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On Positivity and Minimality for Second-Order Holonomic Sequences

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Abstract

An infinite sequence $\langle u_n \rangle_n$ of real numbers is *holonomic* (also known as *P-recursive* or *P-finite*) if it satisfies a linear recurrence relation with polynomial coefficients. Such a sequence is said to be *positive* if each $u_n \geq 0$, and *minimal* if, given any other linearly independent sequence $\langle v_n \rangle_n$ satisfying the same recurrence relation, the ratio $u_n/v_n \to 0$ as $n \to \infty$.

In this paper we give a Turing reduction of the problem of deciding positivity of second-order holonomic sequences to that of deciding minimality of such sequences. More specifically, we give a procedure for determining positivity of second-order holonomic sequences that terminates in all but an exceptional number of cases, and we show that in these exceptional cases positivity can be determined using an oracle for deciding minimality.

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1 Introduction

Holonomic sequences (also known as *P-recursive* or *P-finite* sequences) are infinite sequences of real (or complex) numbers that satisfy a linear recurrence relation with polynomial coefficients. Holonomic sequences play a critical role in many areas of mathematics and computer science—particularly combinatorics, analysis of algorithms, and number theory; see, for instance [31, 6, 7] or the seminal paper [40]. A spectacular application can be found in groundbreaking work by Apéry in the 1970s, who used certain holonomic sequences satisfying the second-order recurrence relation

$$n^3 u_n = (34n^3 - 51n^2 + 27n - 5)u_{n-1} - (n-1)^3 u_{n-2}$$

to prove that $\zeta(3) := \sum_{n=1}^{\infty} n^{-3}$ is irrational [1].

Formally, a holonomic sequence satisfies a recurrence relation of the form:

$$p_{k+1}(n)u_n = p_k(n)u_{n-1} + \cdots + p_1(n)u_{n-k}$$

where $p_{k+1}, \ldots, p_1 \in \mathbb{Q}[n]$ are polynomials with rational coefficients and $p_1 \neq 0$. We define the *order* of the recurrence to be k. Assuming that $p_{k+1}(n) \neq 0$ for each non-negative integer n, the above recurrence uniquely defines an infinite sequence once the initial values $u_{-k+1}, \ldots, u_0$ are specified. By extension, if a holonomic sequence satisifes a recurrence of order k, but no recurrence of smaller order, then we say that the sequence has order k. The class of holonomic sequences who satisfy recurrence relations with constant (rather than polynomial) coefficients are known as *C-finite* sequences. Furthermore, every algebraic sequence of real numbers (i.e., whose ordinary generating function is algebraic) is also holonomic.

The study of identities for holonomic sequences appears frequently in the literature. However, as noted by Kauers and Pillwein, *"in contrast,... almost no algorithms are available for inequalities"* [17]. For example, the *Positivity Problem* (i.e., whether every term of a given sequence is non-negative) for C-finite sequences is only known to be decidable at low orders, and there is strong evidence that the problem is mathematically intractable in general [28, 30]; see also [12, 20, 29]. For holonomic sequences that are not C-finite, very few decision procedures currently exist for Positivity, although several partial results and heuristics are known (see, for example [17, 21, 26, 27, 32, 33, 39]). In particular, in [27], the authors exhibit semi-decision procedures for determining positivity of second-order holonomic sequences for which the degrees of the polynomial coefficients satisfy certain constraints.

Another extremely important property of holonomic sequences is *minimality*; a sequence $\langle u_n \rangle_n$ is a minimal solution if, given any other linearly independent sequence $\langle v_n \rangle_n$ satisfying the same recurrence relation, the ratio u_n/v_n converges to 0. Minimal holonomic sequences play a crucial rôle, among others, in numerical calculations and asymptotics, as noted for example in [11, 4, 5, 8, 9, 10]—see also the references therein. Unfortunately, there is also ample evidence that determining algorithmically whether a given holonomic sequence is minimal is a very challenging task, for which no satisfactory solution is at present known to exist.

One of our main results concerns the relationship between positivity and minimality of sequences $\langle u_n \rangle_{n=-1}^{\infty}$ satisfying second-order polynomial recurrences:[1]

$$p_3(n)u_n = p_2(n)u_{n-1} + p_1(n)u_{n-2}\,. \tag{1}$$

We shall assume throughout that neither p_1 nor p_2 is identically zero. Indeed, if $p_1 \equiv 0$ then $\langle u_n \rangle_{n=-1}^{\infty}$ satisfies a first-order recurrence, while if $p_2 \equiv 0$ then $\langle u_n \rangle_{n=-1}^{\infty}$ is the interleaving of

[1] Indexing the sequence from -1 (rather than the more usual 0) makes no significant mathematical difference, but provides notational expediency in the sequel.

two sequences that satisfy first-order recurrences. But it is trivial to determine the positivity of first-order holonomic sequences.[2] Moreover, by working with a tail of the sequence $\langle u_n \rangle_{n=-1}^{\infty}$ (equivalently, shifting the index n) we can assume that $p_1(n), p_2(n), p_3(n) \neq 0$ for all $n \geq -1$.

Our main contributions are as follows: we characterise the positivity of the sequence $\langle u_n \rangle_{n=-1}^{\infty}$ in (1) in terms of its *inital ratio* u_0/u_{-1}. Specifically, from the recurrence we obtain a single closed subinterval $P \subseteq \mathbb{R}$ such that the sequence is positive if and only if $u_0/u_{-1} \in P$. We moreover show that the endpoints of P can be represented as polynomial continued fractions, allowing them to be computed to arbitrary precision. By approximating the endpoints of P to sufficient accuracy we can decide positivity in all cases except when the initial ratio happens to coincide with an endpoint of P. However, we show that such exceptional cases can be handled using an oracle for deciding minimality. Thus we obtain one of our main results, Theorem 3.1: for second-order holonomic sequences, the Positivity Problem Turing-reduces to the Minimality Problem.

