The exterior algebra and 'Spin' of an orthogonal $\mathfrak{g}$-module

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Introduction

Let $\mathfrak{g}$ be a reductive algebraic Lie algebra over an algebraically closed field $\Bbbk$ of characteristic zero and G is the corresponding connected and simply connected group.

The symmetric algebra of a (finite-dimensional) $\mathfrak{g}$-module $\mathbb{V}$ is the algebra of polynomial functions on the dual space $\mathbb{V}^*$. Therefore one can study the algebra of symmetric invariants using geometry of G-orbits in V^*. In case of the exterior algebra, $\wedge^\bullet\mathbb{V}$, lack of such geometric picture results by now in absence of general structure theorems for the algebra of skew-invariants $(\wedge^\bullet\mathbb{V})^{\mathfrak{g}}$. One may find in the literature several interesting results related to skew-symmetric invariants. We only mention Kostant's computation for cohomology of the nilradical of a parabolic subalgebra in $\mathfrak{g}$ [Ko61] and R. Howe's classification of "skew-multiplicity-free" $\mathfrak{g}$-modules [Ho95, ch. IV]. But the general situation still remains unsatisfactory, and developing of Invariant Theory in the skew-symmetric setting represents an attractive problem.

In this paper, we begin with describing all irreducible orthogonal $\mathfrak{g}$-modules such that $(\wedge^\bullet\mathbb{V})^{\mathfrak{g}}$ is again an exterior algebra. It is shown that in this case either $\mathbb{V}\simeq\mathfrak{g}$ and hence $\mathfrak{g}$ is simple or $\mathfrak{g}\oplus\mathbb{V}$ has a structure of *simple* $\mathbb{Z}_2$-graded Lie algebra, which quickly leads to a short classification, see Table 1. Obviously, none of the symplectic representations (with $\dim\mathbb{V}>2$) can have an exterior algebra of skew-invariants. But the situation for the representations of "general type" is not yet clear.

In case $\mathbb{V}$ is orthogonal, a better understanding of the $\mathfrak{g}$-module structure of $\wedge^\bullet\mathbb{V}$ can be achieved through the notion of *'Spin'* of $\mathbb{V}$. This goes as follows. Let $\pi:\mathfrak{g}\to\mathfrak{so}(\mathbb{V})$ be the corresponding representation. Restricting the spinor representation of $\mathfrak{so}(\mathbb{V})$ to $\mathfrak{g}$ gives us a $\mathfrak{g}$-module, which is denoted by $Spin(\mathbb{V})$. The motivation came from Kostant's result that $Spin(\mathfrak{g})$ is a primary $\mathfrak{g}$-module; namely, $Spin(\mathfrak{g})=2^{[\mathrm{rk}\,\mathfrak{g}/2]}\mathbb{V}_\rho$, the highest weight ρ being the half-sum of the positive roots [Ko61, p. 358]. The main property of $Spin(\mathbb{V})$ is that, depending on parity of $\dim\mathbb{V}$, $\wedge^\bullet\mathbb{V}$ is isomorphic to either $Spin(\mathbb{V})^{\otimes 2}$ or $2{\cdot}Spin(\mathbb{V})^{\otimes 2}$. It is thus interesting to find the orthogonal $\mathfrak{g}$-modules, where $Spin(\mathbb{V})$ has a simple structure. In general, $Spin(\mathbb{V})$, as element of the representation ring, has a numerical factor depending on the zero-weight multiplicity. Omitting this factor yields a $\mathfrak{g}$-module, which is called the *reduced 'Spin'* of $\mathbb{V}$ and denoted by $Spin_0(\mathbb{V})$; e.g. $Spin_0(\mathfrak{g})=\mathbb{V}_\rho$. In a sense, $Spin_0(\mathbb{V})$ behaves better than $Spin(\mathbb{V})$. For, regardless of parity of $\dim\mathbb{V}$, we have $\wedge^\bullet\mathbb{V}\simeq 2^{m(0)}{\cdot}Spin_0(\mathbb{V})^{\otimes 2}$, where $m(0)$ is the zero-weight multiplicity, and $Spin_0(\mathbb{V}_1\oplus\mathbb{V}_2)=Spin_0(\mathbb{V}_1)\otimes Spin_0(\mathbb{V}_2)$.

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An orthogonal $\mathfrak{g}$-module $\mathbb{V}$ is said to be *co-primary*, if $Spin_0(\mathbb{V})$ is irreducible. In sections 2 and 3, a classification of the co-primary modules is obtained. To this end, we give a geometric description of some highest weights of $Spin_0(\mathbb{V})$. These weights are called *extreme*. The assumption that $Spin_0(\mathbb{V})$ has a unique extreme weight imposes strong constraints on the weight structure of $\mathbb{V}$. Using this, one shows that $\mathfrak{g}$ must be simple whenever $\mathbb{V}$ is an irreducible faithful co-primary $\mathfrak{g}$-module, and that any reducible co-primary module is being obtained by iterating the "direct sum" procedure:

$$(\mathfrak{g}_i, \mathbb{V}_i),\ i = 1,2\ \mapsto (\mathfrak{g}_1 \oplus \mathfrak{g}_2, \mathbb{V}_1 \oplus \mathbb{V}_2).$$

It is thus sufficient to classify the irreducible co-primary modules. The resulting list appears to be rather short:

1. $\mathfrak{g}$ is simple and $\mathbb{V} \simeq \mathfrak{g}$;
2. $\mathfrak{g}$ is of type $\mathbf{B}_n$ or $\mathbf{C}_n$ or $\mathbf{F}_4$, and $\mathbb{V}$ is the little adjoint module;
3. $\mathfrak{g} = \mathfrak{so}(\mathbb{W})$ and $\mathbb{V} = \{$the Cartan component of $\mathcal{S}^2\mathbb{W}\}$; $\dim \mathbb{W} = 3, 5, 7, \ldots$.

In case 2, $\mathfrak{g}$ has roots of two lengths and the $\mathfrak{g}$-module whose highest weight is the short dominant root is called *little adjoint* (l.a.). Actually, we give a unified proof for the fact that the l. a. module is co-primary whenever the ratio of root lengths is $\sqrt{2}$. Note that, for $\mathbf{G}_2$, where this ratio is $\sqrt{3}$, the l. a. module is not co-primary. A true reason why this is so is that the l. a. module for $\mathbf{G}_2$ is not the isotropy representation of a symmetric space, whereas this is the case for **B, C**, and **F**. Furthermore, all representations listed above are the isotropy representations of symmetric spaces. A curious coincidence in this regard is the following. Let $\tilde{G}/G$ be a symmetric space and $\tilde{\mathfrak{g}} = \mathfrak{g} \oplus \mathbb{V}$ the corresponding Lie algebra decomposition. Then the $\mathfrak{g}$-module $\mathbb{V}$ is co-primary if and only if $\mathfrak{g}$ is non-homologous to zero in $\tilde{\mathfrak{g}}$. Another by-product of our classifications is that any co-primary module has an exterior algebra of skew-invariants.

Having observed that any co-primary representation is a very specific isotropy representation, one may suggest that $Spin_0(\mathbb{V})$ admits a nice description for *all* symmetric spaces. This is really the case, and a transparent formulation can be given for the inner involutory automorphisms. Let $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ be a $\mathbb{Z}_2$-grading of inner type, i.e., $\mathrm{rk}\,\mathfrak{g}_0 = \mathrm{rk}\,\mathfrak{g}$. As the $\mathfrak{g}_0$-module $\mathfrak{g}_1$ has no zero weight, $Spin(\mathfrak{g}_1) = Spin_0(\mathfrak{g}_1)$. Choose a common Cartan subalgebra $\mathfrak{t}$ for $\mathfrak{g}$ and $\mathfrak{g}_0$, and consider the natural inclusion of the Weyl groups $W_0 \subset W$. Although W_0 is not necessary a parabolic subgroup of W, each coset wW_0 contains a unique element of minimal length (see 4.1). Let $W^0 \subset W$ be the set of such elements. Then the irreducible constituents of the $\mathfrak{g}_0$-module $Spin(\mathfrak{g}_1)$ are parameterized by W^0. Namely, $Spin(\mathfrak{g}_1) = \bigoplus_{w \in W^0} \mathbb{V}_{\lambda_w}$, where $\lambda_w = w^{-1}\rho - \rho_0$ is the highest weight, see section 5. Moreover, the weights λ_w ($w \in W^0$) are distinct and hence $Spin(\mathfrak{g}_1)$ is a multiplicity free $\mathfrak{g}_0$-module. It is worth noting that the above expression for $Spin(\mathfrak{g}_1)$ is equivalent to an identity for root systems that seem to have not been observed before. Let Δ be the root system of $(\mathfrak{g}, \mathfrak{t})$ and let $\Delta^+ = \Delta_0^+ \sqcup \Delta_1^+$ be the partition of the set of positive roots corresponding to the sum $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$. In this situation, one can introduce the "cunning" parity $\tau : W \to \{1, -1\}$, determined by Δ_0^+. If $w \in W_0$, then $\tau(w) = (-1)^{l_0(w)}$, where

$l_0(\cdot)$ is the length in W_0 relative to the set of positive roots Δ_0^+. To extend τ to W, one uses the aforementioned subset W^0 (see section 4). Then the identity reads

$$\sum_{w\in W}\tau(w)e^{w\rho}=\prod_{\alpha\in\Delta_0^+}(e^{\alpha/2}-e^{-\alpha/2})\prod_{\mu\in\Delta_1^+}(e^{\mu/2}+e^{-\mu/2})\ .$$

For the outer involutory automorphisms, the final description of $Spin_0(\mathfrak g_1)$ is almost identical to the previous one, see section 6. However, it requires much more preparations and its proof uses the classification of involutory automorphisms. Our arguments suggest that there should exist interesting connections between cohomology of symmetric spaces, twisted affine Kac–Moody algebras, and $Spin(\mathfrak g_1)$.

The description of the highest weights of $Spin(\mathfrak g_1)$ (for all involutions!) shows that these weights are extreme. This also implies the following claim (see sect. 7):

Let $\Phi(\ ,\ )$ be an invariant bilinear form on $\mathfrak g$ and $\Phi(\ ,\ )_0$ its restriction to $\mathfrak g_0$. Let $c_0\in U(\mathfrak g_0)$ be the Casimir element with respect to $\Phi(\ ,\ )_0$. Then c_0 acts scalarly on $Spin(\mathfrak g_1)$; the value is $(\rho,\rho)-(\rho_0,\rho_0)$, where $(\ ,\ )$ is the W-invariant bilinear form on $\mathfrak t^*$ induced by $\Phi(\ ,\ )$.

A similar result holds for the isotropy representation $\mathfrak h\to\mathfrak{so}(\mathfrak m)$ of non-symmetric space G/H, if $\operatorname{rk}\mathfrak h=\operatorname{rk}\mathfrak g$ and one considers the submodule of $Spin(\mathfrak m)$ generated by the extreme weight vectors.

Recently, B. Kostant obtained a series of nice results for $Spin(\mathfrak g)$ [Ko97]. Since the adjoint representation is one of the isotropy representations of symmetric spaces, our results for $Spin(\mathfrak g_1)$ suggest that many parts of Kostant's theory can be generalized to the setting of arbitrary symmetric spaces.

Main notation. $\mathfrak g$ is a reductive Lie algebra with a fixed triangular decomposition: $\mathfrak g=\mathfrak u^+\oplus\mathfrak t\oplus\mathfrak u^-$. All $\mathfrak g$-modules are assumed to be finite-dimensional.

Δ (resp. Δ^+) is the set of roots (resp. positive roots); $\Pi\subset\Delta^+$ is the set of simple roots; $\Pi=\{\alpha_i\}_{i\in I}$ and φ_i is the fundamental weight corresponding to α_i. For simple Lie algebras, we follow the numeration of the simple roots from [VO88] and [On95].

$\mathcal P$ – the lattice of integral weights, $\mathcal P_+$ – the monoid of dominant integral weights.

$W=N_G(\mathfrak t)/Z_G(\mathfrak t)=N_G(\mathfrak t)/T$ – the Weyl group; for $\beta\in\Delta$, s_β is the reflection in W.

$\mathcal P_{\mathbb Q}=\mathcal P\otimes_{\mathbb Z}\mathbb Q\subset\mathfrak t^*$ and $(\ ,\ )$ is the W-invariant positive-definite scalar product in $\mathcal P_{\mathbb Q}$ determined by a non-degenerate invariant bilinear form $\Phi(\ ,\ )$ on $\mathfrak g$.

If $M\subset\mathcal P$ is any finite set of weights, then $|M|=\sum_{m\in M}m$; $\rho:=\frac12|\Delta^+|$.

If $\lambda\in\mathcal P_+$, then $\mathbb V_\lambda$ stands for the irreducible $\mathfrak g$-module with highest weight λ.

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1 Orthogonal $\mathfrak{g}$-modules with an exterior algebra of skew-invariants

Let $\mathbb{V}$ be a $\mathfrak{g}$-module. Study of the algebra $(\mathcal{S}^\bullet\mathbb{V})^{\mathfrak{g}}$ of symmetric (or polynomial) invariants is the subject of a rich and well-developed theory. In contrast, little is known about the algebra $(\wedge^\bullet\mathbb{V})^{\mathfrak{g}}$ of skew-invariants. The skew-symmetric theory has some parallels to the symmetric case, and many interesting differences. We begin with two observations.

(1.1) Put $n = \dim\mathbb{V}$. Suppose $\mathfrak{g} \subset \mathfrak{sl}(\mathbb{V})$, e.g. $\mathfrak{g}$ is semisimple. Then $\wedge^n\mathbb{V}$ is a trivial $\mathfrak{g}$-module. Therefore $\dim(\wedge^\bullet\mathbb{V})^{\mathfrak{g}} \ge 2$, $\wedge^i\mathbb{V}$ and $\wedge^{n-i}\mathbb{V}$ are isomorphic $\mathfrak{g}$-modules, and the Poincaré polynomial of $(\wedge^\bullet\mathbb{V})^{\mathfrak{g}}$ is symmetric.

