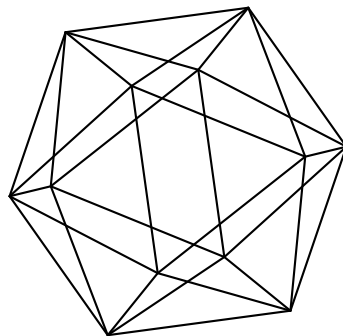


Max-Planck-Institut für Mathematik Bonn

Arithmetic purity of the Hardy-Littlewood property and
geometric sieve for affine quadrics

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ARITHMETIC PURITY OF THE HARDY-LITTLEWOOD PROPERTY AND GEOMETRIC SIEVE FOR AFFINE QUADRICS

YANG CAO AND ZHIZHONG HUANG

ABSTRACT. We establish the Hardy-Littlewood property (*à la Borovoi-Rudnick*) for Zariski open subsets in affine quadrics of the form $q(x_1, \dots, x_n) = m$, where q is a non-degenerate integral quadratic form in $n \geq 3$ variables and m is a non-zero integer. This gives asymptotic formulas for the density of integral points taking coprime polynomial values, which is a quantitative version of the arithmetic purity of strong approximation property off infinity for affine quadrics.

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1. INTRODUCTION

1.1. Arithmetic purity of the Hardy-Littlewood property. Let X be a smooth geometrically integral variety over \mathbb{Q} . Assume that X satisfies strong approximation off the real place \mathbb{R} . An open question first raised by Wittenberg (cf. [32, §2.7 Question 2.11]) asks whether all open subsets $U \subset X$ also satisfy strong approximation off \mathbb{R} , whenever $\text{codim}_X(X - U) \geq 2$. We say that such X satisfies *arithmetic purity of strong approximation* ((**APSA**) for short) off \mathbb{R} (cf. [8, Definition 1.2]). As observed by Minčev (cf. [4, Proposition 2.6]), the condition on the codimension guarantees that there is no cohomological or topological obstruction for U to satisfy strong approximation. Recently in [9, 8], Wittenberg's question was settled in the affirmative for a wide class of semisimple simply connected linear algebraic groups and their homogeneous spaces (with connected stabilizers). We refer to the references therein for an account of proven results towards this question.

In this article, we address an effective and statistic aspect of Wittenberg's question concerning the distribution of integral points in adelic spaces. Our starting point is the work of Borovoi-Rudnick [4]. Now we assume that $X \subset \mathbb{A}_{\mathbb{Q}}^n$ is a smooth quasi-affine geometrically integral variety whose adelic space $X(\mathbf{A})$ is non-empty. Assume moreover that $X(\mathbb{R})$ has no compact connected components. We equip the affine space $\mathbb{A}^n(\mathbb{R}) = \mathbb{R}^n$ with an euclidean norm $\|\cdot\|$. Let $B :=$

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$B_\infty \times B_f \subset X(\mathbf{A})$ be an adelic neighbourhood with $B_\infty \subset X(\mathbb{R})$ a real connected component and $B_f \subset X(\mathbf{A}_f)$ a compact open subset. For any $T > 0$, consider the counting function

$$(1) \quad N_X(B; T) := \#\{\underline{\mathbf{X}} \in X(\mathbb{Q}) \cap B : \|\underline{\mathbf{X}}\| \leq T\}.$$

We are interested in the varieties X for which an asymptotic formula for $N_X(B; T)$ exists for any such B . Assume that X satisfies strong approximation off \mathbb{R} , then we expect that the leading constant should be a product of local densities (depending also on $\|\cdot\|$). However there also exist certain such varieties failing the integral Hasse principle and strong approximation.

To better put forth our qualitative characterization of these varieties, we further assume that X is equipped with a fixed *gauge form* ω_X , i.e. a nowhere zero differential form of top degree. We associate to ω_X a *normalised Tamagawa measure* $m^X = m_\infty^X \times m_f^X$ on $X(\mathbf{A})$, where the real part m_∞^X is defined on $X(\mathbb{R})$, and the finite part m_f^X is defined on $X(\mathbf{A}_f)$ (cf. §2.1 for more details). Let $\delta_X : X(\mathbf{A}) \rightarrow \mathbb{R}_{\geq 0}$ be a locally constant not identically zero function, and let us note

$$B_\infty(T) := \{\underline{\mathbf{X}} \in B_\infty : \|\underline{\mathbf{X}}\| \leq T\}.$$

We say that X is a (*relatively*) *Hardy-Littlewood variety* (with respect to the gauge form ω_X) with density δ_X , after Borovoi-Rudnick [4, Definition 2.2] (see also [11, p. 143]), if for any adelic neighbourhood $B = B_\infty \times B_f$ as before, we have $m_\infty^X(B_\infty(T)) \rightarrow \infty$ and

$$(2) \quad N_X(B; T) \sim \int_{B_\infty(T) \times B_f} \delta_X \, d m^X, \quad T \rightarrow \infty.$$

A Hardy-Littlewood variety X is called *strongly Hardy-Littlewood* if $\delta_X \equiv 1$. That is, for any such B as before, we have

$$(3) \quad N_X(B; T) \sim \int_{B_\infty(T) \times B_f} d m^X = m_\infty^X(B_\infty(T)) m_f^X(B_f), \quad T \rightarrow \infty.$$

We may view the function δ_X as an effective measure of the failure of strong approximation on X . If $X(\mathbb{Q}) = \emptyset$, then clearly X is Hardy-Littlewood with density $\delta_X \equiv 0$. We shall henceforth focus on the case where $X(\mathbb{Q}) \neq \emptyset$. See notably works [11, 13, 14], [4, §6] and more recent ones [26, 16, 5] for various examples of homogeneous spaces satisfying the Hardy-Littlewood property.
1

The following is a natural extension and refinement of the Hardy-Littlewood property in the spirit of Wittenberg's question.

Question 1.1. Assume that X is a Hardy-Littlewood variety with density δ_X . Are all open subsets $U \subset X$ also Hardy-Littlewood (with respect to the gauge form $\omega_X|_U$) with density $\delta_X|_U$, whenever $\text{codim}_X(X - U) \geq 2$?

Definition 1.2. We say that X satisfies *arithmetic purity of the Hardy-Littlewood property* with density δ_X , abbreviated as **(APHL)** with density δ_X , if X is a Hardy-Littlewood variety with density δ_X and Question 1.1 has a positive answer for X .

Similarly, we say that X satisfies *arithmetic purity of the strongly Hardy-Littlewood property*, abbreviated as **(APSHL)**, if X satisfies **(APHL)** with density $\delta_X \equiv 1$.

It is proved by Borovoi-Rudnick [4, Proposition 2.5] (resp. [4, Proposition 2.4]) that being relatively (resp. strongly) Hardy-Littlewood implies a weaker version of (resp. the usual) strong approximation for X and the integral Hasse principle for any integral model of X . In particular,

¹In all known examples, the counting function $N_X(B; T)$ behaves like $T^d(\log T)^e$ (depending on the embedding $X \hookrightarrow \mathbb{A}_{\mathbb{Q}}^n$), where $d > 0, e \geq 0$ are rational numbers.

the condition $\text{codim}_X(X - U) \geq 2$ is necessary for U to be Hardy-Littlewood. Moreover, we shall prove that (cf. Proposition 2.1 in §2.1) this condition guarantees that the restriction of the Tamagawa measure m^X on U is well-defined. To get an instructive idea, let us take \mathcal{X} an integral model of X over \mathbb{Z} , and let \mathcal{Z} be the Zariski closure of $Z := X - U$ in \mathcal{X} and let $\mathcal{U} := \mathcal{X} - \mathcal{Z}$ be the integral model of U . For simplicity we assume that $\mathcal{U}(\widehat{\mathbb{Z}}) := \prod_{p < \infty} \mathcal{U}(\mathbb{Z}_p) \neq \emptyset$. Write

$$\tau_p(\mathcal{U}, \mathcal{X}) := \frac{\#\mathcal{U}(\mathbb{F}_p)}{\#\mathcal{X}(\mathbb{F}_p)}.$$

We may interpret the quantity $\tau_p(\mathcal{U}, \mathcal{X})$ as the probability that an integral point of \mathcal{X} specializes to a point in \mathcal{U} modulo p . By the Lang-Weil estimate (cf. (9) (10) *infra*) to both \mathcal{X} and \mathcal{Z} , we get

$$(4) \quad 1 - \tau_p(\mathcal{U}, \mathcal{X}) = \frac{\#\mathcal{Z}(\mathbb{F}_p)}{\#\mathcal{X}(\mathbb{F}_p)} = O\left(\frac{1}{p^{\text{codim}_X(Z)}}\right),$$

where the implied constant is uniform for any prime p . Roughly speaking, we calculate the finite part measure $m_f^X|_U$ of $\mathcal{U}(\widehat{\mathbb{Z}})$ as being the limit of $m_f^X(\prod_{p < M} \mathcal{U}(\mathbb{Z}_p) \times \prod_{p \geq M} \mathcal{X}(\mathbb{Z}_p))$ as $M \rightarrow \infty$. This turns out to require that infinite product over almost all p of $\tau_p(\mathcal{U}, \mathcal{X})$ is absolutely convergent, which is true whenever $\text{codim}_X(Z) \geq 2$ thanks to (4).

As removing a proper closed subset does not affect the real volume, we expect in particular that if the counting function $N_U(U(\mathbb{R}) \times \mathcal{U}(\widehat{\mathbb{Z}}); T)$ for U is non-zero when $T \rightarrow \infty$, then

$$(5) \quad N_U(U(\mathbb{R}) \times \mathcal{U}(\widehat{\mathbb{Z}}); T) \sim c_{\mathcal{U}, \mathcal{X}} N_X(X(\mathbb{R}) \times \mathcal{X}(\widehat{\mathbb{Z}}); T),$$

where $c_{\mathcal{U}, \mathcal{X}} > 0$ is related the infinite product of $\tau_p(\mathcal{U}, \mathcal{X})$. On the other hand, we do not expect that, amongst other reasons (e.g., the existence of non-constant invertible functions, cf. [4, Lemma 1.5.2]), $N_U(U(\mathbb{R}) \times \mathcal{U}(\widehat{\mathbb{Z}}); T)$ could grow in the same magnitude as $N_X(X(\mathbb{R}) \times \mathcal{X}(\widehat{\mathbb{Z}}); T)$ when $\text{codim}_X(X - U) = 1$, as the infinite product over $\tau_p(\mathcal{U}, \mathcal{X})$ would diverge to 0. For related Schinzel-type conjectures however, see [26, (1.3)–(1.5)].

Integral points of \mathcal{X} lying in such open subsets \mathcal{U} satisfy infinitely many congruence conditions. To derive (2) or (3) for U , this is one difficulty we need to get rid of. For instance, if \mathcal{Z} is defined by two regular functions $f, g \in \mathbb{Z}[\mathcal{X}]$, then $\mathcal{U}(\mathbb{Z})$ consists of precisely the points $\underline{\mathbf{X}} \in \mathcal{X}(\mathbb{Z})$ such that either $p \nmid f(\underline{\mathbf{X}})$ or $p \nmid g(\underline{\mathbf{X}})$ for any prime p , i.e.

$$\mathcal{U}(\mathbb{Z}) = \{\underline{\mathbf{X}} \in \mathcal{X}(\mathbb{Z}) : \gcd(f(\underline{\mathbf{X}}), g(\underline{\mathbf{X}})) = 1\}.$$

1.2. Results on (APHL). Motivated by recent progress [9, 8] on (APSA) for semisimple simply connected groups, the purpose of this article is to go beyond the affine spaces previously considered in [12, 27, 3] and provide a positive answer to Question 1.1 for affine quadrics, by developing a version of *geometric sieve* for them.

1.2.1. Affine spaces. The geometric sieve was first inaugurated by Ekedahl [12] when dealing with $X = \mathbb{A}^n$. Further generalised by Poonen [27, Theorem 3.1] and Bhargava [3, §3], this sieve method has demonstrated surprising applications on the density of square-free polynomial values in various circumstances. Their results indeed prove, on plugging-in the classical Chinese remainder theorem:

Theorem 1.3 (Ekedahl, Poonen, Bhargava). *Affine spaces satisfy (APSHL).*

1.2.2. *Affine quadrics.* Let $q(\mathbf{x}) \in \mathbb{Z}[x_1, \dots, x_n]$ be a non-degenerate integral quadratic form in $n \geq 3$ variables. Suppose that $q(\mathbf{x})$ is indefinite (i.e. isotropic over \mathbb{R}). For $m \in \mathbb{Z}_{\neq 0}$, let us consider the \mathbb{Z} -scheme

$$(6) \quad \mathcal{Q} := (q(\mathbf{x}) = m) \subset \mathbb{A}_{\mathbb{Z}}^n.$$

The smooth geometrically integral \mathbb{Q} -variety

$$(7) \quad \mathbf{Q} := \mathcal{Q} \times_{\mathbb{Z}} \mathbb{Q} \subset \mathbb{A}_{\mathbb{Q}}^n$$

is called an *affine quadric*. In particular, \mathcal{Q} is an integral model of \mathbf{Q} . Recall that affine quadrics with $n \geq 4$ variables satisfy the integral principle and strong approximation, while there is Brauer-Manin obstruction when $n = 3$ (cf. [10, §5.6, §5.8]). We shall always assume in this article that $\mathbf{Q}(\mathbb{Q}) \neq \emptyset$. We remark that rational points on an affine quadric form a *thin* subset of $\mathbb{A}^n(\mathbb{Q})$.

The variety \mathbf{Q} is an affine symmetric space (cf. [4, p. 59], [5, p. 1045]) under the spin group $G := \mathbf{Spin}_q$ (the universal double covering of \mathbf{SO}_q , with the standard almost faithful representation through \mathbf{SO}_q in $\mathbf{GL}_{n, \mathbb{Q}}$). Let $P \in \mathbf{Q}(\mathbb{Q})$ and let H be its stabilizer. Then $H \cong \mathbf{Spin}_q|_{P^\perp}$ is a symmetric subgroup of \mathbf{Spin}_q , where P^\perp is the orthogonal complement of P . The real locus $\mathbf{Q}(\mathbb{R})$ has no compact connected components. Since $n \geq 3$, the group G is always semisimple and simply connected, so is H if $n \geq 4$. Special attention is devoted to the case where $n = 3$ because H is isomorphic to a torus. It is anisotropic over \mathbb{Q} (hence has no non-trivial \mathbb{Q} -characters) precisely when $-m \det(q)$ is not a square, a condition that we shall always assume and denote by $-m \det(q) \neq \square$ in the sequel. The work of Borovoi-Rudnick proves that (cf. [4, §6.4])

- Affine quadrics with $n \geq 4$ variables are strongly Hardy-Littlewood (cf. [4, Theorem 0.3]);
- Affine quadrics with $n = 3$ variables are Hardy-Littlewood with a locally constant density function $\delta_{\mathbf{Q}} : \mathbf{Q}(\mathbb{A}) \rightarrow \{0, 2\}$ (cf. [5, p. 1047-1048], [4, §3], see also §3.2 *infra*).

We recall the well-known asymptotic growth of integral points on affine quadrics (cf. e.g. [23, (2.6)], [11, (1.9)] and [5, p. 1047]). For every connected component $B_\infty \subset \mathbf{Q}(\mathbb{R})$,

$$(8) \quad \#\{\underline{\mathbf{X}} \in \mathcal{Q}(\mathbb{Z}) \cap B_\infty : \|\underline{\mathbf{X}}\| \leq T\} \sim \int_{B_\infty(T) \times \mathcal{Q}(\widehat{\mathbb{Z}})} \delta_{\mathbf{Q}} \, d\mathfrak{m}^{\mathbf{Q}} \sim c_{\mathcal{Q}, B_\infty} T^{n-2},$$

where $c_{\mathcal{Q}, B_\infty} \geq 0$ depends on \mathcal{Q}, B_∞ .

Our main results are the following.

Theorem 1.4. *Let $\mathbf{Q} \subset \mathbb{A}_{\mathbb{Q}}^n$ be the affine quadric (7) with $n \geq 4$ variables. Then \mathbf{Q} satisfies (APSHL).*

Theorem 1.5. *Let $\mathbf{Q} \subset \mathbb{A}_{\mathbb{Q}}^n$ be the affine quadric (7) with $n = 3$ variables. Suppose that the form q is anisotropic over \mathbb{Q} and that $-m \det(q) \neq \square$. Then \mathbf{Q} satisfies (APHL) with density $\delta_{\mathbf{Q}}$.*

1.3. A general strategy to achieve (APHL). In order to prove Theorems 1.4 and 1.5, we develop a road-map, i.e. Theorem 3.1, which comes up with a sufficient criterion towards (APHL) for general affine Hardy-Littlewood varieties X . It consists of two hypotheses. Let us fix \mathcal{X} an affine integral model of X over \mathbb{Z} for convenience of exposition.

The first one is called *strong effective equidistribution condition*. We require finer information on the error terms of the equivalences (2) and (3) when evaluated at certain family of finite adelic neighbourhoods $B_f^{\mathcal{X}}(\xi, l)$ (cf. (20), which we call *congruence neighbourhoods* (Definition 2.2)) associated to $l \geq 2$ and $\xi = (\xi_p)_{p|l} \in \prod_{p|l} X(\mathbb{Q}_p)$. If $\xi \in \prod_{p|l} \mathcal{X}(\mathbb{Z}_p)$, then $B_f^{\mathcal{X}}(\xi, l)$ can be described in terms of congruence conditions: for every $B_\infty \subset X(\mathbb{R})$,

$$(B_\infty \times B_f^{\mathcal{X}}(\xi, l)) \cap \mathcal{X}(\mathbb{Z}) = \{\underline{\mathbf{X}} \in \mathcal{X}(\mathbb{Z}) \cap B_\infty : \text{for every } p \mid l, \underline{\mathbf{X}} \equiv \xi_p \pmod{p^{v_p(l)}}\}.$$

The significant feature is that we require the error terms to have polynomial growth on the level of congruence l , and to be uniform for both l and ξ as well. This formulation is motivated by recent works on counting integral points with congruence conditions due to Nevo-Sarnak [26, §3], Gorodnik-Nevo [17, §6 & §7], and Browning-Gorodnik [5, §2], *et al.* We utilise this hypothesis to deduce the density of integral points on \mathcal{X} that specialize into an arbitrary subset of residues (cf. Proposition 2.5), which has applications in two different directions. On the one hand, we can give an “approximation” of the Tamagawa measure restricted to a given open subset (cf. Corollary 2.6). On the other hand, it turns out to give satisfactory control (cf. Corollary 2.7) for integral points on \mathcal{X} coming from residues in a given closed subset modulo certain primes which are not of polynomial growth.

The second hypothesis is called *geometric sieve condition* in the spirit of Ekedahl’s work [12], for which our formulation is closer to Bhargava’s (cf. Theorem 4.8). For any fixed closed $Z \subset X$ of codimension at least two, let $\mathcal{Z} := \bar{Z} \subset \mathcal{X}$. Informally speaking, this hypothesis requires that integral points on \mathcal{X} arising from residues of \mathcal{Z} of arbitrarily large prime moduli are negligible. Establishing this hypothesis is often challenging in resolving Question 1.1, as we need to sieve out integral points satisfying infinitely many congruence conditions in a thin subset if $X \subset \mathbb{A}^n$ is not of degree one.

Then Theorem 3.1 on the whole says that a Hardy-Littlewood variety X satisfies **(APHL)**, provided that sufficiently many integral models \mathcal{X}_a of X constructed via “rescaling coordinates by $a \in \mathbb{N}_{\neq 0}$ ” (cf. (24) in §2.2 and (42) in §3) all satisfy these two hypotheses. Inspired by various results on counting lattice points in homogeneous spaces in a vast literature, we verify the strong effective equidistribution condition for certain *nice* homogeneous spaces. This implies that to achieve **(APHL)** for such varieties, it remains to show the geometric sieve condition (cf. §3.2).

1.4. A geometric sieve for affine quadrics. The technical core of our paper is the following extension of Ekedahl’s geometric sieve [12] to affine quadrics, which seems to be the first example for affine varieties besides affine spaces.

Theorem 1.6. *Assume that \mathbf{Q} satisfies the hypotheses of Theorems 1.4 and 1.5. Fix an integral model \mathcal{Q} of \mathbf{Q} of the form in (6). For any closed subset $Z \subset \mathbf{Q}$ such that $\text{codim}_{\mathbf{Q}}(Z) \geq 2$, let \mathcal{Z} be the Zariski closure of Z in \mathcal{Q} . Then for any $M \geq 2$, we have*

$$\#\{\underline{\mathbf{X}} \in \mathcal{Q}(\mathbb{Z}) : \|\underline{\mathbf{X}}\| \leq T, \exists p \geq M, \underline{\mathbf{X}} \bmod p \in \mathcal{Z}\} = O\left(T^{n-2} \left(\frac{1}{M^{\text{codim}_{\mathbf{Q}}(Z)-1}} + \frac{1}{\sqrt{\log T}}\right)\right), \quad T \rightarrow \infty$$

where the implied constant depends only on \mathcal{Q} and Z .

The bound above being uniform for M , Theorem 1.6 thus furnishes a satisfactory control for integral points on \mathcal{Q} specializing into \mathcal{Z} modulo any prime larger than arbitrary $M(T)$ which grows to infinity, compared to the order of growth in (8). We would like to mention that the independent work of Browning and Heath-Brown [6] dealing with projective quadrics obtains power saving for the term $\frac{T^{n-2}}{\sqrt{\log T}}$ when $n \geq 5$.

Our strategy of proving Theorem 1.6 unifies radically different ideas on estimating integral points on quadratic hypersurfaces. We start by breaking the prime moduli $[M, \infty]$ into three subfamilies, and then analyse each of them separately. Primes between M and T^α with $\alpha > 0$ small enough to be determined later are dealt with on utilising the effective equidistribution result mentioned in §1.3 (cf. Theorem 4.1). Primes larger than a power of T divided into two parts:

$$T^\alpha < p < T \quad \text{and} \quad p \geq T.$$

We name them respectively “intermediate primes” and “very large primes”. For the intermediate primes (cf. Theorem 4.2), we appeal to uniform estimates for integral points of bounded height on quadrics in [5, §4 §5] developed originally aiming at studying power-free polynomial values on affine quadrics. This slicing argument can be applied straightforward to the case $n \geq 4$ (cf. Proposition 4.6). However it *a priori* does not provide a desired power saving for the case $n = 3$, as was also encountered in [5, p. 1078]. To overcome this difficulty, we make essential use of the assumptions that $-m \det q$ is non-squared and that the form q is \mathbb{Q} -anisotropic in our argument. A consequence is that (cf. Lemma 4.4) any slice contains no line or parabolic conic (i.e. affine plane conic of rank one) and thus contributes few integral points (cf. Proposition 4.5). Note that these assumptions also appear in the results of Liu–Sarnak [23]. For the treatment of very large primes (cf. Theorem 4.7), we make use of Bhargava’s generalised Ekedahl-type geometric sieve [3], and we develop a half-dimensional sieve for affine quadrics (cf. §1.5 below). These two sieve methods are matched together via a fibration argument (cf. §4.3).

1.5. A half-dimensional sieve for affine quadrics. We establish the following auxiliary result on the density of quadratic polynomial values represented by a binary quadratic form. This is an application of the half-dimensional sieve due to Friedlander–Iwaniec [19] [15], which may be of independent interest.

