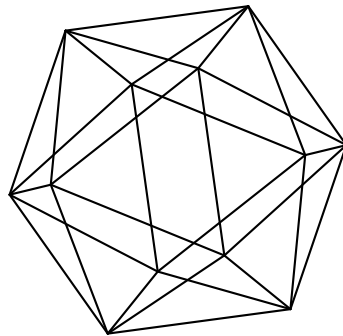


Max-Planck-Institut für Mathematik Bonn

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PARABOLICITY CONJECTURE OF F -ISOCRYSTALS

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ABSTRACT. In this article we prove Crew's parabolicity conjecture of F -isocrystals. For this purpose, we introduce and study the notion of \dagger -hull of a sub- F -isocrystal. On the way, we prove a new Lefschetz theorem for overconvergent F -isocrystals.

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1. INTRODUCTION

1.1. Main results. Let X be a smooth geometrically connected variety over a perfect field k of positive characteristic p and let n be a positive integer. For an overconvergent F^n -isocrystal $(\mathcal{M}^\dagger, \Phi_{\mathcal{M}}^\dagger)$ over X we write $(\mathcal{M}, \Phi_{\mathcal{M}})$ for the associated F^n -isocrystal. Suppose that $(\mathcal{M}, \Phi_{\mathcal{M}})$ admits the slope filtration

$$0 = S_0(\mathcal{M}) \subsetneq S_1(\mathcal{M}) \subsetneq \dots \subsetneq S_m(\mathcal{M}) = \mathcal{M}.$$

If η is a point of X with perfect residue field, Crew defined in [Cre92a] two algebraic *monodromy groups* for \mathcal{M}^\dagger with respect to η , that we denote by $G(\mathcal{M}, \eta)$ and $G(\mathcal{M}^\dagger, \eta)$. The former algebraic group is a subgroup of the latter and they both are subgroups of $\mathrm{GL}(\mathcal{M}_\eta)$, where \mathcal{M}_η is the fibre of \mathcal{M} at η . In this article we prove the following fundamental result about these groups.

Theorem 1.1.1 (Theorem 4.4.13). *The subgroup $G(\mathcal{M}, \eta) \subseteq G(\mathcal{M}^\dagger, \eta)$ is the subgroup of $G(\mathcal{M}^\dagger, \eta)$ stabilising the slope filtration of \mathcal{M}_η . Moreover, when \mathcal{M}^\dagger is semi-simple, $G(\mathcal{M}, \eta)$ is a parabolic subgroup of $G(\mathcal{M}^\dagger, \eta)$.*

This solves the *parabolicity conjecture*, initially proposed as a question in [Cre92a, page 460]. Partial results of this conjecture have been obtained in [Cre92b], [Cre94], [Tsu02], [AD18], and [Tsu19]. Theorem 1.1.1 is a natural enhancement of Kedlaya's full faithfulness theorem, proven in [Ked04].

When k is a finite field and \mathcal{M}^\dagger is semi-simple, the group $G(\mathcal{M}^\dagger, \eta)$ is “the same” as the geometric monodromy group of the semi-simple ℓ -adic *companions* of $(\mathcal{M}^\dagger, \Phi_{\mathcal{M}}^\dagger)$, namely those semi-simple ℓ -adic lisse sheaves with the same L -function as $(\mathcal{M}^\dagger, \Phi_{\mathcal{M}}^\dagger)$ (see [Pál15], [Dri18], and [D'Ad20]). In this particular case, we deduce from Theorem 1.1.1 the following semi-simplicity result.

Theorem 1.1.2 (Theorem 5.1.6). *Let X be a smooth variety over a finite field and $f : A \rightarrow X$ an abelian scheme with constant slopes. If $(\mathcal{M}, \Phi_{\mathcal{M}})$ is the F -isocrystal $R^1 f_{\mathrm{crys}*} \mathcal{O}_{A, \mathrm{crys}}$, the induced F -isocrystal $\mathrm{Gr}_{S_\bullet}(\mathcal{M}, \Phi_{\mathcal{M}})$ is semi-simple. In particular, $R^1 f_{\mathrm{ét}*} \underline{\mathbb{Q}}_p$ is a semi-simple lisse \mathbb{Q}_p -sheaf over X .*

As an additional outcome of the theory developed, we are also able to prove a new diophantine result for abelian varieties.

Theorem 1.1.3 (Theorem 5.2.2). *Let E be a finitely generated field extension of \mathbb{F}_p and let A be an abelian variety over E . The group $A(E^{\mathrm{sep}})[p^\infty]$ is finite in the following two case.*

- (i) *If $\mathrm{End}(A) \otimes_{\mathbb{Z}} \mathbb{Q}_p$ is a division algebra.*
- (ii) *If $\mathrm{End}(A) \otimes_{\mathbb{Z}} \mathbb{Q}$ has no factor of Albert-type IV.*

This theorem enriches the list of known results on the finiteness of separable p -torsion points of abelian varieties (see [Vol95] and [Rös17]). It is worth mentioning that abelian varieties with finite separable p -torsion play an important role in the theory of Brauer–Manin obstructions in positive characteristic (see [PV10]).

As a further consequence of Theorem 1.1.1, we solve a conjecture proposed by Kedlaya in [Ked16, Rmk. 5.14].

Corollary 1.1.4 (Corollary 5.3.1). *Let X be a smooth connected variety over a perfect field k and let $(\mathcal{M}_1^\dagger, \Phi_{\mathcal{M}_1}^\dagger)$ and $(\mathcal{M}_2^\dagger, \Phi_{\mathcal{M}_2}^\dagger)$ be two irreducible overconvergent F^n -isocrystals over X with constant*

slopes. If $(S_1(\mathcal{M}_1), \Phi_{\mathcal{M}_1}|_{S_1(\mathcal{M}_1)})$ and $(S_1(\mathcal{M}_2), \Phi_{\mathcal{M}_2}|_{S_1(\mathcal{M}_2)})$ are isomorphic F^n -isocrystals, then $(\mathcal{M}_1^\dagger, \Phi_{\mathcal{M}_1}^\dagger)$ and $(\mathcal{M}_2^\dagger, \Phi_{\mathcal{M}_2}^\dagger)$ are isomorphic overconvergent F^n -isocrystals.

The conjecture is also proved in [Tsu19] when X has dimension 1 or k is a finite field. Using Corollary 1.1.4 in combination with Langlands reciprocity conjecture for overconvergent isocrystals, proven in [Abe18], and Chebotarev density theorem, we also obtain the following stronger form of the multiplicity one theorem for cuspidal automorphic representations.

Theorem 1.1.5 (Theorem 5.3.3). *Let E be a global field of characteristic p and let \mathbb{A} be its adèle ring. For every positive integer r , the isomorphism class of a $\overline{\mathbb{Q}}_p$ -linear cuspidal automorphic representation of $\mathrm{GL}_r(\mathbb{A})$ is determined by the datum of the Hecke eigenvalues of minimal slope at all but finitely many places of E .*

1.2. †-hull of F -isocrystals. One of the main tools introduced and studied in this article is the notion of †-hull of a sub- F^n -isocrystal.

Definition 1.2.1. Let $(\mathcal{N}, \Phi_{\mathcal{N}}) \subseteq (\mathcal{M}, \Phi_{\mathcal{M}})$ be an inclusion of F^n -isocrystals. The †-hull of $(\mathcal{N}, \Phi_{\mathcal{N}})$ in $(\mathcal{M}, \Phi_{\mathcal{M}})$ is the smallest subobject of $(\mathcal{M}, \Phi_{\mathcal{M}})$ containing $(\mathcal{N}, \Phi_{\mathcal{N}})$ and coming from an overconvergent F^n -isocrystal. We denote it by $(\overline{\mathcal{N}}, \Phi_{\overline{\mathcal{N}}})$.

This notion was introduced by the author in September 2018, during a discussion in Berlin with Abe and Esnault. In that case, it was used as an attempt to better understand the results in [AD18]. A couple of months later, reading [Vel91], the author understood that a certain property on the slopes of the †-hull of an F^n -isocrystal would have implied the parabolicity conjecture. This property was satisfied, for example, by certain F^n -isocrystals coming from p -divisible groups, as a consequence of a local result proven by Tate [Tat67, Prop. 12]. In this article, we prove that the expected property for the †-hull is satisfied in general.

Theorem 1.2.2 (Theorem 4.1.3). *Let X be a smooth variety over a perfect field k and let $(\mathcal{N}, \Phi_{\mathcal{N}}) \subseteq (\mathcal{M}, \Phi_{\mathcal{M}})$ be an inclusion of F^n -isocrystals over X . If \mathcal{M} comes from an overconvergent isocrystal and has constant slopes, then $S_1(\mathcal{N}, \Phi_{\mathcal{N}}) = S_1(\overline{\mathcal{N}}, \Phi_{\overline{\mathcal{N}}})$.*

When X has dimension 1, Theorem 1.2.2 is essentially proven by Tsuzuki in [Tsu19, Prop. 5.8], where [Tat67, Prop. 12] is replaced by a theorem of de Jong (Theorem 4.2.6). In his proof, Tsuzuki also uses the PBQ filtration of an overconvergent F -isocrystal, that he constructs in [Tsu19, Thm. 3.27]. In the proof we propose in this article, instead, we avoid [*ibid.*, Thm. 3.27]. On the other hand, we use Theorem 1.2.2 to recover [*ibid.*, Thm. 3.27] and extend it to arbitrary smooth varieties (Corollary 5.4.2).

For higher dimensional varieties we deduce Theorem 1.2.2 from the case of curves, thanks to a new Lefschetz theorem for overconvergent F^n -isocrystals (Theorem 1.3.1). We postpone for a moment the discussion about this part. Before that, let us briefly explain how to deduce Theorem 1.1.1 from Theorem 1.2.2. We prove first that thanks to Chevalley's theorem, Theorem 1.2.2 implies a variant of Theorem 1.1.1 for some slightly different groups, namely the monodromy groups of F^∞ -isocrystals (Proposition 4.3.2). Then, to pass from this variant to the parabolicity conjecture as stated by Crew, we introduce a third type of monodromy groups, namely the monodromy groups of isocrystals with punctual $\mathbb{Q}_p^{\mathrm{ur}}$ -structure, defined in §3.2.7. In the cases we consider, the latter

monodromy groups are \mathbb{Q}_p^{ur} -forms of Crew's monodromy groups defined in [Cre92a]. Let us see now more in details the technical points we mentioned.

1.3. Lefschetz theorem. The main issue to reduce Theorem 1.2.2 to the case of curves is due to the existence of wild ramification in positive characteristic. One would like to find a smooth connected curve $C \subseteq X$ such that for every overconvergent isocrystal \mathcal{M}^\dagger over X the restriction functor $\langle \mathcal{M}^\dagger \rangle \rightarrow \langle \mathcal{M}^\dagger|_C \rangle$ is an equivalence. This is possible, for example, for local systems in characteristic 0, or for tamely ramified ℓ -adic lisse sheaves in positive characteristic (see [Esn17]). The failure of the existence of such a nice curve for general ℓ -adic lisse sheaves is already clear for \mathbb{A}_k^2 (see [ibid., Lem. 5.4]). On the other hand, if rather than considering all the objects at the same time one focuses on one object at a time, then such a nice curve exists over finite fields both for ℓ -adic lisse sheaves and overconvergent F^n -isocrystals (see [Kat99, Lem. 6 and Thm. 8] and [AE19, Thm. 3.10]). We extend this result to *docile* overconvergent F^n -isocrystals, namely those overconvergent F^n -isocrystals which admit a log-extension with nilpotent residues.

Theorem 1.3.1 (Theorem 4.4.3). *Let $Y \subseteq \mathbb{P}_{k^{\text{alg}}}^d$ be a smooth connected projective variety of dimension at least 2 and let D be a simple normal crossing divisor. If $(\mathcal{M}^\dagger, \Phi_{\mathcal{M}}^\dagger)$ is an overconvergent F^n -isocrystal over $X := Y \setminus D$ docile along D , then there exists a smooth connected curve $C \subseteq X$ such that the restriction functor $\langle \mathcal{M}^\dagger \rangle \rightarrow \langle \mathcal{M}^\dagger|_C \rangle$ is an equivalence of categories.*

This theorem is obtained as a combination of various Lefschetz-type results. One of the main ingredients is Theorem 4.4.2, proven by Abe–Esnault, which gives a class of curves C such that the restriction functor $\langle \mathcal{M}^\dagger \rangle \rightarrow \langle \mathcal{M}^\dagger|_C \rangle$ is fully faithful. To prove Theorem 1.3.1, we show that for at least one of these curves the restriction functor is also essentially surjective. This condition can be tested on rank 1 objects, which have the advantage to come from p -adic characters of the étale fundamental group and are easier to extend from C to X . The difficult part is to impose that the extended characters come from overconvergent F^n -isocrystals. In our proof, this is done by combining Lemma 4.4.6 and Proposition 4.4.7. Interestingly, in order to use Proposition 4.4.7 we need Theorem 1.2.2 for curves. Therefore, the proofs of Theorem 1.2.2 and Theorem 1.3.1 are intrinsically intertwined.

1.4. Punctual \mathbb{Q}_p^{ur} -structures. The other technical issue we have to solve in our article is given by the fact that the field of scalars of the Tannakian category of isocrystals is in general bigger than the field of scalars of the category of F^n -isocrystals. We encounter this issue in the proof of Theorem 1.1.1, which is easier for monodromy groups of objects endowed with an F^n -structure (knowing Theorem 4.1.3), but it is harder for the monodromy groups of the objects without F^n -structure. To jump from one setting to the other we slightly modify both categories. We first replace the category of F^n -isocrystals with the category of F^∞ -isocrystals, namely the 2-colimit of the categories of F^n -isocrystals for various n . If k is big enough, this new category is a \mathbb{Q}_p^{ur} -linear category. At the same time, we construct the category of isocrystals with punctual \mathbb{Q}_p^{ur} -structure (see §3.1.2), which is simply the category of isocrystals endowed with the choice of a \mathbb{Q}_p^{ur} -linear lattice at some fibre. This other Tannakian category is also \mathbb{Q}_p^{ur} -linear.

There is a natural functor between these two categories, obtained thanks to the observation that a finite-rank F^∞ -isocrystal over the spectrum of an algebraically closed field has a natural \mathbb{Q}_p^{ur} -linear lattice (Lemma 3.1.5). The existence of such a lattice might have other consequences that have to be

explored (such as Remark 3.1.7). To prove Theorem 1.1.1 we use the following result, reminiscence of the homotopy exact sequence for the étale fundamental group (the notation is introduced in §3.2.7).

Proposition 1.4.1 (Proposition 3.2.8). *For an overconvergent F^n -isocrystal $(\mathcal{M}^\dagger, \Phi_{\mathcal{M}}^\dagger)$, we have the following commutative diagram of algebraic groups over \mathbb{Q}_p^{ur}*

$$\begin{array}{ccccccc} 1 & \longrightarrow & G(\mathcal{M}, V_{\mathcal{M}}, \eta) & \longrightarrow & G(\mathcal{M}, \Phi_{\mathcal{M}}^\infty, \eta) & \longrightarrow & G(\mathcal{M}, \Phi_{\mathcal{M}}^\infty, \eta)^{\text{cst}} \longrightarrow 1 \\ & & \downarrow & & \downarrow & & \downarrow \\ 1 & \longrightarrow & G(\mathcal{M}^\dagger, V_{\mathcal{M}}, \eta) & \longrightarrow & G(\mathcal{M}^\dagger, \Phi_{\mathcal{M}}^{\dagger, \infty}, \eta) & \longrightarrow & G(\mathcal{M}^\dagger, \Phi_{\mathcal{M}}^{\dagger, \infty}, \eta)^{\text{cst}} \longrightarrow 1. \end{array}$$

The rows are exact, the first two vertical arrows are closed embeddings and the last arrow is a faithfully flat morphism.

1.5. The structure of the article. In §2 we fix most of the notation we use in our article. In §3 we introduce and study isocrystals with punctual \mathbb{Q}_p^{ur} -structure and we prove Proposition 1.4.1. In §4 we prove all the main theorems of our article. In particular, in §4.2 we prove Theorem 1.2.2 over curves, in §4.3 we prove that Theorem 1.2.2 implies a variant of Theorem 1.1.1 and in §4.4 we prove the Lefschetz theorem (Theorem 1.3.1) and we use it to complete the proof of Theorem 1.2.2 in §4.4.12, which is then used to prove Theorem 1.1.1. In §5 we deduce some consequences of the results in §4. More precisely, we first prove Theorem 1.1.2 in §5.1. Subsequently, in §5.2 we prove Theorem 1.1.3, and in §5.3 we prove Kedlaya’s conjecture (Corollary 1.1.4) and its consequence, namely Theorem 1.1.5. Finally, in §5.4, we prove the existence of Tsuzuki’s PBQ filtration for general smooth varieties (Corollary 5.4.2) and we prove a generalisation of Kedlaya’s conjecture (Corollary 5.4.4).

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2. NOTATION

2.1. Tannakian categories. Throughout the article we will freely make use of the notion of *observable functor*, as presented in [DE20, §A], and all the criteria listed there to determine various properties of morphisms of Tannaka groups (e.g. when a morphism is a closed immersion, faithfully flat, etc.). In particular, we recall that by [*ibid.*, Prop. A.3], an exact functor between Tannakian categories $\Psi : \mathbf{C} \rightarrow \mathbf{D}$ is observable if and only if every subquotient of the essential image of Ψ can be embedded into an object in the essential image of Ψ . At the same time, we will use the notion of *scalar extension* of a K -linear Tannakian category with respect to a field extension L/K (possibly

infinite), as presented in [Sta08, Thm. 3.1.3]. Finally, if \mathbf{C} is a Tannakian category, for an object $\mathcal{M} \in \mathbf{C}$ we denote by $\langle \mathcal{M} \rangle$ the Tannakian subcategory¹ of \mathbf{C} spanned by \mathcal{M} .

