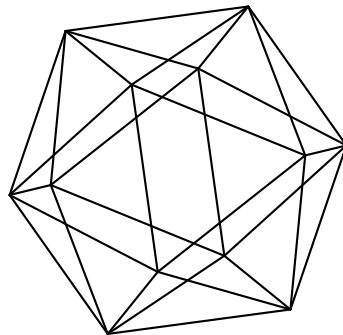


# Max-Planck-Institut für Mathematik Bonn

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Laplace eigenfunctions on the square

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# ON THE BEHAVIOR OF NODAL LINES NEAR THE BOUNDARY FOR LAPLACE EIGENFUNCTIONS ON THE SQUARE

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ABSTRACT. We are interested in the effect of Dirichlet boundary conditions on the nodal length of Laplace eigenfunctions. We study random Gaussian Laplace eigenfunctions on the two dimensional square and find a two terms asymptotic expansion for the expectation of the nodal length in any square of side larger than the Planck scale, along a density one sequence of energy levels. The proof relies on a new study of lattice points in small arcs, and shows that the said expectation is independent of the position of the square, giving the same asymptotic expansion both near and far from the boundaries.

## 1. INTRODUCTION

**1.1. Laplace eigenfunctions on plane domains.** Let  $\Omega \subset \mathbb{R}^2$  be a plane domain with piece-wise real analytic boundaries, and let  $\{\phi_i\}_{i \geq 1}$  be the sequence of Laplace eigenfunctions for the Dirichlet (or Neumann) boundary value problem:

$$\begin{cases} \Delta \phi_i(x) + \lambda_i^2(x) = 0 & x \in \Omega \\ \phi_i(x) = 0 \quad (\text{or} \quad \partial_\nu \phi_i(x) = 0) & x \in \partial\Omega, \end{cases} \quad (1.1)$$

where  $\Delta = \partial_x^2 + \partial_y^2$  for the standard two dimensional (flat) Laplace operator,  $\{\lambda_i\}_{i \geq 1}$  is the discrete spectrum of  $\Delta$ ,  $\partial\Omega$  denotes the boundary of  $\Omega$  and  $\partial_\nu$  the normal derivative. The nodal set  $\phi_i^{-1}(0)$  of an eigenfunction  $\phi_i$  is a smooth curve outside a (possibly infinite) set of points [10]. We are interested in the behavior of the *nodal length*  $\mathcal{L}(\phi_i) := \mathcal{H}(\phi_i^{-1}(0))$ , where  $\mathcal{H}(\cdot)$  is the Hausdorff dimension, near the boundary of the domain  $\partial\Omega$ .

Berry [3] conjectured that high energy Laplace eigenfunctions, on a chaotic surfaces, should behave as superposition of waves with random direction and amplitude, that is as a Gaussian field  $F$  with covariance function

$$\mathbb{E}[F(x)F(y)] = J_0(|x - y|).$$

Subsequently Berry [4] adapted to above model to predict the behavior of the nodal lines in the presence of boundaries. He considered the the random superposition of plane waves

$$G(x_1, x_2) = \frac{1}{\sqrt{J}} \sum_{j=1}^J \sin(x_2 \sin(\theta_j)) \cos(x_1 \cos(\theta_j) + \psi_j),$$

where  $J \rightarrow \infty$  and the  $\theta_j$  and the  $\psi_j$  are random phases, so that the horizontal axis  $x_2 = 0$  serves as a model of the boundaries. Importantly, Berry found that the density of nodal lines near the boundary is smaller than the the density far away from the boundaries and this resulted in a (negative) logarithmic correction term on the expected nodal length.

In the case of plane domains with piece-wise real analytic boundaries (and in the much more general case of real analytic manifolds with boundaries) Donnelly and Fefferman [12] found  $\mathcal{L}(\phi_i)$  to be proportion to  $\lambda_i$ , that is

$$c_\Omega \lambda_i \leq \mathcal{L}(\phi_i) \leq C_\Omega \lambda_i \quad (1.2)$$

for constants  $c_\Omega, C_\Omega > 0$ , thus corroborating both Berry's predictions and a conjecture of Yau asserting that (1.2) holds on any  $C^\infty$  manifold (without boundaries). It worth mentioning that Yau's conjecture was established for the real analytic manifolds [6, 7, 11], whereas, more recently, the optimal lower bound and polynomial upper bound were proved [15, 16, 17] in the smooth case. Unfortunately, results such as (1.2) do not shade any light into the possible effect of boundary conditions on the nodal length.

In order to understand the behavior of the nodal length near the boundaries, we study random Gaussian Laplace eigenfunctions on the 2d square, also known as *boundary adapted Arithmetic Random Waves*. We find a two terms asymptotic expansion for the expectation of the nodal length in squares, of size slightly larger than  $\lambda_i^{-1/2}$ , the Planck scale, both near and far away from the boundaries. In both cases, the said expectation has the same two terms asymptotic expansion showing that the effect of boundaries seem to be uniform in the whole square, even at small scales. Our findings extend previous results obtained by Cammarota, Klurman and Wigman [8], who studied the expectation of the *global* nodal length of Gaussian Laplace eigenfunctions on  $[0, 1]^2$ , with Dirichlet boundary conditions, and complement a similar study on the two dimension round sphere by Cammarota, Marinucci and Wigman [9].

**1.2. Laplace spectrum of  $[0, 1]^2$ .** The Laplace eigenvalues on the square  $[0, 1]^2$  with Dirichlet boundary conditions are given by integer representable as the sum of two squares, that is  $n \in S := \{n \in \mathbb{Z} : n = a^2 + b^2 \text{ for some } a, b \in \mathbb{Z}\}$  and the eigenfunctions can be written explicitly as a Fourier sum

$$f_n(x) = \frac{4}{\sqrt{N}} \sum_{\substack{\xi \in \mathbb{Z}^2 \setminus \sim \\ |\xi|^2 = n}} a_\xi \sin(\pi \xi_1 x_1) \sin(\pi \xi_2 x_2) \quad (1.3)$$

where  $N$  is the number of lattice points on the circle of radius  $n$ ,  $x = (x_1, x_2)$ ,  $\xi = (\xi_1, \xi_2)$ , the  $a_\xi$ 's are complex coefficients and, in order to avoid repetitions, the sum is constrained by the relation  $\xi \sim \eta$  if and only if  $\xi_1 = \pm \eta_1$  and  $\xi_2 = \pm \eta_2$ .

Boundary adapted Arithmetic Random Waves (BARW) are functions  $f_n$  as in (1.3) where the  $a_\xi$ 's are i.i.d. standard Gaussian random variables. Alternatively, BARW are the continuous, *non-stationary*, that its law of  $f_n$  is not invariant under translations by elements of  $\mathbb{R}^2$ , Gaussian field with covariance function

$$r_n(x, y) = \mathbb{E}[f(x)f(y)] = \frac{16}{N} \sum_{\xi \setminus \sim} \sin(\pi \xi_1 x_1) \sin(\pi \xi_2 x_2) \sin(\pi \xi_1 y_1) \sin(\pi \xi_2 y_2). \quad (1.4)$$

For the said model, Cammarota, Klurman and Wigman [8] found that the expectation of the *global* nodal length of  $f_n$  depends on the distribution of the lattice points on the circle of radius  $n$ . Explicitly, they showed that

$$\mathbb{E}[\mathcal{L}(f_n)] = \frac{\sqrt{n}}{2\sqrt{2}} \left( 1 - \frac{1 + \hat{\nu}_n(4)}{16N} + o_{n \rightarrow \infty} \left( \frac{1}{N} \right) \right) \quad (1.5)$$

where the limit is taken along a density one<sup>1</sup> sub-sequence of eigenvalues,

$$\nu_n = \frac{1}{N} \sum_{|\xi|^2 = n} \delta_{\xi/\sqrt{n}} \quad \hat{\nu}_n(4) = \int_{\mathbb{S}^1} \cos(4\theta) d\nu(\theta) \quad (1.6)$$

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<sup>1</sup>A sub-sequence  $S' \subset S$  is of density one if  $\lim_{X \rightarrow \infty} |n \in S' : n \leq X| / |n \in S : n \leq X| = 1$ .

and  $\delta_{\xi/\sqrt{n}}$  is the Dirac distribution at the point  $\xi/\sqrt{n} \in \mathbb{S}^1$ . Moreover, they showed that there exists subsequences of eigenvalues  $n_i$  such that (1.6) holds and  $\hat{\nu}_{n_i}(4)$  attains any value in  $[-1, 1]$ , thus showing that all intermediate “nodal deficiencies” are attainable.

Let  $\mathcal{L}(f_n, s, z) = \text{Vol}\{x \in B(s, z) : f_n(x) = 0\}$ , where  $B(s, z)$  is the box of side  $s > 0$  centered at the point  $z \in [0, 1]^2$ , we prove the following:

**Theorem 1.1.** *Let  $f_n$  be as in (1.3) and  $\varepsilon > 0$ , then there exists a density one sub-sequence of  $n \in S$  such that*

$$\mathbb{E}[\mathcal{L}(f_n, s, z)] = \text{Vol}(B(s, z) \cap [0, 1]^2) \frac{\sqrt{n}}{2\sqrt{2}} \left( 1 - \frac{1 + \hat{\nu}_n(4)}{16N} + o_{n \rightarrow \infty} \left( \frac{1}{N} \right) \right) \quad (1.7)$$

uniformly for  $s > n^{-1/2+\varepsilon}$  and  $z \in [0, 1]^2$ . Moreover, for every  $a \in [-1, 1]$  there exists a subsequence of eigenvalues  $n_i$  such that  $\hat{\nu}_{n_i}(4) \rightarrow a$  and (1.7) hold.

The main new ingredient in the proof of Theorem 1.1 is the study of the distribution of lattice points in small arcs, which we shall now briefly discuss.

**1.3. Semi-correlations and lattice points in shrinking sets.** The study of the nodal length in the Boundary adapted Arithmetic random wave model is intimately connected to the following general result which we shall establish in the next sections.

**Theorem 1.2.** *Let  $\ell > 0$  be an even integer and let  $\bar{v}$  be any fixed directional vector in  $\mathbb{R}^2$ . Let  $P_{\bar{v}} : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  denote the operator of projection on the subspace generated by  $\bar{v}$ . Let  $N = N(n)$  be the number of points  $\xi_i = (\xi_i^1, \xi_i^2) \in \mathbb{Z}^2$  with  $|\xi_i^1|^2 + |\xi_i^2|^2 = n$ . Then for any given  $\varepsilon > 0$  there exists a density one sub-sequence of  $n \in S$  so that the inequality*

$$\left| P_{\bar{v}} \left( \sum_{m=1}^{\ell} \xi_i \right) \right| \leq n^{1/2-\varepsilon} \quad (1.8)$$

has  $\frac{1}{\ell!} N^{\ell/2} + o(N^{\ell/2})$  solutions.

