Max-Planck-Institut für Mathematik
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# New approaches to constructing $p$-adic $L$-functions on classical groups, algebraic differential operators, and BGG

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# New approaches to constructing $p$-adic $L$-functions on classical groups, algebraic differential operators, and BGG

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**Abstract**

Constructing $p$-adic $L$-functions on classical groups relates geometry and arithmetic due to Iwasawa, Coates, Mazur-Manin using Bernoulli-Kummer congruences, modular symbols and Fourier coefficies. Recent approaches due to Eischen-Harris-Li-Skinner use algebraic differential operators on automorphic forms $\boldsymbol{\varphi}$ on unitary groups $U(n,n)$ over CM-fields. Alternatively, BGG method and geometric representation theory can be used with Fourirer coefficients.

*Contents*



## 0.1 $p$-adic $L$-function of $\mathcal{D}(s,\mathbf{f})$

The $p$-adic $L$-function attached to a normalized complex Euler product $\mathcal{D}(s,\mathbf{f})$ are often defined on a $p$-adic weight space $X = X(\mathbb{Z}_p)$ which is $X = Hom_{cont}(\mathbb{Z}_p^*, \mathbb{C}_p^*)$

in the simplest case, and $s \in X$ corresponds to the character $\kappa_s : 1+p \mapsto (1+p)^s$. This function presumably satisfies the conjecture of Coates-Perrin-Riou [CoPe] and it is constructed via *admissible measures of Amice-Vélu*, see also [MTT]. The $p$-ordinary case was treated in [EHLS] via algebraic geometry (method of Katz).

The main application is stated in terms of
the Hodge polygon $P_H(t) : [0, d] \to \mathbb{R}$ and
the Newton polygon $P_N(t) = P_{N,p}(t) : [0, d] \to \mathbb{R}$ of the zeta function $\mathcal{D}(s, \mathbf{f})$ of degree $d = 4n$. Main theorem gives a $p$-adic analytic interpolation of the $L$ values in the form of certain integrals.

For a classical group $G$ over $\mathbb{Q}$ (or over a number field $K$) its $p$-adic weight space is given by $p$-adic characters of the torus $X = Hom_{cont}(T(\mathbb{Z}_p), \mathbb{C}_p)^*$.

## 0.2 Algebraic differential operators in the simplest case of modular forms for $\Gamma = \mathrm{SL}_2(\mathbb{Z})$

*Action of the derivative* $D = \dfrac{1}{2\pi i}\dfrac{d}{dz} = q\dfrac{d}{dq}$(where $q = e^{2\pi i z}$) on a modular form $g = \sum_{n=0}^{\infty} b_n q^n$ is not a modular form, but it is quasi-modular ([DZ], p.59, [MaRo], p.67): the function $f = D^r g = \sum_{n=0}^{\infty} n^r b_n q^n$ satisfies the following transformation law:

$$(cz+d)^{-\ell-2r} D^r g(\gamma z) = \sum_{t=0}^{r} \binom{r}{t} \frac{\Gamma(r+\ell)}{\Gamma(t+\ell)} \left(\frac{1}{2\pi i}\frac{c}{cz+d}\right)^{r-t} D^t g(z)$$

for a modular form $g \in \mathcal{M}_\ell(\Gamma)$ of weight $\ell$, $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma$.
*In order to adjust it to the weight* $\ell + 2r$, let us use $S = \frac{1}{4\pi y}$, $\operatorname{Im} z = \frac{z-\bar{z}}{2i}$, and $\frac{1}{\operatorname{Im}\gamma z} = \frac{|cz+d|^2}{\operatorname{Im} z} = (cz+d)\left(-2ic + \frac{cz+d}{y}\right)$ :

## 0.3 Maass-Shimura differential operator

If $f = D^r g$ where $g \in \mathcal{M}_\ell(\Gamma)$ is a modular form of weight $\ell$, then the transformation law produces also the Maass-Shimura differential operator $\delta_\ell$ to the space of *nearly holomorphic forms* of weight $\ell + 2r$:

$$\delta_\ell^r g(z) = \sum_{t=0}^{r} \binom{r}{t} \frac{\Gamma(r+\ell)}{\Gamma(r-t+\ell)} (-S)^t D^{r-t} g(z), \text{ where } S = \frac{1}{4\pi y},$$

which preserves the rationality of the coefficients of $S$ and $q$. It comes again from the above transformation law of $D^r g$. Notice:
$\delta_\ell(g) = \frac{1}{2\pi i} y^{-\ell} \frac{\partial}{\partial z}(y^\ell g) = \frac{1}{2\pi i}\left(\frac{\partial g(z)}{\partial z} + \frac{\ell}{2iy} g(z)\right) = (D - \ell S)(g)$, which is of

weight $\ell+2$ and its *degree of near holomorphy* (in the variable $S$) is increased by one.
For an integer $r \geq 0$, $\delta_\ell^r := \delta_{\ell+2r-2} \circ \cdots \circ \delta_\ell$ (see also [U14]).

A conceptual explanation of the *algebraicity* comes from the Gauss-Manin connection (due to Grothendieck in higher dimensions see [Gr66], [KaOd68]).

## 0.4 Algebraic differential operators

on automorphic forms on unitary groups. Fix a $\mathcal{O}_{\mathcal{K}}$-algebra $\mathcal{R}$ with inclusion $\iota : \mathcal{R} \to \mathbb{C}$ and a weight representation $\rho = (\rho^+, \rho^-)$ of the maximal compact subgroup $K = U(n) \times U(n)$ of $U = U(n,n)$. Following §8 and 9 of [EE], write an automorphic form in $\mathcal{M}_\rho(\mathcal{R})$ with values in an $\mathcal{R}$-module $V = V^\rho(\mathcal{R}^d)$ on the hermitian space $\mathcal{H}_n = U/K$ as a formal $q$-expansion $f(q) = \sum_{\boldsymbol{\beta} \in H_{\geq 0}} c_{\boldsymbol{\beta}}(\Xi) q^{\boldsymbol{\beta}}$ with vector-valued polynomial coefficients $c_{\boldsymbol{\beta}}(\Xi) \in V^\rho$ of $q^{\boldsymbol{\beta}} = \exp(2\pi i \mathrm{tr}(\boldsymbol{\beta} z))$, $z \in \mathcal{H}_n$, where $\Xi(z) = (i(\bar{z} - {}^t z), i(z^* - z)) = (\xi, \eta)$ (Shimura's notation), $T = \mathbb{C}_n^n$, and $\{e_\nu\}$ a $\mathbb{R}$-rational basis of $T$ over $\mathbb{C}$, $H_{\geq 0}$ is a lattice of hermitian semi-integral non-negative matrices.