(1.2) Let $\mathcal{P}(\mathbb{V})$ be the set of all weights of $\mathbb{V}$ relative to $\mathfrak{t} \subset \mathfrak{g}$ and $\mathbb{V} = \oplus_{\mu\in\mathcal{P}(\mathbb{V})}\mathbb{V}^\mu$ the weight decomposition. Set $m(\mu) = \dim\mathbb{V}^\mu$. Recall that the character of $\mathbb{V}$ is the element of the group algebra $\mathbb{Z}[\mathcal{P}]$ defined by $\mathrm{ch}\,\mathbb{V} = \sum_{\mu\in\mathcal{P}(\mathbb{V})} m(\mu)e^\mu$. Then, t being an indeterminate, we have

$$\sum_{i=1}^{n}(\mathrm{ch}\ \wedge^i\mathbb{V})t^i = \prod_{\mu\in\mathcal{P}(\mathbb{V})}(1+te^\mu)^{m(\mu)}\ .$$

In particular, $\mathrm{ch}\,\wedge^\bullet\mathbb{V} = 2^{m(0)}\prod_{\mu\ne 0}(1+e^\mu)^{m(\mu)}$. Obviously, $\prod_{\mu\ne0}(1+e^\mu)^{m(\mu)}$ must be the character of a $\mathfrak{g}$-module, say $\mathbb{W}$. Hence $1 \le \dim(\wedge^\bullet\mathbb{V})^{\mathfrak{g}} = 2^{m(0)}\cdot\dim\mathbb{W}^{\mathfrak{g}}$ and therefore $\dim(\wedge^\bullet\mathbb{V})^{\mathfrak{g}} \ge 2^{m(0)}$.

It is natural to first describe $\mathfrak{g}$-modules, where the algebra of skew-invariants has a simple structure.

Definition. The algebra $(\wedge^\bullet\mathbb{V})^{\mathfrak{g}}$ is said to be *free* (or an *exterior algebra*), if there exists a graded subspace $P \subset (\wedge^\bullet\mathbb{V})^{\mathfrak{g}}$ such that $(\wedge^\bullet\mathbb{V})^{\mathfrak{g}}$ is the exterior algebra over P.

Suppose $\mathfrak{g} \subset \mathfrak{sl}(\mathbb{V})$ and $(\wedge^\bullet\mathbb{V})^{\mathfrak{g}}$ is an exterior algebra. Let $P = \langle p_1,\dots,p_l\rangle$ with $\deg p_i = d_i$. Then $0 \ne p_1\wedge\ldots\wedge p_l$ must be an element of $\wedge^n\mathbb{V}$. Hence $\sum_i d_i = n$. It follows from the definition that all the d_i's must be odd whenever $l > 1$. However, if $l = 1$, then $d_1 = n$ is allowed to be even. In other words, all 2-dimensional algebras of skew-invariants are proclaimed to be exterior.

Example. Let $\mathbb{V} = \mathfrak{g}$. Then $(\wedge^\bullet\mathfrak{g})^{\mathfrak{g}}$ is free. Here $l = \mathrm{rk}\,\mathfrak{g}$ and $d_i = 2m_i + 1$, where $m_1,\dots,m_l$ are the exponents of $\mathfrak{g}$. A purely algebraic proof of this result was given by Koszul [Kos50].

From now on, $\mathbb{V}$ is an orthogonal $\mathfrak{g}$-module, i.e., we are given a representation $\pi : \mathfrak{g} \to \mathfrak{so}(\mathbb{V})$. In particular, $\mathfrak{g} \subset \mathfrak{sl}(\mathbb{V})$.

1.3 Lemma. *Let $\mathbb{V}$ be an irreducible orthogonal $\mathfrak{g}$-module with $\mathbb{V}^{\mathfrak{g}} = 0$. Suppose $(\wedge^\bullet\mathbb{V})^{\mathfrak{g}}$ is free. Then either $(\wedge^4\mathbb{V})^{\mathfrak{g}} = 0$ or $\dim\mathbb{V} = 4$.*

Proof. Since $(\wedge^1\mathbb{V})^{\mathfrak{g}} = (\wedge^2\mathbb{V})^{\mathfrak{g}} = 0$, any nonzero element of $(\wedge^4\mathbb{V})^{\mathfrak{g}}$ is a generator of $(\wedge^\bullet\mathbb{V})^{\mathfrak{g}}$. □

Let $\mu : \mathbb{V}\times\mathbb{V}^* \to \mathfrak{g}^* \simeq \mathfrak{g}$ be the moment mapping associated with the standard symplectic structure on $\mathbb{V}\times\mathbb{V}^* \simeq T^*(\mathbb{V})$. Identifying $\mathbb{V}$ and $\mathbb{V}^*$, one obtains an *anti-commutative* bilinear mapping $\bar\mu : \mathbb{V}\times\mathbb{V} \to \mathfrak{g}$. Using the $\mathfrak{g}$-invariant symmetric bilinear forms $\Phi(\ ,\ )$ and $(\ ,\ )_{\mathbb{V}}$, one may explicitly define $\bar\mu$ by

$$\textbf{(1.4)} \qquad \Phi(\bar\mu(v_1,v_2),g) := (v_2, g{\cdot}v_1)_{\mathbb{V}}\ ,$$

where $v_1, v_2 \in \mathbb{V}$ and $g{\cdot}v_1$ is a shorthand for $\pi(g)v_1$. This $\bar\mu$ yields an anti-commutative multiplication, denoted by $[\ ,\ ]^{\tilde{}}$, in $\mathfrak{g}\oplus\mathbb{V}$:

$$[(g_1,v_1),(g_2,v_2)]^{\tilde{}} := ([g_1,g_2]+\bar\mu(v_1,v_2), g_1{\cdot}v_2 - g_2{\cdot}v_1)\ .$$

The following assertion is stated in [Co72, p. 152], in the context of compact group representations, as the "Cartan-Kostant theorem". It is an easy part of Kostant's characterization of the isotropy representation of compact homogeneous spaces [loc. cit].

1.5 Proposition. *The multiplication $[\ ,\ ]^{\tilde{}}$ satisfies the Jacobi identity if and only if the skew-symmetric $\mathfrak{g}$-invariant 4-form on $\mathbb{V}$*

$$(v_1,v_2,v_3,v_4) \overset{\kappa}{\mapsto} \Phi(\bar\mu(v_1,v_2),\bar\mu(v_3,v_4)) + \Phi(\bar\mu(v_2,v_3),\bar\mu(v_1,v_4)) + \Phi(\bar\mu(v_3,v_1),\bar\mu(v_2,v_4))$$

is identically equal to zero.

Proof. By bilinearity of $[\ ,\ ]^{\tilde{}}$, it suffices to verify the Jacobi identity for 4 sorts of triples: (i) (g_1,g_2,g_3), (ii) (g_1,g_2,v_1), (iii) (g_1,v_1,v_2), (iv) (v_1,v_2,v_3), where $g_i \in \mathfrak{g}$ and $v_i \in \mathbb{V}$. The Jacobi identity is always satisfied for cases (i)–(iii), because, respectively, $\mathfrak{g}$ is a Lie algebra, $\mathbb{V}$ is a $\mathfrak{g}$-module, and $\bar\mu$ is a homomorphism of $\mathfrak{g}$-modules. For $v_1,v_2,v_3 \in \mathbb{V}$, the identity means that

$$\bar\mu(v_1,v_2){\cdot}v_3 + \bar\mu(v_2,v_3){\cdot}v_1 + \bar\mu(v_3,v_1){\cdot}v_2 = 0 \in \mathbb{V}$$

or

$$(\bar\mu(v_1,v_2){\cdot}v_3 + \bar\mu(v_2,v_3){\cdot}v_1 + \bar\mu(v_3,v_1){\cdot}v_2, v_4)_{\mathbb{V}} = 0$$

for any $v_4 \in \mathbb{V}$. Using Eq. (1.4), one rewrites the last equality as the condition that the mapping $\kappa : \mathbb{V}^{\otimes 4} \to \Bbbk$ is zero. It is also easily seen that κ is skew-symmetric and $\mathfrak{g}$-invariant. □

1.6 Corollary. *If $(\wedge^4\mathbb{V})^{\mathfrak{g}} = 0$, then $\mathfrak{g}\oplus\mathbb{V}$, endowed with multiplication $[\ ,\ ]^{\tilde{}}$, is a $\mathbb{Z}_2$-graded Lie algebra.*

Notice that the condition $(\wedge^4\mathbb{V})^{\mathfrak{g}} = 0$ is not necessary for $\mathfrak{g}\oplus\mathbb{V}$ to be a $\mathbb{Z}_2$-graded Lie algebra. We are going to list all irreducible orthogonal $\mathfrak{g}$-modules $\mathbb{V}$ such that $(\wedge^\bullet\mathbb{V})^{\mathfrak{g}}$ is free.

1.7 Theorem. *Let $\mathfrak{g}$ be semisimple and $\mathbb{V}$ a faithful orthogonal irreducible $\mathfrak{g}$-module. Suppose $(\wedge^\bullet\mathbb{V})^{\mathfrak{g}}$ is free. Then either $\mathfrak{g}$ is simple and $\mathbb{V}\simeq\mathfrak{g}$ or $\tilde{\mathfrak{g}} := \mathfrak{g}\oplus\mathbb{V}$ is a $\underline{\text{simple}}$ $\mathbb{Z}_2$-graded Lie algebra.*

Proof. 1. Assume that $\dim \mathbb{V} \neq 4$. By Lemma 1.3 and Corollary 1.6, it follows that $[\ ,\ ]^{\tilde{}}$ makes $\tilde{\mathfrak{g}}$ a $\mathbb{Z}_2$-graded Lie algebra. Let $\mathfrak{a} \subset \tilde{\mathfrak{g}}$ be an ideal. Then $\mathfrak{a} \cap \mathbb{V}$ and $\mathfrak{a} \cap \mathfrak{g}$ are $\mathfrak{g}$-stable spaces.

(i) If $\mathfrak{a} \cap \mathbb{V} = \mathbb{V}$, then $\mathfrak{a}$ also contains $\bar{\mu}(\mathbb{V}, \mathbb{V}) = [\mathbb{V}, \mathbb{V}]^{\tilde{}}$. Since $\mathbb{V}$ is faithful, $\bar{\mu}(\mathbb{V}, \mathbb{V})$ meets all the simple components of $\mathfrak{g}$. Therefore $\mathfrak{a} = \tilde{\mathfrak{g}}$.

(ii) If $\mathfrak{a} \cap \mathfrak{g} \neq 0$, then, since $\mathbb{V}$ is faithful, $(\mathfrak{a} \cap \mathfrak{g}) \cdot \mathbb{V} \neq 0$. That is, $\mathfrak{a} \cap \mathbb{V} \neq 0$ and we are back in part (i).

(iii) If $\mathfrak{a} \cap \mathbb{V} = 0$ and $\mathfrak{a} \cap \mathfrak{g} = 0$, then the $\mathfrak{g}$-module $\mathfrak{a}$ is isomorphic to its projections to both $\mathbb{V}$ and $\mathfrak{g}$. Hence $pr_{\mathfrak{g}}(\mathfrak{a}) \simeq \mathfrak{a} \simeq pr_{\mathbb{V}}(\mathfrak{a}) \simeq \mathbb{V}$. Therefore $pr_{\mathfrak{g}}(\mathfrak{a})$ is a a simple component of $\mathfrak{g}$. As $\mathbb{V}$ is a faithful $\mathfrak{g}$-module, we conclude that $\mathfrak{g} \simeq \mathbb{V}$ and therefore $\mathfrak{g}$ is simple in this case. Here $\tilde{\mathfrak{g}}$ is the sum of two isomorphic ideals, $\tilde{\mathfrak{g}} \simeq \mathfrak{a} \oplus \mathfrak{a}$. The subalgebra $\mathfrak{g}$, which is isomorphic to $\mathfrak{a}$, is the diagonal in $\tilde{\mathfrak{g}}$, and $\mathbb{V} = \{(x, -x) \mid x \in \mathfrak{a}\}$.

2. Assume that $\dim \mathbb{V} = 4$. Then $\mathfrak{g} \subset \mathfrak{so}_4 = \mathfrak{so}(\mathbb{V})$. Obviously, $\mathfrak{so}_4 \oplus \mathbb{V} \simeq \mathfrak{so}_5$, and one easily verifies that $(\wedge^\bullet \mathbb{V})^{\mathfrak{g}}$ is not free for any proper reductive subalgebra $\mathfrak{g}$ of $\mathfrak{so}_4$. □

As is mentioned above, $(\wedge^\bullet \mathfrak{g})^{\mathfrak{g}}$ is free. Thus, all other irreducible orthogonal modules with free algebra of skew-invariants arise in connection with $\mathbb{Z}_2$-gradings of simple Lie algebras. If $\tilde{\mathfrak{g}} = \mathfrak{g} \oplus \mathbb{V}$ is a simple $\mathbb{Z}_2$-graded Lie algebra, then $(\wedge^\bullet \mathbb{V})^{\mathfrak{g}}$ is isomorphic to $H^*(\tilde{G}/G)$, the cohomology ring of the symmetric space $\tilde{G}/G$ [On95, § 9, n.11]. The cases, where $H^*(\tilde{G}/G)$ is an exterior algebra, are well known, see [On95, § 13, Th. 1]. Note however that our interpretation of "exterior algebras" is a bit wider. In case $\dim H^*(\tilde{G}/G) = 2$, the generator is allowed to be of even degree. The resulting classification is presented in Table 1.

Table 1: The irreducible orthogonal representations with free algebra of skew-invariants

$\mathfrak{g}$	$\mathbb{V}$	$\dim P$	Poincaré polynomial of $(\wedge^\bullet \mathbb{V})^{\mathfrak{g}}$
any simple Lie algebra	$\mathfrak{g}$	$\mathrm{rk}\,\mathfrak{g}$	$\prod_{i=1}^{\mathrm{rk}\,\mathfrak{g}}(1+t^{2m_i+1})$
$\mathfrak{sp}_{2n}$ $(n \geq 2)$	$\mathbb{V}_{\varphi_2}$	$n-1$	$(1+t^5)(1+t^9)\dots(1+t^{4n-3})$
$\mathfrak{so}_{2n+1}$ $(n \geq 2)$	$\mathbb{V}_{2\varphi_1}$	n	$(1+t^5)(1+t^9)\dots(1+t^{4n+1})$
$\mathfrak{sl}_2$	$\mathbb{V}_{4\varphi}$	1	$1+t^5$
$\mathfrak{so}_n$ $(n \geq 5)$	$\mathbb{V}_{\varphi_1}$	1	$1+t^n$
$\mathfrak{sl}_2 \oplus \mathfrak{sl}_2$	$\mathbb{V}_{\varphi} \otimes \mathbb{V}_{\varphi'}$	1	$1+t^4$
$\mathfrak{f}_4$	$\mathbb{V}_{\varphi_1}$	2	$(1+t^9)(1+t^{17})$

If $\mathbb{V}$ is a symplectic $\mathfrak{g}$-module, then $(\wedge^2 \mathbb{V})^{\mathfrak{g}} \neq 0$. Thus, $(\wedge^\bullet \mathbb{V})^{\mathfrak{g}}$ cannot be free unless $\dim \mathbb{V} = 2$. If $\mathbb{V}$ is neither orthogonal nor symplectic, then all known instances of free algebras of skew-invariants are those with $\dim(\wedge^\bullet \mathbb{V})^{\mathfrak{g}} = 2$.