Theorem 1.2 has a simple albeit important geometric interpretation: for almost all  $n \in S$ , cancellations in the vector sum  $\sum_{i=1}^{2\ell} \xi_i$  with  $|\xi_i|^2 = n$  along any given direction  $\bar{v}$  can occur only for trivial reasons, namely when the last  $\ell$  vectors form a cyclic permutation of the first  $\ell$  vectors with opposite signs.

Theorem 1.2 refines and strengthens several important results in the subject. We highlight that in the case when  $P_{\bar{v}}$  is replaced by the identity operator the same conclusion follows from the work of [2], which in turn generalized earlier work by Bombieri and Bourgain [5] where the right hand side of (1.8) was assumed to be identically equal to zero. In our setting, the case when  $\bar{v} = (0, 1)$  and the right hand side being equal to zero has been treated in [8].

To facilitate discussion below, we let  $\xi = (\xi_1, \xi_2)$  with  $|\xi|^2 = n$  and  $\mathcal{M}^t(n, \ell)$  be the number of *semi-correlations*, that is solutions to

$$\xi_t^1 + \dots + \xi_t^\ell = 0 \quad (1.9)$$

for  $t = 1, 2$  and  $\xi^j = (\xi_1^j, \xi_2^j)$  are representations of  $n$  as the sum of two squares. We have the following result, see [8, Theorem 1.3]:

**Lemma 1.3.** *Let  $\ell > 0$  be an even integer then for a density one of  $n \in S$ , we have*

$$\mathcal{M}^t(n, \ell) \leq C_\ell N^{\ell/2} \quad N \rightarrow \infty$$

for some constant  $C_\ell > 0$  and uniformly for  $t = 1, 2$ .

Of particular importance in our study of BARW will be the following special case of Theorem 1.2: let  $K > 0$  be a (large) parameter and let  $\mathcal{V}^t(n, \ell, K)$  be the number of solutions to

$$0 < |\xi_t^1 + \dots + \xi_t^\ell| \leq K \tag{1.10}$$

for  $t = 1, 2$ , then we shall prove the following result.

**Theorem 1.4.** *Let  $\epsilon > 0$  and  $\ell > 0$  be an integer. Then, for a density one of  $n \in S$ , we have*

$$\mathcal{V}^t(n, \ell, n^{1/2-\epsilon}) = \emptyset.$$

We will also need the following simple separation result.

**Lemma 1.5.** *Let  $\epsilon > 0$  then, for a density one of  $n \in S$ , we have*

$$|\xi_t| \geq n^{1/2-\epsilon}$$

for  $t = 1, 2$  and for all  $|\xi|^2 = n$ .

Finally, we construct sequence of circles satisfying the conclusion of Lemma 1.5 and Theorem 1.4 for which we can control the angular distribution:

**Lemma 1.6.** *Let  $\epsilon > 0$  be given. For any  $a \in [-1, 1]$ , there exists a sequence  $\{n_i\}_{i \geq 1}$ , with  $N_{n_i} \rightarrow \infty$  whenever  $i \rightarrow \infty$ , such that  $\widehat{\nu}_{n_i}(4) \rightarrow a$  and the conclusions of Theorem 1.4, Lemma 1.5 hold and moreover  $\mathcal{V}^t(n_j, \ell, n_j^{1/2-\epsilon}) = \emptyset$ .*

**1.4. Notation.** We write  $A \ll B$  or  $A = O(B)$  to designate the existence of a constant  $C > 0$  such that  $A \leq CB$ , we denote the dependence the constant  $C$  depends on some parameter  $\ell$  say, as  $A \ll_\ell B$ . We write  $B(s, z)$  for the box centered at  $z \in \mathbb{T}^2$  of side length  $s > 0$ . For two integers  $m, n$ , we write  $m|n$  if there exists some integer  $k$  such that  $n = km$ .

## 2. PROOF OF SEMI-CORRELATIONS RESULTS

**2.1. Number theoretic preliminaries.** We will need the following two standard results: the first is the following result due to Kubilius [14] about Gaussian primes, which are primes  $\mathcal{P} \subset \mathbb{Z}[i]$  such that  $\mathcal{P} \cap \mathbb{Z} = p$  with  $p \equiv 1 \pmod{4}$ .

**Lemma 2.1** (Kubilius). *Let  $\theta_1, \theta_2 \in [0, 2\pi]$ . Then, the number of Gaussian primes in the sector  $\arg(\mathcal{P}) \in [\theta_1, \theta_2]$  such that  $|\mathcal{P}|^2 \leq X$  is*

$$\frac{2}{\pi}(\theta_1 - \theta_2) \int_2^X \frac{dx}{\log x} + O(X \exp(-c\sqrt{\log X})).$$

The second is Landau's Theorem, see for example [13, Theorem 14.2]: there exists some explicit constant  $c > 0$  such that

$$\#\{n \in S' : n \leq X\} = c \frac{X}{\sqrt{\log X}} (1 + o(1)). \tag{2.1}$$

## 2.2. Proof of Lemma 1.6.

*Proof of Lemma 1.6.* Fix large  $m \geq 1$  and small  $\delta > 0$ . We apply Lemma 2.1 to select infinite sequence of primes  $p_n$  with the property that  $p_n = \pi_n \bar{\pi}_n$ ,  $p_n \equiv 1 \pmod{4}$  and  $|\arg(\pi_n)| \leq \frac{\delta}{100m}$ . Furthermore, we choose large prime  $p$  with

$$|\widehat{\nu}_p(4) - a| \leq \frac{\varepsilon}{2}, \quad (2.2)$$

where  $\hat{\nu}$  is as in (1.6) and consider the numbers of the form  $n = p_n^m p$ . For such  $p_n$  we have

$$|\widehat{\nu}_{p_n}(4) - 1| \leq \frac{\delta}{2m} \quad (2.3)$$

and  $N_n > 2^m$ . From the convolution identity  $\widehat{\nu}_n(4) = (\widehat{\nu}_{p_n}(4))^m \widehat{\nu}_p(4)$  and the triangle inequality deduce the bound

$$|\widehat{\nu}_n(4) - s| \leq m|\widehat{\nu}_{p_n}(4) - 1| + |\widehat{\nu}_p(4) - s| \leq m \cdot \frac{\delta}{2m} + \frac{\delta}{2} = \delta.$$

We claim that inequality (1.10) has only trivial solutions for appropriately chosen values of  $p_n, p$ , which satisfy (2.2) and (2.3).

To this end, we let  $\pi_n = |\pi_n|e^{i\phi}$  and  $p = \pi \cdot \bar{\pi}$  with  $\arg \pi = \alpha$ . For a given point  $\xi_i$  with integer coordinates and  $|\xi_i| = \sqrt{n}$  we write  $\xi_j = \sqrt{n}e^{i(j\phi \pm \alpha + r\frac{\pi}{2})}$  for some  $|j| \leq m$  and  $r = \{0, 1, 2, 3\}$ . In these notation we rewrite (1.10) in the form

$$\left| \sum_{j=1}^{\ell} \varepsilon_j \cos \left( l_j \phi \pm \alpha + \frac{\pi r_j}{2} \right) \right| \ll \frac{1}{n^\epsilon}, \quad (2.4)$$

where  $\varepsilon_j = \{+1, -1\}$  and  $|l_j| \leq m$  for  $1 \leq j \leq \ell$ . The left hand side of (2.4) can therefore be viewed as a trigonometric polynomial

$$F_m(\phi) = \sum_{j=1}^r \cos(m_j \phi) (\alpha_j \cos \alpha + \beta_j \sin \alpha) + \sin(m_j \phi) (\alpha_j^{(1)} \cos \alpha + \beta_j^{(1)} \sin \alpha), \quad (2.5)$$

where  $1 \leq m \leq \ell$  and  $0 \leq m_1 < m_2 < \dots < m_r$  with  $-\ell \leq \alpha_j, \alpha_j^{(1)}, \beta_j, \beta_j^{(1)} \leq \ell$  with the constraints  $|F_m(\phi)| \ll \frac{1}{n^\epsilon}$ . For any fixed  $\ell$ , there are finitely many choices for the coefficients  $\alpha_j, \alpha_j^{(1)}, \beta_j, \beta_j^{(1)}$  and therefore we can select angle  $\alpha$  for which the corresponding prime  $p$  satisfies (2.2) and such that  $a \sin \alpha + b \cos(\alpha) \neq 0$  for all  $a, b \in \mathbb{Z}$  with  $|a| + |b| \neq 0$  and  $|a|, |b| \leq \ell$ . Now since each  $F_r(\phi)$  is a trigonometric polynomial of a total degree at most  $2\ell$ , each non degenerate equation  $F_m(\phi) = 0$  has at most  $2\ell$  solutions. Therefore, the total number of solutions to all such equations is bounded in terms of  $m, \ell$ . Consequently, by adjusting appropriate constants and using the uniform continuity, we may find a point  $|\phi_0| < \frac{\delta}{200m}$ , such that

$$|F_m(\phi)| \gg_m 1$$

for all  $|\phi - \phi_0| \ll \frac{\delta}{200m}$  uniformly for all polynomials defined above. By Lemma 2.1, there are infinitely many primes  $p_n$  with angle  $|\phi_0 - \phi| \ll \frac{\delta}{200m}$ . For such prime  $p_n$ , we have

$$|F_m(\arg \pi_n)| \gg_m 1 \gg \frac{1}{n^\epsilon}$$

for sufficiently large  $p_n$ , which concludes the proof.  $\square$

**2.3. Proof of Theorem 1.4.** We begin by proving Theorem 1.4 in the case of square-free numbers. To this end, for any fixed  $K \geq 1$  we introduce the pre-sieved set

$$\Omega_{M,K} = \{n \leq M, \text{rad}(n) = n, p|n \in S \Rightarrow p \geq K\},$$

where  $\text{rad}(n) = \prod_{p|n} p$ , that is the product over primes dividing  $n$  without multiplicity, and let  $\Omega_M := \Omega_{M,1} = S \cap [1, M]$ . We will need the following lemma borrowed from [5].

**Lemma 2.2.** *For  $m \in \Omega_{M,K}$ , let  $m = p_1 \cdot p_2 \cdots p_r$  be its factorization with  $K < p_1 < p_2 < \cdots < p_r$ . Then as  $M \rightarrow \infty$  we have  $p_s > 2^{s\Phi(s)}$  for  $1 \leq s \leq r$  holds for all  $m \in \Omega_{M,K} \setminus \Omega_{M,K}^{(1)}$ , where the exceptional set  $\Omega_{M,K}^{(1)}$  has cardinality*

$$|\Omega_{M,K}^{(1)}| \leq \eta(K, \Phi) |\Omega_{M,K}|$$

with  $\eta(K, \Phi) \rightarrow 0$  as  $K \rightarrow \infty$ . If  $\Phi(x) = o(\log x)$ , then we can choose  $\eta(K, \Phi) = K^{-1+\delta}$  for every fixed  $\delta > 0$ .

