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PÓLYA–CARLSON DICHOTOMY FOR DYNAMICAL ZETA FUNCTIONS AND TWISTED BURNSIDE-FROBENIUS THEOREM

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Abstract. For the unitary dual map of an automorphism of a torsion-free, finite rank nilpotent group, we prove the Pólya–Carlson dichotomy between rationality and a natural boundary for the analytic behavior of its Artin–Mazur dynamical zeta function. We also establish Gauss congruences for the Reidemeister numbers of the iterations of endomorphism of a group from this class.

Our method is the twisted Burnside–Frobenius theorem, proven in the paper for automorphisms of this class of groups, and a calculation of the Reidemeister numbers via product formula and profinite completions.

1. Introduction

Let G be a group and $\phi : G \to G$ an endomorphism. Two elements $x, y \in G$ are said to be *ϕ-conjugate* or *twisted conjugate,* if and only if there exists $g \in G$ with

$$y = gx\phi(g^{-1}).$$

We will write $\{x\}_\phi$ for the *ϕ-conjugacy* or *twisted conjugacy* class of the element $x \in G$. The number of ϕ-conjugacy classes is called the *Reidemeister number* of an endomorphism ϕ and is denoted by $R(\phi)$. If ϕ is the identity map then the ϕ-conjugacy classes are the usual conjugacy classes in the group G.

Denote by $\widehat{G}$ the *unitary dual* of G, i.e. the space of equivalence classes of unitary irreducible representations of G, equipped with the *hull-kernel* topology, denote by $\widehat{G}_f$ the subspace of the unitary dual formed by irreducible finite-dimensional representations. If $\varphi : G \to G$ is an automorphism, it induces a dual map $\widehat{\varphi} : \widehat{G} \to \widehat{G}$, $\widehat{\varphi}(\rho) = \rho \circ \varphi$. This dual map $\widehat{\varphi}$ define a dynamical system on the unitary dual space $\widehat{G}$ or on its finite-dimensional part $\widehat{G}_f$, because subspace $\widehat{G}_f$ is invariant under the dual map. Denote by $\widehat{\varphi}_f : \widehat{G}_f \to \widehat{G}_f$ the restriction of the dual map $\widehat{\varphi}$ to the finite-dimensional part $\widehat{G}_f$ of the unitary dual space.

In the present paper we prove that the Reidemeister number of an automorphism φ of any nilpotent torsion-free group of finite rank is equal to the number of finite-dimensional fixed points of the induced map $\widehat{\varphi}_f$ on the unitary dual space $\widehat{G}_f$, if one of these numbers is finite. This is so-called twisted Burnside-Frobenius theorem (TBFT), or more precisely TBFT_f, because the initial conjecture [5] supposed all irreducible representations to be

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involved. The conjecture about TBFT was proved in many cases, but failed for an example in [9], which led to the new formulation — TBFT$_f$. In [3] an example of a group that has neither TBFT nor TBFT$_f$ was presented. The most general case of TBFT$_f$ is the case of polycyclic-by-finite groups [7].

Denote by $\mathrm{Fix}(\widehat{\varphi}_f^n)$ the set of fixed points of the dual map $\widehat{\varphi}_f^n$, i.e., *n-periodic points of* $\widehat{\varphi}_f$. Suppose, its cardinality $|\mathrm{Fix}(\widehat{\varphi}_f^n)| < \infty$ for all n. Then the *Artin–Mazur dynamical zeta function* of the map $\widehat{\varphi}_f$ is defined in [3] by

$$AM_{\widehat{\varphi}_f}(z) = \exp\left(\sum_{n=1}^{\infty} \frac{|\mathrm{Fix}(\widehat{\varphi}_f^n)|}{n} z^n\right).$$

The rationality of $AM_{\widehat{\varphi}_f}(z)$ was proven in [3] for finitely generated abelian groups and for finitely generated torsion free nilpotent groups. For ergodic automorphisms of finite dimensional compact connected abelian groups $\widehat{G}_f$ the Pólya–Carlson dichotomy for the Artin–Mazur dynamical zeta function was proven in the work of Bell, Miles, Ward [1]. In this case G itself is an abelian group and it is a subgroup of $\mathbb{Q}^d$, where $d \geq 1$. For endomorphisms of abelian varieties in positive characteristic the dichotomy for the Artin–Mazur dynamical zeta function was proven in [2].

We will fix now some notation and formulate the main results of the paper.

Let $\mu(d)$, $d \in \mathbb{N}$, be the *Möbius function*, i.e.

$$\mu(d) = \begin{cases} 1 & \text{if } d = 1, \\ (-1)^k & \text{if } d \text{ is a product of } k \text{ distinct primes,} \\ 0 & \text{if } d \text{ is not square-free.} \end{cases}$$

We call an endomorphism φ *tame* if the Reidemeister numbers $R(\varphi^n)$ are finite for all $n \in \mathbb{N}$.

A group G is of *finite (Prüfer) rank* $\mathrm{rk}(G)$ if each its finitely generated subgroup has a generating set of cardinality $\leq \mathrm{rk}(G)$ and $\mathrm{rk}(G)$ is a minimal such number.

