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TAME PARAHORIC HIGGS BUNDLES FOR A COMPLEX REDUCTIVE GROUP

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Abstract. For a connected complex reductive group G, we introduce a notion of stability for parahoric $\mathcal{G}_{\boldsymbol{\theta}}$-Higgs bundles over a smooth algebraic curve X, where $\mathcal{G}_{\boldsymbol{\theta}}$ is a parahoric group scheme over X. In the case when the group G is the general linear group GL_n, we show that the stability condition of a parahoric torsor reduces to stability of a parabolic bundle. A correspondence between semistable tame parahoric $\mathcal{G}_{\boldsymbol{\theta}}$-Higgs bundles and semistable tame equivariant G-Higgs bundles allows us to construct the moduli space explicitly. This moduli space is shown to be equipped with a Poisson structure.

To the memory of Professor M. S. Narasimhan

1. Introduction

For a compact Riemann surface X of genus $g \geq 2$, stable vector bundles over X of fixed rank and degree can be characterized in terms of irreducible unitary representations of a certain discrete group. This is the content of the theorem of M. Narasimhan and C. Seshadri [34], which provides a correspondence between such stable bundles and irreducible representations at the level of moduli spaces.

In search of a natural generalization of this remarkable result to the case when X is noncompact, C. Seshadri introduced in [46] an additional layer of structure on the bundles over a smooth irreducible projective curve, which he called a *parabolic structure*, inspired by the work of A. Weil on logarithmic connections with regular singularity at finitely many points [53, §2, Chapter I]. This notion involved the choice of a weighted flag on the fiber over each point from a finite collection of points on the curve. The new objects were called by C. Seshadri *parabolic bundles* and a stability condition in terms of a parabolic bundle degree was introduced analogously to the considerations of D. Mumford in the absence of the parabolic structure; for this notion of stability, the Narasimhan-Seshadri correspondence was subsequently established by V. Mehta and C. Seshadri [33] in this open-curve context, involving fundamental group representations into the group $G = \mathrm{U}(n)$.

The next important step was to extend this correspondence for compact, as well as for non-compact groups G. The case when $G = \mathrm{GL}(n, \mathbb{C})$ carried out by C. Simpson [47] was a landmark in this direction and involved the introduction of filtered objects to clarify the correct version of the bijective correspondence; in particular, stable filtered regular Higgs bundles and stable filtered local systems.

The main objective in the present article is to introduce a notion of stability for Higgs pairs and then construct an algebraic moduli space using GIT methods for general complex reductive groups G, that generalizes the moduli space of C. Simpson in this tame parabolic setting. The language to be used will be that of *parahoric group schemes* in the sense of F. Bruhat-J. Tits [15, 16] for the notion of parahoric weight introduced by P. Boalch [11]. This moduli space will be moreover shown to be Poisson. Before we explain our main considerations leading to the definition of these stable Higgs pairs for the general groups G, it is instructive to review a number of approaches in the literature followed for this problem. Several ideas from these approaches have been adapted in our work.

*Key words: parahoric group scheme, parahoric Higgs bundle, equivariant Higgs bundle, stability, Poisson structure
†MSC2020 Class: 14D23, 32Q26 (Primary), 14L15, 53D17 (Secondary)

1.1. **Background.** In generalizing the notion of a stable parabolic vector bundle to the setting of principal G-bundles for semisimple or reductive structure groups G, a central problem that soon became apparent was to introduce the correct notion of a parabolic weight in order to get a moduli space and a bijective correspondence, which would coincide with the ones of C. Simpson when $G = \mathrm{GL}(n, \mathbb{C})$. In [3], V. Balaji, I. Biswas and D. Nagaraj looked at principal bundles from a Tannakian perspective [20], following the description given by M. Nori [38, 39]. In this sense, principal G-bundles are interpreted as functors from the category of locally free coherent sheaves, and a functor in the parabolic context serves as the right definition that respects the tensor product operation. Even though this definition coincides with the one of C. Seshadri when $G = \mathrm{GL}(n, \mathbb{C})$, it became clear that to a representation of a Fuchsian group into the maximal compact subgroup of G with fixed holonomy around each puncture will not correspond a principal G-bundle in general.

An alternative approach by C. Teleman and C. Woodward [50] involved switching the order of embedding the group G in $\mathrm{GL}(n, \mathbb{C})$ and applying the equivalence with equivariant bundles. The weights in this case, called markings, were defined to lie in a Weyl alcove for the corresponding Lie algebra; this meant though, that one should restrict to a subclass of parabolic G-bundles in order to establish an analog of the Mehta-Seshadri correspondence.

In this same line of an approach, O. Biquard, O. García-Prada and I. Mundet i Riera [8] defined a notion of weight for parabolic principal G-bundles and proved a correspondence in the case of a real reductive group (also including the complex group cases) by using a Donaldson functional and the existence of harmonic reductions in this setting. The notion of weight in [8] involves a choice, for each point in the reduced effective divisor $D \subset X$, of an element in a Weyl alcove $\mathcal{A}$ of the Lie algebra of a fixed maximal torus in a fixed maximal compact subgroup of a non-compact reductive Lie group, with the closure of $\mathcal{A}$ containing 0. The authors allow these elements to lie in a wall of the Weyl alcove, in order to establish the correspondence with parabolic G-local systems having arbitrary fixed holonomy around the points in D. The side-effect of this explicit approach is that under this definition for a parabolic principal G-bundle, to a local system corresponds not a single holomorphic bundle, but rather a class of holomorphic bundles equivalent under gauge transformations with meromorphic singularities.

This defect started to become clear through the work of P. Boalch, who first defined in [11] the notion of weight from the point of view of *parahoric torsors* instead. Parahoric torsors were introduced by G. Pappas and M. Rapoport in [41], and locally these are described as parahoric subgroups of a formal loop group in the sense of F. Bruhat and J. Tits [15, 16]. Several conjectures concerning the moduli space of parahoric torsors were made by G. Pappas and M. Rapoport in [42], most of which have been verified by J. Heinloth in [28], thus generalizing corresponding results by V. Drinfeld and C. Simpson [21] for the moduli stack of principal bundles on a smooth projective curve over an arbitrary field. Then, a weight for a parahoric torsor is a point of the corresponding Bruhat-Tits building in the facet corresponding to the parahoric subgroup of a formal loop group (Definition 1, p. 46 in [11]).

In an independent work but still in this approach, V. Balaji and C. Seshadri [5] introduced a notion of stability for parahoric torsors for a collection of weights chosen from the set of rational one-parameter subgroups of the group G, for G semisimple and simply connected over $\mathbb{C}$. In this case, parahoric torsors over a smooth complex projective curve X of genus $g \geq 2$ are indeed the correct intrinsically defined objects on X associated to a (π, G)-bundle on the upper half plane $\mathbb{H}$, where π is the subgroup of the discontinuous group of automorphisms of $\mathbb{H}$, such that $X = \mathbb{H}/\pi$. V. Balaji and C. Seshadri moreover constructed in *loc. cit.* a moduli space for this notion of stability of parahoric torsors on X and proved, under the assumption that the weights are rational, an analogue of the Mehta-Seshadri correspondence in this context, that is, for the case $G = \mathrm{U}(n)$. Subsequently, V. Balaji, I. Biswas and Y. Pandey extended in [4] the correspondence to the case of real weights using a definition of stability that covers real weights as well.

1.2. **Results.** In this article we introduce a stability condition for tame parahoric Higgs bundles (also parahoric torsors) and construct the moduli space of (semi)stable tame parahoric Higgs bundles for general complex reductive groups G. We give in the sequel more details about the precise statements.

Let X be a smooth algebraic curve with a reduced effective divisor D. Denote by K_X the canonical line bundle over X. Let G be a connected complex reductive group. Fixing a maximal torus T in G, we equip each point $x \in D$ with a rational weight $\theta_x \in Y(T) \otimes_{\mathbb{Z}} \mathbb{Q}$, where $Y(T)$ is the group of one-parameter subgroups of T. Denote by $\boldsymbol{\theta} := \{\theta_x, x \in D\}$ the collection of weights over the points in D. A parahoric Bruhat-Tits group scheme $\mathcal{G}_{\boldsymbol{\theta}}$ is then defined by gluing local parahoric group schemes for formal disks around each point $x \in D$ (see Definition 2.4). A *tame parahoric* $\mathcal{G}_{\boldsymbol{\theta}}$*-Higgs bundle* over X is then defined as a pair (E, φ), where

- E is a $\mathcal{G}_{\boldsymbol{\theta}}$-torsor over X;
- $\varphi \in H^0(X, \mathrm{Ad}(E) \otimes K_X(D))$ is a holomorphic section.

The section φ is called a *tame (parahoric) Higgs field*. Note that the definition of tame parahoric Higgs bundles is a slightly modified version of Z. Yun's definition [55, §4.3], where the Higgs field φ is considered as a section of $\mathrm{Ad}(E)(D)$. Moreover, D. Baraglia, M. Kamgarpour and R. Varma in [6] use a similar definition for a parahoric Higgs bundle for a semisimple and simply connected Lie group G, where the Higgs field is considered as an element in $H^0(X, \mathrm{Ad}(E)^* \otimes K_X)$, where $\mathrm{Ad}(E)^*$ denotes the dual bundle.

V. Balaji and C. Seshadri showed in [5] that parahoric $\mathcal{G}_{\boldsymbol{\theta}}$-torsors over X are equivalent to Γ-equivariant G-principal bundles (called (Γ, G)-*bundles* in this article) over Y, where $Y \to X$ is a Galois covering with Galois group Γ. Based on this work, we generalize the correspondence to Higgs bundles:

Theorem 1.1 (Theorem 3.6)**.** *Let $\mathcal{M}_H(X, \mathcal{G}_{\boldsymbol{\theta}})$ be the stack of tame parahoric $\mathcal{G}_{\boldsymbol{\theta}}$-Higgs bundles, and let $\mathcal{M}_H^{\boldsymbol{\rho}}(Y, \Gamma, G)$ be the stack of tame Γ-equivariant G-principal bundles of type $\boldsymbol{\rho}$, where $\boldsymbol{\rho}$ is a fixed set of representations $\{\rho_y : \Gamma_y \to T, y \in R\}$, for a set of points R of Y. Then we have an isomorphism*

$$\mathcal{M}_H^{\boldsymbol{\rho}}(Y, \Gamma, G) \cong \mathcal{M}_H(X, \mathcal{G}_{\boldsymbol{\theta}})$$

as algebraic stacks.

The notion of stability that we introduce for the tame parahoric Higgs bundles described above, is inspired by the works of A. Ramanathan [43, 44] on the construction of moduli spaces of semistable principal G-bundles over a smooth projective irreducible complex curve. Note that a modified version of this stability condition from an analytic perspective was used in [8] to establish the Hitchin-Kobayashi correspondence in the case of real reductive groups, as briefly reviewed in §1.1. For an appropriate notion of a *parahoric degree* of a $\mathcal{G}_{\boldsymbol{\theta}}$-torsor E (see Definition 4.1), the definition of this Ramanathan-stability is the following:

Definition 1.2 (Definition 4.3)**.** A parahoric $\mathcal{G}_{\boldsymbol{\theta}}$-torsor E is called *R-stable* (resp. *R-semistable*), if for

- any proper parabolic group $P \subseteq G$,
- any reduction of structure group $\varsigma : X \to E/\mathcal{P}_{\boldsymbol{\theta}}$ ($\mathcal{P}_{\boldsymbol{\theta}}$ is a parahoric group constructed from P),
- any nontrivial anti-dominant character $\chi : \mathcal{P}_{\boldsymbol{\theta}} \to \mathbb{G}_m$, which is trivial on the center of $\mathcal{P}_{\boldsymbol{\theta}}$,

one has

$$parh \deg E(\varsigma, \chi) > 0, \quad (\text{resp. } \geq 0).$$

An important remark to make here is that when one considers small weights, this definition coincides with the one of V. Balaji and C. Seshadri in [5, §6], and parahoric $\mathcal{G}_{\boldsymbol{\theta}}$-torsors in this case are precisely parabolic bundles. The notion of Ramanathan-stability for a parahoric $\mathcal{G}_{\boldsymbol{\theta}}$-Higgs bundle (E, φ) now assumes the compatibility of the Higgs field φ (Definition 4.5). With respect to the correspondence between parahoric Higgs bundles and equivariant Higgs bundles, we prove that this correspondence also holds under stability conditions:

Theorem 1.3 (Theorem 4.16)**.** *Let (E, φ) be a tame parahoric $\mathcal{G}_{\boldsymbol{\theta}}$-Higgs bundle over X, and let (F, ϕ) be the corresponding tame (Γ, G)-Higgs bundle over Y. Then, (E, φ) is R-stable (resp. R-semistable) if and only if (F, ϕ) is R-stable (resp. R-semistable).*

This is also the key property to construct the moduli space of tame parahoric $\mathcal{G}_{\boldsymbol{\theta}}$-Higgs bundles, as it implies that constructing this moduli space is equivalent to rather constructing the moduli space of R-stable (R-semistable) (Γ, G)-Higgs bundles:

Theorem 1.4 (Theorem 6.1)**.** *There exists a quasi-projective scheme $\mathfrak{M}_H^{Rss}(X, \mathcal{G}_{\boldsymbol{\theta}})$ as the moduli space for the moduli problem $\mathcal{M}_H^{Rss}(X, \mathcal{G}_{\boldsymbol{\theta}})$ of R-semistable tame $\mathcal{G}_{\boldsymbol{\theta}}$-Higgs bundles.*

In §5, we study further the R-stability condition introduced above. Since we work on general complex reductive groups, the center is generally non-trivial. In the definition of R-stability, we require that the anti-dominant character acts trivially on the center (see §4). However, this condition weakened, in particular not requiring that the anti-dominant character is acting trivially on the center, provides a modified notion of stability which we call *R_μ-stability* (Definitions 5.4 and 5.6). In the case when the group G is $\mathrm{GL}(n, \mathbb{C})$ and for an appropriate choice of μ, we show that this R_μ-stability condition is equivalent to the stability condition for a parabolic Higgs bundle as considered by C. Simpson in [47]. Furthermore, we prove the following relation between R-stability and R_μ-stability:

Proposition 1.5 (Proposition 5.7)**.** *Let E be a parahoric $\mathcal{G}_{\boldsymbol{\theta}}$-torsor that is R-stable (resp. R-semistable). Then there exists a canonical choice of $\mu \in \mathfrak{t}$, depending on the topological type of E, such that E is R_μ-stable (R_μ-semistable).*

We finally show that the moduli space of R-semistable tame parahoric Higgs bundles is equipped with a Poisson structure (see Proposition 7.1). This follows the strategy developed in [32] and involves the construction of an Atiyah sequence inducing a Lie algebroid structure on the tangent space of the moduli space of R-stable tame (Γ, G)-equivariant bundles of type $\boldsymbol{\rho}$. Again, looking back at the case when $G = \mathrm{GL}(n, \mathbb{C})$, our moduli space of R-semistable parahoric $\mathcal{G}_{\boldsymbol{\theta}}$-Higgs bundles over X coincides with the one from [6], where a parahoric Hitchin map was considered and was shown to be a Poisson map with abelian generic fibers; the moduli space considered in [6], however, is viewed as the cotangent space of the stack of parahoric torsors.

1.3. **Applications.** We close this introduction with a discussion about the possible applications and further directions in which the considerations of this article may evolve. In this article, we construct the moduli space of tame parahoric Higgs bundles with respect to a given complex reductive group. For the case when the group G is a real reductive group, an analogous approach as in [8] but for the algebraic moduli space can be used in order to obtain the construction of the moduli space also in this case. Also, this approach can be applied to construct the moduli space of tame parahoric local systems (see Remark 6.11), which is a special case of a Λ-module (see [47] or Definition 6.3).

Secondly, as has already been pointed out in [11, §6], the Dolbeault moduli space of semistable tame parahoric Higgs bundles for general complex reductive groups is expected to provide the correct setup in order to establish a bijective correspondence extending the correspondence of C. Simpson [47], and the existence of this moduli space is given in this paper. Furthermore, in [11, §6] the author includes a table describing the correspondence of the parameters involved in the correspondence, namely the parahoric weights and the eigenvalues of the Higgs field on the one hand, and the weights and monodromy of a logarithmic connection on the other hand. The Riemann-Hilbert correspondence for tame parahoric connections established in [11], also called in that article *"logahoric"* connections, provides the description of the corresponding Betti data as the G-version of the "*$\mathbb{R}$-filtered local systems*" in C. Simpson's work for the $\mathrm{GL}(n, \mathbb{C})$-case [47]; we refer the reader to [11, Remark 2] for this description, as well as to [8, §6] when referring to the parabolic situation. Therefore, we believe that the moduli space of semistable parahoric Higgs bundles constructed here is the correct choice in order to establish the *tame parahoric nonabelian Hodge correspondence* for general complex reductive groups. The construction of the moduli space of tame parahoric $\mathcal{G}_{\boldsymbol{\theta}}$-local systems is obtained analogously to the construction on the Higgs side outlined above (Remark 6.11).