Then a general algebraic operator $\theta(f)$ is defined as above via $\theta(\zeta)(f)$, using $\boldsymbol{\beta}$ and $\Xi$ as formal variables over a cusp: $\theta(\zeta)(f)(q) = \sum_{\beta \in H_{\geq 0}} \zeta(\boldsymbol{\beta}) c_{\boldsymbol{\beta}}(\Xi) q^{\boldsymbol{\beta}}$

This construction allows to treat *vector-valued modular forms as polynomial-valued*, and to prove congruences between them monomial-by-monomial.

Presumably, such operators could be obtained from Verma modules via BGG (to develop), see [BGG75], [Roc80]

## 0.5 Idea of the project: comes from the Zoom Conference Bernstein-75

According to Joseph,
*"This has been a nice and interesting conference. I still prefer a personal contact, so let us hope that soon enough we will be able to travel again".*

Here is my photo taken from France:
Bernstein 75, May 13, 2020, The Weizmann Institute, Rehovot,
transmitted to Joseph thanks to The organizers:
Dmitry Gourevitch and
Andre Reznikov


Figure 1: Bernstein 75, May 13, 2020, The Weizmann Institute, Rehovot

## 0.6 Arithmetical applications of The BGG resolution

In recent years, striking arithmetical applications were found of The BGG resolution, [BGG75], to the $p$-adic variation of automorphic representations, see [AIP15], [U11], [Jon11]. It suggests to look at the $p$-adic variation of the attached auromorphic $L$-functions, via agebraic differential operators acting on holomorphic automorphic forms $\boldsymbol{\varphi}$ on unitary groups $U(n,n)$ over an imaginary quadratic field $\mathcal{K} = \mathbb{Q}(\sqrt{-D_{\mathcal{K}}}) \subset \mathbb{C}$, with possible applications to the special $L$-values $L(s,\boldsymbol{\varphi})$ attached to $\boldsymbol{\varphi}$.

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The attached $p$-adic $L$-function of $\mathcal{D}(s,\mathbf{f})$ satisfies conjecture of Coates-Perrin-Riou [CoPe] and it is constructed via *admissible measures of Amice-Vélu*, see also [MTT]. The $p$-ordinary case was treated in [EHLS] via algebraic geometry (method of Katz).

The main application is stated in terms of the Hodge polygon $P_H(t)$:
$[0,d] \to \mathbb{R}$ and the Newton polygon $P_N(t) = P_{N,p}(t)$:
$[0,d] \to \mathbb{R}$ of the zeta function $\mathcal{D}(s,\mathbf{f})$ of degree $d = 4n$. Main theorem gives a $p$-adic analytic interpolation of the $L$ values in the form of certain integrals.

## 0.7 Outline of the construction of $p$-adic $L$-functions

**Outline of the construction of $p$-adic $L$-functions attached to the families in [AIP15]**

This article we quote throughout the paper.

A basic tool is the use of families of modular distributions on the $p$-adic weight space$\mathcal{W}$, the rigid analytic space over $\mathbb{Q}_p$ associated to the noetherian, complete algebra $\mathbb{Z}_p[\![T(\mathbb{Z}_p)]\!]$, and coming from [AIP15].

This passage imitates the method in [PaTV].

Let $\mathcal{W}$ be the rigid analytic space over $\mathbb{Q}_p$ associated to the noetherian, complete algebra $\mathbb{Z}_p[\![T(\mathbb{Z}_p)]\!]$ described in [AIP15] by

$$\mathcal{W}(\mathbb{C}_p) = \mathrm{Hom}_{cont}(T(\mathbb{Z}_p), \mathbb{C}_p^\times)$$

*Advantages of using matrix coefficients $(v)_{k,l}$:* Introducing more coordinates allows to avoid complicated formulas when dealing with integral/algebraic structures and congruences. Also, computing values of *linear and multilinear forms* could be better organised in such cases as 1) Rankin-Selberg integral of $f$ and $g$ of type $\mathrm{GL}_n \times \mathrm{GL}_m$ or 2) Integrations in the Doubling Method (integrating of $f(z)$ over $z$ against a restriction to the "diagonal" (or block diagonal) $(z, w)$ of a multi-dimensional kernel $\mathcal{E}(z, w)$).

## 0.8 Outline of the construction of $p$-adic $L$-functions

**Decisive stages of the construction**

The techniques in [AIP15] allow to carry out such computations in a relative form, that is for the relative linear and multilinear forms .

1. Applying such relative (multi-) linear forms to the above relative modular distributions

2. Under a non-vanishing condition, one can project the spaces of modular forms in question to a finite-rank relative submodule, using a lowering level operator (Atkin's Hecke operator), as in [PaTV]

3. Comparison of the specialized integrals of the relative-valued distributions to the algebraic part of the scalar-valued ($\mathbb{C}_p$-valued) distributions.
   At this stage, the *classicity criterion* in §5 of [AIP15] plays a crucial role.