Theorem 1.1. *For a tame endomorphism φ of a nilpotent torsion-free group G of finite rank we have the following Gauss congruences*

$$\sum_{d|n} \mu(d) \cdot R(\phi^{n/d}) \equiv 0 \mod n.$$

Theorem 1.2. *The TBFT$_f$ is fulfilled for automorphisms of any nilpotent torsion-free group of finite rank.*

Denote by $\mathbb{P}$ the set of all rational primes; for $p \in \mathbb{P}$, the field of p-adic numbers is denoted by $\mathbb{Q}_p$, the ring of p-adic integers by $\mathbb{Z}_p$, and the p-adic absolute value (as well as its unique extension to a fixed algebraic closure $\overline{\mathbb{Q}_p}$) by $|\cdot|_p$. The absolute value on $\mathbb{C}$ is denoted by $|\cdot|_\infty$.

Theorem 1.3. *Let $\varphi\colon G \to G$ be a tame automorphism of a torsion-free nilpotent group G of finite Prüfer rank. Let c denote the nilpotency class of G and, for $1 \leq k \leq c$, let $\alpha_k : A_k \to A_k$ denote the induced automorphisms of the torsion-free abelian factor groups $A_k = \overline{\gamma}_k(G)/\overline{\gamma}_{k+1}(G)$ of finite rank, $d_k \geq 1$ say, that arise from the isolated lower central series* (3) *of G. Then the following hold.*

1) *For $1 \leq k \leq c$, let*

$$\alpha_{k,\mathbb{Q}}\colon A_{k,\mathbb{Q}} \to A_{k,\mathbb{Q}}$$

denote the extensions of α_k *to the divisible hull* $A_{k,\mathbb{Q}} = \mathbb{Q}\otimes_{\mathbb{Z}} A_k \cong \mathbb{Q}^{d_k}$ *of* A_k*. Let* $\xi_{k,1}, \dots, \xi_{k,d_k}$ *be the eigenvalues of* $\alpha_{k,\mathbb{Q}}$ *in a fixed algebraic closure of the field* $\mathbb{Q}$*, including multiplicities. Set* $L_k = \mathbb{Q}(\xi_{k,1}, \dots, \xi_{k,d_k})$*; for each* $p \in \mathbb{P}$*, fix some embeddings* $L_k \hookrightarrow \overline{\mathbb{Q}}_p$ *and* $L_k \hookrightarrow \mathbb{C}$*.*

Then there exist subsets $I_k(p) \subseteq \{1, \dots, d_k\}$*, for* $p \in \mathbb{P}$*, such that the following hold.*

(i) *For each* $p \in \mathbb{P}$*, the polynomial* $\prod_{i\in I_k(p)}(X - \xi_{k,i})$ *has coefficients in* $\mathbb{Z}_p$*.*

(ii) *For each* $n \in \mathbb{N}$*,*

$$(1)\qquad |\operatorname{Fix}((\widehat{\varphi})^n_f)| = \prod_{k=1}^{c}\prod_{p\in\mathbb{P}}\prod_{i\in I_k(p)} |\xi^n_{k,i} - 1|_p^{-1} = \prod_{k=1}^{c}\prod_{i=1}^{d_k}\left(|\xi^n_{k,i} - 1|_\infty \cdot \prod_{p\in\mathbb{P}}\prod_{i\notin I_k(p)} |\xi^n_{k,i} - 1|_p\right);$$

as this number is a positive integer, $|\xi^n_{k,i} - 1|_p = 1$ *for* $1 \le i \le d_k$ *for almost all* $p \in \mathbb{P}$*.*

2) *Suppose that, for each* $k \in \{1, \dots, c\}$*, the cardinality of the set*

$$\mathbf{P}_k := \left\{ p \in \mathbb{P} \colon \prod_{i\notin I_k(p)} |\xi^n_{k,i} - 1|_p \neq 1 \right\}$$

is finite and that $|\xi_{k,i}|_\infty \neq 1$ *for* $1 \le i \le d_k$*.*

Then the Artin-Mazur dynamical zeta function $AM_{\widehat{\varphi}_f}(z)$ *is either a rational function or it has a natural boundary at its radius of convergence. Furthermore, the latter occurs if and only if* $|\xi_{k,i}|_p = 1$ *for some* $k \in \{1, \dots, c\}$*,* $p \in \mathbf{P}_k$ *and* $i \notin I_k(p)$*.*

The paper is organised as follows.

In Section 2 we remind and prove some properties of Reidemeister numbers, fixed elements of endomorphisms, and endomorphisms of torsion-free abelian groups of finite rank.

In Section 3 we establish Gauss congruences for the Reidemeister numbers of iterations of an endomorphism of a torsion-free nilpotent group of finite rank using a product formula for Reidemeister numbers, which is also proved in the section.

In Section 4 we prove TBFT_f for automorphisms of any nilpotent torsion-free group of finite rank.

In Section 5 we firstly derive a closed formula for the sequence of numbers of fixed points for iterations of the dual map of a tame endomorphism of torsion-free nilpotent groups of finite rank with help of twisted Burside-Frobenius theorem. Our approach is via profinite completions in an analogy to the paper [6]. Then we prove the Pólya–Carlson dichotomy between rationality and the existence of a natural boundary of the Artin–Mazur dynamical zeta function of the dual map $\widehat{\varphi}_f$ for an automorphism of a torsion-free nilpotent group of finite rank.

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The results of Sections 2 and 4 are obtained by E.T., the results of Section 5 are obtained by A.F., the results of Section 3 are obtained by the authors jointly.