2 Preliminaries

2.1 continued fractions

An (infinite) *continued fraction*

$$\operatorname*{K}_{n=1}^{\infty} \frac{a_n}{b_n} := \cfrac{a_1}{b_1 + \cfrac{a_2}{b_2 + \cfrac{a_3}{b_3 + \ddots}}}$$

is defined by an ordered pair of sequences $\langle a_n \rangle_n$ and $\langle b_n \rangle_n$ of complex numbers where $a_n \neq 0$ for each $n \in \mathbb{N}$. Herein we shall always assume that $\langle a_n \rangle_n$ and $\langle b_n \rangle_n$ are real-valued rational functions. A continued fraction *converges* to a value $f = \operatorname{K}(a_n/b_n)$ if its *sequence of approximants* $\langle f_n \rangle_{n=1}^{\infty}$ converges to f in $\hat{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$. The sequence $\langle f_n \rangle_n$ is recursively defined so that

$$f_n = \operatorname*{K}_{m=1}^{n} \frac{a_m}{b_m} := \cfrac{a_1}{b_1 + \cfrac{a_2}{b_2 + \ddots + \cfrac{a_n}{b_n}}}.$$

We respectively call $\langle a_n \rangle_n$ and $\langle b_n \rangle_n$ the sequences of *partial numerators* and *partial denominators* (together the *partial quotients*) of the continued fraction $\operatorname{K}(a_n/b_n)$. Let $\langle A_n \rangle_{n=-1}^{\infty}$ and $\langle B_n \rangle_{n=-1}^{\infty}$ satisfy the recurrence relation $u_n = b_n u_{n-1} + a_n u_{n-2}$ with initial values $A_{-1} = 1, A_0 = 0, B_{-1} = 0$, and $B_0 = 1$. As a pair, $\langle A_n \rangle_{n=-1}^{\infty}$ and $\langle B_n \rangle_{n=-1}^{\infty}$ form a basis for the solution space of the recurrence. We call $\langle A_n \rangle_n$ and $\langle B_n \rangle_n$ the sequences of

[2] The ratio of consecutive terms of a first-order holonomic sequence is a rational function, which has an ultimately constant sign.

canonical numerators and *canonical denominators* of $\mathrm{K}(a_n/b_n)$ because $f_n = A_n/B_n$ for each $n \in \mathbb{N}$.

The following determinant formula is well-known (see, for example, [23, Lemma 4, §IV]).

▶ **Lemma 2.1.** *Suppose that $\langle u_n \rangle_n$ and $\langle v_n \rangle_n$ are both solutions to the recurrence relation $u_n = b_n u_{n-1} + a_n u_{n-2}$. Then*

$$u_n v_{n-1} - u_{n-1} v_n = (u_0 v_{-1} - u_{-1} v_0) \prod_{k=1}^{n} (-a_k).$$

Two continued fractions are *equivalent* if they have the same sequence of approximants. The following theorem is attributed to Seidel in [23, §II.2.2].

▶ **Theorem 2.2.** *The continued fractions $\mathrm{K}(a_n/b_n)$ and $\mathrm{K}(c_n/d_n)$ are equivalent if and only if there exists a sequence $\langle \mathfrak{r}_n \rangle_{n=0}^{\infty}$ with $\mathfrak{r}_0 = 1$ and $\mathfrak{r}_n \neq 0$ for each $n \in \mathbb{N}$ such that $c_n = \mathfrak{r}_n \mathfrak{r}_{n-1} a_n$ and $d_n = \mathfrak{r}_n b_n$ for each $n \in \mathbb{N}$.*

2.2 Śleszyński–Pringsheim continued fractions

A continued fraction $\mathrm{K}_{n=1}^{\infty}(a_n/b_n)$ is a *Śleszyński–Pringsheim continued fraction* if $|b_n| \geq |a_n| + 1$ for each $n \in \mathbb{N}$. As before, let $\langle f_n \rangle_n$ be the sequence of approximants associated with such a continued fraction. The following properties are well-known [23, §I.4]. For the open unit interval $(-1, 1) \subset \mathbb{R}$, $a_n/(b_n + (-1, 1)) \subseteq (-1, 1)$ and we have that $f_n \in (-1, 1)$ for each $n \in \mathbb{N}$. Further, it can be shown that $\langle f_n \rangle_n$ converges to a finite value f with $0 < |f| \leq 1$. We will use the following convergence result, which can be derived from the Śleszyński–Pringsheim Theorem (we reproduce the proof in [24, §3.2.4] below).

▶ **Theorem 2.3.** *Let $\langle f_n \rangle_n$ and $\langle B_n \rangle_n$ be the respective sequences of approximants and canonical denominators for a Śleszyński–Pringsheim continued fraction $\mathrm{K}_{n=1}^{\infty}(a_n/b_n)$ with $a_n < 0$ and $b_n \geq 1 - a_n$ for each $n \in \mathbb{N}$. Then*

$$B_{n+1} > B_n \geq \sum_{k=0}^{n} \prod_{m=1}^{k} (b_m - 1) \geq \sum_{k=0}^{n} \prod_{m=1}^{k} |a_m|,$$

$\langle f_n \rangle_n$ is strictly decreasing, and $-1 < f_n < f_{n-1} < 0$.

Proof. We prove by induction that $\langle B_n \rangle_n$ is a strictly increasing sequence. First, $B_0 - B_{-1} = 1$. Second, for our induction hypothesis, let us assume that $B_{n-1} - B_{n-2} > 0$ and $B_{n-2} \geq 0$. Then, using the recurrence relation and our additional assumptions on the coefficients, we have

$$B_n - B_{n-1} = (b_n - 1)B_{n-1} - (-a_n)B_{n-2} \geq (b_n - 1)(B_{n-1} - B_{n-2}).$$

Repeated application of this technique gives

$$B_n - B_{n-1} \geq (B_{n-1} - B_{n-2})(b_n - 1) \geq (B_0 - B_{-1}) \prod_{m=1}^{n} (b_m - 1) \geq \prod_{m=1}^{n} |a_m| > 0,$$

from which the desired inequalities follow. We apply the determinant formula to the sequences $\langle A_n \rangle_n$ and $\langle B_n \rangle_n$ (see Lemma 2.1) to obtain

$$f_n - f_{n-1} = -\frac{\prod_{k=1}^{n} -a_k}{B_n B_{n-1}} < 0.$$

Thus $\langle f_n \rangle_n$ is a strictly decreasing sequence with $f_1 = a_1/b_1 < 0$. The bounds follow from the aforementioned convergence properties of Śleszyński–Pringsheim continued fractions. ◀

2.3 second-order linear recurrences and continued fractions

Recall that a non-trivial solution $\langle u_n \rangle_{n=-1}^{\infty}$ of the recurrence $u_n = b_n u_{n-1} + a_n u_{n-2}$ is *minimal* provided that, for all other linearly independent solutions $\langle v_n \rangle_{n=-1}^{\infty}$ of the same recurrence, we have $\lim_{n\to\infty} u_n/v_n = 0$. Since the vector space of solutions has dimension two, it is equivalent for a sequence $\langle u_n \rangle_{n=-1}^{\infty}$ to be minimal for there to exist a linearly independent sequence $\langle v_n \rangle_{n=-1}^{\infty}$ satisfying the above property. In such cases the solution $\langle v_n \rangle_n$ is called *dominant*.