We moreover expect that the tame parahoric case treated in this article can be used in order to construct algebraically the moduli space of parahoric Higgs bundles in the case of irregular singularities, thus referring to wild character varieties and the description of corresponding Stokes data (cf. [10, 12, 13]). This space has been constructed analytically in the case when $G = \mathrm{GL}(n, \mathbb{C})$ in [7], and a nonabelian Hodge correspondence for this moduli space was established combining results from [7] and [45]. The construction of moduli spaces of parahoric Higgs bundles for arbitrary complex groups is important also from the point of view of understanding the tamely ramified geometric Langlands correspondence as proposed in the work of S. Gukov and E. Witten [25, 26]. Namely, it is argued that the category of A-branes is equivalent to the derived category of coherent sheaves on the moduli stack of parabolic ${}^{L}G$-local systems, while the category of B-branes is equivalent to the derived category of $\mathcal{D}$ -modules on the moduli stack of parabolic G-bundles.

Notation. Throughout the article, we will be distinguishing the notation between the parahoric Higgs bundles and equivariant Higgs bundles as follows:

	Parahoric	Equivariant
Curve:	X	Y
Local coordinate:	z	w
Coordinate Ring:	A	B
Function field:	K	L
Torsor/Bundle:	E	F
Higgs field:	φ	ϕ
Reduction of structure group:	ς	σ
Character:	κ	χ

2. Parahoric Torsors and Equivariant Bundles

The notion of a parahoric subgroup is similar to that of a parabolic subgroup and can be described using the theory of affine buildings, also known as Bruhat-Tits buildings, originally developed by F. Bruhat and J. Tits in their series of articles [15, 16] (see also [51, 52, 54] for surveys on the structure of affine buildings). The word *parahoric* is a blend word between the words "parabolic" and "Iwahori". An Iwahori subgroup is a subgroup of a reductive algebraic group over a non-archimedean local field, analogous to Borel subgroups of an algebraic group. The seminal work of F. Bruhat and J. Tits *loc. cit.* is extending to the case of reductive algebraic groups over a local field the study of N. Iwahori and H. Matsumoto [29] on the Iwahori subgroups for the Chevalley groups over p-adic fields.

Parahoric group schemes $\mathcal{G}$ and parahoric $\mathcal{G}$-torsors over a smooth projective curve were introduced by G. Pappas and M. Rapoport in [41]. The notion of weight for such torsors was first defined by P. Boalch in [11], while V. Balaji and C. Seshadri in [5] introduced a notion of stability for parahoric $\mathcal{G}$-torsors for a collection of small weights chosen from the set of rational one-parameter subgroups of the group G, assuming that G is semisimple and simply connected.

In this section, we generalize the setup of V. Balaji and C. Seshadri. We set the definition for a parahoric torsor over a complex reductive group with a collection of (arbitrary) rational weights following [5] and [11], and see that parahoric torsors correspond to Γ-equivariant G-bundles, similarly to [5].

2.1. **Parahoric Group Schemes and Parahoric Torsors.** Let G be a connected complex reductive group. We fix a maximal torus T in G. Let $X(T) := \mathrm{Hom}(T, \mathbb{G}_m)$ be the character group and $Y(T) := \mathrm{Hom}(\mathbb{G}_m, T)$ be the group of one-parameter subgroups of T. Let

$$\langle \cdot, \cdot \rangle : Y(T) \times X(T) \to \mathbb{Z}$$

be the canonical pairing, that extends to $\mathbb{Q}$ by tensoring $Y(T)$ with $\mathbb{Q}$.

We will denote by R, the root system with respect to the maximal torus T. Given a root $r \in R$, there is a natural isomorphism

$$Lie(\mathbb{G}_a) \to (Lie(G))_r.$$

This isomorphism induces a natural homomorphism

$$u_r : \mathbb{G}_a \to G,$$

such that $tu_r(a)t^{-1} = u_r(r(t)a)$ for $t \in T$ and $a \in \mathbb{G}_a$. Denote by U_r the image of the homomorphism u_r, which is a closed subgroup.

Fix a co-character with coefficients in $\mathbb{Q}$, $\theta \in Y(T) \otimes_{\mathbb{Z}} \mathbb{Q}$, which we shall call a *weight*. Under differentiation, we can consider θ as an element in $\mathfrak{t}$, which is the Lie algebra of T. Define the integer $m_r(\theta) := \lceil -r(\theta) \rceil$, where $\lceil \cdot \rceil$ is the ceiling function and $r(\theta) := \langle \theta, r \rangle$. We then introduce the following:

Definition 2.1. Let $A := \mathbb{C}[[z]]$ and $K := \mathbb{C}((z))$. With respect to the above data, we define the *parahoric subgroup* G_θ of $G(K)$ as

$$G_\theta := \langle T(A), U_r(z^{m_r(\theta)}A), r \in R \rangle.$$

Denote by $\mathcal{G}_\theta$ the corresponding group scheme of G_θ, which is called the *parahoric group scheme.*

Remark 2.2. An equivalent analytic definition of the parahoric group G_θ was given in [11, §2.1] as

$$G_\theta := \{g \in G(K) \mid z^\theta g z^{-\theta} \text{ has a limit as } z \to 0 \text{ along any ray}\},$$

where $z^\theta := \exp(\theta \log(z))$.

A weight is called *small*, if $r(\theta) \leq 1$ for all roots $r \in R$. If we assume that G is simply connected and semisimple, given any weight θ, the parahoric group G_θ is conjugate to a parahoric subgroup G_{θ_0}, for θ_0 small, where the conjugation is taken in $G(K)$ (see [52], Section 3.1, p. 50); this is also the situation studied in [5].

Remark 2.3. Given a basis t_i of $\mathfrak{t}$, a rational weight θ (considering the corresponding element in $\mathfrak{t}$) can be written as $\theta = \sum \frac{a_i}{d_i} t_i$, where a_i and d_i are integers. Denote by d the least common multiple of d_i. We can assume that the denominators in the coefficients of t_i are equal to d. This integer d corresponds to the order of cyclic group Γ when we discuss the correspondence in §2.3.

The above construction is a local picture of parahoric group schemes. Now we will define the parahoric group schemes globally. Let X be a smooth algebraic curve over $\mathbb{C}$, and we also fix a reduced effective divisor D on X. In fact, the divisor D is a sum of s many distinct points. For each point $x \in D$, we equip it with a rational weight $\theta_x \in Y(T) \otimes_{\mathbb{Z}} \mathbb{Q}$. Denote by $\boldsymbol{\theta} := \{\theta_x, x \in D\}$ the collection of weights over the points in D.

Definition 2.4. Let $\boldsymbol{\theta}$ be a collection of weights over D. We define a group scheme $\mathcal{G}_{\boldsymbol{\theta}}$ over X by gluing the following local data

$$\mathcal{G}_{\boldsymbol{\theta}}|_{X \backslash D} \cong G \times (X \backslash D), \quad \mathcal{G}_{\boldsymbol{\theta}}|_{\mathbb{D}_x} \cong \mathcal{G}_{\theta_x}, x \in D,$$

where $\mathbb{D}_x$ is a formal disc around x. This group scheme $\mathcal{G}_{\boldsymbol{\theta}}$ will be called a *parahoric Bruhat-Tits group scheme.*

By [18, Lemma 3.18], the group scheme $\mathcal{G}_{\boldsymbol{\theta}}$ defined above is a smooth affine group scheme of finite type, flat over X. We will denote by $\mathrm{Bun}(X, \mathcal{G}_{\boldsymbol{\theta}})$, the category of parahoric $\mathcal{G}_{\boldsymbol{\theta}}$-torsors E over X.

Lemma 2.5 (Corollary 4.2.6 in [55])**.** *The category of $\mathcal{G}_{\boldsymbol{\theta}}$-torsors* $\mathrm{Bun}(X, \mathcal{G}_{\boldsymbol{\theta}})$ *has a natural stack structure. More precisely,* $\mathrm{Bun}(X, \mathcal{G}_{\boldsymbol{\theta}})$ *is an algebraic stack locally of finite type.*

2.2. **Equivariant Bundles.** Let Γ be a cyclic group of order d with generator γ, and let G be a connected complex reductive group. Define $B := \mathbb{C}[[w]]$ and $L := \mathbb{C}((w))$. There is a natural Γ-action on $\mathbb{D} := \mathrm{Spec}(B)$, such that $\gamma w = \xi w$, where ξ is a d-th root of unity. We first study the local picture of a (Γ, G)-bundle over $\mathbb{D}$.

Definition 2.6. A Γ-*equivariant* G-*torsor* over $\mathbb{D}$ is a G-torsor (or G-principal bundle) together with a lift of the action of Γ on the total space of F. A Γ-equivariant G-bundle is also called a (Γ, G)-*bundle*.

Since we work on an affine chart, a G-bundle F has the property that

$$F(\mathbb{D}) \cong G(B).$$

Therefore, a (Γ, G)-bundle F is equivalent to a representation $\rho : \Gamma \to G$. Note that Γ is a cyclic group. We can suppose that the representation ρ factors through T under a suitable conjugation. Then, a representation $\rho : \Gamma \to T$ corresponds to an element in $Y(T)$ with order d, that is,

$$\mathrm{Hom}(\Gamma, T) \cong \mathrm{Hom}(X(T), X(\Gamma)) = \mathrm{Hom}(X(T), \mathbb{Z}/d\mathbb{Z}) = Y(T)/d \cdot Y(T).$$

Therefore, a representation $\rho : \Gamma \to T$ uniquely determines a small weight $\theta \in Y(T) \otimes_{\mathbb{Z}} \mathbb{Q}$. One can analogously see that given a small weight, there is a representation ρ as above.

Let Y be a smooth algebraic curve over $\mathbb{C}$ equipped with a Γ-action.

Definition 2.7. A (Γ, G)-*bundle* over Y is a G-torsor (or G-principal bundle) F together with a lift of the action of Γ on the total space of F, which preserves the action of the group G.

We now return to the local picture of a (Γ, G)-bundle over Y. Given $y \in Y$, let Γ_y be the stabilizer group of the point y. Denote by R the set of points in Y, of which the stabilizer groups are nontrivial. As was discussed above, the Γ-action around $y \in R$ is given by a representation $\rho_y : \Gamma_y \to T$, such that

$$\gamma \cdot (u, g) \to (\gamma u, \rho_y(\gamma) g), \quad u \in \mathbb{D}_y, \gamma \in \Gamma_y,$$

where $\mathbb{D}_y$ is a Γ-invariant formal disc around y.

Definition 2.8. We say that a (Γ, G)-bundle F is *of type* $\boldsymbol{\rho} = \{\rho_y, y \in R\}$, if the corresponding representation is ρ_y for each $y \in R$. Denote by $\mathrm{Bun}^{\boldsymbol{\rho}}(Y, \Gamma, G)$ the category of (Γ, G)-bundles of type $\boldsymbol{\rho}$ over Y.

A (Γ, G)-bundle F over Y can be also understood from gluing the following local data. For each $y \in R$, we define $F_y := \mathbb{D}_y \times G$, such that the Γ_y-action is defined as

$$\gamma \cdot (u, g) \to (\gamma u, \rho_y(\gamma) g), \quad u \in \mathbb{D}_y, \gamma \in \Gamma_y,$$

and define $F_0 := (Y \backslash R) \times G$ with the Γ_y-structure

$$\gamma \cdot (u, g) \to (\gamma u, g), \quad u \in Y - R, \gamma \in \Gamma_y.$$

Therefore, a (Γ, G)-torsor F being of type $\boldsymbol{\rho}$, is equivalent to giving (Γ, G)-isomorphisms

$$\Theta_y : F_y|_{\mathbb{D}_y^\times} \to F_0|_{\mathbb{D}_y^\times}, \quad y \in R.$$

Note that given two transition functions Θ_1 and Θ_2, if there exist (Γ, G)-isomorphisms

$$\tau_y : F_y \to F_y, \quad \tau_0 : F_0 \to F_0,$$

such that $\Theta_1 = \tau_0 \Theta_2 \tau_y$, then the corresponding (Γ, G)-bundles are isomorphic. The above observation gives us the following theoretic isomorphism [5, Proposition 3.1.1]

$$\mathrm{Bun}^{\boldsymbol{\rho}}(Y, \Gamma, G) \cong [\prod_{y \in R} G(B) \backslash \prod_{y \in R} G(L) / G(\mathbb{C}[Y \backslash R])]^{\Gamma}.$$

On the other hand, given a reduced effective divisor $R \subseteq Y$ and a collection of cyclic groups $\{\Gamma_y | y \in R\}$, there is a canonical way to define a root stack $\mathscr{X}$ (see [17] for more details). Therefore,

(Γ, G)-bundles of type $\boldsymbol{\rho}$ over Y is equivalent to G-bundles of type $\boldsymbol{\rho}$ over the corresponding root stack $\mathcal{X}$, in other words,

$$\mathrm{Bun}^{\boldsymbol{\rho}}(Y, \Gamma, G) \cong \mathrm{Bun}^{\rho}(\mathcal{X}, G).$$

Since $\mathcal{X}$ is a Deligne-Mumford stack, $\mathrm{Bun}^{\boldsymbol{\rho}}(\mathcal{X}, G)$ has a natural stack structure. Furthermore, it is an algebraic stack locally of finite type by Artin's theorem [2, 27]. This gives us the following lemma.

Lemma 2.9. *The stack* $\mathrm{Bun}^{\boldsymbol{\rho}}(Y, \Gamma, G)$ *is an algebraic stack locally of finite type.*

2.3. **Correspondence.** We first work on the local chart. Let Γ be a cyclic group of order d with generator γ, and there is a natural Γ-action on $B = \mathbb{C}[[w]]$ defined by rotation. Defining a (Γ, G)-torsor F, is equivalent to giving a representation $\rho : \Gamma \to G$. Suppose that the representation ρ factors through T under a suitable conjugation. The representation $\rho : \Gamma \to T$ gives an element $\rho(\gamma)$ in T. Then, we can find an element $\Delta \in T(B)$ such that

$$\Delta(\gamma w) = \rho(\gamma)\Delta(w).$$

Here is a construction of the element Δ. As explained in §2.2, a representation $\rho : \Gamma \to T$ corresponds to a small weight $\theta \in Y(T) \otimes_{\mathbb{Z}} \mathbb{Q}$ with order d. Then, we define

$$\Delta(w) := w^{d \cdot \theta}.$$

Note that the element Δ is not unique. For example, $\Delta(w) = w^{d\cdot(\theta + k \cdot I)}$ also satisfies the condition, where I is the identity element and k is an arbitrary integer. In this case, the weight $\theta + k \cdot I$ is not small in general.

We assume that given a representation ρ, the corresponding small weight is θ and $\Delta(w) := w^{d\cdot\theta}$. Let F be a (Γ, G)-bundle of type ρ over B and denote by $\mathbb{U} := \mathrm{Aut}_{(\Gamma,G)}(F)$ the automorphism group. Given an element $\sigma \in \mathbb{U}$, let $\varsigma := \Delta^{-1}\sigma\Delta$. We then have

$$\varsigma(\gamma w) = \varsigma(w),$$

which means that ς is Γ-invariant. Therefore, it can be descended to an element $G(A)$ by substituting $z = w^d$, where $A = \mathbb{C}[[z]]$. Note that for each root $r \in R$, we have

$$\varsigma(w)_r = \sigma(w)_r w^{-d\cdot r(\theta)},$$

where the subscript r means that the element is in $U_r(B)$. Note that $\sigma(w)_r$ is a holomorphic function, and $\varsigma(w)$ is a Γ-invariant meromorphic function. Substituting $z = w^d$, we have

$$\varsigma(z)_r = \sigma(z)_r z^{-r(\theta)}.$$

Therefore, the order of the pole of $\varsigma(z)_r$ is bounded by $\lceil -r(\theta) \rceil$. Since $\varsigma(w)_r$ is Γ-invariant, we have $\varsigma(z)_r \in U_r(z^{m_r(\theta)}\mathbb{C}[[z]])$ for each $r \in R$. In conclusion, the element $\varsigma(z)$ is in G_θ. The above discussion implies the isomorphism

$$\mathbb{U} \cong G_\theta.$$

This also implies that a (Γ, G)-bundle F of type ρ over $\mathrm{Spec}(B)$ corresponds to a unique $\mathcal{G}_\theta$-torsor E over $\mathrm{Spec}(A)$, and this is an one-to-one correspondence.

