only N2.3 of Lemma 2.1 fits. Hence the second degree n normal word is $z^{n-1}y$ for some $y \in X\setminus\{z\}$.

Hence z^n and $z^{n-1}y$ are linearly independent in A for generic $z, y \in A_1$. Now take such z, y and let $X \in \Omega(A,z,y)$. Since z^n and $z^{n-1}y$ are linearly independent in A and are the two smallest words with respect to $<$, they are the degree n normal words and (G2) follows.

2.5 Proof of Part (G4) of Lemma 2.2

Lemma 2.4. *Let A be a finitely generated degree graded algebra, whose ideal of relations is generated by some homogeneous elements of degree n, where $n \geqslant 3$. Assume also that $\dim A_n = 3$ and that $(\lambda(A,n), \rho(A,n)) \in \{(1,2),(2,1)\}$. Then $\dim A_{n+1} < 3$.*

Proof. Assume the contrary: $\dim A_{n+1} \geqslant 3$. This will allow us to apply Lemma 2.1. Since the two cases $(\lambda(A,n), \rho(A,n)) = (1,2)$ and $(\lambda(A,n), \rho(A,n)) = (2,1)$ are clearly equivalent to each other (passing to the opposite multiplication reduces on case to the other), we can without loss of generality assume that $\lambda(A,n) = 2$ and $\rho(A,n)) = 1$.

Since $\lambda(A,n) = 2$, $\dim xA_{n-1} = 2$ for generic $x \in A_1$. First, we show that $x^2A_{n-2} \neq \{0\}$ for generic $x \in A_1$. Assume the contrary. Then $x^2A_{n-2} = \{0\}$ for all $x \in A_1$. Pick any $a \in A_1$ such that $\dim aA_{n-1} = 2$ and let $X \in \Omega(A,a)$ with $\langle X\rangle$ carrying the LR order $<$. Since $\dim aA_{n-1} = 2$, exactly two degree n normal words start with a. Since $\rho(A,n) = 1$, the (all three) degree n normal words end with pairwise distinct letters. Out of all options provided by Lemma 2.1, only N3.6 and N3.7 fit these requirements. The condition $x^2A_{n-2} = \{0\}$ for all $x \in A_1$ excludes N3.7. Hence the three normal words are given by N3.6: $abab\dots$, $w = baba\dots = baw'$ and $w_3 = abab\dots c$, where $b, c \in X\setminus\{a\}$ are distinct. Now $w = baw'$ and since $x^2A_{n-2} = \{0\}$ for all $x \in A_1$, we have $a^2w' = b^2w' = (a+b)^2w' = 0$. Hence $(ab+ba)w' = 0$ and therefore $w = -abw'$. Since w is normal and $abw' < w$, we have arrived to a contradiction. Thus $x^2A_{n-2} \neq \{0\}$ for a generic $x \in A_1$. Hence for a generic $x \in A_1$ we have that both $\dim xA_{n-1} = 2$ and $\dim x^2A_{n-2} > 0$.

Now pick $a \in A_1$ such that $\dim aA_{n-1} = 2$ and $\dim a^2A_{n-2} > 0$ and let $X \in \Omega(A,a)$ with $\langle X\rangle$ carrying the LR order $<$. Then exactly two of the degree n normal words must start with a and at least one must start with a^2, while the three normal words must end with pairwise distinct letters. Of all options provided by Lemma 2.1 only N3.7 fits and therefore the degree n normal words are a^n, b^n and $a^{n-1}c$, where $b, c \in X$ are such that a, b, c are pairwise distinct.

It follows that z^n, $z^{n-1}y$, x^n form a linear basis in A_n for generic $x, y, z \in A_1$. Hence z^n, y^n are linearly independent and z^n, $z^{n-1}y$ are linearly independent for generic $z, y \in A_1$. Take such z, y and let $X \in \Omega(A,z,y)$ with $\langle X\rangle$ carrying the LR order $<$. Since z^n and $z^{n-1}y$ are two smallest degree n monomials and are linearly independent in A, both are degree n normal words. Since the trio of normal words must be of the form N3.7, the third normal word is x^n for some $x \in X\setminus\{y,z\}$. Since $y^n < x^n$, y^n belongs to the linear span of z^n and $z^{n-1}y$. Since z^n and y^n are linearly independent, it follows that the linear span of z^n and $z^{n-1}y$ coincides with the linear span of z^n and y^n. This happens for generic $y, z \in A_1$. Thus for generic $x, y, z \in A_1$, z^n, $z^{n-1}y$, x^n form a linear basis in A_n and $\mathtt{span}\,\{z^n, x^n\} = \mathtt{span}\,\{z^n, z^{n-1}x\}$. Hence z^n, $z^{n-1}y$ and $z^{n-1}x$ are linearly independent in A_n, yielding $\dim zA_{n-1} = 3$, which contradicts the condition $\lambda(A, n) = 2$. This contradiction completes the proof. $\square$

For the duration of the proof of (G4) of Lemma 2.2 only, we shall introduce the following extra notation. Let A be a finitely generated degree graded algebra and $n \geqslant 3$. We denote

$$\lambda^+(A, n) = \max_{x \in A_1} \dim x^2 A_{n-2} \quad \text{and} \quad \rho^+(A, n) = \max_{x \in A_1} \dim A_{n-2} x^2.$$

Obviously, $\lambda^+(A, n) \leqslant \lambda(A, n)$ and $\rho^+(A, n) \leqslant \rho(A, n)$.

Lemma 2.5. *Let A be a finitely generated degree graded algebra, whose ideal of relations is generated by some homogeneous elements of degree n, where $n \geqslant 3$. Assume also that $\lambda(A, n) = \rho(A, n) = 2$, $\dim A_n = 3$ and $\dim A_{n+1} \geqslant 3$. Then $\max\{\lambda^+(A, n), \rho^+(A, n)\} = 2$.*

Proof. By Lemma 2.3, $x^n \neq 0$ in A for generic $x \in A_1$. Assume the contrary. Then $\dim x^2 A_{n-2} \leqslant 1$ and $\dim A_{n-2} x^2 \leqslant 1$ for all $x \in A_1$. This together with the equality $\lambda(A, n) = \rho(A, n) = 2$ means that

$$\dim xA_{n-1} = \dim A_{n-1}x \text{ and } x^2 A_{n-2} = A_{n-2} x^2 = \texttt{span}\{x^n\} \neq \{0\} \text{ for generic } x \in A_1. \tag{2.8}$$

Hence we can pick a linear basis X in A_1 such that each $x \in X$ satisfies all the properties from (2.8). For $a \in X$ and any degree-lexicographical order on $\langle X \rangle$ for which a is the minimal element of X, a^n must be a normal word. Since $\dim x^2 A_{n-2} \leqslant 1$, $\dim A_{n-2} x^2 \leqslant 1$ for each $x \in A_1$, no other degree n normal word can start or end with a^2. Since $\dim aA_{n-1} = \dim A_{n-1}a = 2$ and a is the minimal letter, there is another normal word either starting or ending with a. The only option from Lemma 2.1 fitting this description is N3.8. Thus there is $b \in X \setminus \{a\}$ such that the degree n normal words are a^n, $abab\dots$ and $baba\dots$. The same three words are normal for any degree-lexicographical order on $\langle X \rangle$ for which a is the minimal element of X and b is second minimal. By the same argument the degree n normal words for any degree-lexicographical order on $\langle X \rangle$ for which b is the minimal element of X and a is second minimal are b^n, $abab\dots$ and $baba\dots$.

As in the proof of the previous lemma, consider two total orders on X such that a is the smallest element and b is second smallest for the first one, while b is the smallest element and a is second smallest for the second one. Now we have four admissible orders on $\langle X \rangle$: the corresponding LR and RL orders. We denote the LR one satisfying $a < b$ by $<_1$, the LR one satisfying $b < a$ by $<_2$, the RL one satisfying $a < b$ by $<_3$ and the RL one satisfying $b < a$ by $<_4$. The degree n words a^n, $abab\dots$ and $baba\dots$ are normal for $<_1$ and $<_3$, while the degree n words b^n, $abab\dots$ and $baba\dots$ are normal for $<_2$ and $<_4$. Since the only words of degree $n+1$ with both degree n subwords being normal are words of the same form, the inequality $\dim A_{n+1} \geqslant 3$ yields that $\dim A_{n+1} = 3$ and that the degree $n+1$ normal (with respect to each of the four orders) words exactly as described above only one letter longer.

Since $a^{n-1}b$ is smaller with respect to $<_1$ than each of the degree n normal words $abab\dots$ and $baba\dots$ and ab^{n-1} is $<_4$ smaller than the same, there exist $p, q \in \mathbb{K}$ such that

$$a^{n-1}b = pa^n \quad \text{and} \quad ab^{n-1} = qb^n.$$

Multiplying the first of the above equalities by b on the right, we get $a^{n-1}b^2 = pa^n b = p^2 a^{n+1}$, while multiplying the second equation by a on the left, we similarly get $a^2 b^{n-1} = q^2 b^{n+1}$. If $n \geqslant 4$, then the only $<_1$ normal word smaller than $a^2 b^{n-2}$ is a^n and the only $<_4$ smaller than $a^2 b^{n-2}$ is b^4. Hence $a^{n-2}b^2$ is a scalar multiple of a^n and $a^2 b^{n-2}$ is a scalar multiple of b^n. Since $a^{n-1}b^2 = pa^n b = p^2 a^{n+1}$, $a^2 b^{n-1} = q^2 b^{n+1}$ and $a^{n+1} \neq 0$, $b^{n+1} \neq 0$, it follows that $a^{n-2}b^2 = p^2 a^n$ and $a^2 b^{n-2} = q^2 b^n$. Iterating these arguments, we get

$$a^{n-j}b^j = p^j a^n \text{ and } a^j b^{n-j} = q^j b^n \text{ for } 2 \leqslant j \leqslant n \tag{2.9}$$

and
$$a^{n+1-j}b^j = p^j a^{n+1} \text{ and } a^j b^{n+1-j} = q^j b^{n+1} \text{ for } 1 \leqslant j \leqslant n-1. \tag{2.10}$$

Let $w = w_1 \dots w_n \in \langle a, b\rangle_n$ be such that $w <_1 abab\dots$. Since the only normal word $<_1$ smaller than w is a^n, $w = \alpha a^n$ for some $\alpha \in \mathbb{K}$. Then $aw = \alpha a^{n+1}$. On the other hand, $aw_1 \dots w_{n-1} <_1 abab\dots$ and therefore $aw_1 \dots w_{n-1} = \beta a^n$ for some $\beta \in \mathbb{K}$. Hence $aw = \beta a^n w_n$. By (2.10), $\beta a^n w_n = \beta a^{n+1}$ if $w_n = a$ and $\beta a^n w_n = p\beta a^{n+1}$ if $w_n = b$. Thus $\beta = \alpha$ if $w_n = a$ and $\beta = p\alpha$ if $w_n = b$. Hence $w = aw_1 \dots w_{n-1}$ if $w_n = a$ and $w = paw_1 \dots w_{n-1}$ if $w_n = b$. Iterating this argument, we get $w = p^{\mathtt{deg}_b w} a^n$, where $\mathtt{deg}_b w$ is the number of b's featuring in w. Similar argument applied to the other three orderings yields

$$\text{for every } w \in \langle a,b\rangle_n,\ \begin{array}{ll} w = p^{\mathtt{deg}_b w} a^n & \text{if } w <_1 abab\dots \text{ or } w <_3 \dots baba, \\ w = q^{\mathtt{deg}_a w} b^n & \text{if } w <_2 baba\dots \text{ or } w <_4 \dots abab. \end{array} \tag{2.11}$$

By (2.10), $a^{n-1}b^2 = p^2 a^{n+1} = q^{n-1}b^{n+1}$. Since $a^{n+1} \neq 0$ and $b^{n+1} \neq 0$, we have that either $p = q = 0$ or $pq \neq 0$. First, we show that the case $p = q = 0$ can not occur. Indeed, assume that $p = q = 0$. Since the degree n words a^n, $abab\dots$ and $baba\dots$ span A_n, $b^n = \alpha a^n + \beta abab \dots + \gamma baba\dots$ for some $\alpha, \beta, \gamma \in \mathbb{K}$. Since both $\{a^n, abab\dots, baba\dots$ and $\{b^n, abab\dots, baba\dots\}$ are linear bases in A_n, $\alpha \neq 0$. Multiplying by a on the left, we get $ab^n = \alpha a^{n+1} + \beta a^2 baba \dots + \gamma ababa\dots$. Since $q = 0$, (2.10) implies that $ab^n = 0$. Since $p = 0$, (2.11) yields $a^2baba\dots = 0$. Hence $\alpha a^{n+1} + \gamma abab \dots = 0$ and $\alpha \neq 0$. Then the degree $n+1$ words a^{n+1} and $abab\dots$ are linearly dependent, which contradicts the fact that they are normal with respect to $<_1$. This contradiction proves that $pq \neq 0$. Now we shall verify that

$$pq = 1 \text{ and } b^n = p^n a^n. \tag{2.12}$$

The equalities (2.10) imply that $ab^n = qb^{n+1} = p^n a^{n+1}$ and $a^2 b^{n-1} = q^2 b^{n+1} = p^{n-1}a^{n+1}$. Since $pq \neq 0$ and a^{n+1}, b^{n+1} are non-zero, $\frac{p^n}{q} = \frac{p^{n-1}}{q^2}$ and therefore $pq = 1$. Furthermore, now $ab^n = qb^{n+1} = p^n a^{n+1}$ yields $b^{n+1} = p^{n+1}a^{n+1}$. If $n \geqslant 4$, (2.9) implies $a^{n-2}b^2 = p^2 a^n = q^{n-2}b^n$. Since $pq = 1$, we have $b^n = p^n a^n$. In order to complete the proof of (2.12), it remains to verify that $b^3 = p^3 a^3$ in the case $n = 3$. Writing b^3 as a linear combination of $<_1$ normal words, we see that $b^3 = \alpha a^3 + \beta aba + \gamma bab$ for some $\alpha, \beta, \gamma \in \mathbb{K}$. Multiplying by a on the left and using (2.10), we get $ab^3 = qb^4 = p^3 a^4$ and $ab^3 = \alpha a^4 + \beta a^2 ba + \gamma abab = (\alpha + p\beta)a^4 + \gamma abab$. Hence $(\alpha + p\beta - p^3)a^4 + \gamma abab = 0$. Since both a^4 and $abab$ are normal words with respect to $<_1$, $\gamma = 0$ and $\alpha + p\beta - p^3 = 0$. Thus $b^3 = (p^3 - p\beta)a^3 + \beta aba$, where $\beta \in \mathbb{K}$. Multiplying this equality by b on the right and using (2.10), we get $b^4 = (p^3 - p\beta)a^3 b + \beta abab = p^2(p^2 - \beta)a^4 + \beta abab$. Since we already know that $b^4 = p^4 a^4$, $\beta(abab - p^2 a^4) = 0$. Since both a^4 and $abab$ are normal words with respect to $<_1$, $\beta = 0$. Hence $b^3 = p^3 a^3$, as required. This completes the proof of (2.12).

In particular, a^n and b^n are proportional. Now recall that this happens for generic a and b in A_1. Hence
$$\mathtt{span}\,\{x^n : x \in A_1\} \text{ is one-dimensional.} \tag{2.13}$$
By scaling b, if necessary, we can turn p into 1. Then $q = 1$ and $b^n = a^n$. Now (2.11) can be rewritten as

$$\text{for every } w \in \langle a,b\rangle_n \text{ different from } abab\dots \text{ and } baba\dots,\ w = a^n \text{ in } A. \tag{2.14}$$

Now (2.14) yields
$$(a+b)^n = (2^n - 2)a^n + abab \dots + baba\dots.$$

Since a^n, $abab\dots$ and $baba\dots$ (being $<_1$ normal words) are linearly independent, a^n and $(a+b)^n$ are linearly independent as well, which contradicts (2.13). This contradiction completes the proof. $\square$

Lemma 2.6. *Let A be a finitely generated degree graded algebra, whose ideal of relations is generated by some homogeneous elements of degree n, where $n \geqslant 3$. Assume also that $\lambda(A,n)=\rho(A,n)=2$, $\dim A_n=3$ and $\dim A_{n+1}\geqslant 3$. Then $\lambda^+(A,n)=\rho^+(A,n)=2$ and $x^2A_{n-2}+A_{n-2}x^2=A_n$ for generic $x\in A_1$.*

Proof. Assume the contrary to the conclusion of the lemma. Then for all $x\in A_1$,

$$\text{either } \dim x^2A_{n-2}<2 \text{ or } \dim A_{n-2}x^2<2 \text{ or } \dim\,(x^2A_{n-2}+A_{n-2}x^2)<3. \tag{2.15}$$

By Lemma 2.5, $\max\{\lambda^+(A,n),\rho^+(A,n)\}=2$. Passing to the opposite multiplication, if necessary, we can without loss of generality assume that $\lambda^+(A,n)=2$. Hence $\dim x^2A_{n-2}=2$ for generic $x\in A_1$. By Lemma 2.3, $x^n\neq 0$ in A for genric $x\in A_1$. Pick $z\in A_1$ such that $\dim z^2A_{n-2}=2$ and $z^n\neq 0$ in A and let $X\in\Omega(A,z)$ with $\langle X\rangle$ carrying the LR order $<$. Since $z^n\neq 0$ in A and z^n is the smallest degree n word, z^n is the smallest degree n normal word. Since $\dim z^2A_{n-2}=2$, the second smallest degree n normal words must start with z^2. Furthermore, the assumption $\lambda(A,n)=\rho(A,n)=2$ implies that the three normal words do NOT start with the same letter or end with the same letter. This leaves us with the options N3.7, N3.12, N3.13 and N3.14 of Lemma 2.1. Since N3.13 and N3.14 are incompatible with (2.15), the only possibilities are N3.7 and N3.12. If N3.7 occurs, we have that

$$z^n,\ y^n \text{ and } z^{n-1}y \text{ form a linear basis in } A_n \text{ for generic } y,z\in A_1. \tag{2.16}$$

If (2.16) fails, then N3.12 is the only option and we have

$$\begin{array}{l} z^n,\ y^n \text{ and } z^{n-1}y \text{ are linearly dependent in } A_n \text{ for all } y,z\in A_1 \text{ and}\\ z^n,\ y^n \text{ and } z^{n-1}x \text{ form a linear basis in } A_n \text{ for generic } x,y,z\in A_1.\end{array} \tag{2.17}$$

If (2.17) is satisfied, then z^n and y^n are linearly independent in A for generic $y,z\in A_1$, z^n and $z^{n-1}y$ are linearly independent in A for generic $y,z\in A_1$, while z^n, y^n and $z^{n-1}y$ are linearly dependent in A_n for all $y,z\in A_1$. It follows that for generic $y,z\in A_1$, the pair z^n,y^n spans the same two-dimensional space as the pair $z^n,z^{n-1}y$. Since for generic $x,y,z\in A_1$, monomials z^n, y^n and $z^{n-1}x$ form a linear basis in A_n, it follows that for generic $x,y,z\in A_1$, words z^n, $z^{n-1}y$ and $z^{n-1}x$ form a linear basis in A_n as well. Hence $\dim zA_{n-1}=3$, which contradicts the equality $\lambda(A,n)=2$. Thus (2.17) can not hold and therefore (2.16) is satisfied. That is, z^n, y^n and $z^{n-1}y$ form a linear basis in A_n for generic $y,z\in A_1$.

First, we verify that $\dim A_{n-2}x^2=2$ for some $x\in A_1$. Assume the contrary. Then $A_{n-2}y^2$ is one-dimensional and is spanned by y^n for generic $y\in A_1$ ($y^n\neq 0$ for generic $y\in A_1$). Thus for generic $y,z\in A_1$, z^n, y^n and $z^{n-1}y$ form a linear basis in A_n and $A_{n-2}y^2$ is spanned by y^n. Take such y and z and let $X\in\Omega(A,z,y)$ with $\langle X\rangle$ carrying the LR order $<$. Since z^n and $z^{n-1}y$ are linearly independent in A and are the two smallest degree n words, z^n and $z^{n-1}y$ are normal words with respect to $<$. Since, as we have already observed, the form of degree n normal words must be either N3.7 or N3.12, the third degree n normal word must be x^n for some $x\in X\setminus\{z\}$. Since z^n, y^n and $z^{n-1}y$ are linearly independent and $y^n\leqslant x^n$ for all $x\in X\setminus\{z\}$, the third normal word is y^n. Hence the three degree n normal words are z^n, y^n and $z^{n-1}y$. Since $z^{n-2}y^2<y^n$, $z^{n-2}y^2=az^n+bz^{n-1}y$ for some $a,b\in\mathbb{K}$. On the other hand, $A_{n-2}y^2$ is spanned by y^n and therefore $z^{n-2}y^2=-cy^n$ for some $c\in\mathbb{K}$. Then $az^n+bz^{n-1}y+cy^n=0$ and therefore $a=b=c=0$. Hence $z^{n-2}y^2=0$. Thus $z^{n-2}y^2=0$ for generic $y,z\in A_1$. Then $z^{n-2}y^2=0$ for all $y,z\in A_1$. Plugging in $y=z$, we get $z^n=0$ for all $z\in A_1$, which is a contradiction. Hence $\dim A_{n-2}x^2=2$ for some $x\in A_1$ and therefore $\dim A_{n-2}x^2=2$ for generic $x\in A_1$. Since $\dim A_{n-1}x=2$ for generic $x\in A_1$, we see that $A_{n-2}x^2=A_{n-1}x$ for generic $x\in A_1$.