2.2. Isocrystals. Let p be a prime number. We fix an algebraic closure of \mathbb{F}_p , denoted by \mathbb{F} , and for $n \in \mathbb{Z}_{>0}$ we write \mathbb{F}_{p^n} for the subfield of \mathbb{F} with p^n elements. For a perfect field k of positive characteristic p we write $W(k)$ for its ring of Witt vectors, $K(k)$ for its fraction field, and σ for the lift of the Frobenius of k . We also write \mathbb{Q}_{p^n} for $K(\mathbb{F}_{p^n})$ and \mathbb{Q}_p^{ur} for the union $\bigcup_{n>0} \mathbb{Q}_{p^n} \subseteq K(\mathbb{F})$. If X is a smooth variety over k , we denote by $\mathbf{Crys}(X)$ the category of crystals of coherent $\mathcal{O}_{X, \text{crys}}$ -modules over the absolute crystalline site of X and by $\mathbf{Isoc}(X)$ the localisation $\mathbf{Crys}(X)_{[1/p]}$. We also denote by $\mathbf{Isoc}^\dagger(X)$ the category of (coherent) overconvergent F -isocrystals over $X/K(k)$ (cf. [Ber96]).

We say that η is a *perfect point* (resp. *geometric point*) of X , if it is an Ω -point for some field extension Ω/k with Ω perfect (resp. algebraically closed). We write ω_η for the exact \otimes -functor $\mathbf{Isoc}(X) \rightarrow \mathbf{Vec}_{K(\Omega)}$ which sends an isocrystal over X to the pullback over $\text{Spec}(\Omega)$. We denote with the same symbol the exact \otimes -functor obtained in the same way from the category $\mathbf{Isoc}^\dagger(X)$. If X is connected, for every $\mathcal{M} \in \mathbf{Isoc}(X)$ we denote by $G(\mathcal{M}, \eta)$ the Tannaka group of $\langle \mathcal{M} \rangle$ with respect to ω_η and by $G(\mathcal{M}^\dagger, \eta)$ the Tannaka group of $\langle \mathcal{M}^\dagger \rangle$ with respect to ω_η . These groups coincide with the groups $\text{DGal}(-, -)$ defined in [Cre92a, §2].

We write $F : X \rightarrow X$ for the absolute Frobenius of X and for a positive integer n , we write $\mathbf{F}^n\text{-Isoc}(X)$ (resp. $\mathbf{F}^n\text{-Isoc}^\dagger(X)$) for the category of pairs $(\mathcal{M}, \Phi_{\mathcal{M}})$ (resp. $(\mathcal{M}^\dagger, \Phi_{\mathcal{M}}^\dagger)$) where $\mathcal{M} \in \mathbf{Isoc}(X)$ (resp. $\mathcal{M}^\dagger \in \mathbf{Isoc}^\dagger(X)$) and $\Phi_{\mathcal{M}}$ is a K -linear isomorphism $(F^n)^*\mathcal{M} \xrightarrow{\sim} \mathcal{M}$ (resp. K -linear isomorphism $(F^n)^*\mathcal{M}^\dagger \xrightarrow{\sim} \mathcal{M}^\dagger$). The category $\mathbf{F}^n\text{-Isoc}(X)$ is canonically equivalent to the category of convergent F^n -isocrystals (see [Ked16, Thm. 2.2]). If $(\mathcal{M}, \Phi_{\mathcal{M}})$ is an F^n -isocrystal we define inductively $\Phi_{\mathcal{M}}^m := \Phi_{\mathcal{M}} \circ F^*(\Phi_{\mathcal{M}}^{m-1})$ for $m > 0$, where $\Phi_{\mathcal{M}}^1 := \Phi_{\mathcal{M}}$ and we say that these F^{nm} -structures are the powers of $\Phi_{\mathcal{M}}$. We write $\mathbf{F}^\infty\text{-Isoc}(X)$ for $2\text{-}\varinjlim_n \mathbf{F}^n\text{-Isoc}(X)$ and $\mathbf{F}^\infty\text{-Isoc}^\dagger(X)$ for $2\text{-}\varinjlim_n \mathbf{F}^n\text{-Isoc}^\dagger(X)$. If $(\mathcal{M}, \Phi_{\mathcal{M}})$ is an F^n -isocrystal, we write $(\mathcal{M}, \Phi_{\mathcal{M}}^\infty)$ for its image in $\mathbf{F}^\infty\text{-Isoc}(X)$ and we use the analogue convention in the overconvergent setting.

We write $\mathbf{Isoc}(X)_F$ (resp. $\mathbf{Isoc}^\dagger(X)_F$) for the subcategory of $\mathbf{Isoc}(X)$ (resp. $\mathbf{Isoc}^\dagger(X)_F$) spanned by those isocrystals (resp. overconvergent isocrystals) which admit some F -structure. The natural “restriction” functor $\alpha : \mathbf{Isoc}^\dagger(X)_F \rightarrow \mathbf{Isoc}(X)_F$ is fully faithful (see [Ked07] and [DE20, Cor. 5.7]). We say that an $\mathcal{M} \in \mathbf{Isoc}(X)_F$ is \dagger -*extendable* if it is in the essential image of α and we write \mathcal{M}^\dagger for the associated overconvergent isocrystal (which is unique up to isomorphism).

By [Kat79, Cor. 2.3.1], [dO99, Thm. 4.1], and [Ked16, Thm. 3.12], after possibly removing a divisor of X , an F^n -isocrystal $(\mathcal{M}, \Phi_{\mathcal{M}})$ has constant slopes. In this case, thanks to [Kat79, Cor. 2.6.2] and [Ked16, Cor. 4.2], it acquires the slope filtration

$$0 = S_0(\mathcal{M}) \subsetneq S_1(\mathcal{M}) \subsetneq \dots \subsetneq S_m(\mathcal{M}) = \mathcal{M}$$

where each graded piece $S_i(\mathcal{M})/S_{i-1}(\mathcal{M})$ is of pure slope $s_i \in \mathbb{Q}$ and $s_1 < s_2 < \dots < s_m$.

¹We will always require that a Tannakian subcategory is closed under the operation of taking subobjects.

3. ISOCRYSTALS WITH PUNCTUAL \mathbb{Q}_p^{ur} -STRUCTURE

3.1. First definitions. Let k be a perfect field of characteristic p and X a smooth variety over k . If one wants to compare the monodromy group of an F^n -isocrystal $(\mathcal{M}, \Phi_{\mathcal{M}})$ over X with the monodromy group of the isocrystal \mathcal{M} one encounters issues related to the field of scalars. Indeed, while the former object lives in $\mathbf{F}^n\text{-Isoc}(X)$, which is a \mathbb{Q}_p^n -linear Tannakian category, the latter is in $\mathbf{Isoc}(X)$, a K -linear Tannakian category where $K := K(k)$. We propose here a possible approach to compare the two groups. This will lead to Proposition 3.2.8.

Hypothesis 3.1.1. Throughout §3, we assume that k is endowed with the choice of an inclusion $\mathbb{F} \subseteq k$.² In particular, for every perfect field extension Ω/k we have a preferred embedding $\mathbb{Q}_p^{\text{ur}} \hookrightarrow K(\Omega)$.

Definition 3.1.2. If η is a geometric point of X , an *isocrystal with (punctual) \mathbb{Q}_p^{ur} -structure* over (X, η) is a pair $(\mathcal{M}, V_{\mathcal{M}})$ where \mathcal{M} is an isocrystal and $V_{\mathcal{M}}$ is a \mathbb{Q}_p^{ur} -linear lattice of $\omega_{\eta}(\mathcal{M})$, namely a \mathbb{Q}_p^{ur} -linear vector subspace $V_{\mathcal{M}} \subseteq \omega_{\eta}(\mathcal{M})$ such that $V_{\mathcal{M}} \otimes_{\mathbb{Q}_p^{\text{ur}}} K(\Omega) = \omega_{\eta}(\mathcal{M})$. A morphism of isocrystals with \mathbb{Q}_p^{ur} -structure $(\mathcal{M}, V_{\mathcal{M}}) \rightarrow (\mathcal{N}, V_{\mathcal{N}})$ is a morphism of isocrystals $f : \mathcal{M} \rightarrow \mathcal{N}$ such that $\omega_{\eta}(f)(V_{\mathcal{M}}) \subseteq \omega_{\eta}(f)(V_{\mathcal{N}})$. We write $\mathbf{Isoc}_{\mathbb{Q}_p^{\text{ur}}}(X, \eta)$ for the category of isocrystals with punctual \mathbb{Q}_p^{ur} -structure over (X, η) .

Lemma 3.1.3. *If X is geometrically connected, the category $\mathbf{Isoc}_{\mathbb{Q}_p^{\text{ur}}}(X, \eta)$ has a natural structure of a \mathbb{Q}_p^{ur} -linear neutral Tannakian category.*

Proof. The \otimes -structure is the one induced by the \otimes -structures of $\mathbf{Isoc}(X)$ and $\mathbf{Vec}_{\mathbb{Q}_p^{\text{ur}}}$, with unit object $\mathbb{1} = (\mathcal{O}_{X/K}, \mathbb{Q}_p^{\text{ur}})$. Thanks to the assumption that X is geometrically connected and k contains \mathbb{F} , we deduce that $\text{End}(\mathbb{1}) = \mathbb{Q}_p^{\text{ur}}$. We claim that $\mathbf{Isoc}_{\mathbb{Q}_p^{\text{ur}}}(X, \eta)$ is rigid. Indeed, for an isocrystal with \mathbb{Q}_p^{ur} -structure $(\mathcal{M}, V_{\mathcal{M}})$, we may take the \mathbb{Q}_p^{ur} -linear subspace $V_{\mathcal{M}^{\vee}} \subseteq \omega_x(\mathcal{M}^{\vee})$ corresponding to those morphisms $f \in \text{Hom}(\omega_x(\mathcal{M}), K)$ such that $f(V) \subseteq \mathbb{Q}_p^{\text{ur}}$. The isocrystal with \mathbb{Q}_p^{ur} -structure $(\mathcal{M}, V_{\mathcal{M}})^{\vee} := (\mathcal{M}^{\vee}, V_{\mathcal{M}^{\vee}})$, endowed with the natural morphisms $\text{ev} : (\mathcal{M}, V_{\mathcal{M}}) \otimes (\mathcal{M}, V_{\mathcal{M}})^{\vee} \rightarrow \mathbb{1}$ and $\delta : \mathbb{1} \rightarrow (\mathcal{M}, V_{\mathcal{M}})^{\vee} \otimes (\mathcal{M}, V_{\mathcal{M}})$, is then a dual of $(\mathcal{M}, V_{\mathcal{M}})$. Finally, we note that $\omega_{\eta, \mathbb{Q}_p^{\text{ur}}} : \mathbf{Isoc}_{\mathbb{Q}_p^{\text{ur}}}(X, \eta) \rightarrow \mathbf{Vec}_{\mathbb{Q}_p^{\text{ur}}}$ defined by $(\mathcal{M}, V_{\mathcal{M}}) \mapsto V_{\mathcal{M}}$ is a fibre functor for $\mathbf{Isoc}_{\mathbb{Q}_p^{\text{ur}}}(X, \eta)$. This ends the proof. \square

Now that we have constructed a neutral Tannakian category of isocrystals with smaller field of scalars, we want to study its interaction with the category of F^n -isocrystals.

Construction 3.1.4. Let Ω/k be an algebraically closed field extension. For an F^n -isocrystal (M, Φ_M) over Ω (of finite-rank), we write $\omega_{\mathbb{Q}_p^{\text{ur}}}(M, \Phi_M) \subseteq M$ for the \mathbb{Q}_p^{ur} -linear vector subspace of vectors $v \in M$ such that $\Phi_M^i v = p^j v$ for some $(i, j) \in \mathbb{Z}_{>0} \times \mathbb{Z}$.

Lemma 3.1.5 (after Dieudonné–Manin). *The vector space $\omega_{\mathbb{Q}_p^{\text{ur}}}(M, \Phi_M)$ is a \mathbb{Q}_p^{ur} -linear lattice of the $K(\Omega)$ -vector space M .*

²One can easily generalise what we say in this section to the case when k does not contain \mathbb{F} . We preferred to avoid this discussion here since the parabolicity conjecture, as we will see, is a “geometric” statement, meaning that we are allowed to replace the field k with its algebraic closure.

Proof. Thanks to Dieudonné–Manin classification, after possibly taking powers of Φ_M , we may assume that $(M, \Phi_M) = (K(\Omega), p^s \sigma)$ for some $s \in \mathbb{Z}$. The result then follows from the observation that $\mathbb{Q}_p^{\text{ur}} = \cup_n K(\Omega) \sigma^n = \text{id}$. \square

Definition 3.1.6. We say that $\omega_{\mathbb{Q}_p^{\text{ur}}}(M, \Phi_M)$ is the *Dieudonné–Manin \mathbb{Q}_p^{ur} -structure* of (M, Φ_M) . The assignment $(M, \Phi_M) \mapsto \omega_{\mathbb{Q}_p^{\text{ur}}}(M, \Phi_M)$ produces a \mathbb{Q}_p^{ur} -linear fibre functor $\omega_{\mathbb{Q}_p^{\text{ur}}} : \mathbf{F}^\infty\text{-Isoc}(\Omega) \rightarrow \mathbf{Vec}_{\mathbb{Q}_p^{\text{ur}}}$. If η is a Ω -point of X , we write

$$\omega_{\eta, \mathbb{Q}_p^{\text{ur}}} : \mathbf{F}^\infty\text{-Isoc}(X) \rightarrow \mathbf{Vec}_{\mathbb{Q}_p^{\text{ur}}}$$

for the composition $\omega_{\mathbb{Q}_p^{\text{ur}}} \circ \eta^*$ and we write

$$\Lambda_\eta : \mathbf{F}^\infty\text{-Isoc}(X) \rightarrow \mathbf{Isoc}_{\mathbb{Q}_p^{\text{ur}}}(X, \eta)$$

for the functor obtained by sending $(\mathcal{M}, \Phi_{\mathcal{M}}^\infty) \mapsto (\mathcal{M}, \omega_{\eta, \mathbb{Q}_p^{\text{ur}}}(\mathcal{M}, \Phi_{\mathcal{M}}^\infty))$. We say that an object in the essential image of Λ_η is an *isocrystal with Dieudonné–Manin \mathbb{Q}_p^{ur} -structure* over (X, η) .

Remark 3.1.7. The existence of the Dieudonné–Manin \mathbb{Q}_p^{ur} -structure of an F -isocrystal over an algebraically closed field has its own interest. For example, thanks to [Ked06], if k is any field of characteristic p , one can associate to a variety X/k the finite-dimensional \mathbb{Q}_p^{ur} -linear vector spaces $\omega_{\mathbb{Q}_p^{\text{ur}}}(H_{\text{rig}}^i(X_{k^{\text{alg}}}/K(k^{\text{alg}})))$ and their variant with compact support. This assignment produces a \mathbb{Q}_p^{ur} -linear cohomology theory with all the desired properties (e.g. Poincaré duality, the Künneth formula, etc.). This solves in a minimal way Serre’s obstruction to the existence of a \mathbb{Q}_p -linear cohomology theory.

3.2. Basic properties. In general, the category $\langle \mathcal{M}, V_{\mathcal{M}} \rangle$ spanned by an isocrystal with \mathbb{Q}_p^{ur} -structure might be quite different from the category spanned by the associated isocrystal \mathcal{M} in $\mathbf{Isoc}(X)$. For example, $(\mathcal{M}, V_{\mathcal{M}})$ might be irreducible even if \mathcal{M} is not, or it might happen that $H^0(\mathcal{M}, V_{\mathcal{M}})$ has lower dimension than $H^0(\mathcal{M})$. In what follows we show that these phenomena do not occur for Dieudonné–Manin \mathbb{Q}_p^{ur} -structures.

Lemma 3.2.1. *If $(\mathcal{M}, \Phi_{\mathcal{M}})$ is an F^n -isocrystal, the maximal trivial³ subobject $\mathcal{T} \subseteq \mathcal{M}$ is kept stable under $\Phi_{\mathcal{M}}$. In particular, if $(\mathcal{M}, V_{\mathcal{M}})$ is the induced isocrystal with Dieudonné–Manin \mathbb{Q}_p^{ur} -structure then*

$$H^0(\mathcal{M}, V_{\mathcal{M}}) \otimes_{\mathbb{Q}_p^{\text{ur}}} K \simeq H^0(\mathcal{M}).$$

Proof. By maximality $\mathcal{T} \subseteq \mathcal{M}$ is preserved by $\Phi_{\mathcal{M}}$, so that it defines an inclusion $(\mathcal{T}, V_{\mathcal{M}}|_{\mathcal{T}}) \subseteq (\mathcal{M}, V_{\mathcal{M}})$ of isocrystals with \mathbb{Q}_p^{ur} -structure. Since \mathcal{T} is trivial, $(\mathcal{T}, V_{\mathcal{M}}|_{\mathcal{T}})$ is a trivial isocrystal with \mathbb{Q}_p^{ur} -structure. This yields the desired result. \square

Lemma 3.2.2. *A semi-simple isocrystal \mathcal{M} admits at most one Dieudonné–Manin \mathbb{Q}_p^{ur} -structure up to isomorphism. In addition, if $V_{\mathcal{M}}$ is such a \mathbb{Q}_p^{ur} -structure, $(\mathcal{M}, V_{\mathcal{M}})$ decomposes as the direct sum $\bigoplus_i (\mathcal{M}_i, V_{\mathcal{M}_i})^{\oplus a_i}$ with $a_i > 0$ and the isocrystals \mathcal{M}_i irreducible and pairwise non-isomorphic.*

Proof. Let $\mathcal{M} = \bigoplus_i \mathcal{M}_i^{\oplus a_i}$ be an isotypic decomposition of \mathcal{M} . Suppose that \mathcal{M} admits an F^n -structure $\Phi_{\mathcal{M}}$ and write $V_{\mathcal{M}}$ for the associated Dieudonné–Manin \mathbb{Q}_p^{ur} -structure. Since $\Phi_{\mathcal{M}}$ permutes the \mathcal{M}_i ’s, each of them admits an F^{nm} -structure for some $m > 0$. Therefore, each \mathcal{M}_i admits a Dieudonné–Manin \mathbb{Q}_p^{ur} -structure $V_{\mathcal{M}_i}$. Let $(\mathcal{T}_i, V_{\mathcal{T}_i})$ be the maximal trivial subobject of

³We say that an object in a Tannakian category is trivial if it is isomorphic to a direct sum $\mathbb{1}^{\oplus m}$ for some $m \geq 0$.