Now we consider the correspondence globally. Let X be a smooth algebraic curve over $\mathbb{C}$ of genus $g \geq 2$ with a fixed reduced effective divisor D. We fix a collection of rational weights $\boldsymbol{\theta} = \{\theta_x, x \in D\}$, where $\theta_x \in Y(T) \otimes_{\mathbb{Z}} \mathbb{Q}$. Denote by d_x the denominator of the rational weight θ_x. The data $(X, D, (d_x)_{x\in D})$ uniquely determine a Galois covering $\pi : Y \to X$ with Galois group Γ such that

- D is the branch locus;
- $R := \pi^{-1}(D)$ is the ramification locus;
- the stabilizer group of $y = \pi^{-1}(x)$ is $\Gamma_y := \Gamma_{d_x}$, where $x \in D$ and Γ_{d_x} is the cyclic group of order d_x.

Let $\boldsymbol{\rho} := \{\rho_y, y \in R\}$ be a collection of representations $\rho_y : \Gamma_y \to T$. Given a (Γ, G)-bundle F over Y of type $\boldsymbol{\rho}$, the restriction $F_y := F|_{\mathbb{D}_y}$ is a (Γ, G)-bundle on a Γ-invariant formal disc $\mathbb{D}_y$. By the discussion above, the (Γ, G)-bundle F_y of type ρ_y corresponds to a unique $\mathcal{G}_{\theta_x}$-torsor of type θ_x on $\mathbb{D}_x$, where $\mathbb{D}_x$ is a formal disc around $x = \pi(y) \in D$ and θ_x is the rational weight corresponding to ρ_y. By gluing the local data $\{(F|_{Y\backslash R})^\Gamma, F_y, y \in R\}$ together, we get a $\mathcal{G}_{\boldsymbol{\theta}}$-torsor E over X, where $\boldsymbol{\theta} := \{\theta_x, x \in D\}$. This correspondence is actually a one-to-one correspondence.

Theorem 2.10 (Theorem 5.3.1 in [5])**.** *With respect to the notation above, there is an isomorphism*

$$\mathrm{Bun}^{\boldsymbol{\rho}}(Y, \Gamma, G) \cong \mathrm{Bun}(X, \mathcal{G}_{\boldsymbol{\theta}})$$

as stacks.

This theorem implies that the correspondence also holds as algebraic stacks locally of finite type.

Now we consider the general case. Let θ be a small weight corresponding to the representation ρ as discussed above. Define

$$\theta' := \theta + \vartheta \in Y(T) \otimes_{\mathbb{Z}} \mathbb{Q},$$

where ϑ is a weight such that $\gamma^\vartheta = I$. Let $\Delta'(w) := w^{\theta'}$ be an element in $T(B)$. We have

$$\Delta'(\gamma w) = (\gamma w)^{\theta+\vartheta} = \gamma^\theta w^{\theta+\vartheta} = \rho(\gamma)\Delta'(w),$$

which means that $\Delta'(w)$ is Γ-equivariant. Applying the same proof as for Theorem 2.10 to $\Delta'(w)$, we get the following isomorphism

$$\mathrm{Bun}^{\boldsymbol{\rho}}(Y, \Gamma, G) \cong \mathrm{Bun}(X, \mathcal{G}_{\boldsymbol{\theta}'}).$$

This observation provides the following proposition.

Proposition 2.11. *With respect to the notation above, there is an isomorphism of stacks*

$$\mathrm{Bun}(X, \mathcal{G}_{\boldsymbol{\theta}}) \cong \mathrm{Bun}(X, \mathcal{G}_{\boldsymbol{\theta}'}).$$

Note that this isomorphism can be also realized in terms of Hecke transformations; we refer to [8, §3.3] for a detailed exposition.

3. Tame Parahoric Higgs Bundles and Tame Equivariant Higgs bundles

In this section, we study tame parahoric Higgs bundles and tame equivariant Higgs bundles. We prove that there is a correspondence between tame parahoric $\mathcal{G}_{\boldsymbol{\theta}}$-Higgs bundles over X and tame Γ-equivariant G-Higgs bundles of type $\boldsymbol{\rho}$ over Y, and this correspondence implies the isomorphism of the corresponding algebraic stacks (Theorem 3.6).

3.1. Tame Parahoric Higgs Bundles.

In this subsection, we define tame parahoric $\mathcal{G}_{\boldsymbol{\theta}}$-Higgs bundles over smooth algebraic curves X following the notation from §2.1.

Let X be a smooth algebraic curve with a given reduced effective divisor D. Denote by K_X the canonical line bundle over X. Let G be a connected reductive complex group together with a set of weights $\boldsymbol{\theta} = \{\theta_x, x \in D\}$. Denote by $\mathcal{G}_{\boldsymbol{\theta}}$ the parahoric group scheme over X and let E be a $\mathcal{G}_{\boldsymbol{\theta}}$-torsor over X, for $\mathrm{Ad}(E)$ the adjoint bundle of E. Note that for a parahoric $\mathcal{G}_{\boldsymbol{\theta}}$-torsor E, we can define its adjoint bundle on each local chart and then glue everything together.

Definition 3.1. A *tame parahoric $\mathcal{G}_{\boldsymbol{\theta}}$-Higgs bundle* over X is a pair (E, φ), where

- E is a $\mathcal{G}_{\boldsymbol{\theta}}$-torsor over X;
- $\varphi \in H^0(X, \mathrm{Ad}(E) \otimes K_X(D))$ is a holomorphic section.

The section φ is called a *tame (parahoric) Higgs field.*

Remark 3.2. In Z. Yun's article [55, §4.3], the tame parahoric Higgs field φ is considered as a section of $\mathrm{Ad}(E)(D)$. In this paper, we slightly modify this definition and the section is taken from $H^0(X, \mathrm{Ad}(E) \otimes K_X(D))$.

Denote by $\mathcal{M}_H(X, \mathcal{G}_{\boldsymbol{\theta}})$ the set of tame parahoric $\mathcal{G}_{\boldsymbol{\theta}}$-Higgs bundles over X, which also has a natural stack structure. Furthermore, we have a natural forgetful morphism of stacks

$$\mathcal{M}_H(X, \mathcal{G}_{\boldsymbol{\theta}}) \to \mathrm{Bun}(X, \mathcal{G}_{\boldsymbol{\theta}}),$$

of which the fiber over a point $E \in \mathrm{Bun}(X, \mathcal{G}_{\boldsymbol{\theta}})$ is a finite module $H^0(X, \mathrm{Ad}(E) \otimes K_X(D))$. Therefore, the forgetful morphism is representable and of finite type. The above discussion provides the following lemma, with an adaptation of Z. Yun's proof for the same result (see [55, Lemma 4.3.5]):

Lemma 3.3. *The stack $\mathcal{M}_H(X, \mathcal{G}_{\boldsymbol{\theta}})$ is an algebraic stack locally of finite type.*

3.2. **Tame Equivariant Higgs Bundles.** Let Y be a smooth algebraic curve. Denote by K_Y the canonical line bundle over Y and let Γ be a finite group together with an action on Y. Denote by R a set of points of Y (also a divisor), such that the stabilizer group Γ_y is nontrivial for any $y \in R$. Let G be a simply connected and semisimple linear algebraic group.

Definition 3.4. Given a set of representations $\boldsymbol{\rho} = \{\rho_y : \Gamma_y \to T, y \in R\}$, a *tame* (Γ, G)*-Higgs bundle of type* ρ over Y is a pair (F, ϕ), where

- F is a (Γ, G)-torsor of type $\boldsymbol{\rho}$ over Y;
- ϕ is a Γ-equivariant holomorphic section in $H^0(Y, \mathrm{Ad}(F) \otimes K_Y(R))$.

The section ϕ is called a *tame Higgs field*.

Denote by $\mathcal{M}_H^{\boldsymbol{\rho}}(Y, \Gamma, G)$ the stack of tame (Γ, G)-Higgs bundles of type $\boldsymbol{\rho}$ over Y. As discussed in §2.2, the data (Y, R, Γ_y) uniquely determines a root stack $\mathcal{X}$. There is a canonical isomorphism of stacks

$$\mathcal{M}_H^{\boldsymbol{\rho}}(Y, \Gamma, G) \cong \mathcal{M}_H^{\boldsymbol{\rho}}(\mathcal{X}, G),$$

where $\mathcal{M}_H^{\boldsymbol{\rho}}(\mathcal{X}, G)$ is the stack of G-torsors over $\mathcal{X}$. Note that $\mathcal{X}$ is also a Deligne-Mumford stack. As an application of Artin's theorem [2], $\mathcal{M}_H^{\boldsymbol{\rho}}(\mathcal{X}, G)$ is an algebraic stack locally of finite type [49, Theorem 5.1]. Then, we have the following lemma.

Lemma 3.5. *The stack $\mathcal{M}_H^{\boldsymbol{\rho}}(Y, \Gamma, G)$ is an algebraic stack locally of finite type.*

3.3. **Correspondence.** We first work on a formal disc $\mathbb{D}_y = \mathrm{Spec}(B)$ around a point $y \in R \subseteq Y$, and we will use the same notation as in §2.3. Let Γ be a cyclic group of order d, and we have a natural Γ-action on $B = \mathbb{C}[[w]]$. Given $\rho : \Gamma \to T$ a representation, denote by $\theta \in Y(T) \otimes_{\mathbb{Z}} \mathbb{Q}$ the rational weight corresponding to ρ. Let $\Delta \in T(B)$ be the element such that $\Delta(\gamma w) = \rho(\gamma)\Delta(w)$.

Let F be a (Γ, G)-torsor of type ρ over B and denote by $\mathrm{Ad}(F)$ the adjoint bundle. Without loss of generality, suppose that $F = G \times \mathbb{D}_y$. Let ϕ be an element in $\mathfrak{g}(\mathbb{C}[[w]]) \cdot \frac{dw}{w}$, which can be considered as a section $H^0(\mathbb{D}_y, \mathrm{Ad}(F) \otimes \Omega^1_{\mathbb{D}}(y))$. This section will be also called a *tame Higgs field.* Assume that ϕ is Γ-equivariant, that is,

$$\phi(\gamma w) = \rho(\gamma)\phi(w)\rho^{-1}(\gamma).$$

Now consider $\varphi := \Delta^{-1}\phi\Delta$ by conjugating with the matrix Δ. Clearly, φ is Γ-invariant:

$$\varphi(\gamma w) = \varphi(w).$$

Therefore, $\varphi(w)$ descends to a section $\mathbb{D}_x \to \mathrm{Ad}(E) \otimes \Omega^1_{\mathbb{D}_x}$ by substituting $z = w^d$, where E is the $\mathcal{G}_\theta$-torsor corresponding to F. For each root $r \in R$, we have

$$\varphi(w)_r = \phi(w)_r w^{-d \cdot r(\theta)},$$

and then, taking $z = w^d$, we get

$$\varphi(z)_r = \phi(z)_r z^{-r(\theta)}.$$

Similar to the discussion in §2.3, the order of the pole of $\varphi(z)_r$ is bounded by $\lceil -r(\theta)\rceil$. Since $\varphi(z)_r$ is Γ-invariant, we have $\varphi(z)_r \in \mathfrak{u}_r(z^{m_r(\theta)}\mathbb{C}[[z]]) \otimes \frac{dz}{z}$ for each $r \in R$. In conclusion, the element $\varphi(z)$ is a section of $\mathrm{Ad}(E) \otimes \Omega^1_{\mathbb{D}_x}(x)$ over $\mathbb{D}_x$. The above, in fact, describes a one-to-one correspondence, thus

$$H^0(\mathbb{D}_y, \mathrm{Ad}(F) \otimes \Omega^1_{\mathbb{D}_y}(y))^\Gamma \cong H^0(\mathbb{D}_x, \mathrm{Ad}(E) \otimes \Omega^1_{\mathbb{D}_x}(x)).$$

We next consider the correspondence globally. The setup is still the same as in §2.3. Let X be a smooth algebraic curve with a fixed reduced effective divisor D. We fix a collection of rational weights $\boldsymbol{\theta} = \{\theta_x, x \in D\}$ and denote by d_x the denominator of the rational weight θ_x. Denote by $\pi : Y \to X$ the Galois covering determined by the above data. Let also $R \subseteq Y$ be the collection of pre-images of the points in D, which is the ramification divisor, and let $\boldsymbol{\rho} = \{\rho_y, y \in R\}$ be the set of representations corresponding to $\boldsymbol{\theta}$. We then have the following

Theorem 3.6. *With the notation above, there is an isomorphism*

$$\mathcal{M}_H^{\boldsymbol{\rho}}(Y, \Gamma, G) \cong \mathcal{M}_H(X, \mathcal{G}_{\boldsymbol{\theta}})$$

as algebraic stacks.

Proof. By Theorem 2.9, the stack of (Γ, G)-equivariant torsors over Y is isomorphic to the stack of $\mathcal{G}_{\boldsymbol{\theta}}$-torsors over X,

$$\mathrm{Bun}^{\boldsymbol{\rho}}(Y, \Gamma, G) \cong \mathrm{Bun}(X, \mathcal{G}_{\boldsymbol{\theta}}).$$

There are two natural forgetful morphisms of stacks

$$\mathcal{M}_H(X, \mathcal{G}_{\boldsymbol{\theta}}) \to \mathrm{Bun}(X, \mathcal{G}_{\boldsymbol{\theta}}) \quad \text{and} \quad \mathcal{M}_H^{\boldsymbol{\rho}}(Y, \Gamma, G) \to \mathrm{Bun}^{\boldsymbol{\rho}}(Y, \Gamma, G),$$

with fibers $H^0(X, \mathrm{Ad}(E) \otimes K_X(D))$ and $H^0(Y, \mathrm{Ad}(F) \otimes K_Y(R))^\Gamma$, for $E \in \mathrm{Bun}(X, \mathcal{G}_{\boldsymbol{\theta}})$ and $F \in \mathrm{Bun}^{\boldsymbol{\rho}}(Y, \Gamma, G)$ the corresponding (Γ, G)-torsor. Therefore, proving that $\mathcal{M}_H^{\boldsymbol{\rho}}(Y, \Gamma, G) \cong \mathcal{M}_H(X, \mathcal{G}_{\boldsymbol{\theta}})$ is equivalent to showing that

$$H^0(X, \mathrm{Ad}(E) \otimes K_X(D)) \cong H^0(Y, \mathrm{Ad}(F) \otimes K_Y(R))^\Gamma,$$

which is already proven at the beginning of this subsection. □

4. Stability Conditions

In this section, we study the stability conditions of tame parahoric $\mathcal{G}_{\boldsymbol{\theta}}$-Higgs bundles and tame Γ-equivariant G-Higgs bundles. We prove that these stability conditions are equivalent. The equivalence of the stability conditions helps us to construct the moduli space of tame parahoric $\mathcal{G}_{\boldsymbol{\theta}}$-Higgs bundles in §6.

The stability conditions we study is a generalization of V. Balaji and C. Seshadri's work [5]. Furthermore, the stability condition of tame parahoric $\mathcal{G}_{\boldsymbol{\theta}}$-Higgs bundles coincides with that in [8], which gives the Hitchin-Kobayashi correspondence.

4.1. **Tame Parahoric Higgs Bundles.** Let G be a connected complex reductive group. Let $\theta \in Y(T) \otimes_{\mathbb{Z}} \mathbb{Q}$ be a rational weight, and denote by $G_\theta \subseteq G(K)$ the parahoric group corresponding to θ. Recall that a parabolic subgroup P of G can be determined by a subset of roots $R_P \subseteq R$. We define the following parahoric group as a subgroup of $P(K)$

$$P_\theta := \langle T(A), U_r(z^{m_r(\theta)}A), r \in R_P \rangle.$$

Denote by $\mathcal{P}_\theta$ the corresponding group scheme over $\mathbb{D} = \mathrm{Spec}(A)$.

Now we consider the global picture. Let X be a smooth algebraic curve with reduced effective divisor D. Let $\boldsymbol{\theta} = \{\theta_x, x \in D\}$ be a collection of rational weights and define the group scheme $\mathcal{P}_{\boldsymbol{\theta}}$ over X by gluing the local data

$$\mathcal{P}_{\boldsymbol{\theta}}|_{\mathbb{D}_x} \cong P \times X \backslash D, \quad \mathcal{P}_{\boldsymbol{\theta}}|_{\mathbb{D}_x} \cong \mathcal{P}_{\theta_x}, x \in D.$$

By [18, Lemma 3.18], the group scheme $\mathcal{P}_{\boldsymbol{\theta}}$ is a smooth affine group scheme of finite type, flat over X and we have that $\mathcal{P}_{\boldsymbol{\theta}} \subseteq \mathcal{G}_{\boldsymbol{\theta}}$.

Let $\kappa : \mathcal{P}_{\boldsymbol{\theta}} \to \mathbb{G}_m$ be a character of group schemes over X. Then, there is a $\mathcal{P}_{\boldsymbol{\theta}}$-action on $\mathcal{G}_{\boldsymbol{\theta}} \times_X \mathbb{G}_m$, which is induced by the inclusion $\mathcal{P}_{\boldsymbol{\theta}} \subseteq \mathcal{G}_{\boldsymbol{\theta}}$ and the character κ. By taking the quotient $\mathcal{G}_{\boldsymbol{\theta}} \times_X \mathbb{G}_m/\mathcal{P}_{\boldsymbol{\theta}}$, we get a $\mathbb{G}_m$-torsor over $\mathcal{G}_{\boldsymbol{\theta}}/\mathcal{P}_{\boldsymbol{\theta}}$, which corresponds to a line bundle over $\mathcal{G}_{\boldsymbol{\theta}}/\mathcal{P}_{\boldsymbol{\theta}}$ and is denoted by $L^{\kappa}_{\boldsymbol{\theta}}$.