Theorem 1.7. *Let $Q_1(\mathbf{x}) \in \mathbb{Z}[x_1, \dots, x_L]$ be a quadratic polynomial in $L \geq 1$ variables, and let $Q_2(\mathbf{y}) \in \mathbb{Z}[y_1, y_2]$ be a binary positive-definite non-degenerate quadratic form.*

- *If $L \geq 2$, assume that the \mathbb{Q} -variety $(Q_1(\mathbf{x}) = 0) \subset \mathbb{A}_{\mathbb{Q}}^L$ is smooth;*
- *If $L = 1$, assume that the \mathbb{Q} -variety*

$$(Q_1(\mathbf{x}) - Q_2(\mathbf{y}) = 0) \subset \mathbb{A}_{\mathbb{Q}}^3$$

is a (smooth) affine quadric, and that it has anisotropic stabilizer.

Then

$$\#\{\mathbf{X} \in \mathbb{Z}^L : \|\mathbf{X}\| \leq T, \exists (u, v) \in \mathbb{Z}^2, Q_1(\mathbf{X}) = Q_2(u, v)\} = O\left(\frac{T^L}{\sqrt{\log T}}\right),$$

where the implied constant depends only on Q_1, Q_2 .

Remark 1.8. Without the condition that the stabilizer be anisotropic, the estimate in Theorem 1.7 is false, as clearly seen from the example $x^2 + 1 = y_1^2 + y_2^2$. See also Remark 5.3.

It would be interesting to ask whether Theorem 1.5 remains true for affine quadrics of dimension two with isotropic stabilizers. A different feature is that the singular series $\prod_{p < \infty} \frac{\#\mathcal{Q}(\mathbb{F}_p)}{p^{\dim \mathcal{Q}}}$ can diverge, and the order of magnitude of $N_{\mathcal{Q}}(\mathbb{Q}(\mathbb{R}) \times \mathcal{Q}(\widehat{\mathbb{Z}}); T)$ can be $T \log T$ instead of T (cf. [11, p. 146]). We likewise ask whether condition that the form q be \mathbb{Q} -anisotropic could be dropped. One may run into the similar pathological phenomenon as in the projective analogue [22].

1.6. Structure of the paper. Section 2 is mostly served for technical preparations. We describe effective equidistribution of integral points with uniform error terms and its applications. In Section 3 we prove Theorem 3.1, and then we deduce Theorems 1.4 and 1.5 as a consequence. Section 4 is entirely devoted to proving Theorem 1.6, i.e. the treatment of “error terms”. Theorem 1.7 is proved in Section 5. More layouts are sketched at the beginning of each section.

1.7. Notation and conventions. Given two real-valued functions f and g with g non-negative, Vinogradov's symbol $f \ll g$ and Landau's symbol $f = O(g)$ both mean that there exists $C > 0$ such that $|f| \leq Cg$. The dependence of C on the variable and on f, g will be specified explicitly. We use these two symbols interchangeably. If f, g are defined over the real numbers and with g nowhere zero, the small "o" notation $f(x) = o(g(x))$ means that $\lim_{x \rightarrow \infty} \frac{f(x)}{g(x)} = 0$. In this article, all implied constants are allowed to depend on the embedding $X \hookrightarrow \mathbb{A}_{\mathbb{Q}}^n$ and the chosen euclidean norm $\|\cdot\|$.

The letter p is always reserved for prime numbers, and ε denotes an arbitrarily small positive parameter that can be rescaled by constant multiples. We write $p^k \parallel n$ for certain $k \in \mathbb{N}, n \in \mathbb{N}_{\neq 0}$ if $p^k \mid n$ and $p^{k+1} \nmid n$. We denote the Möbius function by $\mu(\cdot)$. We write $\Omega(n)$ for the number of prime divisors of $n \in \mathbb{N}_{\neq 0}$. We say that an integer $l \geq 2$ is *sufficiently divisible* if there exists $l_0 \in \mathbb{N}_{\neq 0}$ such that $l_0 \mid l$, where the integer l_0 depends on the setting.

A *variety* over \mathbb{Q} is an integral separated scheme of finite type over \mathbb{Q} . We write $\text{codim}_X(Y)$ for the codimension of a subvariety Y in X . For X a \mathbb{Q} -variety, an *integral model* of X over \mathbb{Z} is a faithfully flat of finite type and separated scheme \mathcal{X} over \mathbb{Z} endowed with an isomorphism $\mathcal{X} \times_{\mathbb{Z}} \mathbb{Q} \cong X$ over \mathbb{Q} . Such an integral model over \mathbb{Z} always exists by [24, Proposition 2.5]. For \mathcal{W} a scheme over \mathbb{Z} , we write $\mathcal{W}(\widehat{\mathbb{Z}}) := \prod_{p < \infty} \mathcal{W}(\mathbb{Z}_p)$.

We frequently use the following version of the Lang-Weil estimate. Let \mathcal{Y} be a separated reduced scheme of finite type of dimension > 0 over \mathbb{Z} . Let $\mathcal{Y}_{\mathbb{Q}} := \mathcal{Y} \times_{\mathbb{Z}} \mathbb{Q}$ be the generic fibre, and for any p , let $\mathcal{Y}_{\mathbb{F}_p} := \mathcal{Y} \times_{\mathbb{Z}} \mathbb{F}_p$. Since $\dim(\mathcal{Y}_{\mathbb{F}_p}) = \dim(\mathcal{Y}_{\mathbb{Q}})$ for almost all p , we have, uniformly for any p ,

$$(9) \quad \#\mathcal{Y}(\mathbb{F}_p) = O(p^{\dim(\mathcal{Y}_{\mathbb{Q}})}).$$

If there exists $L_0 \in \mathbb{N}_{\neq 0}$ such that the scheme \mathcal{Y} is smooth over $\mathbb{Z}[1/L_0]$ with geometrically integral fibres (e.g. when the generic fibre $\mathcal{Y}_{\mathbb{Q}}$ is smooth and geometrically integral), then uniformly for any p ,

$$(10) \quad \#\mathcal{Y}(\mathbb{F}_p) = p^{\dim(\mathcal{Y}_{\mathbb{Q}})} + O(p^{\dim(\mathcal{Y}_{\mathbb{Q}}) - \frac{1}{2}}).$$

In particular, for almost all p , $\mathcal{Y}(\mathbb{Z}_p) \neq \emptyset$. Moreover, for any integer $l \geq 2$ with $(L_0, l) = 1$,

$$(11) \quad \#\mathcal{Y}(\mathbb{Z}/l) \ll l^{\dim(\mathcal{Y}_{\mathbb{Q}})}.$$

All implied constants above depend only on \mathcal{Y} . The estimates of \mathbb{F}_p -points follow from the Weil conjecture [28, Theorem 7.7.1] ignoring finitely many primes. The existence of a \mathbb{Z}_p -point and the estimate of \mathbb{Z}/l -points follow from Hensel's Lemma.

2. HARDY-LITTLEWOOD PROPERTY AND EQUIDISTRIBUTION

The Hardy-Littlewood property signifies that the growth of integral points is quantified by the Tamagawa measure on the adelic space. In §2.1 we first recall the definition of normalised Tamagawa measures on adelic spaces and their restrictions on open subsets. In §2.2 we prove Proposition 2.3, which shows that to deduce the (APHL) property for X , it suffices to prove (2) for congruence neighbourhoods with respect to a specific subfamily of integral models of X . In §2.3 we prove Proposition 2.5, giving explicit error terms for (2) evaluated at a large family of adelic neighbourhoods. In §2.4 we exhibit two different applications of Proposition 2.5.

2.1. Normalised Tamagawa measures. We refer to [4, §1.6] and [31, §2] for details.

Let X be a smooth geometrically integral variety over \mathbb{Q} such that $X(\mathbf{A}) \neq \emptyset$ (which we assume throughout this article). A *gauge form* ω_X on X is a nowhere zero differential form of degree $\dim(X)$. For any place v of \mathbb{Q} , the gauge form ω_X induces a measure m_v^X on $X(\mathbb{Q}_v)$ ([31, §2.2]).

Let \mathcal{X} be an integral model of X over \mathbb{Z} . By the Lang-Weil estimate (10), we can choose an integer l (depending on \mathcal{X}) such that $\prod_{p \nmid l, p < \infty} \mathcal{X}(\mathbb{Z}_p) \neq \emptyset$. According to [31, Theorem 2.2.5], we have, for almost all finite places $v = p < \infty$,

$$(12) \quad m_p^X(\mathcal{X}(\mathbb{Z}_p)) := \int_{\mathcal{X}(\mathbb{Z}_p)} d m_p^X = \frac{\#\mathcal{X}(\mathbb{F}_p)}{p^{\dim X}}.^2$$

A set of *convergence factors* (λ_v) for (m_v^X) is a set of strictly positive real numbers indexed by the places of \mathbb{Q} such that the infinite product

$$(13) \quad \prod_{p \nmid l, p < \infty} \lambda_p^{-1} m_p^X(\mathcal{X}(\mathbb{Z}_p))$$

is absolutely convergent. The *Tamagawa measure* on the adelic space $X(\mathbf{A})$ corresponding to $(\omega_X, (\lambda_v))$ is defined as ([31, §2.3])

$$(14) \quad m^X := \prod_v \lambda_v^{-1} m_v^X.$$

This Tamagawa measure m^X is *normalized* if

$$(15) \quad \lambda_\infty \times \lim_{x \rightarrow \infty} \prod_{p \leq x} \lambda_p = 1.$$

The existence of the limit in (15) (and hence the existence of a normalized Tamagawa measure) is equivalent to the convergence of the infinite product $\prod_{p \nmid l, p < \infty} m_p^X(\mathcal{X}(\mathbb{Z}_p))$, and is independent of the choice of integral models \mathcal{X} of X (cf. [4, §1.6]). Denote by

$$(16) \quad m_f^X := \lambda_\infty^{-1} \prod_{p < \infty} \lambda_p^{-1} m_p^X$$

the finite part measure on $X(\mathbf{A}_f)$. If m^X is normalised with respect to a chosen set of convergence factors, then a different choice of convergent factors satisfying (15) induces the same measure as m^X (resp. m_f^X) on $X(\mathbf{A})$ (resp. $X(\mathbf{A}_f)$).

For any dense open subset $U \subset X$, for every place v , consider $m_v^U := m_v^X|_U$ the restricted v -adic Tamagawa measure on $U_v(\mathbb{Q}_v)$. For $v = \infty$, the set $(X - U)(\mathbb{R}) \subset X(\mathbb{R})$ is closed of lower dimension. Let $\phi : \mathcal{V} \rightarrow X(\mathbb{R})$ be a local diffeomorphism, where $\mathcal{V} \subset \mathbb{R}^{\dim X}$ is an open real neighbourhood. Then for any bounded real neighbourhood $B_{\mathbb{R}} \subset X(\mathbb{R})$, the Lebesgue measure of $\phi^{-1}(B_{\mathbb{R}} \cap (X - U)(\mathbb{R}))$ is zero. Therefore $m_\infty^X(B_{\mathbb{R}} \cap (X - U)(\mathbb{R})) = 0$ and

$$(17) \quad m_\infty^X(B_{\mathbb{R}}) = m_\infty^U(B_{\mathbb{R}} \cap U(\mathbb{R})).$$

The next proposition shows that, the restriction of m^X to U is well-defined, provided $\text{codim}_X(X - U) \geq 2$.

Proposition 2.1. *Suppose that $\text{codim}_X(X - U) \geq 2$. Then (λ_v) is a set of convergence factors for (m_v^U) , and the product measure $m^U := \prod_v \lambda_v^{-1} m_v^U$ is a normalized Tamagawa measure on $U(\mathbf{A})$ corresponding to $(\omega_X|_U, (\lambda_v))$.*

Proof. Let us fix \mathcal{X} an integral model of X over \mathbb{Z} . Let \mathcal{Z} be the Zariski closure of $Z := X - U$ in \mathcal{X} and let $\mathcal{U} := \mathcal{X} - \mathcal{Z}$ be the integral model of U . We fix $l' \in \mathbb{N}_{\neq 0}$ an integer such that \mathcal{X}, \mathcal{U}

²Although it will not be used in the sequel, the real part m_∞^X is closely related to the real Hardy-Littlewood density *à la Siegel*, at least when X is an affine complete intersection. See [4, 0.0.4].

are smooth over $\mathbb{Z}[1/l']$ and $\prod_{p \nmid l', p < \infty} \mathcal{U}(\mathbb{Z}_p) \neq \emptyset$. Take M_0 sufficiently large such that, for any $p \geq M_0$, we have $p \nmid l'$, and by (12),

$$(18) \quad m_p^X(\mathcal{X}(\mathbb{Z}_p)) = \#\mathcal{X}(\mathbb{F}_p)p^{-\dim X}, \quad m_p^X(\mathcal{U}(\mathbb{Z}_p)) = m_p^U(\mathcal{U}(\mathbb{Z}_p)) = \#\mathcal{U}(\mathbb{F}_p)p^{-\dim U}.$$

The Lang-Weil estimates (9) (10) show that

$$\frac{\#\mathcal{Z}(\mathbb{F}_p)}{\#\mathcal{U}(\mathbb{F}_p)} = O\left(\frac{1}{p^{\text{codim}_X(\mathcal{Z})}}\right)$$

uniformly for all p . Therefore for any $p \geq M_0$,

$$(19) \quad \frac{\lambda_p^{-1} m_p^X(\mathcal{X}(\mathbb{Z}_p))}{\lambda_p^{-1} m_p^X(\mathcal{U}(\mathbb{Z}_p))} = \frac{\#\mathcal{X}(\mathbb{F}_p)}{\#\mathcal{U}(\mathbb{F}_p)} = 1 + \frac{\#\mathcal{Z}(\mathbb{F}_p)}{\#\mathcal{U}(\mathbb{F}_p)} = 1 + O\left(\frac{1}{p^{\text{codim}_X(\mathcal{Z})}}\right).$$

This shows that the infinite product

$$\prod_{p \nmid l', p < \infty} \frac{\lambda_p^{-1} m_p^X(\mathcal{X}(\mathbb{Z}_p))}{\lambda_p^{-1} m_p^X(\mathcal{U}(\mathbb{Z}_p))}$$

is absolutely convergent. Since by construction of (λ_v) , (13) is absolutely convergent, we conclude that the infinite product

$$\prod_{p \nmid l', p < \infty} \lambda_p^{-1} m_p^U(\mathcal{U}(\mathbb{Z}_p))$$

is also absolutely convergent, which was to be shown. \square

Normalized Tamagawa measures exist on many homogeneous spaces. For instance, let G be a semisimple and simply connected algebraic group, $H \subset G$ be a connected reductive closed subgroup, and assume that $X \cong G/H$. Then by [31, p. 24 Corollary], there exists a G -invariant gauge form on X . In this case, we can construct a canonical Tamagawa measure m^X on $X(\mathbf{A})$ corresponding to a canonical choice of convergent factors (λ_v) such that m^X is normalised (cf. [4, §1.6.2]). If H is semisimple, we choose $\lambda_v = 1$ for all v , and in general, λ_v are defined in terms of some Artin L -functions.

2.2. Hardy-Littlewood property for congruence neighbourhoods of integral models.

Let $X \subset \mathbb{A}_{\mathbb{Q}}^n$ be a quasi-affine, smooth geometrically integral variety with a normalized Tamagawa measure $m^X = m_{\infty}^X \times m_f^X$ as in §2.1. Recall as in the introduction that we equip the affine space $\mathbb{A}^n(\mathbb{R}) = \mathbb{R}^n$ with an euclidean norm $\|\cdot\|$ in order to define the counting function (1), and we assume that $X(\mathbb{R})$ has no compact connected components.

Let $\mathcal{X} \subset \mathbb{A}_{\mathbb{Z}}^n$ be an integral model of X over \mathbb{Z} . Let $l \geq 2$ be an integer and $\xi := (\xi_p)_{p \mid l} \in \prod_{p \mid l} X(\mathbb{Q}_p) \subset \prod_{p \mid l} \mathbb{Q}_p^n$, (ξ, l) be a collection of local points. For each $p \mid l$, we define the p -adic neighbourhood

$$B_p^{\mathcal{X}}(\xi, l) := X(\mathbb{Q}_p) \cap (\xi_p + p^{v_p(l)}\mathbb{Z}_p^n) \subset \mathbb{Q}_p^n.$$

We then define the finite adelic subset

$$(20) \quad B_f^{\mathcal{X}}(\xi, l) := \prod_{p \mid l} B_p^{\mathcal{X}}(\xi, l) \times \prod_{p \nmid l} \mathcal{X}(\mathbb{Z}_p) \subset X(\mathbf{A}_f).$$

Definition 2.2. We say that $B_f^{\mathcal{X}}(\xi, l)$ is a compact congruence finite adelic neighbourhood of \mathcal{X} , or a *congruence neighbourhood* of \mathcal{X} for short, if it is non-empty and compact (by convention).

Our goal is to show that (cf. Proposition 2.3 below), the validity of formula (2) for all such congruence neighbourhoods $B_f^{\mathcal{X}}(\xi, l)$ ensures the Hardy-Littlewood property for X .

Note that $B_f^{\mathcal{X}}(\xi, l)$ is non-empty if and only if $\prod_{p|l} \mathcal{X}(\mathbb{Z}_p) \neq \emptyset$, and $B_f^{\mathcal{X}}(\xi, l)$ is compact if and only if $B_p^{\mathcal{X}}(\xi, l)$ is closed in $\xi_p + p^{v_p(l)}\mathbb{Z}_p^n$ (hence compact) for each $p|l$ because $\mathcal{X}(\mathbb{Z}_p)$ is compact for any p (cf. the proof of [7, Corollary 3.2]). Any non-empty finite adelic subset $B_f^{\mathcal{X}}(\xi, l)$ of \mathcal{X} of the form (20) becomes a congruence neighbourhood upon replacing l by some of its sufficiently large powers depending on \mathcal{X} and every ξ_p , and leaving ξ unchanged. Hence the family $\{B_f^{\mathcal{X}}(\xi, l)\}$ consisting of all congruence neighbourhoods of \mathcal{X} forms a topological base of $X(\mathbf{A}_f)$.

In many cases, congruence neighbourhoods can equivalently be defined via congruence conditions. For instance, when both X, \mathcal{X} are affine, i.e., both $X \subset \mathbb{A}_{\mathbb{Q}}^n$, $\mathcal{X} \subset \mathbb{A}_{\mathbb{Z}}^n$ are closed, then for any $l \geq 2$, $\bar{\xi} \in \mathcal{X}(\mathbb{Z}/l)$, suppose that there exists ξ a lift of $\bar{\xi}$ in $\prod_{p|l} \mathcal{X}(\mathbb{Z}_p) \subset \prod_{p|l} X(\mathbb{Q}_p)$, we have

$$(21) \quad B_p^{\mathcal{X}}(\xi, l) = \{\underline{\mathbf{X}} \in \mathcal{X}(\mathbb{Z}_p) : \underline{\mathbf{X}} \equiv \bar{\xi} \pmod{p^{v_p(l)}}\}.$$

Each $B_p^{\mathcal{X}}(\xi, l)$ is closed. Hence, if $B_f^{\mathcal{X}}(\xi, l)$ is non-empty, it is a congruence neighbourhood. We will make use of this fact in §3.2. In general if X, \mathcal{X} are quasi-affine, (21) still holds when l is sufficiently divisible (depending on \mathcal{X}).

Any $a \in \mathbb{N}_{\neq 0}$ defines a morphism

$$(22) \quad \begin{aligned} \phi_a : \mathbb{A}_{\mathbb{Z}}^n &\rightarrow \mathbb{A}_{\mathbb{Z}}^n \\ \mathbf{x} &\mapsto a \cdot \mathbf{x}, \end{aligned}$$

and $\phi_{a, \mathbb{Q}} : \mathbb{A}_{\mathbb{Q}}^n \rightarrow \mathbb{A}_{\mathbb{Q}}^n$ is an isomorphism. Let

$$(23) \quad X_a := \phi_a(X) \subset \mathbb{A}_{\mathbb{Q}}^n.$$

Then X_a is isomorphic to X , and is equipped with the normalised Tamagawa measure $m^{X_a} := (\phi_a)_* m^X$. Now we define an integral model \mathcal{X}_a of X_a . Let $\bar{X} \subset \mathbb{A}_{\mathbb{Q}}^n$, $\bar{\mathcal{X}} \subset \mathbb{A}_{\mathbb{Z}}^n$ be their corresponding Zariski closures, and assume that $\bar{\mathcal{X}}$ is defined by a family of polynomials $f_1, \dots, f_r \in \mathbb{Z}[x_1, \dots, x_n]$. Let $\bar{\mathcal{X}}_a \subset \mathbb{A}_{\mathbb{Z}}^n$ be the affine scheme defined by the polynomials

$$a^{\deg f_1} f_1 \left(\frac{1}{a} x_1, \dots, \frac{1}{a} x_n \right), \dots, a^{\deg f_r} f_r \left(\frac{1}{a} x_1, \dots, \frac{1}{a} x_n \right) \in \mathbb{Z}[x_1, \dots, x_n],$$

and let

$$(24) \quad \mathcal{X}_a := \overline{\bar{\mathcal{X}}_a} - \overline{\phi_a(\bar{X} - X)} \subset \mathbb{A}_{\mathbb{Z}}^n.^3$$

Then \mathcal{X}_a is an integral model of X_a and, in fact, $\overline{\bar{\mathcal{X}}_a}$ is the Zariski closure of X_a in $\mathbb{A}_{\mathbb{Z}}^n$. If $\mathcal{X} = \overline{\bar{\mathcal{X}} - X}$ (e.g. when both X, \mathcal{X} are affine), then ϕ_a induces an isomorphism $\mathcal{X} \times_{\mathbb{Z}} \mathbb{Z}[1/a] \cong \mathcal{X}_a \times_{\mathbb{Z}} \mathbb{Z}[1/a]$.

Proposition 2.3. *Let X be as in the beginning of §2.2 and let $\delta_X : X(\mathbf{A}) \rightarrow \mathbb{R}_{\geq 0}$ be a locally constant, not identically zero function. Then the following properties are equivalent.*

- (i) *The variety X is Hardy-Littlewood with density δ_X ;*
- (ii) *For any quasi-affine integral model $\mathcal{X} \subset \mathbb{A}_{\mathbb{Z}}^n$ of X , for any connected component $B_{\infty} \subset X(\mathbb{R})$ and for any congruence neighbourhood $B_f^{\mathcal{X}}(\xi, l)$ of \mathcal{X} , we have*

$$(25) \quad N_X(B_{\infty} \times B_f^{\mathcal{X}}(\xi, l); T) \sim \int_{B_{\infty}(T) \times B_f^{\mathcal{X}}(\xi, l)} \delta_X \, d m^X, \quad T \rightarrow \infty;$$

³The inner bar denotes the Zariski closure in $\mathbb{A}_{\mathbb{Q}}^n$, and the outer bar denotes the closure in $\mathbb{A}_{\mathbb{Z}}^n$.

