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MULTIPLICATIVE FUNCTIONS IN SHORT ARITHMETIC PROGRESSIONS

OLEKSIY KLURMAN, ALEXANDER P. MANGEREL, AND JONI TERÄVÄINEN

Abstract. We study for bounded multiplicative functions f sums of the form

$$\sum_{\substack{n \leq x \\ n \equiv a \pmod q}} f(n),$$

establishing a theorem stating that their variance over residue classes $a \pmod q$ is small as soon as $q = o(x)$, for almost all moduli q, with a nearly power-saving exceptional set of q. This improves and generalizes previous results of Hooley on Barban–Davenport–Halberstam-type theorems for such f, and moreover our exceptional set is essentially optimal unless one is able to make progress on certain well-known conjectures. We are nevertheless able to prove stronger bounds for the number of the exceptional moduli q in the cases where q is restricted to be either smooth or prime, and conditionally on GRH we show that our variance estimate is valid for every q.

These results are special cases of a "hybrid result" that we establish that works for sums of $f(n)$ over almost all short intervals and arithmetic progressions simultaneously, thus generalizing the Matomäki–Radziwiłł theorem on multiplicative functions in short intervals.

We also consider the maximal deviation of $f(n)$ over *all* residue classes $a \pmod q$ in the square root range $q \leq x^{1/2-\varepsilon}$, and show that it is small for "smooth-supported" f, again apart from a nearly power-saving set of exceptional q, thus providing a smaller exceptional set than what follows from Bombieri–Vinogradov-type theorems.

As an application of our methods, we consider Linnik-type problems for products of exactly three primes, and in particular prove a ternary approximation to a conjecture of Erdős on representing every element of the multiplicative group $\mathbb{Z}_p^\times$ as the product of two primes less than p.

To the memory of Christopher Hooley

Contents

1. Main theorems

Let $\mathbb{U} := \{z \in \mathbb{C} : |z| \leq 1\}$ denote the unit disc of the complex plane, and let $f : \mathbb{N} \to \mathbb{U}$ be a 1-bounded multiplicative function. In this paper we study sums of the form

$$\sum_{\substack{n \leq x \\ n \equiv a \pmod q}} f(n) \tag{1}$$

with $(a, q) = 1$ and with the modulus $1 \leq q \leq x$ being very large as a function of x. We call such arithmetic progressions *short*, since the number of terms is $\sim x/q$, which is assumed to grow slowly with x.

Our main results concern the *deviation* of multiplicative functions $f : \mathbb{N} \to \mathbb{U}$ in residue classes in the square-root range $q \leq x^{1/2-\varepsilon}$, as well as their *variance* in residue classes in the full range $q = o(x)$. Here by deviation we mean

$$\max_{a \in \mathbb{Z}_q^\times} \Big| \sum_{\substack{n \leq x \\ n \equiv a \pmod q}} f(n) - \frac{\chi_1(a)}{\phi(q)} \sum_{n \leq x} f(n)\overline{\chi_1}(n) \Big|, \tag{2}$$

where $\mathbb{Z}_q^\times$ the set of invertible residue classes $\pmod q$, and by variance we mean

$$\sideset{}{^*}\sum_{a \pmod q} \Big| \sum_{\substack{n \leq x \\ n \equiv a \pmod q}} f(n) - \frac{\chi_1(a)}{\phi(q)} \sum_{n \leq x} f(n)\overline{\chi_1}(n) \Big|^2, \tag{3}$$

with χ_1 the character $\pmod q$ maximizing $\chi \mapsto \sum_{n \leq x} f(n)\overline{\chi}(n)$ and with $\sum^*_{a(q)}$ denoting a sum over reduced residue classes $\pmod q$. Comparing the sum (1) to the main term $\chi_1(a)/\varphi(q) \cdot \sum_{n \leq x} f(n)\overline{\chi}_1(n)$ is natural, since if f "correlates" with a Dirichlet character χ, then we expect

$$\sum_{\substack{n \leq x \\ n \equiv a \pmod q}} f(n) \approx \frac{\chi(a)}{\phi(q)} \sum_{n \leq x} f(n)\overline{\chi}(n).$$

We develop a systematic approach to estimating weighted character sums $\sum_{n \leq x} f(n)\overline{\chi(n)}n^{it}$ for the wide range of parameters $t, q = O(x)$ and deduce numerous consequences related to (2) and (3).

1.1. **Results for prime moduli.**

For many problems on well-distribution in arithmetic progressions one can obtain stronger results by restricting to prime moduli (see, for example, [12], [6]); the same is true in our setting.

Our first main result concerns the variance (3) in the range where x/q tends to infinity very slowly. It is motivated by the groundbreaking work of Matomäki and Radziwiłł [30], which produces a comparable result for multiplicative functions in short *intervals.*

In the statements of our theorems, for $f : \mathbb{N} \to \mathbb{U}$ and $x, q \geq 1$, we will use the pretentious distance function

$$\mathbb{D}_q(f, g; x) := \Big(\sum_{\substack{p \leq x \\ p \nmid q}} \frac{1 - \mathrm{Re}(f(p)\overline{g(p)})}{p} \Big)^{1/2} \tag{4}$$

of Granville and Soundararajan (see, e.g., [2, p. 3]).

All the constants in this paper implied by the $\ll$ notation will be absolute unless otherwise indicated.

Corollary 1.1. *Let* $1 \leq Q \leq x/10$ *and* $(\log(x/Q))^{-1/200} \leq \varepsilon \leq 1$. *Then there exists a set* $[1, x^{\varepsilon^{200}}] \cap \mathbb{Z} \subset \mathcal{Q}_{x,\varepsilon} \subset [1, x] \cap \mathbb{Z}$ *with* $|[1, Q] \setminus \mathcal{Q}_{x,\varepsilon}| \ll (\log x)^{\varepsilon^{-200}}$, *and such that the following holds.*

Let $p \in \mathcal{Q}_{x,\varepsilon} \cap [1, Q]$ *be a prime. Let* $f : \mathbb{N} \to \mathbb{U}$ *be a multiplicative function. Let* χ_1 *be the character* $\pmod p$ *minimizing the distance* $\inf_{|t| \leq \log x} \mathbb{D}_p(f, \chi(n)n^{it}; x)$. *Then we have*

$$\sideset{}{^*}\sum_{a \pmod p} \Big| \sum_{\substack{n \leq x \\ n \equiv a \pmod p}} f(n) - \frac{\chi_1(a)}{\phi(p)} \sum_{n \leq x} f(n)\overline{\chi_1}(n) \Big|^2 \ll \varepsilon \frac{x^2}{p}. \tag{5}$$

Moreover, assuming GRH, the above estimate holds for all $p \in [1, Q]$.

Remark 1.1. Applying Halász's theorem, we see that in Corollary 1.1 (as well as in our other results that follow) the main term $(\chi_1(a))/\phi(q) \cdot \sum_{n \leq x} f(n)\overline{\chi_1}(n)$ can be deleted from the variance, unless

$$\inf_{|t| \leq \log x} \mathbb{D}_p(f, \chi_1(n)n^{it} : x)^2 \leq 2 \log 1/\varepsilon. \tag{6}$$

In particular, if GRH holds, then by the pretentious triangle inequality we see that (6) can only hold if χ_1 is induced by χ', where χ' is the primitive character of conductor $\leq Q$ that minimizes $\inf_{|t| \leq \log x} \mathbb{D}(f, \chi(n)n^{it}; x)$ (without assuming GRH, the situation is somewhat more complicated; cf. Subsection 3.3).

We refer to Section 3 for a discussion of the strength of this theorem, as well as that of our other theorems.

1.2. **Smooth-supported functions in the square root range.** We are also able to obtain a result on the distribution of multiplicative functions in arithmetic progressions to all residue classes $\pmod q$ in the middle range $q \leq x^{1/2 - o(1)}$. This supports the well-known analogy between results for all moduli in the middle range $q \leq x^{1/2-o(1)}$ and almost all moduli in the large range $x^{1-\varepsilon} \leq q \leq x^{1-o(1)}$ (an example of this analogy is provided by the theorems of Bombieri–Vinogradov and Barban–Davenport–Halberstam).

As in [30], transferring results from the almost all case to the case of all arithmetic progressions requires a bilinear structure in our sums. In our case, we introduce this bilinear structure by considering multiplicative functions f supported on smooth (friable) numbers.

Theorem 1.2. *Let* $x \geq 10$, $(\log x)^{-1/200} \leq \varepsilon \leq 1$, $\eta > 0$, *and* $Q \leq x^{1/2 - 100\eta}$. *There is a set* $[1, x^{\varepsilon^{200}}] \cap \mathbb{Z} \subset \mathcal{Q}_{x,\varepsilon} \subset [1, x] \cap \mathbb{Z}$ *with* $|[1, Q] \setminus \mathcal{Q}_{x,\varepsilon}| \ll Qx^{-\varepsilon^{200}}$ *such that the following holds.*

Let $q \in \mathcal{Q}_{x,\varepsilon} \cap [1, Q]$ *be a prime. Let* $f : \mathbb{N} \to \mathbb{U}$ *be a multiplicative function supported on* x^η*-smooth numbers. Let* χ_1 *be the character* $\pmod q$ *minimizing the distance* $\inf_{|t| \leq \log x} \mathbb{D}(f, \chi(n)n^{it}; x)$. *Then we have*

$$\max_{a \in \mathbb{Z}_q^\times} \Big| \sum_{\substack{n \leq x \\ n \equiv a \pmod q}} f(n) - \frac{\chi_1(a)}{\phi(q)} \sum_{n \leq x} f(n)\overline{\chi_1}(n) \Big| \ll_\eta \varepsilon \frac{x}{q}. \tag{7}$$

Moreover, if $\mathcal{Q}'$ *is any subset of* $[1, Q]$ *whose elements are pairwise coprime, then we have the bound* $|\mathcal{Q}' \setminus \mathcal{Q}_{x,\varepsilon}| \ll (\log x)^{\varepsilon^{-200}}$. *Moreover, assuming GRH,* (7) *holds for all* $q \in [1, Q]$.

1.3. **Results for smooth moduli.** In the context of smooth moduli, our proof methods work better than in the case of prime moduli (see [43], [33] for other works leveraging the smoothness of moduli). Here by smooth moduli we mean those q that are $q^{\varepsilon'}$-smooth. For such q, we are able to unconditionally remove the exceptional set of moduli from Theorem 1.4. When working with composite moduli q with x/q very slowly growing, we need, however, to restrict to moduli that do not have abnormally many small prime divisors. To this end, we make the following definition.

Definition 1.1. We say that an integer $q \geq 1$ is *y-typical* if

$$\sum_{\substack{p|q \\ p \leq z}} 1 \leq \frac{1}{100}\pi(z) \quad \text{for all} \quad z \geq y.$$

We can now state our result for smooth moduli using the concept of $(x/Q)^{\varepsilon^2}$-typical numbers. A simple argument (see Section 9) shows that all $q \leq x$ are such numbers if $Q = o(x/(\log x)^{1/\varepsilon^2})$, and otherwise the number of $q \leq Q$ that are not $(x/Q)^{\varepsilon^2}$-typical is bounded by $\ll Q\exp(-(1/1000 + o(1))(x/Q)^{\varepsilon^2})$.

Theorem 1.3. *Let $1 \leq Q \leq x/10$, $(\log(x/Q))^{-1/200} \leq \varepsilon \leq 1$, and $\varepsilon' = \exp(-\varepsilon^{-4})$. Let $q \leq Q$ be $q^{\varepsilon'}$-smooth and $(x/Q)^{\varepsilon^2}$-typical. Let $f : \mathbb{N} \to \mathbb{U}$ be a multiplicative function. Let $\chi_1 \pmod q$ be the character minimizing the distance $\inf_{|t|\leq \log x} \mathbb{D}_q(f, \chi(n)n^{it}; x)$. Then we have*

$$\sideset{}{^*}\sum_{a \pmod q} \Big| \sum_{\substack{n \leq x \\ n \equiv a \pmod q}} f(n) - \frac{\chi_1(a)}{\varphi(q)} \sum_{n \leq x} f(n)\overline{\chi_1}(n) \Big|^2 \ll \varepsilon\phi(q)\Big(\frac{x}{q}\Big)^2.$$

We note that the need to restrict to typical moduli arises naturally in our proof and is present also in other works (slightly differently formulated), e.g. [27], [10]. See Subsection 3.4 for a discussion of the necessity of this assumption.

We can also generalize Corollary 1.1 from prime moduli to any set of coprime moduli, provided that we restrict to typical moduli (see Theorem 9.4 for a precise statement).

1.4. **General moduli.** We then proceed to state a result for moduli q that are not required to be prime or smooth. In this case we obtain the desired bound for the variance (3) for all typical moduli outside a nearly power-saving exceptional set.

Theorem 1.4. *Let $1 \leq Q \leq x/10$ and $(\log(x/Q))^{-1/200} \leq \varepsilon \leq 1$. Then there exists a set $[1, x^{\varepsilon^{200}}] \cap \mathbb{Z} \subset \mathcal{Q}_{x,\varepsilon} \subset [1, x] \cap \mathbb{Z}$ with $|[1, Q] \setminus \mathcal{Q}_{x,\varepsilon}| \ll Qx^{-\varepsilon^{200}}$ such that the following holds.*

Let $q \in \mathcal{Q}_{x,\varepsilon} \cap [1, Q]$ be $(x/Q)^{\varepsilon^2}$-typical. Let $f : \mathbb{N} \to \mathbb{U}$ be a multiplicative function. Let χ_1 be the character $\pmod q$ minimizing the distance $\inf_{|t|\leq \log x} \mathbb{D}_q(f, \chi(n)n^{it}; x)$. Then we have

$$\sideset{}{^*}\sum_{a(q)} \Big| \sum_{\substack{n \leq x \\ n \equiv a(q)}} f(n) - \frac{\chi_1(a)}{\phi(q)} \sum_{n \leq x} f(n)\overline{\chi_1}(n) \Big|^2 \ll \varepsilon\phi(q)\Big(\frac{x}{q}\Big)^2. \tag{8}$$

Moreover, assuming GRH, (8) holds for all $(x/Q)^{\varepsilon^2}$-typical $q \in [1, Q]$.

1.5. **Hybrid results.** As already mentioned, our results are motivated by the following theorem from [30].

Theorem A (Matomäki–Radziwiłł)**.** *Let $10 \leq h \leq X$, and let $f : \mathbb{N} \to [-1, 1]$ be multiplicative. Then we have*

$$\int_X^{2X} \Big| \sum_{x \leq n \leq x+h} f(n) - \frac{h}{X} \sum_{X \leq n \leq 2X} f(n) \Big|^2 dx \ll \Big(\Big(\frac{\log\log h}{\log h}\Big)^2 + (\log X)^{-1/50}\Big) Xh^2.$$

This was generalized to functions $f : \mathbb{N} \to \mathbb{U}$ that are not n^{it}-pretentious for any t by Matomäki–Radziwiłł–Tao [31]. Our next theorem is a hybrid result that allows us to "interpolate" between Theorem A (in the complex-valued case) and our Theorem 1.4 on multiplicative functions in short arithmetic progressions, thus generalizing both results. This theorem applies to sums of the form

$$\sum_{\substack{x \leq n \leq x+H \\ n \equiv a \pmod q}} f(n)$$

over short intervals and arithmetic progressions, with averaging over $x \in [X, 2X]$ and $a \in \mathbb{Z}_q^\times$, as soon as $H/q \to \infty$.

Theorem 1.5 (A Hybrid theorem)**.** *Let $X \geq h \geq 10$ and $1 \leq Q \leq h/10$. Let $(\log(h/Q))^{-1/200} \leq \varepsilon \leq 1$. Then there is a set $[1, X^{\varepsilon^{200}}] \cap \mathbb{Z} \subset \mathcal{Q}_{X,\varepsilon} \subset [1, X] \cap \mathbb{Z}$ satisfying $|[1, Q] \backslash \mathcal{Q}_{X,\varepsilon}| \ll QX^{-\varepsilon^{200}}$ such that the following holds.*

Let $q \in \mathcal{Q}_{X,\varepsilon} \cap [1, Q]$ be $(h/Q)^{\varepsilon^2}$-typical. Let $f : \mathbb{N} \to \mathbb{U}$ be multiplicative. Let χ_1 be the character $\pmod q$ minimizing the distance $\inf_{|t| \leq X} \mathbb{D}_q(f, \chi(n)n^{it}; X)$, and let $t_\chi \in [-X, X]$ be the point that minimizes[1] $\mathbb{D}_q(f, \chi(n)n^{it}; X)$ for each χ. Then we have

$$\int_X^{2X} \sideset{}{^*}\sum_{a \pmod q} \Big| \sum_{\substack{x < n \leq x+h \\ n \equiv a \pmod q}} f(n) - \frac{\chi_1(a)}{\phi(q)} \left(\int_x^{x+h} v^{it_{\chi_1}}\, dv \right) \frac{1}{X} \sum_{X < n \leq 2X} f(n) \overline{\chi}_1(n) n^{-it_{\chi_1}} \Big|^2 dx \ll \varepsilon \phi(q) X \Big(\frac{h}{q}\Big)^2. \tag{9}$$

Moreover, assuming GRH, the result holds for all $(h/Q)^{\varepsilon^2}$-typical $q \in [1, Q]$.

We note that, for $h \leq \varepsilon^2 X$, one can Taylor expand the integral above to write it as

$$\int_x^{x+h} v^{it_{\chi_1}}\, dv = \begin{cases} O(\varepsilon h), & |t_{\chi_1}| > \varepsilon^{-1} X/h, \\ hx^{it_{\chi_1}} + O(\varepsilon h), & |t_{\chi_1}| < \varepsilon X/h, \\ hx^{it_{\chi_1}} \frac{e^{i\tau} - 1}{i\tau} + O(\varepsilon h), & \tau := |t_{\chi_1}| h/X \in (\varepsilon, \varepsilon^{-1}). \end{cases}$$

In the case of real-valued multiplicative functions $f : \mathbb{N} \to [-1, 1]$, we have a simpler formulation of the result as follows.

Corollary 1.6. *Let the notation be as in Theorem 1.5, and assume additionally that f is real-valued. Then for all $q \in \mathcal{Q}_{X,\varepsilon} \cap [1, Q]$ that are $(h/Q)^{\varepsilon^2}$-typical we have*

$$\int_X^{2X} \sideset{}{^*}\sum_{a \pmod q} \Big| \sum_{\substack{x < n \leq x+h \\ n \equiv a \pmod q}} f(n) - \frac{\chi_1(a)}{\phi(q)} \frac{h}{X} \sum_{X \leq n \leq 2X} f(n) \overline{\chi_1}(n) \Big|^2 dx \ll \varepsilon \phi(q) X \Big(\frac{h}{q}\Big)^2. \tag{10}$$

Moreover, we can take χ_1 to be a real character $\pmod q$. Again, assuming GRH, the result holds for all $(h/Q)^{\varepsilon^2}$-typical $q \in [1, Q]$.

Taking $q = 1$ and h tending to infinity slowly with X, we recover Theorem A (with a smaller power of logarithm), and obtain a form that works for *any* 1-bounded f, whether it be n^{it}-pretentious or not. Taking in turn $Q = o(h)$ and $h = X$, we arrive at a slightly weaker form of our variance result, Theorem 1.4, where we now need to average over $x \in [X, 2X]$.

We can also specialize Corollary 1.6 to $f = \mu$ and to the smaller range $q \leq x^{\varepsilon^{200}}$ to obtain a clean statement, which has recently been used in [41] to obtain applications to ergodic theory.

Corollary 1.7. *Let $A \geq 1$, $\varepsilon > 0$, $X \gg_A 1$, $1 \leq h \leq X^{\varepsilon^{200}}$ and let q be $(h/q)^{\varepsilon^2}$-typical. Then we have*

$$\int_X^{2X} \sideset{}{^*}\sum_{a \pmod q} \Big| \sum_{\substack{x < n \leq x+h \\ n \equiv a \pmod q}} \mu(n) \Big|^2 dx \ll \varepsilon \varphi(q) X \left(\frac{h}{q}\right)^2,$$

except possibly if q is a multiple of a single number $q_0 \geq (\log X)^A$ depending only on A and X.

The exclusion of the multiples of a single modulus is necessary if Siegel zeros exist, as they bias the distribution of μ in residue classes.

[1]If there are several such t_χ, pick any one of them.

2. Applications

A celebrated theorem of Linnik states that the least prime $p \equiv a \pmod q$ is $\ll q^L$ for some absolute constant L and uniformly for $a \in \mathbb{Z}_q^\times$ and $q \geq 1$. The record value to date is $L = 5$, due to Xylouris [42]. For q^δ−smooth moduli (with $\delta = \delta(\varepsilon)$), a better bound of $\ll q^{12/5+\varepsilon}$ is available, this being a result of Chang [3, Corollary 11]. Under GRH, we would have $L = 2+o(1)$ in place of $L = 5$, and assuming a conjecture of Cramér-type, $L = 1+o(1)$ would be the optimal exponent.

We apply the techniques used to prove our main results to make progress on the analogue of Linnik's theorem for E_3 numbers, that is, numbers that are the product of *exactly* 3 primes. We seek bounds on the quantity

$$\mathscr{L}_3(q) := \max_{a\in\mathbb{Z}_q^\times} \min\{n \in \mathbb{N} : \ n \equiv a \pmod q) : \ n \in E_3\}.$$

One can show that under GRH one has $\mathscr{L}_3(q) \ll q^{2+o(1)}$. The E_3 numbers, just like the primes, are subject to the parity problem, and hence one cannot use sieve methods to tackle the problem of bounding $\mathscr{L}_3(q)$ (in contrast, for products of *at most* two primes $\leq x$, it is known that one can find them in every reduced residue class $\pmod q$ for $q \leq x^{1/2+\delta}$ for some $\delta > 0$ by a result of Heath-Brown [15] proved using sieve methods).

We show unconditionally that $\mathscr{L}_3(q) \ll q^{2+o(1)}$ for all smooth moduli and for all but a few prime moduli.

Theorem 2.1. *Let $\varepsilon > 0$, and let $\varepsilon' > 0$ be small enough in terms of ε.*

(i) *For any integer $q \geq 1$ all of whose prime factors are $\leq q^{\varepsilon'}$, for any $a \in \mathbb{Z}_q^\times$, there exists some q-smoooth $n \in E_3$ such that $n \ll q^{2+\varepsilon}$ and $n \equiv a \pmod q$. Thus, $\mathscr{L}_3(q) \ll q^{2+\varepsilon}$.*

(ii) *Let $Q \geq 2$. Then for all but $\ll_\varepsilon 1$ primes $q \in [Q^{1/2}, Q]$, for any $a \in \mathbb{Z}_q^\times$, there exists some q-smoooth $n \in E_3$ such that $n \ll q^{2+\varepsilon}$ and $n \equiv a \pmod q$. Thus, $\mathscr{L}_3(q) \ll q^{2+\varepsilon}$.*

This will be proved in Section 12. Since all the E_3 numbers we detect are q-smooth, our results are connected to the question of representing every element of the multiplicative group $\mathbb{Z}_q^\times$ by using only a bounded number of small primes. This problem was introduced by Erdős, Odlyzko and Sárközy in [5], and there Erdős conjectured that every residue class in $\mathbb{Z}_q^\times$, with q a large prime, has a representative of the form p_1p_2 with $p_1, p_2 \leq q$ primes. As is noted in [40], this remains open, even under GRH. The weaker "Schnirelmann-type" question of representing every residue class in $\mathbb{Z}_q^\times$ as the product of *at most* k primes in $[1, q]$ was studied by Walker [40], who showed[2] that $k = 6$ suffices for all large primes q, and moreover $k = 48$ suffices if we consider products of exactly k primes. Shparlinski [36] then improved on the latter by showing that at most 5 primes suffice for every large integer q. From Theorem 2.1 we deduce the following.

Corollary 2.2 (Ternary version of Erdős's conjecture with bounded exceptional set)**.** *There exists $C > 0$ such that the following holds. For all $Q \geq 2$ and all primes $q \in [Q^{1/2}, Q]$, apart from $\leq C$ exceptions, every element of the multiplicative group $\mathbb{Z}_q^\times$ can be represented as the product of exactly three primes from $[1, q]$.*

We also consider the analogue of Linnik's theorem for the Möbius function. Thus, we aim to bound

$$\mathscr{L}_\mu(q) := \max_{a\in\mathbb{Z}_q^\times} \min\{n \in \mathbb{N} : \ n \equiv a \pmod q) : \ \mu(n) = -1\}.$$

[2] Both in [40] and [36] a stronger result was shown, namely that one can restrict to primes in $[1, q^{1-\eta}]$ for explicitly given values of $\eta > 0$. An inspection of the proof of our Corollary 2.2 shows that there also we could restrict to primes bounded by $q^{1-\eta}$, with $\eta > 0$ small enough.

Since the theorems above give $\mathscr{L}_3(q) \ll q^{2+o(1)}$ for smooth q and all but a few primes q (and since the E_3 numbers we detect are typically squarefree), for such q we clearly have $\mathscr{L}_\mu(q) \ll q^{2+o(1)}$ as well. However, in the case of the Möbius function, we are able to obtain lower bounds of the correct order of magnitude as opposed to just showing the existence of such n.

Proposition 2.3. *Let $\varepsilon > 0$, $Q \geq 2$ Then, for all but $\ll_\varepsilon 1$ primes $q \in [Q^{1/2}, Q]$, we have*

$$\min_{a \in \mathbb{Z}_q^\times} \sum_{\substack{n \leq x \\ n \equiv a \pmod q}} 1_{\mu(n)=-1} \gg_\varepsilon \frac{x}{q}$$

for all $x \geq q^{2+\varepsilon}$. The same holds with $1_{\mu(n)=+1}$ in place of $1_{\mu(n)=-1}$.

Proposition 2.3 is in a sense an arithmetic progression analogue of a short interval result from [30, Corollary 5]. The Linnik-type problem considered above is however more difficult than its short interval analogue, since the current knowledge on zero-free regions of L-functions corresponding to characters of large conductor is somewhat poor. Indeed, unconditionally proving the estimate $\mathscr{L}_3(q) \ll q^{2+o(1)}$ for every q seems out of reach, due to connections between this problem and Vinogradov's conjecture (see Subsection 3.3).

3. Optimality of theorems and previous work

3.1. **Previous results.** The study of the deviations (2) and (3) of f in arithmetic progressions can roughly speaking be divided into three different regimes: the *small moduli* $q \leq x^\varepsilon$, the *middle moduli* $x^\varepsilon \leq q \leq x^{1-\varepsilon}$, and the *large moduli* $x^{1-\varepsilon} \leq q = o(x)$.

In the regime of small moduli, we have Linnik's theorem, which in its quantitative form [25, Theorem 18.6] gives the expected asymptotic formula for the average of μ (or Λ) over $a \pmod q$, valid for *all* $q \leq x^\varepsilon$, $a \in \mathbb{Z}_q^\times$, apart from possibly multiples of a single number q_0 (a Siegel modulus). A far-reaching generalization of this to arbitrary 1-bounded multiplicative functions f was achieved by Balog, Granville and Soundararajan [2]. See also the work [7] of Granville, Harper and Soundararajan for related results.

3.1.1. *Middle moduli.* The middle regime $q = x^\theta$ with $\varepsilon \leq \theta \leq 1-\varepsilon$ (and typically with θ near $1/2$) is arguably the most well-studied one. It includes the celebrated Bombieri–Vinogradov theorem, which for $f = \mu$ (or $f = \Lambda$) can be interpreted as providing cancellation in the deviation (2) for *almost all* $q \leq x^{1/2-\varepsilon}$ and *all* $a \in \mathbb{Z}_q^\times$. A complete generalization of the Bombieri–Vinogradov theorem to arbitrary 1-bounded multiplicative functions was recently achieved by Granville and Shao [8, Theorem 1.2].

The result of Granville and Shao in particular implies the following almost-all result: for all but $\leq Q/(\log x)^{1-\frac{1}{\sqrt{2}}-2\varepsilon}$ choices of $q \in [Q, 2Q] \subset [1, x^{1/2-\varepsilon}]$, we have

$$\max_{a \in \mathbb{Z}_q^\times} \Big| \sum_{\substack{n \leq x \\ n \equiv a \pmod q}} f(n) - \frac{\chi_1(a)}{\phi(q)} \sum_{n \leq x} f(n)\overline{\chi_1}(n) \Big| = o\Big(\frac{x}{q}\Big). \tag{11}$$

In Theorem 1.2, we demonstrated that if f is supported on x^η-smooth numbers, then the size of the exceptional set of $q \leq x^{1/2-\varepsilon}$ in (11) can be reduced to an almost power-saving bound, or even to a power of logarithm in the case of prime moduli. This may be compared with a recent result of Baker [1], which gives an analogous result for $f = \Lambda$, but in the smaller range $q \leq x^{9/40-\varepsilon}$.

3.1.2. *Large moduli.* In the large regime $x^{1-\varepsilon} \leq q = o(x)$, one aims for estimates valid for *almost all* q and for *almost all* $a \in \mathbb{Z}_q^\times$; results of this shape arise from upper bounds for the variance (3). The most classical theorem of this type is the Barban–Davenport–Halberstam theorem [25, Chapter 17], which states that

$$\sum_{q\leq x/(\log x)^B} \sum_{a\in\mathbb{Z}_q^\times} \Big| \sum_{\substack{n\leq x \\ n\equiv a \pmod q}} \mu(n)\Big|^2 \ll_A \frac{x^2}{(\log x)^A}, \tag{12}$$

with $B = B(A)$ explicit (and this result of course has an analogue where μ is replaced with Λ).

The Barban–Davenport–Halberstam theorem was extensively studied by Hooley in a seminal series of publications titled "On the Barban–Davenport–Halberstam theorem", spanning 19 papers. In this series, he significantly improved and generalized the Barban–Davenport–Halberstam bound, and among other things produced an asymptotic formula for the left-hand side of (12), and also with μ replaced by any bounded sequence satisfying a Siegel–Walfisz type assumption. Of this series of papers, the ones related to the aims of the present paper are [18], [19], [20], [21], [22]. In particular, from [18] (where Hooley considers the variance summed over all moduli $q \leq Q$) one can extract the following almost-all result (see also [39] for a related result, proved using the circle method).

Theorem B (Hooley)**.** *Let $\varepsilon > 0$ and $A \geq 1$ be fixed, let $1 \leq Q \leq x$, and let $f : \mathbb{N} \to \mathbb{U}$ be an arbitrary function satisfying the Siegel–Walfisz condition*[3]*. Denote $H := x/Q$. Then, for all $1 \leq q \leq Q$ apart from $\ll Q((\log H)/H + (\log x)^{-A})$ exceptions we have*

$$\sideset{}{^*}\sum_{a \pmod q} \Big| \sum_{\substack{n\leq x \\ n\equiv a \pmod q}} f(n) - \frac{\chi_0(a)}{\phi(q)} \sum_{n\leq x} f(n)\chi_0(n)\Big|^2 \leq \varepsilon\varphi(q)\left(\frac{x}{q}\right)^2, \tag{13}$$

where, for each $q \leq Q$, the character χ_0 is principal modulo q.

By Theorem 1.4, and the fact that the number of moduli $q \leq Q$ that are not H^{ε^2}-typical is $\ll Q\exp(-(1/1000 + o(1))H^{\varepsilon^2})$, the size of the exceptional set here for multiplicative f reduces to $\ll Q\exp(-c_0H^{\varepsilon^2})$. We can at the same time remove the Siegel–Walfisz assumption on f (by replacing χ_0 by another character $\pmod q$). By Theorem 1.4, we can further say that the number of exceptional q that are H^{ε^2}-typical is $\ll Qx^{-\varepsilon^{200}}$. This essentially power-saving bound was not, according to our knowledge, previously available even for $f = \mu$.

We note though that if one is interested in quantitative savings on the right-hand side of Theorem (B), then Hooley's result gives better error terms.

We now discuss some of the key features of Theorem 1.4 when it comes to the strength and optimality of the results.

3.2. **The description and size of the exceptional set.** The set $([1, x] \cap \mathbb{Z}) \setminus \mathcal{Q}_{x,\varepsilon}$ of exceptional moduli present in our main theorems turns out to be completely independent of the function f that we consider, a feature that is not present in the almost-all versions of the Barban–Davenport–Halberstam theorem or Hooley's Theorem B. In fact, we have an explicit description

[3]We say that f satisfies the Siegel–Walfisz condition if for all $1 \leq q \leq x, (a, q) = 1$ we have $\sum_{n\leq x, n\equiv a \pmod q} f(n) = \frac{1}{\phi(q)} \sum_{n\leq x,(n,q)=1} f(n) + O_A(x/(\log x)^A)$; in [20], Hooley works with a slightly more flexible version of this assumption.

of $\mathcal{Q}_{x,\varepsilon}$ in terms of zeros of L-functions $\pmod q$ as

(14)
$$\mathcal{Q}_{x,\varepsilon} := \Big\{q \leq x : \prod_{\substack{\chi \pmod q \\ \mathrm{cond}(\chi) > x^{\varepsilon^{200}}}} L(s,\chi) \neq 0 \quad \text{for} \quad \mathrm{Re}(s) \geq 1 - \frac{\varepsilon^{-100}(\log\log x)}{\log x}, \quad |\mathrm{Im}(s)| \leq 3x\Big\};$$

see Proposition 9.1 and Lemma 8.2 for this. Hence, if GRH (or even a weak version of it) holds, then $\mathcal{Q}_{x,\varepsilon}$ is all of $[1,x] \cap \mathbb{Z}$. From the description (14) and zero density estimates, it is not difficult to see that we have a *structural description* of the exceptional moduli as being multiples of a subset $\mathcal{E}_x \subset [x^{\varepsilon^{200}}, x]$ of integers of size $O((\log x)^{\varepsilon^{-200}})$. This also explains why for prime moduli (Corollary 1.1) we were able to obtain such a strong bound for the exceptional set.

3.3. **Connection to Vinogradov's conjecture and character sums.** For any fixed $\varepsilon > 0$, Theorem 1.4 gives a power-saving bound for the number of exceptional moduli (with the exponent of the saving approaching 0 as $\varepsilon \to 0$). This is essentially best possible, in the sense that replacing the bound $Qx^{-\varepsilon^{200}}$ by $Qx^{-\eta_0}$ for $\eta_0 > 0$ fixed would lead to the proof of some form of Vinogradov's conjecture[4] (which is known under GRH but not unconditionally).