# Contents

# 1 Algebraic representations, see [AIP15], §2.

**Recall: algebraic representations, according [AIP15]**

Let $\mathrm{GL}_g$ be the linear algebraic group of dimension $g$ realized as the group of $g \times g$ invertible matrices. Let $B$ be the Borel subgroup of upper triangular matrices, $T$ the maximal torus of diagonal matrices, and $U$ the unipotent radical of $B$. Let $B^0$ and $U^0$ be the opposite Borel of lower triangular matrices and its unipotent radical. Denote by $X(T)$ the group of characters of $T$ and by $X^+(T)$

its cone of dominant weights with respect to $B$.Identify $X(T)$ with $\mathbb{Z}^g$ via the map that associates to a $g$-uple $(k_1,\dots,k_g)\in\mathbb{Z}^g$ the character

$$\begin{pmatrix} t_1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & t_g \end{pmatrix} \mapsto t_1^{k_1}\dots t_g^{k_g}$$

With this identification, $X^+(T)$ is the cone of elements $(k_1,\dots,k_g)\in\mathbb{Z}^g$ such that $k_1\geq k_2\geq\cdots\geq k_g$. For any $\kappa\in X^+(T)$, we set $V_\kappa=$

$$\{f:\mathrm{GL}_g\to\mathbb{A}^1 \text{ morphism of schemes s.t. } f(gb)=\kappa(b)f(g),\forall(g,b)\in GL_g\times B\}$$

This is a finite-dimensional $\mathbb{Q}_p$-vector space. The group $\mathrm{GL}_g$ acts on $V_\kappa$ by the formula

$$g\cdot f(x)=f(g^{-1}x) \text{ for any } (g,f)\in\mathrm{GL}_g\times V_\kappa.$$

If $L$ is an extension of $\mathbb{Q}_p$, we set $V_{\kappa,L}=V_\kappa\otimes_{\mathbb{Q}_p}L$.

## 1.1 The weight space.

Let $\mathcal{W}$ be the rigid analytic space over $\mathbb{Q}_p$ associated to the noetherian, complete algebra $\mathbb{Z}_p[\![T(\mathbb{Z}_p)]\!]$. $T$ is the split torus of diagonal matrices in $\mathrm{GL}_g$. Let us fix an isomorphism $T\overset{\sim}{\to}\mathbb{G}_m^g$. One obtains an isomorphism $T(\mathbb{Z}_p)\overset{\sim}{\to}T(\mathbb{Z}/p\mathbb{Z})\times(1+p\mathbb{Z}_p)^g$, which implies natural isomorphisms as $\mathbb{Z}_p$-algebras

$$\mathbb{Z}_p[\![T(\mathbb{Z}_p)]\!]\overset{\sim}{\to}(\mathbb{Z}_p[T(\mathbb{Z}/p\mathbb{Z})])[\![(1+p\mathbb{Z}_p)^g]\!]\overset{\sim}{\to}\mathbb{Z}_p[T(\mathbb{Z}/pZ)][\![X_1,X_2,\cdots,X_g]\!]$$

the second isomorphism is obtained by sending $(1,1,\cdots,1+p,1,\cdots,1)$ with $1+p$ on the $i$-th component for $1\leq i\leq g$, to $1+X_i$. It follows that the $\mathbb{C}_p$-points of $\mathcal{W}$ are described by $\mathcal{W}(\mathbb{C}_p)=\mathrm{Hom}_{cont}(T(\mathbb{Z}_p),\mathbb{C}_p^\times)$, and if we denote by $\widehat{T(\mathbb{Z}/p\mathbb{Z})}$ the character group of $T(\mathbb{Z}/p\mathbb{Z})$, the weight space is isomorphic to a disjoint union, indexed by the elements of $\widehat{T(\mathbb{Z}/p\mathbb{Z})}$, of $g$-dimensional open unit polydiscs. More precisely, we have the following explicit isomorphism: $\mathcal{W}\overset{\sim}{\to}\widehat{T(\mathbb{Z}/p\mathbb{Z})}\times\prod_{i=1}^g B(1,1^-)$

$$\kappa\mapsto\big(\kappa|_{T(\mathbb{Z}/p\mathbb{Z})},\kappa((1+p,1,\dots,1)),\kappa((1,1+p,\dots,1)),\dots,\kappa((1,\dots,1,1+p))\big),$$

$$\widehat{T(\mathbb{Z}/p\mathbb{Z})}\times\prod_{i=1}^g B(1,1^-)\ni(\chi,s_1,\cdots,s_g)\leftrightarrow\left((\lambda,x_1,\cdots,x_g)\mapsto\kappa(\lambda)\prod_{i=1}^g s_i^{\frac{\log(x_i)}{\log(1+p)}}\right).$$

## 1.2 The universal character.

If we denote by $\mathcal{O}_\mathcal{W}$ the sheaf of rigid analytic functions on $\mathcal{W}$, we have a natural continuous group homomorphism, obtained as the composition

$$\kappa^{un}:T(\mathbb{Z}_p)\longrightarrow(\mathbb{Z}_p[\![T(\mathbb{Z}_p)]\!])^\times\longrightarrow(\mathcal{O}_\mathcal{W})^\times,$$

called the universal character.

*Definition* (2.2.1 in [AIP15]) Let $w \in \mathbb{Q} > 0$. A character $\kappa \in \mathcal{W}(\mathbb{C}_p)$ is said $w$-analytic if $\kappa$ extends to an analytic map

$$\kappa : T(\mathbb{Z}_p)(1 + p^w \mathcal{O}_{\mathbb{C}_p})^g \to \mathbb{C}_p^\times.$$

It follows from the classical $p$-adic properties of the exponential and the logarithm that any character $\kappa$ is $w$-analytic for some $w > 0$:
*Proposition* 2.2.2 in [AIP15] and [U11], Lemma 3.4.6]). For any quasi-compact open subset $\mathcal{U} \subset \mathcal{W}$, there exists $w_{\mathcal{U}} \in \mathbb{R}_{>0}$ such that the universal character $\kappa^{un} : \mathcal{U} \times T(\mathbb{Z}_p) \to \mathbb{C}_p$ extends to an analytic function $\kappa^{un} : \mathcal{U} \times T(1 + p^{w_{\mathcal{U}}} \mathcal{O}_{\mathbb{C}_p})^g \to \mathbb{C}_p$