(iii) There exists a quasi-affine integral model $\mathcal{X} \subset \mathbb{A}_{\mathbb{Z}}^n$ of X such that, for any connected component $B_{\infty} \subset X(\mathbb{R})$ and for any congruence neighbourhood $B_f^{\mathcal{X}}(\xi, l)$ of \mathcal{X} , the formula (25) holds ;

(iv) There exists a quasi-affine integral model $\mathcal{X} \subset \mathbb{A}_{\mathbb{Z}}^n$ of X such that, for any connected component $B_{\infty} \subset X(\mathbb{R})$, and for any sufficiently divisible $a \in \mathbb{N}_{\neq 0}$ (depending on \mathcal{X}) and any congruence neighbourhood $B_f^{\mathcal{X}_a}(\xi, l)$ of \mathcal{X}_a with $B_f^{\mathcal{X}_a}(\xi, l) \subset \overline{\mathcal{X}_a}(\widehat{\mathbb{Z}})$, we have

$$(26) \quad N_{X_a}(\phi_{a, \mathbb{R}}(B_{\infty}) \times B_f^{\mathcal{X}_a}(\xi, l); T) \sim \int_{\phi_{a, \mathbb{R}}(B_{\infty}(T)) \times B_f^{\mathcal{X}_a}(\xi, l)} \delta_{X_a} \, d m^{X_a},$$

where δ_{X_a} is the locally constant function $\delta_X \circ \phi_a^{-1} : X_a(\mathbf{A}) \rightarrow \mathbb{R}_{\geq 0}$.

Proof. The implications “(ii) \Rightarrow (iii)” and “(ii) \Rightarrow (iv)” are evident, and “(i) \Rightarrow (ii)” just follows from definition.

For “(iii) \Rightarrow (i)”, we need to show: for any connected component $B_{\infty} \subset X(\mathbb{R})$ and any compact open subset $B_f \subset X(\mathbf{A}_f)$, the formula (2) holds. Let \mathcal{X} be as in the assumption satisfying (25). Since B_f is compact, there exist finitely many congruence neighbourhoods $B_f^{\mathcal{X}}(\xi_i, l_i)$, $1 \leq i \leq r$ of \mathcal{X} such that $B_f = \cup_{i=1}^r B_f^{\mathcal{X}}(\xi_i, l_i)$. We can choose a sufficiently divisible $l_0 \in \mathbb{N}_{\neq 0}$ (depending on \mathcal{X} and B_f) with $\prod_{i=1}^r l_i \mid l_0$ such that each $B_f^{\mathcal{X}}(\xi_i, l_i)$ is covered by finitely many congruence neighbourhoods $B_f^{\mathcal{X}}(\xi_j, l_0)$ of \mathcal{X} . Since any two such neighbourhoods $B_f^{\mathcal{X}}(\xi_j, l_0)$ are either equal or disjoint, therefore we conclude that there exist finitely many congruence neighbourhoods $B_f^{\mathcal{X}}(\xi_k, l_0)$, $1 \leq k \leq r'$ such that $B_f = \prod_{k=1}^{r'} B_f^{\mathcal{X}}(\xi_k, l_0)$. Hence, assuming that (25) holds for all adelic neighbourhoods $B_{\infty} \times B_f^{\mathcal{X}}(\xi_k, l_0)$, $1 \leq k \leq r'$, then as $T \rightarrow \infty$,

$$\begin{aligned} N_X(B_{\infty} \times B_f; T) &= \sum_{k=1}^{r'} N_X(B_{\infty} \times B_f^{\mathcal{X}}(\xi_k, l_0); T) \\ &\sim \sum_{k=1}^{r'} \int_{B_{\infty}(T) \times B_f^{\mathcal{X}}(\xi_k, l_0)} \delta_X \, d m^X \\ &= \int_{B_{\infty}(T) \times B_f} \delta_X \, d m^X. \end{aligned}$$

This proves (2), whence X is Hardy-Littlewood with density δ_X .

Now we prove “(iv) \Rightarrow (iii)”. Let \mathcal{X} be as in the assumption. We may assume $\mathcal{X} = \overline{\mathcal{X}} - \overline{(\mathcal{X} - X)}$, which does not affect the definition of \mathcal{X}_a (24). Let $a_0 \in \mathbb{N}_{\neq 0}$ (depending on \mathcal{X}) be such that the assumption of (iv) holds for every \mathcal{X}_a with $a_0 \mid a$. Now for any congruence neighbourhood $B_f^{\mathcal{X}}(\xi, l)$ of \mathcal{X} , we can find $a_l \in \mathbb{N}_{\neq 0}$ (depending on $B_f^{\mathcal{X}}(\xi, l)$) with $l \mid a_l$ and $a_0 \mid a_l$ such that $\phi_{a_l}(\xi) \in \prod_{p \mid l} \overline{\mathcal{X}_{a_l}}(\mathbb{Z}_p)$. Note in particular $\prod_{p \mid l} \overline{\mathcal{X}_{a_l}}(\mathbb{Z}_p) \neq \emptyset$ by the definition of $B_f^{\mathcal{X}}(\xi, l)$. Then it is clear that $\overline{\mathcal{X}_{a_l}}(\widehat{\mathbb{Z}}) \neq \emptyset$, $B_f^{\mathcal{X}_{a_l}}(\phi_{a_l}(\xi), a_l l) \subset \overline{\mathcal{X}_{a_l}}(\widehat{\mathbb{Z}})$ is a congruence neighbourhood of \mathcal{X}_{a_l} , and ϕ_{a_l} induces an isomorphism $B_{\infty} \times B_f^{\mathcal{X}}(\xi, l) \cong \phi_{a_l, \mathbb{R}}(B_{\infty}) \times B_f^{\mathcal{X}_{a_l}}(\phi_{a_l}(\xi), a_l l)$. Therefore since $a_0 \mid a_l$,

applying (26) to $B_f^{X_{a_l}}(\phi_{a_l}(\xi), a_l l)$, as $T \rightarrow \infty$, we have

$$\begin{aligned} N_X(B_\infty \times B_f^X(\xi, l); T) &= N_{X_{a_l}}(\phi_{a_l, \mathbb{R}}(B_\infty) \times B_f^{X_{a_l}}(\phi_{a_l}(\xi), a_l l); a_l T) \\ &\sim \int_{(\phi_{a_l, \mathbb{R}}(B_\infty))(a_l T) \times B_f^{X_{a_l}}(\phi_{a_l}(\xi), a_l l)} \delta_{X_{a_l}} \, d m^{X_{a_l}} \\ &= \int_{B_\infty(T) \times B_f^X(\xi, l)} \delta_X \, d m^X. \end{aligned}$$

This proves (25) for the integral model \mathcal{X} . □

We end this subsection by the following lemma which will be helpful later.

Lemma 2.4. *Let X be as in the beginning of §2.2. Let $\delta_X : X(\mathbf{A}) \rightarrow \mathbb{R}_{\geq 0}$ be a locally constant function. Then we have:*

(i) *For any connected component $B_\infty \subset X(\mathbb{R})$, $\delta_X|_{B_\infty \times X(\mathbf{A}_f)}$ factors through a locally constant function $\delta_{B_\infty} : X(\mathbf{A}_f) \rightarrow \mathbb{R}_{\geq 0}$. In particular, for any compact open $B_f \subset X(\mathbf{A}_f)$, we have*

$$(27) \quad \int_{B_\infty(T) \times B_f} \delta_X \, d m^X = m_\infty^X(B_\infty(T)) \int_{B_f} \delta_{B_\infty} \, d m_f^X.$$

(ii) *Keeping the notation in (i). For any $l_0 \geq 2$ and any compact open subset $B_0 \subset \prod_{p|l_0} X(\mathbb{Q}_p)$, and for any quasi-affine integral model \mathcal{X} of X , on writing*

$$(28) \quad B_{f,0}^X := B_0 \times \prod_{p|l_0} \mathcal{X}(\mathbb{Z}_p) \subset X(\mathbf{A}_f),$$

there exists $l_1 \in \mathbb{N}_{\neq 0}$ depending only on B_∞, B_0 and \mathcal{X} with $l_0 \mid l_1$ such that, $\delta_{B_\infty}|_{B_{f,0}^X}$ factors through the projection $B_{f,0}^X \rightarrow B_0 \times \prod_{p|l_1, p \nmid l_0} \mathcal{X}(\mathbb{Z}_p)$.

Proof. (i) is clear because δ_X is locally constant and B_∞ is connected.

We now prove (ii). By the same arguments for the proof of “(iii) \Rightarrow (i)” of Proposition 2.3, there exists an $l_1 \in \mathbb{N}_{\neq 0}$ depending only on B_∞ and \mathcal{X} with $l_0 \mid l_1$ such that, the compact set $B_0 \times \prod_{p|l_0} \mathcal{X}(\mathbb{Z}_p)$ is a finite union of open subsets of the form $B_f^X(\xi_k, l_1)$, and the locally constant function δ_{B_∞} obtained in (i) is constant on each $B_f^X(\xi_k, l_1)$. Therefore $\delta_{B_\infty}|_{B_{f,0}^X}$ factors through $B_0 \times \prod_{p|l_1, p \nmid l_0} \mathcal{X}(\mathbb{Z}_p)$, whence the statement of (ii). □

2.3. Effective equidistribution with error terms. Let X be as in §2.2 with a fixed integral model \mathcal{X} over \mathbb{Z} . Fix $l_0 \in \mathbb{N}_{\neq 0}$ an integer (depending on \mathcal{X}) such that $\mathcal{X} \times_{\mathbb{Z}} \mathbb{Z}[1/l_0]$ is smooth with geometrically integral fibres over $\mathbb{Z}[1/l_0]$, and that $\prod_{p|l_0} \mathcal{X}(\mathbb{Z}_p) \neq \emptyset$. In the following proposition, we shall consider (30) as an effective version of the asymptotic formula (25) with an explicit uniform error term.

Let us first introduce some more notation. For any integer $l \geq 2$ with $\gcd(l, l_0) = 1$, the residue map

$$(29) \quad \Psi_l := \prod_{p|l} \Psi_p : \prod_{p|l} \mathcal{X}(\mathbb{Z}_p) \rightarrow \prod_{p|l} \mathcal{X}(\mathbb{Z}/p^{v_p(l)}) \cong \mathcal{X}(\mathbb{Z}/l)$$

is surjective by Hensel’s Lemma. Therefore, for any subset $S \subset \mathcal{X}(\mathbb{Z}/l)$, the set $\Psi_l^{-1}(S) \subset \prod_{p|l} \mathcal{X}(\mathbb{Z}_p)$ is compact.

Proposition 2.5. *Let $\delta_X : X(\mathbf{A}) \rightarrow \mathbb{R}_{\geq 0}$ be a locally constant, not identically zero function. Assume that for $B_0 \subset \prod_{p|l_0} X(\mathbb{Q}_p)$ a fixed compact open subset and $B_\infty \subset X(\mathbb{R})$ a fixed connected component, there exist $\sigma_X \geq 0$ and $0 < \beta_X < 1$ such that for any congruence neighbourhood $B_f^\mathcal{X}(\xi, l) \subset B_{f,0}^\mathcal{X}$ (cf. (28)), we have*

$$(30) \quad N_X(B_\infty \times B_f^\mathcal{X}(\xi, l); T) = \int_{B_\infty(T) \times B_f^\mathcal{X}(\xi, l)} \delta_X \, d m^X + O(l^{\sigma_X} m_\infty^X(B_\infty(T))^{1-\beta_X}),$$

where the implied constant of (30) may depend on $B_\infty, \mathcal{X}, B_0$ but it is independent of l and ξ . Then for any integer $l \geq 2$ with $\gcd(l, l_0) = 1$ and any subset $S \subset \mathcal{X}(\mathbb{Z}/l)$, we have

$$(31) \quad N_X(B_\infty \times B_{f,0}^\mathcal{X}(S, l); T) = \int_{B_\infty(T) \times B_{f,0}^\mathcal{X}(S, l)} \delta_X \, d m^X + O(\#S \cdot l^{\sigma_X} \cdot m_\infty^X(B_\infty(T))^{1-\beta_X})$$

where we write

$$(32) \quad B_{f,0}^\mathcal{X}(S, l) := B_0 \times \Psi_l^{-1}(S) \times \prod_{p|l_0} \mathcal{X}(\mathbb{Z}_p) \subset X(\mathbf{A}_f),$$

and the implied constant may depend on $B_\infty, \mathcal{X}, B_0$, but it is independent of l and S .

Proof. Since B_0 is compact, as in the proof of Proposition 2.3 (see also the discussion after Definition 2.2), we can choose an integer l_1 (depending only on B_0, \mathcal{X}) with the same prime factors as l_0 and finitely many congruence neighbourhoods $(B_f^\mathcal{X}(\xi_i, l_1))_{i \in I}$ of \mathcal{X} such that

$$(33) \quad B_0 = \prod_{i \in I} \left(\prod_{p|l_0} B_p^\mathcal{X}(\xi_i, l_1) \right).$$

Now let us fix $l \geq 2$ with $\gcd(l, l_0) = 1$ and $S \subset \mathcal{X}(\mathbb{Z}/l)$. For every $\bar{s} \in S$, we fix a lift $s \in \Psi_l^{-1}(\{\bar{s}\}) \subset \prod_{p|l} \mathcal{X}(\mathbb{Z}_p)$. Then for such s and any $i \in I$, we have $\xi_i \times s \in \prod_{p|l_1 l} \mathcal{X}(\mathbb{Q}_p)$, and $B_f^\mathcal{X}(\xi_i \times s, l_1 l)$ is a congruence neighbourhood of \mathcal{X} . Moreover, thanks to (33), we have

$$B_{f,0}^\mathcal{X}(S, l) = \prod_{i \in I, \bar{s} \in S} B_f^\mathcal{X}(\xi_i \times s, l_1 l).$$

Hence the formula (30) implies

$$\begin{aligned} N_X(B_\infty \times B_{f,0}^\mathcal{X}(S, l); T) &= \sum_{i \in I, \bar{s} \in S} \left(\int_{B_\infty(T) \times B_f^\mathcal{X}(\xi_i \times s, l_1 l)} \delta_X \, d m^X + O((l_1 l)^{\sigma_X} m_\infty^X(B_\infty(T))^{1-\beta_X}) \right) \\ &= \int_{B_\infty(T) \times B_{f,0}^\mathcal{X}(S, l)} \delta_X \, d m^X + O(\#I \cdot \#S \cdot (l_1 l)^{\sigma_X} m_\infty^X(B_\infty(T))^{1-\beta_X}). \end{aligned}$$

Since $\#I, l_1$ only depend on B_0, \mathcal{X} , the desired formula (31) then follows, where the implied constant is only allowed to depend on $B_\infty, \mathcal{X}, B_0$. \square

2.4. Two applications of effective equidistribution. Let $Z \subset X$ be a closed subset with $\text{codim}_X(Z) \geq 2$, and let $U := X - Z$ be the open subset. Let \mathcal{Z} be the Zariski closure of Z in \mathcal{X} as an integral model of Z and let $\mathcal{U} := \mathcal{X} - \mathcal{Z}$ be the integral model of U . By Proposition 2.1, the restriction $m^U := m^X|_{\mathcal{U}}$ is well-defined as a normalised Tamagawa measure on $U(\mathbf{A})$.

Our first application of Proposition 2.5 consists in comparing the counting function of integral points on \mathcal{X} lying in \mathcal{U} modulo any $p < M, p \nmid l_0$, with the Tamagawa measure of the corresponding adelic neighbourhood of $U(\mathbf{A})$, when M is sufficiently large. We shall make use of this result in the deduction of (APHL) in Theorem 3.1.

For any integer $l \geq 2$ with $\gcd(l, l_0) = 1$, recall (29) and take $\mathcal{U}(\mathbb{Z}/l) \subset \mathcal{X}(\mathbb{Z}/l)$. Then

$$(34) \quad \Psi_l^{-1}(\mathcal{U}(\mathbb{Z}/l)) = \prod_{p|l} \mathcal{U}(\mathbb{Z}_p),$$

because $\Psi_p^{-1}(\mathcal{U}(\mathbb{Z}/p^{v_p(l)})) = \mathcal{U}(\mathbb{Z}_p)$ for every $p | l$.⁴ For any $M > 2$, we note

$$(35) \quad \mathfrak{P}_{M, l_0} := \prod_{p|l_0, p < M} p.$$

Hence with the notation (32),

$$(36) \quad \begin{aligned} B_{f,0}^{\mathcal{X}}(\mathcal{U}(\mathbb{Z}/\mathfrak{P}_{M, l_0}), \mathfrak{P}_{M, l_0}) &= B_0 \times \Psi_l^{-1}(\mathcal{U}(\mathbb{Z}/\mathfrak{P}_{M, l_0})) \times \prod_{p \geq M} \mathcal{X}(\mathbb{Z}_p) \\ &= B_0 \times \prod_{p|l_0, p < M} \mathcal{U}(\mathbb{Z}_p) \times \prod_{p \geq M} \mathcal{X}(\mathbb{Z}_p). \end{aligned}$$

Corollary 2.6. *Under the hypothesis of Proposition 2.5, we assume moreover $B_0 \subset \prod_{p|l_0} U(\mathbb{Q}_p)$ and $\prod_{p|l_0} \mathcal{U}(\mathbb{Z}_p) \neq \emptyset$. Then, for any $M > 2$ sufficiently large, we have*

$$\begin{aligned} &N_X(B_\infty \times B_{f,0}^{\mathcal{X}}(\mathcal{U}(\mathbb{Z}/\mathfrak{P}_{M, l_0}), \mathfrak{P}_{M, l_0}); T) \\ &= \int_{B_\infty^U(T) \times B_{f,0}^{\mathcal{U}}} \delta_X|_U \, d m^U + O\left(\mathfrak{P}_{M, l_0}^{\sigma_X + \dim(X)} m_\infty^X(B_\infty(T))^{1-\beta_X} + \frac{m_\infty^X(B_\infty(T))}{M^{\text{codim}_X(Z)-1}}\right), \end{aligned}$$

where $B_{f,0}^{\mathcal{U}} \subset U(\mathbf{A}_f)$ is defined analogously to (28) in terms of \mathcal{U} ,

$$(37) \quad B_\infty^U(T) := B_\infty(T) \cap U(\mathbb{R}),$$

and the implied constant may depend on $B_\infty, B_0, \mathcal{U}, \mathcal{X}$ but it is independent of M .

Proof. On applying Lemma 2.4 (i) to δ_X , let $\delta_{B_\infty} : X(\mathbf{A}_f) \rightarrow \mathbb{R}_{\geq 0}$ be the factorisation of δ_X . Using (27), we then consider

$$I_{\mathcal{X}}(\mathcal{U}(\mathbb{Z}/\mathfrak{P}_{M, l_0})) := \int_{B_\infty(T) \times B_{f,0}^{\mathcal{X}}(\mathcal{U}(\mathbb{Z}/\mathfrak{P}_{M, l_0}), \mathfrak{P}_{M, l_0})} \delta_X \, d m^X = m_\infty^X(B_\infty(T)) \int_{B_{f,0}^{\mathcal{X}}(\mathcal{U}(\mathbb{Z}/\mathfrak{P}_{M, l_0}), \mathfrak{P}_{M, l_0})} \delta_{B_\infty} \, d m_f^X,$$

and

$$(38) \quad I_{\mathcal{U}} := \int_{B_\infty^U(T) \times B_{f,0}^{\mathcal{U}}} \delta_X|_U \, d m^U = m_\infty^U(B_\infty^U(T)) \int_{B_{f,0}^{\mathcal{U}}} \delta_{B_\infty}|_U \, d m_f^U.$$

The formula (17) shows that

$$m_\infty^X(B_\infty(T)) = m_\infty^U(B_\infty^U(T)).$$

Our goal is to compare the finite part integral in $I_{\mathcal{X}}(\mathcal{U}(\mathbb{Z}/\mathfrak{P}_{M, l_0}))$ with the one in $I_{\mathcal{U}}$.

We apply Lemma 2.4 (ii) to the locally constant function δ_{B_∞} , and let $l_1 \geq 2$ (depending only on $B_\infty, B_0, \mathcal{X}$) be an integer satisfying (ii) for B_∞ and B_0 . We take $M_1 > 2$ (depending only on $B_\infty, B_0, \mathcal{X}$) such that $p \geq M_1 \Rightarrow p \nmid l_1$ (for example, we can take $M_1 = l_1 + 1$). Then, for any $M \geq M_1$, when restricted to $B_{f,0}^{\mathcal{X}}$, δ_{B_∞} factors through a locally constant function

$$(39) \quad \delta_{B_\infty}^M : B_{0, M}^{\mathcal{X}} := B_0 \times \prod_{p < M, p|l_0} \mathcal{X}(\mathbb{Z}_p) \rightarrow \mathbb{R}_{\geq 0},$$

⁴Indeed, any $\mathbb{Z}/p^{v_p(l)}$ -point, say P , can be lifted to a \mathbb{Z}_p -point \tilde{P} of \mathcal{X} , by Hensel's lemma. Let \bar{P} be the image of P in $\mathcal{U}(\mathbb{F}_p)$. Then since the open set \mathcal{U} contains the closed point \bar{P} of $\text{Spec}(\mathbb{Z}_p)$, \mathcal{U} must contain its generic point, and therefore $\tilde{P} \in \mathcal{U}$.

i.e., $\delta_{B_\infty}|_{B_{f,0}^\mathcal{X}}$ is the composition of $\delta_{B_\infty}^M$ with projection $B_{f,0}^\mathcal{X} \rightarrow B_{0,M}^\mathcal{X}$. Therefore thanks to (36), on defining $B_{0,M}^\mathcal{U} := B_0 \times \prod_{p|l_0, p < M} \mathcal{U}(\mathbb{Z}_p) \subset B_{0,M}^\mathcal{X}$, we have

$$B_{f,0}^\mathcal{X}(\mathcal{U}(\mathbb{Z}/\mathfrak{P}_{M,l_0}), \mathfrak{P}_{M,l_0}) = B_{0,M}^\mathcal{U} \times \prod_{p \geq M} \mathcal{X}(\mathbb{Z}_p), \quad B_{f,0}^\mathcal{U} = B_{0,M}^\mathcal{U} \times \prod_{p \geq M} \mathcal{U}(\mathbb{Z}_p).$$

On defining moreover the measures

$$\mathfrak{m}_M^\mathcal{X} := \lambda_\infty^{-1} \prod_{p < M} \lambda_p^{-1} \mathfrak{m}_p^\mathcal{X}, \quad \text{and} \quad \mathfrak{m}_M^\mathcal{U} := \mathfrak{m}_M^\mathcal{X}|_{\mathcal{U}},$$

on respectively $\prod_{p < M} X(\mathbb{Q}_p)$ and $\prod_{p < M} U(\mathbb{Q}_p)$, and using (39), we can now compute:

$$I_\mathcal{X}(\mathcal{U}(\mathbb{Z}/\mathfrak{P}_{M,l_0})) = \mathfrak{m}_\infty^\mathcal{X}(B_\infty(T)) \left(\int_{B_{0,M}^\mathcal{U}} \delta_{B_\infty}^M d\mathfrak{m}_M^\mathcal{X} \right) \left(\prod_{p \geq M} \lambda_p^{-1} \mathfrak{m}_p^\mathcal{X}(\mathcal{X}(\mathbb{Z}_p)) \right),$$

and

$$I_\mathcal{U} = \mathfrak{m}_\infty^\mathcal{U}(B_\infty(T)) \left(\int_{B_{0,M}^\mathcal{U}} \delta_{B_\infty}^M|_{\mathcal{U}} d\mathfrak{m}_M^\mathcal{U} \right) \left(\prod_{p \geq M} \lambda_p^{-1} \mathfrak{m}_p^\mathcal{U}(\mathcal{U}(\mathbb{Z}_p)) \right).$$

By definition,

$$(40) \quad \int_{B_{0,M}^\mathcal{U}} \delta_{B_\infty}^M d\mathfrak{m}_M^\mathcal{X} = \int_{B_{0,M}^\mathcal{U}} \delta_{B_\infty}^M|_{\mathcal{U}} d\mathfrak{m}_M^\mathcal{U}.$$

It remains to compare the infinite products. According to (19) as in the proof of Proposition 2.1, if $M > M_0$, on writing $c_p := \frac{\#\mathcal{Z}(\mathbb{F}_p)}{\#\mathcal{U}(\mathbb{F}_p)}$,

$$\prod_{p \geq M} (1 + c_p) = \exp \left(\sum_{p \geq M} \log(1 + c_p) \right) = 1 + O \left(\sum_{p \geq M} \frac{1}{p^{\text{codim}_X(Z)}} \right) = 1 + O \left(\frac{1}{M^{\text{codim}_X(Z)-1}} \right).$$

We also deduce from (38) that $I_\mathcal{U} = O(\mathfrak{m}_\infty^\mathcal{X}(B_\infty(T)))$. Gathering together the computations above, we obtain that, for any $M > \max(M_0, M_1)$,

$$(41) \quad I_\mathcal{X}(\mathcal{U}(\mathbb{Z}/\mathfrak{P}_{M,l_0})) = I_\mathcal{U} \prod_{p \geq M} (1 + c_p) = I_\mathcal{U} + O \left(\frac{I_\mathcal{U}}{M^{\text{codim}_X(Z)-1}} \right) = I_\mathcal{U} + O \left(\frac{\mathfrak{m}_\infty^\mathcal{X}(B_\infty(T))}{M^{\text{codim}_X(Z)-1}} \right).$$

The implied constants above depend only on $B_\infty, B_0, \mathcal{U}, \mathcal{X}$.