Now, using the exact same argument as above with the RL order instead of the LR one, we see that z^n, yz^{n-1} and y^n form a linear basis in A_n for generic $y,z\in A_1$. Now for generic $y,z\in A_1$, both

$\{z^n, y^n, z^{n-1}y\}$ and $\{z^n, y^n, zy^{n-1}\}$ are linear bases in A_n. Take such y, z and let $X \in \Omega(A, z, y)$ with $\langle X\rangle$ carrying the LR order $<$. Then for every $w \in \langle y, z\rangle_n$ containing both y and z, we have $w < y^n$ and therefore $w = pz^n + qz^{n-1}y$ for some $p, q \in \mathbb{K}$. Now, $\{z^n, y^n, zy^{n-1}\}$ is the set of degree n normal words with respect to the RL order $<_1$ on $\langle X\rangle$ corresponding to a total order on X for which y is the minimal element and z is second minimal. Since $w <_1 z^n$, $w = ay^n + bzy^{n-1}$ for some $a, b \in \mathbb{K}$. Since $pz^n + qz^{n-1}y = ay^n + bzy^{n-1}$ and $zy^{n-1} < y^n$, $z^{n-1}y < y^n$, $z^n < y^n$ with y^n being a normal word with respect to $<$, we have $a = 0$. Similarly, $p = 0$ since $zy^{n-1} <_1 z^n$, $z^{n-1}y <_1 z^n$ and $y^n <_1 z^n$. Thus $w = qz^{n-1}y = bzy^{n-1}$. Hence

$$\text{for generic } y, z \in A_1, \text{ every } w \in \langle y, z\rangle_n \text{ containing both } z \text{ and } y \text{ is a scalar multiple of } z^{n-1}y. \tag{2.18}$$

In particular, $yz^{n-1} = \alpha(y, z)z^{n-1}y$ for some $\alpha(y, z) \in \mathbb{K}^*$ for generic y, z ($\alpha(y, z)$ is non-zero generically, since we already know that $yz^{n-1} \neq 0$) for generic $y, z \in A_1$. Next, we show that $\alpha(y, z)$ (generically) does not depend on y. Indeed, $yz^{n-1} = \alpha(y, z)z^{n-1}y$ and $xz^{n-1} = \alpha(x, z)z^{n-1}x$ yields $(x + y)z^{n-1} = \alpha(x + y, z)z^{n-1}(x + y) = \alpha(x, z)z^{n-1}x + \alpha(y, z)z^{n-1}y$ and therefore $z^{n-1}\big((\alpha(x + y, z) - \alpha(x, z))x + (\alpha(x + y, z) - \alpha(x, z))y\big) = 0$. With α being a non-zero rational function on $A_1 \times A_1$, we have $z^{n-1}u = 0$ for generic $z, u \in A_1$ unless α does not depend on the first argument. Thus for generic $z \in A_1$, $yz^{n-1} = \alpha(z)z^{n-1}y$ for generic $y \in A_1$. Hence for generic $z \in A_1$, $yz^{n-1} = \alpha(z)z^{n-1}y$ for all $y \in A_1$. Plugging in $y = z$, we get $\alpha(z) = 1$. Hence $yz^{n-1} = z^{n-1}y$ for generic $z \in A_1$ for all $y \in A_1$. It follows that $yz^{n-1} = z^{n-1}y$ for all $y, z \in A_1$. Now let again $<$ be a left-to-right degree-lexicographical order associated with generic $z, y \in A_1$ in the same way as above. Since the only degree $n + 1$ words with both degree n subwords from the list $\{z^n, y^n, z^{n-1}y\}$ are z^{n+1}, y^{n+1} and z^ny and since $\dim A_{n+1} \geqslant 3$, we have $\dim A_{n+1} \geqslant 3$ and degree $n+1$ normal words with respect to $<$ are z^{n+1}, y^{n+1} and z^ny. Then all the same arguments apply to words of degree $n + 1$ and we have $z^ny = yz^n$. By (2.18), $z^{n-2}yz = az^{n-1}y$ for some $a \in \mathbb{K}$. Hence $z^{n-1}yz = az^ny$. Using the equalities $yz^{n-1} = z^{n-1}y$ and $yz^n = z^ny$, we get $z^{n-1}yz = yz^n = z^ny$. Hence $z^ny = az^ny$ and therefore $a = 1$ and $z^{n-2}yz = z^{n-1}y$. Next, by (2.18), for all $p \in \mathbb{K}$, $z^{n-2}(y + pz)^2$ is a scalar multiple of $z^{n-1}(y + pz)$. That is, for each $p \in \mathbb{K}$, there is $f(p) \in \mathbb{K}$ such that $z^{n-2}(y + pz)^2 = f(p)z^{n-1}(y + pz)$. Using the equality $z^{n-2}yz = z^{n-1}y$, we get $z^{n-2}y^2 = (f(p) - 2p)z^{n-1}y + (pf(p) - p^2)z^n$. Since the left-hand side does not depend on p, there are constants $q, r \in \mathbb{K}$ such that $f(p) - 2p = q$ and $pf(p) - p^2 = r$ for all $p \in \mathbb{K}$. Multiplying the first equation by p and subtracting from the second, we get $p^2 + qp - r = 0$ for all $p \in \mathbb{K}$. Since this is obviously nonsense, we arrive to a contradiction, which completes the proof. □

Now we are ready to prove Part (G4) of Lemma 2.2. Let $n \geqslant 3$ and A be a finitely generated degree-graded algebra such that $\max\{\lambda(A, n), \rho(A, n)\} = 2$, $\dim A_n = 3$ and $\dim A_{n+1} \geqslant 3$. For an ordered basis X in A_1, we denote the corresponding LR and RL orders on $\langle X\rangle$ by $<$ and $\prec$, respectively. The proof will be complete if we show that for generic $X = (x_1, x_2, \dots) \in \Omega(A)$, $\mathrm{NW}_n = \{x_1^n, x_1^{n-1}x_2, x_2x_1^{n-1}\}$ with respect to both $<$ and $\prec$.

By Lemma 2.4, $\lambda(A, n) = \rho(A, n) = 2$. By Lemmas 2.3 and 2.6, for generic $z \in A_1$, we have $z^n \neq 0$, $\dim z^2A_{n-2} = 2$, $\dim A_{n-2}z^2 = 2$ and $z^2A_{n-2} + A_{n-2}z^2 = A_n$. Pick such a z and let $X \in \Omega(A, z)$ with $\langle X\rangle$ equipped with the LR order $<$. Since $z^n \neq 0$ in A and z^n is the smallest degree n word, z^n is the smallest degree n normal word. Since $\dim z^2A_{n-2} = 2$, the second smallest degree n normal words must start with z^2. Since $\lambda(A, n) = 2$, the third degree n normal word can not start with z. Now out of all options provided by Lemma 2.1 only N3.7, N3.12, N3.13 and N3.14 fit the description. It follows that the two normal words starting with z^2 must be z^n and $z^{n-1}y$ for some $y \in X \setminus \{z\}$. In particular, z^n and $z^{n-1}y$ are linearly independent in A for generic z, y.

Now for generic $z, y \in A_1$, we have that z^n and $z^{n-1}y$ are linearly independent in A, $z^n \neq 0$, $\dim z^2A_{n-2} = 2$, $\dim A_{n-2}z^2 = 2$ and $z^2A_{n-2} + A_{n-2}z^2 = A_n$. Pick such z, y and let $X \in \Omega(A, z)$.

We equip $\langle X\rangle$ with the following admissible order $<_0$:

$$\begin{array}{l}\text{for } u,w\in\langle X\rangle,\ u<_0 w \text{ if } \mathtt{deg}\,u<\mathtt{deg}\,w,\\ \text{if } \mathtt{deg}\,u=\mathtt{deg}\,w,\ u<_0 w \text{ provided } u \text{ contains more } z\text{'s (that is, } \mathtt{deg}_z u>\mathtt{deg}_z w),\\ \text{we break the remaining ties by using the left-to-right lexicographical order.}\end{array}\tag{2.19}$$

Since z^n and $z^{n-1}y$ are linearly independent in A and are the two smallest degree n words with respect to $<_0$, they are normal words. Since $\dim\big(z^2A_{n-2}+A_{n-2}z^2\big)=3$, the last degree n normal word must contain at least two z. This excludes N3.7 and N3.12 leaving only N3.13 and N3.14. That is, the third degree n normal word is either yz^{n-1} or xz^{n-1} with $x\in X\setminus\{y,z\}$. Thus we have verified that at least one of the normal word forms N3.13 or N3.14 occurs. If N3.14 occurs, then

$$z^n,\ z^{n-1}y \text{ and } yz^{n-1} \text{ form a linear basis in } A_n \text{ for generic } y,z\in A_1.\tag{2.20}$$

If (2.20) fails, then N3.13 is the only option and we have

$$\begin{array}{l}z^n,\ z^{n-1}y \text{ and } yz^{n-1} \text{ are linearly dependent in } A_n \text{ for all } y,z\in A_1 \text{ and}\\ z^n,\ z^{n-1}y \text{ and } xz^{n-1} \text{ form a linear basis in } A_n \text{ for generic } x,y,z\in A_1.\end{array}\tag{2.21}$$

Now we show that (2.21) is impossible. Indeed, assume that (2.21) is satisfied. Then for generic $y,z\in A_1$, the pairs $z^n,z^{n-1}y$ and z^n,yz^{n-1} span the same two-dimensional space. Hence for generic $x,y,z\in A_1$, the linear spans of $\{z^n,z^{n-1}y,xz^{n-1}\}$ and $\{z^n,z^{n-1}y,z^{n-1}x\}$ coincide. It follows that $\dim zA_{n-1}\geqslant 3$ for generic $z\in A_1$, which contradicts the equality $\lambda(A,n)=2$. Hence (2.20) must be satisfied.

Thus for generic $z,s\in A_1$, z^n, $z^{n-1}s$ and sz^{n-1} form a linear basis in A_n. Pick such z,s and let $X\in\Omega(A,z)$ with $\langle X\rangle$ carrying the LR order $<$. The two smallest degree n words z^n and $z^{n-1}s$ are linearly independent in A and therefore are normal words. Since $\lambda(A,n)=2$, the third normal word must start with a letter different from z. Since sz^{n-1} is the smallest such word and z^n, $z^{n-1}s$ and sz^{n-1} are linearly independent in A, the three normal words with respect to $<$ are z^n, $z^{n-1}s$ and sz^{n-1}. Similar argument shows that with respect to the RL order $\prec$ the triple of degree n normal words is the same. Since all this holds for generic z,s, (G4) of Lemma 2.2 follows.

2.6 Proof of Parts (G5) and (G6) of Lemma 2.2

Lemma 2.7. *Let A be a finitely generated degree graded algebra, whose ideal of relations is generated by some homogeneous elements of degree n, where $n\geqslant 3$. Assume also that* $\dim A_n=3$, $\dim A_{n+1}\geqslant 3$ *and $z\in A_1$. If* $\dim zA_{n-1}=3$, *then* $\dim A_{n+1}=3$ *and the following cancellation rule holds*:

$$\text{if } w\in A_n \text{ and } zw=0 \text{ in } A, \text{ then } w=0 \text{ in } A.\tag{2.22}$$

Furthermore, for every $1\leqslant j\leqslant n-1$,

$$\text{if } w\in A_{n-j} \text{ and } z^jw=0 \text{ in } A, \text{ then } ww_1=0 \text{ in } A \text{ for all } w_1\in A_j.\tag{2.23}$$

If $\dim A_{n-1}z=3$, *then* $\dim A_{n+1}=3$ *and the opposite cancellation rule is satisfied*:

$$\text{if } w\in A_n \text{ and } wz=0 \text{ in } A, \text{ then } w=0 \text{ in } A.\tag{2.24}$$

Furthermore, for every $1\leqslant j\leqslant n-1$,

$$\text{if } w\in A_{n-j} \text{ and } wz^j=0 \text{ in } A, \text{ then } w_1w=0 \text{ in } A \text{ for all } w_1\in A_j.\tag{2.25}$$

Proof. The two parts are obviously equivalent: just pass to the opposite multiplication. Thus it is enough to verify the equality $\dim A_{n+1} = 3$, (2.22) and (2.23) under the assumption $\dim zA_{n-1} = 3$. Pick a linear basis X in A_1 containing z, equip X with a total order for which z is the minimal element and extend this order to the corresponding left-to-right degree-lexicographical order on $\langle X\rangle$. The condition $\dim zA_{n-1} = 3$ implies that all three degree n normal words w_1, w_2 and w_3 must start with z. Out of all options provided by Lemma 2.1, only N3.18, N3.19, N3.21 and N3.23 fit this property. For each of these cases, one easily check that the only degree $n+1$ words for which both degree n subwords are normal are zw_1, zw_2 and zw_3. Since $\dim A_{n+1} \geqslant 3$ it follows that $\dim A_{n+1} = 3$ and zw_1, zw_2 and zw_3 are the degree $n+1$ normal words. Hence the map $w \mapsto zw$ between 3-dimensional spaces A_n and A_{n+1} is invertible and (2.22) follows. Now let $1 \leqslant j \leqslant n-1$ and $y_1, \dots, y_j \in A_1$. Assume that $w \in A_{n-j}$ is such that $z^j w = 0$ in A. Then $z^j wy_1 = 0$ and therefore by (2.22), $z^{j-1}wy_1 = 0$. If $j > 1$, $z^{j-1}wy_1y_2 = 0$ and by (2.22), $z^{j-2}wy_1y_2 = 0$. We repeat the trick until we get $wy_1 \dots y_j = 0$. Since $y_j \in A_1$ are arbitrary, $ww_1 = 0$ in A for each $w_1 \in A_j$. This completes the proof of (2.23). □

Lemma 2.8. *Let A be a finitely generated degree graded algebra, whose ideal of relations is generated by some homogeneous elements of degree n, where $n \geqslant 3$. Assume also that $\dim A_n = 3$, $\dim A_{n+1} \geqslant 3$, $\lambda(A, n) = 3$ and $\dim x^{n-1}A_1 < 3$ for all $x \in A_1$. Then $\rho(A, n) \geqslant 2$ and there exist $z, s \in A_1$ such that z^n, $z^{n-1}s$ and $z^{n-2}st$ are linearly independent in A for some $t \in \{z, s\}$.*

Proof. Since $\lambda(A, n) = 3$, we can pick $z \in A_1$ such that zA_{n-1} is a three-dimensional subspace of A. Consider $X \in \Omega(A, z)$ with $\langle X\rangle$ carrying the LR order. Since $\dim zA_{n-1} = 3$ and z is the smallest letter, all three degree n normal words must start with z. Finally, since $\dim z^{n-1}A_1 < 3$ in A, at least one degree n normal word does not start with z^{n-1}. Only three options from Lemma 2.1 fit these conditions: N3.19, N3.21 and N3.23. Hence the degree n normal words are z^n, $z^{n-1}s$ and $z^{n-2}st$ for some $s \in X \setminus \{z\}$ and $t \in X$. If $t \in \{z, s\}$, then the required linear independence follows from the linear independence of normal words. Moreover two normal words end with either z or s, yielding $\rho(A, n) \geqslant 2$. Thus there is nothing to prove in this case.

It remains to consider the case when for every choice of $z \in A_1$ with $\dim zA_{n-1} = 3$, for every choice of $X \in \Omega(A, z)$ with $\langle X\rangle$ carrying the LR order, the degree n normal words are z^n, $z^{n-1}s$ and $z^{n-2}st$ for pairwise distinct z, s, t (more precisely, we shall show that this case does not occur by arriving to a contradiction). Note that anyway, z^n and $z^{n-1}s$ are linearly independent in A for generic $z, s \in A_1$.

As we have seen, for generic $z, s \in A_1$, $\dim zA_{n-1} = 3$ and z^n and $z^{n-1}s$ are linearly independent in A. For such z, s, let $X \in \Omega(A, z, s)$ with $\langle X\rangle$ carrying the LR order. Since z^n and $z^{n-1}s$ are two smallest degree n words and are linearly independent in A, they are degree n normal words. According to the above observations the third degree n normal word is $z^{n-2}st'$ for some $t' \in X \setminus \{z, s\}$. Since for every $y \in X$, $z^{n-1}y < z^{n-2}st'$, $z^{n-1}y$ is in the linear span of z^n and $z^{n-1}s$. Hence $z^{n-1}x$ is in the linear span of z^n and $z^{n-1}s$ for all $x \in A_1$. In particular, $z^{n-1}t' = pz^{n-1}s + qz^n$ for some $p, q \in \mathbb{K}$. Let $t = t' - ps - qz$. Then $z^{n-1}t = 0$ in A. Since $z^{n-2}s^2 < z^{n-2}st'$, $z^{n-2}sz < z^{n-2}st'$, both $z^{n-2}s^2$ and $z^{n-2}sz$ belong to the linear span of z^n and $z^{n-1}s$. Hence z^n, $z^{n-1}s$ and $z^{n-2}st$ form a linear basis in A_n. Now we slightly change the basis X. We keep all elements of $X \setminus \{t'\}$ as they were and replace t' by t (t takes the place of t' in the total order on X). Since $z^{n-2}s^2$ and $z^{n-2}sz$ are in the linear span of z^n, $z^{n-1}s$ one easily observes that $z^{n-2}st$ is the smallest word not in the said span. Hence for the new X, z^n, $z^{n-1}s$ and $z^{n-2}st$ are degree n normal words with the added bonus of the equality $z^{n-1}t = 0$ in A. By Lemma 2.7, the cancellation rule (2.22) holds.

Since $z^{n-2}s^2$ is in the linear span of z^n and $z^{n-1}s$ and this holds for generic s, we have that $z^{n-2}x^2$ is in the linear span of z^n and $z^{n-1}x$ for generic $x \in A_1$. Since $z^{n-1}x$ is in the linear span of z^n and $z^{n-1}s$ for all $x \in A_1$, it follows that $z^{n-2}x^2$ is in the linear span of z^n and $z^{n-1}s$ for generic $x \in A_1$. Hence $z^{n-2}x^2$ is in the linear span of z^n and $z^{n-1}s$ for all $x \in A_1$. Thus

$z^{n-2}((s+t)^2-s^2-t^2)=z^{n-2}(st+ts)$ is in the linear span of z^n and $z^{n-1}s$. Hence $z^{n-2}ts=-z^{n-2}st+az^{n-1}s+bz^n$ for some $a,b\in\mathbb{K}$ and therefore z^n, $z^{n-1}s$ and $z^{n-2}ts$ form a linear basis in A_n.

Now consider a total order on X for which z is the minimal element t is second smallest and s is the third smallest element and extend this order to the corresponding LR order $<_1$ on $\langle X\rangle$. Since z^n is the smallest degree n word and $z^n\neq 0$ in A, z^n is the smallest degree n normal word. The second smallest degree n word with respect to $<_1$ is $z^{n-1}t$, which vanishes in A, while the third smallest degree n word is $z^{n-1}s$. Since z^n and $z^{n-1}s$ are linearly independent in A, the second degree n normal word with respect to $<_1$ is still $z^{n-1}s$. Since $\dim z^{n-1}A_1<3$, the third $<_1$ degree n normal word can not start with z^{n-1}. Of these the three smallest (in this order) are $z^{n-2}tz$, $z^{n-2}t^2$ and $z^{n-2}ts$. We already know that $z^{n-2}t^2$ is in the linear span of z^n and $z^{n-1}s$. Since $z^{n-1}t=0$, we have $z^{n-1}tz=0$ in A. By (2.22), $z^{n-2}tz=0$ in A. Hence both $z^{n-2}tz$ and $z^{n-2}t^2$ are in the linear span of z^n and $z^{n-1}s$. Since we know that $z^{n-2}ts$ is not in the said span, we have that the third degree n normal word with respect to $<_1$ is $z^{n-2}ts$. Thus the three degree n normal words for $<_1$ are z^n, $z^{n-1}s$ and $z^{n-2}ts$. The only degree $n+1$ words whose degree n subwords are among z^n, $z^{n-1}s$ and $z^{n-2}ts$ are z^{n+1} and z^ns. Hence A_{n+1} is spanned by z^{n+1} and z^ns, which contradicts the inequality $\dim A_{n+1}\geqslant 3$. This contradiction completes the proof. □

Now we are ready to prove Parts (G5) and (G6) of Lemma 2.2. Obviously (G5) and (G6) are equivalent. Indeed they transform to one another when we pass to the opposite multiplication. Thus it suffices to prove (G5). Let $n\geqslant 3$ and let A be a finitely generated degree-graded algebra such that $\lambda(A,n)=3$, $\dim A_n=3$ and $\dim A_{n+1}\geqslant 3$. The proof will be complete if we show that for generic $X=(x_1,x_2,\dots)\in\Omega(A)$ with $\langle X\rangle$ equipped with the LR order $<$, $\mathrm{NW}_n=\{x_1^n,x_1^{n-1}x_2,w\}$ where $w\in\{x_1^{n-1}x_3,x_1^{n-2}x_2x_1,x_1^{n-2}x_2^2\}$.

First, consider the case when $\dim x^{n-1}A_1=3$ for some $x\in A_1$. Then $\dim z^{n-1}A_1=3$ for generic $z\in A_1$. For such a z, for $X\in\Omega(A,z)$ with $\langle X\rangle$ carrying the LR order, all three degree n normal words must start with z^{n-1}. Then by Lemma 2.1 they are of the form z^n, $z^{n-1}y$, $z^{n-1}x$ for some distinct $x,y\in X\setminus\{z\}$. In particular, z^n, $z^{n-1}y$ and $z^{n-1}x$ are linearly independent in A for generic $z,y,x\in A_1$. Then for such z,y,x and for any $X\in\Omega(A,z,y,x)$ with $\langle X\rangle$ carrying the LR order, the three smallest degree n words z^n, $z^{n-1}y$, $z^{n-1}x$ are linearly independent in A and therefore are normal words. Hence for generic $X=(x_1,x_2,\dots)\in\Omega(A)$ with $\langle X\rangle$ equipped with the LR order $<$, $\mathrm{NW}_n=\{x_1^n,x_1^{n-1}x_2,x_1^{n-1}x_3\}$.

It remains to consider the case when $\dim x^{n-1}A_1<3$ for all $x\in A_1$. First, assume that there exist $z,s\in A_1$ for which z^n, $z^{n-1}s$ and $z^{n-2}sz$ are linearly independent in A. Then for generic $z,s\in A_1$, z^n, $z^{n-1}s$ and $z^{n-2}sz$ form a linear basis in A_n. For such z,s and for every $X\in\Omega(A,z,s)$ with $\langle X\rangle$ carrying the LR order $<$, the two smallest degree n words z^n and $z^{n-1}s$ are linearly independent in A and therefore are degree n normal words. Since $\dim z^{n-1}A_1<3$, the third normal word can not start with z^{n-1}. The smallest word, which does not start with z^{n-1} is $z^{n-2}sz$. Since z^n, $z^{n-1}s$ and $z^{n-2}sz$ are linearly independent, $z^{n-2}sz$ is the third degree n normal word. Recall that all this happens for generic z and s. Hence for generic $X=(x_1,x_2,\dots)\in\Omega(A)$ with $\langle X\rangle$ equipped with the LR order $<$, $\mathrm{NW}_n=\{x_1^n,x_1^{n-1}x_2,x_1^{n-2}x_2x_1\}$.

Finally, it remains to deal with the case when $\dim x^{n-1}A_1<3$ for all $x\in A_1$ and z^n, $z^{n-1}s$ and $z^{n-2}sz$ are linearly dependent in A for every $z,s\in A_1$. By Lemma 2.8, there exist $z,s\in A_1$ for which z^n, $z^{n-1}s$ and $z^{n-2}s^2$ are linearly independent in A. Hence z^n, $z^{n-1}s$ and $z^{n-2}s^2$ are linearly independent in A for generic $z,s\in A_1$. For such z,s, let $X\in\Omega(A,z,s)$ with $\langle X\rangle$ carrying the LR order $<$. As in the previous case, the two smallest degree n words z^n and $z^{n-1}s$ are linearly independent in A and therefore are degree n normal words. Since $\dim z^{n-1}A_1<3$, the third normal word can not start with z^{n-1}. The smallest word, which does not start with z^{n-1} is $z^{n-2}sz$ and the second smallest is $z^{n-2}s^2$. Since z^n, $z^{n-1}s$ and $z^{n-2}sz$ are linearly dependent, while z^n, $z^{n-1}s$ and $z^{n-2}s^2$ are linearly independent, the third degree n normal word is $z^{n-2}s^2$. Since this happens for generic z and s, for generic $X=(x_1,x_2,\dots)\in\Omega(A)$ with $\langle X\rangle$ equipped with the LR order $<$, $\mathrm{NW}_n=\{x_1^n,x_1^{n-1}x_2,x_1^{n-2}x_2^2\}$. The proof is now complete.

3 Case $\lambda(A,n)=\rho(A,n)=1$ and Proof of Theorem 1.1

We start with the following technical lemma.