### 1.3 Analytic representations.

Let $\mathsf{I}$ be the Iwahori subgroup of $\mathrm{GL}_g(\mathbb{Z}_p)$ of matrices whose reduction modulo $p$ is upper triangular. Let $N^0$ be the subgroup of $U_0(\mathbb{Z}_p)$ of matrices that reduce to the identity modulo $p$. The Iwahori decomposition is an isomorphism $B(\mathbb{Z}_p) \times N_0 \to \mathsf{I}$. Identify $N_0$ with $(p\mathbb{Z}_p)^{\frac{g(g-1)}{2}} \subset \mathbb{A}_{an}^{\frac{g(g-1)}{2}}$, where $\mathbb{A}^{an}$ denotes the rigid analytic affine line defined over $\mathbb{Q}_p$. For $\varepsilon > 0$, let $N^0_\varepsilon$ be the rigid analytic space

$$\bigcup_{x \in (p\mathbb{Z}_p)^{\frac{g(g-1)}{2}}} B(x, p^{-\varepsilon}) \subset \mathbb{A}_{an}^{\frac{g(g-1)}{2}}$$

Let $L$ be an extension of $\mathbb{Q}_p$ and $\mathcal{F}(N^0, L)$ the ring of $L$-valued functions on $N^0$. We say that a function $f \in \mathcal{F}(N^0, L)$ is $\varepsilon$-analytic if it is the restriction to $N^0$ of a necessarily unique analytic function on $N^0_\varepsilon$ . We denote by $\mathcal{F}^{an}_\varepsilon(N^0, L)$ the set of $\varepsilon$-analytic functions. A function is analytic if it is 1-analytic. We simply denote by $\mathcal{F}^{an}(N^0, L)$ the set of analytic functions. Let $\mathcal{F}^{l-an}(N^0, L)$ be the set of locally analytic functions on $N^0$, i.e., the direct limit of the sets $\mathcal{F}^{\varepsilon-an}(N^0, L)$ for all $\varepsilon > 0$.

$$V^{\varepsilon-an}_{\kappa,L} := \left\{ f : \mathsf{I} \to L | f(ib) = \kappa(b) f(i) \forall (i, b) \in \mathsf{I} \times B(\mathbb{Z}_p), f \in \mathcal{F}^{\varepsilon-an}(N^0, L) \right\}$$

Similarly defined representations of the Iwahori group $\mathsf{I}$ are denoted by $V^{an}_{\kappa,L}$ and $V^{l-an}_{\kappa,L}$.

## 2 The BGG resolution (see [AIP15]

Let $W$ be the Weyl group of $\mathrm{GL}_g$, it acts on $X(T)$.

## 2.1 Recall : the BGG resolution in [AIP15]

We set $\mathfrak{g}$ and $\mathfrak{t}$ for the Lie algebras of $\mathrm{GL}_g$ and $T$. The choice of $B$ determines a system of simple positive roots $\Delta \subset X(T)$.

To any $\alpha \in \Delta$ are associated an element $H_\alpha \in \mathfrak{t}$, elements $X_\alpha \in \mathfrak{g}_\alpha$ and $X_{-\alpha} \in \mathfrak{g}_{-\alpha}$ such that $[X_\alpha, X_{-\alpha}] = H_\alpha$ and a co-root $\alpha^\vee$. Let $s_\alpha \in W$ be the symmetry $\lambda \mapsto \lambda - \langle \lambda, \alpha^\vee \rangle \alpha$. For any $w \in W$ and $\lambda \in X(T)$, we set $w \bullet \lambda = w(\lambda + \rho) - \rho$, where $\rho$ is half the sum of the positive roots. By the main result of [Jon11], for all $\kappa \in X^+(T)$, and any

field extension $L$ of $\mathbb{Q}_p$, we have an exact sequence of $\mathsf{I}$-representations:

$$0 \longrightarrow V_{\kappa,L} \xrightarrow{d_0} V^{an}_{\kappa,L} \xrightarrow{d_1} \bigoplus_{\alpha \in \Delta} V^{an}_{s_\alpha \bullet \kappa, L}$$

To make explicit the differentials, the map $d_0$ is the natural inclusion, the map $d_1$ is the sum of maps $\Theta_\alpha : V^{an}_{\kappa,L} \longrightarrow V^{an}_{s_\alpha \bullet \kappa, L}$ whose definitions is now recalled.

Let $\mathsf{I}$ act on the space of analytic functions on $\mathsf{I}$ by the formula $(i \star f)(j) = f(j \cdot i)$ for any analytic function $f$ and $i, j \in \mathsf{I}$. By differentiating one obtains an action of $\mathfrak{g}$ and hence of the enveloping algebra $U(\mathfrak{g})$ on the space of analytic functions on $\mathsf{I}$. If $f \in V^{an}_{\kappa,L}$, set $\Theta_\alpha(f) = X_{-\alpha}^{\langle \kappa, \alpha^\vee \rangle + 1} \star f$, and show $\Theta_\alpha(f) \in V^{an}_{s_\alpha \bullet \kappa, L}$. First check that $\Theta_\alpha(f)$ is $U(\mathbb{Z}_p)$-invariant.

## 2.2 The BGG resolution-2 in [AIP15]

Enough to prove that $X_\beta \star \Theta_\alpha(f) = 0$ for all $\beta \in \Delta$.
If $\alpha \neq \beta$, this follows easily for $[X_\beta, X_{-\alpha}] = 0$.
If $\alpha = \beta$, one uses the relation

$$[X_\alpha, X_{-\alpha}^{\langle \kappa, \alpha^\vee \rangle}] = (\langle \kappa, \alpha^\vee \rangle + 1) X_{-\alpha}(H_\alpha - \langle \kappa, \alpha^\vee \rangle).$$

We now have $X_\alpha \star \Theta_\alpha(f) = [X_\alpha, X_{-\alpha}^{\langle \kappa, \alpha^\vee \rangle + 1}] \star f =$
$(\langle \kappa, \alpha^\vee \rangle + 1) X_{-\alpha}(H_\alpha - \langle \kappa, \alpha^\vee \rangle) \star f = 0$.
Let us find the weight of $\Theta_\alpha(f)$. For any $t \in \mathsf{T}(\mathbb{Q}_p)$, we have
$t \star \Theta_\alpha(f) = \mathsf{Ad}(t)(X_{-\alpha}^{\langle \kappa, \alpha^\vee \rangle + 1}) t \star f = \alpha^{-\langle \kappa, \alpha^\vee \rangle - 1}(t) \kappa(t)$.
Since $\alpha^{-\langle \kappa, \alpha^\vee \rangle - 1} \kappa = s_\alpha \bullet \kappa$ the map $\Theta_\alpha$ is well definied.