Note that if $\langle u_n \rangle_n$ and $\langle v_n \rangle_n$ are linearly independent solutions of the above recurrence such that u_n/v_n converges in $\hat{\mathbb{R}}$ then the recurrence relation has a minimal solution [23, §IV]. If, in addition, $\langle u_n \rangle_n$ is minimal then all solutions of the form $\langle cu_n \rangle_n$ where $c \neq 0$ are also minimal. If $\langle u_n \rangle_n$ and $\langle v_n \rangle_n$ are respectively minimal and dominant solutions of the recurrence, then together they form a basis of the solution space.

▶ Remark 2.4. When a second-order recurrence relation admits minimal solutions, it is often beneficial (from a numerical standpoint) to provide a basis of solutions where one of the elements is a minimal solution. Such a basis is used to approximate any element of the vector space of solutions: taking $\langle u_n \rangle_n$ and $\langle v_n \rangle_n$ as above, a general solution $\langle z_n \rangle_n$ is given by $z_n = \alpha_1 u_n + \alpha_2 v_n$.

Let $\langle u_n \rangle_{n=-1}^{\infty}$ be a non-trivial solution of the recurrence relation $u_n = b_n u_{n-1} + a_n u_{n-2}$, where $a_n \neq 0$ for all n. If $u_{n-1} \neq 0$ then we can rearrange the relation to obtain

$$-\frac{u_{n-1}}{u_{n-2}} = \cfrac{a_n}{b_n - \cfrac{u_n}{u_{n-1}}} \tag{2}$$

for each $n \in \mathbb{N}$. In the event that $u_{n-2} = 0$ we take the usual interpretation in $\hat{\mathbb{R}}$. Since $\langle u_n \rangle_n$ is non-trivial and $a_n \neq 0$ for each $n \in \mathbb{N}$, the sequence $\langle u_n \rangle_n$ does not vanish at two consecutive indices. Thus if $u_{n-1} = 0$ then $u_{n-2}, u_n \neq 0$ and so both the left-hand and the right-hand sides of the last equation are well-defined in $\hat{\mathbb{R}}$ and are equal to 0. Thus the sequence with terms $-u_n/u_{n-1}$ is well-defined in $\hat{\mathbb{R}}$ for each $n \in \mathbb{N}$. A sequence $\langle t_n \rangle_{n=0}^{\infty}$ where $t_n := -u_n/u_{n-1}$ for each $n \in \mathbb{N}$ and $\langle u_n \rangle_n$ non-trivial is called a *tail sequence*. A tail sequence for $\mathrm{K}(a_n/b_n)$ is wholly determined by its initial value t_0.

Given a convergent continued fraction $\mathrm{K}_{n=1}^{\infty}(a_n/b_n)$ it is easily shown that the sequence $\langle f^{(m)} \rangle_{m=0}^{\infty}$ with terms $f^{(m)} := \mathrm{K}_{n=m+1}^{\infty}(a_n/b_n)$ is a tail sequence. In the literature the sequence $\langle f^{(m)} \rangle_{m=0}^{\infty}$ is the *sequence of tails of* $\mathrm{K}_{n=1}^{\infty}(a_n/b_n)$ [23, §2.1].

The next theorem due to Pincherle [34] connects the existence of minimal solutions for a second-order recurrence to the convergence of the associated continued fraction (see also [8, 23, 3]).

▶ **Theorem 2.5** (Pincherle). *Let $\langle a_n \rangle_{n=1}^{\infty}$ and $\langle b_n \rangle_{n=1}^{\infty}$ be real-valued sequences such that each of the terms a_n is non-zero. First, the recurrence $u_n = b_n u_{n-1} + a_n u_{n-2}$ has a minimal solution if and only if the continued fraction $\mathrm{K}(a_n/b_n)$ converges in $\hat{\mathbb{R}}$. Second, if $\langle u_n \rangle_n$ is a minimal solution of this recurrence then the limit of $\mathrm{K}(a_n/b_n)$ is $-u_0/u_{-1}$. As a consequence, the sequence of canonical denominators $\langle B_n \rangle_{n=-1}^{\infty}$ is a minimal solution if and only if the value of $\mathrm{K}(a_n/b_n)$ is $\infty \in \hat{\mathbb{R}}$.*

▶ Remark 2.6. The convergence properties of continued fractions whose partial quotients are polynomials has long fascinated researchers. It is notable that the sequence of partial denominators in the continued fraction expansion of $\pi = 3 + \mathrm{K}_{n=1}^{\infty}(1/b_n)$ beginning $\langle b_n \rangle_n =$

$\langle 7, 15, 1, 292, 1, 1, 1, 2, 1, 3, 1, \ldots\rangle$ behaves erratically. In contrast, Lord Brouncker (as reported by Wallis in [37][3]) gave a continued fraction expansion for $4/\pi$ as follows:

$$\frac{4}{\pi} = 1 + \mathop{\mathrm{K}}_{n=1}^{\infty} \frac{(2n-1)^2}{2}.$$

Likewise, Apéry's constant $\zeta(3)$ has a continued fraction expansion (see [35])

$$\zeta(3) = \frac{6}{5 + \mathrm{K}_{n=1}^{\infty}(-n^6/(34n^3 + 51n^2 + 27n + 5))}$$

whose partial quotients are ultimately polynomials. Motivated by such constructions, Bowman and Mc Laughlin [2] (see also [25]) coined the term *polynomial continued fraction (PCF)*. A polynomial continued fraction $\mathrm{K}(a_n/b_n)$ has algebraic partial quotients such that for sufficiently large $n \in \mathbb{N}$, a_n and b_n are determined by polynomials in $\mathbb{Q}[n]$.

We call the problem of determining whether a given convergent polynomial continued fraction is equal to a particular algebraic number the *PCF Equality Problem*. The proof of the following corollary is a straightforward application of Theorem 2.5.

▶ **Corollary 2.7.** *The PCF Equality Problem and the Minimality Problem for second-order holonomic sequences are interreducible.*

Proof. A minimality-preserving transformation takes as input a solution $\langle u_n\rangle_n$ of recurrence $p_3(n)u_n = p_2(n)u_{n-1} + p_1(n)u_{n-2}$ and outputs a solution $\langle v_n\rangle_n$, with $v_n = u_n \prod_{j=0}^{n} p_3(j)$, of recurrence $v_n = p_2(n)v_{n-1} + p_1(n)p_3(n-1)v_{n-2}$. Clearly, $\langle u_n\rangle_n$ is a minimal solution if and only if $\langle v_n\rangle_n$ is a minimal solution.