Lemma 3.1. *Let A be a finitely generated degree graded algebra, whose ideal of relations is generated by homogeneous elements of degree $n\geqslant 2$. Let also X be a linear basis in A_1, $a_1,\dots,a_k\in X$. Assume also that for every total order on the set $\{a_1,\dots,a_k\}$ extends to a total order on X in such a way that for both corresponding LR and RL orders on $\langle X\rangle$, the set of degree n normal words is $\{a_p^n:1\leqslant p\leqslant k\}$. Assume also that $\dim A_{n+1}\geqslant k$. Then $\dim A_m=k$ for all $m\geqslant n$.*

Proof. Since a_j^n for $1\leqslant j\leqslant n$ form a linear basis of A_n,

$$w=\sum_{j=1}^{k}\alpha_j(w)a_j^n\ \text{ in } A \text{ for all } w\in\langle X\rangle_n, \tag{3.1}$$

where $\alpha_j(w)\in\mathbb{K}$ are uniquely determined. Now if an element w of $\langle X\rangle$, considered as an element of A is written as a linear combination of normal words (with respect to some admissible order), then the normal words greater than w do not feature (=come with zero coefficients). Now according to the assumption on the existence of orders, we see that

$$\alpha_j(a_pw)=\alpha_j(wa_p)=0\ \text{ if } j\neq p \text{ for every } w\in\langle X\rangle_{n-1}. \tag{3.2}$$

Now using (3.1) and (3.2), we obtain that for all $x,y\in X$ and $w\in\langle X\rangle_{n-1}$, the following equalities hold in A:

$$\begin{aligned}xwy&=\sum_{j=1}^{k}\alpha_j(xw)a_j^ny=\sum_{j=1}^{k}\alpha_j(xw)\alpha_j(a_j^{n-1}y)a_j^{n+1},\\ xwy&=\sum_{j=1}^{k}\alpha_j(wy)xa_j^n=\sum_{j=1}^{k}\alpha_j(wy)\alpha_j(xa_j^{n-1})a_j^{n+1}.\end{aligned}$$

Since $\dim A_{n+1}\geqslant k$, we have that $\dim A_{n+1}=k$ and that a_j^{n+1} for $1\leqslant j\leqslant k$ are all degree $n+1$ normal words with respect to any order with respect to which a_j^n for $1\leqslant j\leqslant k$ are degree n normal words. Furthermore, the above display yields that

$$w=\sum_{j=1}^{k}\alpha'_j(w)a_j^{n+1}\ \text{ in } A \text{ for every } w\in\langle X\rangle_{n+1},$$

where

$$\alpha'_j(xwy):=\alpha_j(xw)\alpha_j(a_j^{n-1}y)=\alpha_j(wy)\alpha_j(xa_j^{n-1})\ \text{ for } x,y\in X,\ w\in\langle X\rangle_{n-1}. \tag{3.3}$$

Note that the validity of the equations (3.3) is equivalent to the fact that all degree $n+1$ overlaps of the leading monomials of the reduced Gröbner basis of the ideal of relations of A resolve (without producing any degree $n+1$ element). The latter is, in turn, the same as the equality $\dim A_{n+1}=k$. This observation, applied one degree further, shows that the equality $\dim A_{n+2}=k$ is equivalent to

$$\alpha'_j(xw)\alpha'_j(a_j^ny)=\alpha'_j(wy)\alpha'_j(xa_j^n)\ \text{ for } 1\leqslant j\leqslant k,\ x,y\in X,\ w\in\langle X\rangle_n. \tag{3.4}$$

However, by (3.3), for $w\in\langle X\rangle_n$ and $x,y\in X$, we have

$$\alpha'_j(xw)=\alpha_j(w)\alpha_j(xa_j^{n-1}),\ \alpha'_j(wy)=\alpha_j(w)\alpha_j(a_j^{n-1}y),$$

while applying the above display to specific w, we get

$$\alpha'_j(xa_j^n) = \alpha_j(xa_j^{n-1}),\ \ \alpha'_j(a_j^n y) = \alpha_j(a_j^{n-1}y).$$

Using the equalities from the above two displays, we immediately see that (3.4) is indeed satisfied and therefore $\dim A_{n+2} = k$ and degree $n+2$ normal words are a_j^{n+2}. That is, we have proved that if $\dim A_{n+1} \geqslant k$, then $\dim A_{n+1} = \dim A_{n+2} = k$ and all assumptions of our lemma are satisfied when n is replaced by $n+1$. Iterating, we get $\dim A_m = k$ for all $m \geqslant n$, which completes the proof. $\square$

Lemma 3.2. *Let A be a finitely generated degree graded algebra, $k, n \in N$, $1 \leqslant k \leqslant 3$ and $n \geqslant \max\{k, 2\}$. Assume also that the ideal of relations of A is generated by homogeneous elements of degree n, $\dim A_n = k$, $\dim A_{n+1} \geqslant k$ and $\lambda(A, n) = \rho(A, n) = 1$. Then $\dim A_m = k$ for all $m \geqslant n$.*

Proof. By Lemma 2.2, for generic $X = (x_1, x_2, \dots) \in \Omega(A)$, $\mathrm{NW}_n = \{x_1^n, \dots, x_k^n\}$ with respect to both the LR order $<$ and the RL order $\prec$ on $\langle X\rangle$. It follows that for generic $X = (x_1, x_2, \dots) \in \Omega(A)$ all conditions of Lemma 3.1 are satisfied (with $a_j = x_j$ for $j \leqslant k$). By Lemma 3.1 $\dim A_m = k$ for all $m \geqslant n$, as required. $\square$

3.1 Proof of Theorem 1.1

Let $n \in N$ and A be a finitely generated degree graded algebra such that $\dim A_n = 1$ and the ideal of relations of A is generated by some homogeneous elements of degree at most n. According to Remark 1.11, we can without loss of generality assume that the ideal of relations of A is generated by homogeneous elements of degree exactly n. If $n = 1$, A is naturally isomorphic as a graded algebra to the algebra $\mathbb{K}[t]$ of polynomials in one indeterminant and therefore $H_A^{[n]} = \overline{1}$. Thus we can assume that $n \geqslant 2$. If $\dim A_{n+1} = 0$, then obviously $H_A^{[n]} = 1\overline{0}$. Next, we assume that $\dim A_{n+1} \geqslant 1$. Since both $\lambda(A, n)$ and $\rho(A, n) = 1$ are between 1 and $\dim A_n$, we have $\lambda(A, n) = \rho(A, n) = 1$. Hence all conditions of Lemma 3.2 with $k = 1$ are satisfied. By Lemma 3.2, $H_A^{[n]} = \overline{1}$. The proof is complete.

4 Case $A_n = z^{n-1}A_1$ for some $z \in A_1$ and proof of Theorem 1.2

Lemma 4.1. *Let $n, k \in \mathbb{N}$, $n \geqslant k$ and let A be a finitely generated degree graded algebra, whose ideal of relations is generated by some homogeneous elements of degree n. Assume also that $z \in A_1$ and $\dim A_n = \dim z^{n-1}A_1 = k$. Then either $\dim A_{n+1} < k$ or $\dim A_m = k$ for all $m \geqslant n$.*

Proof. If $k = 1$, the result follows from the already proven Theorem 1.1. Thus we shall assume that $k \geqslant 2$. Since $z^{n-1}A_1 = A_n$, we immediately have $z^n A_1 = A_{n+1}$. If $z^n = 0$ in A, then $A_{n+1} = \{0\}$ and therefore $\dim A_{n+1} = 0 < k$, as required. Thus for the rest of the proof we can assume that $z^n \neq 0$ in A. Since $\dim A_n = \dim z^{n-1}A_1 = k$ and $z^n \neq 0$ in A, we can pick a k-dimensional subspace L of A_1 such that $z \in L$ and $z^{n-1}L = A_n$. For the sake of brevity denote $V = A_1$ and let F be the tensor algebra of $V = A_1$. Then A naturally interprets as the factor-algebra of the free algebra F by the ideal of relations of A. If $\dim A_{n+1} < k$, there is nothing to prove. Thus we can assume that $\dim A_{n+1} \geqslant k$. Since $A_{n+1} = zA_n = z^n L$ and L is k-dimensional, it follows that $\dim A_{n+1} = k$ and the map $x \mapsto z^n x$ from L to A_{n+1} is a linear isomorphism. Hence there exist unique linear maps $\alpha : F_n \to L$ and $\alpha' : F_{n+1} \to L$ such that

$$\begin{aligned} w &= z^{n-1}\alpha(w) \ \text{ in } A \text{ for all } w \in F_n,\\ w &= z^n\alpha'(w) \ \text{ in } A \text{ for all } w \in F_{n+1}. \end{aligned} \tag{4.1}$$

Note that $\alpha(z^{n-1}x) = \alpha'(z^n x) = x$ for all $x \in L$. Let $M = \{x \in V : \alpha(z^{n-1}x) = 0\}$. Clearly, M is a subspace of V and $M \oplus L = V$. Since $A_{n+1} = zA_n$ and $\dim A_{n+1} = \dim A_n = k$, the map $w \mapsto zw$ from A_n to A_{n+1} is a linear isomorphism. In particular,

$$\text{if } w \in A_n \text{ and } zw = 0 \text{ in } A, \text{ then } w = 0 \text{ in } A. \tag{4.2}$$

Consider the linear map

$$Z : V \to V, \quad Zx = \alpha(z^{n-2}xz).$$

Clearly $Z(V) = L$. Next, if $x \in M$, then $z^{n-1}x = 0$ in A. Hence $z^{n-1}xz = 0$ in A. According to (4.2), $z^{n-2}xz = 0$ in A and therefore $Zx = 0$. That is, Z vanishes on M. Next, let L_0 be the space of $x \in L$ such that $Z^j x = 0$ for some $j \in \mathbb{N}$ (the main eigenspace corresponding to the eigenvalue 0) and let L_+ be the sum of all main eigenspaces corresponding to all non-zero eigenvalues of Z. Then $M \oplus L_0 \oplus L_+ = V$ and each of the spaces M, L_0 and L_+ is invariant for Z. Since $Zz = z$, $z \in L_+$ and therefore $\dim L_0 < k$. Since $n \geqslant k$, it follows that $Z^{n-1}x = 0$ for every $x \in L_0$.

Now let $x, y \in V$ and $w \in F_{n-1}$. Then by (4.1), the following equalities hold in A:

$$\begin{aligned}
&xwy = z^{n-1}\alpha(xw)y = z^n\alpha(z^{n-2}\alpha(xw)y),\\
&xwy = xz^{n-1}\alpha(wy) = z^{n-1}\alpha(xz^{n-1})\alpha(wy) = z^n\alpha(z^{n-2}\alpha(xz^{n-1})\alpha(wy)).
\end{aligned}$$

Hence by (4.1) and the uniqueness of α', we get

$$\alpha'(xwy) = \alpha(z^{n-2}\alpha(xw)y) = \alpha(z^{n-2}\alpha(xz^{n-1})\alpha(wy)) \ \text{ for all } w \in F_{n-1},\ x, y \in V. \tag{4.3}$$

The proof of the fact that $\dim A_m = k$ for all $m \geqslant k$ will be complete if we show that $\dim A_{n+2} = k$. Indeed, then we can simply iterate. In order to prove that $\dim A_{n+2} = k$ it is enough to show that an analog of (4.3) holds one degree higher:

$$\alpha'(z^{n-1}\alpha'(xw)y) = \alpha'(z^{n-1}\alpha'(xz^n)\alpha'(wy)) \ \text{ for all } w \in F_n,\ x, y \in V. \tag{4.4}$$

Indeed, if X is any linear basis in $A_1 = V$ containing z and containing a linear basis Y of L equipped with a total order for which z is the minimal element and every element of $X \setminus Y$ is greater than any element of Y and if $\langle X \rangle$ carries the corresponding left-to-right degree-lexicographical order, then the validity of (4.4) for $w \in \langle X \rangle_n$ such that the words xw and wy are non-normal is exactly the same as resolving of all ambiguities of degree $n+2$, which would imply $\dim A_{n+2} = k$. Thus it remains to verify (4.4). If $w \in F_n$ and $x, y \in V$, then (4.3) yields

$$\alpha'(wy) = \alpha(z^{n-2}\alpha(w)y), \quad \alpha'(xw) = \alpha(z^{n-2}\alpha(xz^{n-1})\alpha(w)) \ \text{ for all } w \in F_{n-1},\ x, y \in V. \tag{4.5}$$

In particular, $\alpha'(zw) = \alpha(w)$ for all $w \in F_n$. Since $\{\alpha(w) : w \in F_n\} = L$, (4.5) now implies that (4.4) is equivalent to

$$\alpha(z^{n-2}\alpha(z^{n-2}\alpha(xz^{n-1})u)y) = \alpha(z^{n-2}\alpha(z^{n-2}\alpha(xz^{n-1})z)\alpha(z^{n-2}uy)) \tag{4.6}$$

for all $x, y \in V$, $u \in L$. Now for each $x \in V$ and $0 \leqslant j \leqslant n-2$, $z^{n-1-j}xz^j = z^{n-1}\alpha(z^{n-1-j}xz^j)$ in A. Hence $z^{n-1-j}xz^{j+1} = z^{n-1}\alpha(z^{n-1-j}xz^j)z$ and therefore by (4.2), $z^{n-2-j}xz^{j+1} = z^{n-2}\alpha(z^{n-1-j}xz^j)z$. It follows that $z^{n-1}\alpha(z^{n-2-j}xz^{j+1}) = z^{n-2}\alpha(z^{n-1-j}xz^j)z$ and by definition of Z, $\alpha(z^{n-2-j}xz^{j+1}) = Z\alpha(z^{n-1-j}xz^j)$. Hence $\alpha(z^{n-1-j}xz^j) = Z^j\alpha(z^{n-1}x)$ for all $x \in V$ and $0 \leqslant j \leqslant n-1$. In particular, $\alpha(xz^{n-1}) = Z^{n-1}\alpha(z^{n-1}x)$. Since $L = \{\alpha(z^{n-1}x) : x \in V\}$, Z^{n-1} vanishes on L_0 and Z^{n-1} is invertible as a linear map from L_+ to itself, $\{\alpha(xz^{n-1}) : x \in V\} = L_+$. Hence (4.6) and therefore (4.4) is equivalent to

$$\alpha(z^{n-2}\alpha(z^{n-2}vu)y) = \alpha(z^{n-2}\alpha(z^{n-2}vz)\alpha(z^{n-2}uy)) \ \text{ for all } y \in V,\ u \in L,\ v \in L_+. \tag{4.7}$$

By definition of α, (4.7) is the same as

$$z^{n-2}\alpha(z^{n-2}vu)y = z^{n-2}\alpha(z^{n-2}vz)\alpha(z^{n-2}uy) \text{ in } A \text{ for all } y \in V,\, u \in L,\, v \in L_+. \tag{4.8}$$

By (4.2), (4.8) is equivalent to (just multiply by z from the left)

$$z^{n-1}\alpha(z^{n-2}vu)y = z^{n-1}\alpha(z^{n-2}vz)\alpha(z^{n-2}uy) \text{ in } A \text{ for all } y \in V,\, u \in L,\, v \in L_+. \tag{4.9}$$

As we have observed above $\alpha(z^{n-1-j}xz^j) = Z^j\alpha(z^{n-1}x)$ for all $x \in V$ and $0 \leqslant j \leqslant n-1$. Since $\alpha(x) = x$ for $x \in L$, it follows that $z^{n-1-j}xz^j = z^{n-1}Z^jx$ for all $x \in L$. Since Z restricted to L_+ is invertible, it follows that $z^{n-1-j}xz^j = (Z^{j+1-n}x)z^{n-1}$ for $0 \leqslant j \leqslant n-1$ and $x \in L_+$, where Z^{-1} is the inverse of the restriction of Z to L_+.

Hence under the assumptions of (4.9), $z^{n-1}\alpha(z^{n-2}vz) = z^{n-2}vz = (Z^{2-n}v)z^{n-1}$ and therefore

$$z^{n-1}\alpha(z^{n-2}vz)\alpha(z^{n-2}uy) = (Z^{2-n}v)z^{n-1}\alpha(z^{n-2}uy) = (Z^{2-n}v)z^{n-2}uy.$$

Next, by the same token $z(Z^{2-n}v)z^{n-2} = z^{n-1}v$ and therefore $z(Z^{2-n}v)z^{n-2}u = z^{n-1}vu$. By (4.2), $(Z^{2-n}v)z^{n-2}u = z^{n-2}vu$. Plugging this into the above display, we get

$$z^{n-1}\alpha(z^{n-2}vz)\alpha(z^{n-2}uy) = z^{n-2}vuy.$$

On the other hand, by definition of α, $z^{n-1}\alpha(z^{n-2}vu) = z^{n-2}vu$ and therefore $z^{n-1}\alpha(z^{n-2}vu)y = z^{n-2}vuy$. This together with the above display confirms the validity of (4.9) and completes the proof. □

4.1 Proof of Theorem 1.2

Let $n \geqslant 2$ and A be a finitely generated degree graded algebra such that $\dim A_n = 2$ and the ideal of relations of A is generated by some homogeneous elements of degree at most n. The proof will be complete if we show that $H_A^{[n]} \in \{\overline{2}, 2\overline{1}, 21\overline{0}, 2\overline{0}\}$. According to Remark 1.11, we can without loss of generality assume that the ideal of relations of A is generated by homogeneous elements of degree exactly n. If $\dim A_{n+1} < 2$, then the result follows from the already proven Theorem 1.1. Then for the rest of the proof we can assume that $\dim A_{n+1} \geqslant 2$. If $\lambda(A,n) = \rho(A,n) = 1$, Lemma 3.2 with $k = 2$ implies that $H_A^{[n]} = \overline{2}$. Thus it remains to consider the case $(\lambda(A,n), \rho(A,n)) \neq (1,1)$. Since both $\lambda(A,n)$ and $\rho(A,n)$ are between 1 and $2 = \dim A_n$, this means that either $\lambda(A,n) = 2$ or $\rho(A,n) = 2$. These two options reduce to one another when we pass to the opposite multiplication. Indeed, the Hilbert series of A and of $A^{\rm opp}$ (being A with the opposite multiplication) coincide. Hence we can without loss of generality assume that $\lambda(A,n) = 2$. By Lemma 2.2, for generic $X = (x_1, x_2, \dots) \in \Omega(A)$ with $\langle X \rangle$ carrying the LR order, $\mathrm{NW}_n = \{x_1^n, x_1^{n-1}x_2\}$. Hence there is $z \in A_1$ such that $\dim z^{n-1}A_1 = 2$. Now Lemma 4.1 with $k = 2$ yields $H_A^{[n]} = \overline{2}$, which completes the proof.

5 Case $\dim A_n = 3$ and $\max\{\lambda(A,n), \rho(A,n)\} = 3$

The main result of this section is the following lemma.

Lemma 5.1. *Let A be a finitely generated degree graded algebra, whose ideal of relations is generated by some homogeneous elements of degree n, where $n \geqslant 3$. Assume also that* $\dim A_n = \max\{\lambda(A,n), \rho(A,n)\} = 3$. *If $n > 3$, then either* $\dim A_{n+1} < 3$ *or* $H_A^{[n]} = \overline{3}$. *If $n = 3$, then either* $\dim A_{n+1} < 3$ *or* $H_A^{[n]} \in \{\overline{3}, 332\overline{1}\}$.

We approach the proof in stages. We start with an enhanced version of Lemma 2.7.

Lemma 5.2. *Let A be a finitely generated degree graded algebra, whose ideal of relations is generated by some homogeneous elements of degree n, where $n \geqslant 3$. Let also $\dim A_n = 3$, $\dim A_{n+1} \geqslant 3$, X be a linear basis in A_1 equipped with a total order, $z = \min X$, $s = \min(X \setminus \{z\})$ and $M = \mathtt{span}\,\{z, s\}$. Finally, assume that with respect to the LR order $<$ on $\langle X\rangle$, the three degree n normal words are either z^n, $z^{n-1}s$, $z^{n-2}st$ with $t \in \{z, s\}$.*

Then for every $x \in A_1$, there exists a unique $\widehat{x} \in M$ such that $z^{n-1}x = z^{n-1}\widehat{x}$ in A. These elements of M satisfy the following property:

$$\widehat{z} = z,\ \widehat{s} = s \text{ and for each } x \in A_1,\ w \in A_{n-1},\ xw = \widehat{x}w \text{ in } A. \tag{5.1}$$

If additionally $\dim A_{n-1}z = 3$, then the following stronger cancellation rule holds: for all non-negative integers j, m, q such that $j + m + q = n$,

$$\text{if } w \in A_m \text{ and } z^{j+q}w = 0 \text{ in } A, \text{ then } w_1ww_2 = 0 \text{ in } A \text{ for all } w_1 \in A_j \text{ and } w_2 \in A_q. \tag{5.2}$$

Furthermore,

$$\text{for all } x_1, \dots, x_n \in A_1,\ x_1 \dots x_n = \widehat{x}_1 \dots \widehat{x}_n \text{ in } A. \tag{5.3}$$

Proof. By Lemma 2.7, $\dim A_{n+1} = 3$ and (2.22), (2.23) hold. In both cases, the two smallest degree n normal words are z^n and $z^{n-1}s$, while the third normal word is greater than $z^{n-1}x$ for every $x \in X$. Hence for each $x \in X$, $z^{n-1}x$ is a linear combination of z^n and $z^{n-1}s$. Hence for every $x \in A_1$, there exists a unique $\widehat{x} \in M$ such that $z^{n-1}x = z^{n-1}\widehat{x}$ in A. Uniqueness immediately yields $\widehat{z} = z$, $\widehat{s} = s$. Let $u = x - \widehat{x}$. Then $z^{n-1}u = 0$ and by (2.23), we have $uw = 0$ in A for each $w \in A_{n-1}$. Hence $xw = \widehat{x}w$ for every $w \in A_{n-1}$, which completes the proof of (5.1).

Now assume additionally that $\dim A_{n-1}z = 3$. Then according to Lemma 2.7, the second cancellation rule (2.24) together with (2.25) hold as well. Hence

$$\text{for any } w \in A_{n-1},\ wz = 0 \text{ in } A \text{ if and only if } zw = 0 \text{ in } A. \tag{5.4}$$

Indeed, by (2.22) and (2.24), both are equivalent to $zwz = 0$. Another immediate corollary of (2.22) and (2.24) is

$$\text{for any } w \in A_n,\ wz = 0 \text{ in } A \text{ if and only if } zw = 0 \text{ in } A. \tag{5.5}$$

Indeed, both are equivalent to $w = 0$.

Let $j + m + q = n$ and $w \in A_m$ be such that $z^{j+q}w = 0$ in A and let $w_1 \in A_j$ and $w_2 \in A_q$. By (2.23), $z^jww_2 = 0$ in A. By (5.4), $ww_2z^j = 0$ in A. Hence by (2.25), $w_1ww_2 = 0$ in A completing the proof of (5.2).

Finally, let $x_1, \dots, x_n \in A_1$. We use induction by j to prove that $z^{n-j}x_1 \dots x_j = z^{n-j}\widehat{x}_1 \dots \widehat{x}_j$ for $1 \leqslant j \leqslant n$. By definition of the hat map $z^{n-1}x_1 = z^{n-1}\widehat{x}_1$, which provides the basis of induction. Now assume that $1 \leqslant j \leqslant n-1$ and we already know that $z^{n-j}x_1 \dots x_j = z^{n-j}\widehat{x}_1 \dots \widehat{x}_j$. By (5.2), the equality $z^{n-1}x_{j+1} = z^{n-1}\widehat{x}_{j+1}$ implies that $z^{n-j}x_1 \dots x_{j+1} = z^{n-j}x_1 \dots x_j\widehat{x_{j+1}}$. Since $z^{n-j}x_1 \dots x_j = z^{n-j}\widehat{x}_1 \dots \widehat{x}_j$, we get $z^{n-j}x_1 \dots x_{j+1} = z^{n-j}\widehat{x}_1 \dots \widehat{x}_{j+1}$. By (2.22), $z^{n-j-1}x_1 \dots x_{j+1} = z^{n-j-1}\widehat{x}_1 \dots \widehat{x}_{j+1}$, completing the induction step and justifying the whole inductive procedure. After the final step, we get (5.3). $\square$

We would also like to make another general observation and introduce some notation useful in the cases when degree n normal words are either z^n, $z^{n-1}s$ and $z^{n-2}st$ with $t \in \{z, s\}$.