# 3 A classicity criterion, §2.5. of [AIP15].

## 3.1 A classicity criterion-1 in [AIP15]

For $1 \leq g-1$, we set $d_i = \begin{pmatrix} p^{-1} 1_{g-i} & 0 \\ 0 & 1_i \end{pmatrix} \in \mathrm{GL}_g(\mathbb{Q}_p)$. The adjoint action of $d_i$ on $\mathrm{GL}_g(\mathbb{Q}_p)$ stabilizes the Borel subgroup $\mathsf{B}$.

The formula $(\delta_i \cdot f)(g) := f(d_i g d_i^{-1})$ defines an action on the space $V_\kappa$ for any $\kappa \in X^+(T)$.

We now define the action on the spaces $V^{\varepsilon-an}_{\kappa,L}$ for any $\kappa \in \mathcal{W}(L)$. We have a well-defined adjoint action of $d_i$ on the group $\mathrm{N}_0$. Let $f \in V^{\varepsilon-an}_{\kappa,L}$ and $j \in \mathsf{I}$. Let

$j = n \cdot b$ be the Iwahori decomposition of $j$. Set $\delta_i f(j) := f(d_i n d_i^{-1} b)$. We hence get operators $\delta_i$ on $V_{\kappa,L}^{\varepsilon-an}$ and $V_{\kappa,L}^{l-an}$ Let $z_{k,l}$ be the $(k,l)$-matrix coefficient on $\mathrm{GL}_g$. If we use the isomorphism $V_{\kappa,L}^{\varepsilon-an} \to \mathcal{F}^{\varepsilon-an}(\mathrm{N}^0, L)$ given by the restriction of functions to $\mathrm{N}^0$, then the operator $\delta_i$ is given by

$$\mathcal{F}^{\varepsilon-an}(\mathrm{N}^0, L) \to \mathcal{F}^{\varepsilon-an}(\mathrm{N}^0, L)$$
$$f \mapsto [(z_{k,l})_{k<l} \mapsto f(p^{n_{k,l}} z_{k,l})]$$

where $n_{k,l} = 1$ if $k \geq g-i+1$ and $l \leq g-i$ and $n_{k,l} = 0$ otherwise. The operator $\delta_i$ is norm decreasing and the operator $\prod_i \delta_i$ on $V_\kappa^{\varepsilon-an}$ is completely continuous.

## 3.2 Advantages of using matrix coefficients $(v)_{k,l}$

Introducing more coordinates allows to avoid complicated formulas when dealing with integral/algebraic structures and congruences. Also, computing values of linear and multilinear forms could be better organised in such cases as
1) Rankin-Selberg integral of $f$ and $g$ for $\mathrm{GL}_n \times \mathrm{GL}_m$ or
2) Integrations in the Doubling Method (integrating of $f(z)$ over $z$ against a restriction to the "diagonal" (or block diagonal) $(z,w)$ of a multi dimensional kernel $\mathcal{E}(z,w)$), or
2') dealing with triple products ("Garrett's type pullback" integral of $\mathcal{E}(\mathrm{diag}[z_1, z_2, z_3])$ against $f_1(z_1) \otimes f_2(z_2) \otimes f_3(z_3)$ of balanced weights $k_1 \geq k_2 \geq k_3$, $k_1 \leq k_2 + k_3$, [Boe-Pa6].

- complicated formulas for the holomorphic projection could be avoided by introducing more coordinates (eg for a (nearly-) holomorphic Eisenstein series multiplied by another form $g$, and projected to the holomorphic characteristic part attached to $f$). Indeed, solving a linear system is often simpler than computing an integral.

- Also, computing of successive integrals in the Doubling Method or proving congruences could be reduced using matrix coefficients $(v)_{k,l}$.

## 3.3 A classicity criterion-2 in [AIP15]

If $\kappa \in X(T)^+$, the map $d_0$ in the exact sequence (2.4.A) is $\delta_i$-equivariant. Regarding the map $d_1$, we have the following variance formula:

$$\delta_i \Theta_\alpha = \alpha(d_i)^{\langle\kappa,\alpha^\vee\rangle+1} \Theta_\alpha \delta_i.$$

Indeed for any $f \in V_{\kappa,L}^{\varepsilon-an}$ , we have

$$\begin{aligned}\delta_i \Theta_\alpha(f) &= d_i \cdot (d_i^{-1} X_{-\alpha}^{\langle\kappa,\alpha^\vee\rangle+1} \star f) \\ &= \alpha(d_i)^{\langle\kappa,\alpha^\vee\rangle+1} d_i \cdot (X_{-\alpha}^{\langle\kappa,\alpha^\vee\rangle+1} d_i^{-1} \star f) \\ &= \alpha(d_i)^{\langle\kappa,\alpha^\vee\rangle+1} \Theta_\alpha(\delta_i f)\end{aligned}$$

Let $\underline{v} = (v_1, \cdots, v_{g-1}) \in \mathbb{R}^{g-1}$. We let $V_\kappa^{l-an,<\underline{v}}$ be the union of the generalized eigenspaces where $\delta_i$ acts by eigenvalues of valuation strictly smaller than $v_i$. We are now able to give the classicity criterion.

### 3.4 A classicity criterion-3 in [AIP15],[BGG2-5p3]

Proposition 2.5.1. Let $\kappa = (k_1, \cdots, k_g) \in X^+(T)$. Set $v_{g-i} = k_i - k_{i+1} + 1$ for $1 \leq i \leq g-1$. Then any element $f \in V_{\kappa,L}^{l-an,<\underline{v}}$ is in $V_{\kappa,L}$.