The next proposition is crucial and estimates the number of solutions (1.10) for almost all admissible integers  $m \in \Omega_{M,K}$ .

**Proposition 2.3.** *Let  $\varepsilon, \delta > 0$  be fixed. If  $K \geq K(\delta)$  and  $M \rightarrow \infty$ , then for all but  $K^{-1+\delta} |\Omega_{M,K}|$  elements  $m \in \Omega_{M,K}$  we have  $\mathcal{V}^t(m, \ell, m^{1/2-\varepsilon}) = \emptyset$ .*

*Proof.* Let  $\tilde{S} \subset S$  such that for every  $n \in \tilde{S}$  we have  $\mathcal{V}^t(m, \ell, m^{1/2-\varepsilon}) \neq \emptyset$ . For any prime  $p$  we write  $p = \pi \cdot \bar{\pi}$  where  $\pi$  is the corresponding Gaussian prime with  $\arg(\pi) \in [0, \pi/2]$ . For any integer  $s \geq 1$  we introduce the set

$$\mathcal{F}_s = \left\{ n \in \Omega_{M,K}, \omega(n) = s, n \in \tilde{S}; \forall d \neq n, d|n \Rightarrow d \in S \setminus \tilde{S} \right\}.$$

Fix  $s \geq 1$  and consider  $n \in \mathcal{F}_s$  with a given factorization  $n = p_1 \cdot p_2 \cdots p_s$ ,  $K < p_1 < p_2 < \cdots < p_s$ . We have that there exist integer points  $\{\xi_j\}_{j=1}^\ell$  with  $\|\xi_j\| = \sqrt{n}$  and  $\varepsilon_j \in \{-1, 0, 1\}$ ,  $1 \leq j \leq \ell$  with

$$\left| \sum_{j=1}^{\ell} \varepsilon_j \text{Re}(\xi_j) \right| \leq (p_1 \cdot p_2 \cdots p_s)^{1/2-\varepsilon}.$$

Each point  $\xi_r$  can be uniquely written as a product  $\xi_r = i^{k_{\xi_r}} \prod_{j \leq s} \pi_{j,r}^*$  where each  $\pi_{j,r}^* \in \{\pi_j, \bar{\pi}_j\}$  and  $k_{\xi_r} \in \{0, 1, 2, 3\}$ . We now regroup the terms in the last expression by collecting  $\pi_s$  and  $\bar{\pi}_s$  into different summands to end up with an equivalent form

$$|\text{Re}(\pi_s A_{s-1}) + \text{Re}(\bar{\pi}_s B_{s-1})| \leq (p_1 \cdot p_2 \cdots p_s)^{1/2-\varepsilon}, \quad (2.6)$$

where each  $A_{s-1}, B_{s-1}$  consists of the sum of at most  $\ell - 1$  terms composed of first  $(s-1)$  Gaussian primes. Let  $\pi_s = |\pi_s| e^{i\phi_s}$ ,  $A_{s-1} = |A_{s-1}| e^{ia_{s-1}}$  and  $B_{s-1} = |B_{s-1}| e^{ib_{s-1}}$ . We rewrite inequality (2.6) in the form

$$||A_{s-1}| \cos(\phi_s + a_{s-1}) + |B_{s-1}| \cos(b_{s-1} - \phi_s)| \leq \frac{(p_1 \cdot p_2 \cdots p_{s-1})^{1/2-\varepsilon}}{|\pi_s|^{2\varepsilon}},$$

which after trigonometric manipulations simplifies to

$$|\cos(\phi_s)(|A_{s-1}| \cos(a_{s-1}) + |B_{s-1}| \cos(b_{s-1})) + \sin(\phi_s)(-|A_{s-1}| \sin(a_{s-1}) + |B_{s-1}| \sin(b_{s-1}))| \quad (2.7)$$

$$\leq \frac{(p_1 \cdot p_2 \cdots p_{s-1})^{1/2-\varepsilon}}{|\pi_s|^{2\varepsilon}}.$$



Let  $\phi_0 \in [0, 2\pi)$  be the angle satisfying

$$\sin(\phi_0) = \frac{|A_{s-1}| \cos(a_{s-1}) + |B_{s-1}| \cos(b_{s-1})}{((|A_{s-1}| \cos(a_{s-1}) + |B_{s-1}| \cos(b_{s-1}))^2 + (|A_{s-1}| \sin(a_{s-1}) - |B_{s-1}| \sin(b_{s-1}))^2)^{1/2}}$$

and

$$\cos(\phi_0) = \frac{-|A_{s-1}| \sin(a_{s-1}) + |B_{s-1}| \sin(b_{s-1})}{((|A_{s-1}| \cos(a_{s-1}) + |B_{s-1}| \cos(b_{s-1}))^2 + (|A_{s-1}| \sin(a_{s-1}) - |B_{s-1}| \sin(b_{s-1}))^2)^{1/2}}.$$

With these notations (2.7) implies the bound

$$\begin{aligned} & |\sin(\phi_s + \phi_0)| \\ & \leq \frac{(p_1 p_2 \dots p_{s-1})^{1/2-\varepsilon}}{|\pi_s|^{2\varepsilon} ((|A_{s-1}| \cos(a_{s-1}) + |B_{s-1}| \cos(b_{s-1}))^2 + (|A_{s-1}| \sin(a_{s-1}) - |B_{s-1}| \sin(b_{s-1}))^2)^{1/2}}. \end{aligned}$$

Since  $n \in \mathcal{F}_s$  we have  $p_1 p_2 \dots p_{s-1} = \tilde{n}|n$  and so by definition  $\tilde{n} \in S \setminus \tilde{S}$ . Therefore,

$$(|A_{s-1}| \cos(a_{s-1}) + |B_{s-1}| \cos(b_{s-1})) = |\operatorname{Re}(A_{s-1} + B_{s-1})| \geq (p_1 \cdot p_2 \dots p_{s-1})^{1/2-\varepsilon}$$

unless  $A_{s-1} = B_{s-1} = 0$ , in which case we end up with a trivial solution. Upon noting that denominator of the above fraction is  $\geq (p_1 p_2 \dots p_{s-1})^{1/2-\varepsilon}$  we deduce

$$|\sin(\phi_s + \phi_0)| \leq \frac{1}{|\pi_s|^{2\varepsilon}} \ll \frac{1}{J^\varepsilon},$$

for some  $J > 0$ . Now for a fixed value of  $\phi_0$ , by convexity we have  $|\sin x| \geq \frac{2}{\pi}x$  for  $x \in [0, \pi/2]$  which for  $q = \{0, 1\}$  yields a measure bound

$$|\phi_s + \phi_0 + q\pi| \ll \frac{1}{|\pi_s|^{2\varepsilon}}. \quad (2.8)$$

We are now ready to estimate the number of  $m \in \Omega_{M,K}$  which give rise to a nontrivial solution of (1.10). Applying Lemma 2.2 allows us to restrict to the case where  $m = p_1 p_2 \dots p_r \in \Omega_{M,K}$ , with  $K < p_1 < p_2 < \dots < p_r$  and  $p_j \geq 2^{j\Phi(j)}$  for any  $1 \leq j \leq r$  and some slowly growing function  $\Phi(x)$  to be determined later. We observe that, for each such  $m$ , there exists unique  $1 \leq s \leq r$  such that the product  $p_1 p_2 \dots p_s \in \mathcal{F}_s$ . Given  $K < p_1 < p_2 < \dots < p_{s-1}$  we can form at most  $2^{\ell(s-1)}$  sums  $A_{s-1}$  and  $B_{s-1}$  and thus produce at most  $2^{\ell(s-1)}$  distinct  $n = p_1 p_2 \dots p_{s-1} p_s \in \Omega_{M,K}$ .

We start by partitioning the range of  $p_s$  into dyadic intervals  $[J, 2J]$  and note that there are at most  $J^{1-\varepsilon'}$  suitable  $p_s$  which satisfy (2.8) for any given  $\varepsilon' > \varepsilon > 0$ . Indeed, we have at most  $\ll \sqrt{J}$  choices for the one coordinate and  $\ll J^{1/2-\varepsilon'}$  choices for the other.

By Lemma 2.2,  $p_s \geq \max\{2^{s\Phi(s)}, K\}$  and therefore the total number of elements in  $\tilde{S} \cap [1, M]$  induced by the elements in  $\mathcal{F}_s$  is at most

$$\begin{aligned} & \ll 2^\ell \sum_{p_1, p_2, \dots, p_{s-1}} \sum_{J \leq \log M} \sum_{\max\{K, 2^{s\Phi(s)}\} \leq p_s \sim J, p_1 \dots p_{s-1} \in \mathcal{F}_s} \left| \left\{ m \leq \frac{M}{p_1 p_2 \dots p_{s-1} p_s}, m \in \Omega_{M,K} \right\} \right| \\ & \ll 2^{ls} \sum_{J \leq \log M} \sum_{\max\{K, 2^{s\Phi(s)}\} \leq p_s \sim J} \mathbf{I}_{m/p_s \in \Omega_{K,M}} \\ & \ll 2^{ls} \sum_{\max\{K, 2^{s\Phi(s)}\} \leq J \leq \log M} \frac{M}{J \sqrt{\log J}} \cdot J^{1-\varepsilon} \end{aligned} \quad (2.9)$$

where the last estimate comes from “conditioning” on at most  $J^{1-\varepsilon}$  possible values of  $p_s$  and the fact that  $m \in \Omega_{K,M}$ . The last sum is clearly bounded above by  $\ll 2^{ls} \frac{M}{\max\{K, 2^{\varepsilon' s \Phi(s)}\}}$ . Since the choice of the function  $\Phi(x)$  is at our disposal as long as  $\Phi(x) = o(\log x)$ , we can follow the same arguments as in [5] verbatim with  $\Phi(x)$  replaced by  $\varepsilon' \Phi(x)$  to arrive at the conclusion.  $\square$

We are now ready to handle the general case.