Lemma 5.3. *Let A be a finitely generated degree graded algebra, whose ideal of relations is generated by some homogeneous elements of degree n, where $n \geqslant 3$. Let also $\dim A_n = 3$, $\dim A_{n+1} \geqslant 3$, X be a linear basis in A_1 equipped with a total order, $z = \min X$, $s = \min(X \setminus \{z\})$ and $M = \mathtt{span}\,\{z, s\}$. Finally, assume that with respect to the left-to-right degree-lexicographical order on $\langle X\rangle$, the three* *degree n normal words are z^n, $z^{n-1}s$, $z^{n-2}st$, where $t\in\{s,z\}$. Denote $L=\mathtt{span}\{st,zs,z^2\}$, $V:=A_1$ and let F be the tensor algebra of V (naturally identified with $\mathbb{K}\langle X\rangle$), making A the quotient of F by the ideal of relations.*

Then there exist a unique map $\alpha:F_n\to L$ (automatically linear) such that

$$w=z^{n-2}\alpha(w)\quad \text{in } A \text{ for all } w\in F_n \tag{5.6}$$

and the following statements are equivalent:

- $\dim A_{n+2}=3$;
- $\dim A_m=3$ *for all* $m\geqslant n$;
- *the equalities*
$$\begin{aligned} z^{n-3}\alpha(sz^{n-1})zy&=z^{n-3}\alpha(sz^{n-1})\alpha(z^{n-1}y),\\ z^{n-3}\alpha(sz^{n-1})sy&=z^{n-3}\alpha(sz^{n-1})\alpha(z^{n-2}sy),\\ z^{n-3}\alpha(sz^{n-2}s)ty&=z^{n-3}\alpha(sz^{n-1})\alpha(z^{n-3}sty)\end{aligned}\tag{5.7}$$
are satisfied in A for every $y\in V$.

Furthermore, if additionally, $\dim A_{n-1}z=3$, then the first two equalities in (5.7) are automatically satisfied, while the validity of the third one for every $y\in V$ is equivalent to its validity for $y\in\{s,z\}$ only. That is, if $\dim A_{n-1}z=3$, then the following statements are equivalent:

- $\dim A_{n+2}=3$;
- $\dim A_m=3$ *for all* $m\geqslant n$;
- *the equalities*
$$\begin{aligned} z^{n-3}\alpha(sz^{n-2}s)tz&=z^{n-3}\alpha(sz^{n-1})\alpha(z^{n-3}stz)\\ z^{n-3}\alpha(sz^{n-2}s)ts&=z^{n-3}\alpha(sz^{n-1})\alpha(z^{n-3}sts)\end{aligned}\tag{5.8}$$
are satisfied in A.

Proof. By assumptions $\dim A_n=3$. Since the only degree $n+1$ words for which both degree n subwords are normal are z^{n+1}, z^ns and $z^{n-1}st$, we have $\dim A_{n+1}\leqslant 3$. Since $\dim A_{n+1}\geqslant 3$, we have $\dim A_{n+1}=3$ and the degree $n+1$ normal words are z^{n+1}, z^ns and $z^{n-1}st$. Note that Lemma 2.7 as well as the first part of Lemma 5.2 apply since all relevant conditions are satisfied. In particular, we can use the cancellation rule (2.22). Consider the linear maps $\pi_0:L\to\mathbb{K}$ and $\pi_1:L\to M$ given by

$$\pi_0(ast+bzs+cz^2)=a,\qquad \pi_1(ast+bzs+csz)=bs+cz,$$

making $v=\pi_0(v)su+z\pi_1(v)$ for every $v\in L$.

Since z^n, $z^{n-1}s$, $z^{n-2}st$ are degree n normal words and z^{n+1}, z^ns, $z^{n-1}st$ are degree $n+1$ normal words and since normal words form a linear basis of A, there are unique linear maps $\alpha:F_n\to L$ and $\alpha':F_{n+1}\to L$ such that

$$\begin{aligned} w&=z^{n-2}\alpha(w)\ \text{ in } A \text{ for all } w\in F_n,\\ w&=z^{n-1}\alpha'(w)\ \text{ in } A \text{ for all } w\in F_{n+1},\end{aligned}\tag{5.9}$$

In particular, (5.6) defines a unique linear map $\alpha:F_n\to L$.

Now let $x,y\in V$ and $w\in F_{n-1}$. Then by (5.9),

$$\begin{aligned} xwy&=z^{n-2}\alpha(xw)y=z^{n-1}\alpha(z^{n-3}\alpha(xw)y),\\ xwy&=xz^{n-2}\alpha(wy)=\alpha_0(wy)xz^{n-2}sz+xz^{n-1}\alpha_1(wy)\\ &=\pi_0(\alpha(wy))z^{n-2}\alpha(xz^{n-2}s)z+z^{n-2}\alpha(xz^{n-1})\pi_1(\alpha(wy))\\ &=\pi_0(\alpha(wy))z^{n-1}\alpha(z^{n-3}\alpha(xz^{n-2}s)z)+z^{n-1}\alpha(z^{n-3}\alpha(xz^{n-1})\pi_1(\alpha(wy))).\end{aligned}$$

Since the map $u \mapsto z^{n-1}u$ from L to A_{n+1} is a linear isomorphism, we get

$$\alpha'(xwy) = \alpha(z^{n-3}\alpha(xw)y) = \pi_0(\alpha(wy))\alpha(z^{n-3}\alpha(xz^{n-2}s)z) + \alpha(z^{n-3}\alpha(xz^{n-1})\pi_1(\alpha(wy))) \quad (5.10)$$

for all $w \in F_{n-1}$ and $x, y \in V$. Hence

$$\begin{aligned} \alpha'(wy) &= \alpha(z^{n-3}\alpha(w)y), \\ \alpha'(xw) &= \pi_0(\alpha(w))\alpha(z^{n-3}\alpha(xz^{n-2}s)t) + \alpha(z^{n-3}\alpha(xz^{n-1})\pi_1(\alpha(w))) \end{aligned} \quad (5.11)$$

for all $x, y \in V$ and $w \in F_n$.

Now the equality $\dim A_{n+2} = 3$ is satisfied if and only if all ambiguities of degree $n+2$ resolve. This happens precisely when the second equality in (5.10) is satisfied with α replaced by α' and n replaced by $n+1$. It is slightly more convenient though to repeat the process of writing the equality (5.10) with w being one degree higher to see that $\dim A_{n+2} = 3$ if and only if the equality

$$z^{n-2}\alpha'(xw)y = z^{n-3}\alpha(xz^{n-1})\alpha'(wy) \quad (5.12)$$

is satisfied in A for every $x, y \in V$ and $w \in F_n$. By (5.11), $\alpha'(xw)$ and $\alpha'(wy)$ depend linearly on w and as far as dependence on w is concerned, they depend on $\alpha(w)$ only. Thus (5.12) for general w is equivalent to (5.12) for $w \in \{z^n, z^{n-1}s, z^{n-2}st\}$ only. Furthermore, by (2.23), nothing changes in (5.12) if we replace x by $\widehat{x}$ as defined in Lemma 5.2. Since the dependence on x is also linear and $\{\widehat{x} : x \in V\} = M = \mathtt{span}\,\{z, s\}$, (5.12) for arbitrary $x \in V$ is the same as (5.12) for $x \in \{z, s\}$. If $x = z$, then from definitions of α and α' together with $\alpha'(zw) = \alpha(w)$ (follows from 5.11) it is easy to see that (5.12) is satisfied. Thus, $\dim A_{n+2} = 3$ if and only if the equalities

$$\begin{aligned} z^{n-2}\alpha'(sz^n)y &= z^{n-3}\alpha(sz^{n-1})\alpha'(z^n y) \\ z^{n-2}\alpha'(sz^{n-1}s)y &= z^{n-3}\alpha(sz^{n-1})\alpha'(z^{n-1}sy) \\ z^{n-2}\alpha'(sz^{n-2}st)y &= z^{n-3}\alpha(sz^{n-1})\alpha'(z^{n-2}sty) \end{aligned} \quad (5.13)$$

are satisfied in A for every $y \in V$. Now by (5.11), $\alpha'(z^n y) = \alpha(z^{n-1}y)$, $\alpha'(z^{n-1}sy) = \alpha(z^{n-2}sy)$, $\alpha'(z^{n-2}sty) = \alpha(z^{n-3}sty)$, $\alpha'(sz^n) = \alpha(z^{n-3}\alpha(sz^{n-1})z)$ and therefore $z^{n-2}\alpha'(sz^n) = z^{n-3}\alpha(sz^{n-1})z$, $\alpha'(sz^{n-1}s) = \alpha(z^{n-3}\alpha(xz^{n-1})s)$ and therefore $z^{n-2}\alpha'(sz^{n-1}s) = z^{n-3}\alpha(xz^{n-1})s$ and finally, $\alpha'(sz^{n-2}st) = \alpha(z^{n-3}\alpha(xz^{n-2}s)t)$ and therefore $z^{n-2}\alpha'(sz^{n-2}st) = z^{n-3}\alpha(xz^{n-2}s)t$. After plugging this in, we see that (5.13) becomes (5.7). Hence (5.7) is equivalent to $\dim A_{n+2} = 3$. Now if the equalities (5.7) are satisfied one can, using (5.11) easily see that the same equalities will be satisfied if we replace α by α' and n by $n+1$. Hence we can iterate the argument to get $\dim A_m = 3$ for all $m \geqslant n$.

Finally, assume that $\dim A_{n-1}z = 3$. Then the second part of Lemma 5.2 kicks in. Since $z^{n-1}y = z^{n-2}\alpha(z^{n-1}y)$ and $z^{n-2}sy = z^{n-2}\alpha(z^{n-2}sy)$, the first two equations in (5.7) follow straight from (5.2). By (5.3) then nothing changes in the equations (5.7) if we replace y by $\widehat{y}$. Hence the validity of the third equation in (5.7) for arbitrary $y \in V$ is the same as for $y \in \{z, s\}$. Thus (5.7) becomes equivalent to (5.8), which completes the proof. □

5.1 Normal words of the form z^n, $z^{n-1}s$ and $z^{n-2}sz$

This, in a way, is the most annoying option. It forces us to separate the cases $n = 3$ and $n > 3$ and the answer differs in these two cases.

Lemma 5.4. *Let $n \in \mathbb{N}$, $n \geqslant 3$ and let A be a finitely generated degree graded algebra, whose ideal of relations is generated by some homogeneous elements of degree n. Assume also that $z, s \in A_1$, z^n, $z^{n-1}s$ and $z^{n-2}sz$ form a linear basis in A_n, $\dim z^{n-1}A_1 < 3$ and $\dim A_{n+1} \geqslant 3$. Let L be the three-dimensional space spanned by sz, zs and z^2. Then there exist unique linear maps $Z : L \to L$ and $S : L \to L$ satisfying*

$$z^{n-2}Z(u) = z^{n-3}uz \quad \text{and} \quad z^{n-2}S(u) = z^{n-3}us \quad \text{for all } u \in L. \quad (5.14)$$

Furthermore,

$$\begin{aligned}&\dim A_{n+2} = 3 \text{ if and only if } (Z^2S - aZSZ - bSZ^2 - cZ^3)(u) = 0 \text{ and}\\ &(SZS - \lambda ZSZ - \mu SZ^2 - \nu Z^3)(u) = 0, \text{ where } u = Z^{n-3}(sz) \in L,\end{aligned} \tag{5.15}$$

where $a, b, c, \lambda, \mu, \nu \in \mathbb{K}$ *are such that*

$$z^{n-3}sz^2 = az^{n-2}sz + bz^{n-1}s + cz^n \text{ and } z^{n-3}szs = \lambda z^{n-2}sz + \mu z^{n-1}s + \nu z^n \text{ in } A. \tag{5.16}$$

Finally, if $n \geqslant 4$, *then* $\dim A_m = 3$ *for all* $m \geqslant n$.

Proof. By assumptions $\dim A_n = 3$. Pick a linear basis X in A_1 containing z and s, equip X with a total order for which z is minimal and s is second minimal and consider the corresponding left-to-right degree-lexicographical order on $\langle X \rangle$, then the degree n normal words are z^n, $z^{n-1}s$ and $z^{n-2}sz$. Indeed, since $\dim z^{n-1}A_1 < 3$, $z^{n-2}sz$ is the smallest word not in the span of z^n and $z^{n-1}s$. Since the only degree $n+1$ words for which both degree n subwords are normal are z^{n+1}, $z^n s$ and $z^{n-1}sz$, we have $\dim A_{n+1} \leqslant 3$. Since $\dim A_{n+1} \geqslant 3$, we have $\dim A_{n+1} = 3$ and the degree $n+1$ normal words are z^{n+1}, $z^n s$ and $z^{n-1}sz$. Note that Lemma 2.7 as well as the first part of Lemma 5.2 apply since all relevant conditions are satisfied. In particular, we can use the cancellation rule (2.22).

Let M, V, F and $\alpha : F_n \to L$ be as in Lemma 5.3 with $t = z$. By the said lemma $\dim A_{n+2} = 3$ if and only if $\dim A_m = 3$ for all $m \geqslant n$ if and only if the equalities (5.7) are satisfied with $t = z$ for all $y \in V$, which read

$$\begin{aligned}z^{n-3}\alpha(sz^{n-1})zy &= z^{n-3}\alpha(sz^{n-1})\alpha(z^{n-1}y)\\ z^{n-3}\alpha(sz^{n-1})sy &= z^{n-3}\alpha(sz^{n-1})\alpha(z^{n-2}sy)),\\ z^{n-3}\alpha(sz^{n-2}s)zy &= z^{n-3}\alpha(sz^{n-1})\alpha(z^{n-3}szy),\end{aligned} \tag{5.17}$$

for all $y \in V$.

According to (5.6), for each $u \in L$, there exist unique $v, w \in L$ such that $z^{n-3}uz = z^{n-2}w$ and $z^{n-3}us = z^{n-2}v$. Hence the formulas $Z(u) = w$ and $S(u) = v$ define unique linear maps $Z, S : L \to L$ satisfying (5.14). Using (5.9) and (2.22), we see that

$$wz^j = z^{n-2}Z^j(\alpha(z^j w)) \quad \text{for } 0 \leqslant j \leqslant n-1 \text{ and } w \in F_{n-j}, \tag{5.18}$$

$$ws = z^{n-2}S(\alpha(zw)) \quad \text{for } w \in F_{n-1}. \tag{5.19}$$

Obviously, $Z(z^2) = z^2$, $Z(zs) = sz$ and $S(z^2) = zs$.

First, we show that the first two equalities in (5.17) are always satisfied. If Z is invertible, then $A_n = z^{n-2}L = z^{n-3}Z(L)z$ and therefore $\dim A_{n-1}z = 3$. In this case the first two equalities in (5.17) are satisfied by Lemma 5.3. On the other hand, if Z is non-invertible, then since $Z(z^2) = z^2$ and $Z(zs) = sz$,

$$Z(L) = L_0, \text{ where } L_0 = Mz = \mathtt{span}\,\{z^2, sz\}.$$

By definition of Z, $z^{n-3}sz^2 = z^{n-2}Z(sz)$. By (2.23) and the definition of α, $zZ(sz)z^{n-3} = sz^n = z^{n-2}\alpha(sz^{n-1})$. Using (2.22), we now see that $z^{n-3}\alpha(sz^{n-1})u = Z(sz)z^{n-3}u$ for every $u \in A_2$. Since $Z(L) = L_0$, there is $v \in M$ such that $Z(sz) = vz$. Thus $z^{n-3}\alpha(sz^{n-1})u = vz^{n-2}u$ for every $u \in A_2$. Hence the first two equalities in (5.17) now read $vz^{n-2}sy = vz^{n-2}\alpha(z^{n-2}sy))$ and $vz^{n-1}y = vz^{n-2}\alpha(z^{n-1}y))$. Both are trivially satisfied according to (5.6). Thus in any case the first two equalities in (5.17) are satisfied. This means that $\dim A_{n+2} = 3$ if and only if

$$z^{n-3}\alpha(sz^{n-2}s)zy = z^{n-3}\alpha(sz^{n-1})\alpha(z^{n-3}szy) \quad \text{for all } y \in V. \tag{5.20}$$

Note that by (5.18), $zsz^{n-2}=z^{n-2}Z^{n-3}(sz)$. Hence $zsz^{n-2}s=z^{n-2}Z^{n-3}(sz)s$ and therefore by (2.22), $sz^{n-2}s=z^{n-3}Z^{n-3}(sz)s=z^{n-2}SZ^{n-3}(sz)$. Also, by (5.18), $\alpha(sz^{n-1})=Z^{n-2}(sz)$ or $sz^{n-1}=z^{n-2}Z^{n-2}(sz)$. We shall record this:

$$sz^{n-2}s=z^{n-2}S(u),\ sz^{n-1}=z^{n-2}Z(u),\quad \text{where } u=Z^{n-3}(sz). \tag{5.21}$$

Next, we shall verify that the validity of (5.20) for all $y\in V$ is equivalent to the validity of the same for $y\in\{z,s\}$. We already know this to be the case if Z is invertible according to Lemma 5.3. Assume now that Z is non-invertible. Then as we have already observed $Z(L)=L_0$, which makes L_0 an invariant subspace for Z. First, assume that the restriction of Z to L_0 is invertible. As we have already shown above, $z^{n-3}\alpha(sz^{n-1})w=Z(sz)z^{n-3}w$ for every $w\in A_2$. Since $Z(sz)\in L_0=Mz$, there is $v\in M$ such that $Z(sz)=vz$. Hence $z^{n-3}\alpha(sz^{n-1})\alpha(z^{n-3}szy)=vz^{n-2}\alpha(z^{n-3}szy)=vz^{n-3}szy$. Since the left-hand side of (5.20) also ends with zy, the equality (5.20) can be written as $wzy=0$, where $w=vz^{n-3}s-z^{n-3}\alpha(sz^{n-2}s)\in A_{n-1}$. By definition of Z, $u_1:=\alpha(wz)\in Z(L)=L_0=Mz$. Since $wz=z^{n-2}u_1$ and $u_1\in L_0$ we can use the invertibility of Z on L_0 and (5.21) to write $wz=Z^{2-n}(u_1)z^{n-2}$, where Z^{2-n} is the $(n-2)^{\text{th}}$ power of the inverse of the restriction of Z to L_0. Since $Z^{2-n}(u_1)\in L_0$, there is $v_1\in M$ such that $Z^{2-n}(u_1)=v_1z$. Hence $wz=v_1z^{n-1}$ in A. Now the equality (5.20) can be written as $v_1z^{n-1}y=0$. Now from invertibility of Z on L_0 it follows that if $v_1z^n=0$, then $v_1z^{n-1}=0$. Hence the validity of (5.20) for $y=z$ (in this case) yields the same for all $y\in V$. In particular, it is enough to verify (5.20) for $y\in\{s,z\}$ only.

The final option is when Z is non-invertible and the restriction of Z to the invariant subspace L_0 is non-invertible as well. Since $Z(zs)=sz$ and $Z(z^2)=z^2$, this means that $Z(sz)$ is a scalar multiple of z^2. That is, $z^{n-3}sz^2=cz^n$ in A for some $c\in\mathbb{K}$ (the numbers a and b in (5.16) equal 0). Performing a linear substitution which leaves every $x\in X\setminus\{s\}$ as it was and replaces s by $s+c_1z$ with an appropriately chosen $c_1\in\mathbb{K}$, we can kill c (without disturbing any of the properties of our algebra). Hence without loss of generality we can assume that $c=0$. Equivalently, $z^{n-3}sz^2=0$ in A or $Z(sz)=Z^2(zs)=0$. If $n\geqslant 4$, then we have $u=0$, where $u=Z^{n-3}(sz)$ and therefore $\alpha(sz^{n-1})=\alpha(sz^{n-2}s)=0$ according to (5.21). Thus (5.20) is satisfied for every $y\in V$. In particular (5.20) for $y\in V$ is trivially equivalent to (5.20) for $y\in\{z,s\}$. If $n=3$, then $u=sz$. Hence by (5.21), (5.20) can be rewritten as $S(sz)zy=Z(sz)\alpha(z^{n-3}szy)$. Since $Z(sz)=0$, (5.20) reads $S(sz)zy=0$. By (5.16) and the definition (5.14) of S and Z, $S(sz)=\lambda sz+\mu zs+\nu z^2$. Thus (5.20) is the same as $\lambda sz^2y+\mu zszy+\nu z^3y=0$. Since $sz^2=zZ(sz)=0$, it further simplifies to $\mu zszy+\nu z^3y=0$. Now the validity of this equality for $y=z$ spells $\nu z^4=0 \iff \nu=0$, while its validity for $y=s$ means $0=\mu zszs+\nu z^3s=\mu\lambda z^2sz+(\mu^2+\nu)z^3s+\mu\nu z^4 \iff \mu\lambda=\mu^2+\nu=\mu\nu=0$. It follows that (5.20) holds for $y\in\{z,s\}$ if and only if $\mu=\nu=0$. On the other hand, if $\mu=\nu=0$, then (5.20) trivially holds for every $y\in V$. This completes the proof of the fact that in every case (5.20) for $y\in V$ is equivalent to (5.20) for $y\in\{z,s\}$.

Hence $\dim A_{n+2}=3$ if and only if

$$\begin{aligned} z^{n-3}\alpha(sz^{n-2}s)z^2&=z^{n-3}\alpha(sz^{n-1})\alpha(z^{n-3}sz^2),\\ z^{n-3}\alpha(sz^{n-2}s)zs&=z^{n-3}\alpha(sz^{n-1})\alpha(z^{n-3}szs). \end{aligned} \tag{5.22}$$

Now by (5.16), $\alpha(z^{n-3}sz^2)=asz+bzs+cz^2$ and $\alpha(z^{n-3}szs)=\lambda sz+\mu zs+\nu z^2$. Using (5.21) and the definitions of Z and S, we now get

$$\begin{aligned} &z^{n-3}\alpha(sz^{n-2}s)z^2=z^{n-3}S(u)z^2=z^{n-1}Z^2S(u),\\ &z^{n-3}\alpha(sz^{n-1})\alpha(z^{n-3}sz^2)=z^{n-3}Z(u)(asz+bzs+cz^2)=z^{n-1}(aZSZ+bSZ^2+cZ^3)(u),\\ &z^{n-3}\alpha(sz^{n-2}s)zs=z^{n-3}S(u)zs=z^{n-1}SZS(u),\\ &z^{n-3}\alpha(sz^{n-1})\alpha(z^{n-3}szs)=z^{n-3}Z(u)(\lambda sz+\mu zs+\nu z^2)=z^{n-1}(\lambda ZSZ+\mu SZ^2+\nu Z^3)(u), \end{aligned}$$

where, as in (5.21), $u=Z^{n-3}(sz)\in L$. Thus (5.22) is equivalent to (5.15). Thus we have proven that $\dim A_{n+2}=3$ if and only if (5.15) is satisfied.