The latter of the two recurrence relations is associated with the polynomial continued fraction $\mathrm{K}(a_n/b_n)$ with partial quotients $b_n = p_2(n)$ and $a_n = p_1(n)p_3(n-1)$ for each $n \in \mathbb{N}$. Note that by our assumption that $p_1(n), p_3(n) \neq 0$ for all $n \geq -1$ (see the Introduction) we have that $a_n \neq 0$, as required in our definition of a continued fraction. By Theorem 2.5, $\langle v_n\rangle_n$ is a minimal solution if and only if $\mathrm{K}(a_n/b_n)$ converges to the limit $-v_0/v_{-1}$. Thus if one has an oracle that can determine the value of a polynomial continued fraction, then one can determine whether $\langle v_n\rangle_n$ is a minimal solution. Since minimality is preserved by this transformation, one can determine whether $\langle u_n\rangle_n$ is a minimal solution.

Conversely, given a polynomial continued fraction $\mathrm{K}(a_n/b_n)$ and an algebraic number $\xi \in \mathbb{R}$, let us construct the holonomic sequence $\langle v_n\rangle_{n=-1}^{\infty}$ as follows. For each $n \in \mathbb{N}$, let $v_n = b_n v_{n-1} + a_n v_{n-2}$ with initial conditions $v_{-1} = 1$ and $v_0 = -\xi$. By Theorem 2.5, the sequence $\langle v_n\rangle_n$ is a minimal solution of the recurrence relation if and only if the continued fraction $\mathrm{K}(a_n/b_n)$ converges to the value $-u_0/u_{-1} = \xi$. Hence if one has an oracle that can determine whether a given holonomic sequence is a minimal solution, then one can test the value of a polynomial continued fraction. ◀

Determining whether a given continued fraction converges has attracted much attention (historical accounts are given in [23, 24]). The following theorem collects together results from the literature; the first statement follows as a consequence of Worpitzky's Theorem (see [24, Theorem 3.29]) and the convergence results in [15], whilst the second statement follows from the Lane–Wall characterisation of convergence [24, Theorem 3.3].

[3] See the translation by Stedall [38].

▶ **Theorem 2.8.** *Let* $\mathbf{K}(\kappa_n/1)$ *be a continued fraction with* $\langle \kappa_n \rangle_n$ *a function in* $\mathbb{Q}(n)$*. If* $\kappa_n < 0$ *for all sufficiently large* $n \in \mathbb{N}$*, then* $\mathbf{K}(\kappa_n/1)$ *converges to a value in* $\hat{\mathbb{R}}$ *if and only if, either*

- $\lim_{n\to\infty} \kappa_n$ *exists and is strictly above* $-1/4$*, or*
- $\lim_{n\to\infty} \kappa_n = -1/4$ *and moreover* $\kappa_n \geq -1/4 - 1/(4n)^2 - 1/(4n \log n)^2$ *for all sufficiently large* n*.*

▶ Remark 2.9. Note that since κ_n is assumed to be a rational function in the above, the eventual inequality $\kappa_n \geq -1/4 - 1/(4n)^2 - 1/(4n \log n)^2$ can be effectively decided, whence convergence of the continued fraction can be ascertained.

The fact that the coefficients $-1/16$ in Theorem 2.8 are best possible is discussed in [14, 13, 23, 24]. For example, if $\kappa_n = -1/4 - \varepsilon/n^2 + \mathcal{O}(1/n^3)$ where $\varepsilon > 1/16$, or $\kappa_n = -1/4 - \varepsilon_1/n + \mathcal{O}(1/n^2)$ where $\varepsilon_1 > 0$, then the continued fraction $\mathbf{K}(\kappa_n/1)$ diverges. We note that later independent work by Kooman and Tijdeman [19, 18] establishes the same results (as a consequence of their results for linear recurrence sequences).

3 Positivity reduces to Minimality

The goal of this section is to show that, for second-order holonomic sequences, the Positivity Problem Turing-reduces to the Minimality Problem; in other words, given an oracle for the Minimality Problem, one can decide the Positivity Problem.

▶ **Theorem 3.1.** *For the class of recurrence relations*

$$u_n = b_n u_{n-1} + a_n u_{n-2} \tag{3}$$

whose coefficients are rational functions in $\mathbb{Q}(n)$*, the Positivity Problem Turing-reduces to the Minimality Problem.*

3.1 reduction argument

Let $\langle u_n \rangle_n$ be a sequence satisfying the second-order relation (1). Recall from the Introduction that we can assume without loss of generality that none of the polynomial coefficients in this recurrence relation has a root $n \geq -1$. Additionally we can assume that $\mathrm{sign}(p_3) = +$ on $\mathbb{N}$. (Herein we denote the sign of a non-zero number by an element of $\{+, -\}$ with the obvious interpretation.) Thus we define the *signature* of a recurrence relation (1) (or its normalisation (3)) as the ordered pair $(\mathrm{sign}(p_2), \mathrm{sign}(p_1))$. It is useful to consider subcases determined by the signature of the recurrence relation $u_n = b_n u_{n-1} + a_n u_{n-2}$. The Positivity Problem is trivial when the signature of the recurrence is either $(+,+)$ or $(-,-)$. It remains to consider the cases $(-,+)$ and $(+,-)$.

Let $\langle u_n \rangle_n$ satisfy a recurrence with signature $(-,+)$. Then a simple substitution argument gives

$$u_{2n} = (b_{2n}b_{2n-1} + a_{2n} + a_{2n-1}b_{2n}/b_{2n-2})u_{2n-2} - (a_{2n-1}a_{2n-2}b_{2n}/b_{2n-2})u_{2n-4}.$$

The sequence of odd terms $\langle u_{2n-1} \rangle_n$ satisfies a similar recurrence relation with signature $(+,-)$. Thus the Positivity Problem for the $(-,+)$ case reduces to determining the Positivity Problem for two recurrences with signature $(+,-)$.