*Proof of Theorem 1.4.* Fix large  $K > 0$  and consider  $\mathcal{P}_K = \prod_{p \leq K} p$ . Each  $n \in \mathbb{N}$  can be written in the form  $n = n_K n_b$  where  $(n_b, \mathcal{P}_K) = 1$  and  $\text{rad}(n_K) | \mathcal{P}_K$ . Since the number of  $n \in \Omega_M$  for which  $p^2 | n$  for some prime  $p \geq K$ , is bounded above by

$$\sum_{p > K} \left| \Omega_{\frac{M}{p^2}} \right| \ll \sum_{p > K} \frac{M}{p^2 \sqrt{\log M}} \ll \frac{M}{K \log K \sqrt{\log M}}$$

and thus give negligible contribution. Consequently, we can restrict ourselves to the set of integers with  $n_b$  being square-free. We now fix  $n_K \in \mathbb{N}$  and count the number of  $n \in \tilde{S} \cap [1, M]$  with  $n_K | n$  and  $(\frac{n}{n_K}, \mathcal{P}_K) = 1$ . More precisely, we would like to count the number of  $n$ , which give nontrivial solutions to

$$\left| \sum_{i=1}^{\ell} \text{Re}(\alpha_i \xi_i) \right| \leq n^{1/2-\varepsilon}$$

with  $\alpha_i | n_K$  and  $\|\xi_i\| = \sqrt{n_b}$  for  $1 \leq i \leq \ell$ . We now follow the proof of Proposition 2.3 regarding  $\alpha_i$  as fixed coefficients. Let  $\Omega_{4,1}(n)$  denote the number of prime divisors  $p = 1 \pmod{4}$  of  $n$  counting multiplicity. We have at most  $2^{\ell \Omega_{4,1}(n_K)}$  choices for the coefficients  $\alpha_i$  and so the number of  $n \in \tilde{S} \cap [1, M]$  induced in this way, after appealing to Proposition 2.3 is bounded above by

$$\begin{aligned} \sum_{\text{rad}(n_K) | \mathcal{P}_K} 2^{\ell \Omega_{4,1}(n_K)} |\Omega_{M/(n_K), K}| &\ll \sum_{\text{rad}(n_K) \in \mathcal{P}_K} \frac{4^{k \Omega_{4,1}(n_K)}}{n_K} \left( (K^{-1+\delta} + o(1)) \frac{M}{\sqrt{\log M}} \right) \\ &\ll \left( \frac{(\log K)^{\ell+1}}{K^{1-\delta}} + o(1) \right) \frac{M}{\sqrt{\log M}}. \end{aligned}$$

The result now follows by letting  $K \rightarrow \infty$ .  $\square$

We now briefly point out the modifications required for the proof of Theorem 1.2.

*Sketch of the proof of Theorem 1.2.* As before, let  $\ell > 0$  be an even integer and let  $\bar{v}$  be our directional vector in  $\mathbb{R}^2$  and set  $\bar{v} = |\bar{v}| e^{i\theta_v}$ . We now follow the notations of Proposition 2.3 and observe that, upon performing rotation by the angle  $-\theta_v$ , our equation (1.8) reduces to

$$\left| \text{Re} \left( \sum_{m=1}^{\ell} e^{-i\theta_v} \xi_m \right) \right| \leq n^{1/2-\varepsilon}.$$

We can now follow the proof of the Proposition 2.3 verbatim and note that equation (2.6) would now take a similar form

$$\left| \text{Re}(e^{-i\theta_v} \pi_s A_{s-1}) + \text{Re}(e^{-i\theta_v} \bar{\pi}_s B_{s-1}) \right| \leq (p_1 \cdot p_2 \cdots p_s)^{1/2-\varepsilon}.$$

This in turn would lead to similar expressions for  $\sin(\phi_0)$  and  $\cos(\phi_0)$  with the corresponding angles  $a_s$  and  $b_s$  replaced with  $a_s - \theta_v$  and  $b_s - \theta_v$ . Crucially, the bound (2.8) remains unchanged which would not affect the rest of the proof.  $\square$

Finally, we conclude with a short proof of Lemma 1.5.

*Proof of Lemma 1.5.* We partition the interval  $[1, N]$  into  $\sim \log N$  dyadic intervals of the form  $[k, 2k]$ . Let  $\xi = (\xi_1, \xi_2)$  with  $|\xi|^2 = \xi_1^2 + \xi_2^2 = m$  and  $k \leq m \leq 2k$ . We observe that if  $|\xi_t| \leq m^{1/2-\epsilon} \leq k^{1/2-\epsilon} \leq 2k^{1/2-\epsilon}$  then the number of integers  $k \leq m \leq 2k$  is upper bounded by  $\ll k^{1/2}k^{1/2-\epsilon} = k^{1-\epsilon}$ , where the first factor comes from the fact that there are at most  $k^{1/2}$  choices for one coordinate and at most  $k^{1/2-\epsilon}$  choices for the other. Summing over all such dyadic intervals we see that the total contribution of such  $\xi$  is  $\ll \log N \cdot N^{1-\epsilon} = o(N/\sqrt{\log N})$ , thus Lemma 1.5 follows from Landau's Theorem 2.1.  $\square$

### 3. FORMULA FOR THE EXPECTATION IN SHRINKING SETS

**3.1. Deterministic grid, reduction to  $n$  square-free.** Let us denote by  $S' := \{n \in S : n \text{ is square-free}\}$ . In this section, we show that, in order to prove Theorem 1.1, it is enough to restrict ourselves to  $n \in S'$ . More precisely, we show that if  $n$  is non-square free, then there exists a deterministic grid where  $f_n(x) = 0$ , see also [8]. However, for most  $n \in S$ , its contribution is negligible compared to main term in Theorem 1.1.

To see this, let  $n \in S$  and write  $n = 2^{\alpha_2} \prod_j p_j^{\alpha_j} \prod_k q_k^{\beta_k}$  where  $p \equiv 1 \pmod{4}$ ,  $q \equiv 3 \pmod{4}$  and the  $\beta_k$ 's are even, and consider the "fix" part

$$Q = 2^{\alpha_2} \prod_k q_k^{\beta_k}. \tag{3.1}$$

Then, letting  $\xi = (\xi_1, \xi_2)$  be any lattice point on the circle  $|\xi|^2 = n$ ,  $Q$  divides both  $\xi_1$  and  $\xi_2$ . Therefore,  $f_n$ , as in (1.3), vanishes on the grid

$$\mathcal{G}(Q) = \cup_{k=1}^{\sqrt{Q}} \{x \in [0, 1]^2 : x_1 = \frac{k}{\sqrt{Q}} \text{ or } x_2 = \frac{k}{\sqrt{Q}}\}.$$

Since, the length of the grid is

$$\mathcal{L}(\mathcal{G}(Q)) = 2(Q - 1),$$

and for almost all  $n \in S$ , thanks to the Erdos-Kac Theorem, see for example [19, Part III Chapter 3], we have  $Q \leq n^{1/4} \ll (\log n)^{O(1)}$ , its contribution is negligible compared to the main term in the statement of Theorem 1.1. Hence, upon rescaling  $\xi \rightarrow \xi/Q$ , from now on, we assume that  $n \in S'$ .

**3.2. Kac-Rice premises.** The aim of this section will be to evaluate the zero density of  $f_n$  as defined in Proposition 3.1 (below) outside a set of "singular" points. We begin with the following, see also [8, Lemma 3.1]:

**Proposition 3.1.** *Let  $n \in S'$  and  $f_n$  be as in (1.3), moreover define the zero density of  $f_n$  to be*

$$K_1(x) = \frac{1}{(2\pi)^{1/2} \sqrt{\text{Var}(f(x))}} \mathbb{E}[|\nabla f_n(x)| | f_n(x) = 0]$$

then

$$\mathbb{E}[\mathcal{L}(f_n, z, s)] = \int_{B(z,s) \cap [0,1]^2} K_1(x) dx.$$

*Proof.* By [1, Theorem 6.3], it is enough to check that the distribution  $f_n(x)$  is non-degenerate for all  $x \in B(z, s)$ , that is

$$\text{Var}(f(x)) = r_n(x, x) = \frac{16}{N} \sum_{\xi \setminus \sim} (\sin(\pi \xi_1 x_1) \sin(\pi \xi_2 x_2))^2 \neq 0 \quad (3.2)$$

for all  $x \in B(z, s)$ . Since the left hand side of (3.2) is a sum of positive terms, if  $\text{Var}(f(x)) = 0$  then  $\xi_1 x_1 = r_1 \in \mathbb{Z}$  or  $\xi_2 x_2 = r_2 \in \mathbb{Z}$  for all  $\xi$ . Now, if  $\xi_1 x_1 = r_1 \in \mathbb{Z}$  and  $\eta_2 x_2 = r_2 \in \mathbb{Z}$ , for  $\xi \neq \eta$  then  $x$  belong to a fine set of points and it does not affect the integral. If  $\xi_i x_i = r_i$  for all  $\xi$  then choose  $\xi = \prod_j \mathcal{P}_j$  and  $\eta = \prod_j \overline{\mathcal{P}_j}$ , where  $\mathcal{P}_j$  are Gaussian primes lying above the primes  $p \equiv 1 \pmod{4}$  dividing  $n$ , to see that  $x_1 | Q$ , with  $Q$  as in (3.1). This contradicts  $n$  being square-free.  $\square$

In order to evaluate the zero density of  $f_n$ , we borrow the following lemma from [8, Lemma 2.2]:

**Lemma 3.2.** *Let  $f_n$  be as in (1.3) and  $x \in [0, 1]^2$ , then*

$$\frac{2}{\pi^2 n} \mathbb{E}[\nabla f_n(x) \cdot \nabla^t f_n(x) | f_n(x) = 0] = I_2 + \Gamma_n(x)$$

where  $\nabla^t$  denotes the gradient transpose,  $I_2$  is the two by two identity matrix and  $\Gamma_n$  is given by

$$\Gamma_n = \frac{8}{nN} \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} - \frac{128}{nN^2 \text{Var}(f(x))} \begin{bmatrix} d_1^2 & d_1 \cdot d_2 \\ d_1 \cdot d_2 & d_2^2 \end{bmatrix}$$

where

$$\begin{aligned} b_{11}(x) &= \sum_{\xi \setminus \sim} \xi_1^2 (\cos(2\pi \xi_1 x_1) - \cos(2\pi \xi_2 x_2) - \cos(2\pi \xi_1 x_1) \cos(2\pi \xi_2 x_2)) \\ b_{12}(x) &= b_{21}(x) = \sum_{\xi \setminus \sim} \xi_1 \xi_2 \sin(2\pi \xi_1 x_1) \cdot \sin(2\pi \xi_2 x_2) \\ b_{22}(x) &= \sum_{\xi \setminus \sim} \xi_2^2 (\cos(2\pi \xi_1 x_1) - \cos(2\pi \xi_2 x_2) - \cos(2\pi \xi_1 x_1) \cos(2\pi \xi_2 x_2)) \\ d_1(x) &= \sum_{\xi \setminus \sim} \xi_1 \sin(2\pi \xi_1 x_1) \cdot \sin(\pi \xi_2 x_2)^2 \\ d_2(x) &= \sum_{\xi \setminus \sim} \xi_2 \sin(2\pi \xi_2 x_2) \cdot \sin(\pi \xi_1 x_1)^2. \end{aligned} \quad (3.3)$$

**3.3. The singular set.** Let  $f_n$  be as in (1.3),  $z \in [0, 1]^2$ ,  $s > 0$  and  $\Gamma_n$  be as in Lemma 3.2. In this section, we want to bound the contributions to  $\mathbb{E}[\mathcal{L}(f_n, z, s)]$  coming from points  $x \in B(z, s)$  where  $\Gamma_n$  is somewhat “large”. More precisely, we divide  $B(z, s)$  into  $O((s \cdot n^{1/2})^2 / \delta^2)$  squares  $Q_i$  of size  $\delta / \sqrt{n}$  for some parameter  $\delta > 0$  to be chosen later, and say that square  $Q_i$  is singular if it contains a point such that

$$\text{Var}(f(x)) =: 1 - s_n(x) > 1 - \gamma \quad \text{or} \quad |\text{Tr}(\Gamma_n)| \geq \gamma \quad \text{or} \quad |\det(\Gamma_n)| \geq \gamma$$

for some  $\gamma > 0$  to be chosen later. We denote by  $\mathcal{Q}_{\text{sing}}$  the union of the singular  $Q_i$ . We then prove the following proposition:

**Proposition 3.3.** *Let  $f_n$  be as in (1.3),  $\ell > 0$  be an even integer and  $K_1$  be as in Proposition 3.1. Then, for a density one of  $n \in S'$  we have*

$$\int_{\mathcal{Q}_{sing}} K_1(x) dx \ll \frac{s^2 \sqrt{n}}{N^{\ell/2-1}}.$$

In order to prove Proposition 3.3 we will need two lemmas. The first is the following deterministic bound on the nodal set of Laplace eigenfunctions on the square, see [18, Proposition 1.5]:

**Lemma 3.4.** *Let  $f_n$  be as in (1.3) and  $\varepsilon > 0$ , then*

$$\mathcal{L}(f_n, z, s) s^{-1} \ll s \sqrt{n} + N$$

uniformly for all  $z \in [0, 1]^2$  and  $s > 0$ .