Now let E be a $\mathcal{G}_{\boldsymbol{\theta}}$-torsor. Given a character κ, we can define a line bundle over $E/\mathcal{P}_{\boldsymbol{\theta}}$, and we use the same notation $L^{\kappa}_{\boldsymbol{\theta}}$ for this line bundle. Let $\varsigma : X \to E/\mathcal{P}_{\boldsymbol{\theta}}$ be a reduction of the structure group. A line bundle $L^{\kappa}_{\boldsymbol{\theta}}(\varsigma) := \varsigma^* L^{\kappa}_{\boldsymbol{\theta}}$ over X can be then defined by pullback in the following diagram

$$\begin{tikzcd} \varsigma^* L^{\kappa}_{\boldsymbol{\theta}} \arrow[r, dotted] \arrow[d, dotted] & L^{\kappa}_{\boldsymbol{\theta}} \arrow[d] \\ X \arrow[r, "\varsigma"] & E/\mathcal{P}_{\boldsymbol{\theta}} \end{tikzcd}$$

If there is no ambiguity, we use the notation $L^{\kappa}_{\boldsymbol{\theta}}$ for the line bundle over X.

Definition 4.1. We define the *parahoric degree* of a $\mathcal{G}_{\boldsymbol{\theta}}$-torsor E with respect to a given reduction ς and a character κ as follows

$$parh \deg E(\varsigma, \kappa) = \deg(L^{\kappa}_{\boldsymbol{\theta}}) + \langle \boldsymbol{\theta}, \kappa \rangle,$$

where $\langle \boldsymbol{\theta}, \kappa \rangle := \sum_{x \in D} \langle \theta_x, \kappa \rangle$.

The following lemma provides an alternative way to understand characters of group schemes.

Lemma 4.2. *With the notation above, there is an isomorphism*

$$\mathrm{Hom}(\mathcal{P}_{\boldsymbol{\theta}}, \mathbb{G}_m) \cong \mathrm{Hom}(P, \mathbb{C}^*)$$

as sets.

Proof. Let $\kappa \in \mathrm{Hom}(\mathcal{P}_{\boldsymbol{\theta}}, \mathbb{G}_m)$ be a character, which is a morphism of schemes over X. Note that

$$\mathcal{P}_{\boldsymbol{\theta}}|_{X \backslash D} \cong P \times X \backslash D, \quad \mathbb{G}_m|_{X \backslash D} \cong \mathbb{C}^* \times X \backslash D.$$

Therefore, restricting the morphism κ to $X \backslash D$, we have

$$\kappa|_{X \backslash D} : \mathcal{P}_{\boldsymbol{\theta}}|_{X \backslash D} \to \mathbb{G}_m|_{X \backslash D}.$$

Since the character κ is a morphism of schemes over X, the restriction $\kappa|_{X \backslash D}$ uniquely determines a character $\chi : P \to \mathbb{C}^*$.

Now we will consider the opposite direction and show that a character $\chi : P \to \mathbb{C}^*$ will uniquely determine a morphism $\kappa : \mathcal{P}_{\boldsymbol{\theta}} \to \mathbb{G}_m$. As a morphism of schemes (or sheaves), we have the following commutative diagram

$$\begin{tikzcd} \mathcal{P}_{\boldsymbol{\theta}}(X \backslash D \cap \mathbb{D}_x) \arrow[r, "\kappa|_{X \backslash D}"] \arrow[d, "\psi_x"] & \mathbb{G}_m(X \backslash D \cap \mathbb{D}_x) \arrow[d, "id"] \\ \mathcal{P}_{\boldsymbol{\theta}}(\mathbb{D}_x \cap X \backslash D) \arrow[r, "\kappa|_{\mathbb{D}_x}"] & \mathbb{G}_m(\mathbb{D}_x \cap X \backslash D), \end{tikzcd}$$

where ψ_x is the transition function defining the group scheme $\mathcal{P}_{\boldsymbol{\theta}}$. Since $\mathbb{G}_m$ is a constant group scheme over X, its transition function is trivial. Since X is connected, the commutativity of the above diagram will uniquely determine the morphism $\kappa|_{\mathbb{D}_x}$. Therefore, a character $\chi \in \mathrm{Hom}(P, \mathbb{C}^*)$ will uniquely determine a morphism $\kappa \in \mathrm{Hom}(\mathcal{P}_{\boldsymbol{\theta}}, \mathbb{G}_m)$. $\square$

Given this lemma, whenever there is no ambiguity we shall be using the same notation χ for characters in $\mathrm{Hom}(\mathcal{P}_{\boldsymbol{\theta}}, \mathbb{G}_m)$ and $\mathrm{Hom}(P, \mathbb{C}^*)$. A character of $\mathcal{P}_{\boldsymbol{\theta}}$ will be called an *anti-dominant character*, if the corresponding character $P \to \mathbb{C}^*$ is anti-dominant.

We introduce the following notion of stability for parahoric $\mathcal{G}_{\boldsymbol{\theta}}$-torsors inspired by the works of A. Ramanathan [43, 44] on the construction of moduli spaces of semistable principal G-bundles over a projective nonsingular irreducible complex curve.

Definition 4.3. A tame parahoric $\mathcal{G}_{\boldsymbol{\theta}}$-torsor E is called *R-stable* (resp. *R-semistable*), if for

- any proper parabolic group $P \subseteq G$,
- any reduction of structure group $\varsigma : X \to E/\mathcal{P}_{\boldsymbol{\theta}}$,
- any nontrivial anti-dominant character $\chi : \mathcal{P}_{\boldsymbol{\theta}} \to \mathbb{G}_m$, which is trivial on the center of $\mathcal{P}_{\boldsymbol{\theta}}$,

one has

$$parh \deg E(\varsigma, \chi) > 0, \quad (\text{resp. } \geq 0).$$

Remark 4.4. In [5, §6], V. Balaji and C. Seshadri consider small weights in their introduction of a notion of semistability and stability for torsors under parahoric Bruhat-Tits group schemes. As pointed out by P. Boalch in [11, p. 31], when the weights in $\boldsymbol{\theta}$ are small, parahoric $\mathcal{G}_{\boldsymbol{\theta}}$-torsors are exactly parabolic bundles. Therefore, V. Balaji and C. Seshadri defined the stability condition for parahoric torsors in terms of parabolic bundles (see [5, Definition 6.3.4]). Our Definition 4.3 is thus a generalization of the one by V. Balaji and C. Seshadri, and works for arbitrary weights.

Now we move to the stability condition for tame parahoric $\mathcal{G}_{\boldsymbol{\theta}}$-Higgs bundles over X. Let (E, φ) be a $\mathcal{G}_{\boldsymbol{\theta}}$-Higgs bundle over X, where $\varphi \in H^0(X, \mathrm{Ad}(E) \otimes K_X(D))$ is a section. A reduction of structure group $\varsigma : X \to E/\mathcal{P}_{\boldsymbol{\theta}}$ is said to be *compatible with the tame Higgs field* φ, if there is a lifting $\varphi' : X \to \mathrm{ad}(\varsigma^* E) \otimes K_Y$, such that the following diagram commutes

$$\begin{tikzcd} & \mathrm{Ad}(\varsigma^* E) \otimes K_X(D) \arrow[d, hook] \\ X \arrow[ur, dotted, "\varphi'"] \arrow[r, "\varphi"] & \mathrm{Ad}(E) \otimes K_X(D) \end{tikzcd}$$

Definition 4.5. A tame parahoric $\mathcal{G}_{\boldsymbol{\theta}}$-Higgs bundle (E, φ) is called *R-stable* (resp. *R-semistable*), if for

- any proper parabolic group $P \subseteq G$,
- any reduction of structure group $\varsigma : X \to E/\mathcal{P}_{\boldsymbol{\theta}}$ compatible with φ,
- any nontrivial anti-dominant character $\chi : \mathcal{P}_{\boldsymbol{\theta}} \to \mathbb{G}_m$, which is trivial on the center of $\mathcal{P}_{\boldsymbol{\theta}}$,

one has

$$parh \deg E(\varsigma, \chi) > 0, \quad (\text{resp. } \geq 0).$$

Remark 4.6. Recall that a classical non-parabolic Higgs bundle (E, φ) over X is stable, if for any φ-invariant subbundle $F \subseteq E$, one has $\frac{\deg F}{\mathrm{rk} F} < \frac{\deg E}{\mathrm{rk} E}$. For (E, φ) a $\mathcal{G}_{\boldsymbol{\theta}}$-Higgs torsor, a reduction of structure group $\varsigma : X \to E/\mathcal{P}_{\boldsymbol{\theta}}$ compatible with the tame Higgs field φ is actually giving a "φ-invariant subbundle".

4.2. **Tame Equivariant Higgs Bundles.** Let Y be a smooth algebraic curve with a Γ-action. Denote by $R \subseteq Y$ the set of points such that the stabilizer group Γ_y is nontrivial for each $y \in R$.

Let F be a (Γ, G)-torsor of type $\boldsymbol{\rho}$ over Y. Let $P \subseteq G$ be a parabolic subgroup. Given a Γ-equivariant reduction $\sigma : Y \to F/P$, the pullback $\sigma^* F$ is a (Γ, P)-torsor over Y. Let $\chi : P \to \mathbb{C}^*$ be a character. We may define a line bundle $L^\chi_{\boldsymbol{\rho}}$ over Y given by the pullback of the following diagram

$$\begin{tikzcd} \sigma^* L^\chi_{\boldsymbol{\rho}} \arrow[r, dotted] \arrow[d, dotted] & L^\chi_{\boldsymbol{\rho}} \arrow[d] \\ Y \arrow[r, "\sigma"] & F/P \end{tikzcd}$$

Definition 4.7. We define the *degree* of a (Γ, G)-bundle F with respect to a given reduction σ and a character χ as

$$\deg F(\sigma, \chi) = \deg(L^\chi_{\boldsymbol{\rho}}).$$

Definition 4.8. A (Γ, G)-bundle F of type $\boldsymbol{\rho}$ is called *R-stable* (resp. *R-semistable*), if for

- any proper parabolic group $P \subseteq G$,

- any reduction of structure group $\varsigma : X \to E/P$,
- any nontrivial anti-dominant character $\chi : P \to \mathbb{C}^*$, which is trivial on the center of P,

one has

$$\deg F(\sigma, \chi) > 0, \quad (\text{resp. } \geq 0).$$

Let (F, ϕ) be a tame (Γ, G)-Higgs bundle over Y. Given a reduction $\sigma : Y \to F/P$, we have a natural morphism $\sigma^* F \to F$ given by the pullback. This induces a natural morphism

$$\mathrm{Ad}(\sigma^* F) = \sigma^* F \times_P \mathfrak{p} \to \sigma^* F \times_G \mathfrak{g} \to F \times_G \mathfrak{g} = \mathrm{Ad}(F).$$

A reduction of structure group $\sigma : Y \to F/P$ is *compatible with* the tame Higgs field ϕ, if there is a lifting $\phi' : Y \to \mathrm{Ad}(\sigma^* F) \otimes K_Y(R)$ such that the following diagram commutes

$$\begin{tikzcd} & \mathrm{Ad}(\sigma^* F) \otimes K_Y(R) \arrow[d, hook] \\ Y \arrow[ur, dotted, "\phi'"] \arrow[r, "\phi"] & \mathrm{Ad}(F) \otimes K_Y(R) \end{tikzcd}$$

Definition 4.9. A tame (Γ, G)-Higgs bundle (F, ϕ) over Y is called *R-stable* (resp. *R-semistable*), if for

- any proper parabolic subgroup P of G,
- any Γ-equivariant reduction of structure group $\sigma : Y \to F/P$ compatible with ϕ,
- any nontrivial anti-dominant character $\chi : P \to \mathbb{C}^*$, which is trivial on the center of P,

one has

$$\deg F(\sigma, \chi) > 0, \quad (\text{resp. } \geq 0).$$

Given a G-bundle F over Y, let $\mathrm{Ad}(F)$ be its adjoint bundle. As a vector bundle, one can consider the μ-stability condition for $\mathrm{Ad}(F)$ defined by the slope μ of vector bundles. A. Ramanathan proved the following result.

Corollary 4.10 (Corollary 3.18 in [43])**.** *The G-bundle F is R-semistable if and only if its adjoint bundle* $\mathrm{Ad}(F)$ *is semistable.*

Remark 4.11. If G is semisimple, then F is R-stable if and only if $\mathrm{Ad}(F)$ is stable. However, if G is a complex reductive linear algebraic group, the equivalence only holds in the semistable case (see [1, Proposition 2.10]).

Let (F, ϕ) be a tame (Γ, G)-Higgs bundle over Y. It is natural to define the adjoint Γ-equivariant Higgs bundle $(\mathrm{Ad}(F), \phi)$. Then, Corollary 4.10 implies the following result at the level of Higgs pairs:

Corollary 4.12. *A tame (Γ, G)-Higgs bundle (F, ϕ) over Y is R-semistable if and only if its adjoint Γ-equivariant Higgs bundle $(\mathrm{Ad}(F), \phi)$ over Y is semistable, where the stability is given by the slope.*

4.3. **Equivalence of Stability Conditions.** In this subsection, we will prove that a tame parahoric $\mathcal{G}_{\boldsymbol{\theta}}$-Higgs bundle (E, φ) over X is R-stable (resp. R-semistable) if and only if the corresponding tame (Γ, G)-Higgs bundle (F, ϕ) over Y is R-stable (resp. R-semistable). By Definition 4.3 and Definition 4.9, we have to show the following correspondences

1. every parabolic subgroup $P \subseteq G$ corresponds to a subgroup scheme $\mathcal{P}_{\boldsymbol{\theta}} \subseteq \mathcal{G}_{\boldsymbol{\theta}}$;
2. there is a one-to-one correspondence between characters $\mathrm{Hom}(P, \mathbb{C}^*)$ and $\mathrm{Hom}(\mathcal{P}_{\boldsymbol{\theta}}, \mathbb{G}_m)$;
3. every reduction of structure group $\varsigma : X \to E/\mathcal{P}_{\boldsymbol{\theta}}$ corresponds to a unique Γ-equivariant reduction of structure group $\sigma : Y \to F/P$;
4. $\deg F(\sigma, \chi) \geq 0$ (resp. $>$) if and only if $parh \deg E(\varsigma, \chi) \geq 0$ (resp. $>$).

The first condition holds from the Definitions in §4.1. Lemma 4.2 gives us the correspondence between characters. The third and fourth conditions will be proved in Lemma 4.13 and 4.14 below.

We review first the construction of a (Γ, G)-torsor F by gluing local data from §2.2. For each $y \in R$, we define $F_y := \mathbb{D}_y \times G$ and define $F_0 := (Y \backslash R) \times G$ together with the Γ-actions

$$\gamma \cdot (u, g) \to (\gamma u, \rho_y(\gamma) g), \quad u \in \mathbb{D}_y, \gamma \in \Gamma_y,$$
$$\gamma \cdot (u, g) \to (\gamma u, g), \quad u \in Y \backslash R, \gamma \in \Gamma_y.$$

By giving (Γ, G)-isomorphisms

$$\Theta_y : F_y|_{\mathbb{D}_y^\times} \to F_0|_{\mathbb{D}_y^\times}, \quad y \in R,$$

we can define a (Γ, G)-torsor F of type $\boldsymbol{\rho}$ over Y. Note that a (Γ, G)-isomorphism Θ_y satisfies the following condition

$$\Theta_y(\gamma w) = \rho(\gamma) \Theta_y(w).$$

A (Γ, G)-torsor is usually not Γ-invariant. However, there is a canonical way to construct a Γ-invariant G-torsor F' based on F. Here is the construction. On each punctured disc $\mathbb{D}_y^\times$, we define a new (Γ, G)-isomorphism

$$\Theta'_y(w) := \Delta(w)^{-1} \Theta_y(w),$$

where Δ is the element in $T(\mathbb{D}_y^\times)$ such that $\Delta(\gamma w) = \rho(\gamma)\Delta(w)$ (see §2.3). Denote by F' the (Γ, G)-torsor given by the isomorphisms Θ'_y. It is easy to check that

$$\Theta'_y(\gamma w) = \Theta'_y(w),$$

which implies that F' is Γ-invariant. Actually, the correspondence we reviewed in §2.3 is given by taking the Γ-invariant of F'.