$(\mathcal{M}_i, V_{\mathcal{M}_i})^\vee \otimes (\mathcal{M}, V_{\mathcal{M}})$. By Lemma 3.2.1, each \mathcal{T}_i is the maximal trivial subobject of $\mathcal{M}_i^\vee \otimes \mathcal{M}$. By construction, we have a tautological inclusion of $(\mathcal{M}_i, V_{\mathcal{M}_i}) \otimes (\mathcal{T}_i, V_{\mathcal{T}_i})$ in $(\mathcal{M}, V_{\mathcal{M}})$ for each i . This gives the desired decomposition. To prove the unicity it is enough to assume \mathcal{M} irreducible. In this case, if $V'_{\mathcal{M}}$ is another Dieudonné–Manin \mathbb{Q}_p^{ur} -structure, we have that $(\mathcal{M}, V_{\mathcal{M}})$ and $(\mathcal{M}, V'_{\mathcal{M}})$ are both irreducible and they admit a non-trivial morphism by Lemma 3.2.1. \square

Lemma 3.2.3. *If $(\mathcal{M}, V_{\mathcal{M}})$ is an isocrystal with Dieudonné–Manin \mathbb{Q}_p^{ur} -structure, the socle $\mathcal{N} \subseteq \mathcal{M}$ admits a (unique) Dieudonné–Manin \mathbb{Q}_p^{ur} -structure $V_{\mathcal{N}}$ and $(\mathcal{N}, V_{\mathcal{N}})$ is a subobject of $(\mathcal{M}, V_{\mathcal{M}})$. Moreover, if $(\mathcal{M}, V_{\mathcal{M}})$ is semi-simple, \mathcal{M} is semi-simple.*

Proof. Let $\Phi_{\mathcal{M}}$ be an F^n -structure of \mathcal{M} . Since $\Phi_{\mathcal{M}}$ preserves \mathcal{N} , it induces a Dieudonné–Manin \mathbb{Q}_p^{ur} -structure $\mathcal{V}_{\mathcal{N}}$ of \mathcal{N} which makes $(\mathcal{N}, V_{\mathcal{N}})$ a subobject of $(\mathcal{M}, V_{\mathcal{M}})$. If $(\mathcal{M}, \Phi_{\mathcal{M}})$ is irreducible, we deduce that $(\mathcal{N}, V_{\mathcal{N}}) = (\mathcal{M}, V_{\mathcal{M}})$. Therefore, by Lemma 3.2.2, the isocrystal \mathcal{N} is irreducible. This concludes the proof. \square

Proposition 3.2.4. *If $(\mathcal{M}, V_{\mathcal{M}})$ is an isocrystal with Dieudonné–Manin \mathbb{Q}_p^{ur} -structure, each irreducible subquotient \mathcal{M}_i of \mathcal{M} admits a (unique) Dieudonné–Manin \mathbb{Q}_p^{ur} -structure $V_{\mathcal{M}_i}$ which makes $(\mathcal{M}_i, V_{\mathcal{M}_i})$ a subquotient of $(\mathcal{M}, V_{\mathcal{M}})$.*

Proof. Let $\Phi_{\mathcal{M}}$ be a F^n -structure of \mathcal{M} inducing $V_{\mathcal{M}}$. By Lemma 3.2.3, after forgetting the \mathbb{Q}_p^{ur} -structure the socle filtration of $(\mathcal{M}, V_{\mathcal{M}})$ is sent to the socle filtration of \mathcal{M} and the various steps of the socle filtration of $(\mathcal{M}, V_{\mathcal{M}})$ have Dieudonné–Manin \mathbb{Q}_p^{ur} -structure. Therefore, after taking the semi-simplification of $(\mathcal{M}, V_{\mathcal{M}})$ with respect to this filtration, the result follows from Lemma 3.2.2. \square

Proposition 3.2.5. *The functor $\Lambda_\eta : \mathbf{F}^\infty\text{-Isoc}(X) \rightarrow \mathbf{Isoc}_{\mathbb{Q}_p^{\text{ur}}}(X, \eta)$ is observable and, if $k = k^{\text{alg}}$, it is fully faithful on unit-root objects.*

Proof. To prove that Λ_η is observable it is enough to notice that by Proposition 3.2.4 every rank 1 subquotient of an isocrystal with Dieudonné–Manin \mathbb{Q}_p^{ur} -structure is itself an isocrystal with Dieudonné–Manin \mathbb{Q}_p^{ur} -structure. If $k = k^{\text{alg}}$ unit-root F^n -isocrystals over k are trivial by Dieudonné–Manin decomposition, which implies that Λ_η is fully faithful thanks to Lemma 3.2.1. \square

Definition 3.2.6. We write $\mathbf{Isoc}_{\mathbb{Q}_p^{\text{ur}}}(X, \eta)_F$ for the Tannakian subcategory of $\mathbf{Isoc}_{\mathbb{Q}_p^{\text{ur}}}(X, \eta)$ spanned by the essential image of Λ_η .

All we said works unchanged for overconvergent isocrystals and we have an observable functor $\Lambda_\eta : \mathbf{F}^\infty\text{-Isoc}^\dagger(X) \rightarrow \mathbf{Isoc}_{\mathbb{Q}_p^{\text{ur}}}^\dagger(X, \eta)$ from overconvergent F^∞ -isocrystals to overconvergent isocrystals with \mathbb{Q}_p^{ur} -structure (defined in the analogous way). We are now ready to present the fundamental result which compares the monodromy groups of F^∞ -isocrystals and isocrystals with Dieudonné–Manin \mathbb{Q}_p^{ur} -structure.

Definition 3.2.7. Let (X, η) be a geometrically connected variety over k endowed with a geometric point η . For an F^n -isocrystal $(\mathcal{M}, \Phi_{\mathcal{M}})$ we write $G(\mathcal{M}, \Phi_{\mathcal{M}}^\infty, \eta)$ for the Tannaka group of $\langle \mathcal{M}, \Phi_{\mathcal{M}}^\infty \rangle$ with respect to $\omega_{\eta, \mathbb{Q}_p^{\text{ur}}}$ and $G(\mathcal{M}, V_{\mathcal{M}}, \eta)$ for the Tannaka group of $\langle \mathcal{M}, V_{\mathcal{M}} \rangle$ with respect to $\omega_{\eta, \mathbb{Q}_p^{\text{ur}}}$, where $V_{\mathcal{M}}$ is the Dieudonné–Manin \mathbb{Q}_p^{ur} -structure induced by $\Phi_{\mathcal{M}}$. We also denote by $G(\mathcal{M}, \Phi_{\mathcal{M}}^\infty, \eta)^{\text{cst}}$ the Tannaka group of constant F^∞ -isocrystals in $\langle \mathcal{M}, \Phi_{\mathcal{M}}^\infty \rangle$, namely those F^∞ -isocrystals coming from $\text{Spec}(k)$. We give analogous definitions in the overconvergent setting.

Proposition 3.2.8. *For a \dagger -extendable F^n -isocrystal $(\mathcal{M}, \Phi_{\mathcal{M}})$, we have the following commutative diagram of algebraic groups over \mathbb{Q}_p^{ur}*

$$\begin{array}{ccccccc} 1 & \longrightarrow & G(\mathcal{M}, V_{\mathcal{M}}, \eta) & \longrightarrow & G(\mathcal{M}, \Phi_{\mathcal{M}}^{\infty}, \eta) & \longrightarrow & G(\mathcal{M}, \Phi_{\mathcal{M}}^{\infty}, \eta)^{\text{cst}} \longrightarrow 1 \\ & & \downarrow & & \downarrow & & \downarrow \\ 1 & \longrightarrow & G(\mathcal{M}^{\dagger}, V_{\mathcal{M}}, \eta) & \longrightarrow & G(\mathcal{M}^{\dagger}, \Phi_{\mathcal{M}}^{\dagger, \infty}, \eta) & \longrightarrow & G(\mathcal{M}^{\dagger}, \Phi_{\mathcal{M}}^{\dagger, \infty}, \eta)^{\text{cst}} \longrightarrow 1. \end{array}$$

The rows are exact, the first two vertical arrows are closed embeddings and the last arrow is a faithfully flat morphism.

Proof. The proof is similar to the proof of [DE20, Cor. 5.12]. To prove the exactness of the two sequences we use [ibid., Prop. A.13]. By Proposition 3.2.5, the functor $\langle \mathcal{M}, \Phi_{\mathcal{M}}^{\infty} \rangle \rightarrow \langle \mathcal{M}, V_{\mathcal{M}} \rangle$ is observable. Let us verify the second main property we need to apply the proposition. For $(\mathcal{N}, \Phi_{\mathcal{N}}^{\infty}) \in \langle \mathcal{M}, \Phi_{\mathcal{M}}^{\infty} \rangle$, we write $(\mathcal{N}, V_{\mathcal{N}})$ for the associated isocrystal with \mathbb{Q}_p^{ur} -structure. By Lemma 3.2.1, if $\mathcal{T} \subseteq \mathcal{N}$ is the maximal trivial subobject, then $(\mathcal{T}, V_{\mathcal{N}}|_{\mathcal{T}})$ is the maximal trivial subobject of $(\mathcal{N}, V_{\mathcal{N}})$. By the maximality of \mathcal{T} , we have that $(\mathcal{T}, (\Phi_{\mathcal{N}}|_{\mathcal{T}})^{\infty})$ is a subobject of $(\mathcal{N}, \Phi_{\mathcal{N}}^{\infty})$. This shows the exactness of the first row. One can argue similarly for the second row. The rest of the statement is a simple check. \square

Remark 3.2.9. Thanks to Theorem 4.1.3 and Corollary 4.4.8 one can also show that the morphism $G(\mathcal{M}, \Phi_{\mathcal{M}}^{\infty}, \eta)^{\text{cst}} \rightarrow G(\mathcal{M}^{\dagger}, \Phi_{\mathcal{M}}^{\dagger, \infty}, \eta)^{\text{cst}}$ is an isomorphism.

3.3. Comparisons. The aim in this section is twofold. On the one hand, we want to compare the category $\mathbf{Isoc}_{\mathbb{Q}_p^{\text{ur}}}(X, \eta)$ and the category $\mathbf{Isoc}(X)$ in order to relate the monodromy groups of isocrystals, as defined in [Cre92a], and the monodromy groups of isocrystals with \mathbb{Q}_p^{ur} -structure (Proposition 3.3.2). In general, the scalar extension of the monodromy group of an isocrystal with \mathbb{Q}_p^{ur} -structure from \mathbb{Q}_p^{ur} to K is bigger than the monodromy group of the associated isocrystal. Instead, for isocrystals with Dieudonné–Manin \mathbb{Q}_p^{ur} -structure, we want to prove that we have the expected base change property. Moreover, we want to compare isoclinic F^{∞} -isocrystals (or rather direct sums of these) and lisse \mathbb{Q}_p^{ur} -sheaves (Proposition 3.3.4).

Lemma 3.3.1. *The functor $\mathbf{Isoc}_{\mathbb{Q}_p^{\text{ur}}}(X, \eta)_F \rightarrow \mathbf{Isoc}(X)$ is $K/\mathbb{Q}_p^{\text{ur}}$ -fully faithful (cf. [Sta08, Def. 1.1.2]) and sends semi-simple objects to semi-simple objects.*

Proof. We first show that the functor is $K/\mathbb{Q}_p^{\text{ur}}$ -fully faithful. This is equivalent to showing that for $(\mathcal{M}, V_{\mathcal{M}}) \in \mathbf{Isoc}_{\mathbb{Q}_p^{\text{ur}}}(X, \eta)_F$, the maximal trivial subobject of \mathcal{M} can be upgraded to a trivial subobject of $(\mathcal{M}, V_{\mathcal{M}})$. By Proposition 3.2.5, every object $(\mathcal{M}, V_{\mathcal{M}}) \in \mathbf{Isoc}_{\mathbb{Q}_p^{\text{ur}}}(X, \eta)_F$ is a subobject of some $(\mathcal{N}, V_{\mathcal{N}}) \in \mathbf{Isoc}_{\mathbb{Q}_p^{\text{ur}}}(X, \eta)_F$ with $V_{\mathcal{N}}$ a Dieudonné–Manin \mathbb{Q}_p^{ur} -structure. By Lemma 3.2.1, if $(\mathcal{T}, V_{\mathcal{T}})$ is the maximal trivial subobject of $(\mathcal{N}, V_{\mathcal{N}})$, then \mathcal{T} is the maximal trivial subobject of \mathcal{N} . The intersection of $(\mathcal{M}, V_{\mathcal{M}})$ and $(\mathcal{T}, V_{\mathcal{T}})$ in $(\mathcal{N}, V_{\mathcal{N}})$ is then an isocrystal with \mathbb{Q}_p^{ur} -structure with underlying isocrystal isomorphic to the maximal trivial subobject of \mathcal{M} , as we wanted. The second part follows from Lemma 3.2.2. \square

Proposition 3.3.2. *The functor $\mathbf{Isoc}_{\mathbb{Q}_p^{\text{ur}}}(X, \eta)_F \otimes_{\mathbb{Q}_p^{\text{ur}}} K \rightarrow \mathbf{Isoc}(X)_F$ is an equivalence of categories. In particular, for an F^n -isocrystal $(\mathcal{M}, \Phi_{\mathcal{M}})$, if $V_{\mathcal{M}}$ is the associated Dieudonné–Manin \mathbb{Q}_p^{ur} -structure, we have $G(\mathcal{M}, \eta) \simeq G(\mathcal{M}, V_{\mathcal{M}}, \eta) \otimes_{\mathbb{Q}_p^{\text{ur}}} K$.*

Proof. This follows from [Sta08, Prop. 3.1.8] thanks to Lemma 3.3.1. \square

Definition 3.3.3. If $k = k^{\text{alg}}$, we write $\mathbf{F}^\infty\text{-Isoc}^{\text{ét}}(X)$ for the Tannakian subcategory of $\mathbf{F}^\infty\text{-Isoc}(X)$ spanned by the isoclinic F^∞ -isocrystals and $\mathbf{Isoc}_{\mathbb{Q}_p^{\text{ur}}}^{\text{ét}}(X, \eta)$ for the Tannakian subcategory of $\mathbf{Isoc}_{\mathbb{Q}_p^{\text{ur}}}(X, \eta)$ spanned by the image of the functor $\mathbf{F}^\infty\text{-Isoc}^{\text{ét}}(X) \rightarrow \mathbf{Isoc}_{\mathbb{Q}_p^{\text{ur}}}(X, \eta)$. We say that the category $\mathbf{Isoc}_{\mathbb{Q}_p^{\text{ur}}}^{\text{ét}}(X, \eta)$ is the category of *étale isocrystals with \mathbb{Q}_p^{ur} -structure*.

Proposition 3.3.4. *There is a natural equivalence $\mathbf{LS}(X, \mathbb{Q}_p^{\text{ur}}) \xrightarrow{\sim} \mathbf{Isoc}_{\mathbb{Q}_p^{\text{ur}}}^{\text{ét}}(X, \eta)$.*

Proof. By [Cre87, Thm. 2.1], for every n there is a fully faithful \mathbb{Q}_p^n -linear \otimes -functor $\mathbf{LS}(X, \mathbb{Q}_p^n) \hookrightarrow \mathbf{F}^n\text{-Isoc}(X)$ with essential image the category of unit-root F^n -isocrystals. This family of \otimes -functors induces a fully faithful \mathbb{Q}_p^{ur} -linear \otimes -functor $\mathbf{LS}(X, \mathbb{Q}_p^{\text{ur}}) \hookrightarrow \mathbf{F}^\infty\text{-Isoc}^{\text{ét}}(X)$. If we postcompose this functor with the functor $\Lambda_\eta : \mathbf{F}^\infty\text{-Isoc}^{\text{ét}}(X) \rightarrow \mathbf{Isoc}_{\mathbb{Q}_p^{\text{ur}}}^{\text{ét}}(X, \eta)$, we get by Proposition 3.2.5 a fully faithful \mathbb{Q}_p^{ur} -linear \otimes -functor $\mathbf{LS}(X, \mathbb{Q}_p^{\text{ur}}) \hookrightarrow \mathbf{Isoc}_{\mathbb{Q}_p^{\text{ur}}}^{\text{ét}}(X, \eta)$. To prove that this functor is essentially surjective it is enough to notice that every isoclinic object in $\mathbf{F}^\infty\text{-Isoc}(X)$ is a tensor product of a unit-root F^∞ -isocrystal by a rank 1 constant F^∞ -isocrystal. \square

Remark 3.3.5. Similarly one can prove that $\mathbf{F}^\infty\text{-Isoc}^{\text{ét}}(X)$ is equivalent to the category of \mathbb{Q} -graded lisse \mathbb{Q}_p^{ur} -sheaves.

Corollary 3.3.6. *Let $(\mathcal{M}, \Phi_{\mathcal{M}})$ be an F^n -isocrystal over X which admits the slope filtration. If $k = k^{\text{alg}}$, an étale path γ joining two geometric points η and η' induces isomorphisms $G(\mathcal{M}, \Phi_{\mathcal{M}}, \eta) \xrightarrow{\sim} G(\mathcal{M}, \Phi_{\mathcal{M}}, \eta')$ and $G(\mathcal{M}, V_{\mathcal{M}}, \eta) \xrightarrow{\sim} G(\mathcal{M}, V_{\mathcal{M}}, \eta')$.*

Proof. Write $(\mathcal{N}, \Phi_{\mathcal{N}})$ for $\text{Gr}_{S_\bullet}(\mathcal{M}, \Phi_{\mathcal{M}})$ and $V_{\mathcal{M}}$ for the Dieudonné–Manin \mathbb{Q}_p^{ur} -structure of $(\mathcal{M}, \Phi_{\mathcal{M}})$ at η . The \mathbb{Q}_p^{ur} -structure $V_{\mathcal{M}}$ is naturally isomorphic to the Dieudonné–Manin \mathbb{Q}_p^{ur} -structure of $(\mathcal{N}, \Phi_{\mathcal{N}})$ at η . Thanks to Proposition 3.3.4, the étale path γ induces an isomorphism between the fibre functors $\omega_{\eta, \mathbb{Q}_p^{\text{ur}}} : \langle \mathcal{N}, V_{\mathcal{N}} \rangle \rightarrow \mathbf{Vec}_{\mathbb{Q}_p^{\text{ur}}}$ and $\omega_{\eta', \mathbb{Q}_p^{\text{ur}}} : \langle \mathcal{N}, V_{\mathcal{N}} \rangle \rightarrow \mathbf{Vec}_{\mathbb{Q}_p^{\text{ur}}}$. This yields the desired result. \square

4. MAIN THEOREMS

4.1. †-hull of F -isocrystals. In order to prove the parabolicity conjecture we chiefly study the †-hulls of sub- F^n -isocrystals.