2. Preliminaries and reminding

First we will remind some properties of Reidemeister classes of extensions. Suppose that a normal subgroup H of G is invariant for an endomorphism $\varphi : G \to G$. Let $p : G \to G/H$ be the natural projection. Denote by $\varphi' : H \to H$ and by $\widetilde{\varphi} : G/H \to G/H$ the induced representations. If H is contained in the center of G, then the extension is called *central.*

The following theorem gather some very useful tools in the field which appeared in [5, 10], see also [7, 11].

Definition 2.1. Denote by $C(\varphi)$ the subgroup of G, formed by elements fixed by $\varphi : G \to G$: $C(\varphi) := \{g \in G \colon \varphi(g) = g\}$.

Theorem 2.2. *Suppose that G, H, and φ are as above and . Then we have the following properties.*

1. Epimorphity: *the projection $G \to G/H$ maps Reidemeister classes of φ onto Reidemeister classes of $\widetilde{\varphi}$, in particular $R(\widetilde{\varphi}) \le R(\varphi)$;*
2. Role of fixed points: *if $\widetilde{\varphi}$ has n fixed points ($|C(\widetilde{\varphi})| = n$), then $R(\varphi') \le R(\varphi) \cdot n$;*
3. Fixed point trivial case: *if $C(\widetilde{\varphi}) = \{e\}$, then each Reidemeister class of φ' is an intersection of the appropriate Reidemeister class of φ and H;*
4. Summation formula: *if $C(\widetilde{\varphi}) = \{e\}$, then $R(\varphi) = \sum_{j=1}^{R} R(\tau_{g_j} \circ \varphi')$, where $g_1, \dots g_R$ are some elements of G such that $p(g_1), \dots, p(g_R)$ are representatives of all Reidemeister classes of $\widetilde{\varphi}$, $R = R(\widetilde{\varphi})$, $\tau_g(x) = gxg^{-1}$;*
5. Product formula: *if additionally the extension is central, then the summation formula becomes the product formula $R(\varphi) = R(\varphi') \cdot R(\widetilde{\varphi})$.*

Theorem 2.3 (see [8])**.** *Suppose that the equivalence class of a finite-dimensional unitary representation $[\rho]$ is $\widehat{\varphi}$-fixed, where $\varphi : G \to G$ is an automorphism with $R(\varphi) < \infty$. Then ρ is finite, i.e. factors through a finite group F: $\rho = \rho' \circ p$, $p : G \to G/H = F$. In this case H can be t aken φ-invariant.*

This implies the following statement.

Lemma 2.4. *G has the $TBFT_f$ for an automorphism φ if and only if G is φ-*conjugacy separable*, i.e. there is a φ-invariant normal subgroup H of finite index in G such that the projection $G \to G/H$ induces a bijection of Reidmeister classes.*

Proof. Indeed, if we have TBFT_f for φ, $R(\varphi) < \infty$, then the $\widehat{\varphi}$-fixed representations are finite and the intersection of their kernels is an invariant subgroup of finite index which can be taken as H. In the other direction the statement is evident. ◻

From the equality

$$yg\varphi(y^{-1})x = ygxx^{-1}\varphi(y^{-1})x = y(gx)(\tau_{x^{-1}} \circ \varphi)(y^{-1}),$$

where $\tau_x(z) := xzx^{-1}$, we obtain the following well-known statement, very useful in the field (see e.g. [7]).

Lemma 2.5. *The shifts of Reidemeister classes of φ are Reidemeister classes of $\tau_{x^{-1}} \circ \varphi$:*

$$\{g\}_\varphi x = \{gx\}_{\tau_{x^{-1}} \circ \varphi}.$$

Similarly, from the equality

$$\begin{aligned} \tau_x(yg\varphi(y^{-1})) &= \tau_x(y)\tau_x(g)x\varphi(y^{-1})x^{-1} = \tau_x(y)\tau_x(g)x\varphi(x^{-1})\varphi(xy^{-1}x^{-1})\varphi(x)x^{-1} \\ &= \tau_x(y)\tau_x(g)\tau_{x\varphi(x^{-1})}\varphi(\tau_x(y^{-1})) \end{aligned}$$

we obtain

$$\tau_x(\{g\}_\varphi) = \{\tau_x(g)\}_\psi, \qquad \psi = \tau_{x\varphi(x^{-1})} \circ \varphi. \tag{2}$$

Lemma 2.6. *We have $TBFT_f$ for an automorphism φ if and only if the stabilizer of each Reidemeister class under right shifts is a subgroup of finite index.*

Proof. If G has φ-conjugacy separability property, then the stabilizer is a pre-image of the corresponding stabilizer in G/H (in the above notation). Then it has a finite index.

Conversely, if the stabilizers are of finite index, take their intersection H, which has finite index too. Moreover, H is φ-invariant and normal. Indeed, since Reidemeister classes are φ-invariant, if h stabilizes a class, i.e., $\{g\}_\varphi h = \{g\}_\varphi$, then $\{g\}_\varphi \varphi(h) = \varphi(\varphi^{-1}\{g\}_\varphi h) = \varphi(\{g\}_\varphi h) = \varphi(\{g\}_\varphi) = \{g\}_\varphi$. Hence H is φ-invariant. Similarly, by (2), applied twice, for the same g and h and an arbitrary $x \in G$, denoting $\psi = \tau_{x^{-1}\varphi(x)} \circ \varphi$, we have

$$\{g\}_\varphi \tau_x(h) = \tau_x(\tau_{x^{-1}}(\{g\}_\varphi)h) = \tau_x(\{\tau_{x^{-1}}(g)\}_\psi h) = \tau_x(\{\tau_{x^{-1}}(g)\}_\psi) = \{g\}_{\psi'},$$

where $\psi' = \tau_{x\varphi(x^{-1})} \circ \psi = \tau_{x\varphi(x^{-1})} \circ \tau_{x^{-1}\varphi(x)} \circ \varphi = \tau_{x\varphi(x^{-1})x^{-1}\varphi(x)} \circ \varphi$. Hence, H is normal.