Indeed, assuming the negation of Vinogradov's conjecture, there exists $\eta > 0$ and infinitely many $x \geq 10$ such that for some prime $x^{\eta - o(1)} \leq q_0 \leq x^{\eta}$ we have $1_{P^+(n) \leq q_0^{\eta}} \cdot \chi_{\mathrm{real}}(n) = 1_{P^+(n) \leq q_0^{\eta}}$ for all n, with χ_{real} the primitive quadratic character $\pmod{q_0}$ and with $P^+(n)$ the largest prime factor of n. Now, for $f_\eta(n) := 1_{P^+(n) \leq q_0^{\eta}}$, by the classical asymptotic formula for smooth numbers (and the fact that q_0 is prime), we have

(15)
$$\begin{aligned} \sum_{n \leq x} f_\eta(n)\chi_0(n) &= (\rho(\eta^{-2}) + o(1))x \\ \sum_{n \leq x} f_\eta(n)\chi_{\mathrm{real}}(n) &= (\rho(\eta^{-2}) + o(1))x, \end{aligned}$$

with χ_0 the principal character $\pmod{q_0}$ and $\rho(\cdot)$ the Dickman function (see Section 5 for its definition), so certainly

(16)
$$\frac{1}{\phi(q_0)} \sum_{\substack{\chi \pmod{q_0} \\ \chi \neq \chi_1}} \Big| \sum_{n \leq x} f_\eta(n)\overline{\chi}(n)\Big|^2 \gg_\eta \frac{x^2}{q_0},$$

for any choice of χ_1. However, by Parseval's identity (in the form of Lemma 7.1), (16) equals to the left-hand side of (8) (with $f = f_\eta$), and thus $q_0 \notin \mathcal{Q}_{x,\varepsilon}$ if ε is small in terms of η.

Note that if $Q = x/\log x$ and $r = q_0 p$ with $p \in [\log x, Q/q_0]$ a prime, then the same argument as above (with $\chi_0(n)$ and $\chi_{\mathrm{real}}(n)$ replaced by $\chi_0(n)1_{(n,r)=1}$ and $\chi_{\mathrm{real}}(n)1_{(n,r)=1}$ in (15)) shows that also $r \notin \mathcal{Q}_{x,\varepsilon}$, meaning that there are $\gg Qx^{-\eta+o(1)}$ exceptional $q \leq Q$ (again with ε small enough in terms of η). Taking $\eta < \eta_0$, this shows that the number of exceptional moduli is in fact not bounded by $\ll Qx^{-\eta_0}$. Thus one cannot generally improve on the exceptional set in Theorem 1.4 without settling Vinogradov's conjecture at the same time.

One could also adapt the argument above to show more strongly that improving the exceptional set would lead to cancellation in smooth character sums. Using arguments from [11], it should further be possible to say that this implies bounds for zeros of L-functions near 1 (and is therefore out of reach).

Similar conclusions apply to the size of the exceptional set in our other main theorems.

[4]Vinogradov's conjecture on the least quadratic nonresidue states that for every $\eta > 0$ and for any prime $q > q_0(\eta)$ there is a quadratic nonresidue $\pmod q$ on the interval $[1, q^{\eta}]$.

3.4. **The restriction to typical moduli.** We now discuss the importance of working with typical moduli in Theorems 1.4 and 1.5. In our proofs, as in the work [30], it is important for us to be able to discard those $n \leq x, n \equiv a \pmod q$ from the sum (8) that have no prime factors from certain long intervals $[P_i, Q_i]$ (with $Q_i \leq h/Q$). However, if q is divisible by all (or most) primes in $[P_i, Q_i]$, then we cannot discard the contribution of such integers. This would then prevent us from factorizing our character sums in the desired way.

While Theorem 1.4 may remain valid for all moduli $q \leq Q$ (under GRH), there seem to be some serious obstacles to proving this. Indeed, Granville and Soundararajan [10] proved a very general uncertainty principle for arithmetic sequences, which roughly speaking says that "multiplicatively interesting" sequences cannot be perfectly distributed in arithmetic progressions. For example, if $f(n) = 1_{(n,r)=1}$ with r having too many small prime factors in the sense that $\sum_{p|r, p \leq \log x} \log p/p \gg \log\log x$, then for each $C > 0$ there exists $y \in (x/4, x)$ and a progression $a \pmod q$ with $(a, q) = 1$ and $q \leq x/(\log x)^C$ and $P^-(q) \gg \log\log x$ such that the mean value of f over $n \leq y, n \equiv a \pmod q$ does not obey the anticipated asymptotic formula. Note that such an $f(n)$ can be $= 1$ for a positive proportion of $n \leq x$, for example if $r = \prod_{(\log x)^{1-\eta} \leq p \leq \log x} p$.

Similarly, if for example f is the indicator of sums of two squares, then the results of [10] imply that f is poorly distributed in some residue classes $a \pmod q$ with $q \leq x/(\log x)^C$.

3.5. **Remarks on improvements.** We finally list a few small improvements to our main theorems that could be obtained with only slight modifications to the proofs.

- In Theorem 1.5, we obtain a quantitative upper bound of the form $(\log(h/Q))^{-c}\phi(q)(x/q)^2$ for small $c > 0$ by choosing $\varepsilon = (\log(h/Q))^{-0.002}$, say. Thus our savings are comparable to those in [30, Theorem 3], and one cannot get larger savings than $((\log\log(h/Q))/\log(h/Q))^2$, since in the proof one restricts to integers having certain typical factorizations. However, if one specializes to the case $f = \mu$ in our main theorems, one can easily adapt the proof to yield savings of the form $\ll (\log(h/Q))^{-2+o(1)}$ by applying the Siegel–Walfisz theorem in place of Hálasz-type estimates. We leave the details to the interested reader.
- As in the work of Granville and Shao [8] on the Bombieri–Vinogradov theorem for multiplicative functions, we could obtain stronger bounds for (8) if we subtracted the contribution of more than one character from the sum of f over an arithmetic progression. Moreover, it follows directly from our proof that if we subtracted the contribution of $\ll (\log x)^{C(\varepsilon)}$ characters, then there would be no exceptional q at all in the theorem. We leave these modifications to the interested reader.

4. Proof ideas

We shall briefly outline some of the ideas that go into the proofs of our main results.

4.0.1. *Proof ideas for the variance results.* We start by discussing the proof of the hybrid result, Theorem 1.5; the proof of our result on multiplicative functions in short progressions, Theorem 1.4, is similar but slightly easier in some aspects.

As in the groundbreaking work of Matomäki–Radziwiłł [30], we begin by applying a suitable version of Parseval's indentity to transfer the problem to estimating an L^2-average of partial sums of f twisted by characters from a family. Of course, since we are working with both intervals and arithmetic progressions, the right family of characters to employ are the *twisted characters* $\{\chi(n)n^{it}\}_{\chi \pmod q, |t| \leq X/h}$; we reduce to obtaining cancellation in

$$\sum_{\chi \pmod q} \int_{t \in T_\chi} \Big| \sum_{X \leq n \leq 2X} f(n)\overline{\chi}(n) n^{-it} \Big|^2 dt,$$

with $T_\chi = [-X/h, X/h]$ if $\chi \neq \chi_1$ and $T_{\chi_1} = [-X/h, X/h] \setminus [t_{\chi_1} - \varepsilon^{-10}, t_{\chi_1} + \varepsilon^{-10}]$, with χ_1 and t_{χ_1} as in the theorem (so $(\chi, t) \mapsto \inf_{|t| \leq X} \mathbb{D}_q(f, \chi, n^{it})$ for $\chi \pmod q$ and $|t| \leq X$ is minimized at (χ_1, t_{χ_1})); this deleted segment of the integral corresponding to the character χ_1 accounts for our main term.

We make crucial use of the Ramaré identity, thus obtaining a factorization[5]

$$\sum_{X\le n\le 2X} f(n)\overline{\chi}(n)n^{-it} \approx \sum_{P_j\le p\le Q_j} f(p)\overline{\chi}(p)p^{-it} \sum_{X/p\le m\le 2X/p} f(m)a_{m,P_j,Q_j}\overline{\chi}(m)m^{-it},$$

with the parameters P_j, Q_j at our disposal, and the approximation being accurate in an L^2-sense (after splitting the p variable into short intervals). Here $a_{m,P_j,Q_j} := \frac{1}{1+\omega_{[P_j,Q_j]}(m)}$ is a well-behaved sequence, behaving essentially like the constant 1 for the purposes of our argument. After having obtained this bilinear structure, we split the "spectrum" $\{\chi \pmod q\} \times [-X/h, X/h]$ into parts depending on which of the sums $\sum_{P_j\le p\le Q_j} f(p)\overline{\chi}(p)p^{-it}$ with $j \le J$ (if any) exhibits cancellation. Different parts of the spectrum are bounded differently by establishing various mean and large values estimates for twisted character sums (see Section 7), in analogy with [30, Section 4] for Dirichlet polynomials.

The outcome of all of this is that we can reduce to the case where the longest of our twisted character sums, $\sum_{P_J\le p\le Q_J} f(p)\overline{\chi}(p)p^{-it}$, has (essentially) *no cancellation* at all. It is this large spectrum case where we significantly deviate from [30]; in that work, the large spectrum is not the most difficult case to deal with, thanks to the Vinogradov–Korobov zero-free region for the Riemann zeta-function. In our setting, in turn, we encounter L-functions $L(s,\chi)$ with χ having very large conductor, and for these L-functions the known zero-free regions are very poor (the best region being the Landau–Page zero-free region $\sigma > 1 - \frac{c_0}{\log(q(|t|+1))}$, valid apart from possible Siegel zeros). Thus, our task is to establish a bound essentially of the form

$$\sup_{\chi \pmod q} \sup_{\substack{|t|\le X \\ \chi=\chi_1 \Longrightarrow |t-t_{\chi_1}|\ge \varepsilon^{-10}}} \Big|\sum_{X\le n\le 2X} f(n)\overline{\chi}(n)n^{-it}\Big| \ll \varepsilon\frac{\phi(q)}{q}X \tag{17}$$

for the *sup norm* of the twisted character sums involved, as well as a proof that the large spectrum set under consideration is extremely small[6], that is,

$$\sup_{P\in[X^{\varepsilon},X]} \Big|\Big\{(\chi,t) \in \{\chi \pmod q\} \times \mathcal{T} :\ |\sum_{P\le p\le 2P} f(p)\overline{\chi}(p)p^{-it}| \ge \frac{\varepsilon P}{\log P}\Big\}\Big| \ll \varepsilon^{-2}, \tag{18}$$

with $\mathcal{T} \subset [-X, X]$ well-spaced. These two bounds are our two key Propositions 8.3 and 8.5 for the proof of the hybrid theorem. We need full uniformity in $|t|, q \le X$, and this makes the proofs somewhat involved: in particular, we need to make use of the work of Koukoulopoulos [29, Lemma 4.2], and the Granville–Harper–Soundararajan pretentious large sieve for the primes [7, Corollary 1.13] (as well as results of Chang [3, Theorem 5] for Theorem 1.3 on smooth moduli) to be able to prove these results. Of course, we cannot prove (17) or (18) for *all* $q \le x$ (see Subsection 3.3). What we instead establish is that (17) and (18) are valid whenever the functions $L(s,\chi)$ for every $\chi \pmod q$ (of large conductor) enjoy a suitable zero-free region (see Proposition 9.2 and Lemma 8.2 for the definition of the region involved). We can then make use of the log-free zero-density estimate for L-functions (Lemma 8.1) to bound the number of bad

[5] Due to the restriction to reduced residue classes $a \pmod q$ in our theorems, we have desirable factorizations for typical integers only if q is not divisible by an atypically large number of small primes, e.g. by almost all of the primes up to $(h/Q)^{0.01}$. This is what results in the need in our main theorems to restrict to typical moduli. This issue of course does not arise in the short interval setting of [30].

[6] One could use moment estimates (e.g. Lemma 7.5) to show that the large values set is $\ll (\log X)^{O_\varepsilon(1)}$ in size; however, in our case that would be a fatal loss, since the saving we get in (17) is at best $1/\log X$ and is therefore not enough to compensate this. In [30], a Halász–Montgomery-type estimate for prime-supported Dirichlet polynomials is established to deal with the large spectrum; our Proposition 8.5 essentially establishes a hybrid version of this, but in a very different regime.

q (and in the case of pairwise coprime moduli, as in Corollary 1.1, the bound is much better thanks to there being no effect from a single bad character inducing many others).

4.0.2. *Proof Ideas for the case of all moduli in the square-root range.* The starting point of the proof of Theorem 1.2 is the simple Lemma 11.4 that allows us to conveniently decompose any x^{η}-smooth number into a product $n = dm$ with an appropriate choice of $d, m \in [x^{1/2-\eta}, x^{1/2+\eta}]$. However, the decoupling of the d and m variables here is somewhat delicate and requires some smooth number estimates. After decoupling the variables (and extracting a further small prime factor), we have introduced a trilinear structure with two variables of almost equal length, which (by Cauchy–Schwarz) means that we can employ the techniques from previous sections to bound mean squares of the resulting character sums.

4.0.3. *Proof Ideas for the Linnik-type results.* For the proof of our Linnik-type results, Theorems 2.1(i)–(ii), we use similar ideas as for Theorem 1.2, with a couple of additions. Since we only need a positive lower bound for the number of $n \equiv a \pmod{q}$ that are E_3 numbers, we can impose the requirement that these n have prime factors from any intervals that we choose. Thanks to this flexibility in the sizes of the prime factors, we can get good bounds for the trilinear sums that arise. A key maneuver here is to count suitable n with the logarithmic weight $1/n$, so that we will be able to utilize a modification of the "Rodosskii bound" from the works of Soundararajan [37] and Harper [14], which establishes cancellation in logarithmically averaged character sums over primes assuming only a very narrow zero-free region. For smooth moduli, we have a suitable zero-free region by a result of Chang [3, Theorem 5], whereas for prime q we apply the log-free zero-density estimate to obtain a suitable region apart from a few bad moduli.

Future work

The arithmetic progression analogue of the Matomäki–Radziwiłł method that forms the basis of this paper is rather flexible, and in particular in a subsequent paper [28] we applied a variant of it over function fields to establish the Matomäki–Radziwiłł theorem for multiplicative functions $f : \mathbb{F}_q[T] \to [-1, 1]$ (which in turn was used to prove the logarithmic two-point Elliott conjecture over function fields). Our methods are not limited to bounded multiplicative functions either, and in a subsequent work we will prove an analogue of Theorem 1.4 for multiplicative functions that are only assumed to be bounded by a k-fold divisor function.

Structure of the paper

We will present the proofs of Theorems 1.4 and 1.5 (as well as Corollary 1.1) in Subsections 9.4 and 9.2, respectively. The necessary lemmas for proving these results are presented in Sections 6 and 7. Section 8 in turn contains two propositions that are key ingredients in the proofs of the main theorems. In Section 10 we prove Theorem 1.3 on smooth moduli. Our result on smooth-supported functions in the square-root range is proved in Section 11. Section 12 in turn contains the proofs of the applications to Linnik-type theorems. We remark that Sections 9, 11 and 12 can all be read independently of each other, but they depend on the work in Section 8.

5. Notation

We use the usual Vinogradov and Landau asymptotic notation $\ll, \gg, \asymp, O(\cdot), o(\cdot)$, with the implied constants being absolute unless otherwise stated. If we write $\ll_\varepsilon, \gg_\varepsilon$ or $O_\varepsilon(\cdot)$, this signifies that the implied constant depends on the parameter ε.

We write $1_S(n)$ for the indicator function of the set S. The functions Λ, ϕ and τ_k are the usual von Mangoldt, Euler phi and k-fold divisor functions, and $\pi(x)$ is the prime-counting function. The symbol $\rho : (0, \infty) \to [0, 1]$ in turn denotes the Dickman function, the unique solution to the delayed differential equation $\rho(u-1) = u\rho'(u)$ for $u > 1$, with the initial data $\rho(u) = 1$ for $0 < u \le 1$; see [17] for further properties of this function.

The symbol p is reserved for primes, whereas j, k, m, n, q are positive integers.

Below we list for the reader's convenience the notation we introduce in later sections.

Nomenclature

$\sum^*_{a(q)}$	A sum over the invertible residue classes $\pmod q$
$\sum^*_{\chi(q)}$	A sum over the primitive characters $\pmod q$
χ_0	The principal character
χ^*	The primitive character inducing the character χ
$\mathrm{cond}(\chi)$	The conductor of the character χ
$\mathbb{Z}_q^\times$	The set of invertible residue classes $\pmod q$
$\Omega_{[P,Q]}(n), \omega_{[P,Q]}(n)$	The number of prime factors of n from $[P, Q]$, with and without multiplicities
$P^+(n), P^-(n)$	The largest and smallest prime factor of n, respectively
$e(x)$	The complex exponential $e^{2\pi ix}$
$\Delta(q, Z)$	Equation (46)
$\Psi_q(X, Y)$	Equation (79)
$\mathbb{D}_q(f, g; x)$	Equation (4)
$\mathbb{D}(f, g; Y, X)$	Equation (23)
$F(\chi)$	Equation (67)
$L_y(s, \chi)$	Equation (20)
$M_q(T)$	Equation (21)
$N(\sigma, T, \chi)$	Equation (32)
$\mathcal{Q}_{x,\varepsilon,M}$	Equation (33)
V_t	Equation (19)

6. Lemmas on multiplicative functions

Throughout this section, given $t \in \mathbb{R}$ we set

$$V_t := \exp\Big(\log(3+|t|)^{2/3}\log\log(3+|t|)^{1/3}\Big). \tag{19}$$

For $y \ge 2$, $\mathrm{Re}(s) > 1$ and a multiplicative function $f : \mathbb{N} \to \mathbb{U}$, we define the truncated Euler product

$$L_y(s, \chi) := \prod_{p>y} \sum_{k \ge 0} \frac{\chi(p)^k}{p^{ks}}. \tag{20}$$

Also recall the definition of the $\mathbb{D}_q$ distance from (4), and let $\mathbb{D} := \mathbb{D}_1$.

Important note. In what follows, we will seek to make all of our estimates as sharp as possible as a function of q, in particular obtaining factors of $\phi(q)/q$ in our estimates wherever possible. While this increases the lengths of some proofs, it is critical in order for us to state our main variance estimates with no loss. We also remark that for the purposes of proving Theorem 1.4 our estimates only require uniformity in the t-aspect for $|t| \le \log x$; however, in order to prove Theorem 1.5 we will need full uniformity in the much larger range $|t| \ll x$.

6.1. **General Estimates for Partial Sums of Multiplicative Functions.**

Lemma 6.1 (A Halász-type inequality)**.** *Let $x \geq 10$ and $1 \leq q, T \leq 10x$. Let $f : \mathbb{N} \to \mathbb{U}$ be a multiplicative function. Then*

$$\frac{1}{x} \sum_{\substack{n \leq x \\ (n,q)=1}} f(n) \ll \frac{\phi(q)}{q}\Big((M_q(T)+1)e^{-M_q(T)} + \frac{1}{\sqrt{T}} + (\log x)^{-1/13}\Big),$$

where

$$M_q(T) = M_q(f;x,T) := \inf_{|t|\leq T} \mathbb{D}_q(f, n^{it}; x)^2. \tag{21}$$

Proof. We may assume that $T \leq \sqrt{\log x}$, since otherwise we can use $M_q(T) \leq M_q(\sqrt{\log x})$ and the fact that $y \mapsto (y+1)e^{-y}$ is decreasing. But then the claim follows[7] from [2, Corollary 2.2]. □

We also need a version of Halász's inequality that is sharp for sums that are restricted to rough numbers (i.e., integers n having only large prime factors). This will be employed in the proof of Lemma 6.6.

Lemma 6.2 (Halász over rough numbers)**.** *Let $2 \leq y \leq x$, and let $f : \mathbb{N} \to \mathbb{U}$ be multiplicative. Then*

$$\frac{1}{x} \sum_{\substack{n \leq x \\ P^-(n) > y}} f(n) \ll \frac{(1 + M(f;(y,x], \frac{\log x}{\log y}))e^{-M(f;(y,x],\frac{\log x}{\log y})}}{\log y} + \frac{1}{\log x},$$

where $M(f;(y,x],T)$ is defined for $T \geq 0$ by

$$M(f;(y,x],T) := \inf_{|t|\leq T} \mathbb{D}(f, n^{it}; y, x)^2 \tag{22}$$

with

$$\mathbb{D}(f, g; y, x) := \Big(\sum_{y<p\leq x} \frac{1 - \mathrm{Re}(f(p)\overline{g(p)})}{p}\Big)^{1/2}. \tag{23}$$

Proof. Without loss of generality, we may assume that $f(p^k) = 0$ for all primes $p \leq y$ and all $k \geq 1$. We may also assume that $y \leq x^{1/2}$, since otherwise the estimate follows trivially from the prime number theorem.

A consequence of [7, Proposition 7.1] (see in particular formula (7.3) there) implies that

$$\sum_{n\leq x} f(n) \ll (1+M)e^{-M}\frac{x}{\log y} + \frac{x}{\log x},$$

where M is defined implicitly via

$$\sup_{|t|\leq \frac{\log x}{\log y}} \Big|\frac{F(1+1/\log x+it)}{1+1/\log x+it}\Big| = e^{-M}\frac{\log x}{\log y},$$

where $F(s) := \prod_{p\leq x}\sum_{k\geq 0} f(p^k)/p^{ks}$ for $\mathrm{Re}(s) > 1$. On the other hand, note that for any $t \in \mathbb{R}$ by the assumption $f(p^k) = 0$ for $p \leq y$ we have

$$|F(1+1/\log x+it)|\frac{\log y}{\log x} \asymp \exp\Big(-\sum_{y<p\leq x}\frac{1-\mathrm{Re}(f(p)p^{-it})}{p}\Big) = e^{-\mathbb{D}^2(f,n^{it};y,x)},$$

[7] In [2, Corollary 2.2], it is assumed that $q \leq \sqrt{x}$, but the same proof works for $q \leq 10x$.

so that

$$e^{-M} \ll \sup_{|t|\leq \frac{\log x}{\log y}} \frac{e^{-\mathbb{D}^2(f,n^{it};y,x)}}{|1+1/\log x+it|} \ll e^{-M(f;(y,x],\frac{\log x}{\log y})}.$$

In particular, $M(f;(y,x],\frac{\log x}{\log y}) \leq M+O(1)$. Since $t\mapsto (1+t)e^{-t}$ is decreasing, it follows that

$$\sum_{n\leq x} f(n) \ll (1+M(f;(y,x],\frac{\log x}{\log y}))e^{-M(f;(y,x],\frac{\log x}{\log y})}\frac{x}{\log y}+\frac{x}{\log x},$$

as claimed. □

In the proof of Theorem 1.5, we will also need the following lemmas.

Lemma 6.3 (Lipschitz Bounds for Multiplicative Functions)**.** *Let $f:\mathbb{N}\to\mathbb{U}$ be multiplicative. Let $1\leq w\leq x^{1/3}$, and let $t_0\in[-\log x,\log x]$ be chosen to minimize $t\mapsto \mathbb{D}(f,n^{it};x)$. Then*

$$\left|\frac{w}{x}\sum_{n\leq x/w} f(n)n^{-it_0}-\frac{1}{x}\sum_{n\leq x} f(n)n^{-it_0}\right| \ll \left(\frac{\log w+(\log\log x)^2}{\log x}\right)^{1-\frac{2}{\pi}}\log\left(\frac{\log x}{\log(ew)}\right).$$

In particular, we have

$$\frac{2w}{x}\sum_{x/(2w)<n\leq x/w} f(n)n^{-it_0}=\frac{2}{x}\sum_{n\leq x/2} f(n)n^{-it_0}+O\left(\left(\frac{\log(ew)}{\log x}\right)^{-1+2/\pi+o(1)}\right).$$

Proof. The first statement is precisely [7, Thm. 1.5]. The second statement follows quickly from two applications of the first. □

Lemma 6.4 (Twisting by n^{it})**.** *Let $f:\mathbb{N}\to\mathbb{U}$ be multiplicative. Let α be any real number. Then for any $x\geq 3$,*

$$\frac{1}{x}\sum_{n\leq x} f(n)n^{i\alpha}=\frac{x^{i\alpha}}{1+i\alpha}\frac{1}{x}\sum_{n\leq x} f(n)+O\left(\frac{\log(2+|\alpha|)}{\log x}\exp\left(\mathbb{D}(f,1;x)\sqrt{(2+o(1))\log\log x}\right)\right).$$

Proof. This follows from [9, Lemma 7.1], combined with

$$\sum_{p\leq x}\frac{|1-f(p)|}{p}\leq(\log\log x+O(1))^{\frac{1}{2}}\left(\sum_{p\leq x}\frac{|1-f(p)|^2}{p}\right)^{\frac{1}{2}}=(\log\log x+O(1))^{\frac{1}{2}}\left(2\sum_{p\leq x}\frac{1-\mathrm{Re}(f(p))}{p}\right)^{\frac{1}{2}},$$

where we applied the Cauchy–Schwarz inequality. □

6.2. **Bounds on Prime Sums of Twisted Dirichlet Characters.**

Lemma 6.5 (A pretentious distance bound)**.** *Let $x\geq 10$, $1\leq q\leq x$, and let χ be any non-principal Dirichlet character modulo q induced by a primitive character χ^* modulo q^*. Then*

$$\inf_{|t|\leq 10x}\mathbb{D}(\chi,n^{it};x)^2\geq\frac{1}{4}\log\left(\frac{\log x}{\log(2q^*)}\right)+\log(q/\phi(q))+O(1).$$

Proof. We may assume that x is larger than any fixed absolute constant, since otherwise the bound is trivial upon choosing the term $O(1)$ appropriately (since $\mathbb{D}(\chi,n^{it};x)^2\geq\sum_{p|q}\frac{1}{p}$). Let $t_0\in[-10x,10x]$ be such that $t\mapsto\mathbb{D}(\chi,n^{it};x)$ is minimized at this point. We split the proof into a few cases.

Case 1. If $|t_0|\leq(\log x)^{10}$, then the claim follows directly from [2, Lemma 3.4].

Case 2. If in turn $|t_0|>(\log x)^{10}$, $q^*\leq V_{10x}$, then we have

$$\mathbb{D}(\chi,n^{it_0};x)^2\geq\left(\frac{1}{3}+o(1)\right)\log\log x-O(1)\geq\frac{1}{4}\log\log x+\log\frac{q}{\phi(q)}$$

by the fact that the zeros of $L(s,\chi)$ with $|\mathrm{Im}(s)| \leq 100x$ all lie in the region $\mathrm{Re}(s) \leq 1 - c/(\log V_{100x})$ (for real zeros corresponding to real characters this follows from Siegel's theorem, whereas for all other zeros this follows from the Vinogradov–Korobov bound).

Case 3. Lastly, assume that $q^* > V_{10x} \geq V_{t_0}$, $|t_0| > (\log x)^{10}$. Let us write $\chi(n) = \chi^*(n)1_{(n,r)=1}$, where χ^* (mod q^*) induces χ and $(r,q^*) = 1$. Let $y := q^* V_{t_0}$; by our assumptions we have $V_{10x} \leq y \leq (q^*)^2$.

We now observe that, since $q^* < y$, we have

$$\mathbb{D}(\chi, n^{it_0}; x)^2 \geq \sum_{p|q^*r} \frac{1}{p} + \mathrm{Re}\Big(\sum_{y<p\leq x} \frac{1-\chi^*(p)p^{-it_0}}{p}\Big) - O(1) - O\Big(\sum_{\substack{p|r\\ p\geq y}} \frac{1}{p}\Big)$$

$$= \log\Big(\frac{\log x}{\log y}\Big) - \log|L_y(1+1/\log x + it_0, \chi^*)| + \log \frac{q}{\phi(q)} - O(1),$$

where on the last line we used [29, Lemma 3.2] of Koukoulopoulos and the crude estimate $\omega(r) \ll \log r \ll \log x = o(y)$.

As $|t_0| \geq (\log x)^{10}$, from [29, Lemma 4.2], we see that $|L_y(1+1/\log x + it, \chi^*)| \asymp 1$ uniformly for $|t| \leq x$, given our choice of y. It follows that

$$\mathbb{D}(\chi, n^{it_0}; x)^2 \geq \log\Big(\frac{\log x}{\log y}\Big) + \log \frac{q}{\phi(q)} - O(1),$$

and recalling that $y \leq (q^*)^2$ the claim follows. □

The following pointwise bound for twisted character sums over primes will be needed in the proof of Proposition 8.5.

Lemma 6.6. *Let $x \geq 10$, $X = x^{(\log x)^{0.01}}$, and $1 \leq q \leq x$. Let h be a smooth function supported on $[1/2, 4]$. Then, for $\varepsilon \in (0,1)$ and for any character χ (mod q) with $\mathrm{cond}(\chi) \leq x^{\varepsilon}$, uniformly in the range $|t| \leq X$ we have*

$$\Big|\sum_n \Lambda(n)\chi(n)n^{-it}h\Big(\frac{n}{x}\Big)\Big| \ll_h x\varepsilon \log^3 \frac{1}{\varepsilon} + \frac{x}{(\log x)^{0.3}} + \frac{x}{t^2+1}. \tag{24}$$

Moreover, the $\frac{x}{t^2+1}$ term can be deleted for all but possibly one non-principal χ (mod q), which has to be real.

Remark 6.1. Without introducing a smooth weight, one could prove (24) with $x/(|t|+1)$ in place of $x/(t^2+1)$. We will however need the $1/(t^2+1)$ decay when we apply Lemma 6.6 in the proof of Proposition 8.5 to ensure that when (24) is summed over a well-spaced set of t the resulting bound is not too large.

Proof. We will split into cases cases depending on the sizes of q and t.

Without loss of generality, we may assume that x is larger than any given constant, $\varepsilon \geq (\log x)^{-0.4}$ and ε is smaller than any fixed constant. If χ is induced by χ^* (mod q^*), we have

$$\sum_n \Lambda(n)\chi(n)n^{-it}h\Big(\frac{n}{x}\Big) = \sum_n \Lambda(n)\chi^*(n)n^{-it}h\Big(\frac{n}{x}\Big) + O_h((\log x)^2),$$

and as the error term is small, may assume that χ is primitive and $q = q^*$.

If χ is the principal character, then χ is identically 1, and in that case (24) follows directly from Perron's formula in the form

$$\sum_n \Lambda(n)n^{-it}h\Big(\frac{n}{x}\Big) = -\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{\zeta'}{\zeta}(s+it)\widetilde{h}(s)x^s\, ds$$

and the Vinogradov–Korobov zero-free region (since the Mellin transform $\widetilde{h}$ of h obeys $|\widetilde{h}(s)| \ll_h 1/(1+|s|^{10})$ for $\mathrm{Re}(s) \in [-100, 100]$ by the smoothness of h).

If we have $2 \le q^*, |t| \le (\log x)^{10}$, then (24) follows straightforwardly from partial summation and the Siegel–Walfisz theorem (with a better bound of $\ll_h x(\log x)^{-100}$).

If in turn $q^* > (\log x)^{10}$, $|t| \le (\log x)^{10}$, we may argue as follows. We apply the explicit formula (proven similarly to [25, Proposition 5.25])

$$\sum_n \Lambda(n)\chi^*(n)n^{-it}h\Big(\frac{n}{x}\Big) = -\sum_{\substack{\rho=\beta+i\gamma:\\ L(\rho,\chi^*)=0\\ |\gamma-it|\le T\\ 0\le\beta\le 1}} x^{\rho-it}\widetilde{h}(\rho-it) + O_h\Big(\frac{x}{T}(\log^3(qx(|t|+2)))\Big),$$

where we choose $T=(\log x)^{100}$ to make the error term small.

Note that by the Landau–Page theorem [38, Theorem II.8.25] we have the zero-free region $L(s,\chi^*)\neq 0$ for $\mathrm{Re}(s)\ge 1-c_0/((\log x^{\varepsilon}(|t|+2)))$ for some constant $c_0>0$, apart from possibly one zero $\rho=\beta$, which has to be real and simple; additionally, such an exceptional zero can only exist for at most one character χ^* of conductor $\le x^{\varepsilon}$, which has to be real and non-principal. Since $\widetilde{h}(s)\ll 1/(1+|s|^{10})$ for $\mathrm{Re}(s)\in[-100,100]$ by the smoothness of h, the contribution of $\rho=\beta$ to this sum is bounded by

$$\ll_h \frac{x}{t^2+1}, \tag{25}$$

which is an admissible contribution. Then, following the analysis in [25, Chapter 18] verbatim (inserting the fact that the contribution of the possible exceptional zero β is bounded by (25)), we deduce

$$\Big|\sum_n \Lambda(n)\chi^*(n)n^{-it}h\Big(\frac{n}{x}\Big)\Big| \ll_h x^{1-c_1/\log(q^*(|t|+2))} + x\frac{\log q^*}{q^*} + \frac{x}{|t|^2+1}, \tag{26}$$

and the last term can be deleted except possibly for one choice of $\chi^*\neq\chi_0$. Since we assumed $|t|\le(\log x)^{10}\le q^*\le x^{\varepsilon}$, we have $\exp(-c_1\frac{\log x}{\log(x^{\varepsilon}(|t|+2)})) \ll \varepsilon^{100}$ and $(\log q^*)/q^*\ll(\log x)^{-9}$, so (26) results in a good enough bound.[8]

We may assume henceforth that $|t|>(\log x)^{10}$. Further, we may assume that $2\le q^*\le x^{1/50000}$. Since t is large, we no longer need the smoothing factor $h(n/x)$, and in fact by partial summation (and the fact that $h'(u)\ll_h 1/(1+u^{10})$) we see that (24) in the regime under consideration follows once we prove

$$\Big|\sum_{n\le x'}\Lambda(n)\chi(n)n^{-it}\Big| \ll \varepsilon\log^3(1/\varepsilon)x' + \frac{x'}{(\log x)^{0.3}} \tag{27}$$

for $x'\in[x/2,4x]$. In what follows, for the sake of notation we denote x' by x.

Put $y=(q^*)^4V_X^{100}$, so that for $q^*\le x^{1/50000}$ we have $y\le x^{1/10000}$. We define

$$\mu_y(n) := \mu(m)1_{P^-(m)>y},$$
$$\log_y m := (\log m)1_{P^-(m)>y},$$

and as in [7, Section 7] we make use of the convolution identity

$$\Lambda(n)1_{P^-(n)>y} = \mu_y\star\log_y(n), \quad n>y.$$

[8]Note that in the case $q=x^{\varepsilon}$, $t=X$, then arguments based on the proof of Linnik's theorem would only give a bound of $x^{1-\varepsilon^{-1}/\log x}(\log X)$ for (26). This is too weak when $\varepsilon=o(1/\log\log x)$, and therefore we need a different argument to handle this case.