Now assume that $n \geqslant 4$. Then by definitions of S and Z it follows that (5.15) is equivalent to

$$z^{n-4}u(sz^2 - azsz - bz^2s - cz^3) = z^{n-4}u(szs - \lambda zsz - \mu z^2s - \nu z^3) = 0 \text{ in } A, \tag{5.23}$$

where $u = Z^{n-3}(sz)$. If Z is invertible, we can apply the second part of Lemma 5.2. Then (5.16) and (5.2) imply (5.23). Hence $\dim A_{n+2} = 3$. If Z is non-invertible and the restriction of Z to L_0 is invertible, then exactly as above (5.23) rewrites as

$$Z^{4-n}(u)z^{n-4}(sz^2 - azsz - bz^2s - cz^3) = Z^{4-n}(u)z^{n-4}(szs - \lambda zsz - \mu z^2s - \nu z^3) = 0 \text{ in } A,$$

where a negative power of Z refers to the inverse of the restriction of Z to L_0 (note that $u \in L_0$). Since $Z^{4-n}(u) \in L_0$, there is $v \in M$ such that $Z^{4-n}(u) = vz$. Hence (5.23) is equivalent to

$$vz^{n-3}(sz^2 - azsz - bz^2s - cz^3) = vz^{n-3}(szs - \lambda zsz - \mu z^2s - \nu z^3) = 0 \text{ in } A,$$

which is obviously satisfied according to (5.16). Hence $\dim A_{n+2} = 3$ in this case as well. Finally, we have already observed above that if Z is non-invertible, the restriction of Z to L_0 is non-invertible and $n \geqslant 4$, then (5.20) is satisfied and therefore $\dim A_{n+2} = 3$.

Thus we have verified that $\dim A_{n+2} = 3$ provided $n \geqslant 4$. By Lemma 5.3, $\dim A_m = 3$ for all $m \geqslant n$, which completes the proof. $\square$

With the same shape of normal words in the case $n = 3$ we have an extra possibility.

Lemma 5.5. *Let A be a finitely generated degree graded algebra, whose ideal of relations is generated by some homogeneous elements of degree* 3*. Assume also that there exist $z, s \in A_1$ such that z^3, z^2s and zsz form a linear basis in A_3. Then there are three mutually exclusive possibilities*:

(1) $\dim A_4 < 3$;
(2) $H_A^{[3]} = \overline{3}$;
(3) $H_A^{[3]} = 32\overline{1}$.

Proof. By assumptions $\dim A_3 = 3$. If there is $x \in A_1$ for which x^2A_1 is 3-dimensional, the result follows from Lemma 4.1 with $n = k = 3$. Thus we can assume that $\dim x^2A_1 < 3$ for all $x \in A_1$. Fix $z, s \in A_1$ for which z^3, z^2s and zsz form a linear basis in A_3. We pick a linear basis X in A_1 containing z and s, equip X with a total order for which z is minimal and s is second minimal and consider the corresponding left-to-right degree-lexicographical order on $\langle X \rangle$. Then the degree 3 normal words are z^3, z^2s and zsz. Since the only degree 4 words for which both degree 3 subwords are normal are z^4, z^3s and z^2sz, we have $\dim A_4 \leqslant 3$. If $\dim A_4 < 3$, there is nothing to prove. Thus we assume $\dim A_4 = 3$ in which case the degree 4 normal words must be z^4, z^3s and z^2sz. Note that Lemma 2.7 as well as the first part of Lemma 5.2 apply since all relevant conditions are satisfied. In particular, we can use the cancellation rule (2.22).

We start by considering the normal word decomposition for zs^2: $zs^2 = rzsz + pz^2s + qz^3$. A linear substitution which leaves every $x \in X \setminus \{s\}$ as it was and replaces s by $s + c_1z$ with an appropriately chosen $c_1 \in \mathbb{K}$, kills r (without disturbing any of the properties of our algebra). Hence without loss of generality we can assume that $r = 0$: $zs^2 = pz^2s + qz^3$. Then $zs^2z = pz^2sz + qz^4$ and by (2.22), $s^2z = pzsz + qz^3$. Similarly, Similarly, $zs^3 = pz^2s^2 + qz^3s = (p^2 + q)z^3s + pqz^4$ and by (2.22), $s^3 = (p^2 + q)z^2s + pqz^3$. Throwing in normal word decompositions for szs and sz^2, we get

$$\begin{array}{l} zs^2 = pz^2s + qz^3, \ sz^2 = azsz + bz^2s + cz^3, \ szs = \lambda zsz + \mu z^2s + \nu z^3, \\ s^2z = pzsz + qz^3, \ s^3 = (p^2 + q)z^2s + pqz^3, \quad \text{where } p, q, a, b, c, \lambda, \mu, \nu \in \mathbb{K}. \end{array} \tag{5.24}$$

The formulas (5.24) allow us to write matrices of the linear maps S and Z as defined by (5.14) in Lemma 5.4 in the basis sz, zs, z^2 (in this order):

$$Z = \begin{pmatrix} a & 1 & 0 \\ b & 0 & 0 \\ c & 0 & 1 \end{pmatrix} \quad \text{and} \quad S = \begin{pmatrix} \lambda & 0 & 0 \\ \mu & p & 1 \\ \nu & q & 0 \end{pmatrix}. \tag{5.25}$$

By Lemma 5.4, the case $\dim A_m = 3$ for all $m \geqslant 3$ happens precisely when (5.15) for $n = 3$ is satisfied. Thus $\dim A_m = 3$ for all $m \geqslant 3$ if and only if

$$(Z^2S - aZSZ - bSZ^2 - cZ^3)(sz) = 0 \text{ and } (SZS - \lambda ZSZ - \mu SZ^2 - \nu Z^3)(sz) = 0.$$

In the matrix form, this statement reads

$$\begin{array}{c}\dim A_m = 3 \text{ for all } m \geqslant 3 \text{ if and only if the first columns of the matrices} \\ Z^2S - aZSZ - bSZ^2 - cZ^3 \text{ and } SZS - \lambda ZSZ - \mu SZ^2 - \nu Z^3 \text{ are zero.}\end{array} \tag{5.26}$$

Now we resolve the degree 4 ambiguities s^4, s^3z, s^2zs, szs^2 and s^2z^2 using the relations (5.24) and the fact that degree 4 normal words are z^4, z^3s and z^2sz. For example, for the ambiguity s^4, we have

$$0 = s(s^3) - (s^3)s = (p^2+q)sz^2s + pqsz^3 - (p^2+q)z^2s^2 - pqz^3s,$$

which we then reduce to a linear combination of z^4, z^3s and z^2sz using the relations (5.24). Since z^4, z^3s and z^2sz, all coefficients of this combination must be zero, yielding three algebraic equations. In total, the five ambiguities in question produce 15 equations, which are

$$\begin{array}{lll}
\lambda a(p^2{+}q){+}(a^2{+}b)pq{=}0, & (\mu a{+}bp{-}p{+}c)(p^2{+}q){+}(ab{-}1)pq{=}0, & (\nu a{+}bq{-}q)(p^2{+}q){+}(a{+}1)cpq{=}0, \\
a(\lambda p{+}aq){+}\mu p{+}bq{-}(p^2{+}q){=}0, & c(\lambda p{+}aq){+}\nu p{+}cq{-}pq{=}0, & b(\lambda p{+}aq){=}0, \\
a(\lambda^2{+}\nu a){+}\lambda\mu{+}(\mu a{-}p)\lambda{+}\nu b{=}0, & b(\lambda^2{+}\nu a){+}(\mu a{-}p)\mu{+}\mu bp{+}\mu c{-}q{=}0, & c(\lambda^2{+}\nu a){+}\lambda\nu{+}(\mu a{-}p)\nu{+}\mu bq{+}\nu c{=}0, \\
\lambda(ap{-}\lambda){+}a^2q{+}bq{=}0, & \mu(ap{-}\lambda){+}p(bp{-}\mu){+}cp{+}abq{-}\nu{=}0, & \nu(ap{-}\lambda){+}q(bp{-}\mu){+}(a{+}1)cq{=}0, \\
a(a\lambda{+}ac{-}p){+}\lambda ab{+}bc{+}a\mu{=}0, & b(a\lambda{+}ac{-}p){+}\mu ab{+}b^2p{+}bc, & c(a\lambda{+}ac{-}p){+}a\nu{+}\nu ab{+}b^2q{+}c^2{-}q{=}0
\end{array} \tag{5.27}$$

We proceed by solving this system of algebraic equations.

Case 1: $bp \neq 0$.

Scaling s (=performing a linear substitution which leaves every $x \in X \setminus \{s\}$ as it was and replaces s by its own non-zero scalar multiple), we can turn p into -1. That is, we may assume that $p = -1$. Then from the equations in the second line of (5.27), we get $\lambda = aq$, $\mu = bq - q - 1$, $\nu = qc + q$.

Case 1a: $bpq \neq 0$.

Then from the first equation in the first line of (5.27), $b = a^2q$ and therefore $\mu = a^2q^2 - q - 1$. Plugging

$$p = -1, \quad b = a^2q, \quad \lambda = aq, \quad \mu = a^2q^2 - q - 1 \quad \text{and} \quad \nu = qc + q$$

into the second equation from the first line and into the second equation of the fourth line of (5.27), we get

$$c(q+1) = -a^3q^3 + a^2q^2 + a^2q - 2q - 1 \quad \text{and} \quad c(q+1) = a^2q^2 + a^2q + aq^2 + 2aq - 2q + a - 1,$$

respectively. Subtracting these equations, we get $a(q+1)^2 = 0$. Since $b = a^2q \neq 0$, we have $a \neq 0$ and therefore $q = -1$. Hence we have

$$p = q = -1, \quad b = -a^2, \quad \lambda = -a, \quad \mu = a^2, \quad \text{and} \quad \nu = -c - 1.$$

Plugging $q = -1$ into $c(q+1) = -a^3q^3 + a^2q^2 + a^2q - 2q - 1$, we get $a^3 = -1$. Plugging the expressions from the above display into the third and second equations in the third row of (5.27), we get $a^4 = a^2$ and $a^2c = -1$, respectively, which together with $a^3 = -1$ yield $a = c = -1$. Plugging this into the above display, we get

$$a = b = c = p = q = -1, \quad \lambda = \mu = 1 \text{ and } \nu = 0. \tag{5.28}$$

Now one easily sees that all 15 equations of (5.27) are satisfied in this case. We have our first solution.

Case 1b: $bp \neq 0$ and $q = 0$.

In this case the equations $\lambda = aq$, $\mu = bq - q - 1$, $\nu = qc + q$ yield

$$q = \lambda = \nu = 0 \text{ and } p = \mu = -1.$$

Plugging these values into (5.27), we see that the system reduces to

$$a + b - c - 1 = 0, \quad (a^2 + b)c = 0, \quad (ac + c + 1)c = 0 \text{ and } ac + c + 1 - a - b = 0,$$

for which the only solutions are

$$p = \mu = -1, \quad q = c = \lambda = \nu = 0 \text{ and } a = 1 - b, \quad \text{where } b \in \mathbb{K}^* \tag{5.29}$$

and we have our second solution. This concludes Case 1.

Case 2: $bq \neq 0$ and $p = 0$.

Scaling s we can turn q into 1. That is, we may assume that $q = 1$. Plugging $p = 0$ and $q = 1$ into the third equation in the second row of (5.27), we get $ab = 0$ and therefore $a = 0$. Plugging $a = p = 0$ and $q = 1$ into the second and the first equations in the second row of (5.27), we get $c = 0$ and $b = 1$, respectively. Plugging $a = p = c = 0$ and $b = q = 1$ into all three equations in the fourth row of (5.27), we get $\lambda^2 = 1$, $\lambda\mu + \nu = 0$ and $\lambda\nu + \mu = 0$. Note that the scaling $s \mapsto -s$ does not change q at all and changes λ into $-\lambda$. Hence by means of this extra scaling, we can reduce the two options $\lambda = \pm 1$ to just $\lambda = 1$. Assuming $\lambda = 1$, the two equations $\lambda\mu + \nu = 0$ and $\lambda\nu + \mu = 0$ become one $\nu = -\mu$. Thus we have

$$p = a = c = 0, \quad b = q = \lambda = 1 \text{ and } \nu = -\mu, \quad \text{where } \mu \in \mathbb{K}. \tag{5.30}$$

One easily sees that all 15 equations of (5.27) are satisfied in this case. We have our third solution.

Case 3: $b \neq 0$ and $p = q = 0$.

Plugging $p = q = 0$ into the first equation of the fourth row in (5.27), we get $\lambda = 0$. Plugging $p = q = \lambda = 0$ into the second equation of the fourth row in (5.27), we get $\nu = 0$. Plugging $p = q = \mu = \nu = 0$ into (5.27), we see that the entire system reduces to

$$\mu(\mu a + c) = 0, \quad a(\mu a + c) + bc = 0, \quad c(a + 1) = 0 \text{ and } \mu a + c + ac = 0.$$

Considering cases $c = 0$ and $c \neq 0$ separately, it is easy to solve this system. This gives three more solutions:

$$p = q = \lambda = \mu = \nu = c = 0 \text{ and } a \in \mathbb{K}, \quad b \in \mathbb{K}^*; \tag{5.31}$$

$$p = q = \lambda = a = \nu = c = 0 \text{ and } \mu \in \mathbb{K}, \quad b \in \mathbb{K}^*; \tag{5.32}$$

$$a = b = -1, \quad p = q = \lambda = \mu = \nu = 0 \text{ and } c \in \mathbb{K}. \tag{5.33}$$

Case 4: $b = 0$ and $apq \neq 0$.

It turns out that we have no solutions in this case. Indeed from the first row of (5.27), we get $p^2 + q \neq 0$. Scaling s, we can turn $p^2 + q$ into 1. Thus we can assume $p^2 + q = 1$. Then $q = 1 - p^2$. Plugging this into the equations of the first row of (5.27), we get

$$\lambda = a(p^3 - p), \quad \mu = \tfrac{2p - p^3 - c}{a} \text{ and } \nu = \tfrac{(1-p^2)(1-(a+1)pc)}{a}.$$

Modulo these expressions, the first equation of the fourth row of (5.27) reduces to $(p^2-1)^3=0$. This yields $p^2=1$ and therefore $q=0$ contradicting the assumptions.

Case 5: $b=q=0$ and $ap\neq 0$.

By scaling s, we can and will assume that $p=1$. Plugging $b=q=0$ and $p=1$ into the first equation in the first row of (5.27), we get $\lambda=0$. Plugging $b=q=\lambda=0$ and $p=1$ into the third equation in the first row of (5.27), we get $\nu=0$. Plugging $b=q=\lambda=\nu=0$ and $p=1$ into the first equation in the second row of (5.27), we get $\mu=1$. Plugging $b=q=\lambda=\nu=0$ and $p=\mu=1$ into the first equation in the fifth row of (5.27), we get $c=0$. Finally, plugging $b=q=\lambda=\nu=c=0$ and $p=\mu=1$ into the second equation in the first row of (5.27), we get $a=1$. Thus

$$a=p=\mu=1 \quad\text{and}\quad b=c=q=\lambda=\nu=0. \tag{5.34}$$

One easily sees that all 15 equations of (5.27) are satisfied in this case. We have our seventh solution.

Case 6: $b=q=p=0$ and $a\neq 0$.

Plugging $b=q=p=0$ into the first equation in the fourth row of (5.27), we get $\lambda=0$. Plugging $b=q=p=\lambda=0$ into the second equation in the fourth row of (5.27), we get $\nu=0$.

Plugging $b=q=p=\lambda=\nu=0$ into (5.27), we see that the entire system reduces to

$$\mu(\mu a+c)=0,\quad ac+\mu=0 \quad\text{and}\quad c(a+1)=0.$$

This is easily seen to produce the following two solutions:

$$a=-1,\quad b=p=q=\lambda=\nu=0 \quad\text{and}\quad \mu=c\in\mathbb{K}; \tag{5.35}$$

$$b=p=q=\lambda=\nu=\mu=c=0 \quad\text{and}\quad a\in\mathbb{K}^*. \tag{5.36}$$

Case 7: $a=b=0$.

The system becomes too simple in this case. We just state what are the last two solutions (leaving the verification to the reader). The path is by, first seeing that $q=0$ and considering the cases $p\neq 0$ (and a scaling of s brings p to 1) and $p=0$. The two solutions emerging are

$$p=c=\mu=1 \quad\text{and}\quad a=b=q=\lambda=\nu=0; \tag{5.37}$$

$$a=b=c=p=q=\lambda=\nu=0 \quad\text{and}\quad \mu\in\mathbb{K}. \tag{5.38}$$

Since Cases 1–7 cover all possibilities, we have described all solutions (modulo scaling of s) of the system (5.27). Now it is a matter of a direct calculation to show that for the matrices Z and S given by (5.25) with $a,b,c,p,q,\lambda,\mu,\nu$ from one of the above eleven solutions (5.28–5.38), the matrix equalities $Z^2S-aZSZ-bSZ^2-cZ^3=0$ and $SZS-\lambda ZSZ-\mu SZ^2-\nu Z^3=0$ (the whole matrices in fact) hold for all solutions except for (5.32) and (5.38), with the latter being the same as (5.32) but with $b=0$. According to (5.26), $\dim A_m=3$ for all $m\geqslant 3$ unless $p=q=\lambda=a=\nu=c=0$.

Thus we may now assume that $p=q=\lambda=a=\nu=c=0$ and the relations (5.24) simplify to

$$zs^2=0,\ sz^2=bz^2s,\ szs=\mu z^2s,\ s^2z=0,\ s^3=0,\quad\text{where } b,\mu\in\mathbb{K}.$$

If either $b=1$ or $\mu=0$, the equalities $Z^2S-aZSZ-bSZ^2-cZ^3=0$ and $SZS-\lambda ZSZ-\mu SZ^2-\nu Z^3=0$ are satisfied anyway yielding $\dim A_m=3$ for all $m\geqslant 3$. Thus we have to assume that $b\neq 1$ and $\mu\neq 0$. Since $\mu\neq 0$, an appropriate scaling of s turns μ into 1 and the relations (5.24) become

$$zs^2=0,\ sz^2=bz^2s,\ szs=z^2s,\ s^2z=0,\ s^3=0,\quad\text{where } b\in\mathbb{K}\setminus\{1\}. \tag{5.39}$$

If we start with (5.39) with $b=0$ and perform the substitution which keeps every $x\in X\setminus\{z,s\}$ as they were and replaces z by $z_{\text{new}}=z-\alpha s$ and s by $s_{\text{new}}=s(1-\alpha)$ with $\alpha\in\mathbb{K}\setminus\{1\}$, then the relations (5.39) written in terms of new z and s (which we now still denote z and s dropping the subscript) look like

$$zs^2 = 0,\ sz^2 = \alpha z^2 s,\ szs = z^2 s,\ s^2 z = 0 \text{ and } s^3 = 0.$$

Basically, this means that substitutions of this type turn $b = 0$ into any $b \in \mathbb{K} \setminus \{1\}$ we like. Hence we can assume that $b = -1$, which yields

$$zs^2 = 0,\ sz^2 = -z^2 s,\ szs = z^2 s,\ s^2 z = 0,\ s^3 = 0. \tag{5.40}$$

In this case Z is invertible and therefore $\dim A_2 z = 3$ and the second part of Lemma 5.2 applies. Then according to (5.3) $\dim A_m = \dim B_m$ for all $m \geqslant 3$, where the algebra B given by just two generators s and z and the relations (5.40). Curiously enough, assuming that $\dim A_4 = 3$ and that $\dim A_m$ is different from 3 for some $m \geqslant 5$, we have almost pinpointed A uniquely up to an isomorphism: algebras A and B may 'differ' only in graded components 1 and 2. Computing the reduced Gröbner basis of the ideal of relations of B, we easily see that it consists of zs^2, $sz^2 + z^2 s$, $szs - z^2 s$, $s^2 z$, s^3 and $z^4 s$, yielding $H_B(t) = 1 + 2t + 4t^2 + 3t^3 + 3t^4 + 2t^5 + t^6 + t^7 + t^8 + \dots$. Hence $H_A^{[3]} = 32\overline{1}$, which completes the proof. □

5.2 Normal words of the form z^n, $z^{n-1}s$ and $z^{n-2}s^2$

Lemma 5.6. *Let $n \in \mathbb{N}$, $n \geqslant 3$ and let A be a finitely generated degree graded algebra, whose ideal of relations is generated by some homogeneous elements of degree n. Assume also that there exist $z, s \in A_1$ such that z^n, $z^{n-1}s$ and $z^{n-2}s^2$ form a linear basis in A_n, $\dim z^{n-1}A_1 < 3$ and $z^{n-2}sz = z^{n-1}s + bz^n$ in A for some $b \in \mathbb{K}^*$. Then either $\dim A_{n+1} < 3$ or $\dim A_m = 3$ for all $m \geqslant n$.*

Proof. If $\dim A_{n+1} < 3$, there is nothing to prove. Thus we can assume that $\dim A_{n+1} \geqslant 3$. Let X be a linear basis in A_1 containing z and s. Equip X with a total order for which $z = \min X$ and $s = \min(X \setminus \{z\})$ and extend it to the left-to-right degree-lexicographical order $<$ on $\langle X \rangle$. Since the two smallest degree n words with respect to this order z^n and $z^{n-1}s$ are linearly independent in A, they are degree n normal words. Since $\dim z^{n-1}A_1 < 3$ and $z^{n-2}sz$ is in the linear span of z^n and $z^{n-1}s$ in A, the smallest degree n word, which does not belong to the said span is $z^{n-2}s^2$ (recall that z^n, $z^{n-1}s$ and $z^{n-2}s^2$ are linearly independent). Hence the three degree n normal words are z^n, $z^{n-1}s$ and $z^{n-2}s^2$. Since the only degree $n+1$ words both degree n subwords of which are normal are z^{n+1}, $z^n s$ and $z^{n-1}s^2$, we have $\dim A_{n+1} = 3$ and the degree $n+1$ normal words are z^{n+1}, $z^n s$ and $z^{n-1}s^2$. Note that by Lemma 2.7, the cancellation rule (2.22) holds. We also denote the tensor algebra of $V := A_1$ by F. Recall that F naturally identified with the free algebra $\mathbb{K}\langle X \rangle$. A scaling of s (=a linear substitution leaving every element of $X \setminus \{s\}$ as it was and replacing s by its own non-zero scalar multiple) allows to turn b into 1. Thus we can assume that $z^{n-2}sz = z^{n-1}s + z^n$. Repeatedly multiplying by z on the right and using the cancellation rule (2.22), we get the following equalities in A:

$$z^{n-1-j}sz^j = z^{n-1}s + jz^n \text{ for } 0 \leqslant j \leqslant n-1 \text{ and } sz^n = z^n s + nz^{n+1}. \tag{5.41}$$

Multiplying $z^{n-2}sz = z^{n-1}s + (n-2)z^n$ by s on the right and using (2.22), we obtain $sz^{n-2}s = z^{n-2}s^2 + (n-2)z^{n-1}s$. Multiplying by z on the right, we get $z^{n-2}s^2 z = sz^{n-2}sz - (n-2)z^{n-1}sz$. Now by (5.41), we get $z^{n-2}s^2 z = sz^{n-1}s + sz^n - (n-2)z^n s - (n-2)z^{n+1}$. Applying (5.41) once again, simplifying and using (2.22), we get

$$z^{n-3}s^2 z = z^{n-2}s^2 + 2z^{n-1}s + 2z^n \text{ in } A. \tag{5.42}$$

Now consider the normal word decomposition of $z^{n-3}s^3$:

$$z^{n-3}s^3 = pz^{n-2}s^2 + qz^{n-1}s + rz^n \text{ in } A, \tag{5.43}$$

where $p, q, r \in \mathbb{K}$. Next, we consider the monomial $sz^{n-3}s^2z$, which we express in two different ways. Multiplying (5.42) by s on the left and simplifying by means of (5.41) and the equality $sz^{n-2}s = z^{n-2}s^2 + (n-2)z^{n-1}s$, we get

$$sz^{n-3}s^2z = sz^{n-2}s^2 + 2sz^{n-1}s + 2sz^n = z^{n-2}s^3 + nz^{n-1}s^2 + 2nz^ns + 2nz^{n+1}.$$

Next, using $z^2sz^{n-3} = z^{n-1}s + (n-3)z^n$ and (2.23), we get $sz^{n-3}s^2z = z^{n-3}s^3z + (n-3)z^{n-2}s^2z$. Now according to (5.43), $sz^{n-3}s^2z = (p+n-3)z^{n-2}s^2z + qz^{n-1}sz + rz^{n+1}$. Simplifying by means of (5.41) and (5.42), we get $sz^{n-3}s^2z = (p+n-3)z^{n-1}s^2 + (2p+q+2n-6)z^ns + (2p+q+r+2n-6)z^{n+1}$. Taking (5.43) into account, we obtain

$$sz^{n-3}s^2z = z^{n-2}s^3 + (n-3)z^{n-1}s^2 + (2p+2n-6)z^ns + (2p+q+2n-6)z^{n+1}.$$

By the above two displays,

$$3z^{n-1}s^2 + 2(3-p)z^ns + (6-2p-q)z^{n+1} = 0 \ \text{ in } A.$$

Since $z^{n-1}s^2$, z^ns and z^{n+1} are linearly independent in A, we see that $3 = 2(3-p) = 6-2p-q = 0$ in $\mathbb{K}$. It follows that $\mathbb{K}$ has characteristic 3 and $p = q = 0$. Note that had we excluded characteristic 3 fields from the start, we would be already done by getting a contradiction. As we did not, we have to proceed. Note that we can kill r by a substitution which leaves every element of $X \setminus \{s\}$ as it was and adds to s an appropriate scalar multiple of z. Assume that we have done this. Then (5.43) becomes

$$z^{n-3}s^3 = 0 \text{ in } A. \tag{5.44}$$

Since we know that $z^{n-2}sz = z^{n-1}s + z^n$ and $z^{n-3}s^2z = z^{n-2}s^2 + 2z^{n-1}s + 2z^n$, monomials $z^{n-2}sz$, $z^{n-3}s^2z$ and z^n are linearly independent in A. Hence $\dim A_{n-1}z = 3$ and we can use both parts of Lemma 5.2.