2 'Spin' of an orthogonal $\mathfrak{g}$-module and its properties

Let $\mathbb{V}$ be a $\Bbbk$-vector space endowed with a non-degenerate quadratic form Q. Denote by $\mathfrak{so}(\mathbb{V})=\mathfrak{so}_Q(\mathbb{V})$ the respective orthogonal Lie algebra and by $\mathcal{C}_Q(\mathbb{V})$ the Clifford algebra of Q. Let $\mathbb{W},\mathbb{W}'$ be maximal Q-isotropic subspaces of $\mathbb{V}$ and $\mathbb{W}\cap\mathbb{W}'=0$. The following relations are well-known in the theory of Clifford algebras (see e.g. [FH96, § 20.1]) :

(i) $\mathcal{C}_Q(\mathbb{V})\simeq \mathrm{End}\,(\wedge^\bullet\mathbb{W})$, if $\dim\mathbb{V}$ is even,

(ii) $\mathcal{C}_Q(\mathbb{V})\simeq \mathrm{End}\,(\wedge^\bullet\mathbb{W})\oplus\mathrm{End}\,(\wedge^\bullet\mathbb{W}')$, if $\dim\mathbb{V}$ is odd.

As $\wedge^\bullet\mathbb{W}$ (or $\wedge^\bullet\mathbb{W}'$) is the underlying space of the spin representation of $\mathfrak{so}(\mathbb{V})$ (in case (i) this representation is the sum of two half-spin representations), we shall write $Spin(\mathbb{V})$ in place of $\wedge^\bullet\mathbb{W}$. The above relations are thought of as isomorphisms of $\mathfrak{so}(\mathbb{V})$-modules. It is well-known (and easily seen) that $\mathcal{C}_Q(\mathbb{V})$ has an $\mathfrak{so}(\mathbb{V})$-stable filtration such that the associated graded algebra is isomorphic to the exterior algebra of $\mathbb{V}$. Since in both cases $Spin(\mathbb{V})$ is a self-dual module, we obtain the following isomorphisms of $\mathfrak{so}(\mathbb{V})$-modules:

(2.1)
$$\begin{aligned}&\wedge^\bullet\mathbb{V}\simeq Spin(\mathbb{V})\otimes Spin(\mathbb{V}), \ \text{if } \dim\mathbb{V} \text{ is even},\\ &\wedge^\bullet\mathbb{V}\simeq 2(Spin(\mathbb{V})\otimes Spin(\mathbb{V})), \ \text{if } \dim\mathbb{V} \text{ is odd}.\end{aligned}$$

Let $\mathfrak{g}$ be a reductive Lie algebra and $\pi:\mathfrak{g}\to\mathfrak{so}(\mathbb{V})$ an orthogonal representation. Using π, one may regard $Spin(\mathbb{V})$ as $\mathfrak{g}$-module. In this way, we obtain a mapping from the set of orthogonal $\mathfrak{g}$-modules to a set of $\mathfrak{g}$-modules: $\mathbb{V}\mapsto Spin(\mathbb{V})$. Of course, the $\mathfrak{g}$-modules of the form $Spin(\mathbb{V})$ must satisfy some constraints; e.g. $\dim Spin(\mathbb{V})$ is a power of 2. Equations (2.1), which can be treated as isomorphisms of $\mathfrak{g}$-modules, suggest that '*Spin*' could be used for better understanding of $\mathfrak{g}$-module structure of the exterior algebra of an orthogonal module.

The point of departure for our considerations is a simple formula for the character of the $\mathfrak{g}$-module $Spin(\mathbb{V})$. Fix some notation, which applies to arbitrary $\mathfrak{g}$-modules (i.e. not necessarily orthogonal ones). Let $\mathcal{P}(\mathbb{V})$ (resp. $\Delta(\mathbb{V})$) denote the set of all (resp. all nonzero) weights of $\mathbb{V}$. For instance, $\Delta(\mathfrak{g})=\Delta$. For $\mu\in\mathcal{P}(\mathbb{V})$, $\mathbb{V}^\mu$ is the corresponding weight space and $m(\mu)=\dim\mathbb{V}^\mu$. If $\mathbb{V}=\mathbb{V}_\lambda$ is irreducible, then the multiplicity is denoted by $m_\lambda(\mu)$. Recall that $\mathbb{V}$ is self-dual if and only if $\Delta(\mathbb{V})=-\Delta(\mathbb{V})$ and $m(\mu)=m(-\mu)$ for all $\mu\in\Delta(\mathbb{V})$.

Given an orthogonal $\mathfrak{g}$-module $\mathbb{V}$, let $\Delta(\mathbb{V})^+$ denote an arbitrary subset such that $\Delta(\mathbb{V})=\Delta(\mathbb{V})^+\sqcup(-\Delta(\mathbb{V})^+)$.

2.2 Lemma. $\mathrm{ch}\, Spin(\mathbb{V})=2^{[m(0)/2]}\displaystyle\prod_{\mu\in\Delta(\mathbb{V})^+}(e^{\mu/2}+e^{-\mu/2})^{m(\mu)}$.

Proof. Using (1.2), one obtains $\mathrm{ch}\,(\wedge^\bullet\mathbb{V})=\displaystyle\prod_{\mu\in\mathcal{P}(\mathbb{V})}(1+e^\mu)^{m(\mu)}=$

$$2^{m(0)}\prod_{\mu\in\Delta(\mathbb{V})^+}[(1+e^\mu)(1+e^{-\mu})]^{m(\mu)}=2^{m(0)}\prod_{\mu\in\Delta(\mathbb{V})^+}(e^{\mu/2}+e^{-\mu/2})^{2m(\mu)}\ .$$

Since $\dim\mathbb{V}-m(0)$ is even, comparing with Eq. (2.1) completes the proof. □

Roughly speaking, Eq. (2.1) asserts that a "square root" of $\wedge^\bullet\mathbb{V}$ is again a $\mathfrak{g}$-module whenever $\mathbb{V}$ is orthogonal. Lemma 2.2 gives a precise form for this. Notice that the transformation from the proof of Lemma can be performed for any self-dual $\mathfrak{g}$-module $\mathbb{V}$. But the respective "square root" does not yield in general the character of a $\mathfrak{g}$-module.

It is convenient to omit the numerical factor in $\mathrm{ch}\, Spin(\mathbb{V})$. The remaining expression is still the character of a $\mathfrak{g}$-module. This module is said to be the *reduced Spin* of $\mathbb{V}$ and we write $Spin_0(\mathbb{V})$ for it:

$$\textbf{(2.3)} \qquad \mathrm{ch}\, Spin_0(\mathbb{V}) = \prod_{\mu\in\Delta(\mathbb{V})^+} (e^{\mu/2}+e^{-\mu/2})^{m(\mu)} \ .$$

Several easy properties of $Spin_0$ are summarized below.

2.4 Proposition. *Let* $\mathbb{V} = \mathbb{V}^{(1)}$ *and* $\mathbb{V}^{(2)}$ *be orthogonal* $\mathfrak{g}$*-modules. Then*

(i) $\dim Spin_0(\mathbb{V}) = 2^{(\dim\mathbb{V}-m(0))/2}$;

(ii) $\wedge^\bullet\mathbb{V} \simeq 2^{m(0)}\cdot Spin_0(\mathbb{V})^{\otimes 2}$;

(iii) $Spin(\mathbb{V}) = Spin_0(\mathbb{V})$ *if and only if* $m(0)\le 1$;

(iv) $Spin_0(\mathbb{V}^{(1)}\oplus\mathbb{V}^{(2)}) \simeq Spin_0(\mathbb{V}^{(1)})\otimes Spin_0(\mathbb{V}^{(2)})$;

(v) $Spin_0(\mathbb{V})$ *is a self-dual* $\mathfrak{g}$*-module.*

Proof. This immediately follows from (2.1), (2.2), and (2.3). □

2.5 Examples. 1. Our consideration of $Spin(\mathbb{V})$ was motivated by the following observation of Kostant, see [Ko61, p. 358] and [Ko97].
Suppose $\mathbb{V}=\mathfrak{g}$ and $\pi = \mathrm{ad}$ is the adjoint representation. Then $\wedge^\bullet\mathfrak{g}\simeq 2^{\mathrm{rk}\,\mathfrak{g}}(\mathbb{V}_\rho\otimes\mathbb{V}_\rho)$. This means that $Spin(\mathfrak{g}) = 2^{[\mathrm{rk}\,\mathfrak{g}/2]}\,\mathbb{V}_\rho$ and $Spin_0(\mathfrak{g}) = \mathbb{V}_\rho$.

2. $\mathfrak{g}=\mathfrak{sl}_2$. We shall write R_d in place of $\mathbb{V}_{d\varphi_1}$. Recall that $R_2=\mathfrak{g}$, $R_d = \mathcal{S}^d R_1$, and R_d is orthogonal if and only if d is even. Let $\mathcal{P}(R_1)=\{\varepsilon,-\varepsilon\}$. Applying Prop. 2.4, we obtain $Spin_0(R_{2d}) = Spin\, R_{2d}$ and $\mathrm{ch}\, Spin\, R_{2d} = \prod_{k=1}^d (e^{k\varepsilon}+e^{-k\varepsilon})$. It is not hard to compute this character for small values of d. Here are first few formulas: $Spin\, R_2 {=} R_1$, $Spin\, R_4{=}R_3$, $Spin\, R_6{=}R_6{+}R_0$, $Spin\, R_8{=}R_{10}{+}R_4$, $Spin\, R_{10}{=}R_{15}{+}R_9{+}R_5$. It is easily seen that if $Spin\, R_{2d} = \oplus_{i\in I} R_{m_i}$, then $Spin\, R_{2(d+1)} \supset \oplus_{i\in I} R_{m_i+d+1}$. Therefore the number of summands is a nondecreasing function of d.

3. Let $\mathbb{W}$ be an arbitrary $\mathfrak{g}$-module. We may regard $\mathbb{V} := \mathbb{W}\oplus\mathbb{W}^*$ as orthogonal $\mathfrak{g}$-module equipped with the quadratic form $Q((w,w^*)) := \langle w|w^*\rangle$, where $(w,w^*)\in\mathbb{V}$ and $\langle\ |\ \rangle$ is the canonical pairing of $\mathbb{W}$ and $\mathbb{W}^*$. Assuming for simplicity that the weights of $\mathbb{W}$ and $\mathbb{W}^*$ are distinct, we see that $\mathcal{P}(\mathbb{W})$ can be taken as $\Delta(\mathbb{V})^+$. Therefore $\mathrm{ch}\, Spin(\mathbb{V}) = \prod_{\mu\in\mathcal{P}(\mathbb{W})} (e^{\mu/2}+e^{-\mu/2})^{m(\mu)} = e^{-\nu}\prod_{\mu\in\mathcal{P}(\mathbb{W})}(1+e^{\mu})^{m(\mu)}$, where $\nu = \frac{1}{2}\sum_{\mu\in\mathcal{P}(\mathbb{W})} m(\mu)\mu$. Whence

$$Spin(\mathbb{W}\oplus\mathbb{W}^*) \simeq \Bbbk_{-\nu}\otimes\wedge^\bullet\mathbb{W}\simeq\Bbbk_\nu\otimes\wedge^\bullet\mathbb{W}^* \ ,$$

where $\Bbbk_{-\nu}$ is 1-dimensional $\mathfrak{g}$-module with character $-\nu$. Obviously, $\nu = 0$ if and only if $\mathfrak{g} \subset \mathfrak{sl}(\mathbb{W})$, e.g. $\mathfrak{g}$ is semisimple. It is not hard to verify that the above formula for $Spin(\mathbb{W} \oplus \mathbb{W}^*)$ remains true for all $\mathbb{W}$.

Definition. An orthogonal $\mathfrak{g}$-module $\mathbb{V}$ is said to be *co-primary*, if $Spin_0(\mathbb{V})$ is irreducible.

In this case $Spin(\mathbb{V})$ is a primary $\mathfrak{g}$-module. We are going to list all co-primary modules for the semisimple Lie algebras. At the moment, the following examples of such modules are known: $\mathbb{V} = \mathfrak{g}$, $\mathfrak{g}$ simple; $\mathbb{V} = R_4$, $\mathfrak{g} = \mathfrak{sl}_2$. As a step towards a classification, we describe another series of co-primary modules.

Let $\mathfrak{g}$ be a simple Lie algebra having two root lengths. We use subscripts 's' and 'l' to mark objects related to short and long roots, respectively. For instance, Δ_s is the set of short roots, $\Delta = \Delta_s \sqcup \Delta_l$, and $\Pi_s = \Pi \cap \Delta_s$. Set $\rho_s = \frac{1}{2}|\Delta_s^+|$ and $\rho_l = \frac{1}{2}|\Delta_l^+|$. As usual, $s_\alpha \in W$ is the reflection corresponding to $\alpha \in \Delta$ and $s_i := s_{\alpha_i}$.

2.6 Lemma. $\rho_s = \sum_{\alpha_i \in \Pi_s} \varphi_i$.

Proof. It is easily seen that $s_i(\rho_s) = \begin{cases} \rho_s, & \text{if } \alpha_i \in \Pi_l \\ \rho_s - \alpha_i, & \text{if } \alpha_i \in \Pi_s \end{cases}$. □

Let $\theta \in \Delta^+$ be the highest root and θ_s the short dominant root. Recall that $\Delta_l = W{\cdot}\theta$, $\Delta_s = W{\cdot}\theta_s$, and $\|\theta\|^2/\|\theta_s\|^2 = 2$ or 3. If $\mu \in \Delta$, then $\mu^\vee := 2\mu/\|\mu\|^2$.