Since by the definition of H any Reidemeister class is a union of some H-cosets, the natural projection to G/H gives rise to a bijection of Reidemeister classes. So we have φ-conjugacy separability and apply Lemma 2.4 to complete the proof. □

Remark 2.7. From Lemma 2.5 it follows that the intersection of stabilizers for φ is the same as for $\tau_g \circ \varphi$.

Lemma 2.8. *Suppose that a group G has finitely many distinct inner automorphisms. Then $TBFT_f$ takes place for G.*

Proof. Indeed, by Lemma 2.5 in this case we have for each Reidemeister class only finitely many distinct sets being its shifts, namely $\leq k$, where k is the number of distinct inner automorphisms. Hence its stabilizer has finite index and we apply Lemma 2.6. □

Now we pass to finite (Prüfer) rank groups.

Lemma 2.9. (Fuchs, see [14, 15.2.3]) *Let ψ be an endomorphism of a torsion-free abelian group A of finite rank. Then ψ is injective if and only if the index of $\mathrm{Im}(\psi)$ is finite.*

Corollary 2.10. *Consider A as in Lemma 2.9. Let $\varphi : A \to A$ be an endomorphism with $R(\varphi) < \infty$. Then the fixed subgroup $C(\varphi)$ is trivial: $C(\varphi) = \{e\}$.*

Proof. One has $R(\varphi) = |\operatorname{Coker}(1-\varphi)| < \infty$. Thus, by Lemma 2.9 for $\psi = 1 - \varphi$, $\{e\} = \operatorname{Ker}(1-\varphi) = C(\varphi)$. □

Also we need the following observation.

Lemma 2.11. *Let G be a finite-by-abelian group, with a finite normal subgroup F and an abelian factor group $A = G/F$ of finite Prüfer rank $\mathrm{rk}(A) < \infty$. Then G has no more than $|F|! \cdot |F|^{\mathrm{rk}(A)}$ distinct inner automorphisms.*

Proof. Denote by p the projection $p : G \to A$. Consider a finitely generated subgroup $A_0 \subseteq A$ with generators $a_1, \dots, a_k$, $k \leq \mathrm{rk}(A)$. Then $G_0 = p^{-1}(A_0)$ has a generating set $F \cup \{g_1, \dots, g_k\}$ of cardinality $|F| + k$, where $g_i \in G_0$ are arbitrary elements such that $p(g_i) = a_i$, $i = 1, \dots, k$. Any inner automorphism is completely defined by a mapping of these generators. Keeping in mind that A is abelian, hence τ_g maps each F-coset to itself, we obtain not more than $|F|! \cdot |F|^k$ possibilities.

Suppose that for the entire G one has $s > |F|! \cdot |F|^{\mathrm{rk}(A)}$ distinct inner automorphisms τ_{y_i}, $y_i \in G$, $i = 1, \dots, s$. So we can find $x_{ij} \in G$ such that $\tau_{y_i}(x_{ij}) \neq \tau_{y_j}(x_{ij})$. Consider a finitely generated A_0 such that $G_0 = p^{-1}A_0$ contains all y_i and x_{ij} (we can take as generators all this elements and all elements of F). Then τ_{y_i} remain distinct as inner automorphisms of G_0. A contradiction with the first part of the proof. □

3. Gauss congruences

For a torsion-free finite rank nilpotent group G of class $c = c(G)$ with the lower central series

$$G = \gamma_1(G) \geq \gamma_2(G) \geq \ldots \geq \gamma_c(G) \geq \gamma_{c+1}(G) = \{e\},$$

consider the *isolated* series

$$G = \overline{\gamma}_1(G) \geq \overline{\gamma}_2(G) \geq \ldots \geq \overline{\gamma}_c(G) \geq \overline{\gamma}_{c+1}(G) = \{e\}, \tag{3}$$

where $\overline{\gamma}_i(G)$ is a subgroup of G such that $\overline{\gamma}_i(G)/\gamma_i(G)$ is the torsion subgroup of $G/\gamma_i(G)$ $(1 \leq i \leq c+1)$. Then (3) is a descending central series of fully invariant subgroups of G such that each factor group $\overline{\gamma}_i(G)/\overline{\gamma}_{i+1}(G)$ $(1 \leq i \leq c)$ is torsion-free (see e.g. [6, Sect. 2.1]).

Definition 3.1. Denote $A_i := \overline{\gamma}_i(G)/\overline{\gamma}_{i+1}(G)$ $(1 \leq i \leq c)$. Denote by $\varphi_i : \overline{\gamma}_i(G) \to \overline{\gamma}_i(G)$ $(1 \leq i \leq c+1)$ and by $\alpha_i : A_i \to A_i$ $(1 \leq i \leq c)$ the induced endomorphisms. In particular, $\varphi_1 = \varphi$.