We have to verify that $\dim A_m = 3$ for all $m \geqslant n$. By Lemma 5.3, for this purpose it suffices to verify the equalities (5.8) with $t = s$, where $\alpha : F_n \to L = \mathtt{span}\,\{s^2, zs, z^2\}$ is given by (5.6). That is we have to verify that

$$\begin{aligned} z^{n-3}\alpha(sz^{n-2}s)sz &= z^{n-3}\alpha(sz^{n-1})\alpha(z^{n-3}s^2z) \\ z^{n-3}\alpha(sz^{n-2}s)s^2 &= z^{n-3}\alpha(sz^{n-1})\alpha(z^{n-3}s^3) \end{aligned} \tag{5.45}$$

are satisfied in A. The equality $sz^{n-2}s = z^{n-2}s^2 + (n-2)z^{n-1}s$ yields $\alpha(sz^{n-2}s) = s^2 + (n-2)zs$, while (5.44) reads $\alpha(z^{n-3}s^3) = 0$. Thus the right-hand side of the second equation in (5.45) vanishes, while the left-hand side equals $z^{n-3}(s^2 + (n-2)zs)s^2 = z^{n-3}s^4 + (n-2)z^{n-2}s^3$, which vanishes as well because of (5.44). Thus the second equality in (5.45) is satisfied.

By (5.41), $\alpha(sz^{n-1}) = zs + (n-1)z^2$, while by (5.42), $\alpha(z^{n-3}s^2z) = s^2 + 2zs + 2z^2$, which together with $\alpha(sz^{n-2}s) = s^2 + (n-2)zs$ allows us to rewrite the first equality in (5.45) as

$$z^{n-3}(s^2 + (n-2)zs)sz = z^{n-3}(zs + (n-1)z^2)(s^2 + 2zs + 2z^2).$$

After opening brackets and simplifying due to (5.41), (5.42) and (5.44), it transforms into $3z^{n-1}s^2 + 6z^ns + 6z^{n+1} = 0$, which holds since $\mathbb{K}$ now has characteristic 3. The proof is complete. $\square$

Lemma 5.7. *Let $n \in \mathbb{N}$, $n \geqslant 3$ and let A be a finitely generated degree graded algebra, whose ideal of relations is generated by some homogeneous elements of degree n. Assume also that there exist $z, s \in A_1$ such that z^n, $z^{n-1}s$ and $z^{n-2}s^2$ form a linear basis in A_n, while z^n, $z^{n-1}s$ and $z^{n-2}sz$ are linearly dependent in A. Then either* $\dim A_{n+1} < 3$ *or* $\dim A_m = 3$ *for all $m \geqslant n$.*

Proof. If $z^{n-1}A_1$ is 3-dimensional, the result follows from Lemma 4.1 with $k = 3$. Thus we can assume that $\dim z^{n-1}A_1 < 3$. Then, by assumptions, z^n and $z^{n-1}s$ form a linear basis in $z^{n-1}A_1$. Let X be a linear basis in A_1 containing z and s. Equip X with a total order for which $z = \min X$ and $s = \min(X\setminus\{z\})$ and extend it to the left-to-right degree-lexicographical order $<$ on $\langle X\rangle$. Since the two smallest degree n words with respect to this order z^n and $z^{n-1}s$ are linearly independent in A, they are degree n normal words. Since $z^{n-1}A_1$ as well as $z^{n-2}sz$ are in the linear span of z^n and $z^{n-1}s$ in A, the smallest degree n word, which does not belong to the said span is $z^{n-2}s^2$ (recall that z^n, $z^{n-1}s$ and $z^{n-2}s^2$ are linearly independent). Hence the three degree n normal words are z^n, $z^{n-1}s$ and $z^{n-2}s^2$. If $\dim A_{n+1} < 3$, there is nothing to prove. Thus we shall assume that $\dim A_{n+1} \geqslant 3$. Since the only degree $n+1$ words both degree n subwords of which are normal are z^{n+1}, $z^n s$ and $z^{n-1}s^2$, we have $\dim A_{n+1} = 3$ and the degree $n+1$ normal words are z^{n+1}, $z^n s$ and $z^{n-1}s^2$. Note that by Lemma 2.7, the cancellation rule (2.22) holds. We also denote the tensor algebra of $V := A_1$ by F. Recall that F naturally identified with the free algebra $\mathbb{K}\langle X\rangle$.

Since z^n, $z^{n-1}s$ and $z^{n-2}sz$ are linearly dependent, while z^n and $z^{n-1}s$ are not, there exist $a, b \in \mathbb{K}$ such that $z^{n-2}sz = az^{n-1}s + bz^n$. If $a = 1$ and $b \neq 0$, the result follows from Lemma 5.6. Thus we can assume that either $b = 0$ or $a \neq 1$. Now if $a \neq 1$, then by a linear substitution, which leaves every generator from $X \setminus \{s\}$ as it was and replaces s by $s + cz$ with appropriately chosen $c \in \mathbb{K}$, we can kill b. Thus for the rest of the proof we can assume that $b = 0$ and therefore

$$z^{n-2}sz = az^{n-1}s \text{ in } A \text{ for some } a \in \mathbb{K}. \tag{5.46}$$

Decomposing via normal words, we can write

$$z^{n-3}s^3 = pz^{n-2}s^2 + qz^{n-1}s + rz^n \text{ in } A \text{ for some } p, q, r \in \mathbb{K}. \tag{5.47}$$

Before proceeding, we would like to provide relations between numbers p, q, r and a, which follow from the assumptions of our lemma.

Repeatedly multiplying $z^{n-1}s$ by z on the right, using (5.46) and cancelling z from the left according to (2.22), we get

$$z^{n-1-j}sz^j = a^j z^{n-1}s \text{ for } 0 \leqslant j \leqslant n \text{ and } sz^n = a^n z^n s \text{ in } A. \tag{5.48}$$

Since by (5.48), $z^{n-1-j}sz^j = a^j z^{n-1}s$, we have $z^{n-1-j}sz^j s = a^j z^{n-1}s^2$ and by (2.22),

$$z^{n-2-j}sz^j s = a^j z^{n-2}s^2 \text{ in } A \text{ for } 0 \leqslant j \leqslant n-2. \tag{5.49}$$

The rest requires considering the cases $a = 0$ and $a \neq 0$ separately. By (5.49), $sz^{n-2}s = a^{n-2}z^{n-2}s^2$. Hence $sz^{n-2}sz = a^{n-2}z^{n-2}s^2z$. On the other hand, by (5.48), $sz^{n-2}sz = asz^{n-1}s = a^n z^{n-1}s^2$. It follows that $a^n z^{n-1}s^2 = a^{n-2}z^{n-2}s^2 z$ and by (2.22), $a^n z^{n-2}s^2 = a^{n-2}z^{n-3}s^2 z$. Hence

$$z^{n-3}s^2 z = a^2 z^{n-2}s^2 \text{ in } A \text{ if } a \neq 0. \tag{5.50}$$

Multiplying by s on the left and using first (5.49) and then (5.47), we get

$$sz^{n-3}s^2 z = a^2 sz^{n-2}s^2 = a^n z^{n-2}s^3 = a^n(pz^{n-1}s^2 + qz^n s + rz^{n+1}).$$

On the other hand, by (5.48), $z^2sz^{n-3} = a^{n-3}z^{n-1}s$ and therefore $z^2sz^{n-3}s = a^{n-3}z^{n-1}s^2$. By (2.22), $zsz^{n-3}s = a^{n-3}z^{n-2}s^2$ and therefore $sz^{n-3}s^2z = a^{n-3}z^{n-3}s^3z$. Using (5.47), we get $sz^{n-3}s^2z = a^{n-3}(pz^{n-2}s^2z + qz^{n-1}sz + rz^{n+1})$ and by (5.48) and (5.50) (here we assume that $a \neq 0$), $sz^{n-3}s^2z = a^{n-1}pz^{n-1}s^2 + a^{n-2}qz^n s + ra^{n-3}z^{n+1}$. Comparing this equality to the above display, we get

$$a^{n-1}(a-1)pz^{n-1}s^2 + a^{n-2}(a^2-1)qz^n s + a^{n-3}(a^3-1)rz^{n+1} = 0 \text{ in } A.$$

Since $z^{n-1}s^2$, $z^n s$ and z^{n+1} are linearly independent in A, we arrive at

$$p(a-1)=q(a^2-1)=r(a^3-1)=0 \ \text{ if } a\neq 0. \tag{5.51}$$

This concludes laying the groundwork.

Case 1: $a\neq 0$. In this case (5.48) and (5.50) imply that $\dim A_{n-1}z=3$. Then by Lemma 5.3 in order to prove that $H_A^{[n]}=\overline{3}$, it suffices to verify the validity of (5.8) with $t=s$, where $\alpha: F_n\to L=\mathtt{span}\,\{s^2,zs,z^2\}$ is given by (5.6). That is, we have to verify (5.45). By (5.48), (5.49), (5.50) and (5.47), $\alpha(sz^{n-2}s)=a^{n-2}s^2$, $\alpha(z^{n-3}s^2z)=a^2s^2$, $\alpha(sz^{n-1})=a^{n-1}zs$ and $\alpha(z^{n-3}s^3)=ps^2+qzs+rz^2$. Now (5.45) reads

$$a^{n-2}z^{n-3}s^3z=a^{n+1}z^{n-2}s^3,\qquad a^{n-2}z^{n-3}s^4=a^{n-1}z^{n-2}s(ps^2+qzs+rz^2).$$

Using (5.47) again, we can rewrite the above display as

$$\begin{aligned}a^{n-2}z^{n-2}(ps^2+qzs+rz^2)z&=a^{n+1}z^{n-1}(ps^2+qzs+rz^2),\\ a^{n-2}z^{n-2}(ps^2+qzs+rz^2)s&=a^{n-1}z^{n-2}s(ps^2+qzs+rz^2).\end{aligned}$$

Simplifying by means of (5.48), (5.49) and (5.50), we see that the problem boils down to the validity of

$$a^2(a-1)pz^{n-1}s^2+a(a^2-1)qz^ns+(a^3-1)rz^{n+1}=0,\quad a^2(a-1)pz^{n-2}s^3+a(a^2-1)qz^{n-1}s^2+(a^3-1)rz^ns=0,$$

which holds due to (5.51). This completes Case 1.

Case 2: $a=0$. By Lemma 5.3, the proof will be complete if we verify (5.7) with $t=s$. That is, we have to show that

$$\begin{aligned}z^{n-3}\alpha(sz^{n-1})zy&=z^{n-3}\alpha(sz^{n-1})\alpha(z^{n-1}y),\\ z^{n-3}\alpha(sz^{n-1})sy&=z^{n-3}\alpha(sz^{n-1})\alpha(z^{n-2}sy),\\ z^{n-3}\alpha(sz^{n-2}s)sy&=z^{n-3}\alpha(sz^{n-1})\alpha(z^{n-3}s^2y)\end{aligned} \tag{5.52}$$

for all $y\in V$. However, since $a=0$, (5.48) and (5.49) imply that $\alpha(sz^{n-1})=\alpha(sz^{n-2}s)=0$. Hence (5.52) in the case $a=0$ is satisfied, which completes Case 2 and the proof of our lemma. □

5.3 Proof of Lemma 5.1

Let A be a finitely generated degree graded algebra, whose ideal of relations is generated by some homogeneous elements of degree n, where $n\geqslant 3$. Assume also that $\dim A_n=\max\{\lambda(A,n),\rho(A,n)\}=3$. Hence either $\lambda(A,n)=3$ or $\rho(A,n)=3$. The two cases are equivalent. Indeed, they are reduced to one another by passing to the opposite multiplication. Thus without loss of generality, we may assume that $\lambda(A,n)=3$. If $\dim A_{n+1}<3$, there is nothing to prove. Hence we shall assume that $\dim A_{n+1}\geqslant 3$. If $\dim x^{n-1}A_1=3$ for some $x\in A_1$, Lemma 4.1 with $k=3$ implies that $H_A^{[n]}=\overline{3}$. Thus for the rest of the proof we can assume that $\dim x^{n-1}A_1<3$ for all $x\in A_1$. Now by Lemma 2.2, for generic $X=(x_1,x_2,\dots)\in\Omega(A)$ with $\langle X\rangle$ carrying the LR order $<$, $\mathrm{NW}_n=\{x_1^n,x_1^{n-1}x_2,w\}$, where $w\in\{x_1^{n-2}x_2x_1,x_1^{n-2}x_2^2\}$. If $w=x_1^{n-2}x_2^2$, Lemma 5.7 (with $z=x_1$ and $s=x_2$) yields $H_A^{[n]}=\overline{3}$. If $w=x_1^{n-2}x_2x_1$ and $n\geqslant 4$, then by Lemma 5.4 with $z=x_1$ and $s=x_2$, $H_A^{[n]}=\overline{3}$. On the other hand, if $n=3$ and $w=x_1^{n-2}x_2x_1=x_1x_2x_1$, then $H_A^{[n]}\in\{\overline{3},332\overline{1}\}$ according to Lemma 5.5 with $z=x_1$ and $s=x_2$. This completes the proof of Lemma maxLR3.

6 Case $\dim A_n=3$ and $\max\{\lambda(A,n),\rho(A,n)\}=2$

The main result of this section is the following lemma. We prove Theorem 1.3 at the end of the section.

Lemma 6.1. *Let A be a finitely generated degree graded algebra, whose ideal of relations is generated by some homogeneous elements of degree n, where $n\geqslant 3$. Assume also that* $\dim A_n=3$, $\dim A_{n+1}\geqslant 3$ *and* $\max\{\lambda(A,n),\rho(A,n)\}=2$. *Then* $H_A^{[n]}=\in\{\overline{3},3\overline{4}\}$.

6.1 Normal words of the form z^n, $z^{n-1}s$ and sz^{n-1}

Lemma 6.2. *Let A be a finitely generated degree graded algebra, whose ideal of relations is generated by some homogeneous elements of degree n, where $n \geqslant 3$. Let X be a linear basis in A_1 and $z, s \in X$ be such that z^n, $z^{n-1}s$ and sz^{n-1} are linearly independent in A, $zA_{n-1} = \mathtt{span}\{z^n, z^{n-1}s\}$ and $A_{n-1}z = \mathtt{span}\{z^n, sz^{n-1}\}$ and A_n is spanned by (not necessarily linearly independent) z^n, $z^{n-1}s$, sz^{n-1} and $sz^{n-2}s$. Then $\dim A_{n+1} \leqslant 4$ and A_{n+1} is spanned by z^{n+1}, $z^n s$, sz^n and $sz^{n-1}s$. Moreover, if $\dim A_{n+1} = 4$, then $\dim A_m = 4$ for all $m > n$. Furthermore, if $\dim A_n = \dim A_{n+1} = 3$, then A_{n+1} is spanned by z^{n+1}, $z^n s$ and sz^n and $\dim A_m = 3$ for all $m \geqslant n$.*

Proof. Let $X \in \Omega(A, z, s)$ with $\langle X \rangle$ carrying the LR order $<$. Since z^n and $z^{n-1}s$ are the two smallest degree n words and z^n, $z^{n-1}s$ are linearly independent in A, they are degree n normal words. Since $zA_{n-1} = \mathtt{span}\{z^n, z^{n-1}s\}$ and z^n, $z^{n-1}s$ and sz^{n-1} are linearly independent in A, the smallest degree n word w for which z^n, $z^{n-1}s$ and w are linearly independent is $w = sz^{n-1}$. Hence sz^{n-1} is a degree n normal word. If z^n, $z^{n-1}s$, sz^{n-1} and $sz^{n-2}s$ are linearly dependent the assumption that these 4 monomials span A_n guarantees that $\dim A_n = 3$ and the degree n normal words are z^n, $z^{n-1}s$ and sz^{n-1}. If z^n, $z^{n-1}s$, sz^{n-1} and $sz^{n-2}s$ are linearly independent, then equalities $zA_{n-1} = \mathtt{span}\{z^n, z^{n-1}s\}$, $A_{n-1}z = \mathtt{span}\{z^n, sz^{n-1}\}$ ensure that $sz^{n-2}s$ is the smallest degree n word for which z^n, $z^{n-1}s$, sz^{n-1} and w are linearly independent and therefore $\dim A_n = 4$ and $sz^{n-2}s$ is the fourth and final normal word.

Thus for each $w \in \langle X \rangle_n$, there exist unique $\alpha(w), \beta(w), \gamma(w), \delta(w) \in \mathbb{K}$ such that

$$w = \alpha(w)z^n + \beta(w)z^{n-1}s + \gamma(w)sz^{n-1} + \delta(w)sz^{n-2}s, \tag{6.1}$$

where we assume that

$$\delta(w) = 0 \text{ for all } w \text{ if } z^n,\ z^{n-1}s,\ sz^{n-1} \text{ and } sz^{n-2}s \text{ are linearly dependent.}$$

Since $zA_{n-1} = \mathtt{span}\{z^n, z^{n-1}s\}$ and $A_{n-1}z = \mathtt{span}\{z^n, sz^{n-1}\}$, we see that

$$\text{for every } w \in \langle X \rangle_{n-1},\ \beta(wz) = \gamma(zw) = \delta(zw) = \delta(wz) = 0. \tag{6.2}$$

Since the only degree $n+1$ words for which both degree n subwords are normal are z^{n+1}, $z^n s$, sz^n and $sz^{n-1}s$, these words span A_{n+1} and therefore $\dim A_{n+1} \leqslant 4$.

Regardless whether $sz^{n-2}s$ is a linear combination of z^n, $z^{n-1}s$ and sz^{n-1} or not, the elements

$$w - \alpha(w)z^n - \beta(w)z^{n-1}s - \gamma(w)sz^{n-1} - \delta(w)sz^{n-2}s$$

for all non-normal degree n words w form the degree n part of the reduced Gröbner basis for the ideal of relations of A. Now if $w \in \langle X \rangle_{n-1}$ and $x, y \in X$, (6.1) and (6.2) easily yield

$$xwy = \alpha'(xwy)z^{n+1} + \beta'(xwy)z^n s + \gamma'(xwy)sz^n + \delta'(xwy)sz^{n-1}s \text{ in } A, \tag{6.3}$$

where

$$\begin{aligned} \alpha'(xwy) &= \alpha(xw)\alpha(z^{n-1}y)+\beta(xw)\alpha(z^{n-2}sy), & \beta'(xwy) &= \alpha(xw)\beta(z^{n-1}y)+\beta(xw)\beta(z^{n-2}sy),\\ \gamma'(xwy) &= \gamma(xw)\alpha(z^{n-1}y)+\delta(xw)\alpha(z^{n-2}sy), & \delta'(xwy) &= \gamma(xw)\beta(z^{n-1}y)+\delta(xw)\beta(z^{n-2}sy). \end{aligned} \tag{6.4}$$

Here we reduced $xwy = (xw)y$. Performing the same procedure with $xwy = x(wy)$, we see that

$$\widetilde{\alpha}(xwy)z^{n+1} + \widetilde{\beta}(xwy)z^n s + \widetilde{\gamma}(xwy)sz^n + \widetilde{\delta}(xwy)sz^{n-1}s = 0 \text{ in } A \tag{6.5}$$

for all $x, y \in X$ and $w \in \langle X \rangle_{n-1}$, where

$$\begin{aligned} \widetilde{\alpha}(xwy) &= \alpha(xw)\alpha(z^{n-1}y) + \beta(xw)\alpha(z^{n-2}sy) - \alpha(wy)\alpha(xz^{n-1}) - \gamma(wy)\alpha(xsz^{n-2}),\\ \widetilde{\beta}(xwy) &= \alpha(xw)\beta(z^{n-1}y) + \beta(xw)\beta(z^{n-2}sy) - \beta(wy)\alpha(xz^{n-1}) - \delta(wy)\alpha(xsz^{n-2}),\\ \widetilde{\gamma}(xwy) &= \gamma(xw)\alpha(z^{n-1}y) + \delta(xw)\alpha(z^{n-2}sy) - \alpha(wy)\gamma(xz^{n-1}) - \gamma(wy)\gamma(xsz^{n-2}),\\ \widetilde{\delta}(xwy) &= \gamma(xw)\beta(z^{n-1}y) + \delta(xw)\beta(z^{n-2}sy) - \beta(wy)\gamma(xz^{n-1}) - \delta(wy)\gamma(xsz^{n-2}) \end{aligned} \tag{6.6}$$

and that the left-hand sides of (6.5) have the same linear spans as the the degree $n+1$ elements of the reduced Gröbner basis (this is how the Buchberger algorithm works). Hence a linear combination g of z^{n+1}, $z^n s$, sz^n and $sz^{n-1}s$ vanishes in A if and only if g is in the linear span of the left-hand sides of (6.5).

First, we shall prove the lemma under the additional assumption that

$$\text{either } \dim A_{n+1}=4 \tag{6.7}$$

$$\text{or } \dim A_n=\dim A_{n+1}=3 \text{ and } sz^{n-1}s=zsz^{n-2}=z^{n-2}sz=0 \text{ in } A. \tag{6.8}$$

It follows (we use $sz^{n-1}s=0$ in the case $\dim A_n=\dim A_{n+1}=3$) that

$$\widetilde{\alpha}(xwy)=\widetilde{\beta}(xwy)=\widetilde{\gamma}(xwy)=0 \ \text{ for all } x,y\in X,\ w\in\langle X\rangle_{n-1}. \tag{6.9}$$

$$\widetilde{\delta}(xwy)=0 \quad \text{for all } x,y\in X,\ w\in\langle X\rangle_{n-1} \text{ if (6.7) is satisfied} \tag{6.10}$$

$$\widetilde{\delta}(xwy)\neq 0 \quad \text{for some } x,y\in X,\ w\in\langle X\rangle_{n-1} \text{ if (6.8) is satisfied.} \tag{6.11}$$

Note that the converse is also true in the following sense. If (6.9) and (6.10) are satisfied, then $\dim A_{n+1}=4$, while if (6.9) and (6.11) are satisfied, then $\dim A_{n+1}=3$ and $sz^{n-1}s=0$.