Finally, we apply Proposition 2.5 with

$$l := \mathfrak{P}_{M,l_0}, \quad S := \mathcal{U}(\mathbb{Z}/\mathfrak{P}_{M,l_0}).$$

The Lang-Weil estimate (11) for \mathcal{U} implies

$$\#S = \#\mathcal{U}(\mathbb{Z}/\mathfrak{P}_{M,l_0}) = O(\mathfrak{P}_{M,l_0}^{\dim(X)}).$$

Therefore using (41), we get,

$$\begin{aligned} N_X(B_\infty \times B_{f,0}^\mathcal{X}(\mathcal{U}(\mathbb{Z}/\mathfrak{P}_{M,l_0}), \mathfrak{P}_{M,l_0}); T) &= I_\mathcal{X}(\mathcal{U}(\mathbb{Z}/\mathfrak{P}_{M,l_0})) + O(\mathfrak{P}_{M,l_0}^{\sigma_X + \dim(X)} \mathfrak{m}_\infty^\mathcal{X}(B_\infty(T))^{1-\beta_X}) \\ &= I_\mathcal{U} + O \left(\mathfrak{P}_{M,l_0}^{\sigma_X + \dim(X)} \mathfrak{m}_\infty^\mathcal{X}(B_\infty(T))^{1-\beta_X} + \frac{\mathfrak{m}_\infty^\mathcal{X}(B_\infty(T))}{M^{\text{codim}_X(Z)-1}} \right), \end{aligned}$$

where the implied constant depends only on $B_\infty, B_0, \mathcal{U}, \mathcal{X}$. This finishes the proof. \square

Our second application is to estimate integral points on \mathcal{X} lying in \mathcal{Z} modulo any single sufficiently big prime. This is used in Section 4.1 for the treatment of error terms.

Corollary 2.7. *Under the hypothesis of Proposition 2.5, we have, uniformly for any prime p_0 sufficiently large,*

$$N_X(B_\infty \times B_{f,0}^X(\mathcal{Z}(\mathbb{F}_{p_0}), p_0); T) = O\left(\frac{m_\infty^X(B_\infty(T))}{p_0^{\text{codim}_X(Z)}} + p_0^{\sigma_X + \dim Z} m_\infty^X(B_\infty(T))^{1-\beta_X}\right),$$

where the implied constant may depend on $B_\infty, B_0, \mathcal{X}, \mathcal{Z}$, but it is independent of p_0 .

Proof. We keep using the constant $M_1 > 2$ (depending only on $B_\infty, B_0, \mathcal{X}$) in the proof of Corollary 2.6 such that, $\delta_X|_{B_\infty \times B_{f,0}^X}$ is the composition of the locally constant function $\delta_{B_\infty}^{M_1}$ (39) with projection $B_\infty \times B_{f,0}^X \rightarrow B_{0,M_1}^X$.

We also recall the constant M_0 in the proof of Proposition 2.1 depending only on \mathcal{X}, \mathcal{U} such that that (18) holds for any $p > M_0$. Consequently, for any such p , thanks to (34), we compute

$$\begin{aligned} m_p^X(\Psi_p^{-1}(\mathcal{Z}(\mathbb{F}_p))) &= m_p^X(\mathcal{X}(\mathbb{Z}_p)) - m_p^X(\Psi_p^{-1}(\mathcal{U}(\mathbb{F}_p))) \\ &= m_p^X(\mathcal{X}(\mathbb{Z}_p)) - m_p^X(\mathcal{U}(\mathbb{Z}_p)) \\ &= \frac{\#\mathcal{X}(\mathbb{F}_p) - \#\mathcal{U}(\mathbb{F}_p)}{p^{\dim X}} = \frac{\#\mathcal{Z}(\mathbb{F}_p)}{p^{\dim X}}. \end{aligned}$$

Therefore, by the Lang-Weil estimates (9) (10),

$$\frac{m_p^X(\Psi_p^{-1}(\mathcal{Z}(\mathbb{F}_p)))}{m_p^X(\mathcal{X}(\mathbb{Z}_p))} = \frac{\#\mathcal{Z}(\mathbb{F}_p)}{\#\mathcal{X}(\mathbb{F}_p)} = O\left(\frac{1}{p^{\text{codim}_X(Z)}}\right),$$

the implied constant depending only on \mathcal{X}, \mathcal{Z} .

For any prime $p_0 \nmid l_0$, consider

$$I_{\mathcal{X}}(\mathcal{Z}(\mathbb{F}_{p_0})) := \int_{B_\infty(T) \times B_{f,0}^X(\mathcal{Z}(\mathbb{F}_{p_0}), p_0)} \delta_X \, d m^X = m_\infty^X(B_\infty(T)) \int_{B_{f,0}^X(\mathcal{Z}(\mathbb{F}_{p_0}), p_0)} \delta_{B_\infty} \, d m_f^X.$$

Then if $p_0 > \max(M_0, M_1)$, following the proof of Corollary 2.6 above, we have

$$\begin{aligned} I_{\mathcal{X}}(\mathcal{Z}(\mathbb{F}_{p_0})) &= m_\infty^X(B_\infty(T)) \left(\int_{B_{0,M_1}^X} \delta_{B_\infty}^{M_1} \, d m_{M_1}^X \right) \left(\prod_{p \geq M_1, p \neq p_0} \lambda_p^{-1} m_p^X(\mathcal{X}(\mathbb{Z}_p)) \right) \lambda_{p_0}^{-1} m_{p_0}^X(\Psi_{p_0}^{-1}(\mathcal{Z}(\mathbb{F}_{p_0}))) \\ &= m_\infty^X(B_\infty(T)) \left(\int_{B_{0,M_1}^X} \delta_{B_\infty}^{M_1} \, d m_{M_1}^X \right) \left(\prod_{p \geq M_1} \lambda_p^{-1} m_p^X(\mathcal{X}(\mathbb{Z}_p)) \right) \frac{\lambda_{p_0}^{-1} m_{p_0}^X(\Psi_{p_0}^{-1}(\mathcal{Z}(\mathbb{F}_{p_0})))}{\lambda_{p_0}^{-1} m_{p_0}^X(\mathcal{X}(\mathbb{Z}_{p_0}))} \\ &= \int_{B_\infty(T) \times B_{f,0}^X} \delta_X \, d m^X \times \frac{m_{p_0}^X(\Psi_{p_0}^{-1}(\mathcal{Z}(\mathbb{F}_{p_0})))}{m_{p_0}^X(\mathcal{X}(\mathbb{Z}_{p_0}))} \\ &= O\left(\frac{m_\infty^X(B_\infty(T))}{p_0^{\text{codim}_X(Z)}}\right), \end{aligned}$$

where the implied constant depends only on $B_\infty, B_0, \mathcal{X}, \mathcal{Z}$.

Finally applying Proposition 2.5 to $l := p_0$ and using again the Lang-Weil estimate (9) for \mathcal{Z} , we get

$$\begin{aligned} N_X(B_\infty \times B_{f,0}^X(\mathcal{Z}(\mathbb{F}_{p_0}), p_0); T) &= I_{\mathcal{X}}(\mathcal{Z}(\mathbb{F}_{p_0})) + O(\#\mathcal{Z}(\mathbb{F}_{p_0}) \cdot p_0^{\sigma_X} m_\infty^X(B_\infty(T))^{1-\beta_X}) \\ &= O\left(\frac{m_\infty^X(B_\infty(T))}{p_0^{\text{codim}_X(Z)}} + p_0^{\sigma_X + \dim Z} m_\infty^X(B_\infty(T))^{1-\beta_X}\right). \end{aligned}$$

This finishes the proof. \square

3. (APHL) FOR AFFINE VARIETIES AND HOMOGENEOUS SPACES

The structure of this section is as follows. In §3.1 we prove Theorem 3.1 which outlines hypotheses (i) and (ii) as a sufficient condition for (APHL). Then in §3.2 we show – Theorem 3.4 – the compatibility of the hypothesis (i) with works on effective equidistribution of integral points on nice affine homogeneous spaces which are proven to be Hardy-Littlewood (Theorems 3.2 & 3.3), and we then state (Corollary 3.5) that to show (APHL) for such varieties it remains to establish the geometric sieve condition. In 3.3 we prove Theorems 1.4 and 1.5.

3.1. A sufficient condition for (APHL). In this subsection, let $X \subset \mathbb{A}_{\mathbb{Q}}^n$ be an affine, smooth geometrically integral variety with a normalized Tamagawa measure $m^X = m_{\infty}^X \times m_f^X$. Let $\delta_X : X(\mathbf{A}) \rightarrow \mathbb{R}_{\geq 0}$ be a locally constant, not identically zero function.

We recall the morphism ϕ_a (22) and the variety X_a (23), together with the measure m^{X_a} and the locally constant function δ_{X_a} defined on $X_a(\mathbf{A})$. We have seen in §2.2 that if

$$\mathcal{X} = \text{Spec}(\mathbb{Z}[x_1, \dots, x_n]/(f_1(\mathbf{x}), \dots, f_r(\mathbf{x})))$$

is an affine integral model of X over \mathbb{Z} , then the integral model \mathcal{X}_a “twisted by $a \in \mathbb{N}_{\neq 0}$ ” for $X_a \cong_{\mathbb{Q}} X$ is

$$(42) \quad \mathcal{X}_a = \text{Spec}(\mathbb{Z}[x_1, \dots, x_n]/(a^{\deg f_1} f_1(a^{-1}\mathbf{x}), \dots, a^{\deg f_r} f_r(a^{-1}\mathbf{x}))).$$

We observe that the condition $\mathcal{X}(\mathbb{Z}) \neq \emptyset$ implies that $\mathcal{X}_a(\mathbb{Z}) \neq \emptyset$ for any $a \in \mathbb{N}_{\neq 0}$. Note also that, if $X(\mathbb{Q}) \neq \emptyset$, then for any sufficiently divisible $a \in \mathbb{N}_{\neq 0}$ (depending on \mathcal{X}), we have $\mathcal{X}_a(\mathbb{Z}) \neq \emptyset$.

Theorem 3.1. *Suppose that X is an affine Hardy-Littlewood variety with density δ_X and $X(\mathbb{Q}) \neq \emptyset$. Assume that there is an affine integral model $\mathcal{X} \subset \mathbb{A}_{\mathbb{Z}}^n$ of X with the following property: for any sufficiently divisible $a \in \mathbb{N}_{\neq 0}$ (depending on \mathcal{X}), the integral model \mathcal{X}_a (cf. (42)) satisfies:*

(i) (Strong effective equidistribution condition) *There exist $\sigma_{\mathcal{X}_a} \geq 0$ and $0 < \beta_{\mathcal{X}_a} < 1$ such that, for any connected component $B_{\infty} \subset X(\mathbb{R})$ and for any congruence neighbourhood $B_f^{\mathcal{X}_a}(\xi, l) \subset \mathcal{X}_a(\widehat{\mathbb{Z}})$ of \mathcal{X}_a ,*

$$(43) \quad \begin{aligned} & N_{X_a}(\phi_{a, \mathbb{R}}(B_{\infty}) \times B_f^{\mathcal{X}_a}(\xi, l); T) \\ &= \int_{\phi_{a, \mathbb{R}}(B_{\infty}(T)) \times B_f^{\mathcal{X}_a}(\xi, l)} \delta_{X_a} \, d m^{X_a} + O(l^{\sigma_{\mathcal{X}_a}} m_{\infty}^X (B_{\infty}(T))^{1-\beta_{\mathcal{X}_a}}), \quad T \rightarrow \infty, \end{aligned}$$

where the implied constant of (43) may depend on $B_{\infty}, \mathcal{X}_a$, but it is independent of l and ξ ;

(ii) (Geometric sieve condition) *For any closed subset $Z \subset X$ with $\text{codim}_X(Z) \geq 2$ and $\mathcal{Z}_a := \overline{Z} \subset \mathcal{X}_a$, there exist continuous functions $f_1, f_2 : \mathbb{R}_{>0} \rightarrow \mathbb{R}_{>0}$ such that $f_i(x) = o(1)$ as $x \rightarrow \infty$ for $i = 1, 2$, such that, for every connected component $B_{\infty} \subset X(\mathbb{R})$, for every $M > 0$,*

$$(44) \quad \begin{aligned} & \#\{\mathbf{X} \in \mathcal{X}_a(\mathbb{Z}) : \mathbf{X} \in \phi_{a, \mathbb{R}}(B_{\infty}(T)), \exists p \geq M, \mathbf{X} \bmod p \in \mathcal{Z}_a\} \\ &= O((f_1(M) + f_2(T)) m_{\infty}^X(B_{\infty}(T))), \quad T \rightarrow \infty, \end{aligned}$$

where the implied constant may depend on $B_{\infty}, \mathcal{X}_a$ and Z , but it is independent of M .

Then the variety X satisfies (APHL) with density δ_X .

Remarks.

- (1) If X in the theorem above is strongly Hardy-Littlewood, i.e. $\delta_X \equiv 1$, then X satisfies (APSHL).

- (2) Our formulation of (44) is compatible with currently known works on geometric sieves [27, 3, 6].

Proof of Theorem 3.1. Step I: Let us fix an open subset $U \subset X$ with $Z := X \setminus U$ and $\text{codim}_X(Z) \geq 2$. The map

$$B_\infty \mapsto B_\infty^U := B_\infty \cap U(\mathbb{R})$$

is a bijection between connected components of $X(\mathbb{R})$ and $U(\mathbb{R})$, because $\text{codim}_X(Z) \geq 2$. We henceforth fix a B_∞ from now on, which amounts to fixing a connected component B_∞^U of $U(\mathbb{R})$. Apply Lemma 2.4 (i) to δ_X and let $\delta_{B_\infty} : X(\mathbf{A}_f) \rightarrow \mathbb{R}_{\geq 0}$ be the resulting locally constant function for B_∞ . We may assume that $B_\infty \cap X(\mathbb{Q}) \neq \emptyset$, because otherwise the assumption that X being Hardy-Littlewood implies that δ_{B_∞} is identically zero on $X(\mathbf{A}_f)$, and the formula (1) for U trivially holds for $B_\infty^U \times B_f^U$, where $B_f^U \subset U(\mathbf{A}_f)$ is any compact open subset.

Let \mathcal{X} be as in the assumption of the theorem. For any $a \in \mathbb{N}_{\neq 0}$, we define the integral model $\mathcal{U}_a := \mathcal{X}_a \setminus \overline{Z} \subset \mathcal{X}_a$ for U_a , where \overline{Z} stands for the Zariski closure of Z in \mathcal{X}_a . Let $a_0 \in \mathbb{N}_{\neq 0}$ be such that the hypotheses (i) and (ii) hold for every \mathcal{X}_a with $a_0 \mid a$. Since $X(\mathbb{Q}) \neq \emptyset$, we can find $a_1 \in \mathbb{N}_{\neq 0}$ (depending on \mathcal{X}) with $a_0 \mid a_1$ such that $\mathcal{X}_{a_1}(\mathbb{Z}) \neq \emptyset$. In particular $\mathcal{X}_a(\mathbb{Z}) \neq \emptyset$ for every $a_1 \mid a$. We take $l_0 \in \mathbb{N}_{\neq 0}$ with $a_1 \mid l_0$ depending only on \mathcal{X}, \mathcal{U} such that $\mathcal{X} \times_{\mathbb{Z}} \mathbb{Z}[1/l_0], \mathcal{U} \times_{\mathbb{Z}} \mathbb{Z}[1/l_0]$ are smooth with geometrically integral fibres over $\mathbb{Z}[1/l_0]$, and that $\prod_{p|l_0} \mathcal{U}(\mathbb{Z}_p) \neq \emptyset$ (by the Lang-Weil estimate (10)). We want to show that, for any a with $l_0 \mid a$, the formula (26) holds for the connected component $\phi_{a,\mathbb{R}}(B_\infty^U)$ of $U_a(\mathbb{R})$ and for any congruence neighbourhood $B_f^{\mathcal{U}_a}(\xi, l)$ (20) of \mathcal{U}_a inside $\mathcal{X}_a(\widehat{\mathbb{Z}})$ ⁵. Upon breaking $B_f^{\mathcal{U}_a}(\xi, l)$ into a finite number of smaller congruence neighbourhoods as in the proof of Proposition 2.3, we may assume $a \mid l$ (hence $l_0 \mid l$), since $\prod_{p|l_0} \mathcal{U}_a(\mathbb{Z}_p) \neq \emptyset$. Granting this, Theorem 3.1 then follows from Proposition 2.3.

Step II: In the remaining of the proof, we work with a fixed \mathcal{X}_a with $l_0 \mid a$ and a fixed congruence neighbourhood $B_f^{\mathcal{U}_a}(\xi_1, l_1) \subset \mathcal{X}_a(\widehat{\mathbb{Z}})$ of \mathcal{U}_a with $a \mid l_1$. Recall that ϕ_a induces an isomorphism $\mathcal{X}_a \times_{\mathbb{Z}} \mathbb{Z}[1/a] \cong \mathcal{X} \times_{\mathbb{Z}} \mathbb{Z}[1/a]$ and the same holds for \mathcal{U}_a . Therefore, $\mathcal{X}_a \times_{\mathbb{Z}} \mathbb{Z}[1/l_1]$ and $\mathcal{U}_a \times_{\mathbb{Z}} \mathbb{Z}[1/l_1]$ are smooth with geometrically integral fibres. By abuse of notation, we shall drop the subscript a in what follows.

If δ_X is identically zero on $B_\infty \times B_f^{\mathcal{U}}(\xi_1, l_1)$, since δ_X is locally constant, there exist compact open subsets $B_i \subset X(\mathbf{A}_f)$ such that $B_f^{\mathcal{U}}(\xi_1, l_1) \subset \cup_i B_i$, and δ_{B_∞} is identically zero on all B_i . The variety X being Hardy-Littlewood implies that $N_X(B_\infty \times B_i; T) = 0$ by (2) and (27), which means that $X(\mathbb{Q}) \cap (B_\infty \times B_i) = \emptyset$ by (1). We henceforth get $U(\mathbb{Q}) \cap (B_\infty^U \times B_f^{\mathcal{U}}(\xi_1, l_1)) \subset X(\mathbb{Q}) \cap (\cup_i (B_\infty \times B_i)) = \emptyset$. In this case, the formula (26) trivially holds.

From now on, we may assume δ_X is not identically zero on $B_\infty \times B_f^{\mathcal{U}}(\xi_1, l_1)$. Then by (17) and (27),

$$(45) \quad \int_{B_\infty^U(T) \times B_f^{\mathcal{U}}(\xi_1, l_1)} \delta_X|_U \, d m^U = m_\infty^X(B_\infty^X(T)) \int_{B_f^{\mathcal{U}}(\xi_1, l_1)} \delta_{B_\infty}|_U \, d m_f^U,$$

where $B_\infty^U(T)$ is defined by (37) and the finite part integral $\int_{B_f^{\mathcal{U}}(\xi_1, l_1)} \delta_{B_\infty}|_U \, d m_f^U$ is non-zero.

We take

$$(46) \quad B_0 := \prod_{p|l_1} B_p^{\mathcal{U}}(\xi_1, l_1) \subset \prod_{p|l_1} \mathcal{X}(\mathbb{Z}_p) \cap \prod_{p|l_1} U(\mathbb{Q}_p),$$

⁵Note that $\overline{\mathcal{U}_a} = \mathcal{X}_a = \overline{\mathcal{X}_a}$ in this affine setting, and $\mathcal{X}_a(\mathbb{Z}) \neq \emptyset$ for any such a .

so that with the notation (28),

$$(47) \quad B_f^{\mathcal{U}}(\xi_1, l_1) = B_{f,0}^{\mathcal{U}} = B_0 \times \prod_{p \nmid l_1} \mathcal{U}(\mathbb{Z}_p).$$

For any M sufficiently large such that $p \geq M \Rightarrow p \nmid l_1$ (e.g. $M \geq l_1 + 1$), by (36), we have

$$(48) \quad \begin{aligned} & N_U(B_\infty^U \times B_f^{\mathcal{U}}(\xi_1, l_1); T) \\ &= \#(\mathcal{U}(\mathbb{Z}) \cap (B_\infty(T) \times B_{f,0}^{\mathcal{U}})) \\ &= N_X(B_\infty \times B_{f,0}^{\mathcal{X}}(\mathcal{U}(\mathbb{Z}/\mathfrak{P}_{M,l_1}), \mathfrak{P}_{M,l_1}); T) \\ &\quad + O(\#\{\underline{\mathbf{X}} \in \mathcal{X}(\mathbb{Z}) \cap B_\infty : \|\underline{\mathbf{X}}\| \leq T, \exists p \geq M, \underline{\mathbf{X}} \bmod p \in \mathcal{Z}\}), \end{aligned}$$

where the implied constant is absolute.