Definition 4.1.1. Let $(\mathcal{N}, \Phi_{\mathcal{N}}) \subseteq (\mathcal{M}, \Phi_{\mathcal{M}})$ be an inclusion of F^n -isocrystals. The †-hull of $(\mathcal{N}, \Phi_{\mathcal{N}})$ in $(\mathcal{M}, \Phi_{\mathcal{M}})$ is the smallest⁴ †-extendable subobject of $(\mathcal{M}, \Phi_{\mathcal{M}})$ containing $(\mathcal{N}, \Phi_{\mathcal{N}})$. We denote it by $(\overline{\mathcal{N}}, \Phi_{\overline{\mathcal{N}}})$.

Notation 4.1.2. Let $(\mathcal{M}, \Phi_{\mathcal{M}})$ be a †-extendable F^n -isocrystal over X with constant slopes. We say that $(\mathcal{M}, \Phi_{\mathcal{M}})$ satisfies $\text{MS}(\mathcal{M}, \Phi_{\mathcal{M}})$ if for every sub- F^n -isocrystal $(\mathcal{N}, \Phi_{\mathcal{N}}) \subseteq (\mathcal{M}, \Phi_{\mathcal{M}})$, the isocrystals $S_1(\mathcal{N})$ and $S_1(\overline{\mathcal{N}})$ are the same. We also say that X satisfies $\text{MS}(X)$ if for every $n > 0$ and every †-extendable F^n -isocrystal $(\mathcal{M}, \Phi_{\mathcal{M}})$ over X , we have that $\text{MS}(\mathcal{M}, \Phi_{\mathcal{M}})$ is true.

Throughout §4 we want to prove the following theorem.

⁴Note that the category of isocrystals is artinian because if $\mathcal{M}' \subseteq \mathcal{M}$ have the same ranks, then $\mathcal{M}' = \mathcal{M}$.

Theorem 4.1.3. *A smooth variety X over a perfect field k satisfies $\text{MS}(X)$.*

We start with some reductions.

Lemma 4.1.4. *To prove $\text{MS}(X)$ we may assume $n = 1$.*

Proof. Let $(\mathcal{N}, \Phi_{\mathcal{N}}) \subseteq (\mathcal{M}, \Phi_{\mathcal{M}})$ be an inclusion of F^n -isocrystals. This induces an inclusion $(\mathcal{N}', \Phi'_{\mathcal{N}'}) \subseteq (\mathcal{M}', \Phi'_{\mathcal{M}'})$ of F -isocrystals, where $\mathcal{M}' := \bigoplus_{i=0}^{n-1} (F^i)^* \mathcal{M}$ and $\mathcal{N}' := \bigoplus_{i=0}^{n-1} (F^i)^* \mathcal{N}$. Since F^* is an autoequivalence of the category of overconvergent isocrystals, we deduce that $\overline{\mathcal{N}'} = \bigoplus_{i=0}^{n-1} (F^i)^* \overline{\mathcal{N}}$. This shows that if $\text{MS}(\mathcal{M}', \Phi'_{\mathcal{M}'})$ is true, then $\bigoplus_{i=0}^{n-1} (F^i)^* \mathcal{N} = S_1(\mathcal{N}') = S_1(\overline{\mathcal{N}'}) = \bigoplus_{i=0}^{n-1} (F^i)^* S_1(\overline{\mathcal{N}})$ (note that $S_1(-)$ commutes with F^*). In turn, this implies that $S_1(\mathcal{N}) = S_1(\overline{\mathcal{N}})$, which yields the desired result. \square

Lemma 4.1.5. *To prove $\text{MS}(X)$ it is enough to check isoclinic sub- F -isocrystals.*

Proof. The proof is by induction on the length of the slope filtration of $(\mathcal{N}, \Phi_{\mathcal{N}})$. Suppose that the slope filtration of $(\mathcal{N}, \Phi_{\mathcal{N}})$ has length m at least 2 and we already know the statement for length at most $m - 1$. It is enough to show that $S_1(\overline{\mathcal{N}}) = S_1(\overline{S_{m-1}(\mathcal{N})})$. Note that if $\overline{S_{m-1}(\mathcal{N})} = \overline{\mathcal{N}}$ there is nothing to prove, so that we may assume $\overline{S_{m-1}(\mathcal{N})} \subsetneq \overline{\mathcal{N}}$. In this case we have to show that the minimal slope of $\overline{\mathcal{N}/S_{m-1}(\mathcal{N})} \neq 0$ is greater than the minimal slope of $\overline{S_{m-1}(\mathcal{N})}$, which by the inductive hypothesis is s_1 . If $\pi : \mathcal{N} \rightarrow \overline{\mathcal{N}/S_{m-1}(\mathcal{N})}$ is the natural projection, then $\pi(\mathcal{N})$ is isoclinic of slope s_m . At the same time, there are no proper \dagger -extendable subobjects of $\overline{\mathcal{N}/S_{m-1}(\mathcal{N})}$ containing $\pi(\mathcal{N})$. By our assumption, the minimal slope of $\overline{\mathcal{N}/S_{m-1}(\mathcal{N})}$ is then equal to the slope of $\pi(\mathcal{N})$, which is greater than s_1 . This concludes the proof. \square

For next lemma we need the following theorem proven by Kedlaya.

Theorem 4.1.6 (Kedlaya). *Let $U \subseteq X$ be a dense open. The following statements are true.*

- (i) *The restriction functor $\mathbf{F}\text{-Isoc}(X) \rightarrow \mathbf{F}\text{-Isoc}(U)$ is fully faithful.*
- (ii) *The restriction functor $\mathbf{F}\text{-Isoc}^\dagger(X) \rightarrow \mathbf{F}\text{-Isoc}^\dagger(U)$ is fully faithful and closed under the operation of taking subquotients.*

Proof. Point (i) is [DK17, Thm. 2.2.3] while point (ii) is [Ked07, Thm. 5.2.1 and Prop. 5.3.1]. \square

Lemma 4.1.7. *If U is a dense open of X , then $\text{MS}(\mathcal{M}|_U, \Phi_{\mathcal{M}|_U})$ implies $\text{MS}(\mathcal{M}, \Phi_{\mathcal{M}})$.*

Proof. Let $(\mathcal{N}, \Phi_{\mathcal{N}})$ be an isoclinic subobject of $(\mathcal{M}, \Phi_{\mathcal{M}})$. We want to prove that the operation of taking \dagger -hulls commutes with the restriction functor to U . We have by definition $\overline{(\mathcal{N}|_U)} \subseteq \overline{\mathcal{N}}|_U$. On the other hand, by Theorem 4.1.6, the \dagger -extendable isocrystal $\overline{(\mathcal{N}|_U)}$ extends to some \dagger -extendable isocrystal \mathcal{M}' over X such that $\mathcal{N} \subseteq \mathcal{M}' \subseteq \mathcal{M}$. Since $\overline{\mathcal{N}} \subseteq \mathcal{M}'$ by definition, we show that $\overline{(\mathcal{N}|_U)} = \overline{\mathcal{N}}|_U$. Thanks to this, if $S_1(\overline{(\mathcal{N}|_U)}) = \mathcal{N}|_U$ then $S_1(\overline{\mathcal{N}})|_U = S_1(\overline{(\mathcal{N}|_U)}) = \mathcal{N}|_U$. This implies that $S_1(\overline{\mathcal{N}}) = \mathcal{N}$. \square

Lemma 4.1.8. *If $f : Y \rightarrow X$ is a finite étale Galois cover the following are true.*

- (i) *For every \dagger -extendable $(\mathcal{M}, \Phi_{\mathcal{M}})$ over X with constant slopes $\text{MS}(f^* \mathcal{M}, f^* \Phi_{\mathcal{M}})$ implies $\text{MS}(\mathcal{M}, \Phi_{\mathcal{M}})$.*
- (ii) *For every \dagger -extendable $(\mathcal{M}, \Phi_{\mathcal{M}})$ over Y with constant slopes $\text{MS}(f_* \mathcal{M}, f_* \Phi_{\mathcal{M}})$ implies $\text{MS}(\mathcal{M}, \Phi_{\mathcal{M}})$.*

Proof. As in Lemma 4.1.7, for (i) it is enough to show that for an isoclinic subobject $(\mathcal{N}, \Phi_{\mathcal{N}}) \subseteq (\mathcal{M}, \Phi_{\mathcal{M}})$ we have that $f^*\overline{\mathcal{N}} = \overline{f^*\mathcal{N}}$. Write G for the Galois group of the cover. The inclusion $f^*\overline{\mathcal{N}} \supseteq \overline{f^*\mathcal{N}}$ follows from the definition of \dagger -hull. On the other hand, if $\mathcal{M}' := \overline{f^*\mathcal{N}}$, then the intersection $\bigcap_{g \in G} g^*\mathcal{M}'$ contains $f^*\mathcal{N}$ and it is \dagger -extendable, thus it is equal to \mathcal{M}' . This implies that $(\mathcal{M}', \Phi_{\mathcal{M}'})$ descends to some \dagger -extendable F -isocrystal over X which contains $(\mathcal{N}, \Phi_{\mathcal{N}})$ and is contained in $(\mathcal{M}, \Phi_{\mathcal{M}})$. Therefore $f^*\overline{\mathcal{N}} = \overline{f^*\mathcal{N}}$, as we wanted.

We prove now (ii). Let $(\mathcal{N}, \Phi_{\mathcal{N}})$ be an isoclinic subobject of $(\mathcal{M}, \Phi_{\mathcal{M}})$. We have that $f^*f_*\mathcal{N} = \bigoplus_{g \in G} g^*\mathcal{N} \subseteq \bigoplus_{g \in G} g^*\mathcal{M} = f^*f_*\mathcal{M}$ so that $\overline{f^*f_*\mathcal{N}} = \bigoplus_{g \in G} g^*\overline{\mathcal{N}}$. Because of this, to prove that $S_1(\overline{\mathcal{N}}) = \overline{\mathcal{N}}$ it is enough to prove that $S_1(\overline{f^*f_*\mathcal{N}}) = \overline{f^*f_*\mathcal{N}}$. By the previous argument we have that $\overline{f^*f_*\mathcal{N}} = f^*\overline{f_*\mathcal{N}}$, so that $S_1(\overline{f^*f_*\mathcal{N}}) = f^*(S_1(\overline{f_*\mathcal{N}}))$ (note that $S_1(-)$ commutes with f^*). On the other hand, by the assumption that $S_1(f_*\mathcal{N}) = f_*\mathcal{N}$ we have that $f^*(S_1(\overline{f_*\mathcal{N}})) = f^*f_*\mathcal{N}$ and this yields the desired result. \square

4.2. The case of curves. In this section we study $\text{MS}(X)$ when X is a curve. In this case, $\text{MS}(X)$ has been essentially proven by Tsuzuki in [Tsu19, Prop. 5.8]. The proof we present here is a shorter variant where we avoid [ibid., Thm. 3.27]. The main ingredient in our case is de Jong's theorem on the existence of the *reverse slope filtration* (also used by Tsuzuki). On the other hand, in Corollary 5.4.2 we extends [ibid., Thm. 3.27] to arbitrary smooth varieties. As in Tsuzuki's proof, the strategy is to first prove an infinitesimal version of $\text{MS}(X)$ and then pass to the global setting. Thanks to the reductions of §4.1, it is enough to treat the case of the affine line.

Consider the Cohen ring

$$\mathcal{O}_{\mathcal{E}} := W((t))^{\wedge} = \left\{ \sum_{i \in \mathbb{Z}} a_i t^i \mid a_i \in W, \lim_{i \rightarrow -\infty} v_p(a_i) = \infty \right\}.$$

This is a complete discrete valuation ring unramified over W with residue field $k((t))$. Let $\mathcal{O}_{\mathcal{E}^\dagger} \subseteq \mathcal{O}_{\mathcal{E}}$ be the subring of series which converge in some annulus $* \leq |t| < 1$. These two rings are both endowed with a Frobenius lift $\varphi(t) = t^p$ and a derivation ∂_t . Write \mathcal{E} and \mathcal{E}^\dagger for the respective fields of fractions.

Definition 4.2.1 ((φ, ∇) -modules). If E is either \mathcal{E} or \mathcal{E}^\dagger , we say that a finite dimensional vector space M over E is a (φ, ∇) -module if it is endowed with a φ -linear isomorphism $\varphi_M : M \xrightarrow{\sim} M$ and an additive morphism $\nabla_{\partial_t} : M \rightarrow M$ which satisfies the Leibniz rule and such that $\nabla_{\partial_t} \circ \varphi_M = p t^{p-1} \varphi_M \circ \nabla_{\partial_t}$.

The category $\mathbf{F}\text{-Isoc}(k((t)))$ is the category of (φ, ∇) -modules over \mathcal{E} and $\mathbf{F}\text{-Isoc}^\dagger(k((t)))$ is the category of (φ, ∇) -modules over \mathcal{E}^\dagger . Thanks to [Ked04, Thm. 5.1], that the functor $\mathbf{F}\text{-Isoc}^\dagger(k((t))) \rightarrow \mathbf{F}\text{-Isoc}(k((t)))$ of extension of scalars is fully faithful. We want to show the following proposition.

Proposition 4.2.2 (Kedlaya, Tsuzuki⁵). *If $N \subseteq M$ is an inclusion of (φ, ∇) -modules over \mathcal{E} and M is \dagger -extendable, then $S_1(N) = S_1(\overline{N})$.*

To prove Proposition 4.2.2 we first need the following construction.

⁵We learned about a proof of Proposition 4.2.2 by Kedlaya, via a private communication. At the same time, the proposition corresponds essentially to [Tsu19, Thm. 2.14].

Construction 4.2.3. Let Q^\dagger be the image of the composition of natural morphisms

$$(M^\dagger)^\vee := \mathrm{Hom}_{\mathcal{E}}^\dagger(M^\dagger, \mathcal{E}^\dagger) \rightarrow \mathrm{Hom}_{\mathcal{E}}(M, \mathcal{E}) \rightarrow \mathrm{Hom}_{\mathcal{E}}(N, \mathcal{E}) =: N^\vee.$$

We have natural maps

$$M^\vee = (M^\dagger)^\vee \otimes_{\mathcal{E}^\dagger} \mathcal{E} \rightarrow Q^\dagger \otimes_{\mathcal{E}^\dagger} \mathcal{E} \rightarrow N^\vee.$$

The first arrow is surjective by construction, the second one is surjective because the morphism $M^\vee \rightarrow N^\vee$ is surjective. Note that even though $Q^\dagger \subseteq N^\vee$, the second map needs not to be injective. Dualizing with respect to \mathcal{E} we get inclusions $N \subseteq Q^\vee \subseteq M$.

Lemma 4.2.4. *The (φ, ∇) -module Q^\vee is the \dagger -hull of N in M . In other words, \overline{N} is the unique submodule of M which contains N and comes from some $\overline{N}^\dagger \subseteq M^\dagger$ such that $(\overline{N}^\dagger)^\vee \rightarrow N^\vee$ is injective.*

Proof. By construction, Q^\vee is \dagger -extendable and it contains N , so that $\overline{N} \subseteq Q^\vee$. On the other hand, we have morphisms $(M^\dagger)^\vee \rightarrow (\overline{N}^\dagger)^\vee \rightarrow N^\vee$, where the first one is surjective. By definition, the morphism $(\overline{N}^\dagger)^\vee \rightarrow N^\vee$ factors through Q^\dagger , which implies that $Q^\vee \subseteq \overline{N}$. \square

We recall now the reverse filtration introduced by de Jong in [deJ98, Prop. 5.5]. For this, we need to introduce two other discrete valuation fields lifting $k((t))^{\mathrm{alg}}$.

Definition 4.2.5. Let $\mathcal{O}_{\mathcal{E}_{\mathrm{alg}}}$ be the ring of Witt vectors of $k((t))^{\mathrm{alg}}$. Every element of $\mathcal{E}_{\mathrm{alg}}$ can be written uniquely as $\sum_{i=0}^{\infty} [f_i] p^i$ where $[f_i]$ is the Teichmüller lift of some $f_i \in k((t))^{\mathrm{alg}}$. Consider the subring $\mathcal{O}_{\mathcal{E}_{\mathrm{alg}}}^\dagger \subseteq \mathcal{O}_{\mathcal{E}_{\mathrm{alg}}}$ of those series such that the t -adic valuations of f_i are bounded below by some linear function in i . The subring $\mathcal{O}_{\mathcal{E}_{\mathrm{alg}}}^\dagger$ is preserved by the Frobenius of $\mathcal{O}_{\mathcal{E}_{\mathrm{alg}}}$. We write $\mathcal{E}_{\mathrm{alg}}^\dagger$ and $\mathcal{E}_{\mathrm{alg}}$ for the fraction fields.

Theorem 4.2.6 ([deJ98, Prop. 5.5]). *For a φ -module M_{alg}^\dagger over $\mathcal{E}_{\mathrm{alg}}^\dagger$ the following statements are true.*

- (i) M_{alg}^\dagger admits a reverse slope filtration, i.e. there exists a filtration

$$0 = S_0^{\mathrm{rev}}(M_{\mathrm{alg}}^\dagger) \subsetneq S_1^{\mathrm{rev}}(M_{\mathrm{alg}}^\dagger) \subsetneq \cdots \subsetneq S_m^{\mathrm{rev}}(M_{\mathrm{alg}}^\dagger) = M_{\mathrm{alg}}^\dagger$$

of φ -modules over $\mathcal{E}_{\mathrm{alg}}^\dagger$ such that $(S_i^{\mathrm{rev}}(M_{\mathrm{alg}}^\dagger)/S_{i-1}^{\mathrm{rev}}(M_{\mathrm{alg}}^\dagger)) \otimes_{\mathcal{E}_{\mathrm{alg}}^\dagger} \mathcal{E}_{\mathrm{alg}}$ is isomorphic to $S_{m-i}(M_{\mathrm{alg}})/S_{m-i-1}(M_{\mathrm{alg}})$.