We come to the final case: recurrences with signature $(+,-)$. Let $\langle A_n \rangle_{n=-1}^{\infty}$ and $\langle B_n \rangle_{n=-1}^{\infty}$ be the canonical solutions as above. In this case $A_1 = a_1 < 0$ and so one can assume that

$u_0 > 0$.[4] It is useful to normalise recurrence (3). Let $\kappa_n := a_n/(b_n b_{n-1})$, and consider

$$w_n = w_{n-1} + \kappa_n w_{n-2}. \tag{4}$$

Then $\langle w_n \rangle_n$ with $w_{-1} = u_{-1}$ and $w_n := u_n/(\prod_{k=0}^{n} b_k)$ is a solution to (4) if and only if $\langle u_n \rangle_n$ is a solution to (3). We note minimality, positivity, and signature $(+,-)$ are invariant under this transformation. Such properties follow from our assumption that each $b_n > 0$ and the equivalence transformations for continued fractions in Theorem 2.2.

In light of this reduction, the next result is an immediate corollary of Theorem 2.5 and Remark 2.9.

▶ **Corollary 3.2.** *Given a recurrence relation of the form* (3)*, it is decidable whether the recurrence admits a minimal solution.*

In the work that follows we split the $(+,-)$ case into subcases depending on whether the limit $\lim_{n\to\infty} \kappa_n$ exists and, if it exists, its value κ. It turns out that such a recurrence relation admits a non-trivial positive solution if and only if the associated continued fraction converges (see Theorem 2.8).

▶ **Lemma 3.3.** *Suppose that the continued fraction* $\mathrm{K}(\kappa_n/1)$ *diverges in* $\hat{\mathbb{R}}$*. Then there are no non-trivial positive solutions to recurrence* (4)*.*

Proof. Suppose, for a contradiction, that $\langle w_n \rangle_n$ is a positive sequence and non-trivial solution of recurrence (4). Notice that two consecutive terms in $\langle w_n \rangle_n$ cannot both vanish since $\langle w_n \rangle_n$ is non-trivial. Furthermore, $w_n > 0$ for all $n \geq 0$ since otherwise $w_n = 0$ and $w_{n+1} = \kappa_{n+1} w_{n-1} < 0$. We first show that the sequence $\langle B_n \rangle_n$ is also positive.

If $w_{-1} = 0$ then $\langle w_n \rangle_n$ is a constant multiple of $\langle B_n \rangle_n$ and we have nothing to show. Otherwise, $w_{-1} > 0$ and let $\langle \breve{w}_n \rangle_n$ be a solution sequence of recurrence (4) such that $\breve{w}_{-1} = w_{-1}$ and $\breve{w}_0 > w_0$. We then have $\breve{w}_n > w_n$ for all $n \in \mathbb{N}$. Indeed, proceeding by induction on n, by Lemma 2.1,

$$\breve{w}_n w_{n-1} - \breve{w}_{n-1} w_n = (\breve{w}_0 w_{-1} - \breve{w}_{-1} w_0) \prod_{k=1}^{n} (-\kappa_k) = (\breve{w}_0 - w_0) w_{-1} \prod_{k=1}^{n} (-\kappa_k) > 0$$

implying that $\breve{w}_n w_{n-1} > \breve{w}_{n-1} w_n$. The induction hypothesis $\breve{w}_{n-1} > w_{n-1}$ (with $n \geq 1$) implies that $\breve{w}_{n-1} w_n > w_{n-1} w_n$ as $w_n > 0$ by assumption. It follows that $\breve{w}_n w_{n-1} > w_{n-1} w_n$, and thus $\breve{w}_n > w_n$.

Notice now that $\langle \breve{w}_n - w_n \rangle_n$ is a positive sequence and non-trivial solution of recurrence (4). For each $n \in \{-1, 0, \ldots\}$ we have $\breve{w}_n - w_n = (\breve{w}_0 - w_0) B_n$ and so conclude that $B_n > 0$ for each $n \in \mathbb{N}$.

Let $\langle f_n \rangle_n$ be the sequence of approximants associated with $\mathrm{K}(\kappa_n/1)$. We now apply Lemma 2.1 to the sequences $\langle A_n \rangle_n$ and $\langle B_n \rangle_n$, and our conclusion that $B_n > 0$ for each $n \in \mathbb{N}$, to obtain

$$f_n - f_{n-1} = -\frac{\prod_{k=1}^{n} -\kappa_k}{B_n B_{n-1}} < 0.$$

Thus $\langle f_n \rangle_n$ is monotonic and therefore convergent in $\hat{\mathbb{R}}$, a contradiction to the divergence of $\mathrm{K}(\kappa_n/1)$. ◀

[4] Indeed, if $u_0 < 0$ then the sequence is not positive; whereas if $u_0 = 0$, then in turn the sequence is either identically zero, or $u_1 < 0$.

In what follows, *'eventually'* statements shall always assume that a property holds for each $n + N$ where $N \in \mathbb{N}$ is a fixed computable constant. Our assumption on the signature means that $a_{n+N} < 0$ and $b_{n+N} > 0$ for each $n \in \mathbb{N}$. Without loss of generality, we can take $N = 0$ in the upcoming statements and results by considering tails of continued fractions as appropriate.

From Theorem 2.8 and Lemma 3.4 (below) we characterise the boundary for a recurrence relation of the form (4) to admit positive solutions. The proof of Lemma 3.4 uses standard analytic tools for continued fractions of limit parabolic type with a particular choice of parameter sequence $\langle g_n \rangle_n$. More general discussions are given in [16, 22, 23].

▶ **Lemma 3.4.** *Suppose that eventually*

$$\kappa_n \geq -1/4 - 1/(4n)^2 - 1/(4n \log n)^2. \tag{5}$$

Then the sequence of approximants of the continued fraction $\mathrm{K}_{n=1}^{\infty}(\kappa_n/1)$ *is strictly decreasing and converges to a finite value.*

Proof. Without loss of generality we assume that (5) holds for each $n \in \mathbb{N}$. Let $g_0 = g_1 = g_2 = 1$ and $g_n := 1/2 + 1/(4n) + 1/(4n \log n)$ for each $n \geq 3$. The continued fractions $\mathrm{K}_{n=1}^{\infty}(\kappa_n/1)$ and

$$g_0 \operatorname*{K}_{n=1}^{\infty} \left(\frac{\kappa_n/(g_{n-1}g_n)}{1/g_n} \right) \tag{6}$$

are equivalent; one can prove this assertion by applying Theorem 2.2 with the transformation choice $\mathfrak{r}_n = 1/(b_n g_n)$ for each $n \in \mathbb{N}$. Then, by assumption, $|\kappa_n| \leq g_{n-1}(1 - g_n)$ for each $n \in \mathbb{N}$. Thus

$$1 - \frac{\kappa_n}{g_{n-1}g_n} = \frac{g_{n-1}g_n - \kappa_n}{g_{n-1}g_n} \leq \frac{1}{g_n}.$$