Step III. Thanks to the hypothesis (i), the formula (30) in Proposition 2.5 holds for any $B_f^{\mathcal{X}}(\xi, l) \subset B_{f,0}^{\mathcal{X}} = B_0 \times \prod_{p \nmid l_1} \mathcal{X}(\mathbb{Z}_p) \subset \mathcal{X}(\widehat{\mathbb{Z}})$ and the assumption of Corollary 2.6 is satisfied for B_0 (46) above. We therefore apply Corollary 2.6 and obtain

$$\begin{aligned} & N_X(B_\infty \times B_{f,0}^{\mathcal{X}}(\mathcal{U}(\mathbb{Z}/\mathfrak{P}_{M,l_1}), \mathfrak{P}_{M,l_1}); T) \\ &= \int_{B_\infty^U(T) \times B_{f,0}^{\mathcal{U}}} \delta_X|_U \, d m^U + O\left(\mathfrak{P}_{M,l_1}^{\sigma_{\mathcal{X}} + \dim(X)} m_\infty^X(B_\infty(T))^{1-\beta_{\mathcal{X}}} + \frac{m_\infty^X(B_\infty(T))}{M^{\text{codim}_X(Z)-1}}\right). \end{aligned}$$

Furthermore, applying the hypothesis (ii) to the error term in (48), we obtain

$$(49) \quad N_U(B_\infty^U \times B_f^{\mathcal{U}}(\xi_1, l_1); T) = \int_{B_\infty^U(T) \times B_{f,0}^{\mathcal{U}}} \delta_X|_U \, d m^U + \text{Er},$$

where

$$\text{Er} = O\left(\mathfrak{P}_{M,l_1}^{\sigma_{\mathcal{X}} + \dim(X)} m_\infty^X(B_\infty(T))^{1-\beta_{\mathcal{X}}} + \frac{m_\infty^X(B_\infty(T))}{M^{\text{codim}_X(Z)-1}} + (f_1(M) + f_2(T)) m_\infty^X(B_\infty(T))\right),$$

and the implied constant depends only on B_∞, \mathcal{X}, Z as well as B_0 which is in turn determined by (ξ_1, l_1) and \mathcal{U} .

As we assume that X is Hardy-Littlewood, $m_\infty^X(B_\infty(T)) \rightarrow \infty$ as $T \rightarrow \infty$ by definition. Take $M = \eta \log(m_\infty^X(B_\infty(T)))$ (*a fortiori* $M \rightarrow \infty$), where $\eta > 0$ depends only on $B_\infty, \mathcal{X}, \mathcal{U}$, so that

$$\mathfrak{P}_{M,l_1}^{\sigma_{\mathcal{X}} + \dim(X)} \leq \exp\left((\sigma_{\mathcal{X}} + \dim(X)) \sum_{p < M} \log p\right) \leq m_\infty^X(B_\infty(T))^{\frac{\beta_{\mathcal{X}}}{2}}.$$

Therefore the hypothesis (ii) implies that

$$\begin{aligned} \text{Er} &= O\left(m_\infty^X(B_\infty(T))^{1-\frac{\beta_{\mathcal{X}}}{2}} + \frac{m_\infty^X(B_\infty(T))}{\log^{\text{codim}_X(Z)-1}(m_\infty^X(B_\infty(T)))} + (f_1(\eta \log(m_\infty^X(B_\infty(T)))) + f_2(T)) m_\infty^X(B_\infty(T))\right) \\ &= o(m_\infty^X(B_\infty(T))). \end{aligned}$$

Returning to (49) and recalling (47), we obtain finally

$$N_U(B_\infty^U \times B_f^{\mathcal{U}}(\xi_1, l_1); T) = \int_{B_\infty^U(T) \times B_{f,0}^{\mathcal{U}}(\xi_1, l_1)} \delta_X|_U \, d m^U + o(m_\infty^X(B_\infty(T))), \quad T \rightarrow \infty,$$

where the implied constant depends only on $B_\infty, \mathcal{X}, \mathcal{U}, (\xi_1, l_1)$. Finally taking (45) into account, we obtain the equivalence (26). The proof of the theorem is thus completed. \square

3.2. Affine homogeneous spaces. In this article we consider homogeneous spaces of the following two types.

Type I: nice algebraic groups. By convention, we call G a *nice algebraic group* if G is a semisimple simply connected \mathbb{Q} -simple linear algebraic group over \mathbb{Q} with $G(\mathbb{R})$ non-compact. We fix faithful representation $G \hookrightarrow \mathrm{SL}_{n,\mathbb{Q}}$. We consider in this way G as a closed subvariety of $\mathrm{SL}_{n,\mathbb{Q}} \subset \mathbb{A}_{\mathbb{Q}}^{n^2}$, and we use euclidean norms from $\mathbb{A}_{\mathbb{Q}}^{n^2}$. According to [31, §2.2], G is endowed with an invariant gauge form, to which we can associate a Tamagawa measure $m^G = m_{\infty}^G \times m_f^G$ (with convergence factors (1)) as a unimodular Haar measure on $G(\mathbf{A})$. The measure m^G induces a (finite) Haar measure m_q^G on $G(\mathbf{A})/G(\mathbb{Q})$. For every lattice Γ of G , i.e., Γ is a discrete cofinite subgroup of $G(\mathbb{R})$, the real part m_{∞}^G induces a (finite) Haar measure m_{∞}^{Γ} on the locally compact topological group $G(\mathbb{R})/\Gamma$. Appealing to works [25, 17], we have

Theorem 3.2. *Nice algebraic groups are strongly Hardy-Littlewood.*

Proof. Fix any such group G . We firstly infer from [25, Theorem 1] and [17, Theorem 2.7] that $m_{\infty}^G(\{\mathbf{x} \in G(\mathbb{R}) : \|\mathbf{x}\| \leq T\}) \rightarrow \infty$ as $T \rightarrow \infty$.⁶ Secondly, [25, Theorem 2] and [17, Theorem 1.2] show in particular that for any $x_0 \in G(\mathbb{Q})$ and any irreducible lattice Γ ,

$$(50) \quad \#\{\gamma \in \Gamma \cdot x_0 : \|\gamma\| \leq T\} \sim \frac{1}{m_{\infty}^{\Gamma}(G(\mathbb{R})/\Gamma)} m_{\infty}^G(\{\mathbf{x} \in G(\mathbb{R}) : \|\mathbf{x}\| \leq T\}), \quad T \rightarrow \infty.^7$$

In order to establish the strongly Hardy-Littlewood property, according to Proposition 2.3, it suffices to prove the formula (25) for $G(\mathbb{R}) \times B_f^G(\xi, l)$, where $B_f^G(\xi, l)$ is a fixed congruence neighbourhood of a fixed integral model \mathcal{G} of G over \mathbb{Z} with $\mathcal{G}(\widehat{\mathbb{Z}}) \neq \emptyset$ and $\xi = (\xi_p) \in \prod_{p|l} \mathcal{G}(\mathbb{Z}_p)$. Consider the congruence subgroup

$$K_f(l) := \prod_{p|l} K_p(l) \times \prod_{p|l} \mathcal{G}(\mathbb{Z}), \quad \text{where } K_p(l) := \{\eta \in \mathcal{G}(\mathbb{Z}_p) : \eta \equiv \mathrm{id} \pmod{p^{v_p(l)}}\},$$

so that

$$(51) \quad B_f^G(\xi, l) = K_f(l) \cdot (\xi \times \prod_{p|l} \mathrm{id}).$$

We shall follow the strategy of the proof of [4, Theorem 4.2]. Let $K(l) := G(\mathbb{R}) \times K_f(l)$ and $\Gamma(l) := K(l) \cap G(\mathbb{Q})$. Then $\Gamma(l)$ is a lattice of $G(\mathbb{R})$. Since $G(\mathbb{R})$ is non-compact, the strong approximation theorem (cf. e.g. [8, Theorem 1.1]) implies that $K(l) \cdot G(\mathbb{Q}) = G(\mathbf{A})$. We therefore have (cf. [4, 4.5])

$$m_q^G(G(\mathbf{A})/G(\mathbb{Q})) = m_q^G(K(l) \cdot G(\mathbb{Q})/G(\mathbb{Q})) = m_{\infty}^{\Gamma(l)}(G(\mathbb{R})/\Gamma(l)) m_f^G(K_f(l)).$$

The left-hand-side is the Tamagawa number of G , which is proved to be 1 (cf. e.g. [4, p. 58]). We therefore conclude that

$$(52) \quad m_{\infty}^{\Gamma(l)}(G(\mathbb{R})/\Gamma(l))^{-1} = m_f^G(K_f(l)).$$

Moreover, $\mathcal{G}(\mathbb{Z})$ is dense in $\mathcal{G}(\widehat{\mathbb{Z}})$. So we can choose $x_0 \in \mathcal{G}(\mathbb{Z})$ such that $x_0 \equiv \xi_p \pmod{p^{v_p(l)}}$ for every $p | l$. Since G is \mathbb{Q} -simple, the lattice $\Gamma(l)$ is irreducible. We now invoke (50) for $\Gamma(l)$, and

⁶In fact they give asymptotic formulas.

⁷To obtain this, we can invoke for example the limits in [25, Theorems 1 & 2] and eliminate the term $T^d (\log T)^e \int_{\mathrm{End}(V)} f \, d\mu_{\infty}$.

we obtain

$$\begin{aligned}
 N_G(G(\mathbb{R}) \times B_f^G(\xi, l); T) &= \#\{\gamma \in \Gamma(l) \cdot x_0 : \|\gamma\| \leq T\} \\
 &\sim m_\infty^{\Gamma(l)}(G(\mathbb{R})/\Gamma(l))^{-1} m_\infty^G(\{\mathbf{x} \in G(\mathbb{R}) : \|\mathbf{x}\| \leq T\}) \\
 &= m_\infty^G(\{\mathbf{x} \in G(\mathbb{R}) : \|\mathbf{x}\| \leq T\}) m_f^G(K_f(l)) \\
 &= m_\infty^G(\{\mathbf{x} \in G(\mathbb{R}) : \|\mathbf{x}\| \leq T\}) m_f^G(B_f^G(\xi, l)),
 \end{aligned}$$

by using (51) and (52). This achieves our goal. □

Type II: nice symmetric spaces. Let G be a connected semisimple simply connected linear algebraic group over \mathbb{Q} such that $G(\mathbb{R})$ has no compact factors. We choose an almost faithful \mathbb{Q} -representation $\iota : G \rightarrow \mathrm{GL}(W)$, where $W \cong \mathbb{Q}^n$ is an n -dimensional \mathbb{Q} -vector space, so that $\iota(G)$ acts on W . In this article, a symmetric space X is isomorphic to a smooth geometrically integral Zariski closed orbit $\iota(G) \cdot v_0$ of a vector $v_0 \in W$, whose stabilizer H is symmetric (i.e., there exists a non-trivial \mathbb{Q} -involution $\sigma \in \mathrm{Aut}(G)$ such that H is the fixed point locus of σ), connected, and has no non-trivial \mathbb{Q} -characters (hence reductive). In particular $X \cong G/H$ and $X(\mathbb{Q}) \neq \emptyset$. We consider X as a closed subvariety of $\mathbb{A}_{\mathbb{Q}}^n$ via $X \cong \iota(G) \cdot v_0 \subset \mathbb{A}_{\mathbb{Q}}^n$. We recall from §2.1 that X is equipped with a normalised Tamagawa measure $m^X = m_\infty^X \times m_f^X$. The non-compactness of every connected component $B_\infty \subset X(\mathbb{R})$ and the unboundness of $m_\infty^X(B_\infty(T)), T \rightarrow \infty$ follow from e.g. [5, Lemma 2.2 & (2.4)].

In [4, §5] Borovoi and Rudnick introduced a locally constant function

$$(53) \quad \delta_X : X(\mathbf{A}) \rightarrow \{0, \#(C(H))\},$$

where $C(H)$ is the dual group of $\mathrm{Pic}(H)$ à la Kottwitz (cf. [4, §3.4]). This function is an indicator of adelic orbits containing rational points. That is, for every adelic orbit $\mathfrak{D}_{\mathbf{A}}$ of $X(\mathbf{A})$ under $G(\mathbf{A})$, (cf. [4, Theorem 3.6])

$$\delta_X|_{\mathfrak{D}_{\mathbf{A}}} > 0 \Leftrightarrow \mathfrak{D}_{\mathbf{A}} \cap X(\mathbb{Q}) \neq \emptyset.$$

Based on works [11, 13, 14] *et al*, Borovoi and Rudnick show that

Theorem 3.3 ([4] Theorems 5.3 & 5.4). *Symmetric spaces are Hardy-Littlewood with density δ_X (53). They are strongly Hardy-Littlewood provided $C(H) = 0$.*

Remark. The density function δ_X (53) can equivalently be defined in terms of the orthogonal locus of the Brauer group $\mathrm{Br}(X)$ (cf. [10, Propositions 2.2 and 2.10]). See the work of Wei–Xu [30] for various explicit formulas in this spirit.

By convention, we say X is a *nice symmetric space* if either

- (1) G is \mathbb{Q} -simple;
- or
- (2) $G \cong \mathrm{SL}_{2,\mathbb{Q}} \times \mathrm{SL}_{2,\mathbb{Q}}$ and H is the diagonal subgroup.⁸

For homogeneous spaces of type (I) or (II), works [26, 17, 5] provide explicit uniform error term estimates for counting integral points with congruence conditions. Based on them, we are now in a position to show:

Theorem 3.4. *Nice algebraic groups and nice homogeneous spaces all satisfy the hypothesis (i) of Theorem 3.1 with respect to an arbitrary affine integral model over \mathbb{Z} (inside of $\mathbb{A}_{\mathbb{Z}}^{n^2}$ or $\mathbb{A}_{\mathbb{Z}}^n$ depending on the type).*

⁸The case (2) corresponds to affine quadrics in four variables of signature (2, 2).

Proof. Let X be such a homogeneous space. Fix \mathcal{X} any affine integral model of X over \mathbb{Z} as above and consider for any $a \in \mathbb{N}_{\neq 0}$ the integral model \mathcal{X}_a of X_a over \mathbb{Z} (cf. (42)), which is also an integral model of X since $X_a \cong_{\mathbb{Q}} X$. As $X(\mathbb{Q}) \neq \emptyset$, then $\mathcal{X}_a(\mathbb{Z}) \neq \emptyset$ if a is sufficiently divisible. Now fix any such \mathcal{X}_a .

Now we fix $B_f^{\mathcal{X}_a}(\xi, l)$ (cf. (20)) a congruence neighbourhood of \mathcal{X}_a , where $l \geq 2$ is an integer and $\xi \in \prod_{p|l} \mathcal{X}_a(\mathbb{Z}_p)$. Let $\bar{\xi} = (\bar{\xi}_p)_{p|l}$ be the image of ξ in $\mathcal{X}_a(\mathbb{Z}/l) \cong \prod_{p|l} \mathcal{X}_a(\mathbb{Z}/p^{v_p(l)})$ under the residue map Ψ_l (29). Since \mathcal{X}_a is affine, each factor $B_p^{\mathcal{X}_a}(\xi, l)$ of $B_f^{\mathcal{X}_a}(\xi, l)$ can equally be defined by means of the congruence conditions in (21).

Building on the works of Nevo-Sarnak [26, Theorem 3.2] and Gorodnik-Nevo [16, Theorem 6.1] for Type I varieties (using (51) (52) as in the proof of Theorem 3.2), and Browning-Gorodnik [5, Corollaries 2.5 & 2.6] for Type II varieties, there exist $\sigma_{\mathcal{X}_a} > 0, 0 < \beta_{\mathcal{X}_a} < 1$ such that, for any connected component $B_\infty \subset X(\mathbb{R})$, we have, uniformly for any such (ξ, l) ,

$$\begin{aligned} N_{X_a}(\phi_{a,\mathbb{R}}(B_\infty) \times B_f^{\mathcal{X}_a}(\xi, l); T) &= \#\{\underline{\mathbf{X}} \in \mathcal{X}_a(\mathbb{Z}) \cap \phi_{a,\mathbb{R}}(B_\infty) : \|\underline{\mathbf{X}}\| \leq T, \text{ for any } p \mid l, \underline{\mathbf{X}} \equiv \bar{\xi}_p \pmod{p^{v_p(l)}}\} \\ &= \#\{\underline{\mathbf{X}} \in \mathcal{X}_a(\mathbb{Z}) \cap \phi_{a,\mathbb{R}}(B_\infty) : \|\underline{\mathbf{X}}\| \leq T, \underline{\mathbf{X}} \equiv \bar{\xi} \pmod{l}\} \\ &= \int_{\phi_{a,\mathbb{R}}(B_\infty)(T) \times B_f^{\mathcal{X}_a}(\xi, l)} \delta_{X_a} \, d m^{X_a} + O(l^{\sigma_{\mathcal{X}_a}} m_\infty^{X_a}(\phi_{a,\mathbb{R}}(B_\infty)(T))^{1-\beta_{\mathcal{X}_a}}) \\ &= \int_{\phi_{a,\mathbb{R}}(B_\infty)(T) \times B_f^{\mathcal{X}_a}(\xi, l)} \delta_{X_a} \, d m^{X_a} + O(l^{\sigma_{\mathcal{X}_a}} m_\infty^X(B_\infty(T))^{1-\beta_{\mathcal{X}_a}}). \end{aligned}$$

Therefore this confirms the hypothesis (i) of Theorem 3.1 for the integral model \mathcal{X} . \square

Corollary 3.5. *Assume that a homogeneous space $X = G/H$ in Theorem 3.4 satisfies the hypothesis (ii) of Theorem 3.1 with respect to a fixed affine integral model (inside of $\mathbb{A}_{\mathbb{Z}}^{n^2}$ or $\mathbb{A}_{\mathbb{Z}}^n$ depending on the type). Then X satisfies (APHL) with density δ_X (53). If moreover H is simply connected, then X satisfies (APSHL).*

Proof. It follows directly from Theorems 3.1, 3.2, 3.3 that X is Hardy-Littlewood with density δ_X . Now let \mathcal{X} be an affine integral model of X , and let $a_1 \in \mathbb{N}_{\neq 0}$ (depending on \mathcal{X}) be such that every integral model \mathcal{X}_a with $a_1 \mid a$ satisfies the hypothesis (ii) of Theorem 3.1. By Theorem 3.4, there exists $a_2 \in \mathbb{N}_{\neq 0}$ (depending on \mathcal{X}) such that every \mathcal{X}_a with $a_2 \mid a$ satisfies the hypothesis (i) of Theorem 3.1. Therefore, \mathcal{X}_a satisfies both hypotheses whenever $a_1 a_2 \mid a$, and hence X satisfies (APHL) with density δ_X by Theorem 3.1. Moreover, [4, Theorem 0.3, Corollary 0.3.3] shows that X is strongly Hardy-Littlewood, provided that H is simply connected. \square

3.3. Affine quadrics. Returning to the affine quadric \mathbf{Q} (6). We recall that, on fixing $P \in \mathbf{Q}(\mathbb{Q})$, \mathbf{Q} is a symmetric space under $G = \mathbf{Spin}_q$ with stabilizer $H \cong \mathbf{Spin}_q|_{P^\perp}$. The group \mathbf{Spin}_q is always \mathbb{Q} -simple if $n \geq 5$ or $n = 3$, whereas when $n = 4$, it can happen that $\mathbf{Spin}_q \cong \mathbf{SL}_{2,\mathbb{Q}} \times \mathbf{SL}_{2,\mathbb{Q}}$, which is the only exceptional case where \mathbf{Spin}_q can possibly be not \mathbb{Q} -simple ([5, Remark 2.4]). In any case, affine quadrics with $n \geq 3$ variables are nice symmetric spaces.

Admitting Theorem 1.6, we can now prove our main theorems stated in the introduction.

Proof of Theorems 1.4 and 1.5. Recall that we assume the form $q(\mathbf{x})$ defining the affine quadric \mathbf{Q} (7) to be \mathbb{R} -isotropic. The real locus $\mathbf{Q}(\mathbb{R})$ of \mathbf{Q} is a hyperboloid. It is connected if $n \geq 4$, and it can be one sheeted or two sheeted when $n = 3$ (depending on H). It follows from (8) that for any connected component $B_\infty \subset \mathbf{Q}(\mathbb{R})$, we have

$$m_\infty^{\mathbf{Q}}(B_\infty(T)) \asymp T^{n-2}.$$

On the other hand, we take an integral model $\mathcal{Q} \subset \mathbb{A}_{\mathbb{Z}}^n$ of the form $q(\mathbf{x}) = m$ (cf. (6)) satisfying the hypotheses of Theorems 1.4 and 1.5. Then for any $a \in \mathbb{N}_{\neq 0}$, on recalling (42), the twisted integral model $\mathcal{Q}_a \subset \mathbb{A}_{\mathbb{Z}}^n$ is defined by the equation

$$q(\mathbf{x}) = a^2 m.$$

In particular $-a^2 m \det q \neq \square$ and hence the hypothesis of Theorem 1.5 is always satisfied for any $a \in \mathbb{N}_{\neq 0}$. Then Theorem 1.6 implies that the hypothesis (ii) of Theorem 3.1 holds for all \mathcal{Q}_a with

$$f_1(x) = \frac{1}{x}, \quad f_2(x) = \frac{1}{\sqrt{\log x}}.$$

Therefore the statements of Theorems 1.4 and 1.5 follow from Corollary 3.5. \square

4. A GEOMETRIC SIEVE FOR AFFINE QUADRICS

In this whole section, we are devoted to proving Theorem 1.6. For this purpose, let us fix for the rest of this section the integral model \mathcal{Q} (6) over \mathbb{Z} for \mathbf{Q} , and a closed subset $Z \subset \mathbf{Q}$ of codimension at least two. Let $\mathcal{Z} := \overline{Z} \subset \mathcal{Q}$. For any $0 < N_1 < N_2 \leq \infty$ sufficiently large, let us define

$$(54) \quad V(T; N_1, N_2) := \#\{\underline{\mathbf{X}} \in \mathcal{Q}(\mathbb{Z}) : \|\underline{\mathbf{X}}\| \leq T, \exists p \in [N_1, N_2], \underline{\mathbf{X}} \bmod p \in \mathcal{Z}\}.$$

Thus for any $M > 0$,

$$\#\{\underline{\mathbf{X}} \in \mathcal{Q}(\mathbb{Z}) : \|\underline{\mathbf{X}}\| \leq T, \exists p \geq M, \underline{\mathbf{X}} \bmod p \in \mathcal{Z}\} = V(T; M, \infty).$$

According to the range of N_1, N_2 , we separate our discussion into three parts, in which (54) is treated using different methods. In §4.1 we derive an upper bound for (54) valid for arbitrary N_1, N_2 , which is satisfactory if we take N_1 growing to infinity as $T \rightarrow \infty$ and $N_2 = T^\alpha$ with $\alpha > 0$ sufficiently small. In §4.2, on taking $N_1 = T^\alpha, N_2 = T$, we deal with the residues coming from intermediate primes, by employing various Serre-type uniform bounds for integral points on quadrics. In §4.3 we match together a generalised geometric sieve and our half-dimensional sieve for affine quadrics so as to derive a satisfactory upper bound for the case $N_1 = T, N_2 = \infty$. In §4.4 we assemble these bounds together and prove Theorem 1.6.

We shall always throughout assume that $-m \det q \neq \square$ if $n = 3$. And we shall emphasise uniformity of dependence in the assumption of all statements. Unless expressly stated to the contrary, all implied constants are only allowed to depend on \mathcal{Q} and Z .