- (ii) If M^\dagger is isoclinic of slope s/r , the φ -module $M_{\mathrm{alg}}^\dagger[p^{1/r}]$ admits a basis of vectors $\{v_1, \dots, v_d\}$ such that $\varphi(v_i) = p^{s/r} v_i$.

Lemma 4.2.7. *Let M^\dagger be a φ -module over \mathcal{E}^\dagger and let N be an isoclinic φ -module over \mathcal{E} of slope s/r . For every morphism $\psi : M \rightarrow N$ of φ -modules, if the restriction of ψ to M^\dagger is injective, then the maximal slope of M is s/r and the rank of $S_m(M)/S_{m-1}(M)$ is smaller or equal than the rank of N .*

Proof. This is a mild generalisation of [Ked07, Lem. 4.2]. Since $\mathcal{E}_{\mathrm{alg}}^\dagger$ is flat over \mathcal{E}^\dagger and $\mathcal{E} \otimes_{\mathcal{E}^\dagger} \mathcal{E}_{\mathrm{alg}}^\dagger \rightarrow \mathcal{E}_{\mathrm{alg}}$ is injective by [Ked07, Prop. 4.1], then $\psi|_{M^\dagger}$ induces an injective morphism

$$\psi' : M_{\mathrm{alg}}^\dagger := M^\dagger \otimes_{\mathcal{E}^\dagger} \mathcal{E}_{\mathrm{alg}}^\dagger \rightarrow N \otimes_{\mathcal{E}^\dagger} \mathcal{E}_{\mathrm{alg}}^\dagger \rightarrow N \otimes_{\mathcal{E}} \mathcal{E}_{\mathrm{alg}}.$$

The restriction of ψ' to $S_1^{\text{rev}}(M_{\text{alg}}^\dagger)$ induces a non-trivial morphism

$$S_1^{\text{rev}}(M_{\text{alg}}^\dagger) \otimes_{\mathcal{E}_{\text{alg}}^\dagger} \mathcal{E}_{\text{alg}} \rightarrow N \otimes_{\mathcal{E}} \mathcal{E}_{\text{alg}}.$$

This implies that the slope of $S_1^{\text{rev}}(M_{\text{alg}}^\dagger)$, which is the maximal slope of M , is s/r . Moreover, by Theorem 4.2.6.(ii), the dimension of the $\mathbb{Q}_p(p^{1/r})$ -vector space $(S_1^{\text{rev}}(M_{\text{alg}}^\dagger)[p^{1/r}])^{\varphi=p^{s/r}}$ is equal to the rank of $S_m(M)/S_{m-1}(M)$. Similarly, by the Dieudonné–Manin decomposition, $(N \otimes_{\mathcal{E}} \mathcal{E}_{\text{alg}}[p^{1/r}])^{\varphi=p^{s/r}}$ is a $\mathbb{Q}_p(p^{1/r})$ -vector space of dimension equal to the rank of N . We then obtain the inequality of ranks thanks to the injectivity of ψ' . \square

4.2.8. *Proof of Proposition 4.2.2.* By Lemma 4.1.5, it is enough to prove the result when N is isoclinic of slope s/r and $\overline{N} = M$. In that case, we have to check that \overline{N} has minimal slope s/r and that the inclusion $N \subseteq S_1(\overline{N})$ is an equality. By Lemma 4.2.4, the morphism $\overline{N}^\vee \rightarrow N^\vee$ satisfies the assumptions of Lemma 4.2.7, thus \overline{N} has minimal slope s/r . Moreover, since

$$\text{rk}(N) = \text{rk}(N^\vee) \geq \text{rk}(S_m(\overline{N}^\vee)/S_{m-1}(\overline{N}^\vee)) = \text{rk}(S_1(\overline{N})),$$

we get $N = S_1(\overline{N})$. \square

4.2.9. Next step is to pass from the local situation to the global situation. Thanks to the reductions in §4.1 it is enough to work with F -isocrystals over \mathbb{A}_k^1 . Let $K\langle u \rangle$ and $K\langle u \rangle^\dagger$ be the rings of convergent and overconvergent series over K . The category $\mathbf{F}\text{-Isoc}(\mathbb{A}_k^1)$ is the category of (φ, ∇) -modules over $K\langle u \rangle$ (defined as in §4.2.1) while $\mathbf{F}\text{-Isoc}^\dagger(\mathbb{A}_k^1)$ is the category of (φ, ∇) -modules over $K\langle u \rangle^\dagger$. We consider the ring homomorphisms $K\langle u \rangle \rightarrow \mathcal{E}$ and $K\langle u \rangle^\dagger \rightarrow \mathcal{E}^\dagger$ mapping $u \mapsto \frac{1}{t}$.

Lemma 4.2.10 ([Tsu19, Lem. 3.1]). *The morphism $K\langle u \rangle^\dagger \rightarrow \mathcal{E}^\dagger$ is flat and $K\langle u \rangle \otimes_{K\langle u \rangle^\dagger} \mathcal{E}^\dagger \simeq \mathcal{E}$.*

Proof. We present here a variant of Tsuzuki's proof. Let A_n be the image of the morphism $W\langle u_1, \dots, u_n \rangle \rightarrow W\langle u \rangle$ sending $u_i \mapsto \bar{u}_i := pu^i$ and $B_n \subseteq \mathcal{O}_{\mathcal{E}}$ the p -adic completion of $W[[t]]\left[\frac{p}{t^n}\right]$. We have that $\varinjlim_n A_n\left[\frac{1}{p}\right] = K\langle u \rangle^\dagger$ and $\varinjlim_n B_n\left[\frac{1}{p}\right] = \mathcal{E}^\dagger$. The morphism $A_n \rightarrow B_n$ mapping $\bar{u}_i \mapsto \frac{p}{t^i}$ factors as

$$A_n \rightarrow A_n[t]/(\bar{u}_i t^i - p)_{1 \leq i \leq n} \rightarrow B_n.$$

The first morphism is a localisation after inverting p and the second one corresponds to the t -adic completion. Therefore, we deduce that $A_n\left[\frac{1}{p}\right] \rightarrow B_n\left[\frac{1}{p}\right]$ is flat for every n . In turn, this implies that $K\langle u \rangle^\dagger \rightarrow \mathcal{E}^\dagger$ is flat. To prove that $K\langle u \rangle \otimes_{K\langle u \rangle^\dagger} \mathcal{E}^\dagger \simeq \mathcal{E}$ it is enough to note that

$$W\langle u \rangle \otimes_{A_n} B_n \simeq W\langle u \rangle[[t]]/(p(ut - 1))$$

for every n . Indeed, after inverting p , we get $(W\langle u \rangle \otimes_{A_n} B_n)\left[\frac{1}{p}\right] \simeq \mathcal{E}$. \square

Lemma 4.2.11 (after Tsuzuki). *Let $N \subseteq M$ be an inclusion of (φ, ∇) -module over $K\langle u \rangle$ where M is \dagger -extendable and has constant slopes. The (φ, ∇) -module $\overline{N} \otimes \mathcal{E}$ is the \dagger -hull of $N \otimes \mathcal{E}$ in $M \otimes \mathcal{E}$.*

Proof. The $K\langle u \rangle$ -module \overline{N} has a similar description as in §4.2.3, where \mathcal{E} and \mathcal{E}^\dagger are replaced with the rings $K\langle u \rangle$ and $K\langle u \rangle^\dagger$. By the analogue of Lemma 4.2.4, the (φ, ∇) -module \overline{N} comes from a (φ, ∇) -module \overline{N}^\dagger over $K\langle u \rangle^\dagger$ such that $(\overline{N}^\dagger)^\vee \rightarrow N^\vee$ is injective. By Lemma 4.2.10, the morphism $(\overline{N}^\dagger \otimes_{K\langle u \rangle^\dagger} \mathcal{E}^\dagger)^\vee \rightarrow (N \otimes_{K\langle u \rangle} \mathcal{E})^\vee$ obtained as the composition of

$$(\overline{N}^\dagger)^\vee \otimes_{K\langle u \rangle^\dagger} \mathcal{E}^\dagger \hookrightarrow N^\vee \otimes_{K\langle u \rangle^\dagger} \mathcal{E}^\dagger \xrightarrow{\sim} N^\vee \otimes_{K\langle u \rangle} \mathcal{E}$$

is injective. This proves what we want thanks to Lemma 4.2.4. \square

Theorem 4.2.12 (after Tsuzuki). *A smooth curve X over a perfect field k satisfies $\text{MS}(X)$.*

Proof. The statement corresponds essentially to [Tsu19, Prop. 5.8]. We propose here a shorter variant. By [Ked05, Thm. 1], there exists a dense open of X which admits a finite étale cover to \mathbb{A}_k^1 . Thanks to Lemma 4.1.7 and Lemma 4.1.8, this implies that we may assume $X = \mathbb{A}_k^1$. Let $N \subseteq M$ be an inclusion of (φ, ∇) -module over $K\langle u \rangle$, where M is \dagger -extendable and has constant slopes. By Lemma 4.2.11, we have that $\overline{N} \otimes \mathcal{E}$ is the \dagger -hull of $N \otimes \mathcal{E}$ in $M \otimes \mathcal{E}$. Therefore, by Proposition 4.2.2,

$$S_1(N) \otimes \mathcal{E} = S_1(N \otimes \mathcal{E}) = S_1(\overline{N} \otimes \mathcal{E}) = S_1(\overline{N}) \otimes \mathcal{E}.$$

This implies that $S_1(N)$ and $S_1(\overline{N})$ have the same slope and the same rank, so that $S_1(N) = S_1(\overline{N})$. This concludes the proof. \square

4.3. Chevalley theorem and filtrations. In this section Hypothesis 3.1.1 is in force. Let $\eta \in X(\Omega)$ be a perfect point of X and let $(\mathcal{M}^\dagger, \Phi_{\mathcal{M}}^\dagger)$ be an overconvergent F^n -isocrystal with constant slopes. Consider the Dieudonné–Manin fibre functor $\omega_{\eta, \mathbb{Q}_p^{\text{ur}}} : \langle \mathcal{M}, \Phi_{\mathcal{M}}^\infty \rangle \rightarrow \mathbf{Vec}_{\mathbb{Q}_p^{\text{ur}}}$ associated to η . Write G for $G(\mathcal{M}^\dagger, \Phi_{\mathcal{M}}^{\dagger, \infty}, \eta)$ and H for $G(\mathcal{M}, \Phi_{\mathcal{M}}^\infty, \eta)$.

Definition 4.3.1. For every e , let $\mathbb{G}_m^{1/e}$ be the torus with character group $\frac{1}{e}\mathbb{Z}$ and $\mathbb{G}_m^{1/\infty} := \varprojlim_i \mathbb{G}_m^{1/e}$. If r is the lcm of the denominators of the slopes of \mathcal{M} , for every $(\mathcal{M}', \Phi_{\mathcal{M}'}^\infty) \in \langle \mathcal{M}, \Phi_{\mathcal{M}}^\infty \rangle$ we denote by $\tilde{S}_{s/r}(\omega_{\eta, \mathbb{Q}_p^{\text{ur}}}(\mathcal{M}'_\eta, \Phi_{\mathcal{M}'}^\infty)) \subseteq \omega_{\eta, \mathbb{Q}_p^{\text{ur}}}(\mathcal{M}'_\eta, \Phi_{\mathcal{M}'}^\infty)$ the \mathbb{Q}_p^{ur} -linear vector subspace of slope at most s/r . This defines an exact \otimes -filtration \tilde{S}_\bullet of $\omega_{\eta, \mathbb{Q}_p^{\text{ur}}}$ indexed by $\frac{1}{r}\mathbb{Z}$, that in turn defines a morphism $\lambda : \mathbb{G}_m^{1/\infty} \twoheadrightarrow \mathbb{G}_m^{1/r} \rightarrow G$ (cf. [Saa72, §2.1.1, page 213]). We say that λ is the *quasi-cocharacter* attached to the slope filtration of \mathcal{M}_η . We denote by $P_G(\lambda)$ the subgroup of G of those \otimes -automorphisms of $\omega_{\eta, \mathbb{Q}_p^{\text{ur}}}$ preserving \tilde{S}_\bullet (as in [ibid., §2.1.3, page 216]).

Proposition 4.3.2. *If $\text{MS}(X)$ is true, $H = P_G(\lambda)$. In particular, if G is a reductive group, H is a parabolic subgroup of G .*

Proof. Since $H \subseteq P_G(\lambda)$, we have to prove that $P_G(\lambda) \subseteq H$. By Chevalley's theorem, there exists an overconvergent F^∞ -isocrystal $(\mathcal{N}^\dagger, \Phi_{\mathcal{N}}^{\dagger, \infty}) \in \langle \mathcal{M}^\dagger, \Phi_{\mathcal{M}}^{\dagger, \infty} \rangle$ and a rank 1 sub- F^∞ -isocrystal $(\mathcal{L}, \Phi_{\mathcal{L}}^\infty) \subseteq (\mathcal{N}, \Phi_{\mathcal{N}}^\infty)$, such that H is the stabiliser of the line

$$L := \omega_{\eta, \mathbb{Q}_p^{\text{ur}}}(\mathcal{L}, \Phi_{\mathcal{L}}) \subseteq \omega_{\eta, \mathbb{Q}_p^{\text{ur}}}(\mathcal{N}, \Phi_{\mathcal{N}}) := V.$$

We have to prove that $P_G(\lambda)$ stabilises L . Let $(\overline{\mathcal{L}}, \Phi_{\overline{\mathcal{L}}})$ be the \dagger -hull of $(\mathcal{L}, \Phi_{\mathcal{L}}) \subseteq (\mathcal{N}, \Phi_{\mathcal{N}})$ and write \overline{L} for $\omega_{\eta, \mathbb{Q}_p^{\text{ur}}}(\overline{\mathcal{L}}, \Phi_{\overline{\mathcal{L}}})$. We denote by s the slope of $(\mathcal{L}, \Phi_{\mathcal{L}})$ and by $V^{\leq s} \subseteq V$ the subspace of slope smaller or equal than s . Since $\text{MS}(X)$ is satisfied, we know that $\mathcal{L} = S_1(\mathcal{L}) = S_1(\overline{\mathcal{L}})$, which implies that $L = \overline{L} \cap V^{\leq s}$. Since $\overline{\mathcal{L}} \subseteq \mathcal{N}$ admits by definition a \dagger -extension, $P_G(\lambda)$ stabilises \overline{L} . On the other hand, $P_G(\lambda)$ stabilises $V^{\leq s}$ because $(\mathcal{N}^\dagger, \Phi_{\mathcal{N}}^{\dagger, \infty})$ is an element in $\langle \mathcal{M}^\dagger, \Phi_{\mathcal{M}}^{\dagger, \infty} \rangle$. This implies that $P_G(\lambda)$ stabilises L , thus $P_G(\lambda) \subseteq H$ as we wanted. If G is reductive, $H = P_G(\lambda)$ is parabolic by [Saa72, Prop. 2.2.5, page 223]. \square

As a first consequence we get next result.

Proposition 4.3.3. *Let X be a smooth geometrically connected variety over k such that $\text{MS}(U)$ is true for every dense open $U \subseteq X$. If $(\mathcal{M}^\dagger, \Phi_{\mathcal{M}}^\dagger)$ is an overconvergent F^n -isocrystal over X , then $\pi_0(G(\mathcal{M}, \Phi_{\mathcal{M}}^\infty, \eta)) = \pi_0(G(\mathcal{M}^\dagger, \Phi_{\mathcal{M}}^{\dagger, \infty}, \eta))$.*

Proof. To prove the statement we may assume that $(\mathcal{M}^\dagger, \Phi_{\mathcal{M}}^{\dagger, \infty})$ is semi-simple by replacing it with the semi-simplification with respect to a Jordan–Hölder filtration. Next step is to show that the map $\pi_0(G(\mathcal{M}, \Phi_{\mathcal{M}}^\infty, \eta)) \rightarrow \pi_0(G(\mathcal{M}^\dagger, \Phi_{\mathcal{M}}^{\dagger, \infty}, \eta))$ is surjective. For this we do not need the previous theorem. Write $\langle \mathcal{M}^\dagger, \Phi_{\mathcal{M}}^{\dagger, \infty} \rangle_{\text{fin}}$ and $\langle \mathcal{M}, \Phi_{\mathcal{M}}^\infty \rangle_{\text{fin}}$ for the Tannakian subcategories of objects with finite monodromy groups. The functor $\langle \mathcal{M}^\dagger, \Phi_{\mathcal{M}}^{\dagger, \infty} \rangle \rightarrow \langle \mathcal{M}, \Phi_{\mathcal{M}}^\infty \rangle$ is fully faithful, thus the functor $\langle \mathcal{M}^\dagger, \Phi_{\mathcal{M}}^{\dagger, \infty} \rangle_{\text{fin}} \rightarrow \langle \mathcal{M}, \Phi_{\mathcal{M}}^\infty \rangle_{\text{fin}}$ is fully faithful and observable, which implies that $\pi_0(G(\mathcal{M}, \Phi_{\mathcal{M}}^\infty, \eta)) \rightarrow \pi_0(G(\mathcal{M}^\dagger, \Phi_{\mathcal{M}}^{\dagger, \infty}, \eta))$ is surjective.