Now we can give a short alternative proof of [6, Theorem 1.4 (1)] in the case of of one endomorphism:

Theorem 3.2. *Suppose that $R(\varphi) < \infty$, where φ is an endomorphism of a torsion-free finite rank nilpotent group G of class c. Then all $R(\alpha_i)$ are finite $(1 \leq i \leq c)$ and for any $k \leq c+1$*

$$R(\varphi_k) = \prod_{i=k}^{c} R(\alpha_i), \text{ in particular, } R(\varphi) = R(\varphi_1) = \prod_{i=1}^{c} R(\alpha_i).$$

Proof. **Way down.** Since $A_1 = \overline{\gamma}_1(G)/\overline{\gamma}_2(G) = G/\overline{\gamma}_2(G)$, we have $R(\alpha_1) \leq R(\varphi) < \infty$. By Lemma 2.10, we obtain $C(\alpha_1) = \{e\}$. Then, by the sum formula from Theorem 2.2, $R(\varphi_2) \leq R(\varphi_1) < \infty$. Thus, we can argue in the same way and obtain $R(\alpha_2) \leq R(\varphi_2) < \infty$, $C(\alpha_2) = \{e\}$ and $R(\varphi_3) \leq R(\varphi_2) \leq R(\varphi_1) < \infty$. Continuing this induction we obtain

$$R(\alpha_i) < \infty, \quad i = 1, \ldots, c, \qquad C(\alpha_1) = C(\alpha_2) = \cdots = C(\alpha_c) = \{e\} \tag{4}$$

and

$$R(\varphi_{c+1}) \leq \cdots \leq R(\varphi_3) \leq R(\varphi_2) \leq R(\varphi_1) < \infty. \tag{5}$$

Way up. Consider $G_i := G/\overline{\gamma}_{i+1}(G)$ and the induced endomorphism $\overline{\varphi_i} : G_i \to G_i$, $i = 1, \ldots, c$. Since $R(\varphi) < \infty$, we have $R(\overline{\varphi_i}) \leq R(\varphi) < \infty$. Now we will prove by induction over i that $C(\overline{\varphi_i}) = \{e\}$ and $R(\overline{\varphi_i}) = \prod_{j=1}^{i} R(\alpha_j)$. Indeed, for $i = 1$, $G_i = A_1$ and the statement is (4). Suppose that the statement is true for $i \leq k-1$ and prove it for k. We have a central extension $A_k \to G_k \to G_{k-1}$ and $\overline{\varphi_{k-1}} : G_{k-1} \to G_{k-1}$ has $C(\overline{\varphi_{k-1}}) = \{e\}$. Then the fixed elements of $\overline{\varphi_k} : G_k \to G_k$ can be situated only in $A_k \subseteq G_k$. But $C(\alpha_k) = \{e\}$ by (4). Thus $C(\overline{\varphi_i}) = \{e\}$. Since the extension is central, we can apply the product formula from Theorem 2.2 and obtain

$$R(\overline{\varphi_k}) = R(\alpha_k) \cdot R(\overline{\varphi_{k-1}}) = R(\alpha_k) \cdot \prod_{j=1}^{k-1} R(\alpha_j) = \prod_{j=1}^{k} R(\alpha_j).$$

So by induction $R(\varphi) = R(\overline{\varphi_c}) = \prod_{j=1}^{c} R(\alpha_j)$. The other cases of the desired formula are its versions for groups of less class. □

Now we can prove the Gauss congruences for our class of groups.

Proof of Theorem 1.1. For iterations of φ we have $R(\varphi^n) = \prod_{j=1}^{c} R((\alpha_j)^n)$. Thus, for any n, $R(\varphi^n) = R(\alpha^n)$, where $\alpha := (\alpha_1, \dots, \alpha_c) : A \to A$ and $A := A_1 \oplus \cdots \oplus A_c$. For an abelian group the congruences are well known [5] and we are done. □

4. Twisted Burnside-Frobenius theorem

In this section we require φ to be an *automorphism* with $R(\varphi) < \infty$. The proof of Theorem 1.2 will be inductive, basing on the following properties of torsion-free nilpotent groups of finite rank: by Theorem 3.2,

$$R(\varphi_k) < \infty, \qquad k = 1, \dots, c, \tag{6}$$

and

$$a_n(G) \le n^{\mathrm{rk}(G)}, \tag{7}$$

where $a_n(G)$ is the number of subgroups of index n in G (see e.g. [13, Lemma 1.4.1]).

Proof of Theorem 1.2. Denote by c the class of G as above. We prove by induction over the class $k = 1, \dots, c$. For abelian groups the result is well known (the Reidemeister class of e is an invariant subgroup and the other ones are its cosets, so it is sufficient to consider the corresponding factor group). Now suppose that the statement is proved for all groups of class $k-1$, in particular for $\overline{\gamma}_{c-k}(G)$ and let us prove the statement for the class k, i.e., for $\overline{\gamma}_{c-k-1}(G)$.

The induction step will be formed by the following two sub-steps:

1) We find a characteristic subgroup H of finite index in $\overline{\gamma}_{c-k}(G)$ such that the projection $\overline{\gamma}_{c-k-1}(G) \to \overline{\gamma}_{c-k-1}(G)/H$ induces a bijection of Reidemeister classes.

2) The finite-by-abelian group $\overline{\gamma}_{c-k-1}(G)/H$ has finitely many inner automorphisms by Lemma 2.11. Applying Lemma 2.8 we complete the proof of the step.