4.1. Extension to prime moduli of polynomial growth. The purpose of section is to prove:

Theorem 4.1. *For any $1 \ll N_1 < N_2 < \infty$, we have*

$$V(T; N_1, N_2) = O\left(\frac{T^{n-2}}{N_1^{\text{codim}_{\mathbf{Q}}(Z)-1}} + T^{(n-2)(1-\beta_{\mathbf{Q}})} N_2^{\frac{(3n-2)(n-1)}{2} + \dim Z + 1}\right).$$

Proof. Recall from §3.3 that we have $\mathbf{Q} \cong G/H$, with

$$\dim G = \frac{n(n-1)}{2}, \quad \dim H = \frac{(n-1)(n-2)}{2}.$$

Let us fix $l_0 \geq 2$ such that \mathcal{Q} is smooth over $\mathbb{Z}[1/l_0]$. Thanks to Theorem 3.4, the hypothesis of Proposition 2.5 is fulfilled for $B_0 := \prod_{p|l_0} \mathcal{Q}(\mathbb{Z}_p)$ with exponent (cf. [5, Corollary 2.6])

$$\sigma_{\mathcal{Q}} = \dim H + 2 \dim G = \frac{(3n-2)(n-1)}{2}.$$

Applying Corollary 2.7, we conclude that for any connected component $B_\infty \subset \mathbf{Q}(\mathbb{R})$, any p_0 sufficiently large with $p_0 \nmid l_0$,

$$\begin{aligned} N_{\mathbf{Q}}(B_\infty \times B_{f,0}^{\mathcal{Q}}(\mathcal{Z}(\mathbb{F}_{p_0}), p_0); T) &\ll \frac{m_\infty^{\mathbf{Q}}(B_\infty(T))}{p_0^{\text{codim}_{\mathbf{Q}}(Z)}} + m_\infty^{\mathbf{Q}}(B_\infty(T))^{1-\beta_{\mathbf{Q}}} p_0^{\sigma_{\mathcal{X}} + \dim Z} \\ &= O\left(\frac{T^{n-2}}{p_0^{\text{codim}_{\mathbf{Q}}(Z)}} + T^{(n-2)(1-\beta_{\mathbf{Q}})} p_0^{\frac{(3n-2)(n-1)}{2} + \dim Z}\right). \end{aligned}$$

For any p , let us now consider

$$(55) \quad V_p(T; \mathcal{Z}) := \#\{\underline{\mathbf{X}} \in \mathcal{Q}(\mathbb{Z}) : \|\underline{\mathbf{X}}\| \leq T, \underline{\mathbf{X}} \bmod p \in \mathcal{Z}\}.$$

Then for N_1 sufficiently large, summing $V_{p_0}(T; \mathcal{Z})$ over all such p_0 's lying in $[N_1, N_2]$, we get the following upper bound for $V(T; N_1, N_2)$:

$$\begin{aligned} V(T; N_1, N_2) &\leq \sum_{N_1 \leq p_0 \leq N_2} V_{p_0}(T; \mathcal{Z}) \\ &= \sum_{N_1 \leq p_0 \leq N_2} \sum_{B_\infty \subset \mathbf{Q}(\mathbb{R})} N_{\mathbf{Q}}(B_\infty \times B_{f,0}^{\mathcal{Q}}(\mathcal{Z}(\mathbb{F}_{p_0}), p_0); T) \\ &= \sum_{N_1 \leq p_0 \leq N_2} O\left(\frac{T^{n-2}}{p_0^{\text{codim}_{\mathbf{Q}}(Z)}} + T^{(n-2)(1-\beta_{\mathbf{Q}})} p_0^{\frac{(3n-2)(n-1)}{2} + \dim Z}\right) \\ &\ll \frac{T^{n-2}}{N_1^{\text{codim}_{\mathbf{Q}}(Z)-1}} + T^{(n-2)(1-\beta_{\mathbf{Q}})} N_2^{\frac{(3n-2)(n-1)}{2} + \dim Z + 1}, \end{aligned}$$

as desired. \square

4.2. Intermediate primes of polynomial range. The goal of this section is to show:

Theorem 4.2. *Assume moreover that the quadratic form q is anisotropic over \mathbb{Q} (and $-m \det q \neq \square$) if $n = 3$. Then for any $\alpha > 0$,*

$$V(T; T^\alpha, T) = O_\alpha\left(\frac{T^{n-2}}{\log T}\right).$$

Our method of obtaining the estimate in Theorem 4.2 is to break the sum into residue classes. For any $l \geq 2$, $\bar{\xi} \in (\mathbb{Z}/l)^n$, let us consider

$$(56) \quad V_l(T; \bar{\xi}) := \#\{\underline{\mathbf{X}} \in \mathcal{Q}(\mathbb{Z}) : \|\underline{\mathbf{X}}\| \leq T, \underline{\mathbf{X}} \equiv \bar{\xi} \bmod l\}.$$

We shall separate our discussion into the cases $n = 3$ and $n \geq 4$. It turns out that we need satisfactory bounds for the quantity (56) with $l = p$, which are uniform respect to p and $\xi \in \mathbb{F}_p^n$, and we need to prove that the contribution from summing over all p in this range is still satisfactory, compared to the order of magnitude T^{n-2} . Our argument is inspired by [5, §5].

4.2.1. Ingredients. We record here a uniform estimate for the growth of integral points on quadratic hypersurfaces due to Browning-Gorodnik. This result is also useful in §5. For $g(\mathbf{x}) \in \mathbb{Q}[x_1, \dots, x_L]$ a polynomial of degree two, the *quadratic part* g_0 of g is the homogeneous degree two part of g . Let $\text{rk}(g_0)$ denote the rank of the quadratic form g_0 .

Theorem 4.3 ([5] Theorem 1.11). *For any $\varepsilon > 0$, we have, uniformly for any irreducible polynomial $g(\mathbf{x}) \in \mathbb{Z}[x_1, \dots, x_L]$ of degree two with $\text{rk}(g_0) \geq 2$,*

$$\#\{\underline{\mathbf{X}} \in \mathbb{Z}^L : \|\underline{\mathbf{X}}\| \leq T, g(\underline{\mathbf{X}}) = 0\} = O_\varepsilon(T^{L-2+\varepsilon}).$$

The implied constant is independent of the polynomial g .

The case of $L = 3$ is a direct consequence of dimension growth bounds obtained by Browning, Heath-Brown and Salberger (cf. [5, Lemma 4.1]).

4.2.2. *Affine quadrics with three variables.* We start by the more involved case $n = 3$. We first show the following basic fact. Recall that we always assume $\mathbf{Q}(\mathbb{Q}) \neq \emptyset$.

Lemma 4.4. *The condition $-m \det q \neq \square$ is equivalent to that \mathbf{Q} does not contain any line over \mathbb{Q} .*

Proof. We first note that $\text{Pic}(\mathbf{Q}_{\overline{\mathbb{Q}}}) = \mathbb{Z}$. To see how $\text{Pic}(\mathbf{Q}_{\overline{\mathbb{Q}}})$ is generated, we recall that a projective quadric surface S in \mathbb{P}^3 over $\overline{\mathbb{Q}}$ has Picard group $\text{Pic}(S) \cong \mathbb{Z}^2$. Any hyperplane section $E \in |\mathcal{O}(1)|$ intersects S at a conic curve C of divisor type $(1, 1)$. So that $\text{Pic}(S - E) \cong \mathbb{Z}^2/\mathbb{Z}(1, 1) \cong \mathbb{Z}$. In particular, if the conic curve C splits into two lines, then each of them generates $\text{Pic}(S - E)$. See [18, II. Examples 6.6.1 & 6.6.2].

So if we compactify \mathbf{Q} into $\overline{\mathbf{Q}} \subset \mathbb{P}^3$ and view \mathbf{Q} as the complement of some hyperplane section in $\overline{\mathbf{Q}}$, then the class of a \mathbb{Q} -line on \mathbf{Q} (if any) generates $\text{Pic}(\mathbf{Q})$, since any \mathbb{Q} -plane through that line intersects \mathbf{Q} at another \mathbb{Q} -line. Let $d := -m \det q$ and fix $P \in \mathbf{Q}(\mathbb{Q})$. Let $H \cong (x^2 - dy^2 = 1)$ be the stabilizer of Spin_q acting on P (cf. §3.3). The group $G = \text{Gal}(\mathbb{Q}(\sqrt{d}/\mathbb{Q}))$ operates on $\hat{H} \cong \text{Pic}(\mathbf{Q}_{\overline{\mathbb{Q}}})$, so that

$$\text{Pic}(\mathbf{Q}) \cong \hat{H}^G.$$

On the one hand, if $d = \square$, then the tangent plane of \mathbf{Q} at P intersects \mathbf{Q} at two lines over \mathbb{Q} (cf. [10, p. 333]). On the other hand, if $d \neq \square$, by [10, p. 331], we have $\text{Pic}(\mathbf{Q}) = 0$. So we conclude that \mathbf{Q} contains no \mathbb{Q} -lines if and only if $d \neq \square$. \square

With this at hand, recalling (56), we now show:

Proposition 4.5. *Assume that $n = 3$. Then under the assumption of Theorem 4.2, uniformly for any $1 \ll p \leq T$ and $\bar{\xi} \in \mathbb{F}_p^3$, we have*

$$V_p(T; \bar{\xi}) \ll_\varepsilon \left(\frac{T}{p}\right)^\varepsilon \left(1 + \frac{T}{p^{\frac{4}{3}}}\right).$$

Proof. We fix $1 \ll p \leq T$ and $\bar{\xi} \in \mathbb{F}_p^3$ in our following arguments. Either $V_p(T; \bar{\xi}) = 0$, for which the desired estimate is evident, or we can find $\underline{\mathbf{X}}_0 \in \mathcal{Q}(\mathbb{Z})$ such that, $\|\underline{\mathbf{X}}_0\| \leq T$, $\underline{\mathbf{X}}_0 \equiv \bar{\xi} \pmod{p}$. On making the change of variables $\underline{\mathbf{X}} = \underline{\mathbf{X}}_0 + p\mathbf{y}$, the new variable $\mathbf{y} \in \mathbb{Z}^3$ satisfies the following equations:

$$(57) \quad \|\mathbf{y}\| \leq \frac{2T}{p}, \quad \mathbf{y} \cdot \nabla q(\underline{\mathbf{X}}_0) + pq(\mathbf{y}) = 0.$$

Since $\underline{\mathbf{X}}_0$ is a smooth point of \mathcal{Q} , we have $\nabla q(\underline{\mathbf{X}}_0) \neq \mathbf{0}$. Moreover $p \nmid \text{gcd}(\underline{\mathbf{X}}_0)$, since otherwise $p \mid m$, which cannot happen for p large enough. Since $\underline{\mathbf{X}}_0 \equiv \bar{\xi} \pmod{p}$, the second equation of (57) implies that \mathbf{y} lies in the lattice

$$\Gamma_{\bar{\xi}}^p := \{\mathbf{x} \in \mathbb{Z}^3 : \mathbf{x} \cdot \nabla q(\bar{\xi}) \equiv 0 \pmod{p}\}$$

of determinant $\gg\ll p$, the implied constants depending only on q and m .⁹ Thus

$$(58) \quad V_p(T; \bar{\xi}) \leq M_{\underline{\mathbf{X}}_0, \bar{\xi}}^p(T),$$

where

$$M_{\underline{\mathbf{X}}_0, \bar{\xi}}^p(T) := \# \left\{ \mathbf{y} \in \Gamma_{\bar{\xi}}^p : \|\mathbf{y}\| \leq \frac{2T}{p}, \mathbf{y} \cdot \nabla q(\underline{\mathbf{X}}_0) + pq(\mathbf{y}) = 0 \right\}.$$

We are led to bounding $M_{\underline{\mathbf{X}}_0, \bar{\xi}}^p(T)$.

Choose a minimal basis $\mathbf{L} = (\mathbf{l}_1, \mathbf{l}_2, \mathbf{l}_3)$ of $\Gamma_{\bar{\xi}}^p$ such that (cf. [5, (5.3)])

$$(59) \quad \|\mathbf{l}_1\| \leq \|\mathbf{l}_2\| \leq \|\mathbf{l}_3\|, \quad \|\mathbf{l}_1\| \|\mathbf{l}_2\| \|\mathbf{l}_3\| \gg\ll p,$$

so that, on making the non-singular change of variables $\mathbf{y} \mapsto \mathbf{L} \cdot \mathbf{z}$, the new variable $\mathbf{z} = (z_1, z_2, z_3) \in \mathbb{Z}^3$ satisfies, by (57) and [5, p. 1075],

$$(60) \quad |z_i| \leq \frac{c_i T}{p \|\mathbf{l}_i\|}, \quad 1 \leq i \leq 3, \quad \tilde{q}(\mathbf{z}) + \mathbf{z} \cdot \mathbf{Y}_0 = 0,$$

where $c_i > 0, 1 \leq i \leq 3$ are absolute constants, and

$$\tilde{q}(\mathbf{z}) = q(\mathbf{L}\mathbf{z}), \quad \mathbf{Y}_0 = p^{-1} \mathbf{L} \nabla q(\underline{\mathbf{X}}_0).$$

Now we slice the second equation of (60) and get for each fixed integer z_3 a resulting polynomial $q_{z_3} \in \mathbb{Z}[z_1, z_2]$. By (60), the total number of z_3 is

$$(61) \quad \ll 1 + \frac{T}{p \|\mathbf{l}_3\|}.$$

We first claim that for any $z_3 = \kappa_1 \in \mathbb{Q}$, $\text{rk}((q_{\kappa_1})_0) = 2$. Indeed, the quadratic part of q_{κ_1} is

$$(q_{\kappa_1})_0 = \tilde{q}(z_1, z_2, 0).$$

The latter, viewed as quadratic form in two varieties, has rank 1 (that is, $\text{rk}((q_{\kappa_1})_0) = 1$) if and only if $(z_3 = 0)$ is the tangent plane at certain point defined over \mathbb{Q} of the projective quadric $(q = 0) \subset \mathbb{P}^2$. By the assumption that q is \mathbb{Q} -anisotropic, we conclude that this is impossible, which proves the claim.

We further claim that for any $z_3 = \kappa_2 \in \mathbb{Q}$, the polynomial q_{κ_2} is irreducible over \mathbb{Q} . Indeed, if the polynomial q_{κ_2} is reducible over \mathbb{Q} for certain $\kappa_2 \in \mathbb{Q}$, that is, it splits into the product of two polynomials f_1, f_2 of degree one, then $(z_3 = \kappa_2) \cap (f_i = 0)$ defines a \mathbb{Q} -line on \mathbf{Q} for $i = 1, 2$ (since the change of variables at each step above is non-singular). This is absurd by Lemma 4.4 as we always assume that $-m \det q \neq \square$.

Consequently, using (60), Theorem 4.3 shows that the contribution from integral points on each quadric $(q_{z_3} = 0)$ with $z_3 \in \mathbb{Z}$ is (as we assume $p \leq T$)

$$\begin{aligned} A_{z_3}^p(T) &:= \#\{(z_1, z_2) \in \mathbb{Z}^2 : |z_i| \ll \frac{T}{p \|\mathbf{l}_i\|}, 1 \leq i \leq 2, q_{z_3}(z_1, z_2) = 0\} \\ &= O_\varepsilon \left(1 + \left(\frac{T}{p \|\mathbf{l}_1\|} \right)^\varepsilon \right) = O_\varepsilon \left(\left(\frac{T}{p} \right)^\varepsilon \right), \end{aligned}$$

⁹in fact $\det(\Gamma_{\bar{\xi}}^p) = p$ for p large enough, cf. [5, p. 1076-1077].

where the implied constant is independent of z_3 . Therefore, taking (61) into account, we obtain an upper bound for $M_{\underline{\mathbf{x}}_0, \bar{\xi}}^p(T)$ as follows:

$$\begin{aligned} M_{\underline{\mathbf{x}}_0, \bar{\xi}}^p(T) &\leq \sum_{|z_3| \ll \frac{T}{p \|\mathbf{l}_3\|}} A_{z_3}^p(T) \\ &\ll_{\varepsilon} \left(1 + \frac{T}{p \|\mathbf{l}_3\|}\right) \times \left(\frac{T}{p}\right)^{\varepsilon} \ll_{\varepsilon} \left(\frac{T}{p}\right)^{\varepsilon} \left(1 + \frac{T}{p^{\frac{4}{3}}}\right), \end{aligned}$$

because $\|\mathbf{l}_3\| \gg p^{\frac{1}{3}}$ by (59). This finishes the proof, thanks to (58). \square

Proof of Theorem 4.2 for the case $n = 3$. We may assume that $Z \neq \emptyset$, hence $\dim Z = 0$. We then have $\#\mathcal{Z}(\mathbb{F}_p) \leq \deg Z$ for every prime p . Employing the estimate (cf. [1, Proposition 10])

$$(62) \quad \sum_{p \leq X} \frac{1}{p^{\sigma}} \ll_{\sigma} \frac{X^{1-\sigma}}{\log X}, \quad 0 < \sigma < 1,$$

and using Proposition 4.5, we sum over all primes in the interval $[T^{\alpha}, T]$ and we get, for any $0 < \varepsilon < \min(\frac{1}{2}, \frac{1}{3}\alpha)$,

$$\begin{aligned} V(T; T^{\alpha}, T) &\leq \sum_{T^{\alpha} \leq p \leq T} \sum_{\bar{\xi} \in \mathcal{Z}(\mathbb{F}_p)} V_p(T; \bar{\xi}) \\ &\ll_{\varepsilon} (\deg Z) \sum_{T^{\alpha} \leq p \leq T} \left(\frac{T^{\varepsilon}}{p^{\varepsilon}} + \frac{T^{1+\varepsilon}}{p^{\frac{4}{3}}}\right) \\ &\ll_{\varepsilon} \frac{T^{\varepsilon} \times T^{1-\varepsilon}}{\log T} + T^{1+\varepsilon-\frac{1}{3}\alpha} = O_{\varepsilon} \left(\frac{T}{\log T}\right). \end{aligned}$$

We fix $\varepsilon > 0$ small enough in terms of α so that the implied constant above depends only on α . This proves the desired upper-bound. \square

4.2.3. Affine quadrics with at least 4 variables. Now we turn to the case $n \geq 4$. The analogous version of Proposition 4.5 as a key input for us is the following estimate obtained by Browning and Gorodnik.

Proposition 4.6 ([5] Proposition 5.1). *Assume $n \geq 4$. Then*

$$V_p(T; \bar{\xi}) \ll_{\varepsilon} \left(\frac{T}{p}\right)^{n-3+\varepsilon} \left(1 + \frac{T}{p^{\frac{n}{n-1}}}\right)$$

holds uniformly for any prime $1 \ll p \leq T$ and for any $\bar{\xi} \in \mathbb{F}_p^n$.

Proof of Theorem 4.2 for the case $n \geq 4$. Recalling the Lang-Weil estimate (9) for \mathcal{Z} , we infer from Proposition 4.5 and (62) that

$$\begin{aligned}
V(T; T^\alpha, T) &\leq \sum_{T^\alpha \leq p \leq T} \sum_{\bar{\xi} \in \mathcal{Z}(\mathbb{F}_p)} V_p(T; \bar{\xi}) \\
&\ll_\varepsilon \sum_{T^\alpha \leq p \leq T} p^{n-3} \times \left(\frac{T}{p}\right)^{n-3+\varepsilon} \left(1 + \frac{T}{p^{\frac{n}{n-1}}}\right) \\
&\ll_\varepsilon \sum_{T^\alpha \leq p \leq T} \left(\frac{T^{n-3+\varepsilon}}{p^\varepsilon} + \frac{T^{n-2+\varepsilon}}{p^{1+\frac{1}{n-1}+\varepsilon}}\right) \\
&\ll_\varepsilon T^{n-3+\varepsilon} \times \frac{T^{1-\varepsilon}}{\log T} + T^{n-2+\varepsilon} \times T^{-\frac{\alpha}{n-1}} \ll_\varepsilon \frac{T^{n-2}}{\log T},
\end{aligned}$$

for any $0 < \varepsilon < \min(\frac{1}{2}, \frac{\alpha}{n-1})$. It remains to fix $\varepsilon > 0$ small enough depending only on α to get the desired dependence for the implied constant. \square

4.3. Treatment of very large primes. The goal of this section is to generalise Ekedahl's geometric sieve [12] to affine quadrics. Our treatment is inspired by a discussion with Tim Browning, to whom we express our gratitude.

Theorem 4.7. *Let $N_1 = T, N_2 = \infty$ in (54). Then*

$$V(T; T, \infty) = O\left(\frac{T^{n-2}}{(\log T)^{\frac{1}{2}}}\right).$$

Proof of Theorem 4.7. Up to a change of variables, we may assume that the \mathbb{Z} -integral model \mathcal{Q} is defined by a diagonal quadratic form:

$$(63) \quad \sum_{i=1}^n a_i x_i^2 = m.$$

This may affect (54) via a different choice of equivalent height functions in terms of the form q , which is clearly negligible. We may assume that $a_{n-1} \cdot a_n > 0$. By multiplying -1 to the equation (63) if necessary we can assume that both of them are > 0 . We can furthermore assume that $a_n = 1$, and all other a_i 's are square-free integers, and we write from now on $a_{n-1} = a > 0$.

Next, upon enlarging the subvariety $Z \subset \mathbf{Q}$ of codimension at least two, we can assume that

$$(64) \quad Z = \mathbf{Q} \cap (f = g = 0) \subset \mathbb{A}_{\mathbb{Q}}^n,$$

where the two polynomials f, g satisfy $f \in \mathbb{Z}[x_1, \dots, x_{n-1}], g \in \mathbb{Z}[x_1, \dots, x_{n-2}]$. Consequently, assuming (64), then there exists $l_0 \in \mathbb{N}_{\neq 0}$ depending only on \mathcal{Q}, f, g such that

$$(65) \quad \mathcal{Z} \times_{\mathbb{Z}} \mathbb{Z}[1/l_0] \simeq \text{Spec} \left(\frac{\mathbb{Z}[1/l_0][x_1, \dots, x_n]}{\sum_{i=1}^n a_i x_i^2 - m, f, g} \right) \subset \mathcal{Q} \times_{\mathbb{Z}} \mathbb{Z}[1/l_0] \subset \mathbb{A}_{\mathbb{Z}[1/l_0]}^n.$$

In order to verify (64), let us consider the maps $\text{pr}_1 : \mathbb{A}^n \rightarrow \mathbb{A}^{n-1}, \text{pr}_2 : \mathbb{A}^{n-1} \rightarrow \mathbb{A}^{n-2}$, and $\text{pr}_3 = \text{pr}_2 \circ \text{pr}_1$, the first (resp. second) being the projection onto the first $(n-1)$ (resp. $(n-2)$) coordinates. Since $\dim(Z) \leq n-3$, its image $\text{pr}_3(Z)$ has codimension at least one in $\mathbb{A}_{\mathbb{Q}}^{n-2}$. Therefore we can choose $g \in \mathbb{Z}[x_1, \dots, x_{n-2}]$ such that $\text{pr}_3(Z) \subseteq Z' := (g = 0) \subset \mathbb{A}_{\mathbb{Q}}^{n-2}$. On the other hand, since the map $\text{pr}_1|_{\mathbf{Q}}$ is affine and finite, the closed subset $\text{pr}_1(Z) \subset \mathbb{A}_{\mathbb{Q}}^{n-1}$ has codimension at least two, and $\text{pr}_1(Z) \subset \text{pr}_2^{-1}(Z') = Z' \times \mathbb{A}_{\mathbb{Q}}^1$, the latter being of codimension one

in $\mathbb{A}_{\mathbb{Q}}^{n-1}$. We can choose $f \in \mathbb{Z}[x_1, \dots, x_{n-1}]$ such that $\text{pr}_1(Z) \subset (f = 0) \cap \text{pr}_2^{-1}(Z')$. This gives $Z \subset \mathbf{Q} \cap (f = g = 0) \subset \mathbb{A}_{\mathbb{Q}}^n$. Note that this above procedure may also be reformulated using the classical elimination theory.