By the prime number theorem, we then see that for any $t \in \mathbb{R}$ we have

$$\begin{aligned}\sum_{n \le x} \Lambda(n)\chi(n)n^{-it} &= \sum_{y^2 < n \le x} \Lambda(n)\chi(n)n^{-it} + O(y^2) \\ &= \sum_{y^2 < md \le x} \mu_y(m)\chi(m)m^{-it} \log_y(d)\chi(d)d^{-it} + O(y^2 + x^{1/3}).\end{aligned}$$

Let $y \le M, D \le x$ be parameters that satisfy $MD = x$, with $D \le x^{1/2}$. Using the hyperbola method, we have

$$\begin{aligned}\sum_{y^2 < n \le x} \Lambda(n)\chi(n)n^{-it} &= T_1 + T_2, \\ T_1 &:= \sum_{m \le M} \mu_y(m)\chi(m)m^{-it} \sum_{y^2/m < d \le x/m} \log_y(d)\chi(d)d^{-it} \\ T_2 &:= \sum_{d \le D} \log_y(d)\chi(d)d^{-it} \sum_{\substack{y^2/d < m \le x/d \\ m > M}} \mu_y(m)\chi(m)m^{-it}.\end{aligned}$$

We first deal with T_2. By Halász's theorem over rough numbers (Lemma 6.2), for each $d \le D$ the inner sum is

$$\ll \frac{x}{d}\Big(\frac{1}{\log y}(N+1)e^{-N} + \frac{1}{\log M}\Big) + \frac{y^2}{d},$$

where we have defined

$$N := \inf_{|u| \le \log x} \sum_{y < p \le M} \frac{1 - \mathrm{Re}(\mu_y(p)\chi(p)p^{-i(t+u)})}{p}.$$

As $D \le x^{1/2}$, $M \ge x^{1/2}$ and $\mu_y(p) = -1_{p>y}$, it follows (as in the proof of Lemma 6.5) that

$$\begin{aligned}N &\ge \inf_{|u| \le \log x} \sum_{y < p \le x} \frac{1 + \mathrm{Re}(\chi(p)p^{-i(t+u)})}{p} + O(1) \\ &\ge \log \frac{\log x}{\log y} + \inf_{|u| \le \log x} \log |L_y(1 + 1/\log x + i(t+u), \chi)| + O(1). \qquad (28)\end{aligned}$$

Now since $y > q^* V_{2X}$, [29, Lemma 4.2] shows that

$$|L_y(1 + 1/\log x + iw, \chi)| \asymp 1. \qquad (29)$$

for χ complex and $|w| \le 2X$, or for χ real and $1 \le |w| \le 2X$. Note that since $|t| \ge (\log x)^{10}$ in (28) by assumption, we have $|t + u| \ge 1$ there, and thus (29) holds in any case for $w = t + u$, $|u| \le \log x$.

The above implies that $N \ge \log((\log x)/(\log y)) - O(1)$, and hence by partial summation and the Selberg sieve we have

$$\begin{aligned}T_2 &\ll \frac{x \log \frac{\log x}{\log y}}{\log x} \sum_{\substack{d \le D \\ P^-(d) > y}} \frac{\log d}{d} + y^2 \log x \\ &\ll \frac{x \log \frac{\log x}{\log y}}{\log x}\Big(\frac{\log D}{\log y} + \int_y^D \Big(\sum_{\substack{d \le u \\ P^-(d) > y}} 1\Big) \log(u/e) \frac{du}{u^2}\Big) + y^2 \log x \\ &\ll x \frac{(\log D)^2 \log \frac{\log x}{\log y}}{(\log x)(\log y)} + y^2 \log x,\end{aligned}$$

for all non-principal characters χ modulo q.

We next estimate T_1. By partial summation, the inner sum in T_1, for each $m \le M$, is

$$\Big| \sum_{\substack{y^2/m<d\le x/m\\ P^-(d)>y}} (\log d)\chi(d)d^{-it}\Big| \ll (\log x) \max_{y\le u_1\le u_2\le x/m} \Big| \sum_{\substack{u_1\le d\le u_2\\ P^-(d)>y}} \chi(d)d^{-it}\Big| := R(m).$$

Recalling that $y = (q^*)^4 V_X^{100}$, we apply [29, Lemma 2.4] to the $R(m)$ terms, obtaining

$$R(m) \ll \frac{\log x}{\log y}\Big((x/m)^{1-1/(30\log y)} + (x/m)^{1-1/(100\log V_t)}\Big),$$

and since $y \ge V_t^{100}$, the second term can be ignored.

Summing over $m \le M$, and using Selberg's sieve to bound the number of integers with $P^-(m) > y$, we conclude that T_1 is bounded by

$$\begin{aligned}\sum_{\substack{m\le M\\ P^-(m)>y}} |R(m)| &\ll x\frac{\log x}{\log y} x^{-1/(30\log y)} \sum_{\substack{m\le M\\ P^-(m)>y}} m^{-1+1/(30\log y)}\\ &\ll x\Big(\frac{\log x}{\log y}\Big)^2\Big(\frac{x}{M}\Big)^{-1/30\log y}.\end{aligned}$$

Putting this all together and recalling $|t| \le X$, we find that

$$\begin{aligned}T_1 &\ll x\Big(\frac{\log x}{\log y}\Big)^2\Big(\frac{x}{M}\Big)^{-1/30\log y},\\ T_2 &\ll x\frac{(\log(x/M))^2\log\frac{\log x}{\log y}}{(\log x)(\log y)} + y^2\log x.\end{aligned}$$

We select $M = x/y^{1000\log(\log x/\log y)} \in [x^{1/2}, x]$ (so in particular $y \le x/M = D \le x^{1/2}$, as required). Then $\log(x/M) = 1000\log y\log\left(\frac{\log x}{\log y}\right)$ and thus, as $q^* \le x^{1/10}$, we have

$$T_1 + T_2 \ll x\Big(\frac{\log y}{\log x}\Big)^{30} + x\frac{\log y}{\log x}\log^3\left(\frac{\log x}{\log y}\right) + (q^*)^8V_X^{200}\log x.$$

If $q^* \le V_X$ then $\log y \le (\log x)^{0.68}$ for large enough x, and hence the bound reduces to $\ll x/(\log x)^{0.3}$. On the other hand, if $V_X < q^* \le x^{1/50000}$ then the above bound becomes $\ll x\frac{\log q^*}{\log x}\log^3\left(\frac{\log x}{\log q^*}\right) \ll \varepsilon\log^3(1/\varepsilon)x$. This completes the proof. □

7. Mean and large values estimates

We begin this section with several standard L^2-bounds for sums twisted by Dirichlet characters, analogous to the mean value theorem for Dirichlet polynomials ([25, Theorem 9.1]), where one twists by the Archimedean characters n^{it} instead. In our statements, care is made to obtain the sharpest possible dependence on q in the upper bounds, in particular in obtaining factors $\frac{\phi(q)}{q}$ wherever relevant, as these will be important in the proofs of the main theorems.

Lemma 7.1. *Let $q, M, N \ge 1$, and let $(a_n)_n$ be any complex numbers. Then*

$$\sum_{\chi \pmod q} \Big| \sum_{M<n\le M+N} a_n\chi(n)\Big|^2 \ll \Big(\phi(q) + \frac{\phi(q)}{q}N\Big) \sum_{\substack{M<n\le M+N\\ (n,q)=1}} |a_n|^2.$$

Proof. This is [32, Theorem 6.2]. □

Lemma 7.2 (L^2 integral large sieve for characters). *Let $T, N, q \geq 1$. Then*

$$\sum_{\chi \pmod q} \int_0^T \Big| \sum_{n \leq N} a_n \chi(n) n^{it} \Big|^2 dt \ll (\phi(q)T + \frac{\phi(q)}{q} N) \sum_{\substack{n \leq N \\ (n,q)=1}} |a_n|^2.$$

Proof. This is a slight sharpening of [32, Theorem 6.4] (more precisely, see (6.14) there). □

For the proof of Lemma 7.5, we will also need a discrete version of the large sieve estimate, in which we sum over well-spaced sets. We say that a set $\mathcal{T} \subset \mathbb{R}$ is *well-spaced* if $t, u \in \mathcal{T}$, $t \neq u$ implies $|t - u| \geq 1$.

Lemma 7.3 (Discrete large sieve for characters). *Let $T, N, q \geq 1$, and let $\mathcal{T} \subset [-T, T]$ be a well-spaced set. Then*

$$\sum_{\chi \pmod q} \sum_{t \in \mathcal{T}} \Big| \sum_{n \leq N} a_n \chi(n) n^{it} \Big|^2 \ll \Big(\phi(q)T + \frac{\phi(q)}{q} N\Big) \log(3N) \sum_{\substack{n \leq N \\ (n,q)=1}} |a_n|^2.$$

Proof. This result, which is a slight sharpening of [32, Theorem 7.4] (taking $\delta = 1$ there), is proved in a standard way by combining Gallagher's Sobolev-type lemma [25, Lemma 9.3] with Lemma 7.2; we leave the details to the reader. □

Lemma 7.4 (Halász–Montgomery large values estimate). *Let $T \geq 1$, $q \geq 2$ and let $\mathcal{E} \subset \{\chi \pmod q\} \times [-T, T]$ be such that $(\chi, t), (\chi, u) \in \mathcal{E}$ implies $|t - u| \geq 1$ or $t = u$. Then*

$$\sum_{(\chi,t) \in \mathcal{E}} \Big| \sum_{n \leq N} a_n \chi(n) n^{it} \Big|^2 \ll \Big(\frac{\phi(q)}{q} N + |\mathcal{E}| (qT)^{1/2} \log(2qT)\Big) \sum_{\substack{n \leq N \\ (n,q)=1}} |a_n|^2.$$

Proof. This is a slight sharpening (paying attention to coprimality with q) of [32, Theorem 8.3] (see especially (8.16), taking $\delta = 1$ and $\sigma_0 = 0$), and is proven much in the same way. We leave the details to the interested reader. □

When it comes to estimating the size of the large values set of a short twisted character sum supported on the primes, the following hybrid version of [30, Lemma 8] will be important.

Lemma 7.5 (Basic large values estimate – prime support). *Let $P, T \geq 2$. Let $\mathcal{T} \subset [-T, T]$ be well-spaced. Let*

$$P_\chi(s) := \sum_{P < p \leq 2P} a_p \chi(p) p^{-s},$$

where $|a_p| \leq 1$ for all $p \leq P$. Then for any $\alpha \in [0, 1]$ we have

$$|\{(\chi, t) \in \{\chi \pmod q\} \times \mathcal{T} : |P_\chi(it)| \geq P^{1-\alpha}\}| \ll (qT)^{2\alpha} \Big(P^{2\alpha} + \exp\Big(100 \frac{\log(qT)}{\log P} \log\log(qT)\Big)\Big).$$

Proof. Without loss of generality, we may assume that P and T are larger than any given constant. Let N be the number of pairs (χ, t) in question and $V := P^{1-\alpha}$; then

$$N \leq V^{-2k} \sum_{\chi \pmod q} \sum_{t \in \mathcal{T}} |P_\chi(it)|^{2k}$$

for any $k \geq 1$. We pick $k = \lceil \frac{\log(qT)}{\log P} \rceil$. Expanding out, we see that

$$B_\chi(s) := P_\chi(s)^k = \sum_{P^k < n \leq (2P)^k} b(n) \chi(n) n^{-s}, \quad \text{where} \quad b(n) = \sum_{\substack{p_1 \cdots p_k = n \\ p_j \in [P, 2P] \ \forall j}} a_{p_1} \cdots a_{p_k}.$$

By the discrete large sieve (Lemma 7.3), we have

$$\sum_{\chi \pmod q} \sum_{t \in \mathcal{T}} |P_\chi(it)|^{2k} = \sum_{\chi \pmod q} \sum_{t \in \mathcal{T}} |B_\chi(it)|^2$$
$$\ll (\phi(q)T + (2P)^k) \log P \sum_{P^k \le n \le (2P)^k} |b(n)|^2.$$

We can then compute the mean square over n as

$$\sum_{P^k \le n \le (2P)^k} |b(n)|^2 \le \sum_{\substack{p_1,\ldots,p_k = q_1 \cdots q_k \\ P \le p_j, q_j \le 2P}} 1 \le k!\Big(\sum_{P<p\le 2P} 1\Big)^k \le \Big(\frac{2P}{\log P}\Big)^k k!.$$

This gives the bound

$$\sum_{\chi \in \Xi} \sum_{t \in \mathcal{T}} |P_\chi(it)|^{2k} \ll k!(\phi(q)T + (2P)^k) \log P \Big(\frac{2P}{\log P}\Big)^k$$
$$= k! \log P \Big(1 + \frac{\phi(q)T}{(2P)^k}\Big)\Big(\frac{4P^2}{\log P}\Big)^k.$$

Multiplying this by V^{-2k} and recalling the choices of V and k, this becomes

$$\ll (qT)^{2\alpha} P^{2\alpha} \Big(\frac{8k}{\log P}\Big)^{k-1}.$$

We certainly have either $\log P \ge 8k$ or $\log P < 8k$, and in the first case we obtain $\ll (qT)^{2\alpha} P^{2\alpha}$ and in the second case we obtain $\ll (qT)^{2\alpha}(e^{20}k)^k$ (since P is large enough), so the claim follows. □

The proofs of the next two lemmas are almost identical to the proofs of the corresponding results in [30] with the following small modifications: one applies Lemma 7.2, rather than the mean value theorem for Dirichlet polynomials; the corresponding Dirichlet polynomials are considered on the zero line rather than the one line; the coefficients are supported on the integers $(n, q) = 1$ which accounts for the extra factor $\phi(q)/q$. We give the proof of one of them to illustrate the changes needed.

Lemma 7.6. *Let $q, T \ge 1$, $2 \le Y_1 \le Y_2$ and $\ell := \left\lceil \frac{\log Y_2}{\log Y_1} \right\rceil$. For a_m, c_p 1-bounded complex numbers, define*

$$Q(\chi, s) := \sum_{Y_1 \le p \le 2Y_1} c_p \chi(p) p^{-s} \quad \text{and} \quad A(\chi, s) := \sum_{X/Y_2 \le m \le 2X/Y_2} a_m \chi(m) m^{-s}.$$

Then

$$\sum_{\chi \pmod q} \int_{-T}^{T} |Q(\chi, it)^\ell A(\chi, it)|^2 dt \ll \frac{\phi(q)}{q} X Y_1 2^\ell \Big(\phi(q)T + \frac{\phi(q)}{q} X Y_1 2^\ell\Big)(\ell+1)!^2.$$

Moreover, we have the same bound for

$$\sum_{\chi \pmod q} |Q(\chi, 0)^\ell A(\chi, 0)|^2$$

when we put $T = 1$ on the right-hand side.

Proof. This is analogous to [30, Lemma 13]. The Dirichlet polynomial $Q(\chi, s)^\ell A(\chi, s)$ has its coefficients supported on the interval

$$[Y_1^\ell \cdot X/Y_2, (2Y_1)^\ell \cdot 2X/Y_2] \subset [X, 2^{\ell+1} Y_1 X].$$

We now apply Lemma 7.2 to arrive at

$$\sum_{\chi \pmod q} \int_{-T}^{T} |Q(\chi,it)^\ell A(\chi,it)|^2 dt \ll (\phi(q)T + \frac{\phi(q)}{q}2^\ell Y_1 X) \sum_{\substack{X\le n\le 2^{\ell+1}Y_1X \\ (n,q)=1}} \Big(\sum_{\substack{n=mp_1\dots p_\ell \\ Y_1\le p_1\dots p_\ell\le 2Y_1, \\ X/Y_2\le m\le 2X/Y_2}} 1\Big)^2.$$

We note that, for each n in the outer sum, we have

$$\sum_{\substack{n=mp_1\dots p_\ell \\ Y_1\le p_1\dots p_\ell\le 2Y_1, \\ X/Y_2\le m\le 2X/Y_2}} 1 \le \ell! \cdot \sum_{\substack{n=mr \\ p|r\Longrightarrow Y_1\le p\le 2Y_1}} 1 := \ell! g(n)$$

where $g(n)$ is a multiplicative function defined by $g(p^k) = k+1$ for $Y_1 \le p \le 2Y_1$ and $g(p^k) = 1$ otherwise. Consequently,

$$\sum_{\chi \pmod q} \int_{-T}^{T} |Q(\chi,it)^\ell A(\chi,it)|^2 dt \ll (\phi(q)T + \frac{\phi(q)}{q}2^\ell Y_1 X)(\ell!)^2 \sum_{\substack{X\le n\le 2^{\ell+1}Y_1X \\ (n,q)=1}} g(n)^2. \tag{30}$$

Shiu's bound [35, Theorem 1] in dyadic ranges yields

$$\sum_{\substack{Y\le n\le 2Y \\ (n,q)=1}} g(n)^2 \ll Y\frac{\phi(q)}{q} \prod_{\substack{p\le Y \\ p\nmid q}} \Big(1 + \frac{|g(p)|^2-1}{p}\Big) \ll Y\frac{\phi(q)}{q}. \tag{31}$$

We now split the right-hand side of (30) into dyadic ranges, apply (31) to each of them and sum the results up to finish the proof. □

Lemma 7.7. *Let $X \ge H \ge 1$, $Q \ge P \ge 1$. Let a_m, b_m, c_p be bounded sequences with $a_{mp} = b_m c_p$ whenever $p \nmid m$ and $P \le p \le Q$. Let Ξ be a collection of Dirichlet characters modulo $q \ge 1$. Let*

$$Q_{v,H}(\chi,s) := \sum_{\substack{P\le p\le Q \\ e^{v/H}\le p\le e^{(v+1)/H}}} c_p\chi(p)p^{-s},$$

and

$$R_{v,H}(\chi,s) := \sum_{Xe^{-v/H}\le m\le 2Xe^{-v/H}} b_m\chi(m)m^{-s}\cdot\frac{1}{1+\omega_{[P,Q]}(m)},$$

for each $\chi \in \Xi$ and $v \ge 0$. Let $\mathcal{T} \subset [-T,T]$ be measurable, and $\mathcal{I} := \{j \in \mathbb{Z} : \lfloor H\log P\rfloor \le j \le H\log Q\}$. Then

$$\sum_{\chi\in\Xi}\int_{\mathcal{T}}\Big|\sum_{n\le X} a_n\chi(n)n^{-it}\Big|^2 dt \ll H\log(Q/P)\sum_{j\in\mathcal{I}}\sum_{\chi\in\Xi}\int_{\mathcal{T}}\Big|Q_{j,H}(\chi,it)R_{j,H}(\chi,it)\Big|^2 dt$$
$$+\frac{\phi(q)}{q}X\Big(\phi(q)T+\frac{\phi(q)}{q}X\Big)\Big(\frac{1}{H}+\frac{1}{P}\Big)+\frac{\phi(q)}{q}X\Big(\sum_{\substack{n\le X \\ (n,q)=1}}|a_n|^2 1_{(n,\mathcal{P})=1}\Big),$$

where $\mathcal{P} := \prod_{P\le p\le Q} p$.

Moreover, the same bound holds for

$$\sum_{\chi\in\Xi}\Big|\sum_{n\le X} a_n\chi(n)\Big|^2$$

when we put $t = 0, T = 1$ and remove the integration on the right-hand side.

Proof. The proof is almost identical to the proof of [30, Lemma 12] (in particular, one uses the Ramaré identity), the only slight difference being that after splitting the sum involving a_n into short sums, one estimates the error terms applying Lemma 7.2, rather than the mean value theorem for Dirichlet polynomials. □

8. Key propositions

The goal of this section is to prove two key propositions, namely Propositions 8.3 and 8.5. For the proofs of both of these propositions, we will need good bounds on the number of Dirichlet characters whose L-functions have a bad zero-free region.

Define the count of zeros of an L-function $L(s,\chi)$ in the rectangle $[\sigma,1]\times[-T,T]$ of the complex plane by

$$N(\sigma,T,\chi):=\sum_{\substack{\rho:\ L(\rho,\chi)=0\\ \mathrm{Re}(\rho)\geq\sigma\\ |\mathrm{Im}(\rho)|\leq T}}1, \tag{32}$$

with multiple zeros counted according to their multiplicities. We will make use of the following bound for $N(\sigma,T,\chi)$.

Lemma 8.1 (Log-free zero-density estimate)**.** *For $Q,T\geq 1$, $\frac{1}{2}\leq\sigma\leq 1$ and $\varepsilon>0$, we have*

$$\sum_{q\leq Q}\ \sideset{}{^*}\sum_{\chi\ (\mathrm{mod}\ q)} N(\sigma,T,\chi)\ll_\varepsilon (Q^2T)^{(\frac{12}{5}+\varepsilon)(1-\sigma)}.$$

Proof. This is well-known (see 'Zeros Result 1 (iv)' in [13]). For $\frac{1}{2}\leq\sigma\leq 4/5$, say, the lemma follows from the work of Huxley [23], whereas in the complementary region we can apply Jutila's log-free zero-density estimate [26] (with $12/5+\varepsilon$ replaced with the better exponent $2+\varepsilon$). □

The log-free zero density estimate is easily employed to obtain the following.

Lemma 8.2. *Let $x\geq 10$, $\varepsilon\in((\log x)^{-0.04},1)$, and $1/(\log\log x)\leq M\leq\varepsilon^{20}\log x/(20\log\log x)$, and define the set*

$$\mathcal{Q}_{x,\varepsilon,M}:=\Big\{q\leq x:\prod_{\substack{\chi\ (\mathrm{mod}\ q)\\ \mathrm{cond}(\chi)>x^{\varepsilon^{20}}}}L(s,\chi)\neq 0\quad\text{for}\quad \mathrm{Re}(s)\geq 1-\frac{M(\log\log x)}{\log x},\quad |\mathrm{Im}(s)|\leq 3x\Big\}. \tag{33}$$

Then for $1\leq Q\leq x$ we have $|[1,Q]\setminus\mathcal{Q}_{x,\varepsilon,M}|\ll Qx^{-\varepsilon^{20}/2}$. Moreover, there exists a set $\mathcal{B}_{x,\varepsilon,M}\subset[x^{\varepsilon^{20}},x]$ of size $\ll(\log x)^{10M}$ such that every element of $[1,x]\setminus\mathcal{Q}_{x,\varepsilon,M}$ is a multiple of some element of $\mathcal{B}_{x,\varepsilon,M}$.

Proof. If $q\leq Q\leq x^{\varepsilon^{20}}$, then trivially $[1,Q]\subseteq\mathcal{Q}_{x,\varepsilon,M}$, so there is nothing to be proved. We may thus assume that $Q>x^{\varepsilon^{20}}$.

Let

$$\mathcal{B}_{x,\varepsilon,M}:=\Big\{q\leq x:\prod_{\substack{\chi\ (\mathrm{mod}\ q)\\ \chi\ \text{primitive}}}L(s,\chi)\neq 0\quad\text{for}\quad \mathrm{Re}(s)\geq 1-\frac{M\log\log x}{\log x},\quad |\mathrm{Im}(s)|\leq 3x\Big\}. \tag{34}$$

By the log-free zero density estimate provided by Lemma 8.1, we have

$$|\mathcal{B}_{x,\varepsilon,M}|\ll(x^3)^{(12/5+0.1)M\log\log x/\log x}\ll(\log x)^{10M}.$$

Since $L(s,\chi)$ and $L(s,\chi')$ have the same zeros if χ and χ' are induced by the same character, we see that every $q \le x$ with $q \notin \mathcal{Q}_{x,\varepsilon,M}$ is a multiple of some element of $\mathcal{B}_{x,\varepsilon,M} \cap [x^{\varepsilon^{20}}, x]$, and each such element has $\le Qx^{-\varepsilon^{20}} + 1$ multiples up to Q. Thus

$$|[1,Q] \setminus \mathcal{Q}_{x,Q,\varepsilon}| \ll (\log x)^{10M} Q x^{-\varepsilon^{20}} \ll Q x^{-\varepsilon^{20}/2},$$

since $M \le \varepsilon^{20}(\log x)/(20 \log\log x)$, $\varepsilon > (\log x)^{-0.04}$, and $Q > x^{\varepsilon^{20}}$. □

Proposition 8.3 (Sup norm bound for twisted sums of a multiplicative function)**.** *Let $x \ge 10$ and $(\log x)^{-1/13} \le \varepsilon \le 1$. Let $f : \mathbb{N} \to \mathbb{U}$ be a multiplicative function. Let (χ_1, t_{χ_1}) be a point minimizing the map $(\chi, t) \mapsto \mathbb{D}_q(f, \chi(n)n^{it}; x)$ among $\chi \pmod q$ and $|t| \le x$. Then, with the notation of Lemma 8.2, for all $q \in \mathcal{Q}_{x,\varepsilon,\varepsilon^{-6}}$ we have*

$$\sup_{\substack{\chi \pmod q \\ \chi \ne \chi_1}} \sup_{|t| \le x} \sup_{y \in [x^{0.1}, x]} \Big| \frac{1}{y} \sum_{n \le y} f(n)\overline{\chi}(n) n^{-it} \Big| \ll \varepsilon \frac{\phi(q)}{q}. \tag{35}$$

In addition, for all $1 \le Z \le x$ and $1 \le q \le x$ we have

$$\sup_{\substack{|t| \le x/2 \\ |t - t_{\chi_1}| \ge Z}} \sup_{y \in [x^{0.1}, x]} \Big| \frac{1}{y} \sum_{n \le y} f(n)\overline{\chi_1}(n) n^{-it} \Big| \ll \frac{\phi(q)}{q} \Big((\log x)^{-1/13} + \frac{1}{\sqrt{Z}} \Big). \tag{36}$$

Remark 8.1. For the proofs of Theorem 1.4 and 1.2, we need a version of this proposition where the infimum over t is over the range $[-\frac{1}{2}\log x, \frac{1}{2}\log x]$, and (χ_1, t_{χ_1}) is taken be a minimizing point of $(\chi, t) \mapsto \mathbb{D}_q(f, \chi(n)n^{it}; x)$ with $|t| \le \log x$. The same proof applies to this case, and we can obtain a similar variant of Corollary 8.4 as well.

Remark 8.2. The same arguments as in Subsection 3.3 show that we cannot prove (35) for *all* $q \le x$ without settling Vinogradov's conjecture at the same time. However, in the smaller regime of $q \le x^{\varepsilon^{20}}$ (which is not the primary concern for our main theorems) the exceptional set of moduli in Proposition 8.3 becomes empty; cf. [2, Lemma 3.4] for a similar result in this smaller range.

Proof. We begin with the first claim. We may assume that x is larger than any fixed absolute constant and that ε is smaller than any fixed absolute constant in what follows.

We now suppose there is a character $\chi \ne \chi_1 \pmod q$ and a real number $t \in [-x, x]$ for which

$$\Big| \sum_{n \le y} f(n)\chi(n) n^{-it} \Big| \ge \varepsilon \frac{\phi(q)}{q} y$$

for some $y \in [x^{0.1}, x]$. Owing to $\varepsilon > (\log x)^{-1/13}$ and the fact that $\sum_{y \le p \le x} \frac{1}{p} \ll 1$, Lemma 6.1 implies that there is a $v \in [-\frac{1}{2}\log x, \frac{1}{2}\log x]$ for which

$$\mathbb{D}_q(f, \chi(n)n^{i(t+v)}; x)^2 \le 1.1 \log(1/\varepsilon) + O(1).$$

According to the definition of χ_1, we also have $\mathbb{D}_q(f, \chi_1(n)n^{it_{\chi_1}}; x)^2 \le 1.1 \log(1/\varepsilon) + O(1)$ with $t_{\chi_1} \in [-x, x]$. As such, the triangle inequality implies that

$$\mathbb{D}_q(\chi_1(n)n^{it_{\chi_1}}, \chi(n)n^{i(t+v)}; x)^2 \le 4.4 \log(1/\varepsilon) + O(1),$$

so

$$\mathbb{D}(\chi_1(n)n^{it_{\chi_1}}, \chi(n)n^{i(t+v)}; x)^2 \le 4.4 \log(1/\varepsilon) + \log(q/\phi(q)) + O(1).$$

Lemma 6.5 gives

$$\mathbb{D}(\chi_1(n)n^{it_{\chi_1}}, \chi(n)n^{i(t+v)}; x)^2 \ge \frac{1}{4} \log\Big(\frac{\log x}{\log(2q^*)} \Big) + \log(q/\phi(q)) + O(1).$$

where $q^* = \mathrm{cond}(\chi_1\overline{\chi})$, so we may assume that $q^* > x^{\varepsilon^{20}}$.

To get a contradiction from this, it is enough to show that for all $q \in \mathcal{Q}_{x,\varepsilon,\varepsilon^{-6}}$ and for all $\xi \pmod q$ of conductor $\in [x^{\varepsilon^{20}}, x]$ we have

$$\sup_{|u|\leq 1.1x} \Big|\sum_{x^{\varepsilon^5}\leq p\leq x} \frac{\xi(p)}{p^{1+iu}}\Big| = o(1). \tag{37}$$

Indeed, once we have this, we deduce

$$\begin{aligned}\mathbb{D}(\chi_1(n)n^{it_{\chi_1}}, \chi(n)n^{i(t+v)}; x)^2 &\geq \sum_{x^{\varepsilon^5}\leq p\leq x} \frac{1-\mathrm{Re}(\chi_1\overline{\chi}(p)p^{i(t_{\chi_1}-t-v)})}{p} + \log(q/\phi(q)) + O(1)\\ &\geq 5\log(1/\varepsilon) + \log(q/\phi(q)) + O(1),\end{aligned}$$

since $\mathrm{cond}(\chi_1\overline{\chi}) \in [x^{\varepsilon^{20}}, x]$ and $\sum_{p|q, p>x^{\varepsilon^5}} 1/p \ll 1$, which contradicts the earlier upper bound.

Note that then by partial summation (37) certainly follows once we show that for $q \in \mathcal{Q}_{x,\varepsilon,\varepsilon^{-6}}$ we have

$$\sup_{|u|\leq 2.1x}\ \sup_{x^{\varepsilon^5}\leq P\leq x} \Big|\sum_{n\leq P} \Lambda(n)\xi(n)n^{-iu}\Big| \ll \frac{P}{(\log P)^{100}}. \tag{38}$$

By Perron's formula, we have

$$\sum_{n\leq P} \Lambda(n)\xi(n)n^{-iu} = -\frac{1}{2\pi i}\int_{1+1/\log x-iT}^{1+1/\log x+iT} \frac{L'}{L}(s+iu,\xi)\frac{P^s}{s}\,ds + O\Big(\frac{P}{(\log P)^{100}}\Big) \tag{39}$$

where $T := (\log x)^{1000}$. Recall that by the definition of $\mathcal{Q}_{x,\varepsilon,\varepsilon^{-6}}$ the function $L(s,\xi)$ has the zero-free region $\mathrm{Re}(s) \geq 1-\sigma_0 := 1-\varepsilon^{-6}(\log\log x)/(\log x)$, $|\mathrm{Im}(s)| \leq 3x$. Utilizing the fact that $|L'/L(s,\chi)| \ll \log^2(q(|t|+2))$ whenever the distance from s to the nearest zero of $L(\cdot,\chi)$ is $\geq \frac{1}{\log(q(|t|+2))}$, and shifting contours to $\mathrm{Re}(s) \geq 1-\sigma_0/2$ and recalling that we may assume $\varepsilon < 0.001$, we indeed obtain for (39) the bound

$$\ll P^{1-\sigma_0/2}(\log x)^3 \ll \frac{P}{(\log P)^{100}},$$

as wanted.

Next, we proceed to the second claim. Suppose $|t - t_{\chi_1}| \geq Z$ and $|t| \leq x$. Let $|u| \leq Z/2$, so that $|t+u-t_{\chi_1}| \geq Z/2$, and $|t+u-t_{\chi_1}| \leq 3x$. By definition of t_{χ_1} and the triangle inequality, we have

$$\begin{aligned}2\mathbb{D}_q(f, \chi_1(n)n^{i(t+u)}; x) &\geq \mathbb{D}_q(f, \chi_1(n)n^{i(t+u)}; x) + \mathbb{D}_q(f, \chi_1(n)n^{it_{\chi_1}}; x)\\ &\geq \mathbb{D}_q(n^{i(t+u-t_{\chi_1})}, 1; x) - O(\log\frac{q}{\phi(q)}).\end{aligned}$$

As in the proof of Lemma 6.1, we conclude that

$$\mathbb{D}_q(f, \chi_1(n)n^{i(t+u)}; x)^2 \geq (\frac{1}{12} - 10\varepsilon)\log\log x.$$

Applying the Halász-type bound of Lemma 6.1 with $T = Z/2$, we derive

$$\Big|\sum_{n\leq y} f(n)\overline{\chi_1}(n)n^{-it}\Big| \ll \frac{\phi(q)}{q}\Big((\log x)^{-1/13} + \frac{1}{\sqrt{Z}}\Big)y,$$

for every $|t| \leq x$ satisfying $|t - t_{\chi_1}| \geq Z$, as claimed. □

In addition to a sup norm bound for twisted character sums, we need a variant of the bound that works with an additional $1/(1+\omega_{[P,Q]}(m))$ weight coming from Lemma 7.7.

Corollary 8.4. *Let $x \geq R \geq 10$, $\varepsilon \in ((\log x)^{-1/13}, 1)$ and $(\log x)^{-0.1} < \alpha < \beta < 1$. Set $P = x^{\alpha}$, $Q = x^{\beta}$, and for $f : \mathbb{N} \to \mathbb{U}$ multiplicative consider the twisted character sum*

$$R(\chi, s) := \sum_{R \leq m \leq 2R} \frac{f(m)\overline{\chi}(m)m^{-s}}{1 + \omega_{[P,Q]}(m)}.$$

Let (χ_1, t_{χ_1}) be a point minimizing the map $(\chi, t) \mapsto \mathbb{D}_q(f, \chi(n)n^{it}; x)$ for $\chi \pmod q$ and $|t| \leq x$. Then, with the notation of Lemma 8.2, for $q \in \mathcal{Q}_{x,\varepsilon,\varepsilon^{-6}}$ we have

$$\sup_{\substack{\chi \pmod q \\ \chi \neq \chi_1}} \sup_{|t| \leq x/2} \sup_{R \in [x^{1/2}, x]} \frac{1}{R}|R(\chi, it)| \ll \Big(\Big(\frac{\beta}{\alpha}\Big)^2 \varepsilon + (20\beta \log \frac{\beta}{\alpha})^{\frac{1}{20\beta}}\Big)\frac{\phi(q)}{q}. \tag{40}$$

Furthermore, for $1 \leq Z \leq (\log x)^{1/10}$ we have

$$\sup_{Z \leq |t - t_{\chi_1}| \leq x} \sup_{R \in [x^{1/2}, x]} \frac{1}{R}|R(\chi_1, it)| \ll \Big(\Big(\frac{\beta}{\alpha}\Big)^2 \frac{1}{\sqrt{Z}} + (20\beta \log \frac{\beta}{\alpha})^{\frac{1}{20\beta}}\Big)\frac{\phi(q)}{q}.$$

Remark 8.3. In the proofs of the main theorems, we will eventually assume $q \in \mathcal{Q}_{x,\varepsilon^6,\varepsilon^{-100}}$ and choose $\alpha = \varepsilon^2$, $\beta = \varepsilon$, so that the bound above becomes $\ll \varepsilon^5 \phi(q)/q \cdot x$.