Proof ([AIP15], p 12. One easily checks that any element $f \in V_{\kappa,L}^{l-an,<\underline{v}}$ is actually analytic because the operators $\delta_i$ increase the radius of analyticity. Using the exact sequence (2.4.A) in [AIP15], we need to see that $d_1 \cdot f = 0$. Let$\alpha$ be the simple positive root given by the character $(t_1, \cdots, t_g) \mapsto t_i t_{i+1}^{-1}$. Since $\delta_{g-i}\Theta_\alpha(f) = p^{k_{i+1}-k_i-1}\Theta_\alpha\delta_{g-i}(f)$, we see that $\Theta_\alpha$ is a generalized eigenvector for $\delta_{g-i}$ for eigenvalues of negative valuation. But the norm of $\delta_{g-i}$ is less than 1, so $\Theta_\alpha(f)$ has to be zero.

### 3.5 Construction of analytic representations.

Recall: Let $\mathsf{I}$ be the Iwahori subgroup of $\mathrm{GL}_g(\mathbb{Z}_p)$ of matrices whose reduction modulo $p$ is upper triangular. Let $N^0$ be the subgroup of $U_0(\mathbb{Z}_p)$ of matrices that reduce to the identity modulo $p$. The Iwahori decomposition is an isomorphism $B(\mathbb{Z}_p) \times N_0 \to \mathsf{I}$. If $f \in V_{\kappa,L}^{an}$, set $\Theta_\alpha(f) = X_{-\alpha}^{\langle\kappa,\alpha^\vee\rangle+1} \star f$. One shows that $\Theta_\alpha(f) \in V_{s_\alpha \bullet \kappa, L}^{an}$. First of all let us check that $\Theta_\alpha(f)$ is $U(\mathbb{Z}_p)$-invariant. It will be enough to prove that $X_\beta \star \Theta_\alpha(f) = 0$ for all $\beta \neq \alpha$. The weight of $\Theta_\alpha(f)$ is $s_\alpha \bullet \kappa$ by checking the action of $t \in T(\mathbb{Q}_p)$:

$$t \star \Theta_\alpha(f) = \mathrm{Ad}(t)(X_{-\alpha}^{\langle\kappa,\alpha^\vee\rangle+1})t \star f = \alpha^{-\langle\kappa,\alpha^\vee\rangle-1}(t)\kappa(t)\Theta_\alpha(f)$$

Since we have $\alpha^{-\langle\kappa,\alpha^\vee\rangle-1}\kappa = s_\alpha \bullet \kappa$ the map $\Theta_\alpha$is well defined.

*A classicity criterion.* For $1 \leq i \leq g-1$, we set $d_i = \begin{pmatrix} g^{-1}1_{g-i} & 0 \\ 0 & 1_i \end{pmatrix} \in \mathrm{GL}_g(\mathbb{Q}_p)$. The adjoint action of $d_i$ on $\mathrm{GL}_g = \mathbb{Q}_p$ stabilizes the Borel subgroup $B$.

The formula $(\delta_i \cdot f)(g) := f(d_i g d_i^{-1})$ defines an action on the space $V_\kappa$ for any $\kappa \in X^+(T)$. Define the action on the spaces $V_{\kappa,L}^{\varepsilon-an}$ for any $\kappa \in \mathcal{W}(L)$.

We have a well-defined adjoint action of $d_i$ on the group $N^0$. Let $f \in V^{\varepsilon-an}$ and $j \in \mathsf{I}$. Let $j = n \cdot b$ be the Iwahori decomposition of $j$. We set $\delta_i f(j) := f(d_i n d_i^{-1} b)$. We get operators $\delta_i$ on $V_{\kappa,L}^{\varepsilon-an}$ and $V_{\kappa,L}^{l-an}$ Let $z_{k,l}$ be the $(k,l)$-matrix coefficient on $\mathrm{GL}_g$. If we use the isomorphism $V_{\kappa,L}^{\varepsilon-an} \to \mathcal{F}^{\varepsilon-an}(N^0, L)$.

### 3.6 Hecke operators

*Hecke operators*, §6 in [AIP15]. An action of the Hecke operators on the space of overconvergent modular forms will be defined and one of these operators that is compact is singled out *Hecke operators outside* $p$, see §6.1.

## 4 Classicity, §7 of [AIP15]

### 4.1 Statement of the main result,§7 of [AIP15]

.

*Classicity*,§7 in [AIP15] *Main result* §7.1. Let $\kappa = (k_1, \cdots, k_g) \in X^+(T)$. We have a series of natural restriction maps,

$$H^0(X_{Iw}, \omega^\kappa) \overset{r_1}{\to} H^0(X_{Iw(p)}(v), \omega^\kappa) \overset{r_2}{\to} H^0(X_{Iw(p)}(v), \omega_w^{\dagger\kappa})$$

and a criterion is established for an element in $H^0(X_{Iw(p)}(v), \omega_w^{\dagger\kappa})$ to be in the image of $r_2 \circ r_1$. Let $a = (a_1, \cdots, a_g) \in \mathbb{R}^g_{\geq 0}$. We set $\mathcal{M}^{\dagger\kappa}(X_{Iw(p)}(v)^{<a}$ for the union of the generalized eigenspaces where $U_{p,i}$ has slope $< a_i$ for $1 \leq i \leq g$.

*Theorem* 7.1.1 in [AIP15]. Let $a = (a_1, \cdots, a_g)$ with $a_i = k_{g-i} - k_{(g-i)+1} + 1$ when $1 \leq i \leq g-1$ and $a_g = k_g - g(g+1)/2$ . Then we have

$$\mathcal{M}^{\dagger\kappa}(X_{Iw(p)}(v))^{<a} \subset H^0(X_{Iw(p)}, \omega^\kappa)$$

The proof of this theorem splits into two parts. *First step* Show that $\mathcal{M}^{\dagger\kappa}(X_{Iw(p)}(v))^{<a} \subset H^0(X_{Iw(p)}(v), \omega^\kappa)$

## 4.2 Applying the main theorem of [BPS]

*Second step* Conclude by applying the main theorem of [BPS] as follows. Since $U_{p,g}$ is a compact operator on $H^0(\mathcal{X}_{Iw(p)}(v); \omega^\kappa)$, for all $a_g \in \mathbb{R}_{\geq 0}$ define $H^0(\mathcal{X}_{Iw(p)}(v); \omega^\kappa)^{<a_g}$, which is the sum of generalized eigenspaces for $U_{p,g}$ with eigenvalues of slope less than $a_g$. *Theorem* 7.1.2 ([BPS]). Let $a_g = k_g - \frac{g(g+1)}{2}$ . Then $H^0(\mathcal{X}_{Iw(p)}(v); \omega^\kappa)^{<a_g}$ is a space of classical forms.