Lemma 4.13. *Let E be a parahoric $\mathcal{G}_{\boldsymbol{\theta}}$-torsor over X. Denote by F the corresponding (Γ, G)-torsor over Y. Then, we have*

$$\mathrm{Hom}(X, E/\mathcal{P}_{\boldsymbol{\theta}}) \cong \mathrm{Hom}^\Gamma(Y, F/G),$$

which describes a one-to-one correspondence between reductions of structure group of E and Γ-equivariant reductions of structure group of F.

Proof. Given a Γ-equivariant reduction $\sigma : Y \to F/G$, we have

$$\sigma(\gamma w) = \rho(\gamma)\sigma(w)\rho(\gamma)^{-1},$$

where $\gamma \in \Gamma$. A Γ-equivariant reduction uniquely determines a Γ-equivariant reduction

$$\sigma' : Y \to F'/G$$

of F', which is the Γ-invariant G-torsor constructed from F. Since F' is Γ-invariant, we have

$$\sigma'(\gamma u) = \sigma'(u).$$

By taking invariants under Γ, we get a section $\varsigma : X \to E/\mathcal{P}_{\boldsymbol{\theta}}$. This process also holds in the other direction. ☐

Lemma 4.14. *Let E be a $\mathcal{P}_{\boldsymbol{\theta}}$-torsor over X and denote by F the corresponding (Γ, G)-bundle of type $\boldsymbol{\rho}$ over Y. Let d be the order of the group Γ. Let $\varsigma : X \to E/\mathcal{P}_{\boldsymbol{\theta}}$ be a reduction of structure group of E, and let $\sigma : Y \to F/P$ be the corresponding Γ-equivariant reduction of structure group of F. Let $\chi : P \to \mathbb{C}^*$ be a character. Then, the following identity holds*

$$d \cdot parh \deg E(\varsigma, \chi) = \deg F(\sigma, \chi).$$

Proof. For simplicity, we assume that $D = \{x\}$ and $R = \{y\}$ are singletons; the proof when D and R have finitely many points is entirely analogous. The stabilizer group of $y \in R$ is Γ, which is a cyclic group of order d.

Recalling the construction of F', we define new transition function $\Theta'_y(w)$ such that

$$\Theta'_y(w) := \Delta(w)^{-1} \Theta_y(w),$$

where $\Theta_y : F_y|_{\mathbb{D}_y^\times} \to F_0|_{\mathbb{D}_y^\times}$ is the transition function of F and Δ is the matrix in $T(\mathbb{D}_y)$ such that $\Delta(\gamma w) = \rho(\gamma)\Delta(w)$. Therefore, Δ contributes to the degree $\sum_{r\in R} d \cdot r(\theta_x)$. Therefore, we have

$$\deg F(\sigma, \chi) = \deg F'(\sigma', \chi) + d \cdot \langle \theta_x, \chi\rangle.$$

Since F' is a Γ-invariant G-bundle, we have

$$d \cdot \deg E(\varsigma, \chi) = \deg F'(\sigma', \chi).$$

By adding the term $d \cdot \langle \theta_x, \chi\rangle$ on both sides of the equation, we finally get

$$\deg F(\sigma, \chi) = d \cdot parh \deg E(\varsigma, \chi),$$

which is the desired identity. □

Theorem 4.15. *Let E be a parahoric $\mathcal{G}_{\boldsymbol{\theta}}$-torsor over X, and let F be the corresponding (Γ, G)-bundle over Y. Then, E is R-stable (resp. R-semistable) if and only if F is R-stable (resp. R-semistable).*

Proof. We will prove that if a (Γ, G)-bundle F is R-stable, then the corresponding parahoric $\mathcal{G}_{\boldsymbol{\theta}}$-torsor E is also R-stable. The other direction can be proved similarly. By Definition 4.3, we have to show that for every maximal parabolic subgroup $P \subseteq G$, every nontrivial anti-dominant character $\chi : \mathcal{P}_{\boldsymbol{\theta}} \to \mathbb{G}_m$ and every reduction of structure group $\varsigma : X \to E/\mathcal{P}_{\boldsymbol{\theta}}$, we have

$$parh \deg E(\varsigma, \chi) > 0.$$

From Lemma 4.2, the character χ can be considered as an anti-dominant character $P \to \mathbb{C}^*$, while Lemma 4.13 provides that the reduction of structure group ς corresponds to a Γ-equivariant reduction $\sigma : Y \to F/P$. Assuming that F is R-stable, this means that

$$\deg F(\sigma, \chi) > 0.$$

By Lemma 4.14, we finally get that

$$parh \deg E(\varsigma, \chi) = \frac{1}{d} \deg F(\sigma, \chi) > 0,$$

and so E is R-stable. □

Theorem 4.16. *Let (E, φ) be a tame parahoric $\mathcal{G}_{\boldsymbol{\theta}}$-Higgs bundle over X, and let (F, ϕ) be the corresponding tame (Γ, G)-Higgs bundle over Y. Then, (E, φ) is R-stable (resp. R-semistable) if and only if (F, ϕ) is R-stable (resp. R-semistable).*

Proof. The only thing we have to show is that a reduction $\varsigma : X \to E/\mathcal{P}_{\boldsymbol{\theta}}$ is compatible with φ if and only if the corresponding Γ-equivariant reduction $\sigma : Y \to E/P$ is compatible with ϕ. We still prove one direction, that if σ is compatible with ϕ, then ς is compatible with φ. The other direction can be proved similarly.

By the assumption, there is a lifting $\phi' : Y \to \mathrm{Ad}(\sigma^* F) \otimes K_Y(R)$, such that the following diagram commutes

$$\begin{tikzcd} & \mathrm{Ad}(\sigma^* F) \otimes K_Y(R) \arrow[d, hook] \\ Y \arrow[ur, dotted, "\phi'"] \arrow[r, "\phi"] & \mathrm{Ad}(F) \otimes K_Y(R) \end{tikzcd}$$

Note that $\sigma^* F$ is a (Γ, P)-bundle, for which the Γ-action is induced from that on F. Therefore, ϕ' is also Γ-equivariant, that is,

$$\phi'(\gamma w) = \rho(\gamma)\phi'(w)\rho(\gamma)^{-1}.$$

By §3.3, the Γ-equivariant tame Higgs field ϕ' will correspond to a tame Higgs field $\varphi' : X \to Ad(\varsigma^* E) \otimes K_X(D)$, where φ' is a lifting of φ, and this correspondence is a one-to-one correspondence. □

Corollary 4.12 and Theorem 4.16 now give the following

Corollary 4.17. *Let (E, φ) be a tame parahoric $\mathcal{G}_{\boldsymbol{\theta}}$-Higgs bundle over X. Denote by (F, ϕ) the corresponding tame (Γ, G)-Higgs bundle over Y. Then, (E, φ) is R-semistable if and only if the adjoint bundle $(\mathrm{Ad}(F), \phi)$ over Y is semistable.*

5. R_μ-stability of Tame Parahoric Higgs Bundles

In this section, we first study tame parahoric GL_n-Higgs bundles in detail as an important example. In §5.2, we introduce the R_μ-stability condition for tame parahoric GL_n-Higgs bundles. Compared to the R-stability condition from Definition 4.3, we do not require that the anti-dominant character acts trivially on the center (see Definition 5.4). We show that for a specific μ, the notion of R_μ-stability coincides with the stability condition for a parabolic Higgs bundle as considered by C. Simpson in [47] (see Proposition 5.5). In §5.3, we generalize the R_μ-stability condition to the case of arbitrary complex reductive groups G, and prove that we can find a canonical μ such that R-stability is equivalent to R_μ-stability (Proposition 5.7).

5.1. **Correspondence.** Let $T \subset \mathrm{GL}$ be the subgroup of diagonal matrices, which is a maximal torus in GL. Let θ be a rational weight in $Y(T) \otimes_{\mathbb{Z}} \mathbb{Q}$. For convenience, we consider θ as an element in $\mathfrak{t}_{\mathbb{Q}}$. Let $\{t_i\}_{1 \leq i \leq n}$ be a basis of $\mathfrak{t}_{\mathbb{Q}}$. Then, $\theta = \sum_{i=1}^{n} \frac{a_i}{d} t_i$, for integers a_i and d, or as a matrix

$$\theta = \begin{pmatrix} \frac{a_1}{d} & & \\ & \ddots & \\ & & \frac{a_n}{d} \end{pmatrix}.$$

Furthermore, we assume that $a_i \leq a_j$, if $i \leq j$. In the case when θ is small, one has that $0 \leq a_i \leq d$. Denote by GL_θ the parahoric subgroup of $\mathrm{GL}(K)$ defined as

$$\mathrm{GL}_\theta := \langle T(A), U_{ij}(z^{\lceil a_j - a_i \rceil} A) \rangle,$$

where U_{ij} is the unipotent group corresponding to the (i, j)-entry. As an example, for $n = 2$ and $\theta = 0 \cdot t_1 + \frac{1}{2} t_2$, the matrix in GL_θ can be written as

$$\begin{pmatrix} A & A \\ zA & A \end{pmatrix}.$$

The corresponding representation $\rho : \Gamma \to T$ is given as

$$\rho(\gamma) = \begin{pmatrix} \xi^{a_1} & & \\ & \ddots & \\ & & \xi^{a_n} \end{pmatrix},$$

where $\xi = e^{\frac{2\pi i}{d}}$. In the local coordinate w, we define the following matrix

$$\Delta(w) := w^\theta = \begin{pmatrix} w^{a_1} & & \\ & \ddots & \\ & & w^{a_n} \end{pmatrix}.$$

Clearly, we have

$$\Delta(\gamma w) = \rho(\gamma) \Delta(w).$$

Let F be a (Γ, GL)-torsor of type θ over $\mathbb{D}_y$, and denote by $\mathbb{U} := \mathrm{Aut}_{(\Gamma, \mathrm{GL})}(F)$ the automorphism of F. We take an element $\sigma \in \mathbb{U}$. Note that in the GL-case, we can consider $\sigma = (\sigma_{ij})$ as a matrix. Define $\varsigma := \Delta^{-1} \sigma \Delta$. Then, we have

$$\varsigma_{ij}(w) = \sigma_{ij}(w) w^{-(a_i - a_j)},$$

and it is easy to check that

$$\varsigma(\gamma w) = \varsigma(w),$$

which means that ς is Γ-invariant. Substituting z by w^d, then $\varsigma_{ij}(z)$ descends to a meromorphic function on $\mathbb{D}_x = \mathrm{Spec}(A)$. The order of the pole of $\varsigma_{ij}(z)$ is bounded by $\lceil \frac{a_j - a_i}{d} \rceil$. Therefore, we have $\varsigma(z) \in \mathrm{GL}_\theta$. The same argument as for Theorem 2.10, provides the following corollary:

Corollary 5.1. *With respect to the above notation, there is an isomorphism*

$$\mathrm{Bun}^{\boldsymbol{\rho}}(Y, \Gamma) \cong \mathrm{Bun}(X, \mathrm{GL}_{\boldsymbol{\theta}})$$

as algebraic stacks.

Now let $\phi = (\phi_{ij})$ be an element in $\mathfrak{gl}(\mathbb{C}[[w]]) \cdot \frac{dw}{w}$, where $\phi_{ij}(w)$ is a holomorphic function corresponding to the (i,j)-entry. The element ϕ can be considered as a tame Higgs field over $\mathbb{D}_y$. Suppose that ϕ is Γ-equivariant, that is, $\phi(\gamma w) = \rho(\gamma)\phi(w)\rho^{-1}(\gamma)$. Define $\varphi := \Delta^{-1}\phi\Delta$. Then, we have

$$\varphi(\gamma w) = \varphi(w),$$

which implies that φ is Γ-invariant. Therefore, φ can be descended to a section $\mathbb{D}_x \to \mathfrak{gl}_\theta \otimes \Omega^1_{\mathbb{D}_x}(x)$ by substituting z by $= w^d$, where $\mathfrak{gl}_\theta$ is the Lie algebra of GL_θ. For each entry, we have

$$\varphi(z)_{ij} = \phi(z)_{ij} z^{-(a_i - a_j)}.$$

Globally, let E be a $\mathrm{GL}_{\boldsymbol{\theta}}$-torsor over X, where $\boldsymbol{\theta} = \{\theta_x, x \in D\}$ is a collection of rational weights over points in the divisor D. Let F be the corresponding Γ-equivariant bundle of type $\boldsymbol{\rho} = \{\rho_y, y \in R\}$ over Y, where R is the pre-image of D. Then, the above discussion implies that there is a one-to-one correspondence between Γ-equivariant tame Higgs fields of F and tame Higgs fields of E,

$$H^0(Y, \mathcal{E}nd(F) \otimes \Omega^1_Y(R))^\Gamma \cong H^0(X, E \otimes \Omega^1_X(D)).$$

As an application of Theorem 3.6, we have the following equivalence.

Corollary 5.2. *One has*

$$\mathcal{M}^{\boldsymbol{\rho}}_H(Y, \Gamma) \cong \mathcal{M}_H(X, \mathrm{GL}_{\boldsymbol{\theta}})$$

as algebraic stacks.

Remark 5.3. V. Mehta and C. Seshadri introduced parabolic bundles to study Γ-equivariant bundles [33]. In fact, there is a correspondence between Γ-equivariant bundles over Y and parabolic bundles over X [9, 35]. Therefore, a parahoric GL_θ-torsor over X can be considered as a parabolic bundle over X. Furthermore, the equivalence can be extended to Higgs bundles [35]. More precisely, there is a correspondence between Γ-equivariant Higgs bundles over Y and parabolic Higgs bundles over X. Thus, a tame parahoric GL_θ-Higgs bundle over Y corresponds to a parabolic Higgs bundle over X. For primary reference on parabolic Higgs bundles, we refer the reader to [14, 24, 47]; examples of parabolic G-Higgs bundles for complex groups G as special parabolic Higgs bundle data are demonstrated in [30, 31].

5.2. **Stability Condition.** In this section, under the assumption that all the weights θ are small, we define the R_μ-stability condition of a parahoric torsor in this GL-case and show that it refers to the parabolic degree of a parabolic bundle.

Let $\boldsymbol{\theta}$ be a set of rational weights over the points in $D \in X$. Let V be a free vector bundle over X, that is, let $V \cong X \times \mathbb{C}^n$. Viewing V as a sheaf, there is a natural GL_θ-action on V. Let $P \subset \mathrm{GL}$ be a parabolic group. Denote by $\mathcal{P}_{\boldsymbol{\theta}}$ the corresponding parahoric group scheme. Let $\kappa \in \mathrm{Hom}(\mathcal{P}_{\boldsymbol{\theta}}, \mathbb{G}_m)$ be a character. By Lemma 4.2, we know that it is equivalent to a character $\chi : P \to \mathbb{C}^*$. Let E be a $\mathcal{GL}_{\boldsymbol{\theta}}$-torsor over X. Let $\varsigma : X \to E/\mathcal{P}_{\boldsymbol{\theta}}$ be a reduction of structure group.

For the calculation below, we assume that the divisor $D = \{x\}$ consists of a single point and so we shall denote $\boldsymbol{\theta} = \theta_x$ for simplicity; the proof for finitely many points in D is then analogous. There is a natural GL-action on $\mathbb{C}^n$, which induces a P-action on this vector space. Note that the differentiation

$d\chi$ is an element in $\mathrm{Hom}(\mathfrak{p}, \mathbb{C}) \subseteq \mathrm{Hom}(\mathfrak{t}, \mathbb{C})$, where $\mathfrak{p}$ is the Lie algebra of P. Denote by $s_\chi \in \mathfrak{t}$ the dual of $d\chi$, and let $\lambda_1, \dots, \lambda_r$ be the eigenvalues of s_χ. We assume that $\lambda_1 < \lambda_2 < \cdots < \lambda_r$. Define

$$V_j := \ker(\lambda_j I - s_\chi)$$

to be the eigenspace of λ_j. In fact, we consider V_j as a trivial vector bundle over X. Then, we have

$$\deg L^\chi_{\theta_x} = \sum_{j=1}^r \lambda_j \deg \mathcal{V}_j,$$

where $\mathcal{V}_j := E \times_{\mathrm{GL}_{\theta_x}} V_j$. In fact, the reduction ς and the character χ determine sub-torsors (or subbundles) of E, and all sub-torsors of E can be constructed in this way.
Suppose that $\theta_x : \mathbb{C}^* \to T$ is given as

$$z \to \begin{pmatrix} z^{\alpha_1} & & \\ & \ddots & \\ & & z^{\alpha_n} \end{pmatrix}.$$

Let $\alpha_1, \dots, \alpha_l$ be all (distinct) eigenvalues of the differentiation $d\theta_x$ such that $\alpha_1 > \alpha_2 > \cdots > \alpha_l$, and denote by A_i the eigenspace of α_i, $1 \leq i \leq l$. Then, we have

$$\langle \theta_x, \chi \rangle = \sum_{i,j} \alpha_i \lambda_j \dim(A_i \cap V_j).$$

For $B_i := A_i \oplus \cdots \oplus A_1$, we have a natural filtration of V_j given by

$$B_l \cap V_j \supseteq B_{l-1} \cap V_j \supseteq \cdots \supseteq B_1 \cap V_j \supseteq \{0\},$$

together with a collection of weights

$$\alpha_l \leq \alpha_{l-1} \leq \cdots \leq \alpha_1.$$

This defines a parabolic structure of $\mathcal{V}_j$ on the fiber of the point x.