Proof. We start with the first claim. We may assume that $1/(20\beta) > \log(\frac{\beta}{\alpha})$, since otherwise the bound offered by the corollary is worse than trivial. For the same reason, we may assume that $\beta \leq 0.01$.

We use the hyperbola method, as in [30, Lemma 3]. Write $m \in [R, 2R]$ uniquely as $m = m_1 m_2$, where $p \mid m_1 \Longrightarrow p \in [P, Q]$ and $(m_2, [P, Q]) = 1$ (meaning that (m, r)=1 for all primes $r \in [P, Q]$). We use Möbius inversion on the m_2 variable in the form $1_{(m_2,[P,Q])} = \sum_{p|d \Longrightarrow p \in [P,Q]} \mu(d) 1_{d|m_2}$ to obtain

(41)

$$|R(\chi, it)| \ll \Big| \sum_{\substack{m_1 \leq x^{0.3} \\ p|m_1 \Longrightarrow p \in [P,Q]}} \frac{f(m_1)\overline{\chi}(m_1)m_1^{-it}}{1 + \omega_{[P,Q]}(m_1)} \sum_{\substack{p|d \Longrightarrow p \in [P,Q] \\ d \leq x^{0.1}}} \mu(d)f(d)\overline{\chi}(d)d^{-it} \sum_{\frac{R}{dm_1} \leq m_2' \leq \frac{2R}{dm_1}} f(m_2')\overline{\chi}(m_2')(m_2')^{-it}\Big|$$

$$+ \sum_{\substack{m_2 \leq 2R/x^{0.3} \\ (m_2,[P,Q])=1 \\ (m_2,q)=1}} \sum_{\substack{x^{0.3} < m_1 \leq 2R/m_2 \\ p|m_1 \Longrightarrow p \in [P,Q] \\ (m_1,q)=1}} 1 + \sum_{\substack{p|m_1 d \Longrightarrow p \in [P,Q] \\ x^{0.1} < d \leq R \\ (m_1 d, q)=1}} \frac{R}{m_1 d}.$$

Put $\Psi(X, Y)$ to denote the number of $n \leq X$ that are Y-smooth. By partial summation the third sum can be bounded by

$$\ll \Big(\sum_{\substack{p|m_1 \Longrightarrow p \in [P,Q] \\ (m_1,q)=1}} \frac{R}{m_1}\Big)\Big(\frac{\Psi(R, x^{\beta})}{x^{\beta}} - \frac{\Psi(x^{0.1}, x^{\beta})}{x^{0.1}} + \int_{x^{0.1}}^{R} \frac{\Psi(u, x^{\beta})}{u}\frac{du}{u}\Big)$$

$$\ll R \prod_{\substack{P \leq p \leq Q \\ p \nmid q}} \Big(1 - \frac{1}{p}\Big)^{-1} \max_{x^{0.1} \leq y \leq R} \frac{\Psi(y, x^{\beta})}{y} \ll \frac{\phi(q)}{q} R \log(\beta/\alpha) \max_{x^{0.1} \leq y \leq R} \frac{\Psi(y, x^{\beta})}{y}.$$

Set $u := \frac{\log y}{\log Q} \geq 1/(10\beta)$. Then inserting the standard upper bound $\Psi(y, x^{\beta}) \ll yu^{-u/2} \ll y\beta^{1/(20\beta)}$ to the estimate above, we conclude that the contribution of the third sum in (41) is acceptable.

The second sum can be treated similarly. Indeed, after swapping the orders of summation and dropping the condition $(m_2, [P,Q]) = 1$, this sum is

$$\leq \sum_{\substack{x^{0.3}<m_1\leq 2R \\ P^+(m_1)\leq Q \\ (m_1,q)=1}} \sum_{m_2\leq 2R/m_1} 1 \ll \sum_{\substack{x^{0.3}<m_1\leq 2R \\ P^+(m_1)\leq Q \\ (m_1,q)=1}} \frac{R}{m_1},$$

which is $O\Big(\frac{\phi(q)}{q}R\beta^{\frac{3}{20\beta}}\Big)$ using the argument above (with $x^{0.3}$ in place of $x^{0.1}$). This is also an acceptable error term.

It remains to estimate the triple sum in absolute values in (41). Note that $R/(dm_1) \geq x^{0.1}$ for all $d \leq x^{0.1}$. Since $\varepsilon > (\log x)^{-1/13}$, from Proposition 8.3 we see that the inner sum over $m_2' \in [R/(dm_1), 2R/(dm_1)]$ is $\ll \varepsilon\phi(q)/q \cdot R/(dm_1)$ whenever $q \in \mathcal{Q}_{x,\varepsilon,\varepsilon^{-6}}$ and $\chi \neq \chi_1$. Thus the sum on the first line of (41) is

$$\begin{aligned} &\ll \varepsilon\frac{\phi(q)}{q}R\Big(\sum_{\substack{m_1\leq x^{0.3} \\ p|m_1\Longrightarrow p\in[P,Q]}} \frac{1}{m_1}\Big)\Big(\sum_{p|d\Longrightarrow p\in[P,Q]}\frac{1}{d}\Big) \\ &\ll \varepsilon\frac{\phi(q)}{q}R\prod_{P\leq p\leq Q}\Big(1-\frac{1}{p}\Big)^{-2} \ll \varepsilon\Big(\frac{\beta}{\alpha}\Big)^2\frac{\phi(q)}{q}R \end{aligned}$$

by Mertens' theorem. The first claim is thus established.

The second claim is proved almost verbatim with the same argument; the only difference is that the pointwise estimate (36), rather than (35), must be used in this case. □

Proposition 8.5 (Sharp large values bound for weighted sums of twisted characters)**.** *Let $x \geq 10$, $(\log x)^{-0.05} \leq \eta \leq 1$ and $(\log x)^{-0.05} < \varepsilon \leq \delta \leq 1/2$. Let $(a_p)_p$ be 1-bounded complex numbers, let $\mathcal{T} \subset [-x,x]$ be a well-spaced set, and let $\mathcal{S} := \{(\chi,t) : \chi \pmod q,\ t \in \mathcal{T}\}$. Define*

$$N_{q,\mathcal{S}} := \sup_{x^\eta\leq P\leq x}\Big|\Big\{(\chi,t)\in\mathcal{S} : \Big|\frac{\log P}{\delta P}\sum_{P\leq p\leq(1+\delta)P} a_p\overline{\chi}(p)p^{-it}\Big| \geq \varepsilon\Big\}\Big|.$$

Then, with the notation of Lemma 8.2, for $q \in \mathcal{Q}_{x,\eta^{1/20}\varepsilon,1000\eta^{-1}}$ we have $N_{q,\mathcal{S}} \ll \varepsilon^{-2}\delta^{-1}$, with the implied constant being absolute.

Moreover, for $1 \leq Q \leq x$ we have $|[1,Q] \setminus \mathcal{Q}_{x,\eta^{1/20}\varepsilon,1000\eta^{-1}}| \ll Qx^{-\eta\varepsilon^{20}/2}$.

Remark 8.4. For proving Theorem 1.4, we only need to establish the result in the special, simpler case that $\mathcal{T} = \{0\}$. In contrast, for Theorem 1.5 we will need the full power of this proposition.

Proof. The part of the proposition involving the size of $\mathcal{Q}_{x,\eta^{1/20}\varepsilon,1000\eta^{-1}}$ follows directly from Lemma 8.2. It thus suffices to prove the bound for $N_{q,\mathcal{S}}$.

We may assume without loss of generality that $\varepsilon > 0$ is smaller than any fixed constant. Let $P \in [x^\eta, x]$ yield the set of largest cardinality that is counted by $N_{q,\mathcal{S}}$, and let $\mathcal{B}_{q,\mathcal{S}}$ denote the set of pairs (χ,t) yielding the large values counted by $N_{q,\mathcal{S}}$. We have

$$\begin{aligned} \frac{\varepsilon\delta P}{\log P}N_{q,\mathcal{S}} &\leq \sum_{(\chi,t)\in\mathcal{B}_{q,\mathcal{S}}}\Big|\sum_{P\leq p\leq(1+\delta)P} a_p\overline{\chi}(p)p^{-it}\Big| = \sum_{(\chi,t)\in\mathcal{B}_{q,\mathcal{S}}} c_{\chi,t}\sum_{P\leq p\leq(1+\delta)P} a_p\overline{\chi}(p)p^{-it} \\ &= \sum_{P\leq p\leq(1+\delta)P} a_p\sum_{(\chi,t)\in\mathcal{B}_{q,\mathcal{S}}} c_{\chi,t}\overline{\chi}(p)p^{-it}, \end{aligned} \tag{42}$$

for suitable unimodular numbers $c_{\chi,t}$. Applying the Cauchy–Schwarz and Brun–Titchmarsh inequalities, an upper bound for this is

$$\begin{aligned} &\ll \Big(\frac{\delta P}{\log P}\Big)^{1/2}\Big(\sum_{P\le p\le (1+\delta)P}\Big|\sum_{(\chi,t)\in\mathcal{B}_{q,\mathcal{S}}} c_{\chi,t}\overline{\chi}(p)p^{-it}\Big|^2\Big)^{1/2}\\ &\le \Big(\frac{\delta P}{\log P}\Big)^{1/2}\Big(\sum_{P\le n\le (1+\delta)P}\frac{\Lambda(n)}{\log P}\Big|\sum_{(\chi,t)\in\mathcal{B}_{q,\mathcal{S}}} c_{\chi,t}\overline{\chi}(n)n^{-it}\Big|^2\Big)^{1/2} \qquad (43)\end{aligned}$$

Let h be a smooth function supported on $[1/2,2]$ with $h(u)=1$ for $u\in[1,3/2]$, and $0\le h(u)\le 1$ for all u. We add the weight $h(n/P)$ to the second sum over p above. Expanding out the square, we see that (43) is

$$\begin{aligned} &\ll \Big(\frac{\delta P}{(\log P)^2}\Big)^{1/2}\Big(\sum_{(\chi_1,t_1)\in\mathcal{B}_{q,\mathcal{S}}}\sum_{(\chi_2,t_2)\in\mathcal{B}_{q,\mathcal{S}}}\Big|\sum_n \Lambda(n)\chi_1\overline{\chi_2}(n)n^{-i(t_1-t_2)}h\Big(\frac{n}{P}\Big)\Big|\Big)^{1/2}\\ &:= \Big(\frac{\delta P}{(\log P)^2}\Big)^{1/2}(S_1+S_2)^{1/2}, \qquad (44)\end{aligned}$$

where S_1 is the sum over the pairs with $\mathrm{cond}(\chi_1\overline{\chi_2})\le x^{\varepsilon^{20}}$ and S_2 is over those pairs with $\mathrm{cond}(\chi_1\overline{\chi_2})> x^{\varepsilon^{20}}$.

Lemma 6.6 tells us that there is some character $\xi_1 \pmod q$ such that whenever $\mathrm{cond}(\chi_1\overline{\chi_2})\le x^{\varepsilon^{20}}$ we have

(45)

$$\Big|\sum_n \Lambda(n)\chi_1\overline{\chi_2}(n)n^{-i(t_1-t_2)}h\Big(\frac{n}{P}\Big)\Big| \ll P\varepsilon^{20}\log^3\Big(\frac{1}{\varepsilon}\Big)+\frac{P}{(\log P)^{0.3}}+\frac{P}{|t_1-t_2|^2+1}1_{\chi_1\overline{\chi_2}\in\{\chi_0,\xi_1\}},$$

with $\chi_0 \pmod q$ the principal character.

Since $|\mathcal{B}_{q,\mathcal{S}}|=N_{q,\mathcal{S}}$ and $\varepsilon^5\ge(\log P)^{-0.3}$, summing (45) over $((\chi_1,t_1),(\chi_2,t_2))\in\mathcal{B}_{q,\mathcal{S}}^2$ shows that the contribution of the characters with small conductor obeys the bound

$$\begin{aligned} S_1 &\ll \varepsilon^5 N_{q,\mathcal{S}}^2 P + \sum_{\chi_1 \pmod q}\sum_{\chi_2 \pmod q}\sum_{t_1,t_2\in\mathcal{T}}\frac{P}{|t_1-t_2|^2+1}1_{\chi_1\overline{\chi_2}\in\{\chi_0,\xi_1\}}1_{(\chi_1,t_1),(\chi_2,t_2))\in\mathcal{B}_{q,\mathcal{S}}^2}\\ &\ll \varepsilon^5 N_{q,\mathcal{S}}^2 P + N_{q,\mathcal{S}}P\sum_{k\in\mathbb{Z}}\frac{1}{k^2+1}\\ &\ll (\varepsilon^5 N_{q,\mathcal{S}}^2+N_{q,\mathcal{S}})P,\end{aligned}$$

since for any k the number of pairs $((\chi_1,t_1),(\chi_2,t_2))\in\mathcal{B}_{q,\mathcal{S}}^2$ with $\chi_1\overline{\chi_2}\in\{\chi_0,\xi_1\}$ and $t_1-t_2\in[k,k+1)$ is $\le 2N_{q,\mathcal{S}}$ (once (χ_1,t_1) has been chosen, there are at most two possibilities for (χ_2,t_2), since $\mathcal{T}$ is well-spaced).

We then consider the contribution of S_2. In this case, it follows similarly to the proof of Proposition 8.3 (cf. the deduction after equation (38)) that for $q\in\mathcal{Q}_{x,\eta^{1/20}\varepsilon,1000\eta^{-1}}$ we have

$$\sup_{\substack{\chi \pmod q\\ \mathrm{cond}(\chi)>P^{\varepsilon^{20}}}}\sup_{|t|\le x}\Big|\sum_n \Lambda(n)\chi(n)n^{-it}h\Big(\frac{n}{P}\Big)\Big| \ll \frac{P}{(\log P)^{100}}.$$

We have $\varepsilon^5>(\log P)^{-100}$, so we find that

$$S_2\ll \varepsilon^5 N_{q,\mathcal{S}}^2 P.$$

Combining the bounds on S_1 and S_2 with (42) and (43), we see that

$$\frac{\varepsilon\delta P}{\log P}N_{q,\mathcal{S}} \ll \frac{\delta^{1/2}P}{\log P}(\varepsilon^{5/2}N_{q,\mathcal{S}}+N_{q,\mathcal{S}}^{1/2}),$$

and since $\varepsilon \le \delta$ and $\varepsilon > 0$ is small enough, we deduce from this that $N_{q,\mathcal{S}} \ll \varepsilon^{-2}\delta^{-1}$, which was to be shown. □

Remark 8.5. From Lemma 8.2, it is clear that if we restrict to a set $\mathcal{Q}' \subset [1,x]$ of pairwise coprime moduli q, then the sizes of the sets of exceptional $q \le x$ in the two propositions are $\ll (\log x)^{10\varepsilon^{-6}}$ and $\ll (\log x)^{10000\eta^{-1}}$, respectively. Moreover, under GRH there are no exceptional moduli.

9. Variance in progressions and short intervals

9.1. **Typical number of prime factors.** Let $\omega_{[P,Q]}(n) := |\{p \mid n :\ p \in [P,Q]\}|$ denote the number of prime factors of n belonging to the interval $[P,Q]$. Moreover, given $Z \ge 1$, define

$$\Delta(q,Z) := \max_{y \ge Z} \frac{\omega_{[y,2y]}(q)}{y/\log y}, \tag{46}$$

which gives the maximal relative density of prime divisors of q on a dyadic interval $\subset [Z,\infty)$. Clearly, if q is Z-typical in the sense of Definition 1.1, then $\Delta(q,Z) \le 1/50 + o(1)$, and if $\Delta(q,Z) \le 1/100$, then q is Z-typical.

Note that since $\omega(q) \le (1+o(1))(\log q)/(\log\log q)$, we have

$$\Delta(q,Z) < 1/100 \quad \text{if} \quad Z \ge 200(\log q), \tag{47}$$

and note that $0 \le \Delta(q,Z) \ll 1$ always.

Moreover, for any fixed $c > 0$ we have

$$|\{q \le Q :\ \Delta(q,Z) > c\}| \ll Q\exp(-(c/10+o(1))Z). \tag{48}$$

Indeed, for any set $\mathcal{P} \subset \mathbb{P} \cap [y,2y]$ of size $\ge \alpha y/\log y$, we have

$$|\{n \le Q :\ \prod_{p\in\mathcal{P}} p \mid n\}| \le \frac{Q}{\prod_{p\in\mathcal{P}} p} \ll xe^{-(\alpha+o(1))y},$$

and there are $\le 2^{(1+o(1))y/\log y}$ such subsets $\mathcal{P}$ for large y, so by the union bound

$$|\{q \le Q :\ \Delta(q,Z) > c\}| \le \sum_{2^j \ge Z} |\{n \le Q :\ \omega_{[2^{j-1},2^j]}(n) \ge \frac{c\cdot 2^j}{10\log(2^j)}\}| \ll \sum_{2^j \ge Z} Qe^{-(c/10+o(1))2^j},$$

and this is $\ll Qe^{-(c/10+o(1))Z}$. This bound is in fact optimal up to a multiplicative factor in the exponential.[9]

From (47) and (48), we deduce the claims made before Theorem 1.3 that all $q \le x$ are $(x/Q)^{\varepsilon^2}$-typical if $Q = o(x/(\log x)^{1/\varepsilon^2})$, and otherwise the number of $q \le x$ that are not $(x/Q)^{\varepsilon^2}$-typical is $\ll \exp(-(1/1000+o(1))(x/Q)^{\varepsilon^2})$.

9.2. **Parseval-type bounds.** We will reduce the proofs of Corollary 1.1 and Theorems 9.4, 1.4 and 1.5 to the following L^2 bounds for (twisted) character sums.

Proposition 9.1. *Let $1 \le Q \le x/10$ and $(\log(x/Q))^{-1/181} \le \varepsilon \le 1$, and let $f : \mathbb{N} \to \mathbb{U}$ be multiplicative. Let χ_1 be the character $\pmod q$ minimizing the distance $\inf_{|t|\le\log x} \mathbb{D}_q(f,\chi(n)n^{it};x)$. Then, with the notation of Lemma 8.2, for $q \in \mathcal{Q}_{x,\varepsilon^6,\varepsilon^{-100}} \cap [1,Q]$ we have*

$$\sum_{\substack{\chi \pmod q \\ \chi \ne \chi_1}} \Big|\sum_{n\le x} f(n)\overline{\chi}(n)\Big|^2 \ll \varepsilon^{1-3\Delta(q,(x/Q)^{\varepsilon})}\Big(\frac{\phi(q)}{q}x\Big)^2. \tag{49}$$

[9]Indeed, if $c \in (0,1)$ is fixed and $Z \le \frac{1}{2}\log Q$, we get a lower bound of $\gg Q\exp(-(2c+o(1))Z)$ for the count of such q by considering those $q \le Q$ that are divisible by $\prod_{Z\le p\le(1+2c)Z} p$.

Moreover, conditionally on GRH, we have $\mathcal{Q}_{x,\varepsilon} = [1,x]\cap\mathbb{Z}$, that is, (49) holds without any exceptional q.

Deduction of Corollary 1.1 and Theorems 9.4, 1.4 from Proposition 9.1. We apply Proposition 9.1 with $\varepsilon^{1.1}$ in place of ε. Note that $\Delta(q,(x/Q)^{\varepsilon^{1.1}}) \le 1/50 + o(1)$ by the assumption that q is $(X/Q)^{\varepsilon^2}$-typical. Therefore, $(\varepsilon^{1.1})^{1-3\Delta(q,(x/Q)^{\varepsilon^{1.1}})} \ll \varepsilon^{1.1\cdot(1-3/50+o(1))} \ll \varepsilon$.

Observe the Parseval-type identity

$$\frac{1}{\varphi(q)}\sum_{\substack{\chi \pmod q\\ \chi\notin\Xi}}\Big|\sum_{n\le x} f(n)\overline{\chi}(n)\Big|^2 = \sum_{a(q)}^{*}\Big|\sum_{\substack{n\le x\\ n\equiv a(q)}} f(n) - \sum_{\chi\in\Xi}\frac{\chi(a)}{\phi(q)}\sum_{n\le x} f(n)\overline{\chi}(n)\Big|^2,$$

valid for $\Xi\subset\{\chi \pmod q\}$. The claim follows from this, since from Lemma 8.2 (see also Remark 8.5) we have the size bounds $|[1,Q]\backslash\mathcal{Q}_{x,(\varepsilon^{1.1})^6,(\varepsilon^{1.1})^{-100}}| \ll Qx^{-\varepsilon^{200}}$ and $|\mathcal{Q}'\backslash\mathcal{Q}_{x,(\varepsilon^{1.1})^6,(\varepsilon^{1.1})^{-100}}| \ll (\log x)^{\varepsilon^{-200}}$ for any $1\le Q\le x$ and any set $\mathcal{Q}'\subset[1,x]$ of pairwise coprime numbers, and moreover under GRH we have $\mathcal{Q}_{x,(\varepsilon^{1.1})^6,(\varepsilon^{1.1})^{-100}} = [1,x]\cap\mathbb{Z}$. □

Proposition 9.2. *Let $1\le Q\le h/10$ and $(\log(h/Q))^{-1/181}\le\varepsilon\le 1$, and let $f:\mathbb{N}\to\mathbb{U}$ be multiplicative. Let χ_1 be the character $\pmod q$ minimizing the distance $\inf_{|t|\le X}\mathbb{D}_q(f,\chi(n)n^{it};X)$, and let $t_{\chi_1}\in[-X,X]$ be a point that minimizes $\mathbb{D}_q(f,\chi_1(n)n^{it};X)$. Let $Z_{\chi_1}=\varepsilon^{-10}$ and $Z_\chi=0$ for $\chi\ne\chi_1$. Then, with the notation of Lemma 8.2, for all $q\in\mathcal{Q}_{X,\varepsilon^6,\varepsilon^{-100}}\cap[1,Q]$ and for $T = X/h\cdot(h/Q)^{0.01\varepsilon}$, we have*

$$\sum_{\chi \pmod q}\int_{\substack{|t-t_\chi|\ge Z_\chi\\ |t|\le T}}\Big|\sum_{X\le n\le 2X} f(n)\overline{\chi_1}(n)n^{-it}\Big|^2 dt \ll \varepsilon^{1-3\Delta(q,(h/Q)^\varepsilon)}\Big(\frac{\phi(q)}{q}X\Big)^2. \tag{50}$$

Moreover, assuming either the GRH or that $Q\le X^{\varepsilon^{150}}$, the exceptional set of q vanishes.

Theorem 1.5 will be deduced from Proposition 9.2, together with the following lemma.

Lemma 9.3. *Let $X,Z\ge 10$, with $1\le Z\le(\log X)^{1/20}$. Let $1\le h\le X$, and let $1\le q\le h/10$. Let $g:\mathbb{N}\to\mathbb{U}$ be multiplicative, and let t_0 be the minimizer of $t\mapsto\mathbb{D}(g,n^{it};X)$ on $|t|\le X$. Then for every $X<x<2X$ with $[x,x+h]\cap[2X(1-Z^{-1/2}),2X(1+Z^{-1/2})]=\emptyset$, we have*

$$\frac{1}{2\pi ih}\int_{t_0-Z}^{t_0+Z}\Big(\sum_{\substack{X<n\le 2X\\ (n,q)=1}} g(n)n^{-it}\Big)\frac{(x+h)^{it}-x^{it}}{t}\,dt$$
$$= 1_{(X,2X)}(x+h)\Big(\frac{1}{X}\sum_{\substack{X<n\le 2X\\ (n,q)=1}} g(n)n^{-it_0}\Big)\cdot\frac{1}{h}\int_x^{x+h} v^{it_0}dv + O\Big(\frac{\phi(q)}{qZ^{1/2}} + (\log X)^{-1/5}\Big).$$

Proof. We note that $\frac{(x+h)^{it}-x^{it}}{it} = \int_x^{x+h} v^{-1+it}dv$, for each $t\in[t_0-Z,t_0+Z]$. Inserting this into the left-hand side in the statement, swapping the orders of integration and making the change of variables $u := t-t_0$, we obtain

$$\frac{1}{2\pi h}\int_x^{x+h} v^{-1+it_0}\Big(\int_{-Z}^{Z} v^{iu}\sum_{\substack{X<n\le 2X\\ (n,q)=1}} g(n)n^{-it_0-iu}\,du\Big)\,dv. \tag{51}$$

Let $M := \min_{|u|\le\frac12\log X}\mathbb{D}_q(g,n^{i(t_0+u)};X)^2$. By Halász' theorem, if $M\ge 0.27\log\log X$, then

$$\sup_{|u|\le Z}\Big|\sum_{\substack{X<n\le 2X\\ (n,q)=1}} g(n)n^{-it_0-iu}\Big| \ll X(1+M)e^{-M} + X/(\log X)^{1-o(1)} \ll X(\log X)^{-0.27+o(1)},$$

in which case the expression (51) can be bounded by

$$\ll \frac{h}{hx}\cdot ZX(\log X)^{-0.27+o(1)} \ll (\log X)^{-1/5}$$

for X sufficiently large, given that $0.27-1/20>1/5$. The claim follows in this case, so we may assume in the sequel that $M<0.27\log\log X$.

Put $g_{t_0}(n) := g(n)n^{-it_0}$. Since $|u|\le Z$, Lemma 6.4 yields

$$\begin{aligned}\sum_{\substack{X<n\le 2X\\(n,q)=1}} g_{t_0}(n)n^{-iu} &= \frac{(2X)^{-iu}}{1-iu}\sum_{\substack{n\le 2X\\(n,q)=1}} g_{t_0}(n) - \frac{X^{-iu}}{1-iu}\sum_{\substack{n\le X\\(n,q)=1}} g_{t_0}(n) + O\Big(\frac{X(\log(2Z))}{\log X}e^{\sqrt{(2+o(1))M\log\log X}}\Big)\\ &= \frac{(2X)^{-iu}}{1-iu}\sum_{n\le 2X} g_{t_0}(n) - \frac{X^{-iu}}{1-iu}\sum_{n\le X} g_{t_0}(n) + O\Big(\frac{X}{(\log X)^{0.265-o(1)}}\Big),\end{aligned}$$

as $\sqrt{27/50}-1<-0.265$. Furthermore, $0.265-1/20>1/5$, so upon inserting this estimate into (51) we obtain

(52)
$$\Big(\sum_{\substack{n\le 2X\\(n,q)=1}} g(n)n^{-it_0}\Big)\int_x^{x+h} v^{-1+it_0}\frac{I(v;2X)}{h}dv-\Big(\sum_{\substack{n\le X\\(n,q)=1}} g(n)n^{-it_0}\Big)\int_x^{x+h} v^{-1+it_0}\frac{I(v;X)}{h}dv+O\Big(\frac{1}{(\log X)^{1/5}}\Big),$$

where for $y\ge 1$ we have defined

$$I(v;y) := \frac{1}{2\pi}\int_{-Z}^{Z} v^{iu}\frac{y^{-iu}}{1-iu}du,$$

Using a standard, truncated version of Perron's formula (e.g., [25, Proposition 5.54], if $y\neq v$ then

$$\begin{aligned}I(v;y) &= \frac{v}{y}\Big(\frac{1}{2\pi i}\int_{\mathrm{Re}(s)=1}\frac{(y/v)^s}{s}ds + O\Big(\frac{y/v}{Z|\log(y/v)|}\Big)\Big)\\ &= \frac{v}{y}\Big(1_{y>v}+\frac{1}{2}1_{y=v}\Big)+O\Big(\frac{1}{Z|\log(y/v)|}\Big).\end{aligned}$$

If $X<x<x+h<2X$, then recalling $[x,x+h]\cap[2X(1-Z^{-1/2}),2X(1+Z^{-1/2})]=\emptyset$ formula (52) becomes

$$\begin{aligned}&(2hX)^{-1}\Big(\sum_{\substack{n\le 2X\\(n,q)=1}} g(n)n^{-it_0}\Big)\int_x^{x+h} v^{it_0}dv + O\Big(\frac{\phi(q)}{qhZ}\int_x^{x+h}\frac{dv}{|\log(2X/v)|}\Big)\\ &= (2hX)^{-1}\Big(\sum_{\substack{n\le 2X\\(n,q)=1}} g(n)n^{-it_0}\Big)\int_x^{x+h} v^{it_0}dv + O\Big(\frac{\phi(q)}{qZ^{1/2}}\Big);\end{aligned}$$

on the other hand, if $2X-h<x\le 2X$ then $I(v;2X)=O(1/Z)$ for all $x\le v\le x+h$, and so the main term above is dropped in this case.

Finally, by Lemma 6.3 we have

$$(2X)^{-1}\Big(\sum_{\substack{n\le 2X\\(n,q)=1}} g(n)n^{-it_0}\Big) = X^{-1}\sum_{\substack{X<n\le 2X\\(n,q)=1}} g(n)n^{-it_0} + O\Big((\log X)^{-1+2/\pi+o(1)}\Big)$$

and thus our main term is

$$1_{(X,2X)}(x+h)(hX)^{-1}\Big(\sum_{\substack{X<n\leq 2X\\(n,q)=1}} g(n)n^{-it_0}\Big)\int_x^{x+h} v^{it_0}dv + O\Big(\frac{\phi(q)}{qZ^{1/2}} + (\log X)^{-1/5}\Big),$$

as claimed. □

Deduction of Theorem 1.5 from Proposition 9.2. We use Proposition 9.2 with $\varepsilon^{1.1}$ in place of ε.

Again by Lemma 8.2, the size of $\mathcal{Q}_{X,\varepsilon^{6.6},\varepsilon^{-110}}$ fulfills the required bounds (and additionally if $q\leq X^{\varepsilon^{200}}$, then automatically $q\in\mathcal{Q}_{x,\varepsilon^{6.6},\varepsilon^{-110}}$). Let $\mathcal{I}:=[2X(1-\varepsilon^{11/2}),2X]\cup[2X(1-\varepsilon^{11/2})-h,2X(1+\varepsilon^{11/2})-h]$. Using the triangle inequality, we can crudely bound the contribution of $x\in\mathcal{I}$ to the integral in (9) to bound it by

$$\sum_{a\ (\mathrm{mod}\ q)}^{*}\int_{[X,2X]\setminus\mathcal{I}}\Big|\sum_{\substack{x<n\leq x+h\\ n\equiv a\ (\mathrm{mod}\ q)}} f(n) - \frac{\chi_1(a)}{\phi(q)}\Big(\int_x^{x+h} v^{it_{\chi_1}}dv\Big)\sum_{X<n\leq 2X} f(n)\overline{\chi_1}(n)n^{-it_{\chi_1}}\Big|^2 dx \quad (53)$$
$$+O\Big(\varepsilon^{11/2}X\phi(q)\Big(\frac{h}{q}\Big)^2\Big).$$

By Lemma 9.3, for $x\in[X,2X]\setminus\mathcal{I}$ the second term inside the square in (53) is

$$=\frac{\chi_1(a)}{\phi(q)}\cdot\frac{1}{2\pi}\int_{t_{\chi_1}-\varepsilon^{-11}}^{t_{\chi_1}+\varepsilon^{-11}}\Big(\sum_{X\leq n\leq 2X} f(n)\overline{\chi_1}(n)n^{-it}\Big)\frac{(x+h)^{it}-x^{it}}{it}dt + O(\varepsilon\Big(\frac{h}{q}\Big)^2).$$

Let us call the main term here $\mathcal{M}(X;x,q,a)$.

By the triangle inequality, this implies that (53) is

$$\ll\sum_{a\ (\mathrm{mod}\ q)}^{*}\int_X^{2X}\Big|\sum_{\substack{x<n\leq x+h\\ n\equiv a\ (\mathrm{mod}\ q)}} f(n)-\mathcal{M}(X;x,q,a)\Big|^2dx+\varepsilon X\phi(q)\Big(\frac{h}{q}\Big)^2.$$

We will now show that the following Parseval-type bound holds: for $1\leq q\leq h\leq X$, we have

$$\sum_{a\ (\mathrm{mod}\ q)}^{*}\int_X^{2X}\Big|\sum_{\substack{x<n\leq x+h\\ n\equiv a\ (\mathrm{mod}\ q)}} f(n)-\mathcal{M}(X;x,q,a)\Big|^2dx\ll\max_{T\geq X/h}\frac{h}{T\phi(q)}\sum_{\chi\ (\mathrm{mod}\ q)}\int_{\substack{|t-t_\chi|\geq Z_\chi\\ |t|\leq T}}|P_{f\overline{\chi}}(it)|^2dt, \quad (54)$$

where $P_g(s):=\sum_{X\leq n\leq 2X}g(n)n^{-s}$ and $Z_\chi=\varepsilon^{-11}$ if $\chi=\chi_1$ and $Z_\chi=0$ otherwise.

Once we have this, the case where the maximum in (54) is attained with $T\geq X/h\cdot(h/Q)^{0.01\varepsilon^{1.1}}$ can be bounded using Lemma 7.2 as

$$\ll\frac{h^2(h/Q)^{-0.01\varepsilon^{1.1}}}{\phi(q)X}\Big(\frac{\phi(q)}{q}X\Big)^2\ll\varepsilon\frac{\phi(q)}{q^2}Xh^2,$$

since $(\log h/Q)^{-1/181}\leq\varepsilon^{1.1}$ certainly implies $(h/Q)^{-0.01\varepsilon^{1.1}}\ll\varepsilon$. This contribution is small enough for Theorem 1.5. If instead $T\in[X/h,X/h\cdot(h/Q)^{0.01\varepsilon^{1.1}}]$, we have $h/(T\phi(q))\ll\frac{h^2}{\phi(q)X}$, so the bound of Proposition 9.2 (with $\varepsilon^{1.1}$ in place of ε) suffices for (54).