Case 1. First of all we recall Bhargava's [3] quantitative version of Ekedahl's geometric sieve [12].

Theorem 4.8 ([3] Lemma 3.1 & Theorem 3.3). *Let $\mathcal{Y} \subset \mathbb{A}_{\mathbb{Z}}^L$ be a subscheme such that the generic fibre $\mathcal{Y}_{\mathbb{Q}} \subset \mathbb{A}_{\mathbb{Q}}^L$ is of codimension k . Then for any $M \gg_{\mathcal{Y}} 1$,*

- (1) $\#\{\underline{\mathbf{X}} \in \mathcal{Y}(\mathbb{Z}) : \|\underline{\mathbf{X}}\| \leq T\} = O_{\mathcal{Y}}(T^{L-k});$
- (2) $\#\{\underline{\mathbf{X}} \in \mathbb{Z}^L : \exists p \geq M, \underline{\mathbf{X}} \bmod p \in \mathcal{Y}\} = O_{\mathcal{Y}}\left(\frac{T^L}{M^{k-1} \log M} + T^{L-k+1}\right).$

We are going to consider three conditions under which we can reduce the problem to the affine space \mathbb{A}^{n-2} via the fibration $\text{pr}_3|_{\mathcal{Q}}$ and adapt Theorem 4.8 to get satisfactory upper bounds. If f is constant in the variable x_{n-1} , then $Z = (Y_1 \times \mathbb{A}_{\mathbb{Q}}^2) \cap \mathbf{Q}$ where

$$Y_1 := (f = g = 0) \subset \mathbb{A}_{\mathbb{Q}}^{n-2}$$

is of codimension two. Otherwise, if f is non-constant in x_{n-1} , denote by $h \in \mathbb{Z}[x_1, \dots, x_{n-2}]$ the leading coefficient of f in x_{n-1} and write $f = hx_{n-1}^c + f'$ with $c > 0$ such that the degree of f' in x_{n-1} is strictly less than c . If $\text{codim}_{\mathbb{A}_{\mathbb{Q}}^{n-2}}(g = h = 0) = 1$, we consider all the irreducible (reduced) components of $(g = 0) \subset \mathbb{A}_{\mathbb{Q}}^{n-2}$, each defined by one of the polynomials g_1, \dots, g_k say. Then either $\text{codim}_{\mathbb{A}_{\mathbb{Q}}^{n-2}}(g_i = h = 0) = 2$ or $(g_i = 0) \subseteq (h = 0)$. In the first case we let $Z_i = (f = g_i = 0) \cap \mathbf{Q}$. In the second case we have $(f = g_i = 0) = (f' = g_i = 0) \subset \mathbb{A}_{\mathbb{Q}}^n$, and we therefore let $Z_i = (f' = g_i = 0) \cap \mathbf{Q}$. So that $Z \subseteq \cup_{i=1}^k Z_i$ (this amounts to enlarging again Z). Thus upon replacing f by f' , considering its respective leading coefficient in x_{n-1} , and finally analysing each component Z_i , we may assume that the subvariety

$$Y_2 := (g = h = 0) \subset \mathbb{A}_{\mathbb{Q}}^{n-2}$$

satisfies $\text{codim}_{\mathbb{A}_{\mathbb{Q}}^{n-2}}(Y_2) = 2$. We then define

$$(66) \quad \mathcal{Y}_1 := \text{Spec}(\mathbb{Z}[x_1, \dots, x_{n-2}]/(f, g)) \subset \mathbb{A}_{\mathbb{Z}}^{n-2}, \text{ and } \mathcal{Y}_2 := \text{Spec}(\mathbb{Z}[x_1, \dots, x_{n-2}]/(g, h)) \subset \mathbb{A}_{\mathbb{Z}}^{n-2}$$

depending on whether f is constant in x_{n-1} or not.

We now extract the set \mathcal{B}_1 consisting of $\underline{\mathbf{X}} \in \mathbb{Z}^{n-2}$ satisfying **at least one of** the following conditions:

- $g(\underline{\mathbf{X}}) = 0$;
- the polynomial f is constant in x_{n-1} and there exists $p \geq T, \underline{\mathbf{X}} \bmod p \in \mathcal{Y}_1$;
- the polynomial f is non-constant in x_{n-1} , and there exists $p \geq T$ such that $p \mid g(\underline{\mathbf{X}})$, and $f(\underline{\mathbf{X}}, x_{n-1})$, as a polynomial in x_{n-1} , is $\in p\mathbb{Z}[x_{n-1}]$. In particular this implies $p \mid h(\underline{\mathbf{X}})$ and thus $\underline{\mathbf{X}} \bmod p \in \mathcal{Y}_2$.

On rearranging the contribution from $V(T; T, \infty)$ by fixing the first $n-2$ variables and summing over integral points on the fibres of $\text{pr}_3|_{\mathcal{Q}}$, we introduce the quantity

$$V_1(T; T, \infty) := \#\{\underline{\mathbf{X}} = (X_1, \dots, X_n) \in \mathcal{Q}(\mathbb{Z}) : \|\underline{\mathbf{X}}\| \leq T, (X_1, \dots, X_{n-2}) \in \mathcal{B}_1\}$$

to bound the overall contribution from the subcases above. Then we have

$$V_1(T; T, \infty) \leq \sum_{\|\underline{\mathbf{X}}\| \leq T, \underline{\mathbf{X}} \in \mathcal{B}_1} H_1(\underline{\mathbf{X}}) \leq \left(\sum_{\substack{\underline{\mathbf{X}} \in \mathbb{Z}^{n-2}, \|\underline{\mathbf{X}}\| \leq T \\ g(\underline{\mathbf{X}}) = 0}} 1 + \sum_{\substack{\text{(if } f \text{ is constant in } x_{n-1}) \\ \underline{\mathbf{X}} \in \mathbb{Z}^{n-2}, \|\underline{\mathbf{X}}\| \leq T \\ \exists p \geq T, \underline{\mathbf{X}} \bmod p \in \mathcal{Y}_1}} 1 + \sum_{\substack{\text{(if } f \text{ is non-constant in } x_{n-1}) \\ \underline{\mathbf{X}} \in \mathbb{Z}^{n-2}, \|\underline{\mathbf{X}}\| \leq T \\ \exists p \geq T, \underline{\mathbf{X}} \bmod p \in \mathcal{Y}_2}} 1 \right) H_1(\underline{\mathbf{X}}),$$

where for any $\underline{\mathbf{X}} \in \mathbb{Z}^{n-2}$,

$$H_1(\underline{\mathbf{X}}) := \#\{(u, v) \in \mathbb{Z}^2 : u^2 + av^2 = m - \sum_{i=1}^{n-2} a_i X_i^2\}.$$

We have clearly $H_1(\underline{\mathbf{X}}) = O_\varepsilon(T^\varepsilon)$ uniformly for any $\|\underline{\mathbf{X}}\| \leq T$. As for the three sums in bracket, one uses Theorem 4.8 ((1) applied to the the closure in $\mathbb{A}_{\mathbb{Z}}^{n-2}$ of the codimension one subvariety $(g = 0) \subset \mathbb{A}_{\mathbb{Q}}^{n-2}$ for the first sum, and (2) applied to $\mathcal{Y}_1, \mathcal{Y}_2 \subset \mathbb{A}_{\mathbb{Z}}^{n-2}$ in the remaining two sums), and gets

$$(67) \quad V_1(T; T, \infty) = O\left(T^{n-3} + \frac{T^{n-2}}{T \log T} + T^{n-3}\right) \times O_\varepsilon(T^\varepsilon) = O_\varepsilon(T^{n-3+\varepsilon}).$$

Case 2. Let us consider from now on the set \mathcal{B}_2 of $\underline{\mathbf{X}} \in \mathbb{Z}^{n-2}$ satisfying **all of** the conditions below:

- $g(\underline{\mathbf{X}}) \neq 0$;
- for any $p \geq T$, $f(\underline{\mathbf{X}}, x_{n-1}) \bmod p$ is a non-zero polynomial in x_{n-1} .

To give an upper bound for the overall contribution from Case 2, we define the quantity $V_2(T; T, \infty)$ as

$$\#\{\underline{\mathbf{X}} = (X_1, \dots, X_n) \in \mathcal{Q}(\mathbb{Z}) : \|\underline{\mathbf{X}}\| \leq T, \exists p \geq T, p \mid \gcd(f(\underline{\mathbf{X}}), g(\underline{\mathbf{X}})), (X_1, \dots, X_{n-2}) \in \mathcal{B}_2\}.$$

We clearly have

$$V_2(T; T, \infty) \leq \sum_{\substack{\underline{\mathbf{X}} \in \mathcal{B}_2, \|\underline{\mathbf{X}}\| \leq T \\ \exists u, v \in \mathbb{Z}, u^2 + av^2 = m - \sum_{i=1}^{n-2} a_i X_i^2}} H_2(\underline{\mathbf{X}}),$$

where for any $\underline{\mathbf{X}} \in \mathbb{Z}^{n-2}$,

$$H_2(\underline{\mathbf{X}}) := \sum_{\substack{p: p \geq T \\ p \mid g(\underline{\mathbf{X}})}} \sum_{\substack{y \in \mathbb{Z}: |y| \leq T \\ p \mid f(\underline{\mathbf{X}}, y)}} \#\{z \in \mathbb{Z} : z^2 = m - ay^2 - \sum_{i=1}^{n-2} a_i X_i^2\}.$$

Under the assumption that $\underline{\mathbf{X}} \in \mathcal{B}_2$, we have,

$$\sum_{\substack{p: p \geq T \\ p \mid g(\underline{\mathbf{X}}) \neq 0}} 1 = O(\deg g),$$

because $g(\underline{\mathbf{X}}) \neq 0$, and $g(\underline{\mathbf{X}}) \ll T^{\deg g}$, so the number of primes $\geq T$ dividing $g(\underline{\mathbf{X}})$ is $\ll \deg g$. Moreover, uniformly for any $p \geq T$,

$$\sum_{\substack{y \in \mathbb{Z}: |y| \leq T \\ p | f(\underline{\mathbf{X}}, y)}} 1 \ll (\deg f) \left(\frac{T}{p} + 1 \right) = O(\deg f),$$

because $f(\underline{\mathbf{X}}, x_{n-1}) \bmod p$ is a non-zero polynomial in x_{n-1} and hence has at most $\deg f$ roots over \mathbb{F}_p . Hence all implied constants above depend only on the polynomials f, g , that is, the variety Z . So

$$H_2(\underline{\mathbf{X}}) \ll O(1) \times O(1) \times O(1) = O(1).$$

Returning to the error term $V_2(T; T, \infty)$. The bound for $H_2(\underline{\mathbf{X}})$ results in

$$V_2(T; T, \infty) \ll \#C(T) \times O(1),$$

where

$$C(T) := \{ \underline{\mathbf{X}} \in \mathbb{Z}^{n-2} : \|\underline{\mathbf{X}}\| \leq T, \exists u, v \in \mathbb{Z}, u^2 + av^2 = m - \sum_{i=1}^{n-2} a_i X_i^2 \}.$$

We are reduced to bounding $\#C(T)$. For this we appeal to Theorem 1.7, whose proof will be given in the next section, by setting $Q_1(\mathbf{x}) = m - \sum_{i=1}^{n-2} a_i x_i^2$ and $Q_2(y_1, y_2) = y_1^2 + ay_2^2$. If $n \geq 4$, then the affine quadric $(Q_1(\mathbf{x}) = 0) \subset \mathbb{A}^{n-2}$ is clearly smooth. When $n = 3$, the condition $-m \det q \neq \square$ is equivalent to the stated one in Theorem 1.7. So all assumptions of Theorem 1.7 are satisfied. We thus obtain

$$(68) \quad V_2(T; T, \infty) \ll \#C(T) \ll \frac{T^{n-2}}{(\log T)^{\frac{1}{2}}}.$$

Finally, thanks to (65) and the construction of $\mathcal{Y}_1, \mathcal{Y}_2$ (66), when $T > l_0$, we obtain the inequality

$$V(T; T, \infty) \leq V_1(T; T, \infty) + V_2(T; T, \infty).$$

Therefore, the bounds obtained in **Case 1** (67) and in **Case 2** (68) complete the proof. \square

4.4. Proof of Theorem 1.6. Keeping the notation in Theorems 4.1, 4.2, 4.7, it suffices to choose an appropriate parameter α to make these error terms satisfactory. We therefore let $\alpha > 0$ satisfy

$$0 < \alpha < \frac{\beta_{\mathbf{Q}}(n-2)}{\frac{(3n-2)(n-1)}{2} + \dim Z + 1},$$

so that

$$\alpha \left(\frac{(3n-2)(n-1)}{2} + \dim Z + 1 \right) + (n-2)(1 - \beta_{\mathbf{Q}}) < n-2.$$

If $M < T^\alpha$, then with the choice $N_1 = M < N_2 = T^\alpha$ in Theorem 4.1, we have

$$V(T; M, T^\alpha) = O_{\alpha, \beta} \left(\frac{T^{n-2}}{M^{\text{codim}_{\mathbf{Q}}(Z)-1}} + T^{\alpha \left(\frac{(3n-2)(n-1)}{2} + \dim Z + 1 \right) + (n-2)(1 - \beta_{\mathbf{Q}})} \right).$$

Therefore, we finally obtain, since $\text{codim}_{\mathbf{Q}}(Z) \geq 2$,

$$\begin{aligned} V(T; M, \infty) &\leq V(T; M, T^\alpha) + V(T; T^\alpha, T) + V(T; T, \infty) \\ &= O\left(\frac{T^{n-2}}{M^{\text{codim}_{\mathbf{Q}}(Z)-1}} + T^{\alpha\left(\frac{(3n-2)(n-1)}{2} + \dim Z + 1\right) + (n-2)(1-\beta_{\mathbf{Q}})} + \frac{T^{n-2}}{\log T} + \frac{T^{n-2}}{(\log T)^{\frac{1}{2}}}\right) \\ &= O\left(\frac{T^{n-2}}{M^{\text{codim}_{\mathbf{Q}}(Z)-1}} + \frac{T^{n-2}}{(\log T)^{\frac{1}{2}}}\right). \end{aligned}$$

If $M \geq T^\alpha$, then

$$V(T; M, \infty) \leq V(T; T^\alpha, T) + V(T; T, \infty) = O\left(\frac{T^{n-2}}{(\log T)^{\frac{1}{2}}}\right).$$

This is also satisfactory because

$$\frac{T^{n-2}}{M^{\text{codim}_{\mathbf{Q}}(Z)-1}} = o\left(\frac{T^{n-2}}{(\log T)^{\frac{1}{2}}}\right).$$

This finishes the proof of Theorem 1.6. \square

5. A HALF-DIMENSIONAL SIEVE FOR AFFINE QUADRICS

The goal of this section is devoted to proving Theorem 1.7. Our strategy is based on an upper-bound version of Brun-type half-dimensional sieve, developed in works [19, 15]. This is recorded in §5.2. Keeping the notation in Theorem 1.7, let us fix throughout this section a quadratic polynomial $Q_1(\mathbf{x}) \in \mathbb{Z}[x_1, \dots, x_L]$, and a primitive positive-definite non-degenerate quadratic form $Q_2(\mathbf{y}) \in \mathbb{Z}[y_1, y_2]$. Let D_{Q_2} be the discriminant of the form Q_2 . We have $D_{Q_2} \leq -3$. In §5.1 we collect some well-known facts about representation of integers by primitive binary quadratic forms of negative discriminant. We shall give full details of the proof of Theorem 1.7 for the case $L \geq 2$ in §5.3, and we indicate necessary modifications in §5.4 for the case $L = 1$, which is a classical result going back to Bernays [2] (see also [15, p. 2 Remarks 2]).

5.1. Representation by binary quadratic forms. Let us define two arithmetic functions $\mathfrak{b}(\cdot)$ and $\mathfrak{b}^*(\cdot)$, characterizing integers represented by the form $Q_2(u, v)$. First define

$$(69) \quad \mathcal{P}_{Q_2} := \left\{ p : \left(\frac{D_{Q_2}}{p} \right) = -1 \right\},$$

where $\left(\frac{\cdot}{p} \right)$ is the Legendre symbol of modulus p . For $n \in \mathbb{Z}$, let

$$(70) \quad \mathfrak{b}(n) := \begin{cases} 1 & \text{if } \exists u, v \in \mathbb{Z}, n = Q_2(u, v); \\ 0 & \text{otherwise,} \end{cases}$$

and

$$(71) \quad \mathfrak{b}^*(n) := \begin{cases} 1 & \text{if } \forall p \mid n, p \notin \mathcal{P}_{Q_2}; \\ 0 & \text{if } \exists p \in \mathcal{P}_{Q_2}, p \mid n. \end{cases}$$

Fix $n \in \mathbb{N}_{\neq 0}$. We recall that (see for example [21, Lemmas 1 & 2]), if there exists $p \in \mathcal{P}_{Q_2}$ such that $n = p^{2k+1}m$, with $k \in \mathbb{N}, m \in \mathbb{N}_{\neq 0}, \gcd(p, m) = 1$, then $\mathfrak{b}(n) = 0$. So, if $\mathfrak{b}(n) = 1$, then for any $p \in \mathcal{P}_{Q_2}, p \mid n$, there exists $k \in \mathbb{N}$, such that $p^{2k} \parallel n$. We conclude from this analysis that if

$\mathfrak{b}(n) = 1$ then $\mathfrak{b}^*(n/r) = 1$ with certain perfect square r dividing n , whose prime divisors (if any) are all in \mathcal{P}_{Q_2} . The function \mathfrak{b}^* is clearly multiplicative, however it is not the case for \mathfrak{b} in general.

5.2. The half-dimensional sieve. Let $\mathcal{A} = (a_i)_{i \in I}$ be a finite sequence of integers indexed by I . For any $r \in \mathbb{N}_{\neq 0}$, let \mathcal{A}_r be the subsequence consisting of elements of \mathcal{A} divisible by r . That is, $\mathcal{A}_r = (a_i)_{i \in I_r}$ with $I_r = \{i \in I : r \mid a_i\}$. Let \mathcal{P} be a subset of prime numbers. For any $z > 1$, define the sifting function

$$S(\mathcal{A}, \mathcal{P}, z) := \#\{i \in I : \gcd(a_i, \prod_{\substack{p \in \mathcal{P} \\ p < z}} p) = 1\}.$$

We will employ the following version of the half-dimensional sieve due to Friedlander and Iwaniec.

Theorem 5.1 ([19] Theorem 1, [15] Lemma 1). *There exists a continuous function $G :]0, \infty[\rightarrow \mathbb{R}_{>0}$ such that, for any multiplicative arithmetic function ρ satisfying*

- $0 \leq \rho(p) < p$ for any $p \in \mathcal{P}$;
- there exists $K > 0$ such that

$$(72) \quad \left| \sum_{\substack{p \in \mathcal{P} \\ p < x}} \frac{\rho(p)}{p - \rho(p)} \log p - \frac{1}{2} \log x \right| \leq K$$

for any $x \geq 2$,

we have, for any $y, z \geq 2$,

$$S(\mathcal{A}, \mathcal{P}, z) \leq \Lambda \left(G(w) + \mathcal{D}_{K,w} (\log y)^{-\frac{1}{5}} \right) \prod_{\substack{p \in \mathcal{P} \\ p < z}} \left(1 - \frac{\rho(p)}{p} \right) + \sum_{\substack{d < y \\ p \mid d \Rightarrow p < z, p \in \mathcal{P}}} |\mu(d) R(d)|,$$

where

$$w := \frac{\log y}{\log z}, \quad \Lambda := \#\mathcal{A} (= \#I), \quad R(d) := \#\mathcal{A}_d - \frac{\rho(d)}{d} \Lambda,$$

and $\mathcal{D}_{K,w} > 0$ depends only on K, w .

5.3. Proof of Theorem 1.7 for the case $L \geq 2$. With the notation in §5.1, our ultimate goal is to deduce the desired non-trivial upper bound for

$$(73) \quad \sum_{\underline{\mathbf{X}} \in \mathbb{Z}^L : \|\underline{\mathbf{X}}\| \leq T} \mathfrak{b}(Q_1(\underline{\mathbf{X}}))$$

via applying Theorem 5.1 to $S(\mathcal{A}, \mathcal{P}, z)$ with appropriately chosen $\rho, \mathcal{A}, \mathcal{P}, z$. Firstly it is convenient to deal separately with the $\underline{\mathbf{X}}$'s with $Q_1(\underline{\mathbf{X}}) = 0$, because $\mathfrak{b}(0) = 1$ but $\mathfrak{b}^*(0) = 0$. For this we use the evident upper bound (Theorem 4.8 (1))

$$(74) \quad B_1(T) := \#\{\underline{\mathbf{X}} \in \mathbb{Z}^L : \|\underline{\mathbf{X}}\| \leq T, Q_1(\underline{\mathbf{X}}) = 0\} \ll T^{L-1}.$$

Now the discussion in §5.1 shows that

$$(75) \quad \sum_{\underline{\mathbf{X}} \in \mathbb{Z}^L : \|\underline{\mathbf{X}}\| \leq T} \mathfrak{b}(Q_1(\underline{\mathbf{X}})) \leq B_1(T) + \sum_r^* \sum_{\substack{\underline{\mathbf{X}} \in \mathbb{Z}^L : \|\underline{\mathbf{X}}\| \leq T \\ r \mid Q_1(\underline{\mathbf{X}})}} \mathfrak{b}^*(Q_1(\underline{\mathbf{X}})/r),$$

where the sum with superscript $*$ means that r is restricted to all non-zero perfect squares whose prime divisors are all in \mathcal{P}_{Q_2} (69).