## 4.3 Recall: Relative BGG resolution-1.

Take $w \in ]\frac{v}{p-1}, 1 - v\frac{p}{p-1}]$. Remark that for such a $w$, the fibers of the morphism $\pi : \mathcal{IW}_w^0 \to \mathcal{X}_{Iw(p)}(v)$ are connected. Consider the cartesian diagram

$$\begin{array}{ccccc}
\mathcal{IW}_w^{0+} & \times & \mathcal{T}_{an}^\times & \longrightarrow & \mathcal{T}_{an}^\times \\
\downarrow^{\pi_1} & & & & \downarrow \\
\mathcal{IW}_w^{0+} & & \longrightarrow & & \mathcal{T}_{an}^\times/U \\
 & & \searrow^{\pi_2} & & \downarrow \\
 & & & & \mathcal{X}_{Iw(p)}(v)
\end{array}$$

An action of the Iwahori subgroup $I$ on $\mathcal{IW}_w^{o+} \times \mathcal{T}_{an}$, and by differentiating obtain an action of the enveloping algebra $U(\mathfrak{g})$ on
$(\pi_2 \circ \pi_1)_* \mathcal{O}_{\mathcal{IW}_w^{o+} \times \mathcal{T}_{an}^\times}$ denoted $\star$. We have already defined an inclusion $d_0 : \omega^\kappa \to \omega_w^{\dagger\kappa}$ . For all $\alpha \in \Delta$, define a map $\Theta_\alpha : \omega^\kappa \to \omega_w^{\dagger s_\alpha \bullet \kappa}$. First define an endomorphism of $(\pi_2 \circ \pi_1)_* \mathcal{O}_{\mathcal{IW}_w^{o+} \times \mathcal{T}_{an}^\times}$ by sending a section $f$ to $X_{-\alpha}^{<\kappa, \alpha^\vee>+1} \star f$. By §2.4,this map restricted to $\omega_w^{\dagger\kappa}$ produces the expected map $\Theta_\alpha$.

## 4.4 Relative BGG resolution, §7.2 of [AIP15]

is a relative version of the theory recalled in §2.4, [AIP15].

*Proposition* 7.2.1 in [AIP15]. There is an exact sequence of sheaves over $\mathcal{X}_{Iw(p)}(v)$ as follows:

$$0 \longrightarrow \omega^{\kappa} \xrightarrow{d_0} \omega_w^{\dagger\kappa} \xrightarrow{d_1} \bigoplus_{\alpha\in\Delta} \omega_w^{\dagger s_\alpha \bullet \kappa}$$

Proof. By the main result of [Jon11], and p.10 in[AIP15], for all $\kappa \in X^+(T)$, and any field extension $L$ of $\mathbb{Q}_p$, we have an exact sequence (2.4.A) of I-representations. Define an automorphism $\kappa \to \kappa'$ of $X(T)$ by $\kappa' = (-k_g, \cdots, -k_1) \in X(T)$.
Tensoring-completing the exact sequence (2.4.A) in [AIP15] (or more precisely its analytic version in [AIP15]) by $\mathcal{O}_{\mathcal{X}_{Iw(p)(v)}}$ we get the following sequence:

$$0 \longrightarrow V_{\kappa',L}\hat{\otimes}\mathcal{O}_{\mathcal{X}_{Iw(p)(v)}} \xrightarrow{d_0\hat{\otimes}1} V^{an}_{\kappa',L}\hat{\otimes}\mathcal{O}_{\mathcal{X}_{Iw(p)(v)}} \xrightarrow{d_1\hat{\otimes}1} \bigoplus_{\alpha\in\Delta} V^{an}_{s_\alpha\bullet\kappa',L}\hat{\otimes}\mathcal{O}_{\mathcal{X}_{Iw(p)(v)}}$$

The image of $d_1$ is closed in $\bigoplus_{\alpha\in\Delta} V^{an}_{s_\alpha\bullet\kappa,L}$ and is a direct factor of an orthonormalizable Banach module by [Jon11].

It follows that there exists an isomorphism of Banach modules splitting the above sequence.

# 5 Families

*Families* §8 in [AIP15] The weight space $\mathcal{W} = \mathrm{Hom}(T(\mathbb{Z}_p), \mathbb{C}_p^\times)$ as defined in §2.2, [AIP15]. For any affinoid open subset $\mathcal{U}$ of $\mathcal{W}$, by Proposition 2.2.2 there exists $w_{\mathcal{U}} > 0$ such that the universal character

$$\kappa^{un} : T(\mathbb{Z}_p) \times \mathcal{W} \to \mathbb{C}_p^\times$$

restricted to $\mathcal{U}$ extends to an analytic character

$$\kappa^{un} : T(\mathbb{Z}_p)(1 + p^{w_{\mathcal{U}}}\mathcal{O}_{\mathbb{C}_p}) \to \mathcal{U} \to \mathbb{C}_p^\times.$$

*Families of overconvergent modular forms*, §8.1 of [AIP15].