Let U be a dense open of X where $(\mathcal{M}^\dagger, \Phi_{\mathcal{M}}^\dagger)$ has constant slopes. Up to replacing η we might assume that it lies in U . We have the following commutative diagram

$$\begin{array}{ccc} \pi_0(G(\mathcal{M}|_U, \Phi_{\mathcal{M}}^\infty|_U, \eta)) & \twoheadrightarrow & \pi_0(G(\mathcal{M}, \Phi_{\mathcal{M}}, \eta)) \\ \downarrow & & \downarrow \\ \pi_0(G(\mathcal{M}^\dagger|_U, \Phi_{\mathcal{M}}^{\dagger, \infty}|_U, \eta)) & \xrightarrow{\sim} & \pi_0(G(\mathcal{M}^\dagger, \Phi_{\mathcal{M}}^\dagger, \eta)). \end{array}$$

Thanks to Theorem 4.1.6, the upper arrow is surjective while the lower one is an isomorphism. To prove the final statement it is enough to prove that the morphism $\pi_0(G(\mathcal{M}|_U, \Phi_{\mathcal{M}}^\infty|_U, \eta)) \rightarrow \pi_0(G(\mathcal{M}^\dagger|_U, \Phi_{\mathcal{M}}^{\dagger, \infty}|_U, \eta))$ is injective, or equivalently that

$$H' := G(\mathcal{M}^\dagger|_U, \Phi_{\mathcal{M}}^{\dagger, \infty}|_U, \eta)^\circ \cap G(\mathcal{M}|_U, \Phi_{\mathcal{M}}^\infty|_U, \eta)$$

is connected. By Proposition 4.3.2, the group H' is a parabolic subgroup of the connected reductive group $G(\mathcal{M}^\dagger|_U, \Phi_{\mathcal{M}}^{\dagger, \infty}|_U, \eta)^\circ$. By [Bor91, Thm. 11.16], this implies that H' is connected, as we wanted. \square

4.4. Lefschetz theorem and proof of Theorem 1.1.1. In this section we deal with the problem of reducing Theorem 4.1.3 to the case of curves. For this aim, we prove a new Lefschetz theorem for overconvergent isocrystals. Subsequently, we use Theorem 4.1.3 to prove Theorem 1.1.1.

Definition 4.4.1. If Y is smooth and proper variety, $D \subseteq Y$ is a simple normal crossing and $X := Y \setminus D$, we say that an overconvergent isocrystal over X is *docile* if it has unipotent monodromy along D (cf. [Ked07, Def. 4.4.2]). We denote by $\mathbf{Isoc}^\dagger(X)_{\text{doc}}$ the category of docile overconvergent isocrystals over X .

In [AE19, Cor. 2.4] the authors proved the following Lefschetz theorem for docile overconvergent isocrystals over perfect fields.

Theorem 4.4.2 (Abe–Esnault). *Let $Y \subseteq \mathbb{P}_k^d$ be a smooth connected projective variety of dimension ≥ 2 and D a simple normal crossing divisor. For every smooth curve $\overline{C} \subseteq Y$ which is a complete intersection of hypersurfaces intersecting transversally D , if we write C for $\overline{C} \setminus D$, the functor*

$$\mathbf{Isoc}^\dagger(X)_{\text{doc}} \rightarrow \mathbf{Isoc}^\dagger(C)_{\text{doc}}$$

is fully faithful.

We explain here how to refine their result and prove the following theorem, which will then be used in §4.4.12 to prove Theorem 4.1.3.

Theorem 4.4.3. *Let $Y \subseteq \mathbb{P}_{k^{\text{alg}}}^d$ be a smooth connected projective variety of dimension ≥ 2 and D a simple normal crossing divisor. If $(\mathcal{M}^\dagger, \Phi_{\mathcal{M}}^\dagger)$ is an overconvergent F^n -isocrystal over $X := Y \setminus D$ docile with respect to D , then there exists a smooth connected curve $C \subseteq X$ intersecting the dense open of X where the slopes of $(\mathcal{M}^\dagger, \Phi_{\mathcal{M}}^\dagger)$ are constant and such that the restriction functor $\langle \mathcal{M}^\dagger \rangle \rightarrow \langle \mathcal{M}^\dagger|_C \rangle$ is an equivalence of categories.*

Remark 4.4.4. In [AE19, Thm. 3.10] the authors improve Theorem 4.4.2 to a stronger statement when k is finite. Their proof does not extend to algebraically closed field (not even $k^{\text{alg}} = \mathbb{F}$) since they crucially use a finiteness result for rank 1 overconvergent F -isocrystals, which ultimately relies on class field theory.

Before proving Theorem 4.4.3 let us present some preliminary results that we will use during the proof. First, we need the following two lemmas on lisse sheaves.

Lemma 4.4.5 (Pink, Serre). *Let X be a smooth connected variety over k and \mathcal{V} a lisse $\overline{\mathbb{Q}}_p$ -sheaf over X . There exists a connected finite étale cover $\tilde{X} \rightarrow X$ with the property that for every smooth connected curve $C \subseteq X$ such that $\tilde{X} \times_X C$ is connected, the restriction functor*

$$\langle \mathcal{V} \rangle \rightarrow \langle \mathcal{V}|_C \rangle$$

is an equivalence of categories.

Proof. This is proven in [Ser89, §10.6] or [Kat90, Key Lemma 8.18.3]. □

Lemma 4.4.6. *Let X be a smooth connected variety over k , $D \subseteq X$ an irreducible divisor and \mathcal{V} a lisse $\overline{\mathbb{Q}}_p$ -sheaf over $X \setminus D$. There exists a dense smooth open $D' \subseteq D$ and a conic closed subscheme $Z \subseteq TX \times_X D'$ of codimension 1 at every fibre which satisfies the following property.*

$R_{\mathcal{V}}(Z)$: *Let $C \subseteq X$ be a smooth curve not contained in D and intersecting D' at some closed point x such that TC_x is not contained in Z_x . For every rank 1 lisse sheaf $\mathcal{L} \in \langle \mathcal{V} \rangle$ ramified at D , $\mathcal{L}|_C$ is ramified at $D \cap C$.*

Proof. This result is a variant of [Dri12, Lem. 5.1]. To prove the lemma we may assume that \mathcal{V} is a direct sum of rank 1 lisse $\overline{\mathbb{Q}}_p$ -sheaves. Let η be a geometric point of $X \setminus D$. Write I for the inertia subgroup of $\pi_1^{\text{ét}}(X \setminus D, \eta)$ associated to D and $\rho : \pi_1^{\text{ét}}(X \setminus D, \eta) \rightarrow \text{GL}(V)$ for the representation attached to \mathcal{V} , where V is a finite-dimensional E -linear vector space for some finite field extension E/\mathbb{Q}_p . Let n be a positive integer such that $p^n \cdot \Pi \cap \rho(I) \subseteq p \cdot \text{im}(I)$. Let Γ be the finite set of morphisms $\gamma : \Pi \rightarrow \mathbb{Z}/p^n\mathbb{Z} \times \Pi_{\text{tors}}$ mapping $\rho(I)$ to a cyclic group G_γ of prime order. For every $\gamma \in \Gamma$, let $f_\gamma : X_\gamma \rightarrow X$ be the finite étale cover such that the image of the restriction of $\gamma \circ \rho$ to the étale fundamental group of $X_\gamma \setminus f_\gamma^{-1}(D)$ is G_γ and let $Y_\gamma \rightarrow X_\gamma \setminus D_\gamma$ be the associated connected G_γ -torsor. Write \bar{Y}_γ for the normalisation of X_γ in the field of functions of Y_γ . Then $\bar{Y}_\gamma \rightarrow X_\gamma$ is a finite cover ramified along D_γ . After shrinking X_γ to an open X'_γ , we may assume that $\bar{Y}'_\gamma := \bar{Y}_\gamma \times_{X_\gamma} X'_\gamma$ is regular and the image of $TY'_\gamma \rightarrow TX'_\gamma$ is a vector subbundle $\tilde{Z}_\gamma \subseteq TX'_\gamma$ of codimension 1. We write $Z_\gamma \subseteq TX$ for the image of \tilde{Z}_γ via the morphism f_γ and D'_γ for its

projection to X . By construction, for every $x \in D'_\gamma$, we have that $Z_{\gamma,x}$ has codimension 1 in TX_x . Write D' for $\bigcap_{\gamma \in \Gamma} D'_\gamma$ and Z for $\bigcup_{\gamma \in \Gamma} Z_\gamma|_{D'}$. We claim that $R_V(Z)$ is satisfied.

Let $C \subseteq X$ be a smooth curve not in D and intersecting D' in a point x and such that TC_x is not in Z_x . By construction, for every $\gamma \in \Gamma$, the induced morphism $C \times_X \bar{Y}'_\gamma \rightarrow C \times_X X_\gamma$ is ramified at the points over x . Therefore, if I_C is the inertia subgroup of the étale fundamental group of $C \setminus D$ at x , it is sent surjectively to G_γ for every $\gamma \in \Gamma$. Via a group-theoretic argument, this implies that I_C is mapped surjectively to $\rho(I)$. This yields the desired result. \square

Secondly, we need the following consequence of Theorem 4.2.12, which helps to describe the subgroup $X^*(G(\mathcal{M}^\dagger)) \subseteq X^*(G(\mathcal{M}))$ in the docile situation.

Proposition 4.4.7. *Let X be a smooth geometrically connected variety over k such that every open U satisfies $\text{MS}(U)$ and let $(\mathcal{M}, \Phi_{\mathcal{M}})$ be a \dagger -extendable F^n -isocrystal. If $(\mathcal{L}, \Phi_{\mathcal{L}}^\infty) \in \langle \mathcal{M}, \Phi_{\mathcal{M}}^\infty \rangle$ is a \dagger -extendable rank 1 F^∞ -isocrystal, then $(\mathcal{L}^\dagger, \Phi_{\mathcal{L}^\dagger}^{\dagger,\infty})$ is in $\langle \mathcal{M}^\dagger, \Phi_{\mathcal{M}^\dagger}^{\dagger,\infty} \rangle$.*

Proof. By Theorem 4.1.6 we may shrink X and assume that $(\mathcal{M}, \Phi_{\mathcal{M}})$ admits the slope filtration. Moreover, we may assume without loss of generality that $(\mathcal{M}^\dagger, \Phi_{\mathcal{M}^\dagger}^\dagger)$ is semi-simple. Let us write G for $G(\mathcal{M}^\dagger \oplus \mathcal{L}^\dagger, \Phi_{\mathcal{M}^\dagger}^{\dagger,\infty} \oplus \Phi_{\mathcal{L}^\dagger}^{\dagger,\infty})$ and G' for $G(\mathcal{M}^\dagger, \Phi_{\mathcal{M}^\dagger}^{\dagger,\infty})$. The inclusion $\langle \mathcal{M}^\dagger, \Phi_{\mathcal{M}^\dagger}^{\dagger,\infty} \rangle \subseteq \langle \mathcal{M}^\dagger \oplus \mathcal{L}^\dagger, \Phi_{\mathcal{M}^\dagger}^{\dagger,\infty} \oplus \Phi_{\mathcal{L}^\dagger}^{\dagger,\infty} \rangle$ induces a surjective morphism $f : G \rightarrow G'$. We want to prove that $N := \ker(f) = 1$. Let $P \subseteq G$ be the subgroup associated to $\langle \mathcal{M} \oplus \mathcal{L}, \Phi_{\mathcal{M}} \oplus \Phi_{\mathcal{L}} \rangle$. By Proposition 4.3.2, since we are assuming that $\text{MS}(X)$ is true, the subgroup P is a parabolic subgroup of G . Thus it contains a maximal torus T of G . Since $\langle \mathcal{M}, \Phi_{\mathcal{M}}^\infty \rangle = \langle \mathcal{M} \oplus \mathcal{L}, \Phi_{\mathcal{M}}^\infty \oplus \Phi_{\mathcal{L}}^\infty \rangle$, the morphism f is an isomorphism when restricted to P , so that $N \cap P = 1$. Therefore, N is a finite group, because it is a normal subgroup of the reductive group G which intersects trivially the maximal torus T . The morphism $\pi_0(f)$ is an isomorphism by Proposition 4.3.3, because both $\pi_0(G)$ and $\pi_0(G')$ are equal to $\pi_0(P)$. This implies that N is contained in G° , the neutral component of G . Moreover, since N is a finite normal subgroup, it is contained in $Z(G^\circ) \subseteq T$. This shows that $N = 1$, as we wanted. \square

Corollary 4.4.8. *If X is a variety as in Proposition 4.4.7, for every overconvergent F^n -isocrystal $(\mathcal{M}, \Phi_{\mathcal{M}})$ over X the morphism $G(\mathcal{M}, \Phi_{\mathcal{M}}^\infty, \eta)^{\text{cst}} \rightarrow G(\mathcal{M}^\dagger, \Phi_{\mathcal{M}^\dagger}^{\dagger,\infty}, \eta)^{\text{cst}}$ is an isomorphism. Equivalently, the left square of the diagram in Proposition 3.2.8 is cartesian.*

We suppose now that X admits a smooth compactification $X \subseteq Y \subseteq \mathbb{P}_k^d$ and $Y \setminus X$ is a simple normal divisor of D . Write $X^*(G(\mathcal{M}, V_{\mathcal{M}}))^{\text{ur}}$ for the group of characters of $G(\mathcal{M}, V_{\mathcal{M}})$ corresponding to unramified rank 1 isocrystals with \mathbb{Q}_p^{ur} -structure.

Corollary 4.4.9. *If X is a variety as in Proposition 4.4.7 and $(\mathcal{M}^\dagger, \Phi_{\mathcal{M}^\dagger}^\dagger)$ is an overconvergent F^n -isocrystal over X docile along D , we have that $X^*(G(\mathcal{M}^\dagger, V_{\mathcal{M}^\dagger}^\dagger)) = X^*(G(\mathcal{M}, V_{\mathcal{M}}))^{\text{ur}}$, where $V_{\mathcal{M}^\dagger}^\dagger$ and $V_{\mathcal{M}}$ are the induced Dieudonné–Manin \mathbb{Q}_p^{ur} -structures.*

Proof. To prove that $X^*(G(\mathcal{M}^\dagger, V_{\mathcal{M}^\dagger}^\dagger)) \subseteq X^*(G(\mathcal{M}, V_{\mathcal{M}}))^{\text{ur}}$ we combine two facts. First, the property of being docile is preserved under the operations of taking direct sum, tensor product, dual, and subquotients in $\mathbf{Isoc}^\dagger(X)$ by [Ked07, Prop. 3.2.20]. Secondly, rank 1 docile overconvergent isocrystals have 0 residues, thus they are unramified. This shows that rank 1 objects in $\langle \mathcal{M}^\dagger, V_{\mathcal{M}^\dagger}^\dagger \rangle$ are unramified.

For the other containment, by Proposition 3.2.4 and Proposition 3.2.5 every rank 1 isocrystal with \mathbb{Q}_p^{ur} -structure in $\langle \mathcal{M}, V_{\mathcal{M}} \rangle$ comes from some F^∞ -isocrystal in $\langle \mathcal{M}, \Phi_{\mathcal{M}}^\infty \rangle$. The same result remains true for overconvergent isocrystals. Thanks to Proposition 4.4.7, if $(\mathcal{L}, \Phi_{\mathcal{L}}^\infty)$ is an unramified (hence \dagger -extendable) rank 1 F^∞ -isocrystal, then $(\mathcal{L}^\dagger, \Phi_{\mathcal{L}^\dagger}^{\dagger, \infty}) \in \langle \mathcal{M}^\dagger, \Phi_{\mathcal{M}^\dagger}^{\dagger, \infty} \rangle$. This concludes the proof. \square

Finally, for Theorem 4.4.3, we need a Bertini's theorem, in the following form.

Theorem 4.4.10 (Bertini's theorem). *Let $Y \subseteq \mathbb{P}_{k^{\text{alg}}}^d$ be a smooth connected projective variety over k^{alg} , D a divisor of Y , D' an open subscheme of D , Z a conic closed subset of $TY \times_Y D'$ which has codimension 1 at each fibre, $U \subseteq Y$ a dense open and $\tilde{U} \rightarrow U$ a connected finite étale cover. There exists a curve $C \subseteq Y$ over k^{alg} satisfying the following conditions.*

- (1) *C is a smooth scheme theoretic complete intersection of hyperplanes intersecting D transversally.*
- (2) *$C \times_U \tilde{U}$ is connected and non-empty.*
- (3) *C intersects each irreducible component D'_i of D' and the image of $TC \times_Y D'_i \rightarrow TY \times_Y D'_i$ is not entirely contained in $Z \times_{D'} D'_i$.*

Proof. By Bertini's theorem, in the form proven in [Jou83, Thm. 6.3], the three conditions correspond to dense opens of the dual of $\mathbb{P}_{k^{\text{alg}}}^d$. The result then follows from an induction on the dimension of Y . \square

4.4.11. *Proof of Theorem 4.4.3.* During the proof we will impose three conditions on $\overline{C} \subseteq Y$. These conditions will ensure that if $C := \overline{C} \setminus D$, the restriction functor $\langle \mathcal{M}^\dagger \rangle \rightarrow \langle \mathcal{M}^\dagger|_C \rangle$ is an equivalence. We first assume that $\overline{C} \subseteq Y$ is a smooth complete intersection of hyperplanes intersecting D transversally (Condition (1)). By Theorem 4.4.2, the restriction functor

$$\mathbf{Isoc}^\dagger(X)_{\text{doc}} \rightarrow \mathbf{Isoc}^\dagger(C)_{\text{doc}}$$

is fully faithful.

Let $U \subseteq X$ be a dense open where \mathcal{M} acquires the slope filtration and such that $D_S := Y \setminus U$ is a divisor (containing D). Write C_U for $C \cap U$ and \mathcal{N} for $\text{Gr}_{S_\bullet}(\mathcal{M}|_U)$. Choose a geometric point η of C_U and write V for the Dieudonné–Manin \mathbb{Q}_p^{ur} -structure of $(\mathcal{N}, \Phi_{\mathcal{N}})$ at η . Let \mathcal{V} be the lisse \mathbb{Q}_p^{ur} -sheaf associated to (\mathcal{N}, V) provided by Proposition 3.3.4 (which does not depend on η). By Lemma 4.4.5, there exists a connected finite étale cover $\tilde{U} \rightarrow U$, with the property that if $\tilde{U} \times_U C_U$ is connected and non-empty, then the functor $\langle \mathcal{N}, V \rangle \rightarrow \langle \mathcal{N}|_{C_U}, V \rangle$ is an equivalence of categories. Let us assume that C_U satisfies this condition (Condition (2)).