The proof of (54) follows closely that of [30, Lemma 14] (here we choose to work on the 0-line rather than on the 1-line for convenience, though). Let us write $\mathcal{I}_\chi:=(t_\chi-Z_\chi,t_\chi+Z_\chi]$, where

Z_χ is as above. We note first of all that[10]

$$\sum_{\substack{x<n\leq x+h \\ n\equiv a \pmod q}} f(n) - \mathcal{M}(X;x,q,a)$$
$$= \sum_{\chi \pmod q} \frac{\chi(a)}{\phi(q)}\Big(\sum_{x<n\leq x+h} f(n)\overline{\chi}(n) - \frac{1}{2\pi}\int_{\mathcal{I}_\chi} P_{f\overline{\chi}}(it)\frac{(x+h)^{it}-x^{it}}{it}dt\Big),$$

so that by Parseval's identity we find

$$\sideset{}{^*}\sum_{a \pmod q} \Big| \sum_{\substack{x<n\leq x+h \\ n\equiv a \pmod q}} f(n) - \mathcal{M}(X;x,q,a)\Big|^2$$
$$= \frac{1}{\phi(q)} \sum_{\chi \pmod q} \Big| \sum_{x<n\leq x+h} f(n)\overline{\chi}(n) - \frac{1}{2\pi i}\int_{\mathcal{I}_\chi} P_{f\overline{\chi}}(it)\frac{(x+h)^{it}-x^{it}}{t}dt\Big|^2.$$

Now, by Perron's formula, whenever $x, x+h$ are not integers, for each χ we have

$$\sum_{x<n\leq x+h} f(n)\overline{\chi}(n) = \frac{1}{2\pi i}\int_{-\infty}^{\infty} P_{f\overline{\chi}}(it)\frac{(x+h)^{it}-x^{it}}{t}dt,$$

so that, if $\mathcal{L}$ is the expression on the left-hand side of (54), we have

$$\mathcal{L} = \frac{1}{\phi(q)} \sum_{\chi \pmod q} \int_X^{2X} \Big|\frac{1}{2\pi i}\int_{\mathbb{R}\setminus\mathcal{I}_\chi} P_{f\overline{\chi}}(it)\frac{(x+h)^{it}-x^{it}}{t}dt\Big|^2 dx.$$

Repeating the trick at the bottom of page 22 of [30], we can find some point $u \in [-h/X, h/X]$ for which

$$\mathcal{L} \ll \frac{1}{\phi(q)} \sum_{\chi \pmod q} \int_X^{2X} \Big|\int_{\mathbb{R}\setminus\mathcal{I}_\chi} P_{f\overline{\chi}}(it)x^{it}\frac{(1+u)^{it}-1}{t}dt\Big|^2 dx.$$

The rest of the proof then follows [30, Lemma 14] almost verbatim (adding a smooth weight to the x integral, expanding the square and swapping the order of integration).

□

9.3. **Proof of hybrid theorem.** We may of course assume in what follows that h is larger than any given absolute constant and $\varepsilon > 0$ is smaller than any given positive constant.

We have shown that it is enough to prove Proposition 9.2, so what we need to show is that

$$\sum_{\chi \pmod q} \int_{\substack{|t-t_\chi|\geq Z_\chi \\ |t|\leq T}} |F(\chi,it)|^2 dt \ll \varepsilon^{1-3\Delta(q,(h/q)^\varepsilon)}\Big(\frac{\phi(q)}{q}X\Big)^2. \tag{55}$$

for $T = X/h \cdot (h/Q)^{0.01\varepsilon}$, where

$$F(\chi,s) := \sum_{X\leq n\leq 2X} f(n)\overline{\chi}(n)n^{-s}.$$

[10]Here the integral over an empty set is interpreted as zero.

As in [30], we restrict to integers with typical factorization. Define the "well-factorable" set $\mathcal{S}$ as follows. For $1 \le Q \le h/10$, $\varepsilon \in ((\log \frac{h}{Q})^{-1/181}, 1)$ and $2 \le j \le J-1$, we define

$$\begin{aligned} &P_1 = (h/Q)^{\varepsilon}, \quad Q_1 = (h/Q)^{1-0.01\varepsilon}, \\ &P_j = \exp\Big(j^{4j}(\log Q_1)^{j-1}\log P_1\Big), \quad Q_j = \exp\Big(j^{4j}(\log Q_1)^{j}\Big), \\ &P_J = X^{\varepsilon^2}, \quad Q_J = X^{\varepsilon}, \end{aligned}$$

with J being chosen minimally subject to the constraint $J^{4J+2}(\log Q_1)^J > (\log X)^{1/2}$. (If $J = 2$, only use the definitions of P_1, Q_1, P_J, Q_J. If $J = 1$, only use the definitions of P_J, Q_J.)

Then let

$$\mathcal{S} := \{n \le x : \omega_{[P_i,Q_i]}(n) \ge 1 \quad \forall i \le J\}.$$

One sees that for $2 \le j \le J$ the inequalities

$$\frac{\log\log Q_j}{\log P_{j-1} - 1} \le \frac{\eta}{4j^2}, \quad \frac{\eta}{j^2}\log P_j \ge 8\log Q_{j-1} + 16\log j \tag{56}$$

hold for fixed $\eta \in (0,1)$ and large enough x (the $j = 2$ case follows from the assumption $\log(h/Q) > \varepsilon^{-100}$, and for the $j = J$ case it is helpful to note that $J \ll \log\log X$ and $P_{J-1} \gg \exp((\log\log X)^{10})$ if $J \ge 3$), and thus the P_j, Q_j satisfy all the same requirements as in [30]. A simple sieve upper bound shows that

$$|[X, 2X] \setminus \mathcal{S}| \ll \sum_{j \le J} X\frac{\log P_j}{\log Q_j} \ll \varepsilon X.$$

We next define

$$\begin{aligned} &Q_{v,H_j}(\chi, s) := \sum_{e^{v/H_j} \le p < e^{(v+1)/H_j}} f(p)\overline{\chi}(p)p^{-s} \\ &R_{v,H_j}(\chi, s) := \sum_{Xe^{-v/H_j} \le m \le 2Xe^{-v/H_j}} \frac{f(m)\overline{\chi}(m)m^{-s}}{1 + \omega_{[P_j,Q_j]}(m)}; \\ &H_1 = H_J = H := \lfloor \varepsilon^{-1} \rfloor, \quad H_j := j^2 P_1^{0.1} \quad \text{for} \quad 2 \le j \le J-1; \\ &\mathcal{I}_j := [\lfloor H_j \log P_j \rfloor, H_j \log Q_j]. \end{aligned} \tag{57}$$

We split the set

$$\mathcal{E} := \{(\chi, t) \in \{\chi \pmod q\} \times [-X, X] : \ |t - t_\chi| \ge Z_\chi, \ |t| \le T\}$$

as $\mathcal{E} = \bigcup_{j \le J-1} \mathcal{X}_j \cup \mathcal{U}$ with

$$\begin{aligned} &\mathcal{X}_1 = \{(\chi, t) \in \mathcal{E} : \ |Q_{v,H_1}(\chi, it)| \le e^{(1-\alpha_1)v/H_1} \quad \forall v \in \mathcal{I}_1\}, \\ &\mathcal{X}_j = \{(\chi, t) \in \mathcal{E} : \ |Q_{v,H_j}(\chi, it)| \le e^{(1-\alpha_j)v/H_j} \quad \forall v \in \mathcal{I}_j\} \setminus \bigcup_{i \le j-1} \mathcal{X}_j, \\ &\mathcal{U} = \mathcal{E} \setminus \bigcup_{j \le J-1} \mathcal{X}_j, \end{aligned}$$

where we take

$$\alpha_j = \frac{1}{4} - \eta\Big(1 + \frac{1}{2j}\Big), \quad \eta = 0.01.$$

We may of course write, for some sets $\mathcal{T}_{j,\chi} \subset [-T, T]$,

$$\mathcal{X}_j = \bigcup_{\chi \pmod q} \{\chi\} \times \mathcal{T}_{j,\chi}.$$

By Lemma 7.7, for each $1 \leq j \leq J-1$ we have

$$\begin{aligned}&\sum_{\chi \pmod q} \int_{\mathcal{T}_{j,\chi}} \Big| \sum_{X \leq n \leq 2X} f(n)\overline{\chi}(n) n^{-it}\Big|^2 dt \\ &\ll H_j \log \frac{Q_j}{P_j} \sum_{v \in \mathcal{I}_j} \sum_{\chi \pmod q} \int_{\mathcal{T}_{j,\chi}} |Q_{v,H_j}(\chi, it)|^2 |R_{v,H_j}(\chi, it)|^2 dt \\ &\quad + \Big(\frac{\phi(q)}{q} X\Big)^2 \Big(\frac{1}{H_j} + \frac{1}{P_j} + \prod_{\substack{P_j \leq p \leq Q_j \\ p \nmid q}} \Big(1 - \frac{1}{p}\Big)\Big),\end{aligned}$$

By our choices of P_j and H_j, the error terms involving $1/H_j$ or $1/P_j$ are $\ll \varepsilon(\phi(q)/q \cdot X)^2$ when summed over $j \leq J-1$, since $\log(h/Q) \geq \varepsilon^{-100}$ by our assumption on the size of ε. After summing over $j \leq J-1$, the error term involving $\prod_{\substack{P_j \leq p \leq Q_j \\ p \nmid q}} (1 - \frac{1}{p})$ becomes

$$\ll \Big(\frac{\phi(q)}{q} X\Big)^2 \sum_{j \leq J-1} \frac{\log P_j}{\log Q_j} \prod_{\substack{p \mid q \\ P_j \leq p \leq Q_j}} \Big(1 + \frac{1}{p}\Big) \ll \Big(\frac{\phi(q)}{q} X\Big)^2 \sum_{j \leq J-1} \frac{\varepsilon}{j^2} \prod_{\substack{p \mid q \\ P_j \leq p \leq Q_j}} \Big(1 + \frac{1}{p}\Big).$$

Using the $\Delta(\cdot)$ function, for $j \leq J-1$ we have

$$\begin{aligned}\prod_{\substack{p \mid q \\ P_j \leq p \leq Q_j}} \Big(1 + \frac{1}{p}\Big) &\ll \exp\Big(\sum_{\substack{p \mid q \\ P_j \leq p \leq Q_j}} \frac{1}{p}\Big) \ll \exp\Big(\sum_{2^k \in [P_j/2, Q_j]} \frac{\Delta(q, P_1)}{(\log 2) k}\Big) \\ &\ll \Big(\frac{\log Q_j}{\log P_j}\Big)^{\frac{1}{\log 2}\Delta(q, P_1)} \ll (j^2 \varepsilon^{-1})^{\frac{1}{\log 2}\Delta(q, (h/Q)^{\varepsilon})}.\end{aligned}$$

Hence, on multiplying by ε/j^2 and summing over $j \leq J-1$, for $\Delta(q, (h/Q)^{\varepsilon}) \leq 1/3$, we get a contribution of $\ll \varepsilon^{1-3\Delta(q,P_1)}$, as desired (since $2(\frac{1}{3\log 2} - 1) < -1$ means that the j sum is convergent). For $\Delta(q, (h/Q)^{\varepsilon}) > 1/3$, in turn, we simply use the triangle inequality to note that the trivial bound $\ll (\phi(q)/q \cdot X)^2$ for (55) coming from Lemma 7.2 (after forgetting the condition $|t - t_\chi| \geq Z_\chi$) is good enough.

Making use of the assumption defining $\mathcal{X}_j$, we have

$$\begin{aligned}&H_j \log \frac{Q_j}{P_j} \sum_{v \in \mathcal{I}_j} \sum_{\chi \pmod q} \int_{\mathcal{T}_{j,\chi}} |Q_{v,H_j}(\chi, it)|^2 |R_{v,H_j}(\chi, it)|^2 dt \\ &\ll H_j \log \frac{Q_j}{P_j} \sum_{v \in \mathcal{I}_j} e^{(2-2\alpha_j)v/H_j} \sum_{\chi \pmod q} \int_{\mathcal{T}_{\chi,j}} |R_{v,H_j}(\chi, it)|^2 dt =: E_j\end{aligned}$$

We thus need to bound E_j as well as the contribution of the pairs $(\chi, t) \in \mathcal{U}$.

Case of $\mathcal{X}_1$. For the pairs in $\mathcal{X}_1$ we crudely extend the t-integral to $[-T, T]$ and apply Lemma 7.2 to arrive at

$$\begin{aligned} E_1 &\ll H_1 \log \frac{Q_1}{P_1} \sum_{v \in \mathcal{I}_1} e^{(2-2\alpha_1)v/H_1} \Big(\frac{\phi(q)}{q} X e^{-v/H_1} + \phi(q) T \Big) \frac{\phi(q)}{q} \cdot X e^{-v/H_1} \\ &\ll \Big(\frac{\phi(q)}{q} X \Big)^2 P_1^{0.01} H_1 \log \frac{Q_1}{P_1} \sum_{v \in \mathcal{I}_1} e^{-2\alpha_1 v/H_1} \\ &\ll \Big(\frac{\phi(q)}{q} X \Big)^2 P_1^{0.01} H_1 \log Q_1 \cdot \frac{1}{P_1^{2\alpha_1}} \cdot \frac{1}{1 - e^{-2\alpha_1/H_1}} \\ &\ll \Big(\frac{\phi(q)}{q} X \Big)^2 \varepsilon^{-1} P_1^{-0.1} H_1^2, \end{aligned}$$

where on the second line we used $Xe^{-v/H} \ge X/Q_1 \ge qTP_1^{0.01}$ by the assumption $T = X/h \cdot P_1^{0.01}$. We see that the contribution of E_1 is small enough, since $H_1 \ll \varepsilon^{-1}$ and $P_1^{-0.1} = (h/Q)^{-0.1\varepsilon} \ll \varepsilon^{10}$.

Case of $\mathcal{X}_j$. Let $2 \le j \le J-1$. We partition

$$\mathcal{X}_j = \bigcup_{r \in \mathcal{I}_{j-1}} \mathcal{X}_{j,r}$$

where $\mathcal{X}_{j,r}$ is the set of $(\chi, t) \in \mathcal{X}_j$ such that r is the minimal index in $\mathcal{I}_{j-1}$ with $|Q_{r,H_{j-1}}(\chi, it)| > e^{(1-\alpha_{j-1})r/H_{j-1}}$. Letting $r_0 \in \mathcal{I}_{j-1}$ and $v_0 \in \mathcal{I}_j$ denote the choices of r and v, respectively, with maximal contribution, we obtain

$$\begin{aligned} E_j &\ll H_j (\log Q_j) |\mathcal{I}_j| |\mathcal{I}_{j-1}| \\ &\quad \cdot \sum_{\chi \pmod q} e^{(2-\alpha_j)v_0/H_j} \int_{-T}^{T} \Big(|Q_{r_0,H_{j-1}}(\chi, it)| / e^{(1-\alpha_{j-1})r_0/H_{j-1}} \Big)^{2\ell_{j,r_0}} |R_{v_0,H_j}(\chi, it)|^2 dt, \end{aligned}$$

where $\ell_{j,r_0} := \lceil \frac{v_0/H_j}{r_0/H_{j-1}} \rceil > 1$.

Using $|\mathcal{I}_{j-1}| \le |\mathcal{I}_j| \ll H_j \log Q_j$, this becomes

$$\begin{aligned} E_j &\ll \Big(H_j \log Q_j \Big)^3 e^{(2-2\alpha_j)v_0/H_j - (2-2\alpha_{j-1})\ell_{j,r_0} r_0/H_{j-1}} \\ &\quad \cdot \sum_{\chi \pmod q} \int_{-T}^{T} |Q_{r_0,H_{j-1}}(\chi, it)^{\ell_{j,r_0}} R_{v_0,H_j}(\chi, it)|^2 dt. \end{aligned}$$

We apply Lemma 7.6 to conclude that

$$\sum_{\chi \pmod q} \int_{-T}^{T} |Q_{r_0,H_{j-1}}(\chi, it)^{\ell_{j,r_0}} R_{v_0,H_j}(\chi, it)|^2 dt \ll \Big(\frac{\phi(q)}{q} X e^{r_0/H_{j-1}} 2^{\ell_{j,r_0}} \Big)^2 ((\ell_{j,r_0} + 1)!)^2.$$

We have $\ell_{j,r_0} \ge \frac{v_0/H_j}{r_0/H_{j-1}}$, whence using $2^\ell (\ell+1)! \ll \ell^\ell$ we get

$$E_j \ll (H_j \log Q_j)^3 \Big(\frac{\phi(q)}{q} X e^{r_0/H_{j-1}} \Big)^2 e^{2(\alpha_{j-1} - \alpha_j)v_0/H_j + 2\ell_{j,r_0} \log \ell_{j,r_0}}. \tag{58}$$

Since $\ell_{j,r_0} \le \frac{v_0/H_j}{r_0/H_{j-1}} + 1$ and $r_0/H_{j-1} \ge \log P_{j-1} - 1$, $v_0/H_j \le \log Q_j$, we have

$$\ell_{j,r_0} \log \ell_{j,r_0} \le \frac{v_0}{H_j} \frac{\log \log Q_j}{\log P_{j-1} - 1} + \log \log Q_j + 1.$$

Thus, (58) is

$$\ll \Big(\frac{\phi(q)}{q}Xe^{r_0/H_{j-1}}\Big)^2 H_j^3(\log Q_j)^5 \exp\Big(\Big(2\frac{\log\log Q_j}{\log P_{j-1}-1}+2(\alpha_{j-1}-\alpha_j)\Big)v_0/H_j\Big).$$

By (56) and the choice of the α_j, we have the inequalities

$$\frac{\log\log Q_{j-1}}{\log P_j - 1} \leq \frac{\eta}{4j^2}, \quad \alpha_{j-1}-\alpha_j \leq -\frac{\eta}{2j^2}, \quad \log Q_j \leq Q_{j-1}^{1/24},$$

so we get

$$\begin{aligned} E_j &\ll \Big(\frac{\phi(q)}{q}X\Big)^2 H_j^3(\log Q_j)^5 Q_{j-1}^2 P_j^{-\frac{\eta}{2j^2}} \\ &\ll \Big(\frac{\phi(q)}{q}X\Big)^2 j^6 P_1^{0.3} Q_{j-1}^{2+5/24} P_j^{-\frac{\eta}{2j^2}} \\ &\ll \Big(\frac{\phi(q)}{q}X\Big)^2 j^6 Q_{j-1}^3 P_j^{-\frac{\eta}{2j^2}}. \end{aligned}$$

Again by (56), we have the inequality

$$\frac{\eta}{j^2}\log P_j \geq 8\log Q_{j-1} + 16\log j,$$

so

$$E_j \ll \Big(\frac{\phi(q)}{q}X\Big)^2 \frac{1}{j^2 Q_{j-1}} \ll \Big(\frac{\phi(q)}{q}X\Big)^2 \frac{1}{j^2 P_1}.$$

Summing over j gives

$$\sum_{2\leq j\leq J-1} E_j \ll \Big(\frac{\phi(q)}{q}X\Big)^2 P_1^{-1},$$

and this is acceptable. It remains to deal with $\mathcal{U}$.

Case of $\mathcal{U}$. Let us write

$$\mathcal{U} = \bigcup_{\chi \pmod q} \{\chi\}\times\mathcal{T}_\chi.$$

By Lemma 7.7 and the definitions of P_J, Q_J and $H := H_J$, we have

$$\begin{aligned} \sum_{\chi \pmod q}\int_{\mathcal{T}_\chi}|F(\chi,it)|^2dt &\ll H\log\frac{Q_J}{P_J}\sum_{v\in\mathcal{I}_J}\sum_{\chi \pmod q}\int_{\mathcal{T}_\chi}|Q_{v,H}(\chi,it)|^2|R_{v,H}(\chi,it)|^2dt \\ &\quad + \Big(\frac{\phi(q)}{q}X\Big)^2\Big(\frac{1}{H}+\frac{1}{P_J}+\varepsilon\prod_{\substack{P_J\leq p\leq Q_J\\ p\mid q}}\Big(1+\frac{1}{p}\Big)\Big). \end{aligned} \tag{59}$$

Since $H = \lfloor \varepsilon^{-1} \rfloor$ and $\prod_{\substack{P_J \le p \le Q_J \\ p|q}} (1+\frac{1}{p}) \ll \varepsilon^{-3\Delta(q,(h/Q)^{\varepsilon})}$ (similarly to the $\mathcal{X}_j$ case), the second term on the right of (59) is $\ll \varepsilon^{1-3\Delta(q,(h/Q)^{\varepsilon})}(\phi(q)/q \cdot X)^2$. Thus we have

$$\sum_{\chi \pmod q} \int_{\mathcal{T}_\chi} |F(\chi, it)|^2 \ll (H \log \frac{Q_J}{P_J})^2 \sum_{\chi \pmod q} \int_{\mathcal{T}_\chi} |Q_{v_0,H}(\chi, it)|^2 |R_{v_0,H}(\chi, it)|^2 dt$$
$$+ \varepsilon^{1-3\Delta(q,(x/Q)^{\varepsilon})} \Big(\frac{\phi(q)}{q} X\Big)^2$$
$$\ll H^2 \varepsilon^2 (\log X)^2 \sum_{\chi \pmod q} \int_{\mathcal{T}_\chi} |Q_{v_0,H}(\chi, it)|^2 |R_{v_0,H}(\chi, it)|^2 \tag{60}$$
$$+ \varepsilon^{1-3\Delta(q,(x/Q)^{\varepsilon})} \Big(\frac{\phi(q)}{q} X\Big)^2$$

for some $v_0 \in [H\varepsilon^2 \log X - 1, H\varepsilon \log X]$, with $H = \lfloor \varepsilon^{-1} \rfloor$.

We discretize the integral, so that the term on the right of (60) is bounded by

$$\ll H^2 \varepsilon^2 (\log X)^2 \sum_{\chi \pmod q} \sum_{t \in \mathcal{T}'_\chi} |Q_{v_0,H}(\chi, it)|^2 |R_{v_0,H}(\chi, it)|^2$$

for some well-spaced set $\mathcal{T}'_\chi \subset \mathcal{T}_\chi \subset [-T, T]$. Let us define the discrete version of $\mathcal{U}$ as

$$\mathcal{U}' = \bigcup_{\chi \pmod q} \{\chi\} \times \mathcal{T}'_\chi.$$

We consider separately the subsets

$$\mathcal{U}_S := \Big\{(\chi, t) \in \mathcal{U}' : |Q_{v_0,H}(\chi, it)| \le \varepsilon^2 \frac{e^{v_0/H}}{v_0}\Big\},$$
$$\mathcal{U}_L := \Big\{(\chi, t) \in \mathcal{U}' : |Q_{v_0,H}(\chi, it)| > \varepsilon^2 \frac{e^{v_0/H}}{v_0}\Big\};$$

note that by the Brun–Titchmarsh inequality the trivial upper bound is $|Q_{v_0,H}(\chi, it)| \ll e^{v_0/H}/v_0$.

We start with the $\mathcal{U}_S$ case. Applying our large values estimate, Lemma 7.5, together with the fact that $qT \ll X^{1+o(1)}$, we have

$$|\mathcal{U}_S| \ll (qT)^{2\alpha_J} (X^{2\varepsilon} + (\log X)^{100\varepsilon^{-2}}) \ll X^{0.49}, \tag{61}$$

since $\alpha_J \le 1/4 - \eta$ and $\eta = 0.01$, so by the Halász–Montgomery inequality for twisted character sums (Lemma 7.4), we have

$$\sum_{(\chi,t) \in \mathcal{U}_S} |Q_{v_0,H}(\chi, it)|^2 |R_{v_0,H}(\chi, it)|^2 \ll \varepsilon^4 \frac{e^{2v_0/H}}{v_0^2} \sum_{(\chi,t) \in \mathcal{U}_S} |R_{v_0,H}(\chi, it)|^2$$
$$\ll \varepsilon^4 \frac{e^{2v_0/H}}{v_0^2} \Big(\frac{\phi(q)}{q} X e^{-v_0/H} + (qT)^{1/2} (\log(2qT)) |\mathcal{U}_S|\Big) \frac{\phi(q)}{q} X e^{-v_0/H}$$
$$\ll \varepsilon^4 \frac{e^{2v_0/H}}{v_0^2} \Big(\frac{\phi(q)}{q} X e^{-v_0/H}\Big)^2 \ll H^{-2} \cdot (\log X)^{-2} \Big(\frac{\phi(q)}{q} X\Big)^2,$$

since $\phi(q)/q \cdot X e^{-v/H} \gg X^{0.999}$ and $v_0 \gg H\varepsilon^2 \log X$. This bound is admissible after multiplying by $H^2 \varepsilon^2 (\log X)^2$.

Now we turn to the $\mathcal{U}_L$ case. We restrict to moduli $q \in \mathcal{Q}_{x,\varepsilon^6,\varepsilon^{-100}}$ and recall that $\varepsilon^7 > (\log X)^{-1/13}$.

By Proposition 8.5 (with $\eta = \varepsilon^2$ and $\delta = e^{1/H} - 1 \asymp 1/H$ and $\varepsilon > 0$ small enough), for $q \in \mathcal{Q}_{x,\varepsilon^8,\varepsilon^{-100}}$, we have $|\mathcal{U}_L| \ll \varepsilon^{-4}H \ll \varepsilon^{-5}$. In addition, by Corollary 8.4 (with $\varepsilon \to \varepsilon^6$, $\alpha \to \varepsilon^2, \beta \to \varepsilon$), for $q \in \mathcal{Q}_{x,\varepsilon^6,\varepsilon^{-100}}$ we have the pointwise bound

$$\sup_{\substack{\chi \pmod q \\ \chi \neq \chi_1}} \sup_{|t| \leq X} |R_{v_0}(\chi, it)|/(Xe^{-v_0/H}) \ll \varepsilon^4 \frac{\phi(q)}{q}.$$

Hence we can bound the contribution of the pairs (χ, t) with $\chi \neq \chi_1$ by

$$\begin{aligned}\sum_{\substack{(\chi,t)\in\mathcal{U}_L \\ \chi\neq\chi_1}} |Q_{v_0,H}(\chi, it)|^2 |R_{v_0,H}(\chi, it)|^2 &\ll |\mathcal{U}_L| \frac{e^{2v_0/H}}{v_0^2} \sup_{\substack{\chi \pmod q \\ \chi \neq \chi_1}} \sup_{|t| \leq X} |R_{v_0,H}(\chi, it)|^2 \\ &\ll \frac{\varepsilon^{-5+8}}{\varepsilon^4 H^2 (\log X)^2} \Big(\frac{\phi(q)}{q} X\Big)^2,\end{aligned}$$

and this multiplied by the factor $H^2\varepsilon^2(\log X)^2$ yields the required bound.

The contribution of $\chi = \chi_1$, in turn, is bounded using Corollary 8.4 in the form that

$$\sup_{Z \leq |t - t_{\chi_1}| \leq x} |R_{v_0,H}(\chi_1, it)|/(Xe^{-v_0/H}) \ll \frac{1}{\sqrt{Z}} \frac{\phi(q)}{q}$$

for $Z = \varepsilon^{-10} \leq (\log X)^{1/13}$ and for q as before. This yields

$$\begin{aligned}\sum_{(\chi_1,t)\in\mathcal{U}_L} |Q_{v_0,H}(\chi_1, it)|^2 |R_{v_0,H}(\chi_1, it)|^2 &\ll |\mathcal{U}_L| \frac{e^{2v_0/H}}{v_0^2} \sup_{Z \leq |t - t_{\chi_1}| \leq X} |R_{v_0,H}(\chi_1, it)|^2 \\ &\ll \frac{\varepsilon^{-5+10}}{\varepsilon^4 H^2 (\log X)^2} \Big(\frac{\phi(q)}{q} X\Big)^2.\end{aligned}$$

This multiplied by $H^2\varepsilon^2(\log X)^2$ produces a good enough bound, finishing the proof of Proposition 9.2, and hence of Theorem 1.5.

Proof of Corollary 1.6. We now briefly describe the modifications needed for obtaining a simpler main term in the case of real-valued multiplicative $f : \mathbb{N} \to [-1, 1]$. We work with the same set of moduli $q \in \mathcal{Q}_{x,\varepsilon^6,\varepsilon^{-100}}$ as in Proposition 9.2. By the triangle inequality and Theorem 1.5, it suffices to show that

(62)
$$\int_X^{2X} \Big| \frac{1}{X} \sum_{X<n\leq 2X} f(n)\overline{\chi_1}(n) - \Big(\frac{1}{h}\int_x^{x+h} v^{it_\chi} dv\Big) \frac{1}{X} \sum_{X<n\leq 2X} f(n)\overline{\chi_1}(n) n^{-it_{\chi_1}} \Big|^2 dx \ll \varepsilon X \Big(\frac{\phi(q)}{q}\Big)^2,$$

and that χ_1 may be taken to be real.

If either χ_1 is complex or $\varepsilon \leq |t_{\chi_1}| \leq X$ then by Halász' theorem and [31, Lemma C.1], we immediately have

$$\max_{u \in \{0, t_{\chi_1}\}} \Big| \frac{1}{X} \sum_{X<n\leq 2X} f(n)\overline{\chi_1}(n) n^{-iu} \Big| \ll (\log X)^{-1/20+o(1)},$$

which is sufficient. Moreover, the same bound holds if $\mathbb{D}(f, \chi_1 n^{it_{\chi_1}}; X) \geq \frac{1}{\sqrt{20}} \log\log X$. Thus, we may assume that χ_1 is real, $|t_{\chi_1}| \leq \varepsilon$, and $\mathbb{D}(f, \chi_1 n^{it_{\chi_1}}; X) < \frac{1}{\sqrt{20}} \log\log X$. In this case, we observe that

$$\frac{1}{h}\int_x^{x+h} v^{it_{\chi_1}} dv = \frac{x^{it_{\chi_1}}}{1 + it_{\chi_1}} \cdot \frac{x}{h}\Big((1 + h/x)^{1+it_{\chi_1}} - 1\Big).$$

Combining this with Lemma 6.4 and the fact that $(x/X)^{it_{\chi_1}} = 1+O(\varepsilon)$, $(x/(2X))^{it_{\chi_1}} = 1+O(\varepsilon)$ for $x \in (X, 2X]$, we obtain

$$\frac{1}{h}\Big(\int_x^{x+h} v^{it_{\chi_1}}\,dv\Big)\frac{1}{X}\sum_{X<n\leq 2X} f(n)\chi_1(n)n^{-it_{\chi_1}}$$
$$= \frac{x}{h}\Big((1+h/x)^{1+it_{\chi_1}}-1\Big)\frac{1}{X}\sum_{X<n\leq 2X} f(n)\chi_1(n) + O\Big(\varepsilon\frac{\phi(q)}{q} + (\log X)^{-1/10}\Big).$$

The integral on the left-hand side of (62) is therefore

$$\ll \Big|\frac{1}{X}\sum_{X<n\leq 2X} f(n)\chi_1(n)\Big|^2 \int_X^{2X}\Big(\frac{x}{h}\Big)^2\Big|\Big((1+h/x)^{1+it_{\chi_1}}-1\Big)-\frac{h}{x}\Big|^2\,dx + \varepsilon\frac{\phi(q)}{q}X$$
$$\ll |t_{\chi_1}|^2 X\Big(\frac{\phi(q)}{q}\Big)^2 + \varepsilon\frac{\phi(q)}{q}X$$
$$\ll \varepsilon X\Big(\frac{\phi(q)}{q}\Big)^2,$$

since $|t_{\chi_1}| \leq \varepsilon$. □

Corollary 1.7 follows quickly from Corollary 1.6.

Proof of Corollary 1.7. We apply Corollary 1.6 with $f = \mu$. Note that the set $\mathcal{Q}_{X,\varepsilon}$ in Corollary 1.6 contains all positive integers $\leq X^{\varepsilon^{200}}$. Now, let c_0 be a small enough absolute constant, and let $m \leq X^{\varepsilon^{200}}$ be a modulus for which $L(s,\chi)$ for some real character $\chi \pmod m$ has a real zero $> 1 - c_0/\log q$ (if it exists). By the Landau–Page theorem and Siegel's theorem, all such m are multiples of a single number $q_0 \gg (\log X)^A$.

It then suffices to show that for q not divisible by q_0 we have

$$\Big|\sum_{n\leq X}\mu(n)\chi(n)\Big| \ll \varepsilon^{1/2}\frac{\varphi(q)}{q}X \tag{63}$$

for all characters $\chi \pmod q$. By Halász's theorem, we have (63) provided that

$$\mathbb{D}_q(\mu,\chi;X)^2 \geq 0.6\log(1/\varepsilon), \tag{64}$$

say. By Mertens's theorem, we can lower bound

$$\mathbb{D}_q(\mu,\chi;X)^2 \geq \sum_{q^{1/\varepsilon}\leq p\leq X}\frac{1+\mathrm{Re}(\chi(p))}{p} = \log\frac{\log X}{\varepsilon^{-1}\log q} + \mathrm{Re}\left(\sum_{q^{1/\varepsilon}\leq p\leq X}\frac{\chi(p)}{p}\right) - O(1). \tag{65}$$

As $q \leq X^{1/\varepsilon^{200}}$, the first term on the right of (65) is $\geq 199\log(1/\varepsilon)$, say. By a quantitative form of Linnik's theorem [25, Theorem 18.6], for any $q^{1/\varepsilon} \leq y \leq X$ and q not divisible by q_0 we have the bound

$$\Big|\sum_{p\leq y}\chi(p)\Big| \ll \varepsilon y/\log y. \tag{66}$$

Using (66) and partial summation, we conclude that the left-hand side of (65) is $\geq 198\log(1/\varepsilon)$, say. Now we obtain (64) and hence (63). □

9.4. **The case of arithmetic progressions.** As mentioned in the introduction, along with Corollary 1.1, we shall establish a slightly more general result.

Theorem 9.4. *Let $\mathcal{Q}' \subset [1, Q]$ be any set of pairwise coprime numbers. Corollary 1.1 continues to hold if the moduli p are required to be $(x/Q)^{\varepsilon^2}$-typical elements of $\mathcal{Q}'$ instead of being prime numbers, with the modification that the right-hand side of* (5) *should be replaced with $\varepsilon\phi(p)(x/p)^2$.*

Proof of Corollary 1.1 and Theorems 9.4, 1.4. It suffices to prove Proposition 9.1. In proving it, we may plainly assume that x is larger than any fixed constant and $\varepsilon > 0$ is smaller than any fixed positive constant.

The proof follows the same lines as that of Proposition 9.2, and we merely highlight the main differences. For $x \geq 10$, $1 \leq Q \leq x/100$ and $(\log(x/Q))^{-1/181} \leq \varepsilon \leq 1$ we let

$$\begin{aligned} &P_1 = (x/Q)^{\varepsilon}, \quad Q_1 = x/Q \\ &P_j = \exp\Big(j^{4j}(\log Q_1)^{j-1}\log P_1\Big), \quad Q_j = \exp\Big(j^{4j+2}(\log Q_1)^{j}\Big), \quad 2 \leq j \leq J-1 \\ &P_J = x^{\varepsilon^2}, \quad Q_J = x^{\varepsilon}, \end{aligned}$$

where $J \geq 1$ is the smallest integer with $J^{4J+2}(\log Q_1)^J > (\log x)^{1/2}$. (If $J = 2$ then only define P_1, Q_1, P_2, Q_2 as above, and if $J = 1$ then just define P_1, Q_1 as above.)