Given the discussion above, we now see

$$\begin{aligned} parh \deg E(\varsigma, \chi) &= \sum_{j=1}^r \lambda_j \deg \mathcal{V}_j + \sum_{i,j} \alpha_i \lambda_j \dim(A_i \cap V_j) \\ &= \sum_{j=1}^r \lambda_j (\deg \mathcal{V}_j + \sum_{i=1}^l \alpha_i \dim(A_i \cap V_j)) \qquad (*) \\ &= \sum_{j=1}^r \lambda_j par \deg \mathcal{V}_j. \end{aligned}$$

Therefore, the parahoric degree in the case GL *coincides with the parabolic degree* of the induced parabolic bundle.

We can find a unique rational weight ϖ such that for any character χ defined as above, we have

$$\langle \varpi, \chi \rangle = \sum_{j=1}^r \lambda_j \dim(V_j).$$

Definition 5.4. Given a rational number μ, a parahoric $\mathrm{GL}_{\boldsymbol{\theta}}$-torsor E is called *R_μ-stable* (resp. *R_μ-semistable*), if for

- any proper parabolic group $P \subseteq G$,
- any reduction of structure group $\varsigma : X \to E/\mathcal{P}_{\boldsymbol{\theta}}$,
- any nontrivial anti-dominant character $\chi : \mathcal{P}_{\boldsymbol{\theta}} \to \mathbb{G}_m$ (not necessarily trivial on the center),

one has

$$parh \deg E(\varsigma, \chi) - \langle \mu\varpi, \chi \rangle \geq 0.$$

Note that this definition can be generalized to define stability in the case of any *real reductive group*; we refer the reader to [8] for more information.

Let E be a parahoric $\mathrm{GL}_{\boldsymbol{\theta}}$-torsor. As was discussed in Remark 5.3, E can be considered as a parabolic bundle, so we shall keep the same notation E to refer to it. The following proposition relates the stability condition of a parahoric $\mathrm{GL}_{\boldsymbol{\theta}}$-Higgs torsor with the stability condition of C. Simpson from [47] for parabolic Higgs bundles.

Proposition 5.5. *Let E be a parahoric $\mathrm{GL}_{\boldsymbol{\theta}}$-torsor. Denote by $\mu := \frac{par \deg E}{rk(E)}$ the parabolic slope of E as a parabolic bundle. Then, E is R_μ-stable (resp. R_μ-semistable) if and only if E is stable (resp. semistable) as a parabolic bundle. Furthermore, let (E,φ) be a tame parahoric $\mathrm{GL}_{\boldsymbol{\theta}}$-Higgs bundle. Then, (E,φ) is R_μ-stable (resp. R_μ-semistable) if and only if (E,φ) is stable (resp. semistable) as a parabolic Higgs bundle.*

Proof. We only give the proof for parahoric $\mathrm{GL}_{\boldsymbol{\theta}}$-torsors, and the proof for the case of parahoric $\mathrm{GL}_{\boldsymbol{\theta}}$-Higgs torsors is, in fact, the same. We follow the same notation as was used in this subsection, and still work for a single point $D = \{x\}$ to simplify exposition. Define

$$W_j := V_j \oplus V_{j-1} \oplus \cdots \oplus V_1, \quad B_i = A_i \oplus A_{i-1} \oplus \cdots \oplus A_1,$$

which are vector spaces or trivial bundles over X. Then, let

$$\mathcal{W}_j := E \times_{\mathrm{GL}_{\theta_x}} W_j.$$

Clearly, we then have

$$\deg \mathcal{W}_j = \deg \mathcal{V}_j + \cdots + \deg \mathcal{V}_1.$$

Therefore, the following equation applies

$$\sum_{j=1}^{r} \lambda_j \deg \mathcal{V}_j = \lambda_r \deg \mathcal{W}_r + \sum_{j=1}^{r-1} (\lambda_j - \lambda_{j+1}) \deg \mathcal{W}_j.$$

Note also that $\mathcal{W}_j$ can be realized as a vector bundle equipped with a natural parabolic structure. Similar calculations then provide the formulas

$$\begin{aligned}\sum_{i,j} \alpha_i \lambda_j \dim(A_i \cap V_j) &= \sum_{i=1}^{l} \left(\lambda_r \dim(A_i \cap W_r) + \sum_{j=1}^{r-1} (\lambda_j - \lambda_{j+1}) \dim(A_i \cap W_j) \right) \\ &= \lambda_r \sum_{i=1}^{l-1} (\alpha_i - \alpha_{i+1}) \dim(B_i) + \sum_{i=1}^{l-1} \sum_{j=1}^{r-1} (\alpha_i - \alpha_{i+1})(\lambda_j - \lambda_{j+1}) \dim(B_i \cap W_j)\end{aligned}$$

and

$$\langle \mu\varpi, \chi \rangle = \mu \sum_{j=1}^{r} \lambda_j \dim(V_j) = \mu \lambda_r \dim(W_r) + \mu \sum_{j=1}^{r-1} (\lambda_j - \lambda_{j+1}) \dim(W_j).$$

Therefore, we have

$$\begin{aligned}
parh \deg E(\varsigma,\chi) - \langle \mu\varpi, \chi\rangle =&\lambda_r \deg \mathcal{W}_r + \sum_{j=1}^{r-1}(\lambda_j-\lambda_{j+1})\deg \mathcal{W}_j \\
&+\lambda_r \sum_{i=1}^{l-1}(\alpha_i-\alpha_{i+1})\dim(B_i) + \sum_{i=1}^{l-1}\sum_{j=1}^{r-1}(\alpha_i-\alpha_{i+1})(\lambda_j-\lambda_{j+1})\dim(B_i\cap W_j) \\
&-\left(\mu\lambda_r\dim(W_r) + \mu\sum_{j=1}^{r-1}(\lambda_j-\lambda_{j+1})\dim(W_j)\right) \\
=&\sum_{r=1}^{j-1}(\lambda_j-\lambda_{j+1})(\deg \mathcal{W}_j + \sum_{i=1}^{l-1}(\alpha_i-\alpha_{i+1})\dim(B_i\cap W_j) - \mu\dim(W_j)) \\
&+\lambda_r\left(\deg \mathcal{W}_r + \sum_{i=1}^{l-1}(\alpha_i-\alpha_{i+1})\dim(B_i) - \mu\dim(W_r)\right) \\
=&\sum_{r=1}^{j-1}(\lambda_j-\lambda_{j+1})(par\deg \mathcal{W}_j - \mu\dim(W_j)) + \lambda_r(par\deg\mathcal{W}_r - \mu\dim(W_r)) \\
=&\sum_{r=1}^{j-1}(\lambda_j-\lambda_{j+1})(par\deg \mathcal{W}_j - \mu\dim(W_j)).
\end{aligned}$$

Note that $\lambda_j - \lambda_{j+1} < 0$. We thus conclude that $parh \deg E(\varsigma,\chi) - \langle \mu\varpi, \chi\rangle \geq 0$ if and only if for any parabolic subbundle $\mathcal{W}$ of E, one has

$$\frac{par\deg\mathcal{W}}{rk(\mathcal{W})} < \frac{par\deg E}{rk(E)}.$$

This exactly means that E is semistable as a parabolic bundle. □

5.3. **R_μ-stability condition for general reductive groups.** In this section, we will define a stability condition for $\mathcal{G}_{\boldsymbol{\theta}}$-torsor with general complex reductive group G. In particular, in the case of $G = \mathrm{GL}_n$ we will recover the R_μ-stability condition in previous section. We have similarly a definition of R_μ-stability of the case GL_n depending on a choice of $\mu \in \mathfrak{t}$.

Definition 5.6. Fixing an element $\mu \in \mathfrak{t}$ in Lie algebra of maximal torus, a parahoric $\mathcal{G}_{\boldsymbol{\theta}}$-torsor E with on X is called R_μ*-stable* (resp. R_μ*-semistable*) if for

- any proper parabolic subgroup P of G,
- any reduction $\varsigma : X \to E/\mathcal{P}_{\boldsymbol{\theta}}$,
- any nontrivial anti-dominant character $\chi : \mathcal{P}_{\boldsymbol{\theta}} \to \mathbb{G}_m$ (not necessarily trivial on the center),

one has

$$parh \deg E(\varsigma,\chi) - \langle \mu, \chi\rangle > 0 \quad (\text{resp. } \geq 0).$$

The relation between these two definitions is given by the following choice of μ:

Proposition 5.7. *Let E be a parahoric $\mathcal{G}_{\boldsymbol{\theta}}$-torsor that is R-stable (resp. R-semistable), then there exists a canonical choice of $\mu \in \mathfrak{t}$, depending on the topological type of E, such that E is R_μ-stable (R_μ-semistable).*

Proof. Since when χ is anti-dominant and trivial on the center $\mathfrak{z}$ there is nothing to prove, we only need to find the pairing of μ with χ nontrivial in $\mathfrak{z}^*$. We find a base of $\mathfrak{z}^*$ and call them $\chi_1, \ldots, \chi_n$. By abuse of notation we can also think of them as anti-dominant characters.

Here we view $\chi_i : \mathcal{G}_{\boldsymbol{\theta}} \to \mathbb{G}_m$ and $\varsigma_0 : X \to E/\mathcal{G}_{\boldsymbol{\theta}} = X$ the parabolic reduction when we choose parabolic subgroup to be G itself. Therefore we may define μ by their actions on χ_i:

$$\langle \mu, \chi_i \rangle := parh \deg E(\varsigma_0, \chi_i).$$

In this case, the parahoric degree is nothing but

$$parh \deg E(\varsigma_0, \chi_i) = \deg(L_{\boldsymbol{\theta}}^{\chi_i}(\varsigma_0)) + \langle \boldsymbol{\theta}, \chi_i \rangle,$$

where $L_{\boldsymbol{\theta}}^{\chi_i}(\varsigma_0)$ is the line bundle over X we defined in §4.1, and thus the definition of μ depends only on the topology of E. Then for every combination $\chi = \sum a_i \chi_i + \delta$, where δ is an anti-dominant element that acts trivially on center, we have

$$parh \deg E(\varsigma, \chi) - \langle \mu, \chi \rangle = \sum_{i=1}^{n} a_i \left(parh \deg E(\varsigma, \chi_i) - \langle \mu, \chi_i \rangle \right) + parh \deg E(\varsigma, \delta)$$

We will show that

$$parh \deg E(\varsigma, \chi_i) - \langle \mu, \chi_i \rangle = 0$$

for each i. Thus the positivity of the number $parh \deg E(\varsigma, \chi) - \langle \mu, \chi \rangle$ depends only on the positivity of $parh \deg E(\varsigma, \delta)$. This will imply the equivalence of stability (semistability).

Thus the only thing remaining is that $parh \deg E(\varsigma_0, \chi_i) = parh \deg E(\varsigma, \chi_i)$ for every reduction $\varsigma : X \to E/\mathcal{P}_{\boldsymbol{\theta}}$. This is because the characters $\chi_i : \mathcal{P}_\theta \to \mathbb{G}_m$ coming from elements in $\mathfrak{z}$ can be lifted to the same ones $\chi_i : \mathcal{G}_{\boldsymbol{\theta}} \to \mathbb{G}_m$. Therefore, we have

$$L_{\boldsymbol{\theta}}^{\chi_i}(\varsigma_0) \cong L_{\boldsymbol{\theta}}^{\chi_i}(\varsigma),$$

and thus

$$parh \deg E(\sigma_0, \chi_i) = parh \deg E(\sigma, \chi_i).$$

This completes the proof. □

Remark 5.8. When $G = \mathrm{GL}_n$, we may find the direct equivalence of R_μ-stability and R-stability by Proposition 5.7. Note that we only have one generator χ_1 of $\mathfrak{z}$ given by the diagonal matrix, which also satisfies $\langle \varpi, \chi_1 \rangle = n$ (see §5.2). With the same idea as in Proposition 5.7, we need to find the element $\mu\varpi$, where μ is a rational number, such that

$$\langle \mu\varpi, \chi_1 \rangle = parh \deg E(\varsigma_0, \chi_1).$$

With the same calculation of formula (∗), we find the following

$$parh \deg E(\varsigma_0, \chi_1) = par\deg(E).$$

Therefore, $\mu = \frac{par \deg(E)}{n}$. This recovers the definition of μ in Proposition 5.5.

6. Moduli Space of Tame Parahoric Higgs Bundles

We proceed next with the construction of the moduli space of R-semistable tame parahoric $\mathcal{G}_{\boldsymbol{\theta}}$-Higgs bundles (Theorem 6.1). Although we only give the construction of the moduli space in the case of Higgs bundles, our approach also works for tame parahoric $\mathcal{G}_{\boldsymbol{\theta}}$-local systems (see Remark 6.11).

6.1. **Main Result.** We define the moduli problem of R-semistable parahoric $\mathcal{G}_{\boldsymbol{\theta}}$-Higgs bundles over X

$$\mathcal{M}_H^{Rss}(X, \mathcal{G}_{\boldsymbol{\theta}}) : (\mathrm{Sch}/\mathbb{C})^{\mathrm{op}} \to \mathrm{Sets}$$

as follows. For each $\mathbb{C}$-scheme S, the set $\widetilde{\mathcal{M}}_H^{Rss}(X, \mathcal{G}_{\boldsymbol{\theta}})(S)$ is defined as the collection of pairs (E, φ) up to isomorphism such that

- E is a $\mathcal{G}_{\boldsymbol{\theta}}$-torsor flat over S;
- $\phi : X_S \to \mathrm{Ad}(E) \otimes \pi_X^* K_X(D)$ is a Γ-invariant section, where $\pi_X : X_S \cong X \times S \to X$ is the natural projection;

- for each point $s \in S$, the restriction $(E|_{X\times s}, \varphi|_{X\times s})$ is an R-semistable $\mathcal{G}_{\boldsymbol{\theta}}$-Higgs bundle over X.

The main theorem in this section is the following:

Theorem 6.1. *There exists a quasi-projective scheme $\mathfrak{M}_H^{Rss}(X,\mathcal{G}_{\boldsymbol{\theta}})$ as the moduli space for the moduli problem $\mathcal{M}_H^{Rss}(X,\mathcal{G}_{\boldsymbol{\theta}})$ of R-semistable tame parahoric $\mathcal{G}_{\boldsymbol{\theta}}$-Higgs bundles.*

We will prove this theorem in the following steps.

(1) There is a natural way to define the moduli problem $\mathcal{M}_{H,\boldsymbol{\rho}}^{Rss}(Y,\Gamma,G)$ of R-semistable tame (Γ,G)-Higgs bundles of type $\boldsymbol{\rho}$ over Y. The moduli problems $\mathcal{M}_H^{Rss}(X,\mathcal{G}_{\boldsymbol{\theta}})$ and $\mathcal{M}_{H,\boldsymbol{\rho}}^{Rss}(Y,\Gamma,G)$ have natural stack structures. By Theorem 3.6 and Theorem 4.16, we have

$$\mathcal{M}_H^{Rss}(X,\mathcal{G}_{\boldsymbol{\theta}}) \cong \mathcal{M}_{H,\boldsymbol{\rho}}^{Rss}(Y,\Gamma,G).$$

Therefore, it is equivalent to construct the moduli space of R-semistable (Γ,G)-Higgs bundles of type $\boldsymbol{\rho}$. Remember that we consider $X = Y/\Gamma$, and X can be understood as the coarse moduli space of the stack $[Y/\Gamma]$. By definition, Γ-equivariant bundles over Y are exactly bundles over $[Y/\Gamma]$. Then, we have the following correspondences.

$$G\text{-Higgs bundles over } [Y/\Gamma] \longleftrightarrow (\Gamma,G)\text{-Higgs bundles over } Y$$

With respect to the above correspondence, it is enough to construct the moduli space of Higgs bundles on $[Y/\Gamma]$.

The above discussion shows that constructing the moduli space of R-semistable tame parahoric $\mathcal{G}_{\boldsymbol{\theta}}$-Higgs bundles over X is equivalent to constructing the moduli space $\mathfrak{M}_{H,\boldsymbol{\rho}}^{Rss}([Y/\Gamma],G)$ of R-semistable tame G-Higgs bundles over $[Y/\Gamma]$, and the following steps are devoted to constructing the moduli space $\mathfrak{M}_{H,\boldsymbol{\rho}}^{Rss}([Y/\Gamma],G)$.