We deduce that the partial numerators and denominators in (6) satisfy the assumptions in Theorem 2.3. Thus the sequence of approximants $\langle f_n \rangle_{n=1}^{\infty}$ associated with (6) is strictly decreasing and converges to a finite value. The desired result follows. ◀

▶ **Lemma 3.5.** *Suppose that* $\langle w_n \rangle_{n=-1}^{\infty}$ *is a solution to* (4) *with signature* $(+, -)$ *such that* (5) *holds for each* $n \in \mathbb{N}$. *Let* $\langle f_n \rangle_n$ *be the sequence of approximants for the associated continued fraction* $\mathrm{K}(\kappa_n/1)$. *Assume that* $w_{-1} > 0$. *Given* $m \in \mathbb{N}$, *we have that* $-w_0/w_{-1} < f_m$ *if and only if* $w_m > 0$.

Proof. Let $\langle A_n \rangle_n$ and $\langle B_n \rangle_n$ be the sequences of canonical numerators and denominators associated with $\mathrm{K}(\kappa_n/1)$. The continued fractions $\mathrm{K}(\kappa_n/1)$ and (6) are equivalent; in addition, the latter is a Śleszyński–Pringsheim continued fraction whose associated sequence of canonical denominators is non-negative (by Theorem 2.3). The transformation between these two continued fractions preserves the positivity property and so we deduce that each term in $\langle B_n \rangle_n$ is also non-negative.

For each $n \in \mathbb{N}$, $w_n = w_{-1}A_n + w_0 B_n$. Since $B_n > 0$, $-w_0/w_{-1} < A_n/B_n = f_n$ if and only if $w_n > 0$, as desired. ◀

We are now in a position to characterise positive solutions to recurrence (4).

▶ **Proposition 3.6.** *Suppose that* $\langle w_n \rangle_{n=-1}^{\infty}$ *is a solution of recurrence* (4) *with signature* $(+, -)$ *such that* (5) *holds for all* $n \in \mathbb{N}$. *First, the continued fraction* $\mathrm{K}_{n=1}^{\infty}(\kappa_n/1)$ *converges to a finite limit* $f < 0$. *Second, the sequence* $\langle w_n \rangle_{n=-1}^{\infty}$ *with* $w_{-1}, w_0 > 0$ *is positive if and only if* $-w_0/w_{-1} \leq f$.

Proof. As observed in the proof of Lemma 3.4, $\mathrm{K}(\kappa_n/1)$ and (6) are equivalent continued fractions. The former converges to a negative value $f \in \mathbb{R}$ because the latter is a Śleszyński–Pringsheim continued fraction that satisfies the assumptions in Theorem 2.3.

Let $\langle w_n \rangle_{n=-1}^{\infty}$ be a solution to recurrence (4). By Lemma 3.5, we have that, for all $n \in \mathbb{N}$, $w_n > 0$ if and only if $-w_0/w_{-1} < f_n$. Moreover, by Theorem 2.3, the sequence $\langle f_n \rangle_n$ is strictly decreasing; it follows that $w_n > 0$ for all $n \in \mathbb{N}$ if and only if $-w_0/w_{-1} \le f$. ◀

The difficulty one encounters when determining positivity arises when $-w_0/w_{-1}$ is equal to the value f. In other words, we can decide positivity of dominant sequences. Indeed, one can always detect if a non-trivial solution $\langle w_n \rangle_n$ is not positive, i.e., $-w_0/w_{-1} > f$ by computing a sufficient number of terms until one finds an $N \in \mathbb{N}$ such that $w_N < 0$. The dominant positive sequences are considered in the following proposition whose proof is delayed to Section 4.

▶ **Proposition 3.7.** *Let $\langle w_n \rangle_{n=-1}^{\infty}$ be a non-trivial solution of* (4) *with signature* $(+,-)$ *and suppose that* (5) *holds for each $n \in \mathbb{N}$. Then one can detect if $-w_0/w_{-1} < f$.*

We deduce that if one can decide whether a holonomic sequence $\langle u_n \rangle_n$ that solves recurrence (3) is minimal, then one can decide whether $\langle u_n \rangle_n$ is a positive solution.

Proof of Theorem 3.1. Assume that one has an oracle for the Minimality Problem for solutions $\langle u_n \rangle_{n=-1}^{\infty}$ to recurrences of the form (3). Note that if $\langle u_n \rangle_{n=-1}^{\infty}$ is *not* positive, this can be substantiated in finite time by simple enumeration. It thus remain to show how one can ascertain positivity. We can assume without loss of generality that the recurrence has signature $(+,-)$. As previously mentioned, the problem of determining the positivity of solutions $\langle u_n \rangle_n$ of (3) is equivalent to the problem of determining the positivity of solutions $\langle w_n \rangle_n$ of (4).

Consider a recurrence relation of the form (4) with signature $(+,-)$. By Theorem 2.8, we can decide whether or not $\mathrm{K}(\kappa_n/1)$ converges. If $\mathrm{K}(\kappa_n/1)$ diverges, then by Lemma 3.3, the recurrence has no non-trivial positive solutions. Suppose now that $\mathrm{K}(\kappa_n/1)$ converges to $f \in \hat{\mathbb{R}}$. Then, by Remark 2.9, inequality (5) holds; by Proposition 3.6, it follows that f is finite and a given solution $\langle w_n \rangle_n$ is positive if and only if $-w_0/w_{-1} \le f$. The condition $-w_0/w_{-1} < f$ is recursively enumerable by Proposition 3.7. Finally, by Theorem 2.5, $-w_0/w_{-1} = f$ if and only if the sequence is minimal, and hence equality of $-w_0/w_{-1}$ and f can be checked by an oracle for Minimality. ◀

3.2 A characterisation of positivity

We end this section by characterising positive solutions to recurrence (3) in terms of the ratio of the initial terms belonging to a certain closed interval. Here we understand a closed interval to be empty, a single point, an interval with finite endpoints, or a half-line (including ∞).