The construction §5 works in families as follows:

*Proposition* 8.1.1.1. There exists a sheaf $\omega_w^{\dagger\kappa^{un}}$ on $\mathcal{X}_{Iw(p)}(v) \times \mathcal{U}$ such that for any weight $\kappa \in \mathcal{U}$, the fiber of $\omega_w^{\dagger\kappa^{un}}$ over $\mathcal{X}_{Iw(p)}(v) \times \{\kappa\}$ is $\omega_w^{\dagger\kappa}$. Proof. Uses the projection $\pi \times 1 : \mathcal{IW}_w^{o+} \to \mathcal{X}_{Iw}(p)(v) \times \mathcal{U}$. Take $\omega_w^{\dagger\kappa^{un}}$ to be the subsheaf of $(\pi \times 1)_*\mathcal{O}_{\mathcal{IW}_w^{o+}\times\mathcal{U}}$ of $(\kappa^{un})'$-invariant sections for the action of $T(\mathbb{Z}_p)$.

# 6 Families in the Unitary Case

### Algebraic differential operators over unitary groups

Algebraic differential operators are described acting on holomorphic automorphic forms $\boldsymbol{\varphi}$ on unitary groups $U(n,n)$ over an imaginary quadratic field $\mathcal{K} = \mathbb{Q}(\sqrt{-D_{\mathcal{K}}}) \subset \mathbb{C}$. Applications are given to special $L$-values $L(s, \boldsymbol{\varphi})$ attached to $\boldsymbol{\varphi}$.

*Contents*

1. The simplest case of modular forms for $\Gamma = \mathrm{SL}_2(\mathbb{Z})$.
2. Algebraic differential operators on symplectic groups.
3. Unitary groups $U(a,b)$ (of signature $(a,b)$, $a+b=n$), and the double group $U(n,n)$.
4. Analytic families of CM-abelian varieties and unitary groups.
5. Algebraic automorphic forms on unitary groups
6. $C^\infty$-differential operators via Shimura's approach.
7. Algebraic differential operators for $U(n,n)$.
8. Applications to critical values of the standard zeta function $\mathcal{L}(s,\boldsymbol{\varphi})$ in the unitary case.
9. Perspectives and examples for $U(n,n)$.

## 6.1 Perspectives and applications

1. The case $U(n,n)$: a striking analogue of Manin-Mazur's result on $p$-adic analytic interpolation of critical values, [Ma73], [MTT], to any imaginary quadraic $\mathcal{K}$, a hermitian Hecke-eigenform of weight $\ell > 2n$, $s_* = n$, $s^* = \ell - n$.
2. Using the Hodge and Newton polygons of an Euler product with a functional equation, for its geometric recognition
3. Link to a new revolutionary tool – Prisms and Prismatic cohomology (by P.Scholze-B.Bhatt [BhSc19], via Kisin-Fargue-Wach-modules and Iwasawa cohomology, using the obtained Iwasawa series.

   Given a formally smooth $\mathbb{Z}_p$-scheme $X$, this cohomology yields a universal $q$-deformation of the de Rham cohomology of $X/\mathbb{Z}_p$ across the map $\mathbb{Z}_p[[q-1]] \xrightarrow{q\to 1} \mathbb{Z}_p$, and the Iwasawa algebra $\mathbb{Z}_p[[q-1]]$ provides a description.
4. Special hypergeometric motives and their L-functions: Asai recognition, see [DPVZ] The generalized hypergeometric functions are often used in arithmetic and algebraic geometry. They come as periods of certain algebraic varieties, and consequently they encode important information about the invariants of these varieties. Euler factors, Newton and Hodge polygons attached to them, provide a tool for their geometric recognition.

## 6.2 An example in the case $U(n,n)$. Hermitian modular group $\Gamma_{n,K}$ and the standard zeta function $\mathcal{Z}(s,\mathbf{f})$ (definitions)

The followng function $\mathcal{Z}(s,\mathbf{f})$ is a special case of Euler products constructed by G. Shimura. Let $\theta = \theta_K$ be the quadratic character attached to $K = \mathbb{Q}(\sqrt{-D_K})$, $n' =$

$\left[\frac{n}{2}\right]$.

$$\Gamma_{n,K} = \left\{ M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{GL}_{2n}(\mathcal{O}_K) | M\eta_n M^* = \eta_n \right\}, \ \eta_n = \begin{pmatrix} 0_n & -I_n \\ I_n & 0_n \end{pmatrix},$$

$$\mathcal{Z}(s, \mathbf{f}) = \left( \prod_{i=1}^{2n} L(2s - i + 1, \theta^{i-1}) \right) \sum_{\mathfrak{a}} \lambda(\mathfrak{a}) N(\mathfrak{a})^{-s},$$

(defined via Hecke's eigenvalues: $\mathbf{f}|T(\mathfrak{a}) = \lambda(\mathfrak{a})\mathbf{f}, \mathfrak{a} \subset \mathcal{O}_K$)

$= \prod_{\mathfrak{q}} \mathcal{Z}_{\mathfrak{q}}(N(\mathfrak{q})^{-s})^{-1}$(an Euler product over primes $\mathfrak{q} \subset \mathcal{O}_K$,

with $\deg \mathcal{Z}_{\mathfrak{q}}(X) = 2n,$ the Satake parameters $t_{i,\mathfrak{q}}, i = 1, \cdots, n$),

$\boxed{\mathcal{D}(s, \mathbf{f}) = \mathcal{Z}(s - \frac{\ell}{2} + \frac{1}{2}, \mathbf{f})}$ (Geometrically normalized standard zeta function

with a functional equation $s \mapsto \ell - s$; rk $= 4n$, and geometric weight $\ell - 1$),

$$\Gamma_{\mathcal{D}}(s) = \prod_{i=0}^{n-1} \Gamma_{\mathbb{C}}(s - i)^2.$$

Main result in the lifted case: Assuming $\ell > 2n$, a $p$-adic interpolation is constructed of all critical values $\mathcal{D}(s, \mathbf{f}, \chi)$ normalized by $\times\Gamma_{\mathcal{D}}(s)/\Omega_{\mathbf{f}}$, in the critical strip $n \leq s \leq \ell - n$ for all $\chi$ mod $p^r$ in both *bounded or unbounded case* , i.e. when the product $\alpha_{\mathbf{f}} = \left( \prod_{\mathfrak{q}|p} \prod_{i=1}^{n} t_{\mathfrak{q},i} \right) p^{-n(n+1)}$ is *not a p-adic unit.*

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