The second is an estimate on the size of the singular set as follows:

**Lemma 3.5.** *Let  $\mathcal{Q}_{sing}$  be as at the beginning of section 3.3 and  $\ell > 0$  be an even integer, then we have*

$$\text{Vol}(\mathcal{Q}_{sing}) \ll_{\ell, \gamma} s^2 \max_{t=1,2} \left( \frac{\mathcal{M}^t(n, \ell)}{N^\ell} + \frac{1}{N^\ell} \sum_{\substack{\xi^1, \dots, \xi^\ell \\ \sum_i \xi_i^t \neq 0}} \frac{1}{s |\sum_i \xi_i^t|} \right),$$

where the sum is subject to the relation  $\xi \sim \eta$  and  $\mathcal{M}^t(n, \ell)$  is as in Lemma 1.3.

*Proof.* Let us first consider squares  $\mathcal{Q}_i$  where  $\text{Var}(f(x)) > 1 - \gamma$ , that is  $|s_n(x)| \leq \gamma$ . Since  $|\nabla s| \leq 100\sqrt{n}$  and  $\mathcal{Q}_i$  has size  $\delta/\sqrt{n}$ , choosing  $\delta$  sufficiently small depending on  $\gamma$ , we may assume that  $|s_n(x)| \leq \gamma/2$  for all  $x \in \mathcal{Q}_i$ . Thus, by Chebyshev's bound, for any even integer  $\ell > 0$ , we have

$$\text{Vol}\{x \in B(z, s) \cap [0, 1]^2 : |s_n(x)| \geq \gamma/2\} \leq \frac{2^\ell}{\gamma^\ell} \int_{B(z, s) \cap [0, 1]^2} |s_n(x)|^\ell dx.$$

Re-writing the definition of  $r_n(x, x)$  in (1.4) using  $\sin(x)^2 = (1/2)(1 - \cos 2x)$ , we see that

$$s_n(x) = \frac{4}{N} \sum_{\xi \setminus \sim} (\cos(2\pi \xi_1 x_1) + \cos(2\pi \xi_2 x_2) - \cos(2\pi \xi_1 x_1) \cos(2\pi \xi_2 x_2)) =: A_n(x_1) + B_n(x_2) + C_n(x).$$

Thus, since  $(a + b + c)^\ell \ll_\ell a^\ell + b^\ell + c^\ell$ , we have

$$\int_{B(z, s) \cap [0, 1]^2} |s_n(x)|^\ell dx \ll_\ell \int_{B(z, s) \cap [0, 1]^2} A_n(x_1)^\ell + B_n(x_2)^\ell + C_n(x)^\ell dx.$$

Let us consider  $A_n(x_1)$ , using the transformation  $x \rightarrow z + sy$ , we have

$$\int_{B(z, s) \cap [0, 1]^2} A_n(x_1)^\ell dx \ll s^2 \int_{a(z)}^{b(z)} A_n(z_1 + sy_1)^\ell dy_1 \leq s^2 \int_{-1/2}^{1/2} A_n(z_1 + sy_1)^\ell dy_1, \quad (3.4)$$

where  $s^{-1}|a(z)|, s^{-1}|b(z)| \leq 1/2$  correspond to the projection of  $B(z, s) \cap [0, 1]^2$  along the  $X$ -axis. Moreover, using the formula  $\cos(x + y) = \cos(x) \cos(y) - \sin(x) \sin(y)$ , we may write

$$\begin{aligned} A_n(z_1 + sy_1) &= \frac{1}{N} \sum_{\xi \setminus \sim} \cos(2\pi \xi_1 z_1) \cos(2\pi \xi_1 sy_1) - \sin(2\pi \xi_1 z_1) \sin(2\pi \xi_1 sy_1) \\ &=: A_n^1(z_1 + sy_1) + A_n^2(z_1 + sy_1). \end{aligned}$$

Thus, using the fact that  $(a + b)^\ell \ll_\ell a^\ell + b^\ell$ , the RHS of (3.4) can be bounded by

$$\int_{-1/2}^{1/2} A_n(z + sy_1)^\ell dy_1 \ll_\ell \int_{-1/2}^{1/2} A_n^1(z + sy_1)^\ell + A_n^2(z + sy_1)^\ell dy_1. \quad (3.5)$$

Let us consider  $A_n^1$ , expanding the  $\ell$ -th power, we have

$$\begin{aligned} \int_{-1/2}^{1/2} A_n^1(z + sy_1)^\ell dy_1 &= \frac{1}{N^\ell} \sum_{\xi^1, \dots, \xi^\ell} \prod_{i=1}^{\ell} \cos(2\pi \xi_1^i z_1) \int_{-1/2}^{1/2} \prod_{i=1}^{\ell} \cos(2\pi \xi_1^i sy_1) dy_1. \\ &\leq \frac{1}{N^\ell} \sum_{\xi^1, \dots, \xi^\ell} \left| \int_{-1/2}^{1/2} \prod_{i=1}^{\ell} \cos(2\pi \xi_1^i sy_1) dy_1 \right| \end{aligned} \quad (3.6)$$

Thanks to the formula  $2^{\ell-1} \prod_{i=1}^{\ell} \cos(a_i) = \sum_{v \in \{-1, 1\}^\ell} \cos(\sum_i v_i a_i)$  which follows by induction using the formula  $\cos(a \cdot b) = (1/2)(\cos(a + b) + \cos(a - b))$ , the inner integral on the right hand side of (3.6) can be rewritten as

$$\int_{-1/2}^{1/2} \prod_{i=1}^{\ell} \cos(2\pi \xi_1^i sy_1) dy_1 = 2^{-\ell+1} \sum_{v \in \{-1, 1\}^\ell} \int_{-1/2}^{1/2} \cos\left(2\pi sy_1 \left(\sum_{i=1}^{\ell} v_i \xi_1^i\right)\right) dy_1$$

Separating the terms with  $\sum_i v_i \xi_1^i = 0$  from the others, bearing in mind that the sum is over  $\ell$ -tuples satisfying the congruence relation  $\xi \sim \eta$  if and only if  $\xi_1 = \pm \eta_1$  and  $\xi_2 = \pm \eta_2$ , we have

$$\begin{aligned} \sum_{\xi^1, \dots, \xi^\ell} \left| \int_{-1/2}^{1/2} \cos\left(2\pi sy_1 \left(\sum_{i=1}^{\ell} v_i \xi_1^i\right)\right) dy_1 \right| &\leq \mathcal{M}^1(n, \ell) + \sum_{\sum \xi_i \neq 0} \left| \int_{-1/2}^{1/2} \cos\left(2\pi sy_1 \left(\sum_{i=1}^{\ell} v_i \xi_1^i\right)\right) dy_1 \right| \\ &= \mathcal{M}^1(n, \ell) + \sum_{\substack{\xi^1, \dots, \xi^\ell \\ \sum_i \xi_1^i \neq 0}} O\left(\frac{1}{s |\sum_i \xi_1^i|}\right) \end{aligned} \quad (3.7)$$

uniformly for all choices of  $v \in \{-1, 1\}^\ell$ . Thus, inserting (3.7) into (3.6), we obtain

$$\int_{-1/2}^{1/2} |A_n^1(z + sy_1)|^\ell dy_1 \leq \frac{\mathcal{M}^1(n, \ell)}{N^\ell} + \frac{1}{N^\ell} \sum_{\substack{\xi^1, \dots, \xi^\ell \\ \sum_i \xi_1^i \neq 0}} O\left(\frac{1}{s |\sum_i \xi_1^i|}\right). \quad (3.8)$$

Inserting (3.8) into (3.5) and using a similar argument to bound the contribution from  $A_n^2(z + sy)$  we have

$$\int_{-1/2}^{1/2} A_n(z + sy_1)^\ell dy_1 \ll_\ell s^2 \max_{t=1, 2} \left( \frac{\mathcal{M}^t(n, \ell)}{N^\ell} + \frac{1}{N^\ell} \sum_{\substack{\xi^1, \dots, \xi^\ell \\ \sum_i \xi_1^i \neq 0}} O\left(\frac{1}{s |\sum_i \xi_1^i|}\right) \right). \quad (3.9)$$

A similar argument bounds the contribution from  $B_n(x)$  and  $C_n(x)$ . Therefore, all in all, we have shown that

$$\int_{B(z,s) \cap [0,1]^2} |s_n(x)|^\ell dx \ll_{l,\epsilon} s^2 \max_{t=1,2} \left( \frac{\mathcal{M}^t(n,\ell)}{N^\ell} + \frac{1}{N^\ell} \sum_{\substack{\xi^1, \dots, \xi^\ell \\ \sum_i \xi_i^\ell \neq 0}} \frac{1}{s^{|\sum_i \xi_i^\ell|}} \right) \quad (3.10)$$

We are left with considering squares with  $|s_n(x)| \leq \gamma$ , but  $|\mathrm{Tr}(\Gamma_n)| \geq \gamma$  or  $|\det(\Gamma_n)| \geq \gamma$ . Again by Chebyshev's bound, for any  $\ell > 0 \in \text{even}$ , we have

$$\mathrm{Vol}\{x \in B(z,s) \cap [0,1]^2 : |\mathrm{Tr}(\Gamma_n)| \geq \gamma/2\} \leq \frac{2^\ell}{\gamma^\ell} \int_{B(z,s) \cap [0,1]^2} |\mathrm{Tr}(\Gamma_n)|^\ell dx.$$