Since $\langle \mathcal{N}, V \rangle \rightarrow \langle \mathcal{N}|_{C_U}, V \rangle$ is an equivalence, it follows that $X^*(G(\mathcal{M}|_U, V)) = X^*(G(\mathcal{M}|_{C_U}, V))$. By Lemma 4.4.6, there exists a dense open $D'_S \subseteq D_S$ and a conic closed subscheme $Z \subseteq TX \times_X D'_S$ of codimension at least 1 at every fibre which satisfies the property $\mathbf{R}_{\mathcal{V}}(Z)$. Suppose that \overline{C} satisfies the assumptions in $\mathbf{R}_{\mathcal{V}}(Z)$ (Condition (3)), then $X^*(G(\mathcal{M}|_U, V))^{\text{ur}} = X^*(G(\mathcal{M}|_{C_U}, V))^{\text{ur}}$.

We know that $\text{MS}(X)$ is true for smooth curves thanks to Theorem 4.2.12. Thus, by Theorem 4.1.6 and Corollary 4.4.9,

$$X^*(G(\mathcal{M}^\dagger|_C, V)) = X^*(G(\mathcal{M}^\dagger|_{C_U}, V)) = X^*(G(\mathcal{M}|_{C_U}, V))^{\text{ur}}.$$

We showed that every rank 1 overconvergent isocrystal with \mathbb{Q}_p^{ur} -structure in $\langle \mathcal{M}^\dagger|_C, V \rangle$ is the restriction of a rank 1 overconvergent isocrystal with \mathbb{Q}_p^{ur} -structure over Y . This implies that the functor $\langle \mathcal{M}^\dagger, V \rangle \rightarrow \langle \mathcal{M}^\dagger|_C, V \rangle$ is observable. By Proposition 3.3.2, we deduce that the restriction functor $\langle \mathcal{M}^\dagger \rangle \rightarrow \langle \mathcal{M}^\dagger|_C \rangle$ is an equivalence. It remains to show that a curve \bar{C} with these three conditions exists, which is guaranteed by Theorem 4.4.10. \square

4.4.12. *Proof of Theorem 4.1.3.* By Theorem 4.2.12, when X is a smooth curve, $\text{MS}(X)$ is true. To prove it in higher dimension, arguing as in Lemma 4.1.8, we may assume $k = k^{\text{alg}}$. In addition, combining the reductions provided by Lemma 4.1.7 and Lemma 4.1.8 and Kedlaya's semi-stable reduction theorem [Ked11, Thm. 2.4.4], we may assume that X admits a compactification as in Theorem 4.4.3 and $(\mathcal{M}^\dagger, \Phi_{\mathcal{M}}^\dagger)$ is an overconvergent F -isocrystal which is docile along D . Applying Theorem 4.4.3, there exists a curve C in X such that the \dagger -extendable subobjects of $(\mathcal{M}, \Phi_{\mathcal{M}})$ are the same as the ones of $(\mathcal{M}|_C, \Phi_{\mathcal{M}}|_C)$. Since we know that $\text{MS}(\mathcal{M}|_C)$ is true, we deduce that $\text{MS}(\mathcal{M})$ is true as well. This ends the proof. \square

As a consequence of Theorem 4.1.3 we prove the parabolicity conjecture.

Theorem 4.4.13. *Let X be a smooth geometrically connected variety over a perfect field k endowed with the choice of a perfect point η . For every overconvergent F^n -isocrystal over X with constant slopes, the subgroup $G(\mathcal{M}, \eta) \subseteq G(\mathcal{M}^\dagger, \eta)$ is the subgroup of $G(\mathcal{M}^\dagger, \eta)$ stabilising the slope filtration of \mathcal{M}_η . Moreover, if \mathcal{M}^\dagger is semi-simple, $G(\mathcal{M}, \eta)$ is a parabolic subgroup of $G(\mathcal{M}^\dagger, \eta)$.*

Proof. By [Cre92a, (2.1.10)], we may assume $k = k^{\text{alg}}$ and that η is a geometric point. By Theorem 4.1.3 and Proposition 4.3.2, we deduce that $G(\mathcal{M}, \Phi_{\mathcal{M}}^\infty, \eta) \subseteq G(\mathcal{M}^\dagger, \Phi_{\mathcal{M}}^{\dagger, \infty}, \eta)$ is the stabiliser of the slope filtration of \mathcal{M}_η . In addition, combining Proposition 3.2.8 and Corollary 4.4.8, we have that $G(\mathcal{M}, \eta) = G(\mathcal{M}^\dagger, \eta) \cap G(\mathcal{M}, \Phi_{\mathcal{M}}^\infty, \eta)$. This proves the first part of the statement. If \mathcal{M}^\dagger is semi-simple, up to replacing $(\mathcal{M}^\dagger, \Phi_{\mathcal{M}}^\dagger)$ with its semi-simplification with respect to a Jordan–Hölder filtration, we may assume that $(\mathcal{M}^\dagger, \Phi_{\mathcal{M}}^\dagger)$ is semi-simple. By Proposition 4.3.2, the subgroup $G(\mathcal{M}, \Phi_{\mathcal{M}}^\infty, \eta) \subseteq G(\mathcal{M}^\dagger, \Phi_{\mathcal{M}}^{\dagger, \infty}, \eta)$ is then parabolic. Since $G(\mathcal{M}, \eta) \subseteq G(\mathcal{M}^\dagger, \eta)$ is a normal subgroup, this implies that $G(\mathcal{M}, \eta) = G(\mathcal{M}^\dagger, \eta) \cap G(\mathcal{M}, \Phi_{\mathcal{M}}^\infty, \eta)$ is a parabolic subgroup of $G(\mathcal{M}^\dagger, \eta)$. \square

5. APPLICATIONS

5.1. **Monodromy over finite fields.** Let X be a smooth connected variety over \mathbb{F}_{p^n} with a rational point x . In this case, the fibre functor $\omega_x : \mathbf{Isoc}(X) \rightarrow \mathbf{Vec}_{\mathbb{Q}_{p^n}}$ induces a \mathbb{Q}_{p^n} -linear fibre functor for the \mathbb{Q}_{p^n} -linear Tannakian category $\omega_x : \mathbf{F}^n\text{-Isoc}(X) \rightarrow \mathbf{Vec}_{\mathbb{Q}_{p^n}}$.

Definition 5.1.1. For $(\mathcal{M}, \Phi_{\mathcal{M}}) \in \mathbf{F}^n\text{-Isoc}(X)$ (resp. $(\mathcal{M}^\dagger, \Phi_{\mathcal{M}}^\dagger) \in \mathbf{F}^n\text{-Isoc}(X)$), we write $G(\mathcal{M}, \Phi_{\mathcal{M}}, x)$ (resp. $G(\mathcal{M}^\dagger, \Phi_{\mathcal{M}}^\dagger, x)$) for the algebraic group of automorphisms of the restriction of ω_x to $\langle \mathcal{M}, \Phi_{\mathcal{M}} \rangle$ (resp. $\langle \mathcal{M}^\dagger, \Phi_{\mathcal{M}}^\dagger \rangle$). The group $G(\mathcal{M}^\dagger, \Phi_{\mathcal{M}}^\dagger, x)$ coincides with the arithmetic monodromy group of $(\mathcal{M}^\dagger, \Phi_{\mathcal{M}}^\dagger)$ defined in [D'Ad20, Def. 3.2.4].

By [AD18, Prop. 2.2.4], for every overconvergent F^n -isocrystal $(\mathcal{M}^\dagger, \Phi_{\mathcal{M}}^\dagger)$ over X , we have, as in Proposition 3.2.8, the following commutative diagram of \mathbb{Q}_p^n -linear algebraic groups

$$(5.1.1) \quad \begin{array}{ccccccc} 1 & \longrightarrow & G(\mathcal{M}, x) & \longrightarrow & G(\mathcal{M}, \Phi_{\mathcal{M}}, x) & \longrightarrow & G(\mathcal{M}, \Phi_{\mathcal{M}}, x)^{\text{cst}} \longrightarrow 1 \\ & & \downarrow & & \downarrow & & \downarrow \\ 1 & \longrightarrow & G(\mathcal{M}^\dagger, x) & \longrightarrow & G(\mathcal{M}^\dagger, \Phi_{\mathcal{M}}^\dagger, x) & \longrightarrow & G(\mathcal{M}^\dagger, \Phi_{\mathcal{M}}^\dagger, x)^{\text{cst}} \longrightarrow 1 \end{array}$$

where the rows are exact. The groups on the right are the Tannaka groups of the category of constant objects in $\langle \mathcal{M}, \Phi_{\mathcal{M}} \rangle$ and $\langle \mathcal{M}^\dagger, \Phi_{\mathcal{M}}^\dagger \rangle$. We write G for $G(\mathcal{M}^\dagger, \Phi_{\mathcal{M}}^\dagger, x)$ and H for $G(\mathcal{M}, \Phi_{\mathcal{M}}, x)$. As in Definition 4.3.1, the slope filtration of \mathcal{M}_x defines a quasi-cocharacter $\lambda : \mathbb{G}_m^{1/\infty} \rightarrow G(\mathcal{M}^\dagger, \Phi_{\mathcal{M}}^\dagger, x)$. Write $P_G(\lambda) \subseteq G$ for the stabiliser of the slope filtration.

Theorem 5.1.2. *If $(\mathcal{M}, \Phi_{\mathcal{M}})$ has constant slopes, $H = P_G(\lambda)$. Moreover, if $(\mathcal{M}^\dagger, \Phi_{\mathcal{M}}^\dagger)$ is semi-simple, H is a parabolic subgroup of G .*

Proof. This follows from Theorem 4.1.3 by arguing as in Proposition 4.3.2. \square

Proposition 5.1.3. *For an overconvergent F^n -isocrystal $(\mathcal{M}^\dagger, \Phi_{\mathcal{M}}^\dagger)$ (possibly with non-constant slopes) $\pi_0(H) = \pi_0(G)$. In addition, the morphism $G(\mathcal{M}, \Phi_{\mathcal{M}}, x)^{\text{cst}} \rightarrow G(\mathcal{M}^\dagger, \Phi_{\mathcal{M}}^\dagger, x)^{\text{cst}}$ is an isomorphism.*

Proof. This follows from Theorem 5.1.2 by arguing as in Proposition 4.3.3 and Proposition 4.4.7. \square

Suppose that $(\mathcal{M}, \Phi_{\mathcal{M}})$ has constant slopes. If we write $(\mathcal{N}, \Phi_{\mathcal{N}})$ for $\text{Gr}_{S_\bullet}(\mathcal{M}, \Phi_{\mathcal{M}})$, there is a functor $\langle \mathcal{M}, \Phi_{\mathcal{M}} \rangle \rightarrow \langle \mathcal{N}, \Phi_{\mathcal{N}} \rangle$ sending $(\mathcal{M}', \Phi_{\mathcal{M}'})$ to $\text{Gr}_{S_\bullet}(\mathcal{M}', \Phi_{\mathcal{M}'})$. This induces the following commutative diagram with exact rows

$$(5.1.2) \quad \begin{array}{ccccccc} 1 & \longrightarrow & G(\mathcal{N}, x) & \longrightarrow & G(\mathcal{N}, \Phi_{\mathcal{N}}, x) & \longrightarrow & G(\mathcal{N}, \Phi_{\mathcal{N}}, x)^{\text{cst}} \longrightarrow 1 \\ & & \downarrow & & \downarrow & & \downarrow = \\ 1 & \longrightarrow & G(\mathcal{M}, x) & \longrightarrow & G(\mathcal{M}, \Phi_{\mathcal{M}}, x) & \longrightarrow & G(\mathcal{M}, \Phi_{\mathcal{M}}, x)^{\text{cst}} \longrightarrow 1. \end{array}$$

Even in this case, the natural morphism $G(\mathcal{N}, \Phi_{\mathcal{N}}, x)^{\text{cst}} \rightarrow G(\mathcal{M}, \Phi_{\mathcal{M}}, x)^{\text{cst}}$ is an isomorphism, since the inclusion $\langle \mathcal{N}, \Phi_{\mathcal{N}} \rangle \hookrightarrow \langle \mathcal{M}, \Phi_{\mathcal{M}} \rangle$ provides an inverse map.

Proposition 5.1.4. *The subgroup $G(\mathcal{N}, \Phi_{\mathcal{N}}, x) \subseteq P_G(\lambda)$ is equal to $Z_G(\lambda)$, the centraliser of the image of λ .*

Proof. The proof is similar to Proposition 4.3.2. By construction, the subgroup $G(\mathcal{N}, \Phi_{\mathcal{N}}, x)$ is in $Z_G(\lambda)$. On the other hand, by Chevalley's theorem, there exists an F^n -isocrystal $(\mathcal{M}', \Phi_{\mathcal{M}'}) \in \langle \mathcal{M}, \Phi_{\mathcal{M}} \rangle$ and a subobject $(\mathcal{L}, \Phi_{\mathcal{L}}) \subseteq \text{Gr}_{S_\bullet}(\mathcal{M}', \Phi_{\mathcal{M}'})$ of rank 1 such that $G(\mathcal{N}, \Phi_{\mathcal{N}}, x)$ is the stabiliser of $L := \omega_x(\mathcal{L}) \subseteq \omega_x(\mathcal{M}') =: V$. Since $(\mathcal{L}, \Phi_{\mathcal{L}})$ has rank 1, it is contained as F^n -isocrystal in the image of the quotient $\pi : S_i(\mathcal{M}') \rightarrow S_i(\mathcal{M}')/S_{i-1}(\mathcal{M}')$ for some i . Write $(\tilde{\mathcal{L}}, \Phi_{\tilde{\mathcal{L}}}) \subseteq (\mathcal{M}', \Phi_{\mathcal{M}'})$ for $\pi^{-1}(\mathcal{L}, \Phi_{\mathcal{L}})$ and $\tilde{L} \subseteq V$ for its fibre at x . If s is the slope of $(\mathcal{L}, \Phi_{\mathcal{L}})$ and $V^s \subseteq V$ is the subspace of slope s , then $L = \tilde{L} \cap V^s$. We deduce that $Z_G(\lambda)$ stabilises L , which gives the desired result. \square

Corollary 5.1.5. *The algebraic group $G(\mathcal{N}, \Phi_{\mathcal{N}}, x)^\circ$ contains a Cartan subgroup of $G(\mathcal{M}^\dagger, \Phi_{\mathcal{M}}^\dagger, x)^\circ$.*

Proof. If T is a maximal torus of $G(\mathcal{M}^\dagger, \Phi_{\mathcal{M}}^\dagger, x)^\circ$ containing the image of λ , the centraliser of T in $G(\mathcal{M}^\dagger, \Phi_{\mathcal{M}}^\dagger, x)$ is contained in $Z_G(\lambda) = G(\mathcal{N}, \Phi_{\mathcal{N}}, x)$. This yields the desired result. \square

Theorem 5.1.6. *Let X be a smooth variety over a finite field \mathbb{F}_{p^n} and $f : A \rightarrow X$ an abelian scheme with constant slopes. If $(\mathcal{M}, \Phi_{\mathcal{M}})$ is the F -isocrystal $R^1 f_{\text{crys}*} \mathcal{O}_{A, \text{crys}}$, the induced F -isocrystal $(\mathcal{N}, \Phi_{\mathcal{N}}) := \text{Gr}_{S_\bullet}(\mathcal{M}, \Phi_{\mathcal{M}})$ is semi-simple. In particular, $R^1 f_{\text{ét}*} \underline{\mathbb{Q}}_p$ is a semi-simple lisse \mathbb{Q}_p -sheaf over X .*

Proof. By étale descent we may assume that X is connected and admits a rational point x . We may also replace $\Phi_{\mathcal{M}}$ with its n -th power. By [Ete02, Thm. 7], the F^n -isocrystal $(\mathcal{M}, \Phi_{\mathcal{M}})$ is \dagger -extendable and, by [D'Ad20, Cor. 3.5.2.(ii)], the monodromy group $G(\mathcal{M}^\dagger, x)$ is a reductive group (note that $(\mathcal{M}^\dagger, \Phi_{\mathcal{M}}^\dagger)$ is pure by the Riemann Hypothesis for abelian varieties). On the other hand, since the action of the p^n -th power Frobenius on the crystalline cohomology groups of A_x is semi-simple (this Frobenius is in the centre of $\text{End}(A_x)$), we get that $G(\mathcal{M}^\dagger, \Phi_{\mathcal{M}}^\dagger, x)^{\text{cst}}$ is a reductive group. We deduce by (5.1.1) that $G(\mathcal{M}^\dagger, \Phi_{\mathcal{M}}^\dagger, x)$ is reductive (see also [Pál15, Thm. 1.2] for a different proof over curves). By Proposition 5.1.4, the group $G(\mathcal{N}, \Phi_{\mathcal{N}}, x)$ is the centraliser of λ in $G(\mathcal{M}, \Phi_{\mathcal{M}}, x)$, thus by [Bor91, Cor. 11.12] it is reductive. This shows that $(\mathcal{N}, \Phi_{\mathcal{N}})$ is semi-simple. To prove that $R^1 f_{\text{ét}*} \underline{\mathbb{Q}}_p$ is semi-simple it is enough to observe that the unit-root F -isocrystal associated to this lisse sheaf corresponds, thanks to [BBM82, Thm. 2.5.6.(ii)], to the first step of the slope filtration of $(\mathcal{M}, \Phi_{\mathcal{M}})$. This concludes the proof. \square

5.2. Separable points of abelian varieties. Let E/\mathbb{F}_p be a finitely generated field extension and let A be an abelian variety over E . We use next consequence of Theorem 4.1.3 to prove a result on torsion E^{sep} -points of A .