▶ **Proposition 3.8.** *Consider a recurrence of the form* (3) *for which the coefficients $\langle a_n \rangle_n$ and $\langle b_n \rangle_n$ have constant sign on $\mathbb{N}$. There exists a closed interval P such that a non-trivial solution $\langle u_n \rangle_n$, with u_{-1}, $u_0 \ge 0$, to the recurrence is positive if and only if $u_0/u_{-1} \in P$. Moreover, the endpoints of the interval can be expressed using polynomial continued fractions.*

Proof. In the cases where the recurrence has signature $(+,+)$ or $(-,-)$, we can set $P = [0,\infty) \cup \{\infty\}$ ($[0,\infty]$ for short) and $P = \emptyset$, respectively.

Consider now a recurrence with signature $(+,-)$. If it is of the form (4), then we have the following: if the associated continued fraction $\mathrm{K}\,(\kappa_n/1)$ diverges, then there are no non-trivial positive solutions by Lemma 3.3, and we set $P = \emptyset$. If it converges to f (in particular (5) holds), we may set $P = [-f, \infty]$ as is immediate from Proposition 3.6.

Assume then that the recurrence is not of the form (4). The transformation from $\langle u_n \rangle_n$ to a solution $\langle w_n \rangle_n$ to a recurrence of the form (4) preserves minimality and positivity. Hence, if the associated continued fraction $\mathrm{K}\,(a_n/b_n)$ does not converge, then there are no non-trivial positive solutions and we may set $P = \emptyset$. If it converges, then so does the continued fraction $\mathrm{K}\,(\kappa_n/1)$; say it converges to f. Then a non-trivial solution to the recurrence is positive if and only if $u_0/u_{-1} = w_0/(b_0 w_{-1}) \in [-f/b_0, \infty]$.

We are left with recurrences (3) with signature $(-,+)$. Let $\langle f_n \rangle_n$ denote the sequence of approximants of the associated continued fraction. One can show by straightforward induction that the sequence $\langle B_n \rangle_n$ is alternating in sign for $n \in \mathbb{N}$: the even terms are positive and the odd terms are negative. Let us write a solution $\langle u_n \rangle_n$, with u_{-1}, $u_0 \geq 0$, as $u_n = u_{-1}A_n + u_0 B_n$. We have $u_n \geq 0$ if and only if $u_0/u_{-1} \geq -A_n/B_n = -f_n$ when n is even and $u_0/u_{-1} \leq -A_n/B_n = -f_n$ when n is odd. Now the continued fraction $\mathrm{K}\,(a_n/b_n)$ is equivalent to $-\mathrm{K}\,(a_n/-b_n)$. By [23, Theorem 2, §III], $\langle -f_{2n} \rangle_n$ is strictly increasing and has finite limit $-f'$, while $\langle -f_{2n-1} \rangle_n$ is strictly decreasing and has finite limit $-f''$. Moreover, $-f' \leq -f''$. It follows that $\langle u_n \rangle_n$ is positive if and only if $u_0/u_{-1} \in [-f', -f'']$.

That the (finite) endpoints of the above intervals can be described using polynomial continued fractions follows from similar minimality-preserving transformations as performed in the proof of Corollary 2.7 and, in the case of the points f', f'', from results in [23, §II.2.4]. ◀

4 Detecting positive and dominant solutions

The goal of this section is to prove Proposition 3.7, as such, we will suppose that (5) holds for each $n \in \mathbb{N}$ in the following. The proof follows from the results in Corollary 4.2 and Corollary 4.6.

Broadly speaking, we describe a semi-algorithm with inputs $\langle w_n \rangle_n$. This semi-algorithm terminates in finite time for sequences that are dominant with output *'input is a positive sequence'* or *'input is not a positive sequence,'* as appropriate. The semi-algorithm is non-terminating when given a minimal solution as an input. In terminating instances, the running time depends upon the distance between $-w_0/w_{-1}$ and $\mathrm{K}_{n=1}^{\infty}(\kappa_n/1)$.

The sequence of approximants $\langle f_n \rangle_{n=1}^{\infty}$ associated with $\mathrm{K}_{n=1}^{\infty}(\kappa_n/1)$ is recursively defined by a composition of linear fractional transformations $f_n := s_1 \circ \cdots \circ s_n(0)$ where $s_n(w) = \kappa_n/(1+w)$ for each $n \in \{1, 2, \ldots\}$ and $w \in \hat{\mathbb{R}}$. The tail sequences of $\mathrm{K}(\kappa_n/1)$ are also recursively defined by linear fractional transformations: given such a tail sequence $\langle t_n \rangle_n$, $s_n^{-1}(t_{n-1}) = t_n$ for each $n \in \mathbb{N}$ (by (2)).

For each n, the linear fractional transformation s_n above has two fixed points $\omega_n^{\pm} := \frac{1}{2}(-1 \pm \sqrt{1+4\kappa_n})$. By (5), $\langle \sqrt{1+4\kappa_n} \rangle_n$ converges to a real value. We split our analysis into two cases depending on whether κ_n converges to $-1/4$. These subcases are common in the literature (cf. [23, §5]) as some of the convergence properties of the continued fraction $\mathrm{K}(\kappa/1)$ (with $\kappa := \lim_{n\to\infty} \kappa_n$) hold for the continued fraction $\mathrm{K}(\kappa_n/1)$. In fact, the subcases of *limit hyperbolic-* and *parabolic type* are named for the classification of the limiting linear fractional transformation $s(w) = \kappa/(1+w)$.

4.1 limit hyperbolic type

A continued fraction $\mathrm{K}(\kappa_n/1)$ is of *limit hyperbolic type* if the finite value $\kappa := \lim_{n\to\infty} \kappa_n$ satisfies $\kappa > -1/4$. In this case the sequences $\langle \omega_n^+ \rangle_n$ and $\langle \omega_n^- \rangle_n$ converge to distinct limits

ω^+ and ω^-, respectively. We shall assume, without loss of generality, that aforementioned eventually statements hold for each $n \in \mathbb{N}$.

The next result is given in the literature. A more general result for asymptotic properties of tail sequences associated with a continued fraction of limit hyperbolic type is given in [24, Theorem 4.13].

▶ **Theorem 4.1.** *Suppose that* $\mathrm{K}(\kappa_n/1)$ *is of limit hyperbolic type. The sequence of tails* $\langle f^{(n)}\rangle_n$ *converges to* ω^+. *A tail sequence* $\langle t_n\rangle_n$ *with* $t_0 \neq f^{(0)}$ *converges to* ω^-.