In analogy to the definitions made in the proof of Proposition 9.1, we also define

$$F(\chi) := \sum_{n \leq x} f(n)\overline{\chi}(n), \tag{67}$$

$$H_1 = H_J = H := \lfloor \varepsilon^{-1} \rfloor, \quad H_j := j^2 P^{0.1} \text{ for } 2 \leq j \leq J-1,$$

$$\mathcal{I}_j := [\lfloor H_j \log P_j \rfloor, H_j \log Q_j] \text{ for } 2 \leq j \leq J-1,$$

$$Q_{v,H_j}(\chi) := \sum_{e^{v/H_j} \leq p < e^{(v+1)/H_j}} f(p)\overline{\chi}(p), \tag{68}$$

$$R_{v,H_j}(\chi) := \sum_{xe^{-v/H_j} \leq m \leq 2xe^{-v/H_j}} \frac{f(m)\overline{\chi}(m)}{1 + \omega_{[P_j,Q_j]}(m)} \text{ for } v \in \mathcal{I}_j, \ 1 \leq j \leq J.$$

Finally, for $q \geq 1$ and $2 \leq j \leq J-1$, let us write

$$\begin{aligned} &\mathcal{X}_1 := \{\chi \neq \chi_1 \ (\mathrm{mod}\ q) : \ |Q_{v,H_1}(\chi)| \leq e^{(1-\alpha_1)v/H_1} \ \forall\, v \in \mathcal{I}_1\}, \\ &\mathcal{X}_j := \{\chi \neq \chi_1 \ (\mathrm{mod}\ q) : \ |Q_{v,H_j}(\chi)| \leq e^{(1-\alpha_j)v/H_j} \ \forall\, v \in \mathcal{I}_j\} \setminus \bigcup_{i \leq j-1} \mathcal{X}_i, \\ &\mathcal{U} := \{\chi \neq \chi_1 \ (\mathrm{mod}\ q)\} \setminus \bigcup_{i \leq J-1} \mathcal{X}_i, \end{aligned}$$

where, as above, we put

$$\alpha_j := \frac{1}{4} - \eta(1 + 1/(2j)) \quad \text{with} \quad \eta = 0.01,$$

for each $1 \leq j \leq J-1$. Similarly to the proof of Proposition 9.2, the proof of Proposition 9.1 (and hence of Theorem 1.4) splits into the cases $\chi \in \mathcal{X}_1, \ldots, \mathcal{X}_{J-1}, \mathcal{U}$, depending on which character sum is small or large.

The introduction of the typical factorizations corresponding to the set $\mathcal{S}$ is handled using Lemma 7.7, as above, which gives

(69)
$$\sum_{\chi \in \mathcal{X}_j} |F(\chi)|^2 \ll H_j \log \frac{Q_j}{P_j} \sum_{v \in I_j} \sum_{\chi \neq \chi_1} |Q_{v,H_j}(\chi) R_{v,H_j}(\chi)|^2 + \Big(\frac{\phi(q)}{q} x\Big)^2 \Big(\frac{1}{H_j} + \frac{1}{P_j} + \prod_{\substack{P_j \leq p \leq Q_j \\ p \nmid q}} (1 - \frac{1}{p})\Big).$$

When summed over $1 \leq j \leq J-1$, the error terms, in light of our parameter choices, are bounded by

$$\Big(\frac{\phi(q)}{q} x\Big)^2 \Big(\varepsilon + \sum_{1 \leq j \leq J} \frac{\log P_j}{\log Q_j} \exp\Big(\sum_{\substack{p|q \\ P_j \leq p \leq Q_j}} \frac{1}{p}\Big)\Big) \ll \varepsilon \Big(\frac{\phi(q)}{q} x\Big)^2 \Big(1 + \varepsilon^{-\frac{\Delta(q,P_1)}{\log 2}} \sum_{1 \leq j \leq J} j^{-\frac{2}{\log 2}(1 - \Delta(q,P_1))}\Big)$$
$$\ll \varepsilon^{1 - 3\Delta(q,(x/Q)^{\varepsilon})} \Big(\frac{\phi(q)}{q} x\Big)^2.$$

Letting E_j denote the main term in (69), we apply the same arguments, but with Lemma 7.1 in place of Lemma 7.2 for $j = 1$, and for $2 \leq j \leq J-1$ we use the second statement of Lemma 7.6, rather than the first. In this way we obtain

$$E_1 \ll \Big(\frac{\phi(q)}{q} x\Big)^2 \varepsilon^{-1} H_1^2 P_1^{-0.1} \ll \varepsilon \Big(\frac{\phi(q)}{q} x\Big)^2$$
$$\sum_{2 \leq j \leq J-1} E_j \ll \Big(\frac{\phi(q)}{q} x\Big)^2 \sum_{2 \leq j \leq J-1} \frac{1}{j^2 Q_{j-1}} \ll \Big(\frac{\phi(q)}{q} x\Big)^2 P_1^{-1},$$

which is sufficient.

In the case of $\mathcal{U}$, we apply Lemma 7.7 once again with the choices P_J and Q_J. As above, we find a $v_0 \in \mathcal{I}_J$ such that

$$\sum_{\chi \in \mathcal{U}} |F(\chi)|^2 \ll H (\log Q_J)^2 \sum_{\chi \in \mathcal{U}} |Q_{v_0,H}(\chi)|^2 |R_{v_0,H}(\chi)|^2 + \varepsilon^{1 - 3\Delta(q,(x/Q)^{\varepsilon})} \Big(\frac{\phi(q)}{q} x\Big)^2,$$

estimating the error term as for the sets $\mathcal{X}_j$, but invoking the specific choices of H_J, P_J and Q_J. As above, we split $\mathcal{U}$ further into the subsets

$$\mathcal{U}_S := \Big\{\chi \neq \chi_1 : |Q_{v_0,H}(\chi)| \leq \varepsilon^2 \frac{e^{v_0/H}}{v_0}\Big\} \cap \mathcal{U}$$
$$\mathcal{U}_L := \Big\{\chi \neq \chi_1 : |Q_{v_0,H}(\chi)| > \varepsilon^2 \frac{e^{v_0/H}}{v_0}\Big\} \cap \mathcal{U}.$$

We combine Lemma 7.5 (with $\mathcal{T} = \{0\}$ this time) with Lemma 7.4 (wherein $\mathcal{E}$ consists of points $(\chi, 0)$) this time, and arguing as in the proof of Proposition 9.2 we obtain that

$$\sum_{\chi \in \mathcal{U}_S} |Q_{v_0,H}(\chi)|^2 |R_{v_0,H}(\chi)|^2 \ll (H \log x)^{-2} \Big(\frac{\phi(q)}{q} x\Big)^2,$$

which, when multiplied by $H(\log Q_J)^2 \ll \varepsilon (H \log x)^2$ yields an acceptable bound.

We treat the $\mathcal{U}_L$ case in essentially the same way as in the proof of Proposition 9.2, and in fact the claim is simpler, as it suffices to combine Propositions 8.5 (with the same parameter choices as in the previous proof) with Corollary 8.4 (taking Remark 8.1 into account). □

10. The case of smooth moduli

In this section, we prove Theorem 1.3 on the variance in arithmetic progressions to all smooth moduli. A key additional ingredient in the proof is the following estimate for short character sums for characters of smooth conductor.

Lemma 10.1. *Let $q, N \geq 1$ with $P^+(q) \leq N^{0.001}$ and $N \geq q^{C/(\log\log q)}$ for a large constant $C > 0$. Then, uniformly for any non-principal character $\chi \pmod q$ and for any $M \geq 1$,*

$$\Big|\sum_{M\leq n\leq M+N} \chi(n)\Big| \ll N\exp\Big(-\frac{1}{4}\sqrt{\log N}\Big). \tag{70}$$

Proof. We note that for $N \geq q$, the estimate (70) follows directly from the Pólya–Vinogradov inequality and thus we can assume that $N < q$. Moreover, (70) holds for primitive $\chi \pmod q$ (in a wider range than stated above and with $\exp(-\sqrt{\log N})$ in place of $\exp(-\frac{1}{4}\sqrt{\log N})$) by a result of Chang [3, Theorem 5]. Indeed, Chang's estimate holds in the regime $\log N > (\log q)^{1-c} + C'\log(2\frac{\log q}{\log q'})\frac{\log q'}{\log q}\frac{\log q}{\log\log q}$ for some $c, C' > 0$ and with $q' = \prod_{p|q} p$, so by $u\log\frac{2}{u} \leq 1$ for $u \leq 1$ this works in the regime $N > q^{C'/\log\log q}$. Let $\chi \pmod q$ be a non-principal character induced by a primitive character $\chi' \pmod{q'}$ with $q' \mid q$. Then by Möbius inversion

$$\sum_{M\leq n\leq M+N}\chi(n) = \sum_{d|q}\mu(d)\chi'(d)\sum_{M/d\leq m\leq (M+N)/d}\chi'(m).$$

Note that in our range $\sqrt{N} \geq q^{0.5C/(\log\log q)}$ and $\sqrt{N}\tau(q) \ll N^{0.9}$, thus taking $C = 10C'$ and using the slightly stronger version of (70) for the primitive character $\chi' \pmod{q'}$, we arrive at

$$\begin{aligned}\sum_{M\leq n\leq M+N}\chi(n) &\ll \sum_{\substack{d|q\\ d\leq\sqrt{N}}}\Big|\sum_{M/d\leq m\leq(M+N)/d}\chi'(m)\Big| + \sum_{\substack{d|q\\ d>\sqrt{N}}}\Big(\frac{N}{d}+1\Big)\\ &\ll N\exp\Big(-\frac{1}{2}\sqrt{\log N}\Big)\sum_{d|q}\frac{1}{d} + \sqrt{N}\tau(q)\\ &\ll N\exp\Big(-\Big(\frac{1}{2}+o(1)\Big)\sqrt{\log N}\Big),\end{aligned}$$

and the result follows. □

Lemma 10.2 below, which uses Lemma 10.1 as an input, allows us to improve on Proposition 8.5 for smooth moduli to obtain good bounds on the frequency of large character sums (mod q) over primes without any exceptional smooth q.

Lemma 10.2. *Let $q \geq P \geq 1$ be integers with $P^+(q) \leq P^{1/10000}$. Suppose also that $q^{1/(\log\log q)^{0.9}} < P < q$. Then for $1 \geq \delta \geq \exp(-(\log P)^{0.49})$ and $V \geq \exp(-(\log P)^{0.49})$ and for any complex numbers $|a_p| \leq 1$, we have*

$$\Big|\Big\{\chi \ (\mathrm{mod}\ q) : \Big|\sum_{P\leq p\leq(1+\delta)P} a_p\chi(p)\Big| \geq V\frac{\delta P}{\log P}\Big\}\Big| \ll (CV^{-1})^{6\log q/\log P}, \tag{71}$$

with the implied constant and $C > 1$ being absolute.

Proof. We begin by noting that, under our assumptions, Lemma 10.1 implies

$$\sum_{n\in I}\chi(n) \ll \delta P\exp(-\frac{1}{10}\sqrt{\log P}) \tag{72}$$

whenever $\chi \pmod q$ is non-principal, and I is an interval of length $\in [P^{0.2}, P]$.
Let R be the quantity on the left-hand side of (71). For any $k \in \mathbb{N}$ we have by Chebychev's inequality

$$\begin{aligned} R &\ll \Big(\frac{\log P}{\delta P}\Big)^{2k} V^{-2k} \sum_{\chi \pmod q} \Big| \sum_{P \leq p \leq (1+\delta)P} a_p \chi(p)\Big|^{2k} \\ &= \Big(\frac{\log P}{\delta P}\Big)^{2k} V^{-2k} \sum_{P \leq p_1, \ldots, p_{2k} \leq (1+\delta)P} a_{p_1} \cdots a_{p_k} \overline{a_{p_{k+1}}} \cdots \overline{a_{p_{2k}}} \sum_{\chi \pmod q} \chi(p_1 \cdots p_k) \overline{\chi}(p_{k+1} \cdots p_{2k}) \\ &\leq \Big(\frac{\log P}{\delta P}\Big)^{2k} V^{-2k} \varphi(q) \sum_{\substack{P \leq p_1, \ldots, p_{2k} \leq (1+\delta)P \\ (p_1 \cdots p_{2k}, q) = 1}} 1_{p_1 \cdots p_k \equiv p_{k+1} \cdots p_{2k} \pmod q}. \end{aligned}$$

We pick $k = \lfloor \frac{3 \log q}{\log P} \rfloor$, so that $3 \leq k \ll \log \log q$. Let $\nu(n)$ be the sieve majorant coming from the linear sieve with sifting level $D = P^{\rho}$ and sifting parameter $z = P^{\rho^2}$, where $\rho > 0$ is a small enough absolute constant (say $\rho = 0.01$). The sieve weight takes the form

$$\nu(n) = \sum_{\substack{d \mid n \\ d \leq P^{\rho}}} \lambda_d$$

for some $\lambda_d \in [-1, 1]$. Then R is bounded by

$$\begin{aligned} R &\ll \Big(\frac{\log P}{\delta P}\Big)^{2k} V^{-2k} \varphi(q) \sum_{\substack{P \leq n_1, \ldots, n_{2k} \leq (1+\delta)P \\ (n_1 \cdots n_{2k}, q) = 1}} \nu(n_1) \cdots \nu(n_{2k}) 1_{n_1 \cdots n_k \equiv n_{k+1} \cdots n_{2k} \pmod q} \\ &= \Big(\frac{\log P}{\delta P}\Big)^{2k} V^{-2k} \sum_{\chi \pmod q} \Big| \sum_{P \leq n \leq (1+\delta)P} \nu(n) \chi(n) \Big|^{2k}. \end{aligned} \tag{73}$$

The contribution of the principal character to the χ sum is

$$\leq \Big(\sum_{P \leq n \leq (1+\delta)P} \nu(n) \Big)^{2k} \ll \Big(\frac{\rho^{-2} \delta P}{\log P}\Big)^{2k}$$

by the linear sieve, and this contribution is admissible by setting $C = \rho^{-2}$ in the lemma. Exchanging the order of summation and applying (72), we have the upper bound

$$\begin{aligned} \sum_{P \leq n \leq (1+\delta)P} \nu(n) \chi(n) = \sum_{d \leq P^{\rho}} \lambda_d \chi(d) \sum_{P/d \leq m \leq (1+\delta)P/d} \chi(m) &\ll P (\log P) \exp\Big(-\frac{1}{10}\sqrt{\log P}\Big) \\ &\ll P \exp\Big(-\frac{1}{15}\sqrt{\log P}\Big). \end{aligned}$$

Hence the contribution of the non-principal characters to the χ sum in (73) is bounded by

$$\ll P^2 \exp\Big(-\frac{2}{15}\sqrt{\log P}\Big) \sum_{\chi \pmod q} \Big| \sum_{P \leq n \leq (1+\delta)P} \nu(n) \chi(n) \Big|^{2(k-1)},$$

and expanding out the moment again, this is

$$\ll P^2 \exp\Big(-\frac{\sqrt{\log P}}{15}\Big)\phi(q) \sum_{\substack{P\le n_1,\dots,n_{2(k-1)}\le(1+\delta)P\\(n_1\cdots n_{2(k-1)},q)=1}} \nu(n_1)\cdots\nu(n_{2(k-1)})1_{n_1\cdots n_{k-1}\equiv n_k\cdots n_{2(k-1)} \pmod q}$$

$$\ll P^2 \exp\Big(-\frac{\sqrt{\log P}}{15}\Big)\phi(q) \sum_{\substack{P\le n_1,\dots,n_{2(k-1)}\le(1+\delta)P\\(n_1\cdots n_{2(k-1)},q)=1}} \tau(n_1)\cdots\tau(n_{2(k-1)})1_{n_1\cdots n_{k-1}\equiv n_k\cdots n_{2(k-1)} \pmod q}.$$

Merging variables, this becomes

$$\begin{aligned}&\ll P^2 \exp\Big(-\frac{\sqrt{\log P}}{15}\Big)\phi(q) \sum_{\substack{m_1,m_2\le(2P)^{k-1}\\ m_1\equiv m_2 \pmod q\\(m_1m_2,q)=1}} \tau_{2(k-1)}(m_1)\tau_{2(k-1)}(m_2)\\ &= P^2 \exp\Big(-\frac{\sqrt{\log P}}{15}\Big)\phi(q) \sum_{\substack{m_1\le(2P)^{k-1}\\(m_1,q)=1}} \tau_{2(k-1)}(m_1) \sum_{\substack{m_2\le(2P)^{k-1}\\ m_2\equiv m_1 \pmod q}} \tau_{2(k-1)}(m_2).\end{aligned}$$

A theorem of Shiu [35] shows that the inner sum is $\ll \frac{(2P)^{k-1}}{\varphi(q)}(\log P^k)^{2(k-1)-1}$, as $q\le(2P)^{0.9(k-1)}$ by our choice of k. Thus the whole expression above is

$$\ll P^{2k}\exp\Big(-\frac{1}{20}\sqrt{\log P}\Big),$$

since $(k\log P)^{2k}\ll \exp((\log P)^{0.01})$. When we multiply this contribution by $(\log P/(\delta P))^{2k}V^{-2k}$ and recall the assumptions $\delta, V\ge \exp(-(\log P)^{0.49})$ and the fact that $k\ll \log\log P$, we see that

$$\begin{aligned}R &\ll (CV^{-1})^{2k} + (\delta^{-1}V^{-1})^{2k}\exp\Big(-\frac{1}{30}\sqrt{\log P}\Big)\Big)\\ &\ll (CV^{-1})^{2k}+1,\end{aligned}$$

which, recalling our choice of k, is what was to be shown. □

Our next lemma improves on Proposition 8.3 for smooth moduli (apart from the t-aspect).

Lemma 10.3. *Let $x\ge 10$, $\kappa>0$ and $2\le P<Q\le x$. Then for all $q\le x$ satisfying $P^+(q)\le q^{\kappa^{100}}$ and for any multiplicative function $f:\mathbb{N}\to\mathbb{U}$, if $\chi_1 \pmod q$ is defined as in Theorem 1.4, we have*

$$\sup_{\substack{\chi \pmod q\\ \chi\ne\chi_1}} \sup_{y\in[x^\kappa,x]} \Big|\frac{1}{y}\sum_{\substack{n\le y\\(n,[P,Q])=1}} f(n)\overline{\chi}(n)\Big| \ll \kappa\left(\frac{\log Q}{\log P}\right)\frac{\phi(q)}{q}. \tag{74}$$

Remark 10.1. We cannot quite make use of zero-free regions corresponding to smooth moduli to prove Lemma 10.3, since Chang's zero-free region in 10.1 for such moduli only applies to non-Siegel zeros (that is, zeros that are not real zeros of L-functions corresponding to non-principal real characters). Instead, we prove the lemma by establishing bounds for $|L(1+it,\chi)|$, as in the low conductor case of Proposition 8.3, and that will yield the asserted result.

Proof. We may assume in what follows that $\kappa>0$ is small enough and fixed (adjusting the implied constant if necessary). We may also assume $q^{\kappa^{100}}\ge 2$, so $\kappa\gg(\log q)^{-0.01}$.

Noting that $n\mapsto 1_{(n,[P,Q])=1}$ is multiplicative, and that for any $g_1,g_2:\mathbb{N}\to\mathbb{U}$ and any $y\ge 2$ we have

$$\mathbb{D}_q(g_1 1_{(\cdot,[P,Q])=1}, g_2; y)^2 \ge \mathbb{D}_q(g_1,g_2;y)^2 - \sum_{P\le p\le Q}\frac{1}{p} = \mathbb{D}_q(g_1,g_2;y)^2 - \log\left(\frac{\log Q}{\log P}\right) + O(1),$$

following the beginning of the proof of Proposition 8.3 almost verbatim, we obtain the result once we prove that

$$\sup_{|t|\leq (\log q)^{0.02}} \mathbb{D}(\xi, n^{it}; x^{\kappa})^2 \geq 4.5 \log \frac{1}{\kappa} + \log \frac{q}{\phi(q)} + O(1)$$

for all non-principal characters $\xi \pmod q$. From the proof of Lemma 6.5, it follows that

$$\mathbb{D}(\xi, n^{it}; x^{\kappa})^2 \geq \log\log x^{\kappa} - \log |L(1+it, \xi)| - O(1),$$

so we need to show that

$$\sup_{|t|\leq (\log q)^{0.02}} |L(1+it,\xi)| \ll \kappa^{5.5} \frac{\phi(q)}{q} \log q$$

for all $q \leq x$ satisfying $P^+(q) \leq q^{\kappa^{100}}$. Partial summation shows that

$$L(1+it,\xi) = \sum_{n\leq q(|t|+1)} \frac{\xi(n)}{n^{1+it}} + O(1).$$

Let $q' = q^{10000\kappa^{100}}$. Then

$$|L(1+it,\xi)| \ll \frac{\phi(q)}{q} \log q' + \Big| \sum_{q'\leq n\leq q(|t|+1)} \frac{\xi(n)}{n^{1+it}} \Big| + 1.$$

The first term on the right-hand side is acceptable. For the second term, we apply partial summation to write it as

$$\sum_{q'\leq n\leq q(|t|+1)} \frac{\xi(n)}{n^{1+it}} = S(q', q(|t|+1))(q(|t|+1))^{-1-it} + (1+it)\int_{q'}^{q(|t|+1)} S(q',u)u^{-2-it}\, du, \tag{75}$$

where

$$S(M,N) := \sum_{M\leq n\leq N} \xi(n).$$

We are now in a position to apply Lemma 10.1 (which is applicable, as $P^+(q) \leq q^{\kappa^{100}} \leq (q')^{0.0001}$), and this allows us to bound the right-hand side of (75) by

$$\ll 1 + (1+|t|)\int_{q'}^{\infty} \frac{1}{u\exp(\frac{1}{10}\sqrt{\log u})}\, du \ll 1 \ll \kappa^{10}\frac{\phi(q)}{q}\log q,$$

since $\phi(q)/q \gg 1/\log\log q$ and $\kappa \geq (\log q)^{-0.01}$. This concludes the proof. □

Corollary 10.4. *Let $x \geq R \geq 10$, $\kappa \in ((\log x)^{-1/3}, 1)$, and $10 \leq P \leq Q/2 \leq x$. Let the twisted character sum $R(\chi, s)$, multiplicative function $f : \mathbb{N} \to \mathbb{U}$ and character $\chi_1 \pmod q$ be defined as in Corollary 8.4. Then for $2 \leq q \leq x$ satisfying $P^+(q) \leq q^{\kappa^{100}}$ we have*

$$\sup_{\substack{\chi \pmod q \\ \chi\neq\chi_1}} \sup_{R\in[x^{1/2},x]} \frac{1}{R}|R(\chi,0)| \ll \kappa\Big(\frac{\log x}{\log P}\Big)^3 \frac{\phi(q)}{q}.$$

Proof. Applying the hyperbola method (similarly as in the proof of Corollary 8.4), we see that

$$\begin{aligned} |R(\chi,0)| \ll \Big| &\sum_{\substack{m_1\leq Rx^{-\kappa} \\ p|m_1 \Longrightarrow p\in[P,Q]}} \frac{f(m_1)\overline{\chi}(m_1)}{1+\omega_{[P,Q]}(m_1)} \sum_{\substack{R/m_1\leq m_2\leq 2R/m_1 \\ (m_2,[P,Q])=1}} f(m_2)\overline{\chi}(m_2)\Big| \\ &+ \sum_{\substack{m_2\leq 2x^{\kappa} \\ (m_2,[P,Q])=1 \\ (m_2,q)=1}} \sum_{\substack{Rx^{-\kappa}<m_1\leq 2R/m_2 \\ p|m_1\Longrightarrow p\in[P,Q]}} 1. \end{aligned} \tag{76}$$

Since $R/m_1 \geq x^\kappa$ holds in the first sum on the right, we can apply Lemma 10.3 to bound this sum by

$$\ll \kappa \frac{\log Q}{\log P}\frac{\phi(q)}{q}\Big(\sum_{\substack{m_1\leq x\\ p\mid m_1\Longrightarrow p\in[P,Q]}}\frac{R}{m_1}\Big)\Big(\sum_{\substack{d\leq x^\kappa\\ p\mid d\Longrightarrow p\in[P,Q]}}\frac{1}{d}\Big) \ll \kappa\frac{\log Q}{\log P}\frac{\phi(q)}{q}R\prod_{P\leq p\leq Q}\Big(1-\frac{1}{p}\Big)^{-2}$$
$$\ll \kappa\frac{\phi(q)}{q}\Big(\frac{\log Q}{\log P}\Big)^3 R.$$

The second sum on the right of (76), in turn, is bounded using Selberg's sieve by

$$\ll \sum_{\substack{m_2\leq 2x^\kappa\\ (m_2,q)=1}}\frac{R}{m_2\log P} \ll \kappa\frac{\log x}{\log P}\frac{\phi(q)}{q}R,$$

as wanted. □

Proof of Theorem 1.3. Inspecting the proof of Theorem 1.5, the result of that theorem holds for any modulus $q \leq x$ satisfying, for $H = \lfloor \varepsilon^{-1}\rfloor$ the bounds

$$\sup_{P\in[x^{\varepsilon^2},x^\varepsilon]}\Big|\Big\{\chi \pmod q : \Big|\sum_{P\leq p\leq Pe^{1/H}} f(p)\chi(p)\Big| \geq \varepsilon^2\frac{P}{H\log P}\Big\}\Big| \ll K(\varepsilon), \tag{77}$$

and for $P = x^{\varepsilon^2}$ and $Q = x^\varepsilon$,

$$\sup_{\substack{\chi \pmod q\\ \chi\neq\chi_1}}\Big|\frac{1}{R}\sup_{R\in[x^{1/2},x]}\sum_{R\leq m\leq 2R}\frac{f(m)\overline{\chi}(m)}{1+\omega_{[P,Q]}(m)}\Big| \ll \frac{\varepsilon}{K(\varepsilon)^{1/2}}\frac{\varphi(q)}{q} \tag{78}$$

for some function $K(\varepsilon) \geq 1$. Indeed, it is only the $\mathcal{U}_L$ case of the proof of Theorem 1.4 where we need to assume something about the modulus q, and the assumptions that we need there are precisely a large values estimate of the form (77) together with a pointwise bound of the type (78).

We then establish (77) and (78). Let $P^+(q) \leq q^{\varepsilon'}$ with $\varepsilon' = \exp(-\varepsilon^{-3})$. Lemma 10.2 (where we take $V = \varepsilon^2/10$ and $\delta = e^{1/H} - 1$) readily provides (77) with $K(\varepsilon) = \varepsilon^{-100\varepsilon^{-2}}$ (assuming as we may that $\varepsilon > 0$ is smaller than any fixed constant).

Corollary 10.4 in turn gives (78) (with the same $K(\varepsilon) = \varepsilon^{-100\varepsilon^{-2}}$ as above) when we take $\kappa = \varepsilon^4 K(\varepsilon)^{-1/2}$ there, which we can do since $P^+(q) \leq q^{\varepsilon'} \leq q^{\kappa^{100}}$. This completes the proof. □

11. All moduli in the square-root range

11.1. **Preliminary lemmas.** For the proof of Theorem 1.2, we need a few estimates concerning smooth and rough numbers to bound the error terms arising from exhibiting good factorizations for smooth numbers in Lemmas 11.5 and 11.7.

Lemma 11.1. *Let $c \in (0,1)$. Let $1 \leq Y \leq X$ and $1 \leq q \leq X^{1-c}$, and let $X^{-c/2} \leq \delta \leq 1$. Then for any reduced residue class a modulo q,*

$$\sum_{\substack{(1-\delta)X<m\leq X\\ P^-(m)>Y, m\equiv a \pmod q}} 1 \ll c^{-1}\frac{\delta X}{\phi(q)\log Y}.$$

Proof. This follows immediately from Selberg's sieve. □

Given $1 \leq q, Y \leq X$, define the counting function of smooth numbers coprime to q as

$$\Psi_q(X,Y) := |\{n \leq X : P^+(n) \leq Y, (n,q) = 1\}|. \tag{79}$$

We have the following estimate for $\Psi_q(X,Y)$ in short intervals.

Lemma 11.2. *Let* $10 \leq Y \leq X$ *and set* $u := \log X/\log Y$*. Assume that* $Y \geq \exp((\log X)^{0.99})$ *and* $\exp(-(\log X)^{0.01}) \leq \delta \leq 1$*. Finally, let* $1 \leq q \leq e^{Y^{1/2}}$*. Then*

$$\Psi_q((1+\delta)X, Y) - \Psi_q(X,Y) \ll \rho(u)\frac{\phi(q)}{q}\delta X.$$

Proof. By the sieve of Eratosthenes, we have

$$\Psi_q((1+\delta)X, Y) - \Psi_q(X,Y) = \sum_{\substack{d|q \\ P^+(d)\leq Y}} \mu(d) \sum_{\substack{\frac{X}{d} \leq m \leq (1+\delta)\frac{X}{d} \\ P^+(m)\leq Y}} 1$$

Let S_1 and S_2 be parts of the sum with $d \leq \exp(10(\log X)^{1/2})$ and $d > \exp(10(\log X)^{1/2})$, respectively. For estimating S_2, we crudely remove the smoothness condition from the m sum and estimate the remaining sum using $1/d \leq \exp(-5(\log X)^{1/2})/\sqrt{d}$ to obtain

$$\begin{aligned} S_2 &\ll \sum_{\substack{d|q \\ d>\exp(10(\log X)^{1/2})}} \frac{X}{d} \ll X\exp(-5(\log X)^{1/2}) \prod_{p|q}\Big(1-\frac{1}{\sqrt{p}}\Big)^{-1} \\ &\ll X\exp(-5(\log X)^{1/2})\exp(3\sqrt{\omega(q)}) \end{aligned}$$

and using $\omega(q) = o(\log q)$ this is certainly $\ll \delta X\rho(u)\frac{\phi(q)}{q}\exp(-\frac{1}{2}(\log X)^{1/2})$ by $u \leq (\log X)^{0.01}$ and the well-known estimate $\rho(u) = u^{-(1+o(1))u}$.

For the S_1 sum, we instead apply [17, Theorem 5.1] (noting that its hypothesis $\delta X/d \geq XY^{-5/12}$ is satisfied) so that we obtain

$$\begin{aligned} S_1 &= \sum_{\substack{d|q \\ d\leq\exp(10(\log X)^{1/2}) \\ P^+(d)\leq Y}} \Big(\mu(d)\frac{\delta X}{d}\rho\Big(u-\frac{\log d}{\log Y}\Big)\Big(1+O\Big(\frac{\log(u+1)}{\log Y}\Big)\Big)\Big) \\ &= \sum_{\substack{d|q \\ P^+(d)\leq Y}} \Big(\mu(d)\frac{\delta X}{d}\rho(u)\Big(1+O\Big(\frac{\log(u+1)}{\log Y}\Big)\Big)\Big) \\ &+ O\Big(\delta X\rho(u-10(\log X)^{1/2}/\log Y)\exp(-(\log X)^{1/2}) + \delta X \sum_{\substack{d|q \\ d\leq\exp(10(\log X)^{1/2}) \\ P^+(d)\leq Y}} \frac{|\rho(u)-\rho(u-\frac{\log d}{\log Y})|}{d}\Big), \end{aligned}$$

where we used the same bound as in the S_2 case to extend the d sum to all $d \mid q, P^+(d) \leq Y$. By the mean value theorem and the identity $u\rho'(u) = -\rho(u-1)$, we have

$$|\rho(u-\frac{\log d}{\log Y}) - \rho(u)| \leq \frac{\log d}{\log Y}\max_{u-10(\log X)^{1/2}/\log Y\leq v\leq u}\frac{\rho(v-1)}{v} \ll \rho(u-2)\frac{(\log X)^{1/2}}{\log Y}, \tag{80}$$

and therefore the expression for S_1 simplifies to

$$S_1 = \delta\rho(u)X\prod_{\substack{p|q \\ p\leq Y}}\Big(1-\frac{1}{p}\Big)\Big(1+O\Big(\sum_{d|q}\frac{1}{d}\cdot\frac{\log(u+1)}{\log Y} + \frac{\rho(u-2)}{\rho(u)}(\log X)^{-0.3}\Big)\Big).$$

Now the proof is completed by recalling that $u \leq (\log X)^{0.01}$ and noting that the product over $p \mid q$ is $\asymp \frac{\phi(q)}{q}$ since $Y \geq \log^2 q$ and that $\rho(u-2) \ll u^3\rho(u)$ by [17, Formulas (2.8) and (2.4)]. □

Corollary 11.3. *Let* $1 \le Y \le X_1 < X_2$ *and* $1 \le q \le X_2$, *with* $Y \ge \exp((\log X_1)^{0.99})$. *Then*

$$\sum_{\substack{X_1 < n \le X_2 \\ P^+(n) \le Y \\ (n,q)=1}} \frac{1}{n} \ll \frac{\phi(q)}{q} \rho(u_1) \log(2X_2/X_1),$$

where $u_1 := (\log X_1)/\log Y$.

Proof. Decompose the interval $(X_1, X_2]$ dyadically. Making use of Lemma 11.2, we find

$$\sum_{\substack{X_1 < n \le 2X_2 \\ P^+(n) \le Y \\ (n,q)=1}} \frac{1}{n} \ll \sum_{X_1 < 2^j \le 4X_2} 2^{-j} \sum_{\substack{2^{j-1} < n \le 2^j \\ P^+(n) \le Y \\ (n,q)=1}} 1 \ll \frac{\phi(q)}{q} \rho(u_1) \sum_{X_1 < 2^j \le 4X_2} 1 \ll \frac{\phi(q)}{q} \rho(u_1) \log(2X_2/X_1),$$

as claimed. □

11.2. **Decoupling of variables.** The proof of Theorem 1.2 is based on obtaining bilinear structure in the sum, coming from the fact that the summation may be restricted to smooth numbers. Certainly any x^η-smooth number $n \in [x^{1-\eta}, x]$ can be written as $n = dm$ with $d, m \in [x^{1/2-\eta}, x^{1/2+\eta}]$, but a typical smooth number has a lot of representations of the above form, and therefore it appears difficult to decouple the d and m variables just from this. The following simple lemma however provides a more specific factorization that does allow decoupling our variables.

Lemma 11.4. *Let* $x \ge 4$, *and let* $n \in [x^{1/2}, x]$ *be an integer. Then* n *can be written uniquely as* dm *with* $d \in [x^{1/2}/P^-(m), x^{1/2})$ *and* $P^+(d) \le P^-(m)$.

Proof. Let $n = p_1 p_2 \cdots p_k$, where $p_1 \le p_2 \le \cdots \le p_k$ are primes. Let $r \ge 1$ be the smallest index for which $p_1 \cdots p_r \ge x^{1/2}$. Then $d = p_1 \cdots p_{r-1}$, $m = p_r \cdots p_k$ works. We still need to show that this is the only possible choice of d and m.