We consider the sequence

$$\mathcal{A}(T) := \{Q_1(\underline{\mathbf{X}})\}_{\underline{\mathbf{X}} \in \mathbb{Z}^L: \|\underline{\mathbf{X}}\| \leq T}.$$

For $r \in \mathbb{N}_{\neq 0}$, we define the subsequence

$$\mathcal{A}(T)_r := \{Q_1(\underline{\mathbf{X}})/r\}_{\substack{\underline{\mathbf{X}} \in \mathbb{Z}^L: \|\underline{\mathbf{X}}\| \leq T \\ r | Q_1(\underline{\mathbf{X}})}}.$$

Consider the arithmetic multiplicative functions ω, ϱ , defined for $N \in \mathbb{N}_{\neq 0}$ by

$$(76) \quad \omega(N) := \#\{\xi \in (\mathbb{Z}/N\mathbb{Z})^L : Q_1(\xi) \equiv 0 \pmod{N}\}, \quad \varrho(N) := \frac{\omega(N)}{N^{L-1}}.$$

Then $\omega(N) \ll_{\varepsilon} N^{L-1+\varepsilon}$. We have, by the Chinese remainder theorem, that for any $N \in \mathbb{N}_{\neq 0}$,

$$(77) \quad \begin{aligned} \#\mathcal{A}(T)_N &= \#\{\underline{\mathbf{X}} \in \mathbb{Z}^L : \|\underline{\mathbf{X}}\| \leq T, r \mid Q_1(\underline{\mathbf{X}})\} \\ &= \sum_{\substack{\xi \in (\mathbb{Z}/N\mathbb{Z})^L \\ Q_1(\xi) \equiv 0 \pmod{N}}} \#\{\mathbf{y} \in (N\mathbb{Z})^L : \|\mathbf{y} + \xi\| \leq T\} \\ &= \omega(N) \left(\frac{2T}{N} + O(1) \right)^L. \end{aligned}$$

By assumption, the affine variety $(Q_1 = 0) \subset \mathbb{A}_{\mathbb{Q}}^L$. Then there exists $l_0 \in \mathbb{N}_{\neq 0}$ such that the affine scheme $\text{Spec}(\mathbb{Z}[x_1, \dots, x_L]/(Q_1(\mathbf{x})))$ is smooth over $\mathbb{Z}[1/l_0]$ with geometrically integral fibres. So by the Lang-Weil estimate (10), for any $p \nmid l_0$, we have

$$\varrho(p) = 1 + O(p^{-\frac{1}{2}}).$$

Let us define

$$(78) \quad \mathcal{P}'_{Q_2} = \mathcal{P}_{Q_2} \setminus \{p : \varrho(p) = p\}.$$

Since there are at most finitely many primes p satisfying $\varrho(p) = p$, the primes in the set \mathcal{P}'_{Q_2} have the same density as those in \mathcal{P}_{Q_2} , namely one half (amongst the prime residues modulo $4D'_{Q_2}$ with $D'_{Q_2} \mid D_{Q_2}$ such that D'_{Q_2} is square-free and D_{Q_2}/D'_{Q_2} is a square (cf. [29, VI. Propositions 5 & 14])). With these notions, for any $r \in \mathbb{N}_{\neq 0}, \lambda > 0$, one has

$$(79) \quad \sum_{\underline{\mathbf{X}} \in \mathbb{Z}^L: \|\underline{\mathbf{X}}\| \leq T} \mathfrak{b}^*(Q_1(\underline{\mathbf{X}})/r) \leq S(\mathcal{A}(T)_r, \mathcal{P}_{Q_2}, T^\lambda) \leq S(\mathcal{A}(T)_r, \mathcal{P}'_{Q_2}, T^\lambda).$$

We next claim that it suffices to deal with sufficiently small r 's in (75), more precisely $r < T^\gamma$ for certain $0 < \gamma < \Delta_L := \frac{1}{4L}$. Whenever there exists $r = q^2 \mid Q_1(\underline{\mathbf{X}}) \neq 0$ for some $\|\underline{\mathbf{X}}\| \leq T$, we always have $r \leq |Q_1(\underline{\mathbf{X}})| \ll T^2$, in other words, $q \leq \sqrt{|Q_1(\underline{\mathbf{X}})|} \ll T$, and so by (77),

$$\begin{aligned} \#\mathcal{A}(T)_{q^2} &\ll \omega(q^2) \times \left(\left(\frac{T}{q^2} \right)^L + 1 \right) \\ &\ll_{\varepsilon} (q^2)^{L-1+\varepsilon} \times \left(\left(\frac{T}{q^2} \right)^L + 1 \right) \\ &\ll_{\varepsilon} \frac{T^L}{q^{2-\varepsilon}} + q^{2L-2+\varepsilon}. \end{aligned}$$

Therefore the contribution from all $r \in [T^\gamma, T^{1+2\Delta_L}]$ is

$$(80) \quad \sum_{r \in [T^\gamma, T^{1+2\Delta_L}]}^* \sum_{\substack{\mathbf{x} \in \mathbb{Z}^L: \|\mathbf{x}\| \leq T \\ r | Q_1(\mathbf{x})}} \mathfrak{b}^*(Q_1(\mathbf{x})/r) \leq \sum_{q \in [T^{\frac{\gamma}{2}}, T^{\frac{1}{2} + \Delta_L}]} \#\mathcal{A}(T)_{q^2} \ll_\varepsilon T^{L - \frac{\gamma}{2} + \varepsilon} + T^{L - \Delta_L + \varepsilon} \ll T^{L - \frac{\gamma}{2} + \varepsilon}.$$

This is satisfactory compared to the expected leading term $\frac{T^L}{\sqrt{\log T}}$. Next, for

$$(81) \quad T^{1+2\Delta_L} < r \ll T^2 \Leftrightarrow T^{\frac{1}{2} + \Delta_L} < q \ll T,$$

we regard the $(L+1)$ -tuple $(\underline{\mathbf{x}}, q) = (X_1, \dots, X_L, q) \in \mathbb{Z}^{L+1}$ as an integral point on the affine quadric

$$Q_s(\mathbf{x}, z) : (Q_1(\mathbf{x}) - sz^2 = 0) \subset \mathbb{A}^{L+1},$$

where s is an auxiliary integer parameter satisfying $0 \neq s \ll T^{1-2\Delta_L}$, thanks to the preassigned bound (81) for q and the fact that if $Q_1(\mathbf{x}) = 0$ then $\mathfrak{b}^*(Q_1(\mathbf{x})/r) = 0$ for any $r \in \mathbb{N}_{\neq 0}$. We want to insert the uniform upper bound estimate in Theorem 4.3 for the quadrics $Q_s(\mathbf{x}, z)$ with $s \neq 0$. Recall that we assume $L \geq 2$, so $\text{rk}(Q_s)_0 \geq 2$ whenever $s \neq 0$, and moreover the quadratic polynomial $Q_1(\mathbf{x}) - sz^2$ is irreducible. Since otherwise $Q_1(\mathbf{x}) - sz^2 = (s_1z + A_1(\mathbf{x}))(s_2z + A_2(\mathbf{x}))$ and this would imply that $Q_1(\mathbf{x}) = A_1(\mathbf{x})A_2(\mathbf{x})$, a contradiction to the assumption that $(Q_1 = 0)$ is smooth. So the hypotheses of Theorem 4.3 are satisfied. We conclude that the contribution of such r 's satisfying (81) is

$$(82) \quad \sum_{T^{1+2\Delta_L} < r \ll T^2}^* \sum_{\substack{\mathbf{x} \in \mathbb{Z}^L: \|\mathbf{x}\| \leq T \\ r | Q_1(\mathbf{x})}} \mathfrak{b}^*(Q_1(\mathbf{x})/r) \ll \sum_{0 \neq s \ll T^{1-2\Delta_L}} \#\{(\underline{\mathbf{x}}, q) \in \mathbb{Z}^{L+1} : \|\underline{(\mathbf{x}, q)}\| \leq T, Q_s(\underline{\mathbf{x}}, q) = 0\} \\ \ll_\varepsilon \sum_{s \ll T^{1-2\Delta_L}} T^{L+1-2+\varepsilon} \ll T^{L-2\Delta_L+\varepsilon}.$$

This is also satisfactory and proves our claim. Gathering together (75) (79), equation (73) now becomes

$$(83) \quad \sum_{\mathbf{x} \in \mathbb{Z}^L: \|\mathbf{x}\| \leq T} \mathfrak{b}(Q_1(\mathbf{x})) \leq B_1(T) + \sum_{r < T^\gamma}^* \sum_{\substack{\mathbf{x} \in \mathbb{Z}^L: \|\mathbf{x}\| \leq T \\ r | Q_1(\mathbf{x})}} \mathfrak{b}^*(Q_1(\mathbf{x})/r) + \sum_{r \geq T^\gamma}^* \sum_{\substack{\mathbf{x} \in \mathbb{Z}^L: \|\mathbf{x}\| \leq T \\ r | Q_1(\mathbf{x})}} \mathfrak{b}^*(Q_1(\mathbf{x})/r) \\ \leq B_2(T) + \sum_{r < T^\gamma}^* S(\mathcal{A}(T)_r, \mathcal{P}'_{Q_2}, T^\lambda),$$

where

$$B_2(T) = B_1(T) + \sum_{r \geq T^\gamma}^* \sum_{\substack{\mathbf{x} \in \mathbb{Z}^L: \|\mathbf{x}\| \leq T \\ r | Q_1(\mathbf{x})}} \mathfrak{b}^*(Q_1(\mathbf{x})/r) = O_\varepsilon(T^{L - \frac{\gamma}{2} + \varepsilon})$$

according to (74) (80) (82), and $0 < \gamma < \Delta_L$, $\lambda > 0$ are to be chosen later. Everything now boils down to the estimation of $S(\mathcal{A}(T)_r, \mathcal{P}'_{Q_2}, T^\lambda)$.

Our task is to apply Theorem 5.1 to each $\mathcal{A}(T)_r$. For this purpose we fix $r \in \mathbb{N}_{\neq 0}$. If $\omega(r) = 0$ then the subsequence $\mathcal{A}(T)_r$ is empty. We therefore assume from now on that $\omega(r) \neq 0$. By taking

a crude estimate on the error term of (77), we get

$$(84) \quad \Lambda_r := \#\mathcal{A}(T)_r = \frac{\omega(r)}{r^L} (2T)^L + O_\varepsilon(r^{L-1+\varepsilon}T^{L-1}).$$

We define arithmetic functions

$$\omega_r(N) := \frac{\omega(rN)}{\omega(r)}, \quad \varrho_r(N) := \frac{\omega_r(N)}{N^{L-1}} = \frac{\varrho(rN)}{\varrho(r)}, \quad N \in \mathbb{N}_{\neq 0}.$$

Let us now verify that the functions ω_r and ϱ_r are multiplicative. Indeed, for any $N \in \mathbb{N}_{\neq 0}$, we factorize $N = N_1N_2, r = r_1r_2$ such that $\gcd(r_1N_1, r_2N_2) = 1$. Then

$$\begin{aligned} \omega_r(N) &= \frac{\omega(r_1r_2N_1N_2)}{\omega(r_1r_2)} = \frac{\omega(r_1N_1)}{\omega(r_1)} \frac{\omega(r_2N_2)}{\omega(r_2)} \\ &= \frac{\omega(r_1N_1)\omega(r_2)}{\omega(r_1)\omega(r_2)} \frac{\omega(r_2N_2)\omega(r_1)}{\omega(r_2)\omega(r_1)} \\ &= \frac{\omega(rN_1)}{\omega(r)} \frac{\omega(rN_2)}{\omega(r)} = \omega_r(N_1)\omega_r(N_2). \end{aligned}$$

By Hensel's lemma, for any $p \nmid l_0$ and $p \mid r$, one has $\omega(pr) = \omega(r)p^{L-1}$, and by the Chinese remainder theorem, for any $p \nmid r$, $\omega(rp) = \omega(r)\omega(p)$. Hence

$$(85) \quad \varrho_r(p) = \begin{cases} \varrho(p) & \text{if } p \nmid r; \\ 1 & \text{if } p \mid r. \end{cases}$$

This implies $\varrho_r(p) = 1 + O(p^{-\frac{1}{2}})$ uniformly for any prime p and any $r \in \mathbb{N}_{\neq 0}$, so by Mertens' first theorem on arithmetic progressions (cf. e.g. [20, Theorem 2.2]), the hypotheses of Theorem 5.1, in particular (72), are satisfied for the arithmetic function ϱ_r and the set \mathcal{P}'_{Q_2} uniformly for any r (that is, the remainder term K in (72) depends only on Q_1, Q_2 and is independent of r). Next we need to evaluate, for each $d \in \mathbb{N}_{\neq 0}$, the cardinality of the subsequence $\mathcal{A}(T)_{rd}$, using (84) and the definition of ω_r, ϱ_r :

$$\begin{aligned} \#\mathcal{A}(T)_{rd} &= \frac{\omega(dr)}{(dr)^L} (2T)^L + O(\omega(dr)T^{L-1}) \\ &= \frac{\omega_r(d)}{d^L} \frac{\omega(r)}{r^L} (2T)^L + O((dr)^{L-1+\varepsilon}T^{L-1}) \\ &= \frac{\varrho_r(d)}{d} \Lambda_r + O((dr)^{L-1+\varepsilon}T^{L-1}). \end{aligned}$$

Define

$$R_r(d) := \#\mathcal{A}(T)_{rd} - \frac{\varrho_r(d)}{d} \Lambda_r.$$

The above computation shows that

$$R_r(d) = O_\varepsilon((dr)^{L-1+\varepsilon}T^{L-1}).$$

On applying Theorem 5.1 to $\varrho_r, \mathcal{A}(T)_r, \mathcal{P}'_{Q_2}$ for each perfect square r with all prime divisors in \mathcal{P}_{Q_2} , with $z = T^\lambda, y = T^\beta$, for $\lambda, \beta > 0$ small enough, we get,

$$\begin{aligned} & S(\mathcal{A}(T)_r, \mathcal{P}'_{Q_2}, T^\lambda) \\ & \leq \left(G(\beta/\lambda) + \mathcal{D}_{\lambda, \beta} (\log T)^{-\frac{1}{5}} \right) \Lambda_r \prod_{\substack{p: p \in \mathcal{P}'_{Q_2} \\ p < T^\lambda}} \left(1 - \frac{\varrho_r(p)}{p} \right) + \sum_{d < T^\beta} |R_r(d)| \\ & = \left(G(\beta/\lambda) + \mathcal{D}_{\lambda, \beta} (\log T)^{-\frac{1}{5}} \right) \left(\frac{\omega(r)}{r^L} \prod_{\substack{p: p \in \mathcal{P}'_{Q_2} \\ p < T^\lambda}} \left(1 - \frac{\varrho_r(p)}{p} \right) \right) (2T)^L + O_\varepsilon \left(\sum_{d < T^\beta} (dr)^{L-1+\varepsilon} T^{L-1} \right), \end{aligned}$$

where $\mathcal{D}_{\lambda, \beta} > 0$ depends only on λ, β, Q_1, Q_2 . Thanks to (85), the leading term in the last expression, up to the factor $\left(G(\beta/\lambda) + \mathcal{D}_{\lambda, \beta} (\log T)^{-\frac{1}{5}} \right) (2T)^L$, can be written as

$$\left(\prod_{\substack{p: p < T^\lambda \\ p \in \mathcal{P}'_{Q_2}} \left(1 - \frac{\varrho(p)}{p} \right) \right) \times \left(\frac{\omega(r)}{r^L} \prod_{\substack{p: p|r \\ p \in \mathcal{P}'_{Q_2}}} \left(\left(1 - \frac{1}{p} \right) \left(1 - \frac{\varrho(p)}{p} \right)^{-1} \right) \right).$$

We are finally in a position to evaluate the sum in (83) as follows. The series $\sum_{r=\square} c_r$, formed by

$$c_r := \frac{\omega(r)}{r^L} \prod_{\substack{p: p|r \\ p \in \mathcal{P}'_{Q_2}}} \left(\left(1 - \frac{1}{p} \right) \left(1 - \frac{\varrho(p)}{p} \right)^{-1} \right) = \frac{\omega(r)}{r^L} \prod_{\substack{p: p|r \\ p \in \mathcal{P}'_{Q_2}}} \frac{p-1}{p-\varrho(p)},$$

converges, because

$$c_r \ll_\varepsilon \frac{r^{L-1+\varepsilon}}{r^L} \times r^\varepsilon \ll_\varepsilon \frac{1}{r^{1-\varepsilon}}.$$

Therefore,

$$\begin{aligned} & (86) \\ & \sum_{\substack{r=\square, r < T^\gamma \\ p|r \Rightarrow p \in \mathcal{P}_{Q_2}}} S(\mathcal{A}(T)_r, \mathcal{P}'_{Q_2}, T^\lambda) \\ & \ll_\varepsilon \left(\left(G(\beta/\lambda) + \mathcal{D}_{\lambda, \beta} (\log T)^{-\frac{1}{5}} \right) (2T)^L \prod_{\substack{p: p < T^\lambda \\ p \in \mathcal{P}'_{Q_2}}} \left(1 - \frac{\varrho(p)}{p} \right) \right) \left(\sum_{\substack{r=\square, r < T^\gamma \\ p|r \Rightarrow p \in \mathcal{P}_{Q_2}}} c_r \right) + \sum_{\substack{r, d \in \mathbb{N}_{\neq 0} \\ r < T^\gamma, d < T^\beta}} (dr)^{L-1+\varepsilon} T^{L-1} \\ & = O_{\gamma, \lambda, \beta} \left(\frac{T^L}{\sqrt{(\log T)}} \left(1 + (\log T)^{-\frac{1}{5}} \right) \right) + O_{\gamma, \lambda, \beta, \varepsilon} (T^{L-1+(\gamma+\beta)L+\varepsilon}). \end{aligned}$$

Upon choosing $\lambda, \gamma, \beta, \varepsilon > 0$ small enough, this finishes the proof of Theorem 1.7 for the case $L \geq 2$. \square

Remark 5.2. It would be interesting to ask whether a lower bound of expected magnitude $\frac{T^L}{\sqrt{\log T}}$ exists for (73), just as was established in [15, Theorem 1] for the case $L = 1$ under some mild assumptions.

5.4. Sketch of proof of Theorem 1.7 for the case $L = 1$. In this case it is equivalent to showing the following. For $b_1, b_2 \in \mathbb{Z}_{\neq 0}$, assume that $-D_{Q_2}b_1b_2 \neq \square$. Then

$$(87) \quad \#\{x \in \mathbb{Z} : |x| \leq T : \exists u, v \in \mathbb{Z}, b_1x^2 + b_2 = Q_2(u, v)\} \ll \frac{T}{\sqrt{\log T}}.$$

We may assume that $b_1 > 0$, since the set above has finite cardinality if $b_1 < 0$ and the upper bound (87) is trivially satisfied. Most reasoning (especially the treatment of error terms) is akin to §5.3. We choose to only outline how the dominant term comes out, which is the major difference between these two cases. Let \mathfrak{D} be the square-free part of $-b_1b_2$.

Let us first assume $\mathfrak{D} = 1$. Upon change of variables we are reduced to the case where $b_1 = 1$ and $b_2 = b^2 = \square$, so that the identity inside (87) is written as

$$(x + b)(x - b) = Q_2(u, v).$$

Since $\gcd(x + b, x - b) \mid 2b$, on defining

$$L_1(x) := x + b, \quad L_2(x) := x - b,$$

and

$$\mathcal{P}_{Q_2}^\Delta := \mathcal{P}_{Q_2} \cup \{p : p \mid 2b\}, \quad \mathfrak{b}^{*\Delta}(n) = \begin{cases} 1 & \text{if } \forall p \mid n, p \notin \mathcal{P}_{Q_2}^\Delta; \\ 0 & \text{if } \exists p \in \mathcal{P}_{Q_2}^\Delta, p \mid n, \end{cases}$$

then our discussion in §5.1 shows that (75) can be modified correspondingly as

$$\sum_{\mathbf{x} \in \mathbb{Z} : \|\mathbf{x}\| \leq T} \mathfrak{b}(Q_1(\mathbf{X})) \leq B_1(T) + \sum_{r^\Delta}^* \sum_{\substack{\mathbf{x} \in \mathbb{Z} : \|\mathbf{x}\| \leq T \\ r^\Delta \mid L_1(\mathbf{X})}} \mathfrak{b}^{*\Delta}(L_1(\mathbf{X})/r^\Delta) + \sum_{r^\Delta}^* \sum_{\substack{\mathbf{x} \in \mathbb{Z} : \|\mathbf{x}\| \leq T \\ r^\Delta \mid L_2(\mathbf{X})}} \mathfrak{b}^{*\Delta}(L_2(\mathbf{X})/r^\Delta),$$

where $B_1(T)$ is the same as (74) and the sums with superscript $*$ are over non-zero squared integers r^Δ such that $p \mid r^\Delta \Rightarrow p \in \mathcal{P}_{Q_2}^\Delta$. The remaining argument is identical and thus omitted.

Let us now discuss the case where $\mathfrak{D} \neq 1$, which means that the polynomial $Q_1(\mathbf{x})$ is irreducible over \mathbb{Q} . In particular the Legendre symbol $\left(\frac{\mathfrak{D}}{\cdot}\right)$ is non-constant. Let us keep using the notation \mathcal{P}_{Q_2} (69) and $\omega(\cdot), \varrho(\cdot)$ (76) defined in §5.3. Then for any $p \nmid 2b_1b_2$,

$$\omega(p) = \varrho(p) = \begin{cases} 2 & \text{if } \left(\frac{\mathfrak{D}}{p}\right) = 1; \\ 0 & \text{otherwise.} \end{cases}$$

Since $-D_{Q_2}b_1b_2 \neq \square$, the map

$$\begin{aligned} \{p \text{ prime} : p \nmid 2b_1b_2D_{Q_2}\} &\longrightarrow \{\pm 1\}^2 \\ p &\longmapsto \left(\left(\frac{D_{Q_2}}{p} \right), \left(\frac{\mathfrak{D}}{p} \right) \right) \end{aligned}$$

is surjective. On the one hand if $|D_{Q_2}\mathfrak{D}| \neq \square$, let D_0 be the square-free part of $D_{Q_2}\mathfrak{D}$. Therefore we have $|D_0| \neq 1$, and so primes in exactly one quarter of the residue classes modulo $4|D_0|$ satisfy

$$(88) \quad \left(\frac{D_{Q_2}}{p} \right) = - \left(\frac{\mathfrak{D}}{p} \right) = -1.$$

On the other hand, if $|D_{Q_2}\mathfrak{D}| = \square$, then in this case primes in exactly one quarter of the residue classes modulo $4|\mathfrak{D}|$ satisfy (88). To summarize, recall \mathcal{P}'_{Q_2} (78) and let

$$\mathcal{P}''_{Q_2} := \mathcal{P}'_{Q_2} - \{p : p \mid 2b_1b_2\}.$$

Then by Mertens' theorem regarding primes in arithmetic progressions, one deduces that

$$\sum_{\substack{p \in \mathcal{P}''_{Q_2} \\ p < x}} \frac{\varrho(p)}{p - \varrho(p)} \log p \sim 2 \sum_{\substack{p: \left(\frac{D_{Q_2}}{p}\right) = -\left(\frac{\varrho}{p}\right) = -1 \\ p < x}} \frac{\log p}{p} \sim 2 \times \frac{1}{4} \log x = \frac{1}{2} \log x.$$

Therefore ϱ satisfies the condition (72) in the half-dimensional sieve (Theorem 5.1). So the dominant term in (86) takes the desired form:

$$T \prod_{\substack{p \in \mathcal{P}''_{Q_2} \\ p < T^\lambda}} \left(1 - \frac{\varrho(p)}{p}\right) \asymp T \prod_{\substack{p: \left(\frac{D_{Q_2}}{p}\right) = -\left(\frac{\varrho}{p}\right) = -1 \\ p < T^\lambda}} \left(1 - \frac{2}{p}\right) \asymp \frac{T}{\sqrt{\log T}}.$$

Remark 5.3. To further clarify Remark 1.8 at this point, note that if $-D_{Q_2}b_1b_2 = \square$, which is equivalent to $D_{Q_2}\mathfrak{D} = \square$, then

$$\left(\frac{D_{Q_2}}{p}\right) = -1 \Leftrightarrow \left(\frac{\mathfrak{D}}{p}\right) = -1.$$

The main term in the sifting function now grows like

$$T \prod_{\substack{p \in \mathcal{P}'_{Q_2} \\ p < T^\lambda}} \left(1 - \frac{\varrho(p)}{p}\right) = T \prod_{\substack{p \in \mathcal{P}_{Q_2} \\ p < T^\lambda}} \left(1 - \frac{2}{p}\right) \asymp T,$$

which does not give the log saving.

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