(2) Given a tame (Γ,G)-Higgs bundle (F,ϕ), we may consider its adjoint Γ-equivariant Higgs bundle $(\mathrm{Ad}(F),\phi)$. A. Ramanathan proved that a principal G-bundle F is R-semistable if and only if its adjoint bundle $\mathrm{Ad}(F)$ is semistable (see [43, Corollary 3.18] and [22, Theorem 2.2]). This property can be generalized to G-Higgs bundles (see Corollary 4.12 and 4.17). In conclusion, the R-stability condition for a (Γ,G)-Higgs bundle is equivalent to the stability condition of $(\mathrm{Ad}(F),\phi)$. Thus, we will construct the moduli space $\mathfrak{M}_H^{ss}([Y/\Gamma])$ of semistable Higgs bundles over $[Y/\Gamma]$.

(3) F. Nironi in [37] defined the $\mathcal{E}$-stability condition for sheaves over $[Y/\Gamma]$, where $\mathcal{E}$ is a generating sheaf. With a good choice of $\mathcal{E}$, the $\mathcal{E}$-stability condition for bundles on $[Y/\Gamma]$ is equivalent to the stability of Γ-equivariant bundles on Y, where the stability condition is defined by the bundle slope. Therefore, the existence of the moduli space of $\mathcal{E}$-semistable Higgs bundles on $[Y/\Gamma]$ will imply the existence of the moduli space $\mathfrak{M}_H^{ss}([Y/\Gamma])$ in the second step.

Note, lastly, that the construction of the moduli space of Higgs bundles on smooth projective varieties (over $\mathbb{C}$) was first given by C. Simpson [48], and this construction was generalized by the second author in [49] to the case of (tame) projective Deligne-Mumford stacks (over any algebraically closed field); we refer the reader to the aforementioned articles for further information.

The above discussion is the basic idea to construct the moduli space $\mathfrak{M}_H^{Rss}(X,\mathcal{G}_{\boldsymbol{\theta}})$.

In §6.2 below, we construct first the moduli space $\mathfrak{M}_H^{\mathcal{E}ss}([Y/\Gamma])$ of $\mathcal{E}$-semistable Higgs bundles over $[Y/\Gamma]$. With a good choice of $\mathcal{E}$, it gives us the moduli space of semistable Higgs bundles over $[Y/\Gamma]$, which gives the construction of the moduli space $\mathfrak{M}_H^{ss}([Y/\Gamma])$ of semistable adjoint Higgs bundles $(\mathrm{Ad}(F),\phi)$. In §6.3, we start with $\mathfrak{M}_H^{ss}([Y/\Gamma])$ and construct the moduli space $\mathfrak{M}_{H,\boldsymbol{\rho}}^{Rss}([Y/\Gamma],G)$ of R-semistable tame G-Higgs bundles (of type $\boldsymbol{\rho}$) over $[Y/\Gamma]$. As we discussed above, the moduli space $\mathfrak{M}_{H,\boldsymbol{\rho}}^{Rss}([Y/\Gamma],G)$ is exactly the moduli space $\mathfrak{M}_H^{Rss}(X,\mathcal{G}_{\boldsymbol{\theta}})$ of R-semistable $\mathcal{G}_{\boldsymbol{\theta}}$-Higgs bundles over X.

6.2. **Moduli Space of Higgs Bundles on Quotient Stacks.** In this section, we give the construction of the moduli space $\mathfrak{M}_H^{\mathcal{E}ss}([Y/\Gamma])$ of $\mathcal{E}$-semistable Higgs bundles on quotient stacks $[Y/\Gamma]$, where $\mathcal{E}$ is a generating sheaf.

Note that a Higgs field $\phi : F \to F \otimes \Omega^1_{\mathcal{X}}$ is equivalent to a morphism $\mathrm{Sym}(T_{\mathcal{X}}) \to \mathcal{E}nd(F)$. Denote by $\Lambda = \mathrm{Sym}(T_{\mathcal{X}})$ the corresponding sheaf of differential graded algebras (see [48, 49]). Therefore, Higgs bundles are a special case of Λ-modules, and the moduli space of $\mathcal{E}$-semistable Higgs bundles (that is, GL-Higgs bundles) $\mathfrak{M}_H^{\mathcal{E}ss}([Y/\Gamma])$ on $[Y/\Gamma]$ is constructed in the same way as the moduli space of Λ-modules. We will briefly review the construction of the moduli space of Λ-modules and refer the reader to [49] for more details.

Let $\mathcal{X} := [Y/\Gamma]$ be the quotient stack, and denote by $\pi : \mathcal{X} \to X$ the coarse moduli space. A locally free sheaf $\mathcal{E}$ is a *generating sheaf*, if for any coherent sheaf F on $\mathcal{X}$, the morphism

$$\theta_{\mathcal{E}}(F) : \pi^*\pi_*\mathcal{H}om(\mathcal{E}, F) \otimes \mathcal{E} \to F$$

is surjective. By [40, Proposition 5.2], there exists a generating sheaf $\mathcal{E}$ for $\mathcal{X}$ in our case. A very important property of the generating sheaf is that the functor

$$\begin{aligned} F_{\mathcal{E}} : \mathrm{Coh}(\mathcal{X}) &\to \mathrm{Coh}(X) \\ F &\mapsto \pi_*\mathcal{H}om(\mathcal{E}, F) \end{aligned}$$

induces a closed immersion of quot-schemes (see [40, Lemma 6.2])

$$\begin{aligned} F_{\mathcal{E}} : \mathrm{Quot}(G, \mathcal{X}, P) &\to \mathrm{Quot}(F_{\mathcal{E}}(G), X, P) \\ [G \to F] &\mapsto [F_{\mathcal{E}}(G) \to F_{\mathcal{E}}(F)], \end{aligned}$$

where G is a coherent sheaf over $\mathcal{X}$ and P is an integer polynomial as the "Hilbert polynomial". This property implies that $\mathrm{Quot}(G, \mathcal{X}, P)$ is a projective scheme. Therefore, we can construct the moduli space of coherent sheaves on $\mathcal{X}$ with respect to a "good" stability condition. This "good" stability condition is called the $\mathcal{E}$-stability. First, we define the *modified Hilbert polynomial*. Let F be a coherent sheaf on $\mathcal{X}$. The *modified Hilbert polynomial* $P_{\mathcal{E}}(F, m)$ is defined as

$$P_{\mathcal{E}}(F, m) = \chi(\mathcal{X}, F \otimes \mathcal{E}^\vee \otimes \pi^*\mathcal{O}_X(m)), \quad m \gg 0.$$

Definition 6.2. A pure coherent sheaf F over $\mathcal{X}$ is *$\mathcal{E}$-semistable* (resp. *$\mathcal{E}$-stable*), if for every proper subsheaf $F' \subseteq F$ we have

$$p_{\mathcal{E}}(F') \leq p_{\mathcal{E}}(F) \quad (\text{resp. } p_{\mathcal{E}}(F') < p_{\mathcal{E}}(F)),$$

where $p_{\mathcal{E}}(\bullet)$ is the reduced modified Hilbert polynomial.

Let Λ be a sheaf of graded algebras over $\mathcal{X}$. A *coherent Λ-sheaf* F is a coherent sheaf (with respect to the $\mathcal{O}_{\mathcal{X}}$-structure) over $\mathcal{X}$ together with a left Λ-action. A subsheaf $F' \subseteq F$ is a *Λ-subsheaf*, if we have $\Lambda \otimes F' \subseteq F'$. There are several ways to understand "an action of Λ". Usually an action of Λ on F means that we have a morphism

$$\Lambda \to \mathcal{E}nd(F).$$

Equivalently, this morphism can be interpreted as

$$\Lambda \otimes F \to F.$$

The above morphism induces a morphism $\mathrm{Gr}_1(\Lambda) \otimes F \to F$ naturally. If $\mathrm{Gr}_1(\Lambda)$ is a locally free sheaf, then it corresponds to a morphism $F \to F \otimes \mathrm{Gr}_1(\Lambda)^*$.

Definition 6.3. A Λ-sheaf F is *$\mathcal{E}$-semistable* (resp. *$\mathcal{E}$-stable*), if F is a pure coherent sheaf and for any Λ-subsheaf $F' \subseteq F$ with $0 < \mathrm{rk}(F') < \mathrm{rk}(F)$, we have

$$p_{\mathcal{E}}(F') \leq p_{\mathcal{E}}(F), \quad (\text{resp. } <).$$

Now we are ready to construct the moduli space of p-semistable Λ-sheaves. Let k be a positive integer. We consider the quot-scheme $Q_1 := \mathrm{Quot}(\Lambda_k \otimes V \otimes G, \mathcal{X}, P)$, which parameterizes quotients $[\Lambda_k \otimes V \otimes G \to F]$ such that

- P is an integer polynomial taken as the modified Hilbert polynomial;
- V is a $\mathbb{C}$-vector space of dimension $P(N)$, where N is a large enough positive integer;
- G is $\pi^*\mathcal{O}_X(-N)$.

We construct the moduli space using the following steps.

(1) There exists a closed subscheme $Q_2 \subseteq Q_1$ such that any point $[\rho : \Lambda_k \otimes V \otimes G \to F]$ *admits a factorization.* More precisely, the quotient ρ has the following factorization

$$\begin{tikzcd} \Lambda_k \otimes V \otimes G \arrow[rr, "\rho"] \arrow[dr, "1\otimes\rho'"'] & & \mathcal{F} \\ & \Lambda_k \otimes F \arrow[ur, "\phi_k"'] & \end{tikzcd}$$

such that
- the induced morphism $\rho' : V \otimes G \to F$ is an element in $\mathrm{Quot}(V \otimes G, \mathcal{X}, P)$;
- $\phi_k : \Lambda_k \otimes F \to F$ is a morphism.

This condition gives a Λ_k-structure on the coherent sheaf F.

(2) Let $[\rho : \Lambda_k \otimes V \otimes G \to F] \in Q_2$ be a point. Denote by $[\rho' : V \otimes G \to F]$ the quotient in the factorization of ρ. The morphism ρ also induces a morphism $\Lambda_1 \otimes V \otimes G \to F$. Denote by K the kernel of the quotient map

$$0 \to K \to V \otimes G \to F \to 0.$$

Denote by $Q_3 \subseteq Q_2$ the closed subscheme of Q_2 such that the induced map $\Lambda_1 \otimes K \to \Lambda_1 \otimes V \otimes G \to F$ is trivial.

(3) As discussed in the last step, a point $[\rho : \Lambda_k \otimes V \otimes G \to F] \in Q_3$ induces a morphism $\Lambda_1 \otimes F \to F$. This morphism also induces the following ones

$$\underbrace{(\Lambda_1 \otimes \cdots \otimes \Lambda_1)}_{j} \otimes F \to F,$$

for each positive integer j. Denote by K_j the kernel of the surjection

$$\underbrace{\Lambda_1 \otimes \cdots \otimes \Lambda_1}_{j} \to \Lambda_j \to 0.$$

Thus, we have a natural map

$$K_j \otimes F \to F.$$

For each positive integer j, there exists a closed subscheme $Q_{4,j} \subseteq Q_3$ such that if $[\rho] \in Q_{4,j}$, then the corresponding map $K_j \otimes F \to F$ is trivial.

(4) Denote by $Q_{4,\infty}$ the intersection of $Q_{4,j}$, for $j \geq 1$.

(5) The above steps show that a quotient $[\rho] \in Q_{4,\infty}$ gives a Λ-structure on F, and this Λ-structure will induce a Λ_k-structure on F. Note that this induced Λ_k-structure may not be the same as the given one. However, there is a closed subscheme $Q_5 \subseteq Q_{4,\infty}$ such that these two structures are the same.

(6) Let $Q_6 \subseteq Q_5$ be the open subscheme such that if $[\rho : \Lambda_k \otimes V \otimes G \to F] \in Q_6$, then we have $V \cong H^0(Y, F(N))$.

(7) There is an open subset $Q^{ss}_\Lambda \subseteq Q_6$ such that if $[\rho : \Lambda_k \otimes V \otimes G \to F] \in Q^{ss}_\Lambda$, then F is a p-semistable Λ-sheaf.

With respect to the above construction, the subset

$$Q^{ss}_\Lambda \subseteq \mathrm{Quot}(\Lambda_k \otimes V \otimes G, \mathcal{X}, P)$$

is a quasi-projective scheme, which parameterizes Λ-modules with modified Hilbert polynomial P. There is an induced $\mathrm{SL}(V)$-action on Q^{ss}_Λ. Given a point

$$[\rho : \Lambda_k \otimes V \otimes G \to F] \in \mathrm{Quot}(\Lambda_k \otimes V \otimes G, \mathcal{X}, P),$$

if F is $\mathcal{E}$-semistable (resp. $\mathcal{E}$-stable), then it is also semistable (resp. stable) in the sense of GIT [49, Lemma 6.4]. Define $\mathfrak{M}^{\mathcal{E}ss}_{\Lambda}(\mathcal{X}, P) := Q^{ss}_{\Lambda}/\mathrm{SL}(V)$.

Theorem 6.4 (Theorem 6.8 in [49]). *The quasi-projective scheme $\mathfrak{M}^{\mathcal{E}ss}_{\Lambda}(\mathcal{X}, P)$ is the coarse moduli space of $\mathcal{E}$-semistable Λ-sheaves with modified Hilbert polynomial P over $\mathcal{X}$, and the geometric points of $\mathfrak{M}^{\mathcal{E}ss}_{\Lambda}(\mathcal{X}, P)$ represents the equivalence classes of $\mathcal{E}$-semistable Λ-sheaves, where the equivalence is given by the Jordan-Hölder filtration and known as the S-equivalence.*

There is a well-known correspondence between Γ-equivariant bundles over Y and parabolic bundles over the coarse moduli space X [35]. It was observed by F. Nironi in [37], that with a good choice of the generating sheaf $\mathcal{E}$, the $\mathcal{E}$-stability of coherent sheaves over $[Y/\Gamma]$ is equivalent to the stability of the corresponding parabolic bundle over X.

Lemma 6.5 (§7.2 in [37]). *There exists a generating sheaf $\mathcal{E}$ over $[Y/\Gamma]$, such that the $\mathcal{E}$-stability of coherent sheaves over $[Y/\Gamma]$ is equivalent to the stability of the corresponding parabolic bundles over X.*

On the other hand, the stability of parabolic bundles over X is equivalent to the stability of Γ-equivariant bundles over Y. Therefore, we have the following lemma.

Lemma 6.6. *There exists a generating sheaf $\mathcal{E}$ over $[Y/\Gamma]$ such that the $\mathcal{E}$-stability of coherent sheaves over $[Y/\Gamma]$ is equivalent to the stability of the corresponding Γ-equivariant bundles over Y.*

As a special case, the moduli space $\mathfrak{M}^{\mathcal{E}ss}_{H}(\mathcal{X}, P)$ of $\mathcal{E}$-semistable Higgs bundles over $\mathcal{X}$ is exactly the moduli space of semistable Γ-equivariant Higgs bundles over Y.

Corollary 6.7. *There exists a generating sheaf $\mathcal{E}$ such that the moduli space of $\mathcal{E}$-semistable Higgs bundles over $\mathcal{X}$ is isomorphic to the moduli space of semistable Γ-equivariant Higgs bundles over Y, that is,*

$$\mathfrak{M}^{\mathcal{E}ss}_{H}(\mathcal{X}, P) \cong \mathfrak{M}^{ss}_{H}([Y\backslash\Gamma], P).$$

6.3. **Moduli Space of R-semistable Equivariant G-Higgs Bundles.** We first review two important lemmas. Let A, A' be two algebraic groups, and let $\rho : A' \to A$ be a homomorphism. Let $\mathcal{S}$ be a set of isomorphism classes of principal A-bundles on Y. Let $\mathscr{E} \to \mathcal{R} \times X$ be a family of principal A-bundles in $\mathcal{S}$. Suppose that an algebraic group H acts on T by $\sigma : H \times T \to T$, and we have an isomorphism $\widetilde{\sigma} : H \times \mathscr{E} \cong (\sigma \times id_X)^*\mathscr{E}$. The family $\mathscr{E}$ is a *H-universal family*, if the following conditions hold.

(1) For any family of principal A-bundles $\mathscr{F} \to S \times X$ and any point $s \in S$, there exists an open neighbourhood of $s \in S$, and a morphism $f : U \to \mathcal{R}$ such that $\mathscr{F}|_{U\times X} \cong (f \times 1_X)^*\mathscr{E}$.
(2) Given two morphisms $f_1, f_2 : S \to \mathcal{R}$ and an isomorphism $\varphi : \mathscr{E}_{f_1} \cong \mathscr{E}_{f_2}$, there exists a unique morphism $h : S \to H$ such that $f_2 = \sigma \circ (f_1 \times h)$ and $\varphi = (f_1 \times h \times 1_X)^*(\widetilde{\sigma})$.

We consider the functor

$$\widetilde{\Gamma}(\rho, \mathscr{E}) : (\mathrm{Sch}/\mathcal{R}) \to \mathrm{Sets},$$

such that for each $\mathcal{R}$-scheme S, $\widetilde{\Gamma}(\rho, \mathscr{E})(S)$ is the set of pairs (E, τ), where E is a principal A'-bundles over $S \times X$, and $\tau : \rho_*(E) \to \mathscr{E}_S$ is an isomorphism.