Let d and m be as in the lemma. Since $dm = p_1 \cdots p_k$ and $P^+(d) \le P^-(m)$, there exists $r \ge 1$ such that $d = p_1 \cdots p_{r-1}$, $m = p_r \cdots p_k$, and by the condition on the size of d we must have $p_1 \cdots p_{r-1} < x^{1/2}$, $p_1 \cdots p_{r-1} \ge x^{1/2}/p_r$. There is exactly one suitable r, namely the smallest r with $p_1 \cdots p_r \ge x^{1/2}$. □

We need to be able to control the size of the $P^-(m)$ variable, since if it is very small then so is $P^+(d)$, leading to character sums over very sparse sets. The next lemma says that for typical $n \le x$ the corresponding $P^-(m)$ is reasonably large, even if n is restricted to an arithmetic progression.

In what follows, set

$$\theta_j := \eta(1-\varepsilon^2)^j \quad \text{for all} \quad j \ge 0, \tag{81}$$

and let

$$J := \lceil \varepsilon^{-2} \log\log(1/\varepsilon) \rceil \tag{82}$$

so that for small $\varepsilon > 0$ we have

$$\theta_J \asymp_\eta 1/\log(1/\varepsilon) \quad \text{and} \quad \rho(1/\theta_J) \ll (1/\theta_J)^{0.5/\theta_J} \ll \varepsilon^{100}.$$

We have $J \le 2\varepsilon^{-2} \log\log(1/\varepsilon)$ as long as $\varepsilon > 0$ is small enough in terms of η.

Lemma 11.5 (Restricting to numbers with specific factorizations)**.** *Let* $x \ge 10$, $\eta \in (0, 1/10)$ *and* $(\log x)^{-1/100} \le \varepsilon \le 1$. *Let* θ_j *be given by* (81) *and* J *given by* (82), *and define*

$$\mathcal{S}_J := \bigcup_{0 \le j \le J} \{n \le x : \; n = dm, \; d \in (x^{1/2-\theta_{j+1}}, x^{1/2}), \; P^+(d) \le x^{\theta_{j+1}}, \; P^-(m) \in (x^{\theta_{j+1}}, x^{\theta_j}]\}.$$

Let $q \leq x^{1/2-100\eta}$. Then for $(a,q)=1$ we have

$$\sum_{\substack{n \leq x \\ n \equiv a \pmod q \\ P^+(n) \leq x^{\eta}}} (1 - 1_{\mathcal{S}_J}(n)) \ll_{\eta} \varepsilon \frac{x}{q}.$$

Proof. We may assume that ε is smaller than any fixed function of η. In what follows, let $n \leq x$, $P^+(n) \leq x^{\eta}$ and $n \equiv a \pmod q$ with $(a,q)=1$.

Owing to Lemma 11.4, we know that we may write any n as above uniquely in the form $n = dm$ with $P^+(d) \leq P^-(m)$ and $d \in [x^{1/2}/P^-(m), x^{1/2})$. Let us further denote by $\mathcal{T}_j$ the set of n as above for which $P^-(m) \in (x^{\theta_{j+1}}, x^{\theta_j}]$, so that every n belongs to a unique set $\mathcal{T}_j$ with $j \geq 0$. We claim that $n \in \mathcal{S}_J$ unless one of the following holds:
(i) n has a divisor $d \geq x^{1/2-\eta}$ with $P^+(d) \leq x^{\theta_J}$ and $P^-(n/d) \geq P^+(d)$;
(ii) There exist two (not necessarily distinct) primes $p_1, p_2 > x^{\theta_{J+1}}$ with $p_1 p_2 \mid n$ and $1 \leq p_1/p_2 \leq x^{\varepsilon^2}$;
(iii) For some $0 \leq j \leq J$, we can write $n = rs$ with $r \in [x^{1/2-\theta_j}, x^{1/2-\theta_{j+1}}]$, $P^+(r) \leq x^{\theta_j}$, $P^-(s) \in (x^{\theta_{j+1}}, x^{\theta_j}]$.
Indeed, if $n \leq x$, $P^+(n) \leq x^{\eta}$ and none of (i), (ii), (iii) holds, then letting j be the index for which $n \in \mathcal{T}_j$, we have $j \leq J$ (by negation of (i)) and in the factorization $n = dm$ of n we have the conditions $P^-(m) \in (x^{\theta_{j+1}}, x^{\theta_j}]$, $P^+(d) \leq x^{\theta_{j+1}}$ (by negation of (ii) and the fact that $\theta_j - \theta_{j+1} \leq \varepsilon^2$), and $d \in (x^{1/2-\theta_{j+1}}, x^{1/2}]$ (by negation of (iii)), so that $n \in \mathcal{S}_J$.

Applying Lemma 11.1, the contribution of (i) is

$$\ll \sum_{\substack{x^{1/2-\eta} \leq d \leq x^{1/2} \\ P^+(d) \leq x^{\theta_J} \\ (d,q)=1}} \sum_{\substack{m \leq x/d \\ P^-(m) \geq P^+(d) \\ m \equiv a d^{-1} \pmod q}} 1 \ll \eta^{-1} \sum_{\substack{x^{1/2-\eta} \leq d \leq x^{1/2} \\ P^+(d) \leq x^{\theta_J} \\ (d,q)=1}} \frac{x/d}{\phi(q)(\log P^+(d))}$$
$$\ll_{\eta} \sum_{k \geq \log(1/\theta_J)-1} e^{-k} \sum_{\substack{x^{1/2-\eta} \leq d \leq x^{1/2} \\ P^+(d) \in [x^{e^{-k-1}}, x^{e^{-k}}] \\ (d,q)=1}} \frac{1}{d \log x} \cdot \frac{x}{\phi(q)}.$$

Set $u_0 := (\log x)^{0.01}$. The contribution of the terms with $e^k \leq u_0$ can be bounded using Lemma 11.2, and $\rho(u) \ll u^{-u}$ (see [17, (2.6)]), yielding a contribution of

$$\ll \sum_{k \geq \log(1/\theta_J)-1} e^{-k} \rho(e^k/3) \frac{x}{q} \ll \sum_{k \geq \log(1/\theta_J)-1} e^{-(k-\log 3)e^{-k}/3} \ll \varepsilon^{100} \frac{x}{q},$$

since $\theta_J \gg_{\eta} \log(1/\varepsilon)$. The remaining terms with $e^k > u_0$ can be estimated trivially using Corollary 11.3, giving

$$\ll \eta^{-1} \sum_{k \geq 0.01 \log\log x} e^{-k} \rho(u_0/3) \frac{x}{q} \ll_{\eta} \varepsilon \frac{x}{q}.$$

Denoting $M = \theta_{J+1}\varepsilon^{-2}$ and applying the prime number theorem, the contribution of (ii) in turn is bounded by

$$\begin{aligned}\sum_{M\le k\le \varepsilon^{-2}} \sum_{p_1,p_2\in[x^{(k-1)\varepsilon^2},x^{(k+1)\varepsilon^2}]} \sum_{\substack{n\le x\\ n\equiv a \pmod q\\ p_1p_2\mid n}} 1 &\ll \frac{x}{q}\sum_{M\le k\le\varepsilon^{-2}}\Big(\sum_{p\in[x^{(k-1)\varepsilon^2},x^{(k+1)\varepsilon^2}]}\frac{1}{p}\Big)^2\\ &\ll \frac{x}{q}\sum_{M\le k\le\varepsilon^{-2}}\Big(\log\Big(\frac{k+1}{k-1}\Big)+(\log x)^{-100}\Big)^2\\ &\ll \frac{x}{q}\sum_{M\le k\le\varepsilon^{-2}}\Big(\frac{1}{k^2}+(\log x)^{-100}\Big)\\ &\ll \frac{x}{qM},\end{aligned}$$

and by the definition of M and the fact that $\theta_{J+1}\ll_\eta 1/\log(1/\varepsilon)$, this is $\ll_\eta \varepsilon\frac{x}{q}$.

Lastly, by Lemma 11.1 and Corollary 11.3, for any fixed $0\le j\le J$, the contribution of (iii) is

$$\begin{aligned}&\sum_{\substack{x^{1/2-\theta_j}\le r\le x^{1/2-\theta_{j+1}}\\ P^+(r)\le x^{\theta_j}\\ (r,q)=1}} \sum_{\substack{s\le x/r\\ P^-(s)\in[x^{\theta_{j+1}},x^{\theta_j}]\\ s\equiv ar^{-1} \pmod q}} 1 \le \sum_{\substack{x^{1/2-\theta_j}\le r\le x^{1/2-\theta_{j+1}}\\ P^+(r)\le x^{\theta_j}\\ (r,q)=1}} \sum_{\substack{p\in[x^{\theta_{j+1}},x^{\theta_j}]\\ p\nmid q}} \sum_{\substack{s'\le x/(pr)\\ P^-(s')\ge x^{\theta_{j+1}}\\ s'\equiv a(pr)^{-1} \pmod q}} 1\\ &\ll \eta^{-1}\sum_{\substack{x^{1/2-\theta_j}\le r\le x^{1/2-\theta_{j+1}}\\ P^+(r)\le x^{\theta_j}\\ (r,q)=1}} \sum_{p\in[x^{\theta_{j+1}},x^{\theta_j}]} \frac{x/(pr)}{\phi(q)\theta_{j+1}(\log x)}\\ &\ll_\eta \sum_{\substack{x^{1/2-\theta_j}\le r\le x^{1/2-\theta_{j+1}}\\ P^+(r)\le x^{\theta_j}\\ (r,q)=1}} \frac{x}{\phi(q)r\theta_{j+1}(\log x)}\Big(\log\frac{\theta_j}{\theta_{j+1}}+(\log x)^{-100}\Big)\\ &\ll_\eta \frac{\theta_j-\theta_{j+1}}{\theta_{j+1}}\log\frac{\theta_j}{\theta_{j+1}}\rho(3/\theta_j)\frac{x}{q}+\frac{x}{q(\log x)^{99}}.\end{aligned}$$

Here the second term is certainly small enough. Using $\rho(u)\ll u^{-1}$, $\log(1+v)\ll v$ and formulas (81) and (82), the first term summed over $0\le j\le J$ is crudely bounded by

$$\ll_\eta \sum_{0\le j\le J}(\theta_j-\theta_{j+1})^2\frac{x}{q}\ll_\eta J\varepsilon^4\ll_\eta \varepsilon^{1.9}\frac{x}{q}.$$

Therefore we have proved the assertion of the lemma. □

We further wish to split the d and m variables into short intervals to dispose of the cross-condition $dm\le x$ on their product. This is achieved in the following lemma.

Lemma 11.6 (Separating variables)**.** *Let $x\ge 10$, $\eta\in(0,1/10)$ and $(\log x)^{-1/100}\le\varepsilon\le 1$. Let θ_j be given by (81), and let $H:=\lfloor\varepsilon^{-1.1}\rfloor$. For each $0\le j\le J$ (with J given by (82)), write*

$$\begin{aligned}\mathcal{I}_j &:= \{u\in\mathbb{Z}:\ H\theta_{j+1}\log x\le u\le H\theta_j\log x-1\},\\ \mathcal{K}_j &:= \{v\in\mathbb{Z}:\ (1/2-\theta_{j+1})H\log x\le v\le \frac{1}{2}H\log x-1\}.\end{aligned}\tag{83}$$

Define the set

$$\mathcal{S}_J' := \bigcup_{0\le j\le J}\bigcup_{u\in\mathcal{I}_j, v\in\mathcal{K}_j}\{n = pdm',\ p\in(e^{u/H},e^{(u+1)/H}],\ d\in(e^{v/H},e^{(v+1)/H}],\ m'\le xe^{-(u+v+2)/H},\ P^+(d)\le x^{\theta_{j+1}},\ P^-(m')>x^{\theta_j}\}.$$

Then we have

$$\sum_{\substack{n\le x\\ n\equiv a \pmod q\\ P^+(n)\le x^\eta}}(1-1_{\mathcal{S}_J'}(n)) \ll_\eta \varepsilon\frac{x}{q}.$$

Proof. By Lemma 11.5, it suffices to prove the claim with $1_{\mathcal{S}_J}(n)-1_{\mathcal{S}_J'}(n)$ in place of $1-1_{\mathcal{S}_J'}(n)$. We have $\mathcal{S}_J'\subset\mathcal{S}_J$, since for $n\in\mathcal{S}_J$ we have a unique way to write it, for some $0\le j\le J$, as $n=dm$ with $P^+(d)\le x^{\theta_{j+1}}$, $P^-(m)\in(x^{\theta_{j+1}},x^{\theta_j}]$, and we may further write $m=pm'$, so that $p\in(x^{\theta_{j+1}},x^{\theta_j}]$ and $P^-(m')>p$.

Now, if we define $u_j^{(1)},u_j^{(2)}$ as the endpoints of the discrete interval $\mathcal{I}_j$, and similarly $v_j^{(1)},v_j^{(2)}$ as the endpoints of $\mathcal{K}_j$, we see that $n\in\mathcal{S}_J$ belongs for unique $0\le j\le J$, $u\in\mathcal{I}_j$, $v\in\mathcal{K}_j$ to the set in the definition of $\mathcal{S}_J'$, unless one of the following holds for the factorization $n=pdm'$ of n:
(i) We have $p\in[e^{(u_j^{(i)}-1)/H},e^{(u_j^{(i)}+1)/H}]$ or $d\in[e^{(v_j^{(k)}-1)/H},e^{(v_j^{(k)}+1)/H}]$ for some $i\in\{1,2\}$ and $0\le j\le J$;
(ii) We have $p\in[e^{u/H},e^{(u+1)/H}]$, $d\in[e^{v/H},e^{(v+1)/H}]$, $m'\in[xe^{-(u+v+2)/H},xe^{-(u+v)/H}]$ for some $u\in\mathcal{I}_j$, $v\in\mathcal{K}_j$ and $0\le j\le J$.
(iii) We have $P^-(m')\in(x^{\theta_{j+1}},x^{\theta_j})$. Condition (iii) clearly leads to condition (ii) in the proof of Lemma 11.5 holding, so its contribution is $\ll_\eta \varepsilon x/q$.

We are left with the contributions of (i) and (ii). They are bounded similarly, so we only consider (ii).

For given j,u,v, Lemmas 11.1 and 11.2 tell us that the contribution of (ii) is

$$\begin{aligned}&\sum_{\substack{e^{u/H}\le p\le e^{(u+1)/H}\\ p\nmid q}}\ \sum_{\substack{e^{v/H}\le d\le e^{(v+1)/H}\\ P^+(d)\le x^{\theta_{j+1}}\\ (d,q)=1}}\ \sum_{\substack{xe^{-(u+v+2)/H}\le m'\le xe^{-(u+v)/H}\\ P^-(m')\ge x^{\theta_{j+1}}\\ m'\equiv a(pd)^{-1} \pmod q}} 1\\
&\ll \frac{\eta^{-1}}{H}\sum_{e^{u/H}<p\le e^{(u+1)/H}}\ \sum_{\substack{e^{v/H}<d\le e^{(v+1)/H}\\ P^+(d)\le x^{\theta_{j+1}}\\ (d,q)=1}}\frac{xe^{-(u+v)/H}}{\phi(q)\theta_{j+1}\log x}\\
&\ll_\eta \frac{1}{H^2\theta_{j+1}}\rho\Big(\frac{1/2-\eta}{\theta_{j+1}}\Big)\frac{x}{uq\log x},\end{aligned}$$

where the second $1/H$ factor arose from summation over d and the $1/u$ factor arose from the summation over p. Summing this over $u\in\mathcal{I}_j$, $v\in\mathcal{K}_j$ and $0\le j\le J$ and recalling that $|\mathcal{I}_j|\ll(\theta_j-\theta_{j+1})H(\log x)$, $|\mathcal{K}_j|\ll\theta_jH\log x$ and $\rho(y)\ll y^{-2}$ yields a bound of

$$\ll_\eta \sum_{0\le j\le J}(\theta_j-\theta_{j+1})\theta_j(H\log x)^2\cdot\frac{1}{H^2}\frac{1}{H\log^2 x}\cdot\frac{x}{q}\ll_\eta\frac{\varepsilon^2J}{H}\cdot\frac{x}{q}\ll_\eta\varepsilon\frac{x}{q}$$

by the definitions of H and J. ∎

Now that we have decoupled the variables, we may transfer to characters and obtain a trilinear sum. For $u \in \mathcal{I}_j, v \in \mathcal{K}_j$ and $H = \lfloor \varepsilon^{-1.1} \rfloor$, write

$$\begin{aligned} P_u(\chi) &= \sum_{e^{u/H} < p \le e^{(u+1)/H}} f(p)\chi(p) \\ D_v(\chi) &= \sum_{\substack{e^{v/H} < d \le e^{(v+1)/H} \\ P^+(d) \le x^{\theta_{j+1}}}} f(d)\chi(d), \\ M_{u,v}(\chi) &= \sum_{\substack{m \le x/e^{(u+v+2)/H} \\ P^-(m) > x^{\theta_j}}} f(m)\chi(m). \end{aligned} \tag{84}$$

Then we have the following.

Lemma 11.7. *Let $x \ge 10$, $\eta \in (0, 1/10)$, $\varepsilon \in ((\log x)^{-1/200}, 1)$, $q \le x^{1/2-100\eta}$, and let $f : \mathbb{N} \to \mathbb{U}$ be a multiplicative function supported on x^η-smooth numbers. Letting χ_1 be as in Theorem 1.2, and recall the definitions in* (83)*. Then for $(a, q) = 1$ we have*

$$\begin{aligned} &\Big| \sum_{\substack{n \le x \\ n \equiv a \pmod q}} f(n) - \frac{\chi_1(a)}{\phi(q)} \sum_{n \le x} f(n)\overline{\chi_1}(n) \Big| \\ &\le \frac{1}{\phi(q)} \sum_{\chi \ne \chi_1 \pmod q} \sum_{0 \le j \le J} \sum_{u \in \mathcal{I}_j} \sum_{v \in \mathcal{K}_j} |P_u(\overline{\chi})||D_v(\overline{\chi})||M_{u,v}(\overline{\chi})| + O_\eta\Big(\frac{\varepsilon x}{q}\Big). \end{aligned}$$

Proof. Applying Lemma 11.6 to both f and $f\overline{\chi_1}$ and observing that the union of sets in the definition of $\mathcal{S}'_J$ is disjoint, we see that the left-hand side in the statement is

$$\begin{aligned} &\Big| \sum_{0 \le j \le J} \sum_{\substack{u \in \mathcal{I}_j \\ v \in \mathcal{K}_j}} \sum_{e^{u/H} < p \le e^{(u+1)/H}} \sum_{\substack{e^{v/H} < d \le e^{(v+1)/H} \\ P^+(d) \le x^{\theta_{j+1}}}} \sum_{\substack{m \le x e^{-(u+v+2)/H} \\ P^-(m) > x^{\theta_j}}} f(p)f(d)f(m)\xi_q(mdp) \Big| \\ &+ O_\eta\Big(\frac{\varepsilon x}{q}\Big), \end{aligned}$$

where

$$\xi_q(n) := 1_{n \equiv a \pmod q} - \frac{\chi_1(a)}{\phi(q)} \overline{\chi_1}(n).$$

Making use of the orthogonality of characters, and then applying the triangle inequality, the main term here is (omitting the summation ranges for brevity)

$$\begin{aligned} &\Big| \sum_{0 \le j \le J} \sum_{\substack{u \in \mathcal{I}_j \\ v \in \mathcal{K}_j}} \sum_{\chi \ne \chi_1 \pmod q} \frac{\chi(a)}{\phi(q)} \Big(\sum_p f(p)\overline{\chi}(p) \Big) \Big(\sum_{\substack{d \\ P^+(d) \le x^{\theta_{j+1}}}} f(d)\overline{\chi}(d) \Big) \Big(\sum_{\substack{m \\ P^-(m) > x^{\theta_j}}} f(m)\overline{\chi}(m) \Big) \Big| \\ &\le \frac{1}{\phi(q)} \sum_{0 \le j \le J} \sum_{u \in \mathcal{I}_j} \sum_{v \in \mathcal{K}_j} \sum_{\chi \ne \chi_1 \pmod q} |P_u(\overline{\chi})||D_v(\overline{\chi})||M_{u,v}(\overline{\chi})|, \end{aligned}$$

and the claim follows. □

11.3. **The main proof.** Let $\eta > 0$. Suppose henceforth that the multiplicative function $f : \mathbb{N} \to \mathbb{U}$ is supported on x^η-smooth integers. Our task is to prove Theorem 1.2, i.e., to obtain cancellation in the deviation

$$\max_{a \in \mathbb{Z}_q^\times} \Big| \sum_{\substack{n \le x \\ n \equiv a \pmod q}} f(n) - \frac{\chi_1(a)}{\phi(q)} \sum_{n \le x} f(n)\overline{\chi_1}(n) \Big|.$$

In what follows, let $(\log x)^{-1/200} \le \varepsilon \le 1$, and let θ_j and J be given by (81) and (82), and recall the notation of (83) and (84).

According to Lemma 11.7, we can restrict ourselves to bounding the product of character sums present in that lemma. Taking the maximum over $(u,v) \in \mathcal{I}_j \times \mathcal{K}_j$ there, it suffices to prove that

$$\sum_{j \le J} \frac{(\theta_j - \theta_{j+1}) H^2 (\log x)^2}{\phi(q)} \sum_{\chi \neq \chi_1 \pmod q} |P_{u_j}(\overline{\chi})||D_{v_j}(\overline{\chi})||M_{u_j,v_j}(\overline{\chi})| \ll_\eta \varepsilon \frac{x}{q},$$

where for each $0 \le j \le J$ the numbers $u_j \in \mathcal{I}_j$, $v_j \in \mathcal{K}_j$ are chosen so that they give maximal contribution.

In analogy with the proofs of Theorems 1.4 and 1.5, for each $j \le J$ we define[11] the sets $\mathcal{X}^{(j)}$ and $\mathcal{U}_L^{(j)}$ by

$$\mathcal{X}^{(j)} := \{\chi \neq \chi_1 \pmod q : |P_{u_j}(\overline{\chi})| \le \varepsilon^3 e^{u_j/H}/u_j\}$$
$$\mathcal{U}^{(j)} := \{\chi \neq \chi_1 \pmod q\} \backslash \mathcal{X}^{(j)}.$$

11.3.1. *Case of* $\mathcal{X}^{(j)}$. For a given $0 \le j \le J$, consider the contribution from $\mathcal{X}^{(j)}$. Applying Cauchy–Schwarz, we have

$$\begin{aligned} &\frac{1}{\phi(q)} \sum_{\chi \in \mathcal{X}^{(j)}} |P_{u_j}(\overline{\chi})||D_{v_j}(\overline{\chi})||M_{u_j,v_j}(\overline{\chi})| \\ &\le \Big(\frac{1}{\phi(q)} \sum_{\chi \in \mathcal{X}^{(j)}} |M_{u_j,v_j}(\chi)|^2\Big)^{1/2} \Big(\frac{1}{\phi(q)} \sum_{\chi \in \mathcal{X}^{(j)}} |P_{u_j}(\overline{\chi})|^2 |D_{v_j}(\overline{\chi})|^2\Big)^{1/2}. \end{aligned}$$

We begin by bounding the first bracketed sum. We do not use Lemma 7.1 directly for this, since that would lose one factor of $\log x$ that comes from the sparsity of the m variable in the definition of $M_{u_j,v_j}(\chi)$. Instead, we expand the square and apply orthogonality, which shows that the first bracketed sum is bounded by

$$\Big(\sum_{\substack{m_1, m_2 \le x e^{-(u_j+v_j)/H} \\ P^-(m_1), P^-(m_2) \ge x^{\theta_j} \\ m_1 \equiv m_2 \pmod q}} 1\Big)^{1/2}.$$

Taking the maximum over m_1 and summing over the m_2 variable, and applying Lemma 11.1 (recalling that $x e^{-(u_j+v_j)/H}/q \gg x^\eta$), this is

$$\ll \frac{\eta^{-1}}{\theta_j \phi(q)^{1/2} \log x} x e^{-(u_j+v_j)/H}.$$

To treat the remaining bracketed expression, we use the pointwise bound from the definition of $\mathcal{X}^{(j)}$, and then use Lemma 7.1 to $D_{v_j}(\overline{\chi})$, giving

$$\Big(\frac{1}{\phi(q)} \sum_{\chi \in \mathcal{X}^{(j)}} |P_{u_j}(\overline{\chi})|^2 |D_{v_j}(\overline{\chi})|^2\Big)^{1/2} \ll \Big(\frac{\varepsilon^6}{\phi(q)} e^{2u_j/H}/u_j^2 \sum_{\chi \in \mathcal{X}^{(j)}} |D_{v_j}(\overline{\chi})|^2\Big)^{1/2}$$

$$\ll \Big(\frac{\varepsilon^6}{\phi(q)} e^{2u_j/H}/u_j^2 \Big(\phi(q) + \frac{\phi(q)}{q} e^{v_j/H}\Big) \Big(\Psi_q(e^{(v_j+1)/H}, x^{\theta_{j+1}}) - \Psi_q(e^{v_j/H}, x^{\theta_{j+1}})\Big)\Big)^{1/2}. \tag{85}$$

[11] We only need to split the χ spectrum into two sets here, as opposed to many sets in the proof of Theorem 1.4. This is owing to the fact that $P_{u_j}(\chi)$ already has length $\gg q^\varepsilon$, and thus our large values estimates for it are effective. The reason we are allowed to take $P_{u_j}(\chi)$ so long here (unlike in our previous proofs) is that we are assuming $q \le x^{1/2-100\eta}$. If we only assumed that $q = o(x^{1/2})$, we would have to perform an iterative decomposition as in the preceding sections.

By Lemma 11.2,

$$\Psi_q(e^{(v_j+1)/H}, x^{\theta_{j+1}}) - \Psi_q(e^{v_j/H}, x^{\theta_{j+1}}) \ll \rho(1/3\theta_j)\frac{\phi(q)}{q}e^{v_j/H}/H.$$

Inserting this into (85), and using $e^{v_j/H}/q \geq 1$ for any $v_j \in \mathcal{K}_j$, results in the bound

$$\ll \varepsilon^3\Big(\frac{\phi(q)}{q^2}\rho(1/3\theta_j)H^{-1}\Big)^{1/2} e^{(u_j+v_j)/H}/u_j.$$

Combining this with the contribution from $M_{u_j,v_j}(\chi)$ yields the upper bound

$$\frac{1}{\phi(q)}\sum_{\chi\in\mathcal{X}^{(j)}}|P_{u_j}(\overline{\chi})||D_{v_j}(\overline{\chi})||M_{u_j,v_j}(\overline{\chi})|$$
$$\ll_\eta \varepsilon^3 H^{-1/2}\frac{\rho(1/(3\theta_j))^{1/2}}{\theta_{j+1}}\frac{x}{u_j(\log x)q}.$$

Recalling $H = \lfloor \varepsilon^{-1.1}\rfloor$, $\rho(u) \ll u^{-2}$ and $\theta_j - \theta_{j+1} \ll \varepsilon^2$ this expression multiplied by $(\theta_j - \theta_{j+1})H^2(\log x)^2$ is

$$\ll_\eta \varepsilon^3(\theta_j-\theta_{j+1})H^2(\log x)^2\cdot H^{-1/2}\rho(1/(3\theta_j))^{1/2}\frac{x}{\theta_{j+1}(\log x)^2 q} \ll_\eta \varepsilon^{3.3}\frac{x}{q}.$$

Finally summing this over $0 \leq j \leq J$, the bound we obtain is

$$\ll_\eta J\varepsilon^{3.3}\frac{x}{q} \ll_\eta \varepsilon^{1.2}\frac{x}{q},$$

which is good enough.

11.3.2. *Case of* $\mathcal{U}^{(j)}$. It remains to consider the contributions from $\mathcal{U}^{(j)}$. We restrict to $q \in \mathcal{Q}_{x,\varepsilon^{9.5},\varepsilon^{-100}}$ with the notation of Lemma 8.2. As in the proof of Theorem 1.4, that set satisfies the desired size bound $|[1,Q]\setminus \mathcal{Q}_{x,\varepsilon^{9.5},\varepsilon^{-100}}| \ll Qx^{-\varepsilon^{200}}$ (since $9.5\cdot 20 < 200$), and for any set $\mathcal{Q}' \subset [1,x]$ of coprime integers the set $\mathcal{Q}_{x,\varepsilon^{9.5},\varepsilon^{-100}}$ intersects it in $\ll (\log x)^{\varepsilon^{-200}}$ points (and under GRH we have $\mathcal{Q}_{x,\varepsilon^{9.5},\varepsilon^{-100}} = [1,x]\cap\mathbb{Z}$). We also recall that in Theorem 1.2 the character $\chi_1 \pmod q$ is such that $\inf_{|t|\leq x}\mathbb{D}_q(f,\chi_j(n)n^{it};x)$ is minimal.

By Proposition 8.5 (with $\delta := e^{1/H}-1 \asymp 1/H$), for q as above we have $|\mathcal{U}^{(j)}| \ll \varepsilon^{-6}H \ll \varepsilon^{-7.1}$, since $P_u(\chi)$ has length $\gg x^{\theta_J}$ and $\theta_J \gg_\eta 1/(\log\frac{1}{\varepsilon})$.

Furthermore, applying Proposition 8.3 (and Remark 8.1) to $f(n)1_{P^+(n)\leq x^{\theta_j}}$ (and recalling $q \in \mathcal{Q}_{x,\varepsilon^{9.5},\varepsilon^{-100}}$), we see that[12]

$$|D_{v_j}(\overline{\chi})| = \Big|\sum_{e^{v_j/H}\leq d\leq e^{(v_j+1)/H}} f(d)\overline{\chi}(d)1_{P^+(d)\leq x^{\theta_{j+1}}}\Big| \ll \varepsilon^{9.5}\frac{\phi(q)}{q}e^{v_j/H} \tag{86}$$

for all $\chi \in \mathcal{U}^{(j)}$, except possibly for the $\chi = \chi^{(j)}$ that minimizes the pretentious distance $\inf_{|t|\leq\log x}\mathbb{D}_q(f,\chi(n)1_{P^+(n)\leq x^{\theta_{j+1}}}n^{it};x)$. We argue that $\chi^{(j)}$ must be the character χ_1 of Theorem 1.2, in which case $\chi^{(j)} \notin \mathcal{U}^{(j)}$ and we can ignore this character.

By applying Lemma 6.1, we see that either

$$\inf_{|t|\leq\log x}\mathbb{D}_q^2(f,\chi^{(j)}(n)1_{P^+(n)\leq x^{\theta_{j+1}}}n^{it};x) \leq 1.01\log(1/\varepsilon^{9.5}) + O(1)$$

[12]Note that the saving of $\varepsilon^{9.5}$ is much better than the trivial saving (which we do not need to exploit here) that comes from the fact that d is supported on x^{θ_j}-smooth numbers. The trivial saving would only be better if θ_j is roughly of size $1/\log(1/\varepsilon)$ or smaller, but as we shall see the contribution of these large values of the index j is small in any case by trivial estimation.

or else (86) holds without any exceptional characters. We may assume we are in the former case, and then by $\theta_{j+1} \geq \theta_{J+1} \gg_\eta 1/(\log(1/\varepsilon))$ and trivial estimation we obtain

$$\inf_{|t|\leq \log x} \mathbb{D}_q^2(f, \chi^{(j)}(n)n^{it}; x) \leq 1.1\log(1/\varepsilon^{9.5}) + O_\eta(1).$$

But we have the same for χ_1 in place of $\chi^{(j)}$ by the minimality of χ_1. Thus, assuming that $\chi^{(j)} \neq \chi_1$ and following the argument of Proposition 8.3 verbatim, we obtain a contradiction. This means that we may assume from now on that (86) holds for all $\chi \in \mathcal{U}^{(j)}$ and $0 \leq j \leq J$.

Now we take the maximum over $\chi \in \mathcal{U}^{(j)}$ in the sum that we are considering and apply the Brun–Titchmarsh inequality to $P_{u_j}(\overline{\chi})$ and Lemma 11.1 to $M_{u_j,v_j}(\overline{\chi})$ to bound

$$\begin{aligned}\frac{1}{\phi(q)}\sum_{\chi\in\mathcal{U}^{(j)}} |P_{u_j}(\overline{\chi})||D_{v_j}(\overline{\chi})||M_{u_j,v_j}(\overline{\chi})| &\ll \frac{\varepsilon^{-7.1+9.5}}{q} e^{u_j/H}/u_j \cdot e^{v_j/H} x e^{-(u_j+v_j)/H} \frac{\eta^{-1}}{\theta_{j+1}\log x} \\ &\ll_\eta \varepsilon^{2.4} \frac{x}{q\theta_{j+1}^2 H(\log x)^2},\end{aligned}$$

and this multiplied by $(\theta_j - \theta_{j+1})H^2(\log x)^2$ and summed over $0 \leq j \leq J$ (recalling that $\theta_J \gg_\eta 1/\log(1/\varepsilon))$ produces the bound

$$\ll_\eta \varepsilon^{2.4}(\log\frac{1}{\varepsilon})^2 H \sum_{0\leq j\leq J} (\theta_j - \theta_{j+1})\frac{x}{q} \ll_\eta \varepsilon^{1.2}\frac{x}{q}.$$

This completes the proof of Theorem 1.2.

12. A Linnik-type result

In this section, we prove our Linnik-type theorems stated in Section 2. As in the proof of Theorem 1.4, we employ the Matomäki–Radziwiłł method in arithmetic progressions.

Our main propositions for this section concern products of exactly three primes of the form

$$E_3^* := \{n = p_1p_2p_3 : \ P_j^{1-\varepsilon} \leq p_j \leq P_j,\ j \in \{1,2,3\}\}, \quad P_1 = q^{1000\varepsilon}, P_2 = P_3 = q. \tag{87}$$

Proposition 12.1 (E_3^* numbers in progressions to smooth moduli)**.** *For every small enough $\varepsilon > 0$ there exists $\eta(\varepsilon) > 0$ such that the following holds.*

Let $q \geq 2$ with $P^+(q) \leq q^{\eta(\varepsilon)}$. There exists a real character $\xi \pmod q$ such that for all a coprime to q we have

$$\begin{aligned}\sum_{\substack{n\in E_3^* \\ n\equiv a \pmod q}} \frac{1}{n} = \frac{1+O(\varepsilon)}{\phi(q)} &\sum_{P_1^{1-\varepsilon}\leq p_1\leq P_1}\ \sum_{P_2^{1-\varepsilon}\leq p_2\leq P_2}\ \sum_{P_3^{1-\varepsilon}\leq p_3\leq P_3} \frac{1}{p_1p_2p_3} \\ &+ \frac{\xi(a)}{\varphi(q)} \sum_{P_1^{1-\varepsilon}\leq p_1\leq P_1}\ \sum_{P_2^{1-\varepsilon}\leq p_2\leq P_2}\ \sum_{P_3^{1-\varepsilon}\leq p_3\leq P_3} \frac{\xi(p_1p_2p_3)}{p_1p_2p_3}\end{aligned} \tag{88}$$

with E_3^*, P_1, P_2, P_3 as in (87).