2.7 Lemma. *Suppose $\|\theta\|^2/\|\theta_s\|^2 = 2$ and $\mu \in \Delta_s$. Then $(\rho + \rho_s, \mu^\vee)$ is even.*

Proof. Let $\mu = \sum_{\alpha_i \in \Pi_s} n_i\alpha_i + \sum_{\alpha_j \in \Pi_l} m_j\alpha_j$. Then $\mu^\vee = \sum_{\alpha_i \in \Pi_s} n_i\alpha_i^\vee + 2\sum_{\alpha_j \in \Pi_l} m_j\alpha_j^\vee$. Therefore $(\rho + \rho_s, \mu^\vee) = (2\rho_s + \rho_l, \mu^\vee) = (2\rho_s, \sum_{\alpha_i \in \Pi_s} n_i\alpha_i^\vee) + (\rho_l, 2\sum_{\alpha_j \in \Pi_l} m_j\alpha_j^\vee) = 2(\sum_i n_i + \sum_j m_j)$. □

The following assertion can be proved using classification, but we give a unified proof.

2.8 Proposition.

(i) $\dim \mathbb{V}_{\theta_s} = (h+1)m_{\theta_s}(0)$, *where h is the Coxeter number of $\mathfrak{g}$;*

(ii) $m_{\theta_s}(0) = \#\Pi_s$.

Proof. (i) It is clear that $\mathcal{P}(\mathbb{V}_{\theta_s}) = \{0\} \cup \Delta_s$. Moreover, $m_{\theta_s}(\alpha) = 1$ for all $\alpha \in \Delta_s$. Applying Freudenthal's multiplicity formula to $m_{\theta_s}(0)$, we obtain

$$(\theta_s + 2\rho, \theta_s)m_{\theta_s}(0) = 2\sum_{\alpha\in\Delta^+}\sum_{t\ge 1} m_{\theta_s}(t\alpha)(t\alpha, \alpha) = 2\sum_{\alpha\in\Delta_s^+} m_{\theta_s}(\alpha)(\alpha,\alpha) = 2\sum_{\alpha\in\Delta_s^+}(\alpha,\alpha)\ .$$

Whence

$$(1 + (\rho, \theta_s^\vee))m_{\theta_s}(0) = \#\Delta_s = \dim \mathbb{V}_{\theta_s} - m_{\theta_s}(0)\ .$$

As $\theta_s^\vee$ is the highest root in the dual root system $\Delta^\vee$, we have $(\rho, \theta_s^\vee) = h - 1$.

(ii) By part (i), we have $m_{\theta_s}(0) = \dfrac{\dim \mathbb{V}_{\theta_s} - m_{\theta_s}(0)}{h} = \dfrac{\#\Delta_s}{h}$. Let $c \in W$ be a Coxeter element associated with Π. It is known that each orbit of c in Δ has cardinality h and contains a unique simple root, see [BOU, ch.VI, § 1, Prop. 33]. Hence $\#\Delta_s = h(\#\Pi_s)$. $\square$

Some authors call $\mathbb{V}_{\theta_s}$ the *little adjoint module*. To a great extent, properties of $\mathbb{V}_{\theta_s}$ are similar with properties of $\mathfrak{g}$.

2.9 Theorem. *Suppose* $\|\theta\|^2/\|\theta_s\|^2 = 2$. *Then* $\wedge^\bullet \mathbb{V}_{\theta_s} \simeq 2^{\#\Pi_s} \cdot (\mathbb{V}_{\rho_s} \otimes \mathbb{V}_{\rho_s})$.

Proof. By Proposition 2.8, we have $\mathrm{ch}\, \mathbb{V}_{\theta_s} = \#\Pi_s + \sum_{\alpha\in\Delta_s} e^\alpha$. Therefore

$$\mathrm{ch}\, \wedge^\bullet \mathbb{V}_{\theta_s} = 2^{\#\Pi_s} \prod_{\alpha\in\Delta_s} (1+e^\alpha) = 2^{\#\Pi_s} \prod_{\alpha\in\Delta_s^+} (e^{\alpha/2} + e^{-\alpha/2})^2 .$$

Thus, the statement of theorem is equivalent to that

$$\textbf{(2.10)} \qquad \mathrm{ch}\, \mathbb{V}_{\rho_s} = \prod_{\alpha\in\Delta_s^+} (e^{\alpha/2} + e^{-\alpha/2}) = e^{\rho_s} \prod_{\alpha\in\Delta_s^+} (1+e^{-\alpha}).$$

By Weyl's character formula

$$\textbf{(2.11)} \qquad \mathrm{ch}\, \mathbb{V}_{\rho_s} = \frac{\sum_{w\in W} \varepsilon(w) e^{w(\rho+\rho_s)}}{\sum_{w\in W} \varepsilon(w) e^{w\rho}} = \frac{\sum_{w\in W} \varepsilon(w) e^{w(\rho+\rho_s)}}{\prod_{\alpha\in\Delta^+} (e^{\alpha/2} - e^{-\alpha/2})} .$$

Here $\varepsilon(w) = (-1)^{l(w)}$, where $l(w)$ is the length of w with respect to Δ^+. Take $\alpha \in \Delta_s^+$. We are going to prove that $1+e^{-\alpha}$ divides $\mathrm{ch}\, \mathbb{V}_{\rho_s}$ in $\mathbb{Z}[\mathcal{P}]$. Since $\mathrm{ch}\, \mathbb{V}_{\rho_s}$ is W-invariant, it is enough to consider the case in which α is simple, i.e., $\alpha \in \Pi_s$. Actually, we shall prove that $1+e^{-\alpha}$ divides the numerator in Eq. (2.11). For this, we show how to group together the summands of the numerator. Let $W^\alpha = \{w \in W \mid w^{-1}\alpha \in \Delta^+\}$. Then W is the disjoint union of pairs $\{s_\alpha w, w\}$ $(w \in W^\alpha)$. Consider the corresponding pairs of summands in the numerator of (2.11). Since $\alpha \in \Pi_s$, we have $\varepsilon(s_\alpha w) = -\varepsilon(w)$ and

$$\varepsilon(w) e^{w(\rho+\rho_s)} + \varepsilon(s_\alpha w) e^{s_\alpha w(\rho+\rho_s)} = \varepsilon(w) e^{w(\rho+\rho_s)} (1 - e^{-n\alpha}) ,$$

where $n = (w(\rho+\rho_s), \alpha^\vee) = (\rho+\rho_s, (w^{-1}\alpha)^\vee)$. By the definition of W^α, n is positive. The divisibility will follow from the fact that n is even. But this is just Lemma 2.7.

Since $\mathbb{Z}[\mathcal{P}]$ is factorial and the factors $1+e^{-\alpha}$ $(\alpha \in \Delta_s^+)$ are coprime (see [BOU, ch. VI, § 3, Lemma 1]), $e^{\rho_s} \prod_{\alpha\in\Delta_s^+} (1+e^{-\alpha})$ divides $\mathrm{ch}\, \mathbb{V}_{\rho_s}$. The quotient is a W-invariant element of $\mathbb{Z}[\mathcal{P}]$. Comparing the maximal terms in both expressions, we see that the quotient must be equal to 1. $\square$

2.12 Corollary. *1.* $Spin_0(\mathbb{V}_{\theta_s}) = \mathbb{V}_{\rho_s}$ *;*
2. $\dim(\wedge^\bullet \mathbb{V}_{\theta_s})^{\mathfrak{g}} = 2^{\#\Pi_s}$.

2.13 Examples. To realize the scope of Theorem 2.9, we look at all simple Lie algebras with two root lengths.

1. $\mathfrak{g}=\mathfrak{sp}_{2n}$. Here $\theta=2\varphi_1$, $\theta_s=\varphi_2$, and $\rho_s=\varphi_1+\ldots+\varphi_{n-1}$. Thus

$$\wedge^\bullet\mathbb{V}_{\varphi_2}=2^{n-1}{\cdot}(\mathbb{V}_{\varphi_1+\ldots+\varphi_{n-1}})^{\otimes 2}\quad\text{and}\quad Spin_0(\mathbb{V}_{\varphi_2})=\mathbb{V}_{\varphi_1+\ldots+\varphi_{n-1}}\ .$$

2. $\mathfrak{g}=\mathfrak{f}_4$. Here $\theta=\varphi_4$, $\theta_s=\varphi_1$, and $\rho_s=\varphi_1+\varphi_2$. Thus

$$\wedge^\bullet\mathbb{V}_{\varphi_1}=4(\mathbb{V}_{\varphi_1+\varphi_2})^{\otimes 2}\quad\text{and}\quad Spin_0(\mathbb{V}_{\varphi_1})=\mathbb{V}_{\varphi_1+\varphi_2}\ .$$

3. $\mathfrak{g}=\mathfrak{so}_{2n+1}$. Here $\theta=\varphi_2$, $\theta_s=\varphi_1$, and $\rho_s=\varphi_n$. In this case $Spin_0(\mathbb{V}_{\varphi_1})=Spin(\mathbb{V}_{\varphi_1})=\mathbb{V}_{\varphi_n}$ and the formula of Theorem 2.9 is nothing but the second equality in Eq. (2.1). Hence the theorem also yields another approach to defining 'Spin' of an orthogonal representation.

4. $\mathfrak{g}=\mathfrak{g}_2$. Here $\|\theta\|^2/\|\theta_s\|^2=3$ and Theorem 2.9 does not apply. In this case $\rho_s=\theta_s=\varphi_1$ and $\theta=\varphi_2$. An explicit (easy) computation with characters shows that

$\wedge^\bullet\mathbb{V}_{\varphi_1}=2{\cdot}(\mathbb{V}_{\varphi_1}\oplus\mathbb{1})^{\otimes 2}$, i.e., $Spin(\mathbb{V}_{\varphi_1})=\mathbb{V}_{\varphi_1}\oplus\mathbb{1}$. Hence $\mathbb{V}_{\theta_s}$ is not co-primary.

Here $\mathbb{1}$ stands for the trivial 1-dimensional module.

(2.14) Another proof of Theorem 2.9. Making use of Weyl's character formula, we interpret Eq. (2.10) as Weyl's denominator identity for the dual root system.
Recall that $\Delta=\Delta_l\sqcup\Delta_s$ and we assume that $\|\theta\|^2/\|\theta_s\|^2=2$. The dual root system is therefore isomorphic to $\tilde\Delta:=\Delta_l\sqcup 2\Delta_s$. Here $(\tilde\Delta)_l=2\Delta_s$ and $(\tilde\Delta)_s=\Delta_l$. Since $\widetilde{W}\simeq W$, Weyl's denominator identity for $\tilde\Delta$ reads

$$\sum_{w\in W}\varepsilon(w)e^{w\tilde\rho}=\prod_{\alpha\in\tilde\Delta^+}(e^{\alpha/2}-e^{-\alpha/2})\ .$$

We have $\tilde\rho=\rho+\rho_s$ on the left hand side and

$$\prod_{\alpha\in\Delta_l^+}(e^{\alpha/2}-e^{-\alpha/2}){\cdot}\prod_{\mu\in\Delta_s^+}(e^{\mu}-e^{-\mu})=\prod_{\alpha\in\Delta^+}(e^{\alpha/2}-e^{-\alpha/2})\prod_{\alpha\in\Delta_s^+}(e^{\mu/2}+e^{-\mu/2})$$

on the right hand side. Hence dividing Weyl's identity by $\prod_{\alpha\in\Delta^+}(e^{\alpha/2}-e^{-\alpha/2})$ yields $\mathrm{ch}\,\mathbb{V}_{\rho_s}=\prod_{\mu\in\Delta_s^+}(e^{\mu/2}+e^{-\mu/2})$.

Remark. The previous argument suggests a proper analogue of (2.10) for the exceptional Lie algebra $\mathfrak{g}_2$. Here the dual root system is isomorphic to $\tilde\Delta:=\Delta_l\sqcup 3\Delta_s$ and a similar transformation proves that $\mathrm{ch}\,\mathbb{V}_{2\rho_s}=\prod_{\mu\in\Delta_s^+}(e^{\mu}+1+e^{-\mu})$.

3 Classification of co-primary $\mathfrak{g}$-modules

In this section, $\mathfrak{g}$ is a semisimple Lie algebra and $\mathbb{V}$ an orthogonal $\mathfrak{g}$-module. From Eq. (2.3) it is clear that $\mathbb{V}^{\mathfrak{g}}$ has no affect on $Spin_0(\mathbb{V})$. We may therefore assume that $\mathbb{V}^{\mathfrak{g}}=0$.

3.1 Proposition. *Suppose $\mathbb{V}$ is co-primary. Then there exist decompositions $\mathfrak{g}=\mathfrak{g}_1\oplus\ldots\oplus\mathfrak{g}_s$, $\mathbb{V}=\mathbb{V}_1\oplus\ldots\oplus\mathbb{V}_s$ such that*

(i) *Each* $\mathfrak{g}_i$ *is a (semisimple) ideal of* $\mathfrak{g}$,

(ii) $\mathfrak{g}_i$ *acts trivially on* $\mathbb{V}_j$ $(i \neq j)$,

(iii) $\mathbb{V}_i$ *is an irreducible orthogonal co-primary* $\mathfrak{g}_i$*-module.*

Proof. Assume that $\mathbb{V} = \mathbb{V}_1 \oplus \mathbb{V}_2$, where $\mathbb{V}_1$ and $\mathbb{V}_2$ are orthogonal $\mathfrak{g}$-modules. It follows from the assumptions and Proposition 2.4(iv) that the $\mathfrak{g}$-module $Spin_0(\mathbb{V}_1) \otimes Spin_0(\mathbb{V}_2)$ is irreducible. Since both factors are non-trivial, the only possibility for this is that $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$, where $\mathfrak{g}_i$ acts trivially on $\mathbb{V}_j$ $(i \neq j)$ and $\mathbb{V}_i$ is a co-primary $\mathfrak{g}_i$-module $(i = 1, 2)$. Repeating this procedure, we obtain a decomposition satisfying (i) and (ii), where each $\mathbb{V}_i$ is orthogonal co-primary and is not a sum of two proper orthogonal $\mathfrak{g}_i$-submodules. Then either $\mathbb{V}_i$ is irreducible or $\mathbb{V}_i = \mathbb{W}_i \oplus \mathbb{W}_i^*$, where $\mathbb{W}_i$ is already irreducible. In the second case, we have $Spin(\mathbb{V}_i) \simeq \wedge^\bullet \mathbb{W}_i$ (see Example 2.5(3)). It is easily seen that the $\mathfrak{g}_i$-module $\wedge^\bullet \mathbb{W}_i$ is never primary, i.e., $Spin_0(\mathbb{V}_i)$ can not be irreducible here. □

Whenever $(\mathfrak{g}, \mathbb{V})$ admits a decomposition satisfying conditions (i) and (ii) of the Proposition, this will be denoted by $(\mathfrak{g}, \mathbb{V}) = (\mathfrak{g}_1, \mathbb{V}_1) \oplus \ldots \oplus (\mathfrak{g}_s, \mathbb{V}_s)$.
Notice that if each $\mathbb{V}_i$ is irreducible, then all the summands in the above decomposition are uniquely determined.