However, as we may assume that  $|s_n(x)| \leq \gamma$ , we perform the asymptotic expansion

$$\frac{1}{1 - s_n(x)} = 1 + O(s_n(x)^2),$$

in the formula for  $\mathrm{Tr}(\Gamma_n)$  and observe that bounding moments of  $|\mathrm{Tr}(\Gamma_n)|$  again reduces to computations similar to moments of  $s_n(x)$ , which we therefore omit. Similarly, we can bound moments of  $|\det(\Gamma_n)|$  and Lemma 3.5 follows from (3.10).  $\square$

We are finally ready to prove Proposition 3.3:

*Proof of Proposition 3.3.* Let  $Q$  be a singular square, then Proposition 3.1 and Lemma 3.4, applied with  $s = n^{-1/2}\delta$ , imply that

$$\int_Q K_1(x) dx = \mathbb{E}[\mathcal{L}(f_n, Q)] \lesssim \frac{\delta N}{\sqrt{n}}.$$

Thus, Lemma 3.5, bearing in mind that each singular square is counted  $n$ -times, and taking  $\delta, \gamma > 0$  to be two small, fixed constants, gives

$$\int_{\mathcal{Q}_{\mathrm{sing}}} K_1(x) dx \ll s^2 \frac{\sqrt{n}}{N^{\ell-1}} \max_{t=0,1} \left( \mathcal{M}^t(n,\ell) + \sum_{\substack{\xi^1, \dots, \xi^\ell \\ \sum_i \xi_i^\ell \neq 0}} \frac{1}{s^{|\sum_i \xi_i^\ell|}} \right) \quad (3.11)$$

where  $\ell > 0$  is an even integer. Hence, in light of the fact that  $N \ll n^\epsilon$  for all  $\epsilon > 0$ , Proposition 3.3 follows from (3.11) together with Lemma 1.3 and Theorem 1.4.  $\square$

#### 4. PROOF OF THEOREM 1.1

In order to complete the proof of Theorem 1.1, we need to evaluate the integral of  $K_1$ , as in Proposition 3.1, outside the singular set. This will be the content of the next section:

**4.1. Asymptotic expansion outside the singular set.** The following proposition follows from Lemma 3.2 and a standard calculations about the expectation of a two dimensional Gaussian random variable, see also [8, Proposition 2.7]:

**Proposition 4.1.** *Let  $K_1(x)$  be as in Proposition 3.1, then for  $x \in [0,1]^2 \setminus \mathcal{Q}_{\mathrm{sing}}$  we have*

$$K_1(x) = \frac{\sqrt{n}}{2\sqrt{2}} + L_n(x) + \Upsilon_n(x)$$

where

$$L_n(x) = \frac{\pi\sqrt{n}}{4\sqrt{2}} \left( s_n(x) + \frac{\text{Tr } \Gamma_n}{2} + \frac{3}{4}s_n^2 + \frac{1}{4}s_n(x) \cdot \text{Tr } \Gamma_n - \frac{\text{Tr}(\Gamma_n^2)}{16} - \frac{(\text{Tr } \Gamma_n)^2}{32} \right)$$

and

$$|\Upsilon_n(x)| \ll \sqrt{n} (|s_n(x)|^3 + |\Gamma_n(x)|^3)$$

Therefore, in order to evaluate the integral of  $K_1$ , we will need the following lemma:

**Lemma 4.2.** *Let  $\varepsilon > 0$  and write  $\mathcal{S} = \text{Vol}(B(z, s) \cap [0, 1]^2)$ . There exists a density one of  $n \in S'$  such that*

$$\begin{aligned} \mathcal{S}^{-1} \int_{B(z,s) \cap [0,1]^2} L(x) dx &= -\frac{\pi(1 + \hat{\nu}_n(4))}{32\sqrt{2}} \cdot \frac{\sqrt{n}}{N} + O\left(\frac{\sqrt{n}}{N^2}\right) \\ \mathcal{S}^{-1} \int_{B(z,s) \cap [0,1]^2} |\Upsilon_n(x)| dx &\ll \frac{\sqrt{n}}{N^2} \end{aligned}$$

uniformly for all  $s > n^{-1/2+\varepsilon}$  and  $z \in [0, 1]^2$ .

We observe that Lemma 4.2 follows from the following lemma via an immediate computation:

**Lemma 4.3.** *Let  $\varepsilon > 0$  and write  $\mathcal{S} = \text{Vol}(B(z, s) \cap [0, 1]^2)$ . There exists a density one of  $n \in S'$  such that*

(1)

$$\mathcal{S}^{-1} \int_{B(z,s) \cap [0,1]^2} s_n(x) dx \ll n^{-\varepsilon/2}$$

(2)

$$\mathcal{S}^{-1} \int_{B(z,s) \cap [0,1]^2} \text{Tr } \Gamma_n(x) dx = -\frac{6}{N} + O(N^{-2})$$

(3)

$$\mathcal{S}^{-1} \int_{B(z,s) \cap [0,1]^2} s_n^2(x) dx = \frac{5}{N} + O(n^{-\varepsilon/2})$$

(4)

$$\mathcal{S}^{-1} \int_{B(z,s) \cap [0,1]^2} s_n(x) \cdot \text{Tr } \Gamma_n(x) dx = \frac{2}{N} + O(N^{-2})$$

(5)

$$\mathcal{S}^{-1} \int_{B(z,s) \cap [0,1]^2} \text{Tr}(\Gamma_n^2(x)) dx = \frac{4}{N} [1 + 2^5 \sum_{\xi \sim} \xi_1^4] + O(N^{-2})$$

(6)

$$\mathcal{S}^{-1} \int_{B(z,s) \cap [0,1]^2} (\text{Tr } \Gamma_n(x))^2 dx = \frac{4}{N} [2^6 \sum_{\xi \sim} \xi_1^4 - 3] + O(N^{-2})$$

(7)

$$\mathcal{S}^{-1} \int_{B(z,s) \cap [0,1]^2} s_n^3(x) dx \ll N^{-2}$$

(8)

$$\mathcal{S}^{-1} \int_{B(z,s) \cap [0,1]^2} (\text{Tr } \Gamma_n(x))^3 dx \ll N^{-2}$$



uniformly for all  $s > n^{-1/2+\varepsilon}$  and  $z \in [0, 1]^2$ .

Indeed, we have the following:

*Proof of Lemma 4.2 given Lemma 4.3.* By Lemma 4.3, with the same notation, we

$$\mathcal{S}^{-1} \int_{B(z,s) \cap [0,1]^2} s_n^3(x) dx \ll N^{-2} \quad \mathcal{S}^{-1} \int_{B(z,s) \cap [0,1]^2} (\text{Tr } \Gamma_n(x))^3 dx \ll N^{-2}.$$

and the second part of Lemma 4.2 follows upon noticing that, using the inequality  $|ab| \leq a^2 + b^2$ , the off diagonal entries of  $\Gamma_n$  can be bounded by the diagonal ones. The first part of Lemma 4.2 follows from the remaining asymptotic formulas in Lemma 4.3 together with the following identity:

$$11 - 2^7 \sum_{\xi \sim} \xi_1^4 = \hat{\nu}_n(4).$$

□

The proof of Lemma 4.3, being similar to the proof of Lemma 3.5 will be given in Appendix A.

**4.2. Proof of Theorem 1.1.** We are now in the position to prove Theorem 1.1

*Proof of Theorem 1.1.* Let  $\varepsilon > 0$  and  $\ell > 0$  be an even integer, thanks to Proposition 3.1 and Proposition 3.3, with the same notation, uniformly for all  $s > n^{-1/2+\varepsilon}$  and  $z \in [0, 1]^2$ , we have

$$\begin{aligned} \mathbb{E}[\mathcal{L}(f_n, z, s)] &= \int_{B(z,s) \cap [0,1]^2} K_1(x) dx = \int_{B(z,s) \cap [0,1]^2 \setminus \mathcal{Q}_{\text{sing}}} K_1(x) dx + \int_{\mathcal{Q}_{\text{sing}}} K_1(x) dx \\ &= \int_{B(z,s) \cap [0,1]^2 \setminus \mathcal{Q}_{\text{sing}}} K_1(x) dx + O\left(\frac{s^2 n^{1/2}}{N^{\ell/2-1}}\right). \end{aligned} \quad (4.1)$$

Using Proposition 4.1, with the same notation, we have

$$\int_{B(z,s) \cap [0,1]^2 \setminus \mathcal{Q}_{\text{sing}}} K_1(x) dx = \int_{B(z,s) \cap [0,1]^2 \setminus \mathcal{Q}_{\text{sing}}} \left( \frac{\sqrt{n}}{2\sqrt{2}} + L_n(x) + \Gamma_n(x) \right) dx \quad (4.2)$$

Assuming the conclusion of Lemma 1.3 and Theorem 1.4, bearing in mind that  $s_n(x) = O(1)$  and  $\Gamma_n(x) = O(1)$ , thanks to Lemma 3.5, we may extend the integral on the RHS of (4.2) to the whole of  $B(z,s) \cap [0, 1]^2$  at a cost of an error term of size at most  $O_\delta\left(\frac{s^2 n^{1/2}}{N^{\ell/2-1}}\right)$  to find

$$\int_{B(z,s) \cap [0,1]^2 \setminus \mathcal{Q}_{\text{sing}}} K_1(x) dx = \text{Vol}(B(z,s) \cap [0, 1]^2) \frac{\sqrt{n}}{2\sqrt{2}} \left( 1 - \frac{1 + \hat{\nu}_n(4)}{16N} + o\left(\frac{1}{N}\right) \right) + O\left(\frac{s^2}{N^{\ell/2-1}}\right). \quad (4.3)$$

Hence, Theorem 1.1 follows upon inserting (4.3) into (4.1) and taking  $\ell = 6$ , say. □

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## APPENDIX A. PROOF OF LEMMA 4.3

To prove Lemma 4.3 we will use the following:

**Lemma A.1.** *Let  $s > 0$ ,  $z \in [0, 1]^2$ ,  $i = 1, 2$  and write  $\mathcal{S} = \text{Vol}(B(z, s) \cap [0, 1]^2)$ . We have the following bounds:*

(1)

$$\sum_{\xi \sim} \mathcal{S}^{-1} \int_{B(z, s) \cap [0, 1]^2} \cos(2\pi\xi_i x_i) dx \ll \sum_{\xi \sim} \frac{1}{|\xi_i s|}$$

(2)

$$\sum_{\xi \sim} \mathcal{S}^{-1} \int_{B(z, s) \cap [0, 1]^2} \cos(2\pi\xi_i x_i)^2 dx = \frac{1}{2} \cdot \frac{N}{4} + O\left(\sum_{\xi \sim} \frac{1}{|\xi_i s|}\right)$$