It remains to find a subgroup as in 1). Choose some representatives $g_1, \dots, g_{R(\alpha_{c-k-1})}$ in $\overline{\gamma}_{c-k-1}(G)$, which go to representatives of all Reidemeister classes in A_{c-k-1} under the projection $\overline{\gamma}_{c-k-1}(G) \to A_{c-k-1}$. Then it is sufficient to find a subgroup H such that $\overline{\gamma}_{c-k}(G) \to \overline{\gamma}_{c-k}(G)/H$ induces a bijection of Reidemeister classes of $\tau_{g_i} \circ \varphi_{c-k}$, for any g_i, where $\tau_g(h) = ghg^{-1}$ as above.

Indeed, to have the desired bijection it is sufficient to have the property: for any Reidemeister class $\{sg_i\}_\varphi$ $(s \in \overline{\gamma}_{c-k}(G))$ of φ_{c-k-1} (we write simply φ as the index instead of φ_{c-k-1}) there is a shift of H (a coset) Hg contained in this class. We may consider specifically $g = sg_i$. Passing from a verification of an inclusion of Hg in the class of φ_{c-k-1} (i.e. $Hg \subseteq \{g\}_\varphi$) to a verification of an inclusion of H into the shifted class (i.e. $H \subseteq \{g\}_\varphi g^{-1}$), which is the Reidemeister class of e for $\tau_g \circ \varphi_{c-k-1}$ (by Lemma 2.5) and by Theorem 2.2 (the fixed-point trivial case) this is the same as to control that $H \subseteq \{e\}_{\psi'}$, where $\psi' = \tau_g \circ \varphi_{c-k}$. All these automorphisms have finite Reidemeister numbers and for each of them the TBFT_f is true by the induction supposition. So we have the appropriate normal $\tau_{sg_i} \circ \varphi_{c-k}$-invariant subgroup $H(sg_i)$ of finite index $I(sg_i)$, which is the stabilizer (with respect to right shifts) of all Reidemeister classes of $\tau_{sg_i} \circ \varphi_{c-k}$ (see the proof of Lemma 2.6). Since $\tau_{sg_i} \circ \varphi_{c-k} = \tau_s \circ (\tau_{g_i} \circ \varphi_{c-k})$ and $s \in \overline{\gamma}_{c-k}(G)$, we have $H(sg_i) = H(g_i)$ by Remark 2.7. So in fact we have only finitely many subgroups $H_i = H(g_i)$ of index $J_i < \infty$ $(i = 1, \dots, R(\alpha_{c-k-1}))$, each is $\tau_{sg_i} \circ \varphi_{c-k}$-invariant, normal in $\overline{\gamma}_{c-k}(G)$, and $H_i \subseteq \{e\}_{\tau_{sg_i} \circ \varphi_{c-k}} \subseteq \{e\}_{\tau_{sg_i} \circ \varphi_{c-k-1}}$ for any $s \in \overline{\gamma}_{c-k}(G)$. For each J_i we have finitely many subgroups of this index, namely $\le (J_i)^{\mathrm{rk}(G)}$ by (7), so their intersection $H_i^{\cap}$ is a characteristic subgroup of finite index in $\overline{\gamma}_{c-k}(G)$, in particular, normal in $\overline{\gamma}_{c-k-1}(G)$ and φ_{c-k-1}-invariant, because φ is an automorphism. It remains to take $H := \cap_i H_i^\cap$. □

Remark 4.1. From this theorem we obtain another proof of the Gauss congruences in the case of an automorphism φ.

5. Pólya-Carlson dichotomy

In this section we prove a Pólya–Carlson dichotomy between rationality and a natural boundary for the analytic behaviour of the Artin–Mazur dynamical zeta function of the dual map $\widehat{\varphi}_f$ for a torsion-free, finite rank nilpotent group automorphism φ.

We remind the definition of a natural boundary

Definition 5.1. Suppose that an analytic function F is defined somehow in a region D of the complex plane. If there is no point of the boundary ∂D of D over which F can be analytically continued, then ∂D is called a *natural boundary* for F .

The following results are needed to have more ready access to the theory of linear recurrence sequences.

Lemma 5.2. *(cf. [1]) Let $Z(z) = \sum_{n=1}^\infty |\operatorname{Fix}(\widehat{\varphi}_f^n)| z^n$. If $AM_{\widehat{\varphi}_f}(z)$ is rational then $Z(z)$ is rational. If $AM_{\widehat{\varphi}_f}(z)$ has an analytic continuation beyond its circle of convergence, then so does $Z(z)$ too. In particular, the existence of a natural boundary at the circle of convergence for $Z(z)$ implies the existence of a natural boundary for $AM_{\widehat{\varphi}_f}(z)$.*

Proof. This follows from the fact that $Z(z) = z \cdot AM_{\widehat{\varphi}_f}(z)'/AM_{\widehat{\varphi}_f}(z)$. □

One of the important links between the arithmetic properties of the coefficients of a complex power series and its analytic behaviour is given by the Pólya–Carlson theorem [15].

Pólya–Carlson Theorem. *A power series with integer coefficients and radius of convergence 1 is either rational or has the unit circle as a natural boundary.*

The *places* of a field $\mathbb{K}$ are the equivalence classes of absolute values on $\mathbb{K}$. When $char(\mathbb{K}) = 0$, the infinite places are the archimedean ones. All other places are said to be finite. Given a finite place of $\mathbb{K}$, it corresponds to a unique discrete valuation v whose precise value group is $\mathbb{Z}$. The corresponding normalised absolute value $|\cdot|_v = |\mathcal{R}_v|^{-v(\cdot)}$, where $\mathcal{R}_v$ is the residue class field of v. For any set of places S, we write $|x|_S = \prod_{v\in S} |x|_v$.