Now the numbers $\alpha'(xwy)$, $\beta'(xwy)$, $\gamma'(xwy)$ and $\delta'(xwy)$ satisfying (6.3) (with $\delta'(xwy)=0$ in the case (6.8)) are uniquely determined (and, of course, given by (6.4) except for δ' in the case (6.8)) and all conditions of the lemma are satisfied with n replaced by $n+1$. In order to show that $\dim A_m=4$ for all $m\geqslant n+1$ in the case (6.7), it suffices to verify that $\dim A_{n+2}=4$ (then we can iterate the argument). For the same reason, in order to show that $\dim A_m=3$ for all $m\geqslant n$ in the case (6.8), it suffices to verify that $\dim A_{n+2}=3$ and $sz^ns=0$ in A (the equalities $zsz^{n-1}=z^{n-1}sz=0$ follow from $zsz^{n-2}=z^{n-2}sz=0$ trivially). We shall do just that (in both of the above cases) thereby proving that the conclusion of our lemma is satisfied provided (6.7) or (6.8) holds.

In order to show that $\dim A_{n+2}=4$ in the case (6.7), it is now sufficient to verify that (6.9) and (6.10) are satisfied if the degree of w is increased by 1 and $\alpha,\beta,\gamma,\delta$ are replaced by $\alpha',\beta',\gamma',\delta'$, respectively. Similarly, in the case (6.8), in order to show that $\dim A_{n+2}=3$ and $sz^ns=0$, it is sufficient to verify that (6.9) and (6.11) are satisfied if the degree of w is increased by 1 and $\alpha,\beta,\gamma,\delta$ are replaced by $\alpha',\beta',\gamma',\delta'$.

Now, (6.9) for monomials xwy with $x,y\in X$ and $w\in\langle X\rangle_n$ (degree increased by 1) reads

$$\begin{aligned}
&\alpha'(xw)\alpha'(z^ny)+\beta'(xw)\alpha'(z^{n-1}sy)-\alpha'(wy)\alpha(xz^n)-\gamma'(wy)\alpha(xsz^{n-1})=0,\\
&\alpha'(xw)\beta'(z^ny)+\beta'(xw)\beta'(z^{n-1}sy)-\beta'(wy)\alpha'(xz^{n1})-\delta'(wy)\alpha'(xsz^{n-1})=0,\\
&\gamma'(xw)\alpha'(z^ny)+\delta'(xw)\alpha'(z^{n-1}sy)-\alpha'(wy)\gamma'(xz^n)-\gamma'(wy)\gamma'(xsz^{n-1})=0,
\end{aligned} \tag{6.12}$$

(6.10) in the same situation becomes

$$\gamma'(xw)\beta'(z^ny)+\delta'(xw)\beta'(z^{n-1}sy)-\beta'(wy)\gamma'(xz^n)-\delta'(wy)\gamma'(xsz^{n-1})=0, \tag{6.13}$$

while naturally, (6.11) becomes the opposite:

$$\begin{gathered}
\gamma'(xw)\beta'(z^ny)+\delta'(xw)\beta'(z^{n-1}sy)-\beta'(wy)\gamma'(xz^n)-\delta'(wy)\gamma'(xsz^{n-1})\neq 0\\
\text{for some } \ x,y\in X \text{ and } w\in\langle X\rangle_n.
\end{gathered} \tag{6.14}$$

Using the original (6.9), (6.6) and (6.4), we see that

$$\begin{aligned}
&\alpha'(wy)=\alpha(w)\alpha(z^{n-1}y)+\beta(w)\alpha(z^{n-2}sy), && \alpha'(xw)=\alpha(w)\alpha(xz^{n-1})+\gamma(w)\alpha(xsz^{n-2}),\\
&\beta'(wy)=\alpha(w)\beta(z^{n-1}y)+\beta(w)\beta(z^{n-2}sy), && \beta'(xw)=\beta(w)\alpha(xz^{n-1})+\delta(w)\alpha(xsz^{n-2}),\\
&\gamma'(wy)=\gamma(w)\alpha(z^{n-1}y)+\delta(w)\alpha(z^{n-2}sy), && \gamma'(xw)=\alpha(w)\gamma(xz^{n-1})+\gamma(w)\gamma(xsz^{n-2}).
\end{aligned} \tag{6.15}$$

Applying (6.15) in few particular cases and using (6.2), we get

$$\begin{array}{lll} \alpha'(z^n y) = \alpha(z^{n-1}y) & \alpha'(xz^n) = \alpha(xz^{n-1}), & \beta'(z^n y) = \beta(z^{n-1}y), \\ \gamma'(xz^n) = \gamma(xz^{n-1}), & \alpha'(z^{n-1}sy) = \alpha(z^{n-2}sy) & \alpha'(xsz^{n-1}) = \alpha(xsz^{n-2}), \\ \beta'(z^{n-1}sy) = \beta(z^{n-2}sy) & \gamma'(xsz^{n-1}) = \gamma(xsz^{n-2}). & \end{array} \tag{6.16}$$

Plugging (6.15) and (6.16) into (6.12), we see that (6.12) is trivially satisfied (in both cases): everything cancels out. If $\dim A_{n+1} = 4$ (condition (6.7)), then we similarly have

$$\delta'(wy) = \gamma(w)\beta(z^{n-1}y) + \delta(w)\beta(z^{n-2}sy), \quad \delta'(xw) = \beta(w)\gamma(xz^{n-1}) + \delta(w)\gamma(xsz^{n-2}),$$

using which together with (6.15) and (6.16), we see that (6.13) is also satisfied, thus proving the lemma in the case $\dim A_{n+1} = 4$.

Now if $\dim A_n = \dim A_{n+1} = 3$ and $sz^{n-1}s = z^{n-2}sz = zsz^{n-2} = 0$ in A (condition (6.8)), then $\delta(w) = 0$ for all $w \in \langle X \rangle_{n-1}$ and $\delta'(w) = 0$ for $w \in \langle X \rangle_n$, while $\widetilde{\delta}(xwy) = \gamma(xw)\beta(z^{n-1}y) - \beta(wy)\gamma(xz^{n-1})$ is not identically zero. As we already have (6.12), it remains to verify (6.14). Assume the contrary. Then $\gamma'(xw)\beta'(z^n y) - \beta'(wy)\gamma'(xz^n) = 0$ for all $x, y \in X$ and $w \in \langle X \rangle_n$. Using (6.15) and (6.16), we can rewrite this equality as

$$\gamma(w)\gamma(xsz^{n-2}))\beta(z^{n-1}y) - \beta(w)\beta(z^{n-2}sy))\gamma(xz^{n-1}) = 0.$$

Since for $w = sz^{n-1}$, $\gamma(w) = 1$ and $\beta(w) = 0$, while for $w = z^{n-1}s$, $\gamma(w) = 0$ and $\beta(w) = 1$, the above equality yields

$$\gamma(xsz^{n-2})\beta(z^{n-1}y) = \beta(z^{n-2}sy)\gamma(xz^{n-1}) = 0 \quad \text{for all } x, y \in X.$$

Since $\beta(z^{n-1}s) = \gamma(sz^{n-1}) = 1$, the above display is equivalent to

$$\gamma(xsz^{n-2}) = \beta(z^{n-2}sy) = 0 \quad \text{for all } x, y \in X.$$

Using these equalities in addition to (6.15), we get $\beta(wy) = \beta'(zwy) = \alpha(zw)\beta(z^{n-1}y)$ and $\gamma(xw) = \gamma'(xwz) = \alpha(wz)\gamma(xz^{n-1})$. Hence $\widetilde{\delta}(xwy) = \gamma(xz^{n-1})\beta(z^{n-1}y)(\alpha(wz) - \alpha(zw))$. On the other hand, by (6.15), $\alpha'(zwz) = \alpha(wz) + \gamma(wz)\alpha(zsz^{n-2}) = \alpha(zw) + \beta(zw)\alpha(z^{n-2}sz)$. Since by (6.8), $zsz^{n-2} = z^{n-2}sz = 0$, it follows that $\alpha(zw) = \alpha(wz)$. Plugging this into the last expression for $\widetilde{\delta}(w)$, we get $\widetilde{\delta}(w) = 0$ for all w. This contradiction completes the proof of lemma under the extra assumption (6.8).

Since (6.7) covers the case $\dim A_{n+1} = 4$, for the remainder of the proof we can assume that $\dim A_n = \dim A_{n+1} = 3$. The rest of the proof is the reduction to the case (6.8). Since $\dim A_n = 3$, the degree n normal words are z^n, $z^{n-1}s$ and sz^{n-1} and $\delta(w) = 0$ for all $w \in \langle X \rangle_n$. Since $\dim A_{n+1} = 3$, the set NW_{n+1} of degree $n+1$ normal words is a 3-element subset of $\{z^{n+1}, z^n s, sz^n, sz^{n-1}s\}$. We shall see that only one of the four possibilities can actually occur. Still we have to consider them all. Before doing this recall that $zA_{n-1} = \mathtt{span}\,\{z^n, z^{n-1}s\}$ and $A_{n-1}z = \mathtt{span}\,\{z^n, sz^{n-1}\}$ and therefore $zA_{n-2}z = \mathtt{span}\,\{z^n\}$. Hence both $z^{n-2}sz$ and zsz^{n-2} are scalar multiples of z^n. Performing a linear substitution, which does not change any element of $X \setminus \{s\}$ and adds to s an appropriate scalar multiple of z, we can turn $z^{n-2}sz$ into zero. Thus we can without loss of generality assume that

$$z^{n-2}sz = 0 \quad \text{and} \quad zsz^{n-2} = az^n \quad \text{in } A, \text{ where } a \in \mathbb{K}. \tag{6.17}$$

Case 1: $\mathrm{NW}_{n+1} = \{z^n s, sz^n, sz^{n-1}s\}$. This is only possible if $z^{n+1} = 0$ in A and

$$\begin{array}{l} \widetilde{\beta}(xwy) = \widetilde{\gamma}(xwy) = \widetilde{\delta}(xwy) = 0 \quad \text{for all } x, y \in X \text{ and } w \in \langle X \rangle_n, \\ \text{while } \widetilde{\alpha}(xwy) \neq 0 \text{ for some } x, y \in X \text{ and } w \in \langle X \rangle_n. \end{array} \tag{6.18}$$

Writing $sz^{n-2}s$, $z^{n-2}s^2$ and s^2z^{n-2} as linear combinations of normal words, we get $sz^{n-2}s = \lambda_1 z^n + \lambda_2 z^{n-1}s + \lambda_3 sz^{n-1}$, $s^2z^{n-2} = \lambda_4 z^n + \lambda_5 sz^{n-1}$ and $z^{n-2}s^2 = \lambda_6 z^n + \lambda_7 z^{n-1}s$ with $\lambda_j \in \mathbb{K}$, where the last two expressions are one term shorter because $zA_{n-1} = \mathtt{span}\{z^n, z^{n-1}s\}$ and $A_{n-1}z = \mathtt{span}\{z^n, sz^{n-1}\}$. Then $0 = sz^{n-2}sz = \lambda_1 z^{n+1} + \lambda_2 z^{n-1}sz + \lambda_3 sz^n = \lambda_3 sz^n$. Since sz^n is a normal word, $\lambda_3 = 0$. Hence, $az^ns = zsz^{n-2}s = \lambda_1 z^{n+1} + \lambda_2 z^n s$. Since z^ns is a normal word, $\lambda_2 = a$. Thus $sz^{n-2}s = \lambda_1 z^n + az^{n-1}s$. Now $\lambda_1 sz^n + asz^{n-1}s = s^2z^{n-2}s = \lambda_4 z^ns + \lambda_5 sz^{n-1}s$ and therefore $\lambda_1 sz^n + (a-\lambda_5)sz^{n-1}s - \lambda_4 z^ns = 0$. Since sz^n, z^ns and $sz^{n-1}s$ are normal words, we have $\lambda_1 = \lambda_4 = 0$ and $\lambda_5 = a$. Hence $sz^{n-2}s = az^{n-1}s$ and $s^2z^{n-2} = asz^{n-1}$. Finally, $az^{n-1}s^2 = sz^{n-2}s^2 = \lambda_6 sz^n + \lambda_7 sz^{n-1}s$. Since sz^n, z^ns and $sz^{n-1}s$ are normal words and $z^{n+1} = 0$, we have $\lambda_6 = \lambda_7 = 0$. Summarizing, we get

$$z^{n+1} = z^{n-2}s^2 = z^{n-2}sz = 0,\ zsz^{n-2} = az^n,\ sz^{n-2}s = az^{n-1}s \text{ and } s^2z^{n-2} = asz^{n-1}. \tag{6.19}$$

Since $\delta(w) = 0$, by (6.6), the equation in (6.18) reads

$$\begin{aligned}
&\alpha(xw)\beta(z^{n-1}y) + \beta(xw)\beta(z^{n-2}sy) - \beta(wy)\alpha(xz^{n-1}) = 0,\\
&\gamma(xw)\alpha(z^{n-1}y) - \alpha(wy)\gamma(xz^{n-1}) - \gamma(wy)\gamma(xsz^{n-2}) = 0,\\
&\gamma(xw)\beta(z^{n-1}y) - \beta(wy)\gamma(xz^{n-1}) = 0.
\end{aligned} \tag{6.20}$$

Plugging in $x = s$, $x = z$, $y = s$ or $y = z$ one at a time and using (6.2), (6.19) where appropriate, we obtain

$$\begin{aligned}
&\alpha(sw)\beta(z^{n-1}y) + \beta(sw)\beta(z^{n-2}sy) = 0, \quad \gamma(sw)\alpha(z^{n-1}y) - \alpha(wy) - a\gamma(wy) = 0,\\
&\beta(wy) = \gamma(sw)\beta(z^{n-1}y) = \alpha(zw)\beta(z^{n-1}y) + \beta(zw)\beta(z^{n-2}sy),\\
&\alpha(xw) = \beta(ws)\alpha(xz^{n-1}), \quad \alpha(ws)\gamma(xz^{n-1}) + \gamma(ws)\gamma(xsz^{n-2}) = 0,\\
&\gamma(xw) = \beta(ws)\gamma(xz^{n-1}) = \alpha(wz)\gamma(xz^{n-1}) + \gamma(wz)\gamma(xsz^{n-2}), \quad \beta(wz)\alpha(xz^{n-1}) = 0.
\end{aligned} \tag{6.21}$$

Plugging in $x = s$, $x = z$, $y = s$ or $y = z$ once again, we get

$$\alpha(zw) = \alpha(wz) = \beta(ws) = \gamma(sw), \quad \alpha(sw) = 0, \quad \alpha(ws) + a\gamma(ws) = 0. \tag{6.22}$$

Since $\alpha(zw) = \alpha(wz)$, we have $0 = \alpha(z^{n-2}sz) = \alpha(zsz^{n-2}) = a$ and therefore $a = 0$ and $\alpha(ws) = 0$. Now using (6.22) and (6.21), we get

$$\alpha(xw) = \beta(ws)\alpha(xz^{n-1}) = \alpha(zw)\alpha(xz^{n-1}) \text{ and } \alpha(wy) = \gamma(sw)\alpha(z^{n-1}y) = \alpha(zw)\alpha(z^{n-1}y).$$

Since $\alpha(zw) = \alpha(wz)$ and $\alpha(sw) = 0$, we have $\alpha(z^{n-2}sy) = \alpha(syz^{n-2}) = 0$. Similarly, $\alpha(xsz^{n-2}) = \alpha(z^{n-2}xs) = 0$. Using these equalities and the above display, we obtain

$$\begin{aligned}
&\widetilde{\alpha}(xwy) = \alpha(xw)\alpha(z^{n-1}y) + \beta(xw)\alpha(z^{n-2}sy) - \alpha(wy)\alpha(xz^{n-1}) - \gamma(wy)\alpha(xsz^{n-2})\\
&= \alpha(xw)\alpha(z^{n-1}y) - \alpha(wy)\alpha(xz^{n-1}) = \alpha(zw)\alpha(xz^{n-1})\alpha(z^{n-1}y) - \alpha(zw)\alpha(z^{n-1}y)\alpha(xz^{n-1}) = 0,
\end{aligned}$$

which contradicts the second condition in (6.18). This contradiction shows that Case 1 does not occur.

Case 2: NW_{n+1} is $\{z^ns, z^{n+1}, sz^{n-1}s\}$, $\{sz^n, z^{n+1}, sz^{n-1}s\}$ or $\{z^ns, z^{n+1}, sz^n\}$. Since for $1 \leqslant j \leqslant n-2$, $z^{n-1-j}sz^j$ belongs to $zA_{n-2}z$, it is a scalar multiple of z^n: $z^{n-1-j}sz^j = b_jz^n$. Hence $z^{n-1-j}sz^{j+1} = b_jz^{n+1} = b_{j+1}z^{n+1}$. Since z^{n+1} is non-zero in A, $b_j = b_{j+1}$. Hence (6.17) yields that $z^{n-1-j}sz^j = 0$ for $1 \leqslant j \leqslant n-2$ and therefore $a = 0$. Hence (6.17) now reads

$$z^{n-2}sz = zsz^{n-2} = 0 \text{ in } A. \tag{6.23}$$

Case 2a: NW_{n+1} is either $\{z^ns, z^{n+1}, sz^{n-1}s\}$ or $\{sz^n, z^{n+1}, sz^{n-1}s\}$. Again, we shall show that this case does not occur by obtaining a contradiction. Since the two options reduce to each other by passing to the opposite multiplication, we can assume that $\mathrm{NW}_{n+1} = \{z^ns, z^{n+1}, sz^{n-1}s\}$. This can only occur if sz^n is a linear combination of z^{n+1} and $z^n s$ in A, while $z^n s$, z^{n+1} and $sz^{n-1}s$ are linearly independent in A.

As in Case 1, we write $sz^{n-2}s$, $z^{n-2}s^2$ and s^2z^{n-2} as linear combinations of normal words: $sz^{n-2}s=\lambda_1z^n+\lambda_2z^{n-1}s+\lambda_3sz^{n-1}$, $s^2z^{n-2}=\lambda_4z^n+\lambda_5sz^{n-1}$ and $z^{n-2}s^2=\lambda_6z^n+\lambda_7z^{n-1}s$ with $\lambda_j\in\mathbb{K}$. Then by (6.23), $0=zsz^{n-2}s=\lambda_1z^{n+1}+\lambda_2z^ns+\lambda_3zsz^{n-1}=\lambda_1z^{n+1}+\lambda_2z^ns$. Since z^{n+1} and z^ns are normal, $\lambda_1=\lambda_2=0$ and therefore, denoting $\lambda_3=a$ (for aesthetic reasons), we have $sz^{n-2}s=asz^{n-1}$. Hence $0=sz^{n-2}sz=asz^n$. Next, since $asz^n=0$, $a\lambda_4z^{n+1}=a\lambda_4z^{n+1}+a\lambda_5sz^n=as^2z^{n-1}=s^2z^{n-2}s=\lambda_4z^ns+\lambda_5sz^{n-1}s$, which implies $\lambda_4=\lambda_5=0$ and therefore $s^2z^{n-2}=0$. Finally, $asz^{n-1}s=sz^{n-2}s^2=\lambda_6sz^n+\lambda_7sz^{n-1}s$, which yields $\lambda_7=a$ and $\lambda_6sz^n=0$. Summarizing, we get

$$z^{n-2}sz=zsz^{n-2}=s^2z^{n-2}=0,\ sz^{n-2}s=asz^{n-1},\ z^{n-2}s^2=az^{n-1}s+bz^n,\ asz^n=bsz^n=0, \tag{6.24}$$

where $b=\lambda_6\in\mathbb{K}$. Multiplying $z^{n-2}s^2=az^{n-1}s+bz^n$ by s on the right, we get $z^{n-2}s^3=+az^{n-1}s^2+bz^ns=(a^2+b)z^ns+abz^{n+1}$. If $n\geqslant 4$, $z^{n-3}s^3$ starts with z and therefore is a linear combination of $z^{n-1}s$ and z^n: $z^{n-3}s^3=pz^{n-1}s+qz^n$. Multiplying by z on the left, we get $z^{n-2}s^3=pz^ns+qz^{n+1}$ and therefore (z^ns and z^{n+1} are normal words), $p=a^2+b$ and $q=ab$. Hence $z^{n-3}s^3=(a^2+b)z^{n-1}s+abz^n$. Iterating this procedure, we get that for $1\leqslant j\leqslant n-1$ $z^{n-j}s^j=p_jz^{n-1}s+q_jz^n$ and $zs^n=p_nz^ns+q_nz^{n+1}$, where $(p_1,q_1)=(1,0)$, $(p_2,q_2)=(a,b)$ and $p_{j+1}=ap_j+q_j$, $q_{j+1}=bp_j$. Writing s^n as a linear combination of normal words, we have $s^n=fsz^{n-1}+gz^{n-1}s+hz^n$, where $f,g,h\in\mathbb{K}$. Multiplying by z on the left and using the equality $zsz^{n-2}=0$, we have $p_nz^ns+q_nz^{n+1}=zs^n=gz^ns+hz^{n+1}$. Since z^ns and z^{n+1} are normal, $g=p_n$ and $h=q_n$. Hence $s^n=fsz^{n-1}+p_nz^{n-1}s+q_nz^n$. Since s^n commutes with s,

$$0=fs^2z^{n-1}+p_nsz^{n-1}s+q_nsz^n-fsz^{n-1}s-p_nz^{n-1}s^2-q_nz^ns.$$

By (6.24), we can rewrite the above display as

$$(p_n-f)sz^{n-1}s+q_nsz^n-(ap_n+q_n)z^ns-bp_nz^{n+1}=0.$$

If $(a,b)\neq(0,0)$, then by (6.24), $sz^n=0$ and we have $(p_n-f)sz^{n-1}s-(ap_n+q_n)z^ns-bp_nz^{n+1}=0$. Since $sz^{n-1}s$, z^ns and z^{n+1} are normal words, we get $f=p_n$, $bp_n=0$ and $q_n+ap_n=0$. From the recurrent formula for p_j,q_j, we have

$$\begin{pmatrix}p_n\\ q_n\end{pmatrix}=\begin{pmatrix}a&1\\ b&0\end{pmatrix}^{n-1}\begin{pmatrix}1\\ 0\end{pmatrix}.$$

Now we show that $b=0$. Indeed, if $b\neq0$, then the equations $bp_n=0$ and $q_n+ap_n=0$ yield $p_n=q_n=0$ and therefore the matrix in the above display must be non-invertible, which yields $b=0$, providing a contradiction. Thus $b=0$. Hence by the above display, $p_n=a^{n-1}$ and $q_n=0$. Then the equation $q_n+ap_n=0$ reads $a^n=0$ ensuring that $a=0$ as well. Hence $a=b=0$ and (6.24) can be rewritten as

$$z^{n-2}sz=zsz^{n-2}=s^2z^{n-2}=sz^{n-2}s=z^{n-2}s^2=0. \tag{6.25}$$

Since sz^n is a linear combination of z^ns and z^{n+1}, $sz^n+pz^ns+qz^{n+1}=0$ in A for some $p,q\in\mathbb{K}$. Then

$$\begin{array}{l}\widetilde{\alpha}(xwy)=q\widetilde{\gamma}(xwy),\ \widetilde{\beta}(xwy)=p\widetilde{\gamma}(xwy),\ \widetilde{\delta}(xwy)=0\ \text{ for all } x,y\in X \text{ and } w\in\langle X\rangle_n,\\ \text{while } \widetilde{\gamma}(xwy)\neq0 \text{ for some } x,y\in X \text{ and } w\in\langle X\rangle_n.\end{array} \tag{6.26}$$