(3)

$$\sum_{\xi, \eta} \mathcal{S}^{-1} \int_{B(z, s) \cap [0, 1]^2} \cos(2\pi\xi_i x_i) \cos(2\pi\eta_i x_i) dx = \frac{1}{2} \cdot \frac{N}{4} + O\left(\frac{1}{\sum_{\xi \sim} |\xi_i s|} + \sum_{\xi \neq \eta} \frac{1}{s|\xi_i - \eta_i|}\right)$$

(4)

$$\sum_{\xi \sim} \mathcal{S}^{-1} \int_{B(z, s) \cap [0, 1]^2} \cos(2\pi\xi_i x_i)^3 dx \ll \sum_{\xi \sim} |\xi_i s|$$

(5)

$$\sum_{\xi \sim} \mathcal{S}^{-1} \int_{B(z, s) \cap [0, 1]^2} \cos(2\pi\xi_i x_i)^4 dx = \frac{3}{8} \cdot \frac{N}{4} + O\left(\sum_{\xi \sim} \frac{1}{|\xi_i s|}\right)$$

*Proof.* Through the proof we will write  $\mathcal{B} = \tilde{B}(z) = s^{-1} \cdot (B(z, s) \cap [0, 1]^2 - z)$ , that the image of  $B(z, s) \cap [0, 1]^2$  under the homothety defined by scaling by translation by  $-z$  and scaling by  $s^{-1}$ . Moreover, we denote by  $a_i = a_i(z)$  and  $b_i = b_i(z)$  for  $i = 1, 2$  the coordinate of the projection of (the corners of)  $\mathcal{B}(z)$  along the  $X$  and  $Y$  axis respectively. Using the transformation  $x \rightarrow z + sy$ , we have

$$\sum_{\xi \sim} \mathcal{S}^{-1} \int_{B(z, s) \cap [0, 1]^2} \cos(2\pi\xi_i x_i) dx = \frac{s^2}{\mathcal{S}} \sum_{\xi \sim} \int_{\tilde{B}} \cos(2\pi\xi_i(z_i + sy_i)) dy_i$$

Since  $s^2/\mathcal{S} = O(1)$ , using the formula  $\cos(a+b) = \cos(a)\cos(b) - \sin(a)\sin(b)$ , we obtain

$$\begin{aligned} \sum_{\xi \sim} \mathcal{S}^{-1} \int_{B(z, s) \cap [0, 1]^2} \cos(2\pi\xi_i x_i) dx &\ll \sum_{\xi \sim} \left| \int_{a_i}^{b_i} \cos(2\pi\xi_i sy_i) dy_i \right| + \sum_{\xi \sim} \left| \int_{a_i}^{b_i} \sin(2\pi\xi_i sy_i) dy_i \right| \\ &\ll \sum_{\xi \sim} \frac{1}{|\xi_i s|}, \end{aligned}$$

this concludes the proof of (1).

Using the formula  $2 \cos(a)^2 = 1 + \cos(2a)$ , we may rewrite (2) as

$$\begin{aligned} \sum_{\xi \setminus \sim} \mathcal{S}^{-1} \int_{B(z,s) \cap [0,1]^2} \cos(2\pi \xi_i x_i)^2 dx &= \frac{1}{2} \cdot \frac{N}{4} + \sum_{\xi \setminus \sim} \int_{B(z,s) \cap [0,1]^2} \cos(4\pi \xi_i x) dx \\ &= \frac{1}{2} \cdot \frac{N}{4} + O\left(\sum_{\xi \setminus \sim} \frac{1}{|\xi_i s|}\right), \end{aligned} \quad (\text{A.1})$$

where we have bounded the error term using a similar bound to the one used to obtain (1). This proves (2).

Separating diagonal terms from the others, and using (A.1), (3) becomes

$$\begin{aligned} &\sum_{\xi, \eta} \mathcal{S}^{-1} \int_{B(z,s) \cap [0,1]^2} \cos(2\pi \xi_i x_i) \cos(2\pi \eta_i x_i) dx = \\ &= \frac{1}{2} \cdot \frac{N}{4} + \sum_{\xi \neq \eta} \mathcal{S}^{-1} \int_{B(z,s) \cap [0,1]^2} \cos(2\pi \xi_i x_i) \cos(2\pi \eta_i x_i) dx + O\left(\sum_{\xi \setminus \sim} \frac{1}{|\xi_i s|}\right). \end{aligned} \quad (\text{A.2})$$

Using the formula  $2 \cos(a) \cos(b) = \cos(a+b) + \cos(a-b)$ , the second term in (A.2) is at most

$$\begin{aligned} &\sum_{\xi \neq \eta} \left| \int_{\tilde{B}} \cos(2\pi(\xi_i + \eta_i)(z_i + sy_i)) dy \right| + \left| \int_{\tilde{B}} \cos(2\pi(\xi_i - \eta_i)(z_i + sy_i)) dy \right| \\ &\ll \sum_{\xi \neq \eta} \frac{1}{s|\xi_i - \eta_i|}, \end{aligned}$$

this concludes the proof of (3).

Writing  $4 \cos(a)^3 = 3 \cos(a) + \cos(3a)$ , (4) becomes

$$\sum_{\xi \setminus \sim} \mathcal{S}^{-1} \int_{B(z,s) \cap [0,1]^2} \cos(2\pi \xi_i x_i)^3 dx \ll \sum_{\xi \setminus \sim} \frac{1}{|\xi_i s|},$$

this concludes the proof of (4).

Finally, using the fact that  $8 \cos(a)^4 = 2(1 + \cos(2a))^2 = 3 + 4 \cos(2a) + \cos(4a)$  we obtain

$$\sum_{\xi \setminus \sim} \mathcal{S}^{-1} \int_{B(z,s) \cap [0,1]^2} \cos(2\pi \xi_i x_i)^4 dx = \frac{3}{8} \cdot \frac{N}{4} + O\left(\sum_{\xi \setminus \sim} \frac{1}{|\xi_i s|}\right),$$

as required.  $\square$

We are finally ready to prove Lemma 4.3

*Proof of Lemma 4.3.* Through the proof, we may assume that the conclusion of Lemma 1.3 and Theorem 1.4 hold for some fixed  $\varepsilon > 0$  and  $\ell \geq 6$ . Moreover, we will use the notation introduced in the proof of Lemma A.1. By the definition of  $s_n$  and Lemma A.1

part (1), we have

$$\begin{aligned} \mathcal{S}^{-1} \int_{B(z,s) \cap [0,1]^2} s_n(x) dx &= \frac{1}{N} \sum_{\xi \setminus \sim} \mathcal{S}^{-1} \int_{B(z,s) \cap [0,1]^2} \cos(2\pi\xi_1 x_1) + \cos(2\pi\xi_2 x_2) dx \\ &\quad - \frac{1}{N} \sum_{\xi \setminus \sim} \mathcal{S}^{-1} \int_{B(z,s) \cap [0,1]^2} \cos(2\pi\xi_1 x_1) \cos(2\pi\xi_2 x_2) dx \\ &\ll \frac{1}{N} \sum_{\xi \setminus \sim} \frac{1}{|\xi_1 s|} \ll n^{-\epsilon/2}, \end{aligned} \quad (\text{A.3})$$

this proves (1).

Invoking Lemma A.1 parts (2) and (3), we see that

$$\mathcal{S}^{-1} \int_{B(z,s) \cap [0,1]^2} s_n^2(x) = \frac{5}{N} + O(n^{-\epsilon/2}) \quad (\text{A.4})$$

$$\mathcal{S}^{-1} \int_{B(z,s) \cap [0,1]^2} b_{11}(x) dx \ll N n^{1-\epsilon/2} \quad (\text{A.5})$$

$$\mathcal{S}^{-1} \int_{B(z,s) \cap [0,1]^2} b_{22}(x) dx \ll N n^{1-\epsilon/2}, \quad (\text{A.6})$$

this proves (3).

We now begin the proof of (2), first we observe that

$$\sum_{\xi \setminus \sim} \xi_i \xi_j = \frac{n}{2} \frac{N}{4} \delta_{ij}. \quad (\text{A.7})$$

Moreover, separating diagonal terms from the off-diagonal ones, using  $2 \sin(a)^2 = 1 - \cos(2a)$ ,  $8 \sin(a)^4 = 3 - 3 \cos(2a)$  and Lemma A.1 part (1), we obtain

$$\begin{aligned} &\mathcal{S}^{-1} \int_{B(z,s) \cap [0,1]^2} d_1^2(x) dx \\ &= \mathcal{S}^{-1} \int_{B(z,s) \cap [0,1]^2} \sum_{\xi, \eta} \xi_1 \eta_1 \sin(2\pi\xi_1 x_1) \sin(2\pi\eta_1 x_1) \sin(\pi\xi_2 x_2)^2 \sin(\pi\eta_2 x_2)^2 dx \\ &= \frac{3}{16} \sum_{\xi \setminus \sim} \xi_1^2 + O \left( \sum_{\xi \neq \eta} \xi_1 \eta_1 s \cdot \mathcal{S}^{-1} \left| \int_{a_1 s}^{b_1 s} \sin(2\pi\xi_1 x_1) \sin(2\pi\eta_1 x_1) dx_1 \right| + \max_{t=1,2} \sum_{\xi \setminus \sim} \frac{n}{s|\xi_t|} \right). \end{aligned} \quad (\text{A.8})$$

Thus, using  $2 \sin(a) \sin(b) = \cos(a-b) + \cos(a+b)$ , Lemma A.1 part (1) and Theorem 1.4, we obtain

$$\mathcal{S}^{-1} \int_{B(z,s) \cap [0,1]^2} d_1^2(x) dx = \frac{3nN}{2^7} + O \left( N \sum_{\xi \neq \eta} \frac{n}{s|\xi_1 - \eta_1|} + N n^{1-\epsilon/2} \right) = \frac{3nN}{2^7} + O(N n^{1-\epsilon/2}). \quad (\text{A.9})$$

Similarly we get

$$\mathcal{S}^{-1} \int_{B(z,s) \cap [0,1]^2} d_2^2(x) dx = \frac{3nN}{2^7} + O(N n^{1-\epsilon/2}).$$

Observe that similar computations to (A.9) give  $\mathcal{S}^{-1} \int_{B(z,s) \cap [0,1]^2} d_i^4(x) \ll n^2 N^2$  for  $i = 1, 2$ , therefore using the Cauchy-Schwartz inequality and (A.4), we get

$$\mathcal{S}^{-1} \int_{B(z,s) \cap [0,1]^2} d_1^2(x) s_n(x) dx = \mathcal{S}^{-1} \int_{B(z,s) \cap [0,1]^2} d_2^2(x) s_n(x) dx \ll n \quad (\text{A.10})$$

Using (A.6) and (A.7), the bound (A.10) and the expansion  ${}^2 \text{Var } f(x)^{-1} = 1 + O(s_n(x))$ , we get

$$\begin{aligned} \mathcal{S}^{-1} \int_{B(z,s) \cap [0,1]^2} \text{Tr}(\Gamma_n(x)) dx &= -\frac{2^7 \mathcal{S}^{-1}}{nN^2} \int_{B(z,s) \cap [0,1]^2} \frac{1}{\text{Var } f(x)} [d_1^2(x) + d_2^2(x)] dx + O(n^{-\epsilon}) \\ &= -\frac{2^7 \mathcal{S}^{-1}}{nN^2} \int_{B(z,s) \cap [0,1]^2} [d_1^2(x) + d_2^2(x)] [1 + O(s_n(x))] dx + O(N^{-2}) \\ &= -\frac{6}{N} + O\left(\frac{1}{N^2}\right), \end{aligned} \quad (\text{A.11})$$

this concludes the proof of (2).