Lemma 6.8 (Lemma 4.8.1 in [44]). *If $\rho : A' \to A$ is injective, then the functor $\widetilde{\Gamma}(\rho, \mathscr{E})$ is representable by a quasi-projective $\mathcal{R}$-scheme.*

Denote by $\mathcal{R}_1$ the quasi-projective $\mathcal{R}$-scheme representing $\widetilde{\Gamma}(\rho, \mathscr{E})$, and $\mathcal{E}_1 \to \mathcal{R}_1 \times X$ the universal family corresponding to the universal element in $\widetilde{\Gamma}(\rho, \mathscr{E})(\mathcal{R}_1)$.

Lemma 6.9 (Lemma 4.10 in [44]). *Let $\rho : A' \to A$ be a homomorphism of algebraic groups. Let $\mathscr{E} \to \mathcal{R} \times X$ be a H-universal family for a set $\mathcal{S}$ of principal G-bundles. Suppose that the functor $\widetilde{\Gamma}(\rho, \mathscr{E})$ is representable by a scheme $\mathcal{R}_1$. Then, we have*

(1) *The group H acts on $\mathcal{R}_1$ in a natural way, and $\mathcal{R}_1$ is a H-universal family for the set of principal A'-bundles, which give A-bundles in $\mathcal{S}$ by extending the structure group by $\rho : A' \to A$.*
(2) *If ρ is injective, then there exists a universal family $\mathscr{E}_1 \to \mathcal{R}_1 \times X$ of principal A'-bundles, which corresponds to the universal element in $\widetilde{\Gamma}(\rho, \mathscr{E})(\mathcal{R}_1)$.*

Now we will construct the moduli space $\mathfrak{M}_H^{Rss}([Y\backslash\Gamma], G)$ of R-semistable tame (Γ, G)-Higgs bundles over Y. Let $G \to \mathrm{GL}(\mathfrak{g})$ be the adjoint representation. Then, for every tame (Γ, G)-Higgs bundles (F, ϕ) over Y, we can associate an adjoint Higgs bundle $(\mathrm{Ad}(F), \phi)$. By Theorem 6.4 and Corollary 6.7, there exists a moduli space of semistable Γ-equivariant (adjoint) Higgs bundles over Y with Hilbert polynomial P. Denote by $R := \mathfrak{M}_H^{ss}([Y\backslash\Gamma], P)$ the moduli space and $\mathscr{E} \to \mathcal{R} \times Y$ the universal family. Recall that the moduli space is defined as $R := \mathrm{Quot}/\mathrm{SL}(V)$, where Quot is a quot-scheme. Then, there is a natural $\mathrm{GL}(V)$-action on R. Now we follow Ramanathan's approach [44] to construct the moduli space $\mathfrak{M}_H^{Rss}([Y\backslash\Gamma], G)$. We omit the fixed Hilbert polynomial P for simplicity.

$$\begin{tikzcd}
\mathrm{Aut}(\mathfrak{g}) \arrow[r] & \mathrm{Aut}(\mathfrak{g}) \times \mathbb{C}^* \arrow[r, hook] & \mathrm{GL}(\mathfrak{g}) \\
\mathrm{Ad}(G) = G/Z \arrow[u, hook] & G/G' \times G/Z \arrow[l] & G \arrow[l]
\end{tikzcd}$$

(1) Let $\mathbb{C}^* \times \mathrm{Aut}(\mathfrak{g}) \hookrightarrow \mathrm{GL}(\mathfrak{g})$ be the natural inclusion. By Lemma 6.9, we get a universal family for the set of $(\mathbb{C}^* \times \mathrm{Aut}(\mathfrak{g}))$-bundles, of which the associated $\mathrm{GL}(\mathfrak{g})$-bundles are semistable. Denote by
$$\mathscr{E}_1 \to \mathcal{R}_1 \times Y$$
the universal family of $(\mathbb{C}^* \times \mathrm{Aut}(\mathfrak{g}))$-bundles in this case.
(2) Given an $(\mathbb{C}^* \times \mathrm{Aut}(\mathfrak{g}))$-bundle F, if the associated line bundle $F(\mathbb{C}^*)$ is trivial, the $(\mathbb{C}^* \times \mathrm{Aut}(\mathfrak{g}))$-bundle F has a natural reduction structure of $\mathrm{Aut}(\mathfrak{g})$. By the universal property of the Picard scheme $\mathrm{Pic}(Y)$, the associated family
$$\mathscr{E}_1(\mathbb{C}^*) \to \mathcal{R}_1 \times Y$$
corresponds to a morphism $f : \mathcal{R}_1 \to \mathrm{Pic}(Y)$. Let $\mathcal{R}_1' = f^{-1}([\mathcal{O}_X])$. Then, the family
$$\mathscr{E}_1' := \mathscr{E}_1|_{\mathcal{R}_1'} \to \mathcal{R}_1' \times Y$$
is a $\mathrm{GL}(V)$-universal family for $\mathrm{Aut}(\mathfrak{g})$-bundles, of which associated Γ-equivariant bundles $\mathrm{GL}(\mathfrak{g})$-bundles are semistable.
(3) Note that $\mathrm{Ad}(G) = G/Z \hookrightarrow \mathrm{Aut}(\mathfrak{g})$ is injective. By Lemma 6.8, the functor $\widetilde{\Gamma}(\mathrm{Ad}, \mathscr{E}_1')$ is representable. Let
$$\mathscr{E}_2 \to \mathcal{R}_2 \times Y$$
be the $\mathrm{GL}(V)$-universal family of G/Z-bundles, of which the associated Γ-equivariant $\mathrm{GL}(\mathfrak{g})$-bundles are semistable.
(4) In this step, we will construct a universal family for $(G/G' \times G/Z)$-bundles, where G' is the derived group. Since G is reductive, G/G' is a torus. We assume $G/G' \cong \mathbb{C}^l$. It is well-known that a $\mathbb{C}^*$-bundle is a line bundle, and $\mathrm{Pic}(Y)$ classifies all line bundles over Y. Therefore, $\prod^l \mathrm{Pic}(Y)$ parameterizes all G/G'-bundles. Denote by $\mathscr{P} \to \mathrm{Pic}(Y)$ the Poincaré bundle. We consider the following family
$$(\underbrace{\mathscr{P} \times_Y \cdots \times_Y \mathscr{P}}_{l}) \times_Y \mathscr{E}_2 \to (\prod^{l} \mathrm{Pic}(Y) \times \mathcal{R}_2) \times Y$$
of $(G/G' \times G/Z)$-bundles. We define
$$\mathcal{E}_2' := (\underbrace{\mathscr{P} \times_Y \cdots \times_Y \mathscr{P}}_{l}) \times_Y \mathscr{E}_2$$

and

$$\mathcal{R}'_2 := (\prod^{l} \mathrm{Pic}(Y) \times \mathcal{R}_2).$$

Then, $\mathscr{E}'_2$ is a $\mathrm{GL}(V)$-universal family of $(G/G' \times G/Z)$-bundles.

(5) Now we consider the natural projection $\rho : G \to G/G' \times G/Z$. The functor $\widetilde{\Gamma}(\rho, \mathscr{E}'_2)$ is representable by a scheme $\mathcal{R}_3$ (see [44, Lemma 4.15.1]). Denote by $\mathscr{E}_3 \to \mathcal{R}_3 \times Y$ the $\mathrm{GL}(V)$-universal family of (Γ, G)-Higgs bundles, of which the associated $\mathrm{GL}(\mathfrak{g})$-Higgs bundles are semistable. By Corollary 4.12, a tame G-Higgs bundle (F, ϕ) is R-semistable if and only if the adjoint bundle $(\mathrm{Ad}(F), \phi)$ is p-semistable. Therefore, the scheme $\mathcal{R}_3$ parameterizes R-semistable tame (Γ, G)-Higgs bundles over Y.

The above discussion gives the following proposition.

Proposition 6.10. *There exists a coarse moduli space $\mathfrak{M}_H^{Rss}([Y \backslash \Gamma], G)$ of R-semistable (Γ, G)-Higgs bundles over Y.*

As we explained at the beginning of this section, this then implies that there exists a moduli space of R-semistable tame parahoric $\mathcal{G}_{\boldsymbol{\theta}}$-Higgs bundles over X.

Remark 6.11. We briefly discuss the construction of the moduli space of tame parahoric $\mathcal{G}_{\boldsymbol{\theta}}$-local systems. Similar to tame parahoric $\mathcal{G}_{\boldsymbol{\theta}}$-Higgs bundles, tame parahoric $\mathcal{G}_{\boldsymbol{\theta}}$-local systems over X correspond to tame (Γ, G)-local systems over Y (see [4]). Therefore, it is equivalent to construct the moduli space of R-semistable tame G-local systems over the stack $[Y/\Gamma]$, where R-semistability of tame G-local systems can be defined in a similar way as in §4. Note that the (integrable) connections can be understood as a special case of a sheaf of graded algebras (see [48, 49]). Therefore, the moduli space of semistable tame local systems over $[Y/\Gamma]$ exists (see again [48, 49]), and analogously to the construction provided in this section one gets the construction of the moduli space of R-semistable tame parahoric $\mathcal{G}_{\boldsymbol{\theta}}$-local systems.

7. Poisson Structure on the Moduli Space of Tame Parahoric Higgs Torsors

In this section, we will construct a Poisson structure on the moduli space of tame parahoric Higgs bundles. By Theorem 6.1, we have an isomorphism

$$\mathfrak{M}_H^{Rss}(X, \mathcal{G}_{\boldsymbol{\theta}}) \cong \mathfrak{M}_{H,\boldsymbol{\rho}}^{Rss}([Y/\Gamma], G).$$

Therefore, it is equivalent to work on the moduli space of (Γ, G)-Higgs bundles over Y. The authors studied Poisson structures on the moduli space of Higgs bundles over stacky curves in [32], and we use a similar approach to construct the Poisson structure here.

Before we demonstrate the construction of the Poisson structure on $\mathfrak{M}_{H,\boldsymbol{\rho}}^{Rss}([Y/\Gamma], G)$, we first review some results on Lie algebroids and Poisson structures. Let $\mathfrak{M}$ be a projective (or quasi-projective) scheme over $\mathbb{C}$ together with a proper and free group action $K \times \mathfrak{M} \to \mathfrak{M}$. Then, we have a natural projection $\pi : \mathfrak{M} \to \mathfrak{M}/K$, which induces

$$0 \to T_{\mathrm{orb}}\mathfrak{M} \to T\mathfrak{M} \to \pi^* T(\mathfrak{M}/K) \to 0.$$

This exact sequence gives us a natural surjective morphism $T\mathfrak{M}/K \to T(\mathfrak{M}/K)$, which is the *anchor map*. Then, we have

$$0 \to \mathrm{Ad}(\mathfrak{M}) \to T\mathfrak{M}/K \to T(\mathfrak{M}/K) \to 0,$$

which is known as the *Atiyah sequence*. The Atiyah sequence induces a *Lie algebroid* structure on $T\mathfrak{M}/K$. By [19, Theorem 2.1.4], the total space $(T\mathfrak{M}/K)^*$ has a Poisson structure.

Let $\mathcal{X} = [Y/\Gamma]$, and denote by X the coarse moduli space of $\mathcal{X}$ with the natural morphism $\pi : \mathcal{X} \to X$. For simplicity, we use the following notation in the sequel

- $\mathfrak{M}_H(\mathcal{X})$: moduli space $\mathfrak{M}_{H,\boldsymbol{\rho}}^{Rss}([Y/\Gamma], G)$ of R-semistable (Γ, G)-Higgs bundles of type $\boldsymbol{\rho}$ over Y;

- $\mathfrak{M}^0_H(\mathcal{X})$: moduli space of R-stable (Γ, G)-Higgs bundles (F, ϕ) of type $\boldsymbol{\rho}$ over Y such that F is R-stable;
- $\mathfrak{M}(\mathcal{X})$: moduli space of R-stable (Γ, G)-bundles of type $\boldsymbol{\rho}$ over Y;
- $\mathfrak{M}(X)$: moduli space of R-stable G-bundles over X.

$$\begin{tikzcd} & \mathfrak{M}(\mathcal{X}) \times \mathcal{X} \arrow[dl, "\mu_1"'] \arrow[dr, "\nu_1"] & \\ \mathfrak{M}(\mathcal{X}) & & \mathcal{X} \end{tikzcd} \qquad \begin{tikzcd} & \mathfrak{M}(X) \times X \arrow[dl, "\mu_2"'] \arrow[dr, "\nu_2"] & \\ \mathfrak{M}(X) & & X \end{tikzcd}$$

Given a G-bundle F over $\mathcal{X}$, the G-bundle $\pi^*\pi_*F$ is a Γ-invariant G-bundle over $\mathcal{X}$. Then, we have a natural morphism

$$\mathrm{Ad}(F) \hookrightarrow \mathrm{Ad}(\pi^*\pi_*F).$$

For each $\rho_y \in \boldsymbol{\rho}$, it gives a parabolic subgroup $P_y \subseteq G$. Denote by $P_y = L_y N_y$ the Levi factorization. Then, we have

$$0 \to \mathrm{Ad}(F) \to \mathrm{Ad}(\pi^*\pi_*F) \to \prod_{y \in R} \mathfrak{n}_y \otimes \mathcal{O}_y \to 0.$$

Suppose that F is R-stable. By the deformation theory of G-bundles, the tangent space $T\mathfrak{M}(\mathcal{X})$ at $[F]$ is given by

$$T_{[F]}\mathfrak{M}(\mathcal{X}) \cong H^1(\mathcal{X}, \mathrm{Ad}(F)),$$

where F is stable. Similarly, we have

$$T_{[\pi_*F]}\mathfrak{M}(X) \cong H^1(\mathcal{X}, \mathrm{Ad}(\pi^*F)).$$

By Grothendieck duality over stacky curves (see [36]), we have

$$T^*_{[F]}\mathfrak{M}(\mathcal{X}) \cong H^0(\mathcal{X}, \mathrm{Ad}(F) \otimes \omega_{\mathcal{X}}).$$

Taking an element $\phi \in H^0(\mathcal{X}, \mathrm{Ad}(F) \otimes \omega_{\mathcal{X}})$, we get the following isomorphism

$$T_\phi T^*_{[F]}\mathfrak{M}(\mathcal{X}) \cong T_{(F,\phi)}\mathfrak{M}_H(\mathcal{X}),$$

where $T_{(F,\phi)}\mathfrak{M}_H(\mathcal{X}) \cong \mathbb{H}^1(\mathrm{Ad}(F) \xrightarrow{\mathrm{Ad}(\phi)} \mathrm{Ad}(F) \otimes \omega_{\mathcal{X}})$ (see [23, §2.3]).

Now let $\mathscr{F}$ and $\mathscr{E}$ be universal families over $\mathfrak{M}(\mathcal{X})$ and $\mathfrak{M}(X)$ respectively. The above discussion gives the following short exact sequence.

$$0 \to \mathrm{Ad}(\mathscr{F}) \to \mathrm{Ad}(\mathscr{E}) \to \prod_{y \in R} \mathfrak{n}_y \otimes \mathcal{O}_{\nu_1^{-1}(y)} \to 0,$$

which induces

$$0 \to \mathscr{A}d \to R^1(\mu_1)_*\mathrm{Ad}(\mathscr{F}) \to R^1(\mu_1)_*\mathrm{Ad}(\mathscr{E}) \to 0.$$

This sequence is an Atiyah sequence and so we have a Lie algebroid structure on $R^1(\mu_1)_*\mathrm{Ad}(\mathscr{E})$. Note that

$$R^1(\mu_1)_*\mathrm{Ad}(\mathscr{F}) \cong T\mathfrak{M}(\mathcal{X}) \quad \text{and} \quad R^1(\mu_1)_*\mathrm{Ad}(\mathscr{E}) \cong T\mathfrak{M}(X).$$

Then, $(T\mathfrak{M}(\mathcal{X}))^*$ has a Poisson structure.
Moreover, it holds that

$$T(R^1(\mu_1)_*\mathrm{Ad}(\mathscr{F}))^* \cong T\mathfrak{M}_H(\mathcal{X}),$$

thus by [19, Theorem 2.1.4] the moduli space $\mathfrak{M}^0_H(\mathcal{X})$ is equipped with a Poisson structure. Since $\mathfrak{M}^0_H(\mathcal{X})$ is dense in $\mathfrak{M}_H(\mathcal{X})$, there exists a Poisson structure on $\mathfrak{M}_H(\mathcal{X})$. We have proven the following:

Proposition 7.1. *There exists a Poisson structure on the moduli space $\mathfrak{M}^{Rss}_H(X, \mathcal{G}_{\boldsymbol{\theta}})$ of R-semistable tame parahoric $\mathcal{G}_{\boldsymbol{\theta}}$-Higgs bundles over X.*

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