Corollary 5.2.1. *If $(\mathcal{M}, \Phi_{\mathcal{M}})$ is a \dagger -extendable F^n -isocrystal over a smooth variety X over k , every isoclinic subobject of maximal slope is \dagger -extendable.*

Proof. After shrinking X we may assume that the slopes of $(\mathcal{M}, \Phi_{\mathcal{M}})$ are constant. By Theorem 4.1.3, a subobject $(\mathcal{N}, \Phi_{\mathcal{N}}) \subseteq (\mathcal{M}, \Phi_{\mathcal{M}})$ of maximal slope is equal to its \dagger -hull. This yields the desired result. \square

Theorem 5.2.2. *Let A be an abelian variety over E . The group $A(E^{\text{sep}})[p^\infty]$ is finite in the following two cases.*

- (i) *If $\text{End}(A) \otimes_{\mathbb{Z}} \mathbb{Q}_p$ is a division algebra.*
- (ii) *If $\text{End}(A) \otimes_{\mathbb{Z}} \mathbb{Q}$ has no factor of Albert-type IV.*

Proof. Suppose that the group $A(E^{\text{sep}})[p^\infty]$ is infinite, then the Barsotti–Tate group $A_{E^{\text{sep}}}[p^\infty]/E^{\text{sep}}$ admits $\mathbb{Q}_p/\mathbb{Z}_p$ as a subgroup. Let $\tilde{\mathcal{H}}$ be the maximal constant Barsotti–Tate subgroup of $A_{E^{\text{sep}}}[p^\infty]$. By Galois descent, $\tilde{\mathcal{H}}$ descends to some étale Barsotti–Tate group \mathcal{H}/E . Let X be a smooth connected variety over \mathbb{F}_p with function field E . After shrinking X , we may assume that A/E admits a smooth model $\mathcal{A} \rightarrow X$ with constant Newton polygon. Let $(\mathcal{M}, \Phi_{\mathcal{M}})$ be the crystalline Dieudonné module of $\mathcal{A}[p^\infty]$ and $(\mathcal{N}, \Phi_{\mathcal{N}})$ the crystalline Dieudonné module of the model of \mathcal{H} in $\mathcal{A}[p^\infty]$. The F -isocrystal $(\mathcal{N}, \Phi_{\mathcal{N}})$ is a quotient of $(\mathcal{M}, \Phi_{\mathcal{M}})$. As \mathcal{H} is étale, the F -isocrystal $(\mathcal{N}, \Phi_{\mathcal{N}})$ is unit-root. Since $(\mathcal{M}, \Phi_{\mathcal{M}})$ is \dagger -extendable, by Corollary 5.2.1 the isocrystal $(\mathcal{N}, \Phi_{\mathcal{N}})$ is \dagger -extendable as well.

Let us first prove that $\text{End}(A) \otimes_{\mathbb{Z}} \mathbb{Q}_p$ contains a non-trivial idempotent. Thanks to the theory of weights for overconvergent F -isocrystals, the F -isocrystal $(\mathcal{M}, \Phi_{\mathcal{M}})$ is semi-simple in the category of overconvergent F -isocrystal (see the proof of Theorem 5.1.6). Thus $(\mathcal{N}, \Phi_{\mathcal{N}})$ is actually a proper direct summand of $(\mathcal{M}, \Phi_{\mathcal{M}})$. This implies that $\text{End}(\mathcal{M}, \Phi_{\mathcal{M}})$ contains some non-trivial idempotent. Thanks to [deJ98, Thm. 2.6], we deduce that the same is true for $\text{End}(A) \otimes_{\mathbb{Z}} \mathbb{Q}_p$.

We prove now that if we further assume A geometrically simple, then the centre of the neutral component $G(\mathcal{M}^{\dagger}, \Phi_{\mathcal{M}}^{\dagger})^{\circ}$ is of dimension at least 2. This shows that $\text{End}(A) \otimes_{\mathbb{Z}} \mathbb{Q}$ is of Albert-type IV thanks to [CT20, Lem. 10.5]. By [D'Ad20, Prop. 3.3.4], up to replacing E with a finite field extension (and X with some finite étale cover) we may assume that $G(\mathcal{M}^{\dagger}, \Phi_{\mathcal{M}}^{\dagger})$ is connected. By [D'Ad20, Cor. 3.4.8], the rank of the group of *twist classes* $\mathbb{X}(\mathcal{M}^{\dagger}) \subseteq \overline{\mathbb{Q}}_p^{\times} / \mu_{\infty}(\overline{\mathbb{Q}}_p)$ (cf. Def. 3.4.3 in [ibid.]) is equal to the dimension of the centre of $G(\mathcal{M}^{\dagger}, \Phi_{\mathcal{M}}^{\dagger})$. In addition, the determinant of $(\mathcal{M}^{\dagger}, \Phi_{\mathcal{M}}^{\dagger})$ has twist class $[p^g]$, where g is the dimension of A . Write $[a]$ for a twist class of $(\mathcal{N}^{\dagger}, \Phi_{\mathcal{N}}^{\dagger})$. If we prove that $[a]$ is not a \mathbb{Q} -multiple of $[p^g]$, we deduce that $G(\mathcal{M}^{\dagger}, \Phi_{\mathcal{M}}^{\dagger})^{\circ}$ is of dimension at least 2, as we wanted. Suppose by contradiction that $\frac{a^n}{p^{mg}}$ is a root of unity for some $(m, n) \in \mathbb{Z} \times \mathbb{Z}_{>0}$. Since a is a p -adic unit, we have that $m = 0$. This implies that a is a root of unity itself. On the other hand, by the Riemann Hypothesis, a is a Weil number of weight 1. This leads to a contradiction. \square

Remark 5.2.3. In the proof of Theorem 5.2.2 one could deduce that the unit-root F -isocrystal $(\mathcal{N}, \Phi_{\mathcal{N}})$ is \dagger -extendable by replacing Corollary 5.2.1 with [Rös17, Thm. 1.2.(a)]. It is also worth mentioning that Helm [Hel20] recently found an example of an ordinary abelian variety over a certain finitely generated field E with no isotrivial factors and such that $A(E^{\text{sep}})[p^{\infty}]$ is infinite.

5.3. Kedlaya's conjecture. Thanks to Theorem 4.1.3, we are also able to prove a conjecture proposed by Kedlaya.

Corollary 5.3.1 (Kedlaya's conjecture). *Let X be a smooth connected variety over a perfect field k and let $(\mathcal{M}_1^{\dagger}, \Phi_{\mathcal{M}_1}^{\dagger})$ and $(\mathcal{M}_2^{\dagger}, \Phi_{\mathcal{M}_2}^{\dagger})$ be two irreducible overconvergent F^n -isocrystals over X with constant slopes. If there exists an isomorphism $\epsilon : (S_1(\mathcal{M}_1), \Phi_{\mathcal{M}_1}|_{S_1(\mathcal{M}_1)}) \xrightarrow{\sim} (S_1(\mathcal{M}_2), \Phi_{\mathcal{M}_2}|_{S_1(\mathcal{M}_2)})$, then $(\mathcal{M}_1^{\dagger}, \Phi_{\mathcal{M}_1}^{\dagger}) \simeq (\mathcal{M}_2^{\dagger}, \Phi_{\mathcal{M}_2}^{\dagger})$.*

Proof. Write $(\mathcal{N}, \Phi_{\mathcal{N}})$ for $(S_1(\mathcal{M}_1), \Phi_{\mathcal{M}_1}|_{S_1(\mathcal{M}_1)})$ and $\iota : (\mathcal{N}, \Phi_{\mathcal{N}}) \hookrightarrow (\mathcal{M}_1, \Phi_{\mathcal{M}_1})$ for the tautological inclusion. If $(\mathcal{M}, \Phi_{\mathcal{M}})$ is the direct sum $(\mathcal{M}_1, \Phi_{\mathcal{M}_1}) \oplus (\mathcal{M}_2, \Phi_{\mathcal{M}_2})$, we have an inclusion $(\iota, \epsilon) : (\mathcal{N}, \Phi_{\mathcal{N}}) \hookrightarrow (\mathcal{M}, \Phi_{\mathcal{M}})$. By Theorem 4.1.3, the isocrystal $\overline{\mathcal{N}}$ is a proper subobject of \mathcal{M} since $S_1(\overline{\mathcal{N}}) = \mathcal{N} \neq \mathcal{N} \oplus \mathcal{N} = S_1(\mathcal{M})$. In turn, by the assumption, this implies that $(\overline{\mathcal{N}}^{\dagger}, \Phi_{\overline{\mathcal{N}}}^{\dagger})$ is irreducible. On the other hand, by construction, $(\overline{\mathcal{N}}^{\dagger}, \Phi_{\overline{\mathcal{N}}}^{\dagger})$ admits non-zero morphisms to $(\mathcal{M}_1^{\dagger}, \Phi_{\mathcal{M}_1}^{\dagger})$ and $(\mathcal{M}_2^{\dagger}, \Phi_{\mathcal{M}_2}^{\dagger})$. Therefore we get

$$(\mathcal{M}_1^{\dagger}, \Phi_{\mathcal{M}_1}^{\dagger}) \simeq (\overline{\mathcal{N}}^{\dagger}, \Phi_{\overline{\mathcal{N}}}^{\dagger}) \simeq (\mathcal{M}_2^{\dagger}, \Phi_{\mathcal{M}_2}^{\dagger}),$$

as we wanted. \square

In addition, we can prove next related result.

Corollary 5.3.2. *Let X be a smooth connected variety over a perfect field k . If $(\mathcal{M}^{\dagger}, \Phi_{\mathcal{M}}^{\dagger})$ is an irreducible overconvergent F^n -isocrystal with constant slopes, then $(S_1(\mathcal{M}), \Phi_{\mathcal{M}}|_{S_1(\mathcal{M})})$ is irreducible.*

Proof. Let $(\mathcal{N}, \Phi_{\mathcal{N}}) \subseteq (S_1(\mathcal{M}), \Phi_{\mathcal{M}}|_{S_1(\mathcal{M})})$ be an irreducible sub- F^n -isocrystal. Since $(\mathcal{M}^\dagger, \Phi_{\mathcal{M}}^\dagger)$ is irreducible, the \dagger -hull of $(\mathcal{N}, \Phi_{\mathcal{N}})$ is equal to $(\mathcal{M}^\dagger, \Phi_{\mathcal{M}}^\dagger)$. By Theorem 4.1.3 we deduce that $\mathcal{N} = S_1(\overline{\mathcal{N}}) = S_1(\mathcal{M})$. This yields the desired result. \square

Combining Kedlaya's conjecture and Abe's Langlands correspondence we deduce the following p -adic refinement of the strong multiplicity one theorem for cuspidal automorphic representations.

Theorem 5.3.3. *Let X be a smooth connected curve over a finite field, let \mathbb{A} be its adèle ring and let r be a positive integer. The isomorphism class of a $\overline{\mathbb{Q}}_p$ -linear cuspidal automorphic representation π of $\mathrm{GL}_r(\mathbb{A})$ is determined by the datum of the Hecke eigenvalues of minimal slope at all but finitely many closed points of X .*

Proof. By [Abe18], for any such π there exists an irreducible $\overline{\mathbb{Q}}_p$ -linear overconvergent F -isocrystal $(\mathcal{M}^\dagger, \Phi_{\mathcal{M}}^\dagger)$ defined over a certain dense open of X which corresponds to π in the sense of Langlands. After shrinking X we may assume that $(\mathcal{M}^\dagger, \Phi_{\mathcal{M}}^\dagger)$ has constant slopes. By Corollary 5.3.1, the F -isocrystal $(S_1(\mathcal{M}), \Phi_{\mathcal{M}}|_{S_1(\mathcal{M})})$ determines the isomorphism class of \mathcal{M} . On the other hand, by [Cre87, Thm. 2.1], the isoclinic F -isocrystal $S_1(\mathcal{M})$ is induced by a $\overline{\mathbb{Q}}_p$ -linear continuous representation ρ of the Weil group of X . Moreover, by Corollary 5.3.2, ρ is irreducible. Thanks to Chebotarev's density theorem for the étale fundamental group of X , since ρ is semi-simple, its isomorphism class is determined by the Frobenius eigenvalues at all but finitely many points. By construction, these eigenvalues are the same as the Frobenius eigenvalues of $(\mathcal{M}^\dagger, \Phi_{\mathcal{M}}^\dagger)$ of minimal slope. Since π and $(\mathcal{M}^\dagger, \Phi_{\mathcal{M}}^\dagger)$ correspond in the sense of Langlands, we obtain the desired result. \square

5.4. PBS filtration and \dagger -compactifications. In this last section we extend [Tsu19, Thm. 3.27] to general smooth varieties and we introduce the notion of \dagger -compactification of an F^n -isocrystals, which is used to state Corollary 5.4.4, a stronger form of Corollary 5.3.1.

Definition 5.4.1. We say that a \dagger -extendable F^n -isocrystal over X is *pure in bounded subobjects* or simply PBS if for every connected open $U \subseteq X$, the isoclinic subobjects of the restriction to U have minimal generic slope⁶.

Corollary 5.4.2. *Let \mathcal{M} be a \dagger -extendable F^n -isocrystal over a smooth connected variety over a perfect field k . There exists a unique filtration $0 = P_0(\mathcal{M}) \subsetneq P_1(\mathcal{M}) \subsetneq \cdots \subsetneq P_r(\mathcal{M}) = \mathcal{M}$ of \dagger -extendable F^n -isocrystals such that each quotient $P_i(\mathcal{M})/P_{i-1}(\mathcal{M})$ is PBS with minimal generic slope $t_i \in \mathbb{Q}$ and $t_1 > t_2 > \cdots > t_r$.*

Proof. After shrinking the variety in order to have constant slopes, we construct $P_{r-1}(\mathcal{M})$ as the \dagger -hull of the sum of all the isoclinic subobjects of \mathcal{M} which do not have minimal generic slope. By Theorem 4.1.3, we deduce that $S_1(\mathcal{M}) \cap P_{r-1}(\mathcal{M}) = 0$. Therefore, if $\mathcal{M} \neq 0$ then the quotient $\mathcal{M}/P_{r-1}(\mathcal{M})$ is not 0. On the other hand, $\mathcal{M}/P_{r-1}(\mathcal{M}) \neq 0$ is PBS by construction. The result then follows by an induction on the rank of \mathcal{M} . \square

Definition 5.4.3. If $(\mathcal{N}, \Phi_{\mathcal{N}})$ is an F^n -isocrystal, we say that a \dagger -extendable F^n -isocrystal $(\mathcal{M}, \Phi_{\mathcal{M}})$ endowed with an inclusion $\iota_{\mathcal{M}} : (\mathcal{N}, \Phi_{\mathcal{N}}) \hookrightarrow (\mathcal{M}, \Phi_{\mathcal{M}})$ such that $\overline{\mathcal{N}} = \mathcal{M}$ is a \dagger -compactification of $(\mathcal{N}, \Phi_{\mathcal{N}})$. A morphism $\psi : (\mathcal{M}_1, \Phi_{\mathcal{M}_1}) \rightarrow (\mathcal{M}_2, \Phi_{\mathcal{M}_2})$ of \dagger -compactifications is a morphism of

⁶This is the dual of Tsuzuki's notion of PBQ overconvergent F^n -isocrystals

F^n -isocrystals such that $\psi \circ \iota_{\mathcal{M}_1} = \iota_{\mathcal{M}_2}$. We say that a weakly final object of the category of \dagger -compactifications of \mathcal{N} is a *minimal \dagger -compactification* of \mathcal{N} .

Corollary 5.4.4. *If $(\mathcal{N}, \Phi_{\mathcal{N}})$ is isoclinic and it can be embedded into a \dagger -extendable F^n -isocrystal, then it admits a minimal \dagger -compactification.*

Proof. Since the category of F^n -isocrystals is noetherian, it is enough to prove that for every pair $(\mathcal{M}_1, \Phi_{\mathcal{M}_1}), (\mathcal{M}_2, \Phi_{\mathcal{M}_2})$ of \dagger -compactifications of $(\mathcal{N}, \Phi_{\mathcal{N}})$, we can find isomorphic \dagger -compactifications $(\mathcal{M}'_1, \Phi_{\mathcal{M}'_1})$ and $(\mathcal{M}'_2, \Phi_{\mathcal{M}'_2})$ with (surjective) morphisms $(\mathcal{M}_i, \Phi_{\mathcal{M}_i}) \rightarrow (\mathcal{M}'_i, \Phi_{\mathcal{M}'_i})$. Write $(\mathcal{M}, \Phi_{\mathcal{M}})$ for $(\mathcal{M}_1, \Phi_{\mathcal{M}_1}) \oplus (\mathcal{M}_2, \Phi_{\mathcal{M}_2})$ and endow $(\mathcal{N}, \Phi_{\mathcal{N}})$ with the “diagonal” inclusion in $(\mathcal{M}, \Phi_{\mathcal{M}})$ induced by $\iota_{\mathcal{M}_1}$ and $\iota_{\mathcal{M}_2}$. Write $(\overline{\mathcal{N}}, \Phi_{\overline{\mathcal{N}}})$ for the \dagger -hull of $(\mathcal{N}, \Phi_{\mathcal{N}})$ in $(\mathcal{M}, \Phi_{\mathcal{M}})$. We consider the isocrystals $\mathcal{M}'_i := \mathcal{M}_i / (\mathcal{M}_i \cap \overline{\mathcal{N}})$ endowed with their natural F^n -structures $\Phi_{\mathcal{M}'_i}$ induced by $\Phi_{\mathcal{M}_i}$.

We claim that the induced morphisms $\iota_{\mathcal{M}'_i} : \mathcal{N} \rightarrow \mathcal{M}'_i$ are injective. To prove this, we may shrink X in order to acquire the slope filtration. Since $(\mathcal{N}, \Phi_{\mathcal{N}})$ is isoclinic, by Theorem 4.1.3 the subobject $\overline{\mathcal{N}} \cap \iota_{\mathcal{M}_i}(\mathcal{N}) \subseteq \overline{\mathcal{N}}$ lies in $S_1(\overline{\mathcal{N}}) = \mathcal{N}$. This implies that $\mathcal{M}_i \cap \overline{\mathcal{N}} \cap \iota_{\mathcal{M}_i}(\mathcal{N}) = \mathcal{M}_i \cap \mathcal{N} = 0$, as we wanted. To end the proof we notice that

$$\mathcal{M}_1 / (\mathcal{M}_1 \cap \overline{\mathcal{N}}) \simeq \mathcal{M} / \overline{\mathcal{N}} \simeq \mathcal{M}_2 / (\mathcal{M}_2 \cap \overline{\mathcal{N}}),$$

so that $(\mathcal{M}'_1, \Phi_{\mathcal{M}'_1})$ and $(\mathcal{M}'_2, \Phi_{\mathcal{M}'_2})$, endowed with the morphisms $\iota_{\mathcal{M}'_1}$ and $\iota_{\mathcal{M}'_2}$, satisfy the desired properties. □

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