▶ **Corollary 4.2.** *Suppose that* $\mathrm{K}(\kappa_n/1)$ *is of limit hyperbolic type. One can detect if a solution sequence* $\langle w_n\rangle_n$ *of recurrence* (4) *is positive and dominant.*

Proof. Let $\langle t_n\rangle_n$ be the tail sequence associated with a non-trivial solution $\langle w_n\rangle_n$. By Theorem 4.1, a tail sequence $\langle t_n\rangle_n$ associated with a dominant solution converges to ω^- in the limit, whilst the tail sequence $\langle f^{(n)}\rangle_n$ associated with a minimal solution converges to ω^+ in the limit.

There is a computable $N \in \mathbb{N}$ such that for all $n \geq N$, the two fixed points of s_n^{-1} are separated: $\omega_n^- < (\omega^- + \omega^+)/2 = -1/2 < \omega_n^+$. If $m > N$ and $t_m < -1/2$ then $\langle t_{n+m}\rangle_n$ is bounded from above by $-1/2$. This is established by induction. The base case is ensured by the assumption $t_m < -1/2$. Now let $n \geq m$ such that $t_n < -1/2$, we have that $t_{n+1} = s_{n+1}^{-1}(t_n) = \frac{\kappa_{n+1}}{t_n} - 1$. Assume first that $\omega_{n+1}^- < t_n$, then $t_{n+1} = s_{n+1}^{-1}(t_n) \leq t_n < -1/2$ as t_n lies between the two fixed points of s_{n+1}. Otherwise, if $\omega_{n+1}^- \geq t_n$, then

$$t_{n+1} = s_{n+1}^{-1}(t_n) = \frac{\kappa_{n+1}}{t_n} - 1 \leq \frac{\kappa_{n+1}}{\omega_{n+1}^-} - 1 = s_{n+1}^{-1}(\omega_{n+1}^-) = \omega_{n+1}^- < -1/2$$

which completes the induction step.

Thus we can detect if a tail sequence is associated with a dominant solution. Moreover, this observation allows us to detect whether a dominant solution is positive in finite time. ◀

4.2 limit parabolic type

A continued fraction $\mathrm{K}(\kappa_n/1)$ is of *limit parabolic type* if $\lim_{n\to\infty} \kappa_n = -1/4$. In this case both $\langle \omega_n^+\rangle_n$ and $\langle \omega_n^-\rangle_n$ converge to $-1/2$.

The limit parabolic case is subtler than the limit hyperbolic case; this is best illustrated by the following result: all tail sequences converge to the same limit (see the general case [24, Theorem 4.17]).

▶ **Theorem 4.3.** *Let* $\mathrm{K}(\kappa_n/1)$ *be a continued fraction of limit parabolic type such that* (5) *holds for each* $n \in \mathbb{N}$. *Each tail sequence* $\langle t_n\rangle_n$ *associated with* $\mathrm{K}(\kappa_n/1)$ *converges to* $-1/2$.

From our assumption that (5) holds for each $n \in \mathbb{N}$, we have bounds on the sequence of tails $f^{(n-1)} := \mathrm{K}_{m=n}^{\infty}(\kappa_m/1)$ by the following generalisation of Worpitzky's Theorem (see, for example, [24, Theorem 3.30]).

▶ **Theorem 4.4.** *Let* $\langle \kappa_n\rangle_n$ *be a sequence such that* (5) *holds for* $n \in \mathbb{N}$. *Then* $\mathrm{K}(\kappa_n/1)$ *converges to a finite value* f *with* $0 < |f| \leq 1$ *and* $|f^{(n)}| \leq g_n$ *for each* n *where* $g_0 = 1$ *and* $g_n := 1/2 + 1/(4n) + 1/(4n \log n)$ *for each* $n \in \mathbb{N}$.

The inequalities given in the proof of the next lemma follow from the observation that $s_n^{-1}\colon (-\infty, 0) \to (-1, \infty)$ given by $s_n^{-1}(w) = -1 + \kappa_n/w$ is a monotonic bijection.

▶ **Lemma 4.5.** *Suppose that* $\mathrm{K}(\kappa_n/1)$ *is of limit parabolic type such that* (5) *holds for each* $n \in \mathbb{N}$*. Let* $\langle t_n \rangle_n$ *be a tail sequence such that* $f^{(0)} - t_0 > 0$*. Then there exists an* $N \in \mathbb{N}$ *such that* $t_N < -g_N < f^{(N)}$*.*

Proof. First, note that one can deduce from Theorem 4.4 that $-g_n < f^{(n)} < 0$ for each $n \in \mathbb{N}$ (otherwise there is an m such that $f^{(m+1)} < -g_{m+1}$). Now let $\langle t_n \rangle_n$ be a tail sequence associated with the continued fraction $\mathrm{K}_{n=1}^{\infty}(\kappa_n/1)$ such that $f^{(0)} - t_0 > 0$. Suppose, for a contradiction, that $-g_n < t_n$ for each $n \in \mathbb{N}$. Thus $\sum_{n=1}^{\infty} \prod_{k=1}^{n} \big(-\frac{1+t_n}{t_n}\big)$ diverges to ∞ by comparison with $\sum_{n=1}^{\infty} \prod_{k=1}^{n} \big(\frac{1-g_n}{g_n}\big)$; the latter series is known to diverge as $\mathrm{K}_{n=1}^{\infty}\big(\frac{-g_{n-1}(1-g_n)}{1}\big)$ is a convergent continued fraction (the full argument, which is beyond the scope of this paper, is given in [36]). However, by Waadeland's Tail Theorem [36, Theorem 1], divergence of $\sum_{n=1}^{\infty} \prod_{k=1}^{n} \big(-\frac{1+t_n}{t_n}\big)$ implies that $t_0 = \mathrm{K}_{n=1}^{\infty}(\kappa_n/1) = f^{(0)}$, which contradicts our assumption that $f^{(0)} - t_0 > 0$. ◀

▶ **Corollary 4.6.** *Suppose that* $\mathrm{K}(\kappa_n/1)$ *is of limit parabolic type such that* (5) *holds for each* $n \in \mathbb{N}$*. One can detect if a solution sequence* $\langle w_n \rangle_n$ *of recurrence* (4) *is positive and dominant.*

Proof. Let $\langle t_n \rangle_n$ be the tail sequence associated with a non-trivial solution $\langle w_n \rangle_n$. If $\langle w_n \rangle_n$ is dominant and positive, one has $f^{(0)} - t_0 > 0$ by Proposition 3.6. Moreover, by Lemma 4.5, there exists an $N \in \mathbb{N}$ such that for $t_N < -g_N < f^{(N)}$. Hence one can use the threshold of $-g_N$ to detect whether a solution sequence is dominant and positive. ◀

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