For the proof of the main theorem of this section we use the following result of Bell, Miles and Ward [1, Lem. 17]; one of the ingredients in its proof is the Hadamard quotient theorem.

Lemma 5.3 (Lemma 17 in [1])**.** *Let S be a finite list of places of algebraic number fields and, for each $v \in S$, let ξ_v be a non-unit root in the appropriate number field such that $|\xi_v|_v = 1$. Then the function*

$$F(z) = \sum_{n=1}^{\infty} f(n) z^n,$$

where $f(n) = \prod_{v\in S} |\xi_v^n - 1|_v$ for $n \geq 1$, has the unit circle as a natural boundary.

Let $\varphi\colon G \to G$ be an endomorphism of a group G. Then φ induces a continuous endomorphism $\overline{\varphi}\colon \overline{G} \to \overline{G}$ of the profinite completion $\overline{G}$ of G, and a natural map

$$\mathcal{R}(\varphi) \to \mathcal{R}(\overline{\varphi}), \quad [x]_\varphi \mapsto [\iota x]_{\overline{\varphi}},$$

where $\iota : G \to \overline{G}$ is the completion map.

Lemma 5.4. [4, Sec. 5.3.2], [3], [6]. *In the situation described above, the following properties hold.*

1. *If $R(\varphi) < \infty$, then the natural map $\mathcal{R}(\varphi) \to \mathcal{R}(\overline{\varphi})$ is surjective.*
2. *If G is abelian and $R(\varphi) < \infty$ then the natural map $\mathcal{R}(\varphi) \to \mathcal{R}(\overline{\varphi})$ is bijective.*

Finally we deduce the main result about Pólya–Carlson dichotomy stated in the introduction.

Proof of Theorem 1.3. For a tame automorphism φ, Theorems 1.2 and 3.2 imply that

$$|\operatorname{Fix}((\widehat{\varphi})^n_f)| = R(\varphi^n) = \prod_{k=1}^{c} R(\alpha_k^n), \qquad \text{for } n \in \mathbb{N}.$$

To prove the assertion 1) and the formula (1) we follow closely the proofs of Proposition 3.4 and Theorem 1.4 in [6]. Using Lemma 5.4(2), we may pass to the profinite completion of α_k. For $1 \le k \le c$, the profinite completion $\overline{A_k}$ of the abelian group A_k and its endomorphism $\overline{\alpha_k}$ decompose as direct products

$$\iota\colon A_k \to \overline{A_k} = \prod_{p \in \mathbb{P}} (\overline{A_k})_p \qquad \text{and} \qquad \overline{\alpha_k} = \prod_{p \in \mathbb{P}} (\overline{\alpha_k})_p,$$

where, for each prime p, the Sylow pro-p subgroup of $\overline{A_k}$ is the pro-p completion $(\overline{A_k})_p$ of A_k, equipped with the endomorphism $(\overline{\alpha_k})_p\colon (\overline{A_k})_p \to (\overline{A_k})_p$. Lemma 5.4(2) shows that

$$R(\alpha_k^n) = \prod_{p \in \mathbb{P}} R((\overline{\alpha_k})_p^n), \qquad \text{for } n \in \mathbb{N};$$

in particular, $R(\alpha_k^n) < \infty$ implies that $R((\overline{\alpha_k})_p^n) = 1$, for almost all $p \in \mathbb{P}$. Hence the product is only formally infinite.

Fix a prime $p \in \mathbb{P}$. The pro-p group $(\overline{A_k})_p$ is torsion-free, abelian and of rank at most d_k, hence $(\overline{A_k})_p \cong \mathbb{Z}_p^{d_k(p)}$, where $d_k(p) = \operatorname{rk}((\overline{A_k})_p) \le d_k$. Then there exist a subset $I_k(p) \subseteq \{1, \ldots, d_k\}$ such that the endomorphism $(\overline{\alpha_k})_p$ has eigenvalues $\xi_{k,i}$, $i \in I_k(p)$ (i.e. a part of eigenvalues of $\alpha_{k,\mathbb{Q}}$. In particular, the coefficients of the characteristic polynomial $\prod_{i \in I_k(p)} (X - \xi_{k,i})$ of $(\overline{\alpha_k})_p$ belong to $\mathbb{Z}_p$ (see [6, Prop. 3.4] for more detail).

Finally,

$$R((\overline{\alpha_k})_p^n) = |\operatorname{Coker}((\overline{\alpha_k})_p^n - 1)| = |\det((\overline{\alpha_k})_p^n - 1)|_p = \prod_{i \in I_k(p)} |\xi_{k,i}^n - 1|_p^{-1}.$$

Taking the product over all primes p and then over $1 \le k \le c$, we arrive to the first equality in (1). Using the adelic formula (see e.g. [16]) $|a|_\infty \prod_{p \in \mathbb{P}} |a|_p = 1$, for $a \in \mathbb{Q} \smallsetminus \{0\}$, we obtain the second equality in (1).

Thus it remains to prove the assertion 2).