3.2 Lemma. *If* $\mathbb{V}_\lambda$ *is an irreducible co-primary* $\mathfrak{g}$*-module, then* $m_\lambda(0) \neq 0$.

Proof. If $m_\lambda(0) = 0$, then $\wedge^\bullet \mathbb{V}_\lambda \simeq Spin_0(\mathbb{V}_\lambda)^{\otimes 2}$, see 2.4(ii). Since $\dim(\wedge^\bullet \mathbb{V}_\lambda)^{\mathfrak{g}} \geq 2$, the Schur lemma shows that $Spin_0(\mathbb{V}_\lambda)$ cannot be irreducible. □

It follows from the above two assertions that $\mathcal{P}(\mathbb{V})$ lies in the root lattice whenever $\mathbb{V}$ is co-primary.

Let us present an explicit way for finding some irreducible constituents of $Spin_0(\mathbb{V})$. To write an expression for $\mathrm{ch}\, Spin_0(\mathbb{V})$ in (2.3), we exploited an arbitrary 'half' $\Delta(\mathbb{V})^+$ of $\Delta(\mathbb{V})$. However a clever choice of $\Delta(\mathbb{V})^+$ will provide us with a maximal term in $\mathrm{ch}\, Spin_0(\mathbb{V})$ and hence with a highest weight. Take $\nu \in \mathcal{P}_+$ such that $(\nu, \mu) \neq 0$ for all $\mu \in \Delta(\mathbb{V})$. Put $\Delta(\mathbb{V})^+_\nu = \{\mu \in \Delta(\mathbb{V}) \mid (\mu, \nu) > 0\}$. A subset of such form is said to be a *dominant half* of $\Delta(\mathbb{V})$. Set $\Lambda_\nu := \frac{1}{2}\sum_\mu m(\mu)\mu$, where μ ranges over $\Delta(\mathbb{V})^+_\nu$.

3.3 Lemma. Λ_ν *is a highest weight of* $Spin_0(\mathbb{V})$.

Proof. We show that Λ_ν is dominant and it is a maximal element in $\mathcal{P}(Spin_0(\mathbb{V}))$. Note that the first part is not tautological. We exploit formula 2.3 with $\Delta(\mathbb{V})^+_\nu$:

$$\mathrm{ch}\, Spin_0(\mathbb{V}) = \prod_{\mu \in \Delta(\mathbb{V})^+_\nu} (e^{\mu/2} + e^{-\mu/2})^{m(\mu)} .$$

This shows that e^{Λ_ν} occurs in $\mathrm{ch}\, Spin_0(\mathbb{V})$ with coefficient 1, $(\nu, \Lambda_\nu) = \max\limits_{\mu \in \mathcal{P}(Spin_0(\mathbb{V}))} (\nu, \mu)$, and Λ_ν is the unique element of $\mathcal{P}(Spin_0(\mathbb{V}))$, where the maximal value is attained. Let

Λ'_ν be the dominant representative in $W\cdot\Lambda_\nu$. Then $\Lambda'_\nu \in \mathcal{P}(Spin_0(\mathbb{V}))$ and $\Lambda'_\nu - \Lambda_\nu = \sum_{\alpha_i\in\Pi} n_i\alpha_i$ with $n_i \geq 0$. Therefore $(\nu, \Lambda'_\nu) \geq (\nu, \Lambda_\nu)$ and hence $\Lambda'_\nu = \Lambda_\nu$. $\square$

(3.4) The highest weights of $Spin_0(\mathbb{V})$ of the form Λ_ν are said to be *extreme*. It is easy to describe all dominant halfs of $\Delta(\mathbb{V})$ and hence all extreme weights of $Spin_0(\mathbb{V})$. Consider the Weyl chamber $C := \mathbb{Q}_+\mathcal{P}_+ \subset \mathcal{P}_\mathbb{Q}$ and its interior C^o. Let H_μ denote the hyperplane in $\mathcal{P}_\mathbb{Q}$ orthogonal to $\mu \in \mathcal{P}$. Recall that C^o is the connected component[2] of $\mathcal{P}_\mathbb{Q} \setminus \cup_{\gamma\in\Delta} H_\gamma$, containing dominant weights. Then the hyperplanes H_μ $(\mu \in \Delta(\mathbb{V}))$ cut C in smaller chambers. When ν varies inside of such a 'small' chamber the corresponding extreme weight does not change. We thus obtain a bijection

$$\{\text{extreme weights of } Spin(\mathbb{V})\} \leftrightarrow \{\text{connected components of } C^o \setminus \bigcup_{\mu\in\Delta(\mathbb{V})} H_\mu\} .$$

In particular, $Spin_0(\mathbb{V})$ has a unique extreme weight if and only if $\Delta(\mathbb{V})$ has a unique dominant half if and only if none of the hyperplanes H_μ cuts C^o.

3.5 Lemma. *Suppose $\Delta(\mathbb{V})$ lies in the root lattice. Then:*

none of the hyperplanes H_μ $(\mu \in \Delta(\mathbb{V}))$ cuts C^o $\iff$ $\Delta(\mathbb{V}) \subset \cup_{\alpha\in\Delta}\mathbb{Z}\alpha$.

Proof. "$\Leftarrow$" This is obvious.

"$\Rightarrow$" Assume that $M := \Delta(\mathbb{V}) \setminus \cup_{\alpha\in\Delta}\mathbb{Z}\alpha \neq \varnothing$. Let $\mu \in M \cap \mathcal{P}_+$ be an element closest to 0. Write μ as sum of positive roots with positive integral coefficients $\mu = \sum_{i=1}^d k_i\gamma_i$ $(\gamma_i \neq \gamma_j)$ and so that $\sum_i k_i$ is minimal over all such presentations. Then $\gamma_i + \gamma_j$ is not a root, i.e., $(\gamma_i, \gamma_j) \geq 0$. Therefore $(\mu, \gamma_1) > 0$ and hence $\mu - \gamma_1 \in \Delta(\mathbb{V})$. As $\|\mu - \gamma_1\| < \|\mu\|$, we obtain $\mu - \gamma_1 \in \cup_{\alpha\in\Delta^+}\mathbb{N}\alpha$. Thus, $k_1 = 1$, $d = 2$ and, by symmetry, $\mu = \gamma_1 + \gamma_2$. Since $(\gamma_1, \gamma_2) \geq 0$ and $\gamma_1 + \gamma_2$ is not a multiple of a root, it is easily seen that $(\gamma_1, -\gamma_2)$ is a basis of the root system $\Delta \cap (\mathbb{Q}\gamma_1 + \mathbb{Q}\gamma_2)$. Therefore $(\gamma_1, -\gamma_2)$ is W-conjugate to a pair of simple roots (α_i, α_j) (see [BOU, ch. VI, § 1,Prop. 24]). Thus, $\alpha_i - \alpha_j \in \Delta(\mathbb{V})$ and $H_{\alpha_i - \alpha_j}$ cuts C^o. $\square$

3.6 Proposition. *Let $\mathbb{V}$ be a co-primary faithful irreducible $\mathfrak{g}$-module. Then*

$\Delta(\mathbb{V}) \subset \cup_{\alpha\in\Delta}\mathbb{Z}\alpha$ *and* $\mathfrak{g}$ *is simple.*

Proof. By Lemma 3.2, $\Delta(\mathbb{V})$ lies in the root lattice. Therefore the first claim readily follows from (3.4) and Lemma 3.5. Assume that $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$ is a sum of two ideals. Then $\mathbb{V} = \mathbb{V}_1 \otimes \mathbb{V}_2$, where $\mathbb{V}_i$ is a non-trivial $\mathfrak{g}_i$-module. Obviously, if $\mu_i \in \Delta(\mathbb{V}_i)$ $(i = 1, 2)$, then $\mu_1 + \mu_2 \in \Delta(\mathbb{V})$ and it is not a multiple of a root of $\mathfrak{g}$. $\square$

Now, we are ready to state a classification.

3.7 Theorem. (i) *Let $\mathfrak{g}$ be semisimple and $\mathbb{V}$ a faithful orthogonal $\mathfrak{g}$-module with $\mathbb{V}^\mathfrak{g} = 0$. Suppose $\mathbb{V}$ is co-primary. Then*

$$(\mathfrak{g}, \mathbb{V}) = (\mathfrak{g}_1, \mathbb{V}_1) \oplus \ldots \oplus (\mathfrak{g}_s, \mathbb{V}_s) ,$$

[2] Strictly speaking, use of the term "connected component" is correct only for the *real* vector space $\mathcal{P}_\mathbb{R}$.

where each $\mathfrak{g}_i$ is simple and $\mathbb{V}_i$ is irreducible and co-primary. Each weight of $\mathbb{V}$ is a multiple of a root of $\mathfrak{g}$.

(ii) *If $\mathfrak{g}$ is simple and $\mathbb{V} = \mathbb{V}_\lambda$ is irreducible and co-primary, then the pair $(\mathfrak{g}, \lambda)$ is one of the following:*

(a) $\mathfrak{g}$ is any and $\lambda = \theta$.

(b) $\mathfrak{g} \in \{\mathfrak{so}_{2n+1}, \mathfrak{sp}_{2n}, \mathfrak{f}_4\}$ and $\lambda = \theta_s$.

(c) $\mathfrak{g} = \mathfrak{so}_{2n+1}$, $\lambda = 2\theta_s = 2\varphi_1$ $(n \ge 2)$.

(d) $\mathfrak{g} = \mathfrak{sl}_2$, $\lambda = 4\varphi_1$.

Proof. (i) By Proposition 3.1, such a decomposition with irreducible and co-primary summands $\mathbb{V}_i$ exists. The other assertions are proved in Proposition 3.6.

(ii) By part (i), we have $\lambda \in \{k\theta, k\theta_s \mid k \in \mathbb{N}\}$.

Let $\mathrm{rk}\,\mathfrak{g} = 1$. It follows from Example 2.5(2) that the only co-primary $\mathfrak{sl}_2$-modules are $\mathfrak{sl}_2 = R_2$ and R_4.

Let $\mathrm{rk}\,\mathfrak{g} \ge 2$. Consider the following possibilities.

- $\lambda = 2\theta$. Take $\alpha_i \in \Pi$ such that $(\alpha_i, \theta) \ne 0$. Then $2\theta - \alpha_i$ is a weight of $\mathbb{V}_{2\theta}$, which is not a multiple of a root. Thus, $\mathbb{V}_{2\theta}$ is not co-primary.
- $\mathfrak{g} = \mathfrak{sp}_{2n}$ or $\mathfrak{f}_4$ and $\lambda = 2\theta_s$. If α_i is the unique simple root such that $(\alpha_i, \theta_s) \ne 0$, then $2\theta_s - \alpha_i \in \Delta(\mathbb{V}_{2\theta_s})$ is not a multiple of a root.
- $\mathfrak{g} = \mathfrak{so}_{2n+1}$ and $\lambda = 3\theta_s = 3\varphi_1$. Here $3\varphi_1 - \alpha_1$ is not proportional to a root.
- $\mathfrak{g} = \mathfrak{g}_2$. We have already shown in Example 2.13(4) that $\mathbb{V}_{\theta_s}$ is not co-primary.

Obviously, if $\mathbb{V}_{k\lambda}$ is not co-primary, then the same holds for any $m \ge k$. Thus, comparing with results of section 2, we see that the only unclear case is *ii(c)*. Our proof that this module is co-primary is similar to the first proof of Theorem 2.9. It will be given in the next proposition, where we also compute the reduced *Spin* of $\mathbb{V}_{2\theta_s}$. $\square$

3.8 Proposition. *Let $\mathfrak{g} = \mathfrak{so}_{2n+1}$. Then*

1. $Spin_0(\mathbb{V}_{2\varphi_1}) = \mathbb{V}_{\rho+2\varphi_n}$;

2. $\wedge^\bullet \mathbb{V}_{2\varphi_1} = 2^n \cdot (\mathbb{V}_{\rho+2\varphi_n})^{\otimes 2}$.

Proof. First, we describe the weight structure of the $\mathfrak{g}$-module $\mathbb{V}_{2\varphi_1}$. This is easy, since $\mathbb{V}_{2\varphi_1}$ is the Cartan (highest) component in $\mathcal{S}^2\mathbb{V}_{\varphi_1}$. Here $\mathcal{P}(\mathbb{V}_{2\varphi_1}) = \{0\} \cup \Delta \cup 2\Delta_s$. Hence $\Delta(\mathbb{V})^+ = \Delta^+ \cup 2\Delta_s^+$. The non-zero weights are of multiplicity 1, and $m_{2\varphi_1}(0) = n$. Therefore, making use of Eq. (2.3), we obtain

$$\mathrm{ch}\, Spin_0(\mathbb{V}_{2\varphi_1}) = \prod_{\alpha\in\Delta^+}(e^{\alpha/2} + e^{-\alpha/2}) \prod_{\alpha\in\Delta_s^+}(e^{\alpha} + e^{-\alpha}) = e^{\rho+2\varphi_n} \prod_{\alpha\in\Delta^+}(1 + e^{-\alpha}) \prod_{\alpha\in\Delta_s^+}(1 + e^{-2\alpha}) =$$

$$= e^{\rho+2\varphi_n} \prod_{\alpha\in\Delta_l^+}(1 + e^{-\alpha}) \prod_{\alpha\in\Delta_s^+}(1 + e^{-\alpha} + e^{-2\alpha} + e^{-3\alpha}) .$$

On the other hand,

$$\mathrm{ch}\, \mathbb{V}_{\rho+2\varphi_n} = \frac{\sum_{w\in W} \varepsilon(w) e^{w(\rho+\rho+2\varphi_n)}}{\prod_{\alpha\in\Delta^+}(e^{\alpha/2} - e^{-\alpha/2})} . \qquad \textbf{(3.9)}$$

Since $\mathrm{ch}\, Spin_0(\mathbb{V}_{2\varphi_1})$ and $\mathrm{ch}\,\mathbb{V}_{\rho+2\varphi_n}$ have the same maximal term $e^{\rho+2\varphi_n}$, it suffices to prove that each factor in the last expression for $\mathrm{ch}\, Spin_0(\mathbb{V}_{2\varphi_1})$ divides $\mathrm{ch}\,\mathbb{V}_{\rho+2\varphi_n}$, i.e., the numerator in Eq. (3.9).

The same procedure, as in the proof of Theorem 2.9, reduces the problem to proving that, for any $w \in W^\alpha$,

$$(w(2\rho+2\varphi_n), \alpha^\vee) \quad \begin{cases} \text{is even} & , \text{ if } \alpha \in \Pi_l \\ \text{is divisible by } 4 & , \text{ if } \alpha \in \Pi_s \end{cases} .$$

That is, we need actually to verify that $(w(\rho+\varphi_n), \alpha^\vee)$ is even whenever α is short. As $\varphi_n = \rho_s$ for our $\mathfrak{g}$, this is just Lemma 2.7.

2. This is a formal consequence of part 1, see Proposition 2.4. □

Having obtained the list of all irreducible co-primary modules in Theorem 3.7(ii), it is worth looking it through again in order to find out common features and latent regularities for the representations in question.

First, item *(ii)d* in (3.7) can be thought of as starting point for the series in *(ii)c*. Indeed, $V_{2\varphi_1}$ is the Cartan component in $\mathcal{S}^2\mathbb{V}_{\varphi_1}$ and $\mathbb{V}_{\varphi_1}$ is the tautological module for $\mathfrak{so}_{2n+1}$ $(n \ge 2)$, whereas $\mathfrak{sl}_2$-module R_4 is the Cartan component in $\mathcal{S}^2 R_2$ and R_2 is the tautological module for $\mathfrak{so}_3$. Thus, the list consists of three groups of representations:

1. $(\mathfrak{g}, \mathbb{V}_\theta = \mathfrak{g})$;
2. $(\mathfrak{g}, \mathbb{V}_{\theta_s})$, where $\mathfrak{g}$ is of type $\mathbf{B}$, $\mathbf{C}$, or $\mathbf{F}$;
3. $\mathfrak{g} = \mathfrak{so}(\mathbb{W})$ and $\mathbb{V} = \mathcal{S}^2_0(\mathbb{W})$, where $\dim \mathbb{W} = 3, 5, 7, \dots$

The second (more interesting) observation is that, for all items $(\mathfrak{g}, \mathbb{V})$ in the list, $\mathfrak{g} \to \mathfrak{so}(\mathbb{V})$ is the isotropy representation of an *irreducible* symmetric space. In other words, $\tilde{\mathfrak{g}} := \mathfrak{g} \oplus \mathbb{V}$ has a structure of irreducible $\mathbb{Z}_2$-graded semisimple Lie algebra. More precisely, $\tilde{\mathfrak{g}}$ is simple for items 2 and 3, and $\tilde{\mathfrak{g}} \simeq \mathfrak{g} \oplus \mathfrak{g}$ for item 1. Furthermore, it follows from the well-known classification of symmetric spaces that items 1–3 correspond exactly to the cases, where $\mathfrak{g}$ is *non-homologous to zero*[3] in $\tilde{\mathfrak{g}}$. The class of homogeneous spaces $\tilde{G}/G$ (not necessarily symmetric ones) such that $\tilde{G}, G$ are connected and $\mathfrak{g}$ is non-homologous to zero in $\tilde{\mathfrak{g}}$ has many nice descriptions. We refer the reader to [On95, § 13, n.2] for a thorough treatment in the context of homogeneous spaces of compact Lie groups. In the symmetric case, yet another characterization is that this happens if and only if $\mathfrak{g}$ is determined by a *diagram involutory automorphism* of $\tilde{\mathfrak{g}}$. An explicit description of the diagram automorphisms of simple Lie algebras is found in [Ka90, § 7.9, 7.10]. The third observation is that any irreducible co-primary module occurs in Table 1 in section 1, i.e., it has a free algebra of skew-invariants.

These observations give us some hope that the reduced *Spin* of the isotropy representation of an *arbitrary* symmetric spaces might have some interesting properties. This is really the case and we turn to such considerations in the following sections.

[3]this means that the canonical map of homology spaces $H_*(\mathfrak{g}) \to H_*(\tilde{\mathfrak{g}})$ is injective.

4 Some auxiliary results

In this section, we prove an auxiliary result on Weyl groups and recall some standard facts on involutions of simple Lie algebras.

Let W be the Weyl corresponding to a reduced root system Δ with a set of positive roots Δ^+. Let $w \mapsto l(w)$ be the length function on W determined by Δ^+. Recall that l can be defined as $l(w) = \#\{\alpha \in \Delta^+ \mid w(\alpha) \in \Delta^-\}$. Consider an arbitrary subset $\Delta_0 \subset \Delta$ which is a root system in its own right, but is not necessarily closed in Δ. That is, it is allowed that $\alpha + \beta \in \Delta \setminus \Delta_0$ for some $\alpha, \beta \in \Delta_0$. It is easily seen that such a phenomenon can only occur if Δ has roots of different length. As a sample of such non-closed subset, we mention $\Delta_0 = \Delta_s$. Nevertheless, W_0, the Weyl group of Δ_0, is always identified with a subgroup of W. Clearly, $\Delta_0^+ := \Delta_0 \cap \Delta^+$ can be taken as set of positive roots for Δ_0.

4.1 Proposition. *1. Any coset $wW_0 \subset W$ contains a unique representative of minimal length. Denoting by W^0 the set of minimal length representatives, we have $W^0 = \{w \in W \mid w(\Delta_0^+) \subset \Delta^+\}$.*
2. The mapping $W^0 \times W_0 \to W$ $((w^o, w_o) \mapsto w_o(w^o)^{-1})$ is a bijection.

Proof. 1. Set $W' = \{w \in W \mid w(\Delta_0^+) \subset \Delta^+\}$. Then $W^0 \subset W'$. Indeed, assume that $w \in W^0$ and $w(\beta) \in \Delta^-$ for some $\beta \in \Delta_0^+$. Then $ws_\beta(\beta) \in \Delta^+$ and it follows from [BGG, 2.3] that $l(ws_\beta) < l(w)$. But this contradicts the fact that $w \in W_0$. Obviously, each coset contains elements of minimal length and hence elements from W'. Assume that $u, v \in W' \cap vW_0$. Then $u = vw$ for some $w \in W_0$. If $w \neq e$, then $w(\beta) \in \Delta_0^-$ for some $\beta \in \Delta_0^+$. Whence $u(\beta) = v(w(\beta)) \in \Delta^-$, which contradicts the assumption. Thus, each coset contains a unique element of W', $W^0 = W'$, and we are done.

2. Obvious. □

Remark. If Δ_0 is generated by a part of the basis $\Pi \subset \Delta^+$ (i.e., $\Delta_0 \cap \Pi$ is a basis of Δ_0), then W_0 is a parabolic subgroup of W. In this case the Proposition is well known and, moreover, the relation $l(w^o w_0) = l(w^o) + l(w_o)$ holds, see e.g. [Hu95, 1.10]. However this relation does not hold in general.
For $w_o \in W_0$, let $l_0(w_o)$ denote the length of w_o in W_0. That is, $l_0(w_o) = \#\{\mu \in \Delta_0^+ \mid w_o(\mu) \in \Delta_0^-\}$. If W_0 is a parabolic subgroup, then $l_0(w_o) = l(w_o)$, but in general we have only "$\le$". The usual *determinant* or *parity* for the elements of W is defined by $\varepsilon(w) = (-1)^{l(w)}$. Making use of the above bijection, one may introduce a parity depending on Δ_0. By Prop. 4.1(2), each element $w \in W$ has a unique presentation $w = w_o(w^o)^{-1}$, where $w_o \in W_0$ and $w^o \in W^0$. Set $l_0(w) := l_0(w_o)$ and $\tau(w) := (-1)^{l_0(w)}$. So, if $w = w_o$, then $\tau(w_o)$ is nothing but the usual parity on W_0, which will be denoted by $\varepsilon_0(w_o)$. Therefore one may say that τ is the extension of the parity ε_0 to W determined by the 'section' W^0. The function $w \in W \mapsto \tau(w) \in \{1, -1\}$ is said to be the *cunning parity* on W, determined by Δ_0^+ (or W_0). It is convenient to give an expression for $l_0(w)$, and hence for $\tau(w)$, where w_o is not explicitly mentioned.

4.2 Lemma. $l_0(w) = \#\{\alpha \in \Delta^- \mid w(\alpha) \in \Delta_0^+\}$.

Proof. Let $w = w_o(w^o)^{-1}$, as above. Consider the subsets

$$M_1 = \{\alpha \in \Delta^- \mid w(\alpha) \in \Delta_0^+\} \quad \text{and} \quad M_2 = \{\mu \in \Delta^- \mid w_o(\mu) \in \Delta_0^+\}.$$

Since Δ_0 is W_0-stable, $M_2 \subset \Delta_0^-$ and therefore $l_0(w_o) = \#M_2$. By Prop. 4.1(1), we have $(w^o)^{-1}M_1 \cap \Delta_0^+ = \varnothing$. Since $w_o((w^o)^{-1}M_1) \subset \Delta_0^+$, we see that $(w_o)^{-1}M_1 \subset \Delta_0^-$. Thus $(w^o)^{-1}M_1 \subset M_2$. Similarly, one proves the opposite containment. Thus, $l_0(w) = \#M_2 = \#M_1$, and we are done. □

(4.3) Classes of involutory automorphisms. Here $\mathfrak{g}$ is a simple Lie algebra.

Given an involutory automorphism Θ of $\mathfrak{g}$, consider the $\mathbb{Z}_2$-grading $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$, where $\mathfrak{g}_i = \{x \in \mathfrak{g} \mid \Theta(x) = (-1)^i x\}$. The reductive subalgebra $\mathfrak{g}_0$ is called *symmetric*. The involutory automorphisms fall into three classes:

a) $\mathrm{rk}\,\mathfrak{g} = \mathrm{rk}\,\mathfrak{g}_0$ and $\mathfrak{g}_0$ is semisimple;

b) $\mathrm{rk}\,\mathfrak{g} = \mathrm{rk}\,\mathfrak{g}_0$ and $\mathfrak{g}_0$ has 1-dimensional centre;

c) $\mathrm{rk}\,\mathfrak{g} > \mathrm{rk}\,\mathfrak{g}_0$.

In cases a) and b), Θ is inner and, accordingly, both $\mathfrak{g}_0$ and the $\mathbb{Z}_2$-grading are said to be *of inner type*. It is well known that the $\mathfrak{g}_0$-module $\mathfrak{g}_1$ is irreducible in cases a) and c), and is the sum of two dual submodules in case b). However, $\mathfrak{g}_1$ is orthogonal in all three cases and one may consider the $\mathfrak{g}_0$-module $Spin(\mathfrak{g}_1)$. An important feature of this situation is that all nonzero weights of $\mathfrak{g}_1$ are of multiplicity 1. This is clear in the equal rank cases, and can also be proved for c). An invariant theoretic proof of this uses Lemma 3.4 in [Ka80] and the fact that the linear group $G_0 \to GL(\mathfrak{g}_1)$ is *visible*.

5 Spin($\mathfrak{g}_1$) for the inner involutory automorphisms

In this section, $\mathfrak{g}$ is simple and $\mathfrak{g}_0$ is a symmetric subalgebra of inner type. Retain for $\mathfrak{g}$ the previous notation such as $\mathfrak{t}$, Δ, Δ^+, ρ, C, etc. Since $\mathrm{rk}\,\mathfrak{g} = \mathrm{rk}\,\mathfrak{g}_0$, we may assume that $\mathfrak{t}$ is a Cartan subalgebra in both $\mathfrak{g}$ and $\mathfrak{g}_0$. Let Δ_0 be the root system of $(\mathfrak{g}_0, \mathfrak{t})$ and Δ_1 the set of weights of the $\mathfrak{g}_0$-module $\mathfrak{g}_1$. Then $\Delta = \Delta_0 \sqcup \Delta_1$ and Δ_0 is a closed subset of Δ. We regard $\Delta_0^+ := \Delta^+ \cap \Delta_0$ as set of positive roots for $\mathfrak{g}_0$. Note also that Δ_1 contains a distinguished 'half' $\Delta_1^+ = \Delta^+ \cap \Delta_1$. Then Prop. 4.1 applies to the Weyl groups $W_0 \subset W$ and one obtains the "minimal length" subset $W^0 \subset W$.

Our aim is to describe the $\mathfrak{g}_0$-module $Spin_0(\mathfrak{g}_1)$. As $\mathfrak{g}_1$ has no zero weight, we have $Spin(\mathfrak{g}_1) = Spin_0(\mathfrak{g}_1)$. As a first step, we find all extreme weights of $Spin(\mathfrak{g}_1)$. Recall from (3.3), (3.4) that each dominant half of Δ_1 determines an extreme weight for $Spin(\mathfrak{g}_1)$. According to that discussion, one has to take the dominant chamber C_0 for $\mathfrak{g}_0$ and cut it up by the hyperplanes orthogonal to the roots of Δ_1. Clearly, each small chamber is isomorphic to C. Since there are $\#W$ chambers for $\mathfrak{g}$ and $\#W_0$ chambers for $\mathfrak{g}_0$, we obtain the partition of C_0 in $\#(W/W_0)$ small chambers. Then any weight inside of a small chamber determines a dominant half of Δ_1 and an extreme weight. In the following proposition we give a formula for these extreme weights.

5.1 Proposition.
1. The set of hyperplanes H_μ ($\mu \in \Delta_1$) cuts C_0 in $\#(W/W_0)$ small chambers;
2. The collection of ($\mathfrak{g}_0$-dominant) weights $w^{-1}\rho$ ($w \in W^0$) contains representatives of all small chambers in C_0.
3. The extreme weight of $Spin(\mathfrak{g}_1)$ corresponding to $w^{-1}\rho$ is $\lambda_w := w^{-1}\rho - \rho_0$.

Proof. 1. This is proved in the previous paragraph.
2 & 3. If $\alpha \in \Delta_0^+$ and $w \in W^0$, then $w\alpha \in \Delta^+$, see Prop. 4.1(1). Therefore $w^{-1}\rho$ is $\mathfrak{g}_0$-dominant. Since the number of these weights is $\#(W/W_0)$, as required, it suffices to verify that the corresponding dominant halfs are different.

By definition, the dominant half of Δ_1 associated with $w^{-1}\rho$ is

$$(\Delta_1)_w^+ = \{\mu \in \Delta_1 \mid (w^{-1}\rho, \mu) > 0\} = \{\mu \in \Delta_1 \mid w\mu \in \Delta^+\} .$$

Because all weight multiplicities in $\mathfrak{g}_1$ are equal to 1, the corresponding extreme weight is $\lambda_w := \frac{1}{2}|(\Delta_1)^+_{w^{-1}\rho}|$. Set $M_w = \{\mu \in \Delta_1^+ \mid w\mu \in \Delta^+\}$ and $\overline{M}_w = \Delta_1^+ \setminus M_w$. Then $\Delta^+ = \Delta_0^+ \sqcup M_w \sqcup \overline{M}_w$ and

$$\rho = \rho_0 + \tfrac{1}{2}|M_w| + \tfrac{1}{2}|\overline{M}_w| .$$

Since $w \in W^0$, we obtain

$$w^{-1}\rho = \rho_0 + \tfrac{1}{2}|M_w| - \tfrac{1}{2}|\overline{M}_w| .$$

Note also that $|(\Delta_1)_w^+| = |M_w| - |\overline{M}_w|$. Whence $\lambda_w = w^{-1}\rho - \rho_0$. Thus, we have obtained the required number of different extreme weights. □

In the remainder of the section, notation $\mathbb{V}_\lambda$ refers to a $\mathfrak{g}_0$-module.

5.2 Theorem. *Let $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ be a $\mathbb{Z}_2$-grading of inner type. Then*

$$Spin_0(\mathfrak{g}_1) = Spin(\mathfrak{g}_1) = \bigoplus_{w \in W^0} \mathbb{V}_{\lambda_w} .$$

Proof. It follows from the preceding exposition that

$$\bigoplus_{w \in W^0} \mathbb{V}_{\lambda_w} \subset Spin_0(\mathfrak{g}_1) = Spin(\mathfrak{g}_1) .$$

Since $Spin(\mathfrak{g}_1)$ is self-dual, $\dim(Spin(\mathfrak{g}_1)^{\otimes 2})^{\mathfrak{g}_0}$ is greater than or equal to the number of irreducible summands of $Spin(\mathfrak{g}_1)$. Therefore the desired equality is equivalent to that $\dim(Spin(\mathfrak{g}_1)^{\otimes 2})^{\mathfrak{g}_0} = \#W^0$. Recall the main property of 'Spin' in our situation:

$$\wedge^\bullet \mathfrak{g}_1 \simeq Spin(\mathfrak{g}_1)^{\otimes 2} .$$

Hence the question about $\mathfrak{g}_0$-invariants is being translated in the setting of exterior algebras. Assuming that $\Bbbk = \mathbb{C}$, we can exploit de Rham cohomology with complex coefficients. It is well known that $(\wedge^\bullet \mathfrak{g}_1)^{\mathfrak{g}_0}$ is isomorphic to $H^*(G/G_0)$, the cohomology ring of the symmetric space G/G_0 [On95, §9 n.11], and that $\dim H^*(G/G_0) = \#(W/W_0)$ [On95, §13 n.3]. This completes the proof. $\square$

5.3 Example. Let Θ be a 'Hermitian' involutory automorphism, i.e., $\mathfrak{g}_0$ has a 1-dimensional centre and $\mathfrak{g}_1 \simeq \mathbb{W} \oplus \mathbb{W}^*$, where $\mathbb{W}$ is a faithful irreducible $\mathfrak{g}_0$-module. This is just case 4.3(b). Then $\Delta(\mathbb{W}) = \mathcal{P}(\mathbb{W}) = \Delta_1^+$ and, according to Example 2.5(3), $Spin(\mathfrak{g}_1) \simeq \Bbbk_{\rho_1} \otimes \wedge^\bullet \mathbb{W}^*$, where $\rho_1 = \frac{1}{2}|\Delta_1^+|$. It then follows from Theorem 5.2 that

$$\wedge^\bullet \mathbb{W}^* = \Bbbk_{-\rho_1} \otimes Spin(\mathfrak{g}_1) = \bigoplus_{w \in W^0} \mathbb{V}_{w\rho - \rho} \ .$$

Or, equivalently, $\{\rho - w\rho \mid w \in W^0\}$ is the set of all highest weights for the $\mathfrak{g}_0$-module $\wedge^\bullet \mathbb{W}$. This result was obtained by Kostant (see [Ko61, 8.2]) as application of his results on the cohomology of the nilpotent radical of a parabolic subalgebra of $\mathfrak{g}$. In this situation, $\mathbb{W}$ is the Abelian nilpotent radical of the parabolic subalgebra $\mathfrak{g}_0 \oplus \mathbb{W}$. So, the concept of 'Spin' and Theorem 5.2 yield another generalization of Kostant's result.

Purists may condemn the above proof of Theorem 5.2, since it invokes the cohomology theory of compact Lie groups over $\mathbb{C}$. Fortunately, there exists also a rather simple and purely algebraic proof. We shall show that the equality in 5.2 is equivalent to an identity in $\mathbb{Z}[\mathcal{P}]$, which is a variation of the Weyl denominator formula. Recall from section 4 the cunning parity $\tau(w)$ for $w \in W$, determined by W_0.

5.4 Theorem. *Let $\Delta = \Delta_0 \sqcup \Delta_1$ be the partition corresponding to a $\mathbb{Z}_2$-grading of $\mathfrak{g}$ of inner type. Then*

$$\sum_{w \in W} \tau(w) e^{w\rho} = \prod_{\alpha \in \Delta_0^+} (e^{\alpha/2} - e^{-\alpha/2}) \prod_{\mu \in \Delta_1^+} (e^{\mu/2} + e^{-\mu/2}) \ .$$

Proof. The fact that Δ_0 and Δ_1 originate from an inner involutory automorphism can alternatively be stated as follows:

$(*)$ if $\alpha \in \Delta_i$, $\beta \in \Delta_j$, and $\alpha + \beta \in \Delta$, then $\alpha + \beta \in \Delta_{i+j}$,

where, of course, $i, j \in \mathbb{Z}/2\mathbb{Z}$. Let $\mathcal{Q} \subset \mathcal{P}$ be the root lattice. For $\mathbb{Z}[\mathcal{Q}]$, with basis e^α $(\alpha \in \mathcal{Q})$, one has a version of Weyl's denominator identity:

$$\sum_{w \in W} \varepsilon(w) e^{w\rho - \rho} = \prod_{\alpha \in \Delta^+} (1 - e^{-\alpha}) \ .$$

Consider the second copy of $\mathbb{Z}[\mathcal{Q}]$, with basis q^α $(\alpha \in \mathcal{Q})$, and the equality in $\mathbb{Z}[\mathcal{Q}] \otimes \mathbb{Z}[\mathcal{Q}]$:

$$\textbf{(5.5)} \qquad \sum_{w \in W} \varepsilon(w) q^{w\rho - \rho} e^{w\rho - \rho} = \prod_{\alpha \in \Delta^+} (1 - q^{-\alpha} e^{-\alpha}) \ .$$

Take the specialization of this identity such that $q^\alpha \to \begin{cases} 1, \ \alpha \in \Delta_0 \\ -1, \ \alpha \in \Delta_1 \end{cases}$. It has to be verified that one obtains a well-defined homomorphism $(\mathcal{Q}, +) \to \{1, -1\} \simeq \mathbb{Z}/2\mathbb{Z}$. In other words, if $\nu = \sum_{i \in I} \mu_i$ is a sum of roots then the number of summands lying in Δ_1 should

have the same parity for all such presentations. Indeed, assume that $\sum_{i\in I}\mu_i = \sum_{j\in J}\beta_j$. We argue by induction on $\#I+\#J$. Since $(\sum_{i\in I}\mu_i, \sum_{j\in J}\beta_j) > 0$, there exist i_0, j_0 such that $(\mu_{i_0}, \beta_{j_0}) > 0$. Hence $\mu_{i_0} - \beta_{j_0}$ is a root and $\sum_{i\in I\setminus\{i_0\}}\mu_i + (\mu_{i_0} - \beta_{j_0}) = \sum_{j\in J\setminus\{j_0\}}\beta_j$. We conclude by applying the inductive hypothesis to this equality and using $(*)$.
Thus, the specialization is well-defined and we obtain $\prod\limits_{\alpha\in\Delta_0^+}(1-e^{-\alpha})\prod\limits_{\mu\in\Delta_1^+}(1+e^{-\mu})$ at the right hand side of Eq. (5.5). It is easily seen that $w\rho-\rho = -|\Delta(w)|$, where $\Delta(w) = \{\alpha\in\Delta^+ \mid w^{-1}\alpha\in\Delta^-\} = \Delta^+\cap w(\Delta^-)$. Therefore $q^{w\rho-\rho}$ specializes to $(-1)^n$, where $n = \#(\Delta_1^+\cap w\Delta^-)$. Recall that $\varepsilon(w) = (-1)^{l(w)}$ and $l(w) = \#(\Delta^+\cap w\Delta^-)$. Thus the resulting sign on the left hand side is $(-1)^{\#(\Delta_0\cap w\Delta^-)}$, which is just $\tau(w)$ by Lemma 4.2. This completes the proof of the theorem. $\square$

(5.6) Another proof of theorem 5.2. By Weyl's character formula for $\mathfrak{g}_0$-modules and Prop. 5.1(3),

$$\mathrm{ch}\,\mathbb{V}_{\lambda_w} = \frac{\sum_{\tilde w\in W_0}\varepsilon_0(\tilde w)e^{\tilde w(\rho_0+\lambda_w)}}{\prod_{\alpha\in\Delta_0^+}(e^{\alpha/2}-e^{-\alpha/2})} = \frac{\sum_{\tilde w\in W_0}\varepsilon_0(\tilde w)e^{\tilde w w^{-1}\rho}}{\prod_{\alpha\in\Delta_0^+}(e^{\alpha/2}-e^{-\alpha/2})} .$$

Hence

$$\mathrm{ch}\Big(\bigoplus_{w\in W^0}\mathbb{V}_{\lambda_w}\Big) = \frac{\sum_{w\in W^0}\sum_{\tilde w\in W_0}\varepsilon_0(\tilde w)e^{\tilde w w^{-1}\rho}}{\prod_{\alpha\in\Delta_0^+}(e^{\alpha/2}-e^{-\alpha/2})} .$$

By the very definition of $\tau(w)$ (see section 4) and Prop. 4.1(2), it follows that the numerator is equal to $\sum_{w\in W}\tau(w)e^{w\rho}$. Whence, by Theorem 5.4,

$$\mathrm{ch}\Big(\bigoplus_{w\in W^0}\mathbb{V}_{\lambda_w}\Big) = \prod_{\mu\in\Delta_1^+}(e^{\mu/2}+e^{-\mu/2}) = \mathrm{ch}\,Spin(\mathfrak{g}_1) .$$

5.7 Examples. 1. $\mathfrak{g} = \mathfrak{so}_{2n+1}$, $\mathfrak{g}_0 = \mathfrak{so}_{2n}$. Here $\mathfrak{g}_1 \simeq \mathbb{V}_{\varphi_1}$ is the tautological $\mathfrak{so}_{2n}$-module and $\#(W/W_0) = 2$. Let $\{\varepsilon_1,\dots,\varepsilon_n\}$ be the standard basis of $\mathfrak{t}^*$ so that $\Delta = \{\pm\varepsilon_i\pm\varepsilon_j, \pm\varepsilon_i \mid 1\le i,j\le n, i\ne j\}$. Here $\Delta_0^+ = \{\varepsilon_i\pm\varepsilon_j\,(i<j), \varepsilon_i\}$ and $\Delta_0 = \Delta_l$. Then $W^0 = \{id, w_n\}$, where $w_n(\varepsilon_i) = \varepsilon_i$ $(i\le n-1)$ and $w_n(\varepsilon_n) = -\varepsilon_n$. Since $\Delta_1 = \Delta(\mathbb{V}_{\varphi_1}) = \{\pm\varepsilon_1,\dots,\pm\varepsilon_n\}$, the corresponding dominant halfs are $(\Delta_1)^+_{id} = \{\varepsilon_1,\dots,\varepsilon_n\}$ and $(\Delta_1)^+_{w_n} = \{\varepsilon_1,\dots,\varepsilon_{n-1},-\varepsilon_n\}$, and the corresponding extreme weights are φ_n and φ_{n-1}. Thus, $Spin(\mathbb{V}_{\varphi_1}) = \mathbb{V}_{\varphi_{n-1}}\oplus\mathbb{V}_{\varphi_n}$ and $\wedge^{\bullet}\mathbb{V}_{\varphi_1} = (\mathbb{V}_{\varphi_{n-1}}\oplus\mathbb{V}_{\varphi_n})^{\otimes 2}$. Notice that the last equality is nothing but the first equality in Eq. (2.1).

2. $\mathfrak{g} = \mathfrak{f}_4$, $\mathfrak{g}_0 = \mathfrak{so}_9$. Here $\mathfrak{g}_1 \simeq \mathbb{V}_{\varphi_4}$ and $\#(W/W_0) = 3$. In the standard notation for $\mathfrak{f}_4$, we have $\Delta^+ = \{\varepsilon_i\pm\varepsilon_j\,(i{<}j),\ \varepsilon_i,\ \frac12(\varepsilon_1\pm\varepsilon_2\pm\varepsilon_3\pm\varepsilon_4)\}$. Then $\Delta_0^+ = \Delta_l^+\sqcup\{\varepsilon_1,\varepsilon_2,\varepsilon_3,\varepsilon_4\}$. An explicit computation shows that $W^0 = \{id, w', w''\}$, where

$$w' : \begin{cases} \varepsilon_1 \mapsto \frac12(\varepsilon_1+\varepsilon_2+\varepsilon_3+\varepsilon_4) \\ \varepsilon_2 \mapsto \frac12(\varepsilon_1+\varepsilon_2-\varepsilon_3-\varepsilon_4) \\ \varepsilon_3 \mapsto \frac12(\varepsilon_1-\varepsilon_2+\varepsilon_3-\varepsilon_4) \\ \varepsilon_4 \mapsto \frac12(\varepsilon_1-\varepsilon_2-\varepsilon_3+\varepsilon_4) \end{cases} \text{ and } w'' : \begin{cases} \varepsilon_1 \mapsto \frac12(\varepsilon_1+\varepsilon_2+\varepsilon_3-\varepsilon_4) \\ \varepsilon_2 \mapsto \frac12(\varepsilon_1+\varepsilon_2-\varepsilon_3+\varepsilon_4) \\ \varepsilon_3 \mapsto \frac12(\varepsilon_1-\varepsilon_2+\varepsilon_3+\varepsilon_4) \\ \varepsilon_4 \mapsto \frac12(\varepsilon_1-\varepsilon_2-\varepsilon_3-\varepsilon_4) \end{cases} .$$

(One may notice that any $w\in W^0$ must preserve $(\Delta_0)^+_l = \Delta_l^+$, Δ_l being the root system of type $\mathbf{D}_4$. Hence w takes $\varepsilon_2-\varepsilon_3$ to itself and permutes somehow $\varepsilon_1-\varepsilon_2$, $\varepsilon_3-\varepsilon_4$, and $\varepsilon_3+\varepsilon_4$.) Whence

$\lambda_{id}=\rho-\rho_0=2\varepsilon_1$,

$\lambda_{w'}=(w')^{-1}\rho-\rho_0=\varepsilon_1+\varepsilon_2+\varepsilon_3$,

$\lambda_{w''}=(w'')^{-1}\rho-\rho_0=(3\varepsilon_1+\varepsilon_2+\varepsilon_3+\varepsilon_4)/2$.

Thus

$$Spin(\mathbb{V}_{\varphi_4})=\mathbb{V}_{2\varphi_1}\oplus\mathbb{V}_{\varphi_3}\oplus\mathbb{V}_{\varphi_1+\varphi_4}.$$

6 Spin($\mathfrak{g}_1$) for the outer involutory automorphisms

In this section $\mathfrak{g}$ is simple and Θ is outer. Since $\mathrm{rk}\,\mathfrak{g}_0<\mathrm{rk}\,\mathfrak{g}$, there is no clear relation between roots and Weyl groups of the two algebras, and the approach of section 5 seems to fail completely. Yet, it appears to be possible to describe $Spin(\mathfrak{g}_1)$ in a similar fashion, but with some complications. Another price is that we have to exploit case-by-case arguments several times.

(6.1) Associated diagram involutory automorphism of $\mathfrak{g}$. By a result of Steinberg, Θ keeps stable a Borel subalgebra and a Cartan subalgebra in it. Therefore we may (and shall) assume that $\Theta\mathfrak{t}=\mathfrak{t}$ and $\Theta\mathfrak{u}^+=\mathfrak{u}^+$. Then Θ also preserves Δ^+ and Π, as subsets of $\mathfrak{t}^*$. In particular, Θ induces an involution of the Dynkin diagram. Associated with this involution, one has the specific involutory automorphism of $\mathfrak{g}$, which is called the *diagram involutory automorphism* and denoted by $\overline{\Theta}$. Roughly speaking, $\overline{\Theta}$ performs the same involution on Π, as Θ, and transforms 'well' the Chevalley generators of $\mathfrak{g}$[4]. We are going to compare properties of the $\mathbb{Z}_2$-gradings

$$\mathfrak{g}=\mathfrak{g}_0\oplus\mathfrak{g}_1 \quad\text{and}\quad \mathfrak{g}=\mathfrak{g}_{\bar 0}\oplus\mathfrak{g}_{\bar 1}$$

arising from Θ and $\overline{\Theta}$. By construction, $\Theta|_{\mathfrak{t}}=\overline{\Theta}|_{\mathfrak{t}}$. Therefore Θ and $\overline{\Theta}$ act identically on Δ^+ and $\mathfrak{t}_0:=\mathfrak{t}^\Theta$ is a Cartan subalgebra for both $\mathfrak{g}_0$ and $\mathfrak{g}_{\bar 0}$. Let us organize notation for roots and weights of the symmetric subalgebras in question:

- Δ_0 (resp. $\Delta_{\bar 0}$) is the root system of $\mathfrak{g}_0$ (resp. $\mathfrak{g}_{\bar 0}$) relative to $\mathfrak{t}_0$;
- Δ_1 (resp. $\Delta_{\bar 1}$) is the set of non-zero weights of the $\mathfrak{g}_0$-module $\mathfrak{g}_1$ (resp. $\mathfrak{g}_{\bar 0}$-module $\mathfrak{g}_{\bar 1}$) relative to $\mathfrak{t}_0$.

Since all these sets are defined with respect to a common Cartan subalgebra, $\Delta_0\cup\Delta_1=\Delta_{\bar 0}\cup\Delta_{\bar 1}$ and, more precisely, the totality of weights occurring in $\{\Delta_0,\Delta_1\}$ is the same as in $\{\Delta_{\bar 0},\Delta_{\bar 1}\}$. Because $\mathfrak{t}_0$ contains regular elements of $\mathfrak{g}$ (see e.g. [Ka90, 8.1(b)]), none of the roots of $\mathfrak{g}$ vanishes on $\mathfrak{t}_0$. Therefore the above totality of weights consists of all restricted roots. Moreover, since the non-zero weights of $\mathfrak{t}_0$ in $\mathfrak{g}_1$ (or $\mathfrak{g}_{\bar 1}$) are of multiplicity 1,

$$\#\Delta=\#\Delta_0+\#\Delta_1=\#\Delta_{\bar 0}+\#\Delta_{\bar 1}.$$

[4] Explicit formulas for the diagram automorphisms of all simple Lie algebras are written in [Ka90, § 7.9,7.10].

Warning. Unlike section 5, elements of Δ_0 and Δ_1 have not much in common with roots of $\mathfrak{g}$. Actually, we do not need Δ in this section.

The next assertion follows from the classification.

Fact. The fixed point subalgebra of an outer involutory automorphism always has roots of different length, with $\|long\|^2/\|short\|^2=2$.

This applies to both $\mathfrak{g}_0$ and $\mathfrak{g}_{\bar 0}$ and, as in section 2, we use the subscripts 's' and 'l' to denote the objects related to short and long roots in Δ_0 and $\Delta_{\bar 0}$. A close look to the classification list reveals important features of this situation.

(6.2) Suppose $\Theta\ne\overline{\Theta}$. Then $\begin{cases}\mathfrak{g}_{\bar 0} \text{ is simple and } \mathfrak{g}_{\bar 1} \text{ is the little adjoint module for } \mathfrak{g}_{\bar 0};\\ \Delta_{\bar 1}=(\Delta_{\bar 0})_s;\\ \Delta_0\subset\Delta_{\bar 0} \text{ and } (\Delta_0)_s=(\Delta_{\bar 0})_s.\end{cases}$

In fact, there are 7 series of outer involutory automorphisms of simple Lie algebras. They form three pairs $(\Theta,\overline{\Theta})$ and one "isolated" diagram involutory automorphism, where (6.2) is not satisfied. The relevant data for all these series are presented in Table 2.

Table 2: The outer involutory automorphisms

	Θ		$\overline{\Theta}$		
$\mathfrak{g}$	$\mathfrak{g}_0$	$\mathfrak{g}_1$	$\mathfrak{g}_{\bar 0}$	$\mathfrak{g}_{\bar 1}$	$\#W_{\bar 0}/W_0$
$\mathfrak{sl}_{2n}$	$\mathfrak{so}_{2n}$	$\mathbb{V}_{2\varphi_1}$	$\mathfrak{sp}_{2n}$	$\mathbb{V}_{\varphi_2}$	2
$\mathfrak{so}_{2n+2m+2}$	$\mathfrak{so}_{2n+1}\oplus\mathfrak{so}_{2m+1}$	$\mathbb{V}_{\varphi_1}\otimes\mathbb{V}'_{\varphi_1}$	$\mathfrak{so}_{2n+2m+1}$	$\mathbb{V}_{\varphi_1}$	$\binom{n+m}{m}$
$\mathfrak{e}_6$	$\mathfrak{sp}_8$	$\mathbb{V}_{\varphi_4}$	$\mathfrak{f}_4$	$\mathbb{V}_{\varphi_1}$	3
$\mathfrak{sl}_{2n+1}$	—	—	$\mathfrak{so}_{2n+1}$	$\mathbb{V}_{2\varphi_1}$	—

It follows from (6.2) that $\Delta_{\bar 0}\cup\Delta_{\bar 1}=\Delta_{\bar 0}$. Thus, everything lies in $\Delta_{\bar 0}$. Therefore a choice of the set of positive roots $\Delta_{\bar 0}^+$ determines $\Delta_{\bar 1}^+$, Δ_0^+, and Δ_1^+ as well. Of course, we choose $\Delta_{\bar 0}^+$ so that it is the image of Δ^+ under the projection $\mathfrak{t}^*\to(\mathfrak{t}_0)^*$.
Then $\{\Delta_0^+,\Delta_1^+\}$ and $\{\Delta_{\bar 0}^+,\Delta_{\bar 1}^+\}$ are two presentations for the totality of all restricted *positive* roots of $\mathfrak{g}$. Set $\rho_i=\frac12|\Delta_i^+|$ and $\rho_{\bar i}=\frac12|\Delta_{\bar i}^+|$ $(i=0,1)$. Recall that $\rho=\frac12|\Delta^+|$ and therefore $\Theta\rho=\rho$. That is, $\rho\in(\mathfrak{t}^*)^\Theta\simeq\mathfrak{t}_0^*$. It then follows from the above discussion that

(6.3) $$\rho=\rho_0+\rho_1=\rho_{\bar 0}+\rho_{\bar 1}=\rho_{\bar 0}+(\rho_{\bar 0})_s\ .$$

Let W_0 and $W_{\bar 0}$ be the Weyl groups of $\mathfrak{g}_0$ and $\mathfrak{g}_{\bar 0}$, respectively. Although $\Delta_0\subset\Delta_{\bar 0}$, $\mathfrak{g}_0$ is not a subalgebra of $\mathfrak{g}_{\bar 0}$ (if $\Theta\ne\overline{\Theta}$). In other words, Δ_0 is a non-closed subset of $\Delta_{\bar 0}$. Nevertheless, Prop. 4.1 applies to $W_0\subset W_{\bar 0}$ and one obtains the subset $W'\subset W_{\bar 0}$ consisting of the elements of minimal length in the cosets $\{wW_0\}$. Equivalently, $W'=\{w\in W_{\bar 0}\mid w(\Delta_0^+)\subset\Delta_{\bar 0}^+\}$. Below, we consider the Weyl chambers C_0 and $C_{\bar 0}$, and the hyperplanes H_μ $(\mu\in\Delta_1)$. They are regarded as subsets of the rational span of $\Delta_{\bar 0}$ in $\mathfrak{t}_0^*$.

6.4 Proposition.

1. The set of hyperplanes H_μ ($\mu \in \Delta_1$) cuts C_0 in $\#(W_{\bar 0}/W_0)$ small chambers;
2. The collection of ($\mathfrak{g}_0$-dominant) weights $w^{-1}\rho_{\bar 0}$ ($w \in W'$) contains representatives of all small chambers in C_0.
3. The extreme weight of $Spin_0(\mathfrak{g}_1)$ corresponding to $w^{-1}\rho_{\bar 0}$ is $\lambda_w := w^{-1}(\rho_0+\rho_1)-\rho_0 = w^{-1}\rho-\rho_0$.

Proof. To a great extent, the proof is parallel to the proof of Prop. 5.1.
1. The union $\Delta_0 \cup \Delta_1$ coincides with $\Delta_{\bar 0}$. Therefore each small chamber is isomorphic to $C_{\bar 0}$. Comparing the total number of chambers, we see that C_0 splits into $\#(W_{\bar 0}/W_0)$ small chambers.

2 & 3. By the definition of W', it follows that $w^{-1}\rho_{\bar 0}$ is $\mathfrak{g}_0$-dominant. So, we have the required number of dominant weights and it suffices to verify that the corresponding extreme weights of $Spin_0(\mathfrak{g}_1)$ are different.

Given $w \in W'$, the dominant half of Δ_1 associated with $w^{-1}\rho_{\bar 0}$ is

$$(\Delta_1)^+_w := \{\mu \in \Delta_1 \mid (w^{-1}\rho_{\bar 0}, \mu) > 0\} = \{\mu \in \Delta_1 \mid w\mu \in \Delta^+_{\bar 0}\}$$

and the corresponding extreme weight is $\lambda_w := \frac{1}{2}|(\Delta_1)^+_w|$.
Set $M_w = \{\mu \in \Delta^+_1 \mid w\mu \in \Delta^+_{\bar 0}\}$ and $\overline{M}_w = \Delta^+_1 \setminus M_w$. Then

$$\rho_{\bar 0} + \rho_{\bar 1} = \rho_0 + \rho_1 = \rho_0 + \tfrac{1}{2}|M_w| + \tfrac{1}{2}|\overline{M}_w| \ .$$

Since $w \in W'$, we have

$$w^{-1}(\rho_{\bar 0} + \rho_{\bar 1}) = \rho_0 + \tfrac{1}{2}|M_w| - \tfrac{1}{2}|\overline{M}_w| \ .$$

Noting that $|(\Delta_1)^+_w| = |M_w| - |\overline{M}_w|$, we obtain $\lambda_w = w^{-1}(\rho_{\bar 0}+\rho_{\bar 1}) - \rho_0$. Obviously, these weights are different, and we are done. □

In the next theorem, $\mathbb{V}_\lambda$ denotes a $\mathfrak{g}_0$-module.

6.5 Theorem. *Let $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