We are now going to prove (4). First, we observe that

$$\begin{aligned} \mathcal{S}^{-1} \int_{B(z,s) \cap [0,1]^2} s_n(x) b_{11}(x) dx &= \mathcal{S}^{-1} \int_{B(z,s) \cap [0,1]^2} s_n(x) b_{11}(x) dx \\ &= \frac{4}{N} \sum_{\xi, \eta} \mathcal{S}^{-1} \int_{B(z,s) \cap [0,1]^2} \xi_1^2 (\cos(2\pi \xi_1 x_1) - \cos(2\pi \xi_2 x_2) - \cos(2\pi \xi_1 x_1) \cos(2\pi \xi_2 x_2)) \cdot \\ &\quad \cdot (\cos(2\pi \eta_1 x_1) + \cos(2\pi \eta_2 x_2) - \cos(2\pi \eta_1 x_1) \cos(2\pi \eta_2 x_2)) dx \end{aligned} \quad (\text{A.12})$$

Using Lemma A.1 parts (1), (2), (3) and (A.7), we have

$$\mathcal{S}^{-1} \int_{B(z,s) \cap [0,1]^2} s_n(x) b_{11}(x) dx = \frac{4}{N} \sum_{\xi \setminus \sim} \frac{\xi_1^2}{4} + O(n^{1-\epsilon}) = \frac{n}{8} + O(n^{1-\epsilon/2}) \quad (\text{A.13})$$

Therefore, since

$$\begin{aligned} \mathcal{S}^{-1} \int_{B(z,s) \cap [0,1]^2} s_n(x) \cdot \text{Tr } \Gamma_n(x) dx &= \frac{8}{nN} \mathcal{S}^{-1} \int_{B(z,s) \cap [0,1]^2} s_n(x) [b_{11}(x) + b_{22}(x)] dx - \\ &\quad \frac{2^7}{nN^2} \mathcal{S}^{-1} \int_{B(z,s) \cap [0,1]^2} \frac{s_n}{\text{Var}(f(x))} [d_1^2(x) + d_2^2(x)] dx, \end{aligned} \quad (\text{A.14})$$

the bound (A.10) together with the asymptotic relation (A.13) give

$$\mathcal{S}^{-1} \int_{B(z,s) \cap [0,1]^2} s_n(x) \cdot \text{Tr } \Gamma_n(x) dx = \frac{2}{N} + O(N^{-2}).$$

This concludes the proof of (4).

<sup>2</sup>Since  $|\Gamma_n(x)| = O(1)$ , using Lemma 3.5 together with Theorem 1.4 as in the proof of Theorem 1.1, we may assume that  $x \in B(z, s) \cap [0, 1]^2 \setminus \mathcal{Q}_{\text{sing}}$ .

To prove (5), upon recalling that  $\text{Var}(f(x))^{-1/2} = 1 + O(s_n(x))$ , we observe that

$$\begin{aligned} [\text{Tr}(\Gamma_n(x))]^2 &= \frac{8^2}{n^2 N^2} [b_{11}^2(x) + b_{22}^2(x) + 2b_{11}(x)b_{22}(x)] \\ &\quad + \frac{128^2}{n^2 N^4} [d_1^4(x) + d_2^4(x) + 2d_1^2(x)d_2^2(x)][1 + O(s_n(x))] \\ &\quad - 2\frac{8}{nN} \frac{128}{nN^2} [b_{11}(x) + b_{22}(x)][d_1^2(x) + d_2^2(x)][1 + O(s_n(x))]. \end{aligned} \quad (\text{A.15})$$

Using Lemma A.1 parts (2) and (3) and Theorem 1.4, we have

$$\begin{aligned} \mathcal{S}^{-1} \int_{B(z,s) \cap [0,1]^2} b_{11}^2(x) dx &= \mathcal{S}^{-1} \int_{B(z,s) \cap [0,1]^2} b_{22}^2(x) dx = \frac{5}{4} \sum_{\xi \setminus \sim} \xi_1^4 + O(n^{1-\epsilon/2}) \\ \mathcal{S}^{-1} \int_{B(z,s) \cap [0,1]^2} b_{22}(x)b_{11}(x) dx &= -\frac{3}{4} \sum_{\xi \setminus \sim} \xi_1^2 \xi_2^2 + O(n^{1-\epsilon/2}) \end{aligned} \quad (\text{A.16})$$

Moreover, we observe that

$$\mathcal{S}^{-1} \int_{B(z,s) \cap [0,1]^2} d_1^4(x) dx = \sum_{\xi^1, \xi^2, \xi^3, \xi^4} \mathcal{S}^{-1} \int_{B(z,s) \cap [0,1]^2} \prod_{i=1}^4 \xi_1^i \sin(2\pi \xi_1^i x_1) \sin(2\pi \xi_2^i x_2)^2. \quad (\text{A.17})$$

Thus, separating the terms with  $\xi^1 = \dots = \xi^4$  and the terms with  $\xi^1 = \xi^2$  and  $\xi^3 = \xi^4$  from the rest, arguing as in (A.8) and bearing in mind that  $|\xi_t| \leq n^{1/2}$ , we obtain

$$\begin{aligned} \mathcal{S}^{-1} \int_{B(z,s) \cap [0,1]^2} d_1^4(x) dx &= \frac{105}{1024} \sum_{\xi \setminus \sim} \xi_1^4 + \frac{9}{256} \sum_{\xi \neq \eta} \xi_1^2 \eta_1^2 + O\left( \frac{\sum_{\substack{\xi^1, \dots, \xi^4 \\ \sum_i \xi_1^i \neq 0}} \frac{n^2}{s |\sum_i \xi_1^i|}} \right) \\ &= \frac{105}{1024} \sum_{\xi \setminus \sim} \xi_1^4 + \frac{9}{256} \sum_{\xi \neq \eta} \xi_1^2 \eta_1^2 + O(n^{2-\epsilon/2}), \end{aligned}$$

where, in the last line, we have used Theorem 1.4. Similar computations give

$$\begin{aligned} \mathcal{S}^{-1} \int_{B(z,s) \cap [0,1]^2} d_2^4(x) dx &= \frac{105}{1024} \sum_{\xi \setminus \sim} \xi_1^4 + \frac{9}{256} \sum_{\xi \neq \eta} \xi_1^2 \eta_1^2 + O(n^{2-\epsilon/2}) \\ \mathcal{S}^{-1} \int_{B(z,s) \cap [0,1]^2} d_1^2(x) d_2^2(x) dx &= \frac{25}{1024} \sum_{\xi \setminus \sim} \xi_1^2 \xi_2^2 + \frac{9}{256} \sum_{\xi \neq \eta} \xi_2^2 \eta_1^2 + O(n^{2-\epsilon/2}) \\ \mathcal{S}^{-1} \int_{B(z,s) \cap [0,1]^2} b_{ii}(x) d_j^2(x) dx &\ll n^2 N^2 \quad i = 1, 2 \end{aligned} \quad (\text{A.18})$$

Finally, bearing in mind that  $n = \xi_1^2 + \xi_2^2$  and  $\sum_{\xi \setminus \sim} \xi_1^2 = nN/8$ , we have

$$\sum_{\xi \setminus \sim} \xi_1^2 \xi_2^2 = \frac{n^2 N}{8} - \sum_{\xi \setminus \sim} \xi_1^4$$

and, for  $i, j = 1, 2$ ,

$$\sum_{\xi \neq \eta} \xi_i^2 \eta_j^2 = \sum_{\xi} \xi_i^2 \sum_{\eta} (\eta_j^2 - \xi_i^2) = \frac{n^2 N}{8} - \sum_{\xi \setminus \sim} \xi_1^4.$$

Thus, (5) follows inserting (A.16), (A.17) and (A.18) into (A.15).

To see (6) we observe that, for symmetric matrix  $A = a_{ij}$ ,  $i, j = 1, 2$ ,  $\text{Tr}(A^2) = a_1^2 + 2a_{12}^2 + a_2^2$ , thus

$$\begin{aligned} \text{Tr}(\Gamma_n^2(x)) &= \left( \frac{8}{nN} b_{11}(x) - \frac{128}{nN^2 \text{Var}(f(x)) d_1^2(x)} \right)^2 \\ &+ 2 \left( \frac{8}{nN} b_{12}(x) - \frac{128}{nN^2 \text{Var}(f(x)) d_1(x) d_2(x)} \right)^2 + \left( \frac{8}{nN} b_{22}(x) - \frac{128}{nN^2 \text{Var}(f(x)) d_2^2(x)} \right)^2. \end{aligned} \quad (\text{A.19})$$

Thanks to Lemma A.1 parts (2) and (3), we have

$$\mathcal{S}^{-1} \int_{B(z,s) \cap [0,1]^2} b_{12}^2(x) dx = \frac{1}{4} \sum_{\xi \setminus \sim} \xi_1^2 \xi_2^2 + O \left( \sum_{\xi \neq \eta} \frac{n^2}{s |\xi_t - \eta_t|} \right) \quad (\text{A.20})$$

and

$$\mathcal{S}^{-1} \int_{B(z,s) \cap [0,1]^2} b_{12}(x) d_1(x) d_2(x) dx \ll n^2 N.$$

Therefore, part (6) follows from inserting (A.16), (A.18) and (A.20) into (A.19). Finally, separating diagonal terms from the off-diagonal ones, we observe that

$$\mathcal{S}^{-1} \int_{B(z,s) \cap [0,1]^2} s_n(x)^3 dx = \frac{4^3}{N^3} \sum_{\xi \setminus \sim} \left( -\frac{3}{4} \right) + O \left( \frac{1}{N^3} \sum_{\xi^1, \xi^2, \xi^3} \frac{1}{s |\xi_i^1 + \xi_i^2 + \xi_i^3|} \right) \ll N^{-2},$$

where in the last line we have used Theorem 1.4. Similarly, we have

$$\mathcal{S}^{-1} \int_{B(z,s) \cap [0,1]^2} \text{Tr}(\Gamma_n(x))^3 dx \ll N^{-2}.$$

Thus, we have proved parts (7) and (8), and hence Lemma 4.3.  $\square$

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