For $1 \le k \le c$ and $p \in \mathbf{P}_k$ we write $S_k(p) = \{1, \ldots, d_k\} \smallsetminus I_k(p)$ and $S_k^*(p) = \{i \in S_k(p) \mid |\xi_{k,i}|_p \ne 1\}$. We set

$$b = \prod_{k=1}^{c} \prod_{p \in \mathbf{P}_k} \prod_{i \in S_k^*(p)} \max\{|\xi_{k,i}|_p, 1\}.$$

Then for $i \in S_k^*(p)$, $|\xi_{k,i}^n|_p = |\xi_{k,i}|_p^n \ne 1$ and $|\xi_{k,i}^n - 1|_p = \max\{|\xi_{k,i}^n|_p, 1\} = \max\{|\xi_{k,i}|_p^n, 1\} = \max\{\xi_{k,i}, 1\}^n$ (see e.g. [12, p. 6]). From this and from (1) we deduce that, for $n \in \mathbb{N}$,

$$|\operatorname{Fix}((\widehat{\varphi})^n_f)| = g(n) \cdot f(n),$$

where

$$g(n)=\prod_{k=1}^{c}\prod_{i=1}^{d_k}|\xi_{k,i}^n-1|_\infty\cdot b^n \text{ and } f(n)=\prod_{k=1}^{c}\prod_{p\in\mathbf{P}_k}\prod_{i\in S_k(p)\smallsetminus S_k^*(p)}\left|(\xi_{k,i})^n-1\right|_p. \tag{8}$$

Now we will rearrange the product $\prod_{i=1}^{d_k}|\xi_{k,i}^n-1|_\infty$. Complex eigenvalues $\xi_{k,i}$ in the spectrum of $\alpha_{k,\mathbb{Q}}$ appear in pairs with their complex conjugate $\overline{\xi_{k,i}}$. For a complex number λ one has

$$|\lambda^n-1|\cdot|\overline{\lambda}^n-1|=|\lambda^n-1|^2=(\lambda^n-1)\cdot(\overline{\lambda}^n-1).$$

If $\xi_{k,i}$ are real eigenvalues of $\alpha_{k,\mathbb{Q}}$ then we have $|\xi_{k,i}^n-1|_\infty=\delta_{1,k,i}^n-\delta_{2,k,i}^n$, where $\delta_{1,k,i}=\max\{|\xi_{k,i}|_\infty,1\}$ and $\delta_{2,k,i}=\frac{\xi_{k,i}}{\delta_{1,k,i}}$. Suppose that $\lambda_1,\overline{\lambda_1},\dots,\lambda_s,\overline{\lambda_s}$ are all complex eigenvalues and $\xi_{k,i(t)}$, $t=1,\dots,T$, are the real ones. Then the above two observations show that

$$\prod_{i=1}^{d_k}|\xi_{k,i}^n-1|_\infty=\sum\pm(\mu_1\nu_1\cdots\mu_s\nu_s\delta_{\epsilon(1),k,i(1)}\cdots\delta_{\epsilon(T),k,i(T)})^n, \tag{9}$$

where μ_i is λ_i or 1, ν_i is $\overline{\lambda_i}$ or 1, $\epsilon(i)$ is 1 or 2. Hence, taking the product over k and incorporating b we obtain

$$g(n)=\sum_{j\in J}c_jw_j^n, \tag{10}$$

where J is a finite index set, $c_j=\pm1$ and $w_j\in\mathbb{C}\smallsetminus\{0\}$, $j\in J$. Thus, the Artin-Mazur dynamical zeta function can be written as

$$AM_{\widehat{\varphi}_f}(z)=\exp\left(\sum_{j\in J}c_j\sum_{n=1}^{\infty}\frac{f(n)(w_jz)^n}{n}\right).$$

If $S_k(p)\setminus S_k^*(p)=\varnothing$ for all $p\in\mathbf{P}_k$ and all $k=1,...,c$, then $f(n)\equiv1$, and it follows immediately that the Artin-Mazur dynamical zeta function $AM_{\widehat{\varphi}_f}(z)$ is a rational function.

Now suppose that $S_k(p)\setminus S_k^*(p)\neq\varnothing$ for some $k\in\{1,\dots,c\}$, $p\in\mathbf{P}_k$. As noted in Lemma 5.2, we need only to exhibit a natural boundary at the circle of convergence for

$$\sum_{j\in J}\sum_{n=1}^{\infty}f(n)(w_jz)^n$$

to exhibit one for zeta function $AM_{\widehat{\varphi}_f}(z)$. Moreover, $\limsup_{n\to\infty}f(n)^{1/n}=1$ as it is evident from the definition (8), because the involved eigenvalues satisfy $|\xi_{k,i}|=1$. So for each $j\in J$, the series

$$\sum_{n=1}^{\infty}f(n)(w_jz)^n$$

has $|w_j|^{-1}$ as its radius of convergence.

Since $|\xi_{k,i}|_\infty\neq1$ for $i=1,\dots,d_k$, $k=1,\dots,c$, the equality (9) and the explicit form of $\delta_{1,k,i}$ and $\delta_{2,k,i}$ imply that there is a dominant term w_m in the expansion (10) of the form $|w_m|=\prod_{k=1}^{c}\prod_{i=1}^{d_k}\max\{|\xi_{k,i}|_\infty,1\}\cdot b$, such that $|w_m|>|w_j|$ for all $j\neq m$.

Since $|w_m|^{-1}<|w_j|^{-1}$ for all $j\neq m$, this means that it suffices to show that the circle of convergence $|z|=|w_m|^{-1}$ is a natural boundary for $\sum_{n=1}^{\infty}f(n)(w_mz)^n$. But this is the case precisely when $\sum_{n=1}^{\infty}f(n)z^n$ has the unit circle as a natural boundary, and this was handled in Lemma 5.3. □

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