Using (6.6) and $\delta(w)=0$, we see that the equality in (6.26) reads

$$\begin{array}{l}\alpha(xw)\beta(z^{n-1}y)+\beta(xw)\beta(z^{n-2}sy)-\beta(wy)\alpha(xz^{n-1})-p\gamma(xw)\alpha(z^{n-1}y)+p\alpha(wy)\gamma(xz^{n-1})+p\gamma(wy)\gamma(xsz^{n-2})=0,\\ \alpha(xw)\alpha(z^{n-1}y)+\beta(xw)\alpha(z^{n-2}sy)-\alpha(wy)\alpha(xz^{n-1})-\gamma(wy)\alpha(xsz^{n-2})-q\gamma(xw)\alpha(z^{n-1}y)+q\alpha(wy)\gamma(xz^{n-1})+q\gamma(wy)\gamma(xsz^{n-2})=0,\\ \gamma(xw)\beta(z^{n-1}y)-\beta(wy)\gamma(xz^{n-1})=0.\end{array} \tag{6.27}$$

Plugging in $x = s$, $x = z$, $y = s$ or $y = z$ and using (6.2), (6.25) where appropriate, we obtain

$$\begin{gathered}\beta(wy)=\gamma(sw)\beta(z^{n-1}y)=\alpha(zw)\beta(z^{n-1}y)+\beta(zw)\beta(z^{n-2}sy),\ \ \alpha(sw)\alpha(z^{n-1}y)+\beta(sw)\alpha(z^{n-2}sy)-q\gamma(sw)\alpha(z^{n-1}y)+q\alpha(wy)=0,\\ \alpha(sw)\beta(z^{n-1}y)+\beta(sw)\beta(z^{n-2}sy)-p\gamma(sw)\alpha(z^{n-1}y)+p\alpha(wy)=0,\ \ p(\gamma(xw)-\alpha(wz)\gamma(xz^{n-1})-\gamma(wz)\gamma(xsz^{n-2}))=0,\\ \alpha(xw)=\beta(ws)\alpha(xz^{n-1})-p\alpha(ws)\gamma(xz^{n-1})-p\gamma(ws)\gamma(xsz^{n-2}),\ \ \alpha(wy)=\alpha(zw)\alpha(z^{n-1}y)+\beta(zw)\alpha(z^{n-2}sy),\\ \alpha(ws)\alpha(xz^{n-1})+\gamma(ws)\alpha(xsz^{n-2})-q\alpha(ws)\gamma(xz^{n-1})-q\gamma(ws)\gamma(xsz^{n-2})=0,\ \ \gamma(xw)=\beta(ws)\gamma(xz^{n-1}),\\ \alpha(xw)-\alpha(wz)\alpha(xz^{n-1})-\gamma(wz)\alpha(xsz^{n-2})-q\gamma(xw)+q\alpha(wz)\gamma(xz^{n-1})+q\gamma(wz)\gamma(xsz^{n-2})=0.\end{gathered} \tag{6.28}$$

Plugging in $x = s$, $x = z$, $y = s$ or $y = z$ once again, we get

$$\alpha(zw) = \alpha(wz) = \beta(ws) = \gamma(sw),\ \alpha(sw) = \alpha(ws) = 0. \tag{6.29}$$

Since $\alpha(zw) = \alpha(wz)$ and $\alpha(sw) = \alpha(ws) = 0$, $\alpha(z^{n-2}sy) = \alpha(syz^{n-2}) = 0$ and $\alpha(xsz^{n-2}) = \alpha(z^{n-2}xs) = 0$. By (6.28), $\gamma(xw) = \beta(ws)\gamma(xz^{n-1})$ and therefore $\gamma(xsz^{n-2}) = \beta(sz^{n-2}s)\gamma(xz^{n-1}) = 0$ since $sz^{n-2}s = 0$ according to (6.25). Hence (6.28) simplifies to

$$\begin{aligned}&\alpha(xw) = \alpha(wz)\alpha(xz^{n-1}) + q\gamma(xw) - q\alpha(wz)\gamma(xz^{n-1}) = \beta(ws)\alpha(xz^{n-1}) - p\alpha(ws)\gamma(xz^{n-1}),\\ &\beta(wy) = \gamma(sw)\beta(z^{n-1}y) = \alpha(zw)\beta(z^{n-1}y) + \beta(zw)\beta(z^{n-2}sy),\ \ \gamma(xw) = \beta(ws)\gamma(xz^{n-1}),\\ &\alpha(sw)\beta(z^{n-1}y) + \beta(sw)\beta(z^{n-2}sy) - p\gamma(sw)\alpha(z^{n-1}y) + p\alpha(wy) = 0,\ \ \alpha(wy) = \alpha(zw)\alpha(z^{n-1}y),\\ &\alpha(sw)\alpha(z^{n-1}y) + \beta(sw)\alpha(z^{n-2}sy) - q\gamma(sw)\alpha(z^{n-1}y) + q\alpha(wy) = 0,\\ &p(\gamma(xw) - \alpha(wz)\gamma(xz^{n-1})) = 0,\ \ \alpha(ws)\alpha(xz^{n-1}) = q\alpha(ws)\gamma(xz^{n-1}),\end{aligned} \tag{6.30}$$

In particular, $\alpha(wy) = \alpha(zw)\alpha(z^{n-1}y)$. Now by (6.6),

$$\widetilde{\gamma}(xwy) = \gamma(xw)\alpha(z^{n-1}y) - \alpha(wy)\gamma(xz^{n-1}) - \gamma(wy)\gamma(xsz^{n-2}).$$

Plugging in $\alpha(wy) = \alpha(zw)\alpha(z^{n-1}y)$, $\gamma(xw) = \beta(ws)\gamma(xz^{n-1})$ and $\gamma(xsz^{n-2}) = 0$, we get $\widetilde{\gamma}(xwy) = \gamma(xz^{n-1})\alpha(z^{n-1}y)(\beta(ws) - \alpha(zw))$. Since by (6.29), $\beta(ws) = \alpha(zw)$, we see that $\widetilde{\gamma}(xwy) = 0$ for all x, y, w, which contradicts (6.26). This contradiction leaves us the following option only.

Case 2b: $\mathrm{NW}_{n+1} = \{sz^n, z^ns, z^{n+1}\}$. Unlike all previous cases, this one is real, meaning there are algebras with this property. In this case, we just have to show that (6.8) is satisfied, thus reducing to the case already dealt with. In view of (6.23), the only missing condition is $sz^{n-1}s = 0$ in A. Thus the proof will be complete if we show that $sz^{n-1}s = 0$ in A.

Again, we write $sz^{n-2}s$, $z^{n-2}s^2$ and s^2z^{n-2} as linear combinations of normal words: $sz^{n-2}s = \lambda_1z^n + \lambda_2z^{n-1}s + \lambda_3sz^{n-1}$, $s^2z^{n-2} = \lambda_4z^n + \lambda_5sz^{n-1}$ and $z^{n-2}s^2 = \lambda_6z^n + \lambda_7z^{n-1}s$ with $\lambda_j \in \mathbb{K}$. Then by (6.23), $0 = zsz^{n-2}s = \lambda_1z^{n+1} + \lambda_2z^ns + \lambda_3zsz^{n-1} = \lambda_1z^{n+1} + \lambda_2z^ns$, yielding $\lambda_1 = \lambda_2 = 0$. Similarly, $0 = sz^{n-2}sz = \lambda_1z^{n+1} + \lambda_2z^{n-1}sz + \lambda_3sz^n = \lambda_1z^{n+1} + \lambda_3sz^ns$ and therefore $\lambda_1 = \lambda_3 = 0$. Hence $\lambda_1 = \lambda_2 = \lambda_3 = 0$ and $sz^{n-2}s = 0$. Thus $0 = s^2z^{n-2}s = \lambda_4z^ns + \lambda_5sz^{n-1}s$ and $0 = sz^{n-2}s^2 = \lambda_6sz^n + \lambda_7sz^{n-1}s$. That is, $0 = \lambda_4z^ns + \lambda_5sz^{n-1}s = \lambda_6sz^n + \lambda_7sz^{n-1}s$. Since sz^n and z^ns are normal words, if $\lambda_5\lambda_7 \neq 0$, we have $\lambda_4 = \lambda_6 = 0$. Similarly, if $\lambda_5 = 0$, then $\lambda_4 = 0$, while if $\lambda_7 = 0$, then $\lambda_6 = 0$. Thus we must have one of the following options with $a \in \mathbb{K}$ and $b \in \mathbb{K}^*$:

$$z^{n-2}sz = zsz^{n-2} = sz^{n-2}s = 0,\ s^2z^{n-2} = asz^{n-1},\ z^{n-2}s^2 = bz^{n-1}s,\ sz^{n-1}s = 0, \tag{6.31}$$

$$z^{n-2}sz = zsz^{n-2} = sz^{n-2}s = 0,\ s^2z^{n-2} = 0,\ z^{n-2}s^2 = az^n+bz^{n-1}s,\ sz^{n-1}s = -\tfrac{a}{b}z^ns, \tag{6.32}$$

$$z^{n-2}sz = zsz^{n-2} = sz^{n-2}s = 0,\ z^{n-2}s^2 = 0,\ s^2z^{n-2} = az^n+bsz^{n-1},\ sz^{n-1}s = -\tfrac{a}{b}z^ns, \tag{6.33}$$

$$z^{n-2}sz = zsz^{n-2} = sz^{n-2}s = z^{n-2}s^2 = s^2z^{n-2} = 0. \tag{6.34}$$

If (6.31) is satisfied, then $sz^{n-1}s = 0$ and there is nothing to prove. Next, assume that (6.32) is satisfied. Considering normal word expansions and consecutively multiplying by s on either side in the same way as we have done in Case 2a, starting with $s^2z^{n-2} = 0$, we get $s^jz^{n-j} = 0$ for

$1 \leqslant j \leqslant n-1$. If we start with $z^{n-2}s^2 = az^n + bz^{n-1}s$, we arrive to $z^{n-j}s^j = a_j z^n + b_j z^{n-1}s$ and $zs^n = a_n z^{n+1} + b_n z^n s$, where

$$\begin{pmatrix} b_j \\ a_j \end{pmatrix} = \begin{pmatrix} b & 1 \\ a & 0 \end{pmatrix}^{j-1} \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

In particular, $s^{n-1}z = 0$ and $zs^{n-1} = a_{n-1}z^n + b_{n-1}z^{n-1}s$. Hence $0 = zs^{n-1}z = a_{n-1}z^{n+1} + b_{n-1}z^{n-1}sz = a_{n-1}z^{n+1}$ and therefore $a_{n-1} = 0$. If $s^n = pz^n + qz^{n-1}s + rsz^{n-1}$, then $0 = s^n z = pz^{n+1} + qz^{n-1}sz + rsz^n = pz^{n+1} + rsz^n$ and $p = r = 0$. Hence $s^n = qz^{n-1}s$. Then $qz^n s = zs^n = a_n z^{n+1} + b_n z^n s$. Thus $a_n = 0$ and $q = b_n$. Then $b_n z^{n-1}s = s^n$ commutes with s and therefore $0 = b_n(z^{n-1}s^2 - sz^{n-1}s) = b_n\bigl(az^{n+1} + \bigl(b + \frac{a}{b}\bigr)z^n s\bigr)$. Since $z^n s$ and z^{n+1} are normal words and $b \neq 0$, it follows that $b_n = 0$. Since $a_n = b_n = 0$, the matrix in the above display is degenerate and therefore $a = 0$. Then $b_j = b^{j-1}$ for all j. Hence $0 = b_n = b^{n-1}$ and therefore $b = 0$, which is a contradiction. Thus the case (6.32) does not occur. Similarly, the case (6.33) does not occur. Alternatively, cases (6.32) and (6.33) transform to one another if we pass to the opposite multiplication. Hence the only remaining option is (6.34).

Assume that (6.34) is satisfied. Since $sz^{n-1}s$ is a linear combination of sz^n, $z^n s$ and z^{n+1}, there are $p, q, r \in \mathbb{K}$ such that $sz^{n-1}s + pz^n s + rsz^n + qz^{n+1}$ in A. The assumptions of our case ensure that

$$\begin{array}{l} \widetilde{\alpha}(xwy) = q\widetilde{\delta}(xwy),\ \widetilde{\beta}(xwy) = p\widetilde{\delta}(xwy),\ \widetilde{\gamma}(xwy) = r\widetilde{\delta}(xwy)\ \text{ for all } x, y \in X,\ w \in \langle X\rangle_n, \\ \text{while } \widetilde{\delta}(xwy) \neq 0 \text{ for some } x, y \in X \text{ and } w \in \langle X\rangle_n. \end{array} \tag{6.35}$$

According to (6.6), the equations in 6.35 read

$$\begin{array}{l} \alpha(xw)\beta(z^{n-1}y)+\beta(xw)\beta(z^{n-2}sy)-\beta(wy)\alpha(xz^{n-1})-p\gamma(xw)\beta(z^{n-1}y)+p\beta(wy)\gamma(xz^{n-1})=0, \\ \alpha(xw)\alpha(z^{n-1}y)+\beta(xw)\alpha(z^{n-2}sy)-\alpha(wy)\alpha(xz^{n-1})-\gamma(wy)\alpha(xsz^{n-2})-q\gamma(xw)\beta(z^{n-1}y)+q\beta(wy)\gamma(xz^{n-1})=0, \\ \gamma(xw)\alpha(z^{n-1}y)-\alpha(wy)\gamma(xz^{n-1})-\gamma(wy)\gamma(xsz^{n-2})-r\gamma(xw)\beta(z^{n-1}y)+r\beta(wy)\gamma(xz^{n-1})=0. \end{array} \tag{6.36}$$

Plugging in $x = s$, $x = z$, $y = s$ or $y = z$ one at a time and using (6.2), (6.34) where appropriate, we obtain

$$\begin{array}{l} \alpha(xw)=\alpha(wz)\alpha(xz^{n-1})+\gamma(wz)\alpha(xsz^{n-2})=\beta(ws)\alpha(xz^{n-1})+p\gamma(xw)-p\beta(ws)\gamma(xz^{n-1}), \\ \beta(wy)=\alpha(zw)\beta(z^{n-1}y)+\beta(zw)\beta(z^{n-2}sy), \quad \gamma(xw)=\alpha(wz)\gamma(xz^{n-1})+\gamma(wz)\gamma(xsz^{n-2}), \\ \alpha(wy)=\alpha(zw)\alpha(z^{n-1}y)+\beta(zw)\alpha(z^{n-2}sy), \quad \alpha(sw)\beta(z^{n-1}y)+\beta(sw)\beta(z^{n-2}sy)=p(\gamma(sw)\beta(z^{n-1}y)-\beta(wy)), \\ \alpha(sw)\alpha(z^{n-1}y)+\beta(sw)\alpha(z^{n-2}sy)=q(\gamma(sw)\beta(z^{n-1}y)-\beta(wy)), \quad \gamma(sw)\alpha(z^{n-1}y)-\alpha(wy)=r(\gamma(sw)\beta(z^{n-1}y)-\beta(wy)), \\ \alpha(ws)\alpha(xz^{n-1})+\gamma(ws)\alpha(xsz^{n-2})=q(\beta(ws)\gamma(xz^{n-1})-\gamma(xw)), \quad \alpha(ws)\gamma(xz^{n-1})+\gamma(ws)\gamma(xsz^{n-2})=r(\beta(ws)\gamma(xz^{n-1})-\gamma(xw)). \end{array} \tag{6.37}$$

Plugging in $x = s$, $x = z$, $y = s$ or $y = z$ again, we get

$$\alpha(zw) = \alpha(wz) = \beta(ws) = \gamma(sw),\ \alpha(ws) = \alpha(sw) = 0. \tag{6.38}$$

In particular, $\alpha(xsz^{n-2}) = \alpha(z^{n-2}xs) = 0$ and $\alpha(z^{n-2}sy) = \alpha(syz^{n-2}) = 0$. Plugging all this back into (6.37), we get

$$\begin{array}{l} \alpha(xw) = \alpha(wz)\alpha(xz^{n-1}) = \beta(ws)\alpha(xz^{n-1}) + p\gamma(xw) - p\beta(ws)\gamma(xz^{n-1}), \\ \alpha(wy) = \alpha(zw)\alpha(z^{n-1}y) = \gamma(sw)\alpha(z^{n-1}y) - r\gamma(sw)\beta(z^{n-1}y) + r\beta(wy), \\ \beta(wy) = \alpha(zw)\beta(z^{n-1}y) + \beta(zw)\beta(z^{n-2}sy), \quad \gamma(xw) = \alpha(wz)\gamma(xz^{n-1}) + \gamma(wz)\gamma(xsz^{n-2}), \\ q(\beta(wy) - \gamma(sw)\beta(z^{n-1}y)) = q(\gamma(xw) - \beta(ws)\gamma(xz^{n-1})) = 0, \\ \beta(sw)\beta(z^{n-2}sy) = p(\gamma(sw)\beta(z^{n-1}y) - \beta(wy)), \quad \gamma(ws)\gamma(xsz^{n-2}) = r(\beta(ws)\gamma(xz^{n-1}) - \gamma(xw). \end{array} \tag{6.39}$$

If $q \neq 0$, then $\beta(wy) = \gamma(sw)\beta(z^{n-1}y))$ and $\gamma(xw) = \beta(ws)\gamma(xz^{n-1})$. Since by (6.6),

$$\widetilde{\delta}(xwy) = \gamma(xw)\beta(z^{n-1}y) - \beta(wy)\gamma(xz^{n-1}),$$

in the case $q \neq 0$, we have $\widetilde{\delta}(xwy) = \beta(z^{n-1}y)\gamma(xz^{n-1})(\beta(ws) - \gamma(sw)) = 0$ according to (6.38), yielding a contradiction with (6.35). Thus $q = 0$. In any case, plugging $\beta(wy) = \alpha(zw)\beta(z^{n-1}y) + \beta(zw)\beta(z^{n-2}sy)$ and $\gamma(xw) = \alpha(wz)\gamma(xz^{n-1}) + \gamma(wz)\gamma(xsz^{n-2})$ from (6.39) into the above display and using $\alpha(zw) = \alpha(wz)$, we get

$$\widetilde{\delta}(xwy) = \gamma(wz)\gamma(xsz^{n-2})\beta(z^{n-1}y) - \beta(zw)\beta(z^{n-2}sy)\gamma(xz^{n-1}). \tag{6.40}$$

Plugging $w = z^{n-2}s$ into the equality $\alpha(wy) = \gamma(sw)\alpha(z^{n-1}y) - r\gamma(sw)\beta(z^{n-1}y) + r\beta(wy)$ from (6.39) and using (6.34), we get $\alpha(z^{n-2}sy) = r\beta(z^{n-2}sy)$. Since we already know that $\alpha(z^{n-2}sy) = 0$, we have $r\beta(z^{n-2}sy) = 0$. Similarly, plugging $w = sz^{n-2}$ into $\alpha(xw) = \beta(ws)\alpha(xz^{n-1}) + p\gamma(xw) - p\beta(ws)\gamma(xz^{n-1})$, we get $p\gamma(xsz^{n-2}) = 0$. Plugging $w = z^{n-2}s$ into $\beta(sw)\beta(z^{n-2}sy) - p\gamma(sw)\beta(z^{n-1}y) + p\beta(wy) = 0$, we get $p\beta(z^{n-2}sy) = 0$, while plugging $w = sz^{n-2}$ into $\gamma(ws)\gamma(xsz^{n-2}) + r\gamma(xw) - r\beta(ws)\gamma(xz^{n-1}) = 0$, we get $r\gamma(xsz^{n-2}) = 0$. If $(p, r) \neq (0, 0)$ it now follows that $\gamma(xsz^{n-2}) = \beta(z^{n-2}sy) = 0$ and therefore $\widetilde{\delta}(xwy) = 0$ according (6.40). Since this contradicts (6.35), we must have $p = r = 0$. Hence $p = q = r = 0$ and therefore $sz^{n-1}s = 0$ in A. By (6.34), we also have $zsz^{n-2} = z^{n-2}sz = 0$. Thus all conditions of (6.8) are satisfied and, as we have already shown, $\dim A_m = 3$ for all $m \geqslant n$. This completes the proof. $\square$

6.2 Proof of Lemma 6.1

Let A be a finitely generated degree graded algebra, whose ideal of relations is generated by some homogeneous elements of degree n, where $n \geqslant 3$. Assume also that $\dim A_n = 3$, $\dim A_{n+1} \geqslant 3$ and $\max\{\lambda(A, n), \rho(A, n)\} = 2$. By Lemma 2.2, for generic $X = (x_1, x_2, \dots) \in \Omega(A)$, $\mathrm{NW}_n = \{x_1^n, x_1^{n-1}x_2, x_2x_1^{n-1}\}$ with respect to the LR order $<$. Thus we can choose a basis $X = (x_1, x_2, \dots)$ with this property and denote $z = x_1$ and $s = x_2$. Since $\lambda(A, n) \leqslant 2$ and z^n, $z^{n-1}s$ are linearly independent in A, $zA_{n-1} = \mathtt{span}\{z^n, z^{n-1}s\}$. Since $\rho(A, n) \leqslant 2$ and z^n, sz^{n-1} are linearly independent in A, $A_{n-1}z = \mathtt{span}\{z^n, sz^{n-1}\}$. Now all conditions of Lemma 6.2 are satisfied. By Lemma 6.2, $H_A^{[n]} = \in \{\overline{3}, 3\overline{4}\}$, which completes the proof.

6.3 Proof of Theorem 1.3

Let $n \geqslant 3$ and A be a finitely generated degree graded algebra such that $\dim A_n = 3$ and the ideal of relations of A is generated by some homogeneous elements of degree at most n. The proof will be complete if we show that $H_A^{[n]} \in \{3\overline{4}, \overline{3}, 332\overline{1}, 3\overline{2}, 32\overline{1}, 321\overline{0}, 32\overline{0}, 3\overline{1}, 31\overline{0}, 3\overline{0}\}$. And that $H_A^{[n]} \neq 332\overline{1}$ provided $n \geqslant 4$. By Remark 1.11, we can without loss of generality assume that the ideal of relations of A is generated by some homogeneous elements of degree exactly n. If $\dim A_{n+1} < 3$, the result follows from the already proven Theorem 1.2. Thus it remains to consider the case $\dim A_{n+1} \geqslant 3$.

If $\lambda(A, n) = \rho(A, n) = 1$, then by Lemma 3.2, $H_A^{[n]} = \overline{3}$. In the case $\max\{\lambda(A, n), \rho(A, n)\} = 3$, Lemma 5.1 guarantees that $H_A^{[n]} = \overline{3}$ provided $n \geqslant 4$ and that $H_A^{[n]} \in \{\overline{3}, 332\overline{1}\}$ if $n = 3$. The only remaining option is $\max\{\lambda(A, n), \rho(A, n)\} = 2$. By Lemma 6.1, $H_A^{[n]} \in \{\overline{3}, 3\overline{4}\}$, which completes the proof of our main theorem.

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