Proposition 12.2 (E_3^* numbers in progressions to prime moduli)**.** *For every small enough $\varepsilon > 0$ there exists $M(\varepsilon) > 0$ such that the following holds.*

Let $q \geq 2$. Suppose that the product $\prod_{\chi \pmod q} L(s,\chi)$ has the zero-free region $\mathrm{Re}(s) \geq 1 - \frac{M(\varepsilon)}{\log q}$, $|\mathrm{Im}(s)| \leq (\log q)^3$. Then for all a coprime to q we have

$$\sum_{\substack{n\in E_3^* \\ n\equiv a \pmod q}} \frac{1}{n} = \frac{1+O(\varepsilon)}{\phi(q)} \sum_{P_1^{1-\varepsilon}\leq p_1\leq P_1}\ \sum_{P_2^{1-\varepsilon}\leq p_2\leq P_2}\ \sum_{P_3^{1-\varepsilon}\leq p_3\leq P_3} \frac{1}{p_1p_2p_3} \tag{89}$$

with E_3^*, P_1, P_2, P_3 as in (87).

We shall Theorem 2.1(i)–(ii) from these two propositions at the end of the section.

12.1. **Auxiliary Lemmas.** In order to prove these propositions, we shall need a result of Chang [3, Theorem 10], giving an improved zero-free region for $L(s,\chi)$ when the conductor of χ is smooth.

Lemma 12.3 (Zero-free region for L-functions to smooth moduli)**.** *Suppose $q \geq 2$ and $P^+(q) \leq q^\kappa$ with $C/(\log\log(10q)) < \kappa < 0.001$ for large enough $C > 0$. Then the product $\prod_{\chi \pmod q} L(s,\chi)$ obeys the zero-free region*

$$\mathrm{Re}(s) \geq 1 - \frac{c\kappa^{-1}}{\log q}, \quad |\mathrm{Im}(s)| \leq q$$

for some constant $c > 0$, apart from possibly a single zero β. If β exists, then it is real and simple and corresponds to a unique real character $\pmod q$.

Proof. This was proved by Chang in [3, Theorem 10], apart from possible Siegel zeros[13]. Indeed, in that theorem it was shown that, apart from Siegel zeros, $L(s,\chi)$ has the zero-free region

$$\mathrm{Re}(s) > 1 - c\min\left\{\frac{1}{\log P^+(q)}, \frac{\log\log d'}{(\log d')\log(2\frac{\log d}{\log d'})}, \frac{1}{(\log(dT))^{1-c'}}\right\}, \quad |\mathrm{Im}(s)| \leq T,$$

where d is a modulus such that χ is induced by a primitive character $\pmod d$ and $d' = \prod_{p|d} p$. We take $T = q$ and note that the middle term in the minimum is $\gg \frac{\log\log d}{\log d} \geq \frac{\log\log q}{\log q}$, and this produces the zero-free region of the lemma apart from Siegel zeros.

What we still need to show is that there cannot exist two real zeros β_1, β_2 corresponding to two distinct real characters $\chi_1, \chi_2 \pmod q$ and violating our zero-free region. For this, we follow the proof of [24, Lemma 12]. We may assume that q is larger than any given constant, since otherwise the Vinogradov–Korobov zero-free region is good enough.

By Lemma 10.1 (and our smoothness assumption on q), we have the twisted character sum estimate

$$\Big|\sum_{n\in I}\chi(n)n^{-it}\Big| \ll N\exp\Big(-\frac{1}{4}\sqrt{\log N}\Big) \tag{90}$$

for any interval I of length $N \in [\exp(C\frac{\log q}{\log\log q}), q)$ and for $N > P^+(q)^{1000}$. Applying partial summation to the definition of $L(s,\chi)$, splitting this infinite sum into the ranges $[1, q^{1000\kappa}]$ $[q^{1000\kappa}, q^2]$ and (q^2, ∞) (cf. [24, Proof of Lemma 8]), and estimating the first range trivially, the second range using (90) and the third range using Pólya–Vinogradov, we deduce

$$|L(s,\chi)| < q^{10000\kappa\eta}, \quad |\mathrm{Re}(s)| \geq 1-\eta,\ |\mathrm{Im}(s)| \leq q,\ \eta := \frac{1}{(\log q)^{1/2}\log\log q};$$

note that the trivial bound is $\ll q^\eta$, and it is crucial to beat this in what follows.

Let $\theta := \frac{10^{-7}\kappa^{-1}}{\log q}$, and let $\sigma_0 := 1+5\theta$. Assume for the sake of contradiction that $\min\{\beta_1,\beta_2\} > 1-\theta$. By comparing the L-function corresponding to the principal character $\chi_0 \pmod q$ with the Riemann zeta function (cf. [24, Proof of Lemma 11]), we find

$$\frac{L'}{L}(\sigma_0, \chi_0) \geq \frac{1}{1-\sigma_0} - 2\log\log(3q).$$

Another observation is that $\chi_3 := \chi_1\chi_2$ is a real, non-principal character and $1+\chi_1(n)+\chi_2(n)+\chi_3(n) = (1+\chi_1(n))(1+\chi_2(n)) \geq 0$ for all n.

[13] In [3], it was not fully specified what is meant by Siegel zeros, so we assume the weakest possible interpretation that for every real, non-principal character $\pmod q$ there can be one zero of the corresponding L-function that violates the zero-free region, with these zeros being real and simple.

By [24, Lemma 10], this gives

$$0 < -\sum_{i=0}^{3} \frac{L'}{L}(\sigma_0, \chi_i) \le 2\log\log(3q) + 3 \cdot \frac{4}{\eta}\log M + \frac{1}{\sigma_0 - 1} - \frac{1}{\sigma_0 - \beta_1} - \frac{1}{\sigma_0 - \beta_2}, \tag{91}$$

where M is such that $|L(s,\chi_i)| \le M|L(\sigma_0,\chi_i)|$ whenever $|s-\sigma_0| \le \eta$, and additionally we need to have $\sigma_0 - \eta/2 < 1-\theta$ (which clearly holds in our case).

Note then that, as in [24, Proof of Lemma 11], a trivial triangle inequality estimate gives $|L(\sigma_0,\chi)^{-1}| < \frac{1}{5\theta}$, so we can take $M := q^{10000\kappa\eta}/(5\theta) \le q^{10001\kappa\eta}$ above. In particular, we have $(\log M)/\eta \le 10001\kappa \log q$.

Inserting our bound on M and the lower bounds on β_1, β_2 into (91) and estimating $\log\log(3q)$ crudely results in

$$0 < \frac{1}{100\theta} + \frac{1}{10\theta} + \frac{1}{5\theta} - \frac{1}{6\theta} - \frac{1}{6\theta} < 0,$$

which is a contradiction, as desired. □

We will also need the following mean value estimate for sums over small sets of characters.

Lemma 12.4 (Halász–Montgomery type estimate over primes)**.** *Let $q \ge 1$ be an integer, and let Ξ be a set of characters $\pmod q$. Then for $k \in \{2,3\}$, $\eta > 0$, $2 \le R < \sqrt{N}$, and for any complex numbers a_p, we have the estimate*

$$\sum_{\chi \in \Xi} \Big|\sum_{p \le N} a_p \chi(p)\Big|^2 \ll_{k,\eta} \Big(\frac{N}{\log R} + N^{1-1/k} q^{(k+1)/4k^2+\eta} |\Xi| R^{2/k}\Big) \sum_{p \le N} |a_p|^2.$$

Proof. This is a result of Schlage-Puchta [34, Theorem 3]. □

In the proof of Theorem 2.1(i)–(ii), we will need pointwise estimates for logarithmically weighted character sums assuming only a narrow zero-free region. By a simple Perron's formula argument, we can obtain cancellation in

$$\sum_{P \le p \le P^{1+\kappa}} \frac{\chi(p)}{p} \tag{92}$$

for $\chi \ne \chi_0 \pmod q$, $\kappa > 0$ fixed, and $P \in [q^\varepsilon, q]$ if we assume a zero-free region of the form $\mathrm{Re}(s) > 1 - 3\frac{\log\log q}{\log P}$, $|\mathrm{Im}(s)| \le q$ for $L(s,\chi)$; the need for this zero-free region comes from pointwise estimation of $|\frac{L'}{L}(s,\chi)| \ll \log^2(q(|t|+2))$ which costs us two logarithms (in the region where we are $\gg \frac{1}{\log(q(|t|+1))}$ away from any zeros). However, here we must argue differently, since we are only willing to assume a zero-free region of the form $\mathrm{Re}(s) > 1 - \frac{M(\varepsilon)}{\log P}$, $|\mathrm{Im}(s)| \le q$ (which we know for smooth moduli apart from Siegel zeros). To do so, we exploit the logarithmic weight $1/p$ in the sum over $P \le p \le P^{1+\kappa}$, which allows us to insert a carefully chosen smoothing. Variant of such an argument is known as a Rodosskii bound in the literature.

Lemma 12.5 (A Rodosskii-type bound)**.** *Let $q \ge 2$, $\varepsilon > 0$, $\kappa > 0$, and let $\chi \pmod q$ be a non-principal character. Suppose that $L(s,\chi) \ne 0$ for $\mathrm{Re}(s) > 1 - \frac{\kappa^{-2}}{\log q}$, $|\mathrm{Im}(s)| \le (\log q)^3$. Then, provided that $P \ge q^\kappa \gg_\kappa 1$, we have*

$$\sup_{|t| \le (\log q)^3/2} \Big|\sum_{P \le p \le P^{1+\varepsilon}} \frac{\chi(p)}{p^{1+it}}\Big| \le C_0 \kappa \tag{93}$$

with $C_0 > 0$ an absolute constant.

Proof. This is a slight modification of results proved by Soundararajan [37, Lemma 4.2] and by Harper [14, Rodosskii Bound 1]; in those bounds there is the nonnegative function $(1 - \mathrm{Re}(\chi(p)p^{-it}))/p$ in place of $\chi(p)/p^{1+it}$ in (93), and consequently only lower bounds of the correct order of magnitude are needed in those results. We will choose a more elaborate smoothing to obtain asymptotics (up to $O(\kappa)$) for (93). Also note that our range of $|t|$ is smaller than in the works mentioned above, but correspondingly the zero-free region is assumed to a lower height.

We may assume without loss of generality that $\kappa < \varepsilon/10 < 1/10$, since otherwise the trivial Mertens bound for (93) is good enough. By splitting the interval $[P, P^{1+\varepsilon}]$ into $\ll \varepsilon/\kappa$ intervals of the form $[y, y^{1+\kappa}]$, it suffices to show that

$$\sup_{|t|\le(\log q)^3/2} \Big| \sum_{y\le p\le y^{1+\kappa}} \frac{\chi(p)\log p}{p^{1+it}} \Big| \ll \kappa^2 \log y \tag{94}$$

uniformly for $y \in [P, P^2]$.

We introduce the continuous, nonnegative weight function

$$g(u) = \begin{cases} \kappa^{-2}u, & u \in [0, \kappa^2] \\ 1, & u \in [\kappa^2, \kappa - \kappa^2] \\ \kappa^{-2}(\kappa - u), & u \in [\kappa - \kappa^2, \kappa], \\ 0, & u \notin [0, \kappa]; \end{cases}$$

in other words, g is a trapezoid function. We further define the weight function

$$W(p) = W_{y,\kappa}(p) = g\Big(\frac{\log \frac{p}{y}}{\log y}\Big) \log y.$$

Since $W(p) = \log y$ for $p \in [y^{1+\kappa^2}, y^{1+\kappa-\kappa^2}]$, and $0 \le W(p) \le \log y$ everywhere, by estimating the contribution of $p \in [y, y + y^{1+\kappa^2}] \cup [y^{1+\kappa-\kappa^2}, y^{1+\kappa}]$ trivially, it suffices to show that

$$\sup_{|t|\le(\log q)^3/2} \Big| \sum_{p} \frac{\chi(p)W(p)\log p}{p^{1+it}} \Big| \ll \kappa^2 \log^2 y. \tag{95}$$

Let χ^* be the primitive character that induces χ. Since the contribution of $p \mid q$ to the sum in (95) is negligible, and we can replace $\log p$ with the von Mangoldt function, from Perron's formula we see that

$$\sum_p \frac{\chi(p)W(p)\log p}{p^{1+it}} = -\frac{1}{2\pi i}\int_{-i\infty}^{i\infty} \frac{L'}{L}(1 + it + s, \chi^*)\widetilde{W}(s)\,ds + O(\kappa^2 \log y),$$

where

$$\widetilde{W}(s) := \int_0^\infty W(x)x^{s-1}\,dx = \kappa^{-2}\frac{y^{(1+\kappa)s} - y^{(1+\kappa-\kappa^2)s} - y^{(1+\kappa^2)s} + y^s}{s^2} \tag{96}$$

is the Mellin transform of W.

Shifting the contours to the left, and noting that $\widetilde{W}(s)$ is entire and $|\widetilde{W}(s)| \ll \frac{\kappa^{-2}}{|s|^2}$ for $\mathrm{Re}(s) \le 0$, we reach

$$\sum_p \frac{\chi(p)W(p)\log p}{p^{1+it}} = -\sum_\rho \widetilde{W}(\rho - 1 - it) + O(1), \tag{97}$$

where the sum is taken over all nontrivial zeros of $L(s, \chi^*)$. Since $|t| \le \frac{(\log q)^3}{2}$, we can truncate the ρ sum to end up with

$$\sum_p \frac{\chi(p)W(p)\log p}{p^{1+it}} = -\sum_{|\mathrm{Im}(\rho)|\le(\log q)^3} \widetilde{W}(\rho - 1 - it) + O(1).$$

Let $A := \kappa^{-2}$. Thanks to our assumption on zero-free regions, we clearly have

$$|\widetilde{W}(\rho - 1 - it)| \ll \frac{\kappa^{-2} y^{-\frac{A}{\log q}}}{|\rho - 1 - it|^2},$$

and consequently

$$\Big|\sum_p \frac{\chi(p)W(p)\log p}{p^{1+it}}\Big| \ll \kappa^{-2} y^{-\frac{A}{\log q}} \sum_{|\mathrm{Im}(\rho)|\le (\log q)^3} \frac{1}{|1+it-\rho|^2} + 1. \tag{98}$$

We now note that for any zero $\rho = \beta + i\gamma$, with $|\gamma| \le (\log q)^3$ we must have $\beta \le 1 - \frac{A}{\log q}$, and so

$$\frac{1}{|1+it-\rho|^2} \ll \frac{1}{|1+1/\log q + it - \rho|^2} \ll \frac{\log q}{A}\mathrm{Re}\Big(\frac{1}{1+1/\log q + it - \rho}\Big).$$

Thus we can estimate

$$\Big|\sum_p \frac{\chi(p)W(p)\log p}{p^{1+it}}\Big| \ll \kappa^{-2} y^{-\frac{A}{\log q}} \cdot \frac{\log q}{A} \sum_\rho \mathrm{Re}\Big(\frac{1}{1+1/\log q + it - \rho}\Big) + 1.$$

Recall that $y \ge P \ge q^{1/\sqrt{A}}$. We can use the Hadamard factorization theorem in the form given in [4, Chapter 12] on the right-hand side of the above formula, and estimate $|\frac{L'}{L}(1+1/\log q)| \ll \log q$, to see that

$$\Big|\sum_p \frac{\chi(p)W(p)\log p}{p^{1+it}}\Big| \ll \kappa^{-2} e^{-\sqrt{A}} A (\log y)^2 + 1 \ll \kappa^2 \log^2 y$$

by our choice of A. This finishes the proof of the lemma. □

12.2. **Proof of Propositions 12.1 and 12.2.**

Proof of Proposition 12.1. We may assume that $\varepsilon > 0$ is small enough and q is large enough in terms of ε, since we must have $q^{\varepsilon'} \ge 2$, and we are free to choose the dependence of ε' on ε. We shall show that if q is such that we have the zero-free region $L(s,\chi) \ne 0$ for $\mathrm{Re}(s) \ge 1 - \varepsilon^{-100}/\log q$, $|\mathrm{Im}(s)| \le (\log q)^3$ for all $\chi \pmod q$ apart from possibly one real character ξ, then (88) holds[14]. This zero-free region is in particular satisfied for those q that satisfy $P^+(q) \le q^{\eta(\varepsilon)}$ with small enough $\eta(\varepsilon) > 0$.

By the orthogonality of characters, we have

$$\sum_{\substack{n\in E_3^* \\ n\equiv a \pmod q}} \frac{1}{n} = \sum_{\chi\in\{\chi_0,\xi\} \pmod q} \frac{\chi(a)}{\phi(q)} P_1(\overline{\chi})P_2(\overline{\chi})P_3(\overline{\chi}) + \sum_{\substack{\chi \pmod q \\ \chi\ne\chi_0,\xi}} \frac{\chi(a)}{\phi(q)} P_1(\overline{\chi})P_2(\overline{\chi})P_3(\overline{\chi}),$$

where we have defined

$$P_j(\chi) := \sum_{P_j^{1-\varepsilon} \le p \le P_j} \frac{\chi(p)}{p}, \quad j \in \{1,2,3\}.$$

In the above expression, in the term corresponding to χ_0 we can replace χ_0 by 1 at the cost of $O((\log q)/q^{\varepsilon})$.

We employ the Matomäki–Radziwiłł method as in our other proofs. Let

$$\mathcal{X} := \{\chi \ne \chi_0, \xi \pmod q : \ |P_1(\overline{\chi})| \le P_1^{-0.01}\},$$
$$\mathcal{U}_S := \{\chi \ne \chi_0, \xi \pmod q\} \setminus \mathcal{X}.$$

Unlike in the earlier sections, there is no $\mathcal{U}_L$ case to analyze, owing to the fact that for $\chi \in \mathcal{U}_S$ we already have some cancellation in $|P_1(\chi)|$ by Lemma 12.5 and our assumption on q.

[14]If this bad ξ does not exist, let ξ be any non-principal real character in what follows.

The case of $\mathcal{X}$ is handled similarly to our other proofs. Indeed, by Cauchy–Schwarz, we have

$$\sum_{\chi\in\mathcal{X}}|P_1(\overline{\chi})||P_2(\overline{\chi})||P_3(\overline{\chi})| \ll P_1^{-0.01}\Big(\sum_{\chi\in\mathcal{X}}|P_2(\overline{\chi})|^2\Big)^{1/2}\Big(\sum_{\chi\in\mathcal{X}}|P_3(\overline{\chi})|^2\Big)^{1/2}.$$

By the mean value theorem for character sums (Lemma 7.1) and the fact that $P_1 = q^{1000\varepsilon}$, $P_2 = P_3 = q$, this is

$$\ll q^{-10\varepsilon}\phi(q)\Big(\sum_{P_2^{1-\varepsilon}\le p_2\le P_2}\frac{1}{p_2^2}\Big)^{1/2}\Big(\sum_{P_3^{1-\varepsilon}\le p_3\le P_3}\frac{1}{p_3^2}\Big)^{1/2} \ll q^{-\varepsilon},$$

say, since $\phi(q)/(P_2P_3)^{\frac{1}{2}(1-\varepsilon)} \ll q^{\varepsilon}$.

The remaining case to consider is that of $\mathcal{U}_S$. Note that, combining the assumed zero-free region for $L(s,\chi)$, $\chi\neq\xi \pmod q$ with Lemma 12.5 we see that $|P_1(\chi)|\ll\varepsilon^2$ for all $\chi\in\mathcal{U}_S$.

From Lemma 7.5, which bounds the number of large values taken by a prime-supported character sum, we have the size bound

$$|\mathcal{U}_S| \le |\{\chi \pmod q:\ |P_1(\overline{\chi})| > P_1^{-0.01}\}| \ll q^{0.05}.$$

Introducing the dyadic sums

$$P_{j,v}(\chi) := \sum_{\substack{e^v\le p\le e^{v+1}\\ P_j^{1-\varepsilon}\le p\le P_j}}\frac{\chi(p)}{p},\quad v\in I_j := [(1-\varepsilon)\log P_j, \log P_j],$$

the upper bound on $|P_1(\chi)|$ above and Cauchy–Schwarz give

$$\sum_{\chi\in\mathcal{U}_S}|P_1(\overline{\chi})||P_2(\overline{\chi})||P_3(\overline{\chi})| \ll \varepsilon^2\sum_{v_1,v_2\in I_2}\sum_{\chi\in\mathcal{U}_S}|P_{2,v_1}(\overline{\chi})||P_{3,v_2}(\overline{\chi})|$$

$$\ll \varepsilon^2(\varepsilon\log q)^2\Big(\sum_{\chi\in\mathcal{U}_S}|P_{2,v_1'}(\overline{\chi})|^2\Big)^{1/2}\Big(\sum_{\chi\in\mathcal{U}_S}|P_{3,v_2'}(\overline{\chi})|^2\Big)^{1/2}$$

for some $v_1', v_2'\in I_2$ (since as $P_2 = P_3$ we have $I_2 = I_3$). It remains to be shown that

$$\sum_{\chi\in\mathcal{U}_S}|P_{j,v}(\overline{\chi})|^2 \ll \frac{1}{\log^2 q}$$

for $j\in\{2,3\}$, since then we get a bound of $\ll\varepsilon^4$ for the sum over $\chi\in\mathcal{U}_S$, and this (multiplied by the $1/\phi(q)$ factor) can be included in the error term in (88).

For this purpose, we apply Lemma 12.4, which is a sharp inequality of Halász–Montgomery-type for character sums over primes[15]. We take $N = e^{v+1}$, $|\Xi| = |\mathcal{U}_L| \ll q^{0.05}$, $k=3$, $R = N^{0.0001}$, $a_p = \frac{1}{p}1_{p\in[e^v,e^{v+1}]\cap[P_j^{1-\varepsilon},P_j]}$ in that lemma. Since the term $N^{2/3}q^{1/9}|\Xi|R^{2/3}$ appearing in Lemma 12.4 is smaller than the other term $\frac{N}{\log R}$ for our choice of parameters, we get a bound of $\ll e^v/v\cdot\frac{1}{ve^v}\ll\frac{1}{\log^2 q}$, as desired. This completes the analysis of the $\mathcal{U}_S$ case, so Proposition 12.1 follows. □

Proof of Proposition 12.2. The proof of Proposition 12.2 is similar to that of Proposition 12.1, except that there are no exceptional characters arising. The proof of (88) goes through for any q for which $L(s,\chi)\neq 0$ whenever $\mathrm{Re}(s) > 1-\varepsilon^{-100}/q$, $|\mathrm{Im}(s)|\le(\log q)^3$ and $\chi\neq\xi \pmod q$. Moreover, since under the assumption of Proposition 12.2 the exceptional character ξ does not exist (that is, the above holds for all $\chi \pmod q$), we can delete the term involving ξ from (88), giving (89). This gives Proposition 12.2. □

[15]For this estimate to work, it is crucial that the character sums $P_{j,v}(\chi)$ are long enough in terms of q; in particular, we need them to have length $\gg q^{1/3+\varepsilon}$.

12.3. **Deductions of Linnik-type theorems.** Corollary 2.2 is a direct consequence of Theorem 2.1(i) (by fixing $\varepsilon > 0$ in its statement). Hence, it suffices to prove Theorem 2.1(i)–(ii).

Proof of Theorem 2.1(ii). It suffices to show that for all but $\ll_\varepsilon 1$ primes $q \in [Q^{1/2}, Q]$ the right-hand side of (89) is > 0; indeed, then the smallest q-smooth E_3 number in the progression $a \pmod q$ is $\leq q^{2+1000\varepsilon}$ (and since $\varepsilon > 0$ is arbitrarily small, this is good enough).

In view of Proposition 12.2, it suffices to show that $\prod_{\chi \pmod q} L(s,\chi)$ obeys the zero-free region $\mathrm{Re}(s) \geq 1 - \frac{M(\varepsilon)}{\log q}$, $|\mathrm{Im}(s)| \leq (\log q)^3$ required by that proposition.

Since q is a prime, all the characters $\pmod q$ apart from the principal one are primitive. Moreover, the zeros of the L-function corresponding to the principal character are the same as the zeros of the Riemann zeta function, so we have the Vinogradov–Korobov zero-free region for this L-function. It therefore suffices to consider the L-functions corresponding to primitive characters. By the log-free zero density estimate (Lemma 8.1), we immediately see that $\prod^*_{\chi \pmod q} L(s,\chi)$ has the required zero-free region for all but $\ll \exp(100M(\varepsilon))$ prime moduli $q \in [Q^{1/2}, Q]$, so we have the claimed result. □

Proof of Theorem 2.1(i). Fixing $\delta > 0$, we will show that if $P^+(q) \leq q^{\varepsilon'}$ with ε' very small in terms of δ, then the least product of exactly three primes in every reduced residue class $a \pmod q$ is $\ll q^{2+\delta}$.

Let $\varepsilon > 0$ be very small in terms of δ. By Lemma 12.3, we have the zero-free region required by Proposition 12.2 whenever $P^+(q) \leq q^{\eta(\varepsilon)}$ with $\eta(\varepsilon) > 0$ small enough, apart from possibly a single zero β, which is real and simple and corresponds to a single real character $\pmod q$.

If this exceptional zero β does not exist, then from Proposition 12.2 we obtain a positive lower bound for the left-hand side of (88). Therefore, we can assume that β exists. This is a real zero of an L-function $\pmod q$, and we write the zero as $\beta = 1 - \frac{c}{\log q}$ with $c > 0$. By a result of Heath-Brown [16, Corollary 2] on Linnik's theorem and Siegel zeros, if $c \leq c_0(\delta)$ for a suitably small function $c_0(\delta)$, then the least prime in any arithmetic progression $a \pmod q$ with $(a,q) = 1$ is $\ll q^{2+\delta/2}$, and thus also the least $n \equiv a \pmod q$ with exactly three prime factors obeys the same bound (indeed, if $p_1, p_2 \ll \log^2 q$ are chosen to be primes not dividing q and $p \ll q^{2+\delta/2}$ is a prime $\equiv a(p_1p_2)^{-1} \pmod q$, then $p_1p_2p \ll q^{2+\delta}$ and $p_1p_2p \equiv a \pmod q$). Thus we have proved the theorem if $\beta \geq 1 - \frac{c_0(\delta)}{\log q}$, so henceforth we will assume we are in the opposite case.

According to Proposition 12.1, it suffices to show that

$$\Big|\sum_{P_3^{1-\varepsilon} \leq p \leq P_3} \frac{\xi(p)}{p}\Big| \leq (1-\sqrt{\varepsilon}) \sum_{P_3^{1-\varepsilon} \leq p \leq P_3} \frac{1}{p},$$

since then the left-hand side of (88) is > 0 for $\varepsilon > 0$ small enough.

Following the exact same argument as in the proof of Lemma 12.5, and introducing the same weight function $W = W_{y,\kappa}$ with $y \in [P_3^{1-\varepsilon}, P_3]$ and $\kappa = \varepsilon^{10}$ (and using (97)), it is enough to show that

$$\Big|\sum_{\rho} \widetilde{W}(\rho - 1)\Big| \leq (1 - 10\sqrt{\varepsilon})(\log^2 y) \sum_{y^{1-\kappa} \leq p \leq y} \frac{1}{p},$$

where the sum is over the nontrivial zeros of $L(s,\xi)$. Just as in Lemma 12.5, the contribution of all the zeros $\rho \neq \beta$ is $\ll \varepsilon(\log^2 y)\sum_{y^{1-\kappa} \leq p \leq y} \frac{1}{p}$ as long as $P^+(q) \leq q^{\eta_1(\varepsilon)}$ with $\eta_1(\varepsilon)$ small enough. What remains to be shown then is that

$$|\widetilde{W}(\beta - 1)| \leq (1 - 11\sqrt{\varepsilon})(\log^2 y) \sum_{y^{1-\kappa} \leq p \leq y} \frac{1}{p}. \tag{99}$$

We recall that $\beta \le 1 - \frac{c_0(\delta)}{2\log y}$, and denote

$$F(u) := \widetilde{W}(-\frac{u}{\log y}) = \kappa^{-2}\frac{e^{-au} - e^{-bu} - e^{-cu} + e^{-u}}{u^2}\log^2 y,$$

where $a = 1+\kappa$, $b = 1+\kappa-\kappa^2$, $c = 1+\kappa^2$ and the value at $u = 0$ is interpeted as the limit as $u \to 0$. We compute using L'Hôpital's rule that $\widetilde{W}(0) = F(0) = \kappa(1-\kappa)\log^2 y$, and differentiation shows that F is decreasing, so $\widetilde{W}$ is increasing. Moreover, F' is increasing and $F'(u) = (\kappa/2\cdot(-2+\kappa+\kappa^2)+O(\kappa u))\log^2 y$ for $|u| \le 1$. Thus, by the mean value theorem applied to F we have

$$\widetilde{W}(\beta-1) \le \widetilde{W}(-\frac{c_0(\delta)}{2\log y}) = F(\frac{c_0(\delta)}{2}) \le F(0) + \frac{c_0(\delta)}{2}F'(\frac{c_0(\delta)}{2}) \le \kappa(1-\kappa-c_0(\delta)/4)\log^2 y,$$

since $\delta > 0$ is small. We further have $1-\kappa-c_0(\delta)/4 \le 1-100\sqrt{\varepsilon}$ if $\varepsilon > 0$ (and hence κ) is small enough in terms of δ, so that (99) holds by the Mertens bound. This completes the proof. □

Proof of Proposition 2.3. The proof of Proposition 2.3 follows along similar lines as those above, so we merely sketch it, indicating the required modifications; we outline the lower bound for n with $\mu(n) = -1$; the corresponding estimate for $\mu(n) = +1$ is proved in the analogous way.

When considering numbers n with $\mu(n) = -1$, we restrict to those n that belong to the set

$$\mathcal{S} := \{n \in \mathbb{N} :\ \Omega_{[P_j,Q_j]}(n) = 1,\ j \in \{1,2\}\}$$

with $P_1 = x^{\varepsilon/10}$, $Q_1 = x^{\varepsilon/5}$, $P_2 = x^{1/2-\varepsilon}$, $Q_2 = x^{1/2-\varepsilon/2}$; this introduces essentially the same factorization patterns for our n as in the case of products of exactly three primes. By writing $1_{\mu(n)=-1} = \frac{1}{2}(\mu^2(n)-\mu(n))$, it suffices to bound

$$\sum_{\substack{n\le x\\ n\equiv a \pmod q}} \mu^2(n)1_{\mathcal{S}}(n) \gg \varepsilon\frac{x}{q}, \qquad \Big|\sum_{\substack{n\le x\\ n\equiv a \pmod q}} \mu(n)1_{\mathcal{S}}(n)\Big| \ll \varepsilon^2\frac{x}{q}.$$

We concentrate on the latter bound (the former is similar but easier). Write $n = p_1p_2m$ with $p_j \in [P_j, Q_j]$, $m \le \frac{x}{p_1p_2}$. As in the previous sections,we can easily get rid of the cross condition on the variables by splitting into short intervals, so applying orthogonality of characters it suffices to show that

$$\frac{1}{\varphi(q)}\sum_{\substack{\chi \pmod q\\ \chi\neq\chi_0}}\Big|Q_{v_1,H}(\chi)Q_{v_2,H}(\chi)R_{v_1+v_2,H}(\chi)\Big| \ll \frac{\varepsilon^2 x}{H^3(\log Q_1)(\log Q_2)q}, \tag{100}$$

uniformly for $v_i \in I_i$, where we have defined

$$Q_{v,H}(\chi) := \sum_{e^{v/H}\le p<e^{(v+1)/H}}\chi(p), \quad R_{v,H}(\chi) := \sum_{m\le x/e^{v/H}}\mu(m)\chi(m)1_{\mathcal{T}}(m),$$
$$I_i = [H\log P_i, H\log Q_i], \quad H = \lfloor\varepsilon^{-3}\rfloor,$$

and $\mathcal{T}$ is the set of numbers coprime to all the primes in $[P_j, Q_j]$ for $j \in \{1,2\}$. We split our considerations into the cases

$$\mathcal{X} := \{\chi \neq \chi_0 \pmod q :\ |Q_{v_1,H}(\overline{\chi})| \le e^{0.99v_1/H}\}$$
$$\mathcal{U}_S := \{\chi \neq \chi_0 \pmod q :\ |Q_{v_1,H}(\chi)| \le \varepsilon^{20}e^{v_1/H}/v_1\} \setminus \mathcal{X}$$
$$\mathcal{U}_L := \{\chi \neq \chi_0 \pmod q\} \setminus (\mathcal{X}\cup\mathcal{U}_S).$$

The case of $\mathcal{X}$ is easy and is handled just as in the proof of Proposition 12.1. The case of $\mathcal{U}_S$ is also handled similarly as in that proposition, except that we also need a Halász–Montgomery estimate for $\sum_{\chi\in\mathcal{U}_S}|R_{v_1+v_2,H}(\chi)|^2$. This bound takes the same form as Lemma 12.4, but is proved simply by applying duality and the Burgess bound (since $R_{v_1+v_2,H}(\chi)$ is a sum over the integers rather than over the primes). Finally, the $\mathcal{U}_L$ set is small in the sense that $|\mathcal{U}_L| \ll \varepsilon^{-43}$ by Proposition 8.5 whenever we have a zero-free region of the form $\prod_{\chi \pmod q} L(s,\chi) \neq 0$ for $\mathrm{Re}(s) > 1 - \frac{M(\varepsilon)}{\log q}$, $|\mathrm{Im}(s)| \leq 3q$ with $M(\varepsilon)$ large enough. It thus suffices to prove that

$$\sup_{\chi \neq \chi_0 \pmod q} |R_{v_1+v_2,H}(\chi)| \ll \varepsilon^{60} \frac{\phi(q)}{q} x e^{-v_1-v_2},$$

and by Lemma 6.1 this reduces to the bound

$$\sup_{\substack{\chi \pmod q \\ \chi \neq \chi_0}} \inf_{|t| \leq (\log q)^3/2} \sum_{\substack{p \leq x \\ p \nmid q}} \frac{1 + \mathrm{Re}(\chi(p)p^{-it})}{p} \geq 61 \log(1/\varepsilon) + O(1). \tag{101}$$

At first, a direct application of Lemma 6.1 reduces to proving (101) with $\chi(p)p^{-it}1_{\mathcal{S}}(p)$ in place of $\chi(p)p^{-it}$, but since $\log Q_j / \log P_j \ll 1$ by our choices, the contribution of those p with $1_{\mathcal{S}}(p) \neq 1$ is negligible in (101).

Restricting the sum in (101) to $p \in [x^\kappa, x]$ with $\kappa = \varepsilon^{61}$, we indeed obtain (101) from Lemma 12.5, as long as we have the zero-free region mentioned above. This zero-free region is indeed available by Lemma 8.1 for all but $\ll_\varepsilon 1$ primes $q \in [Q^{1/2}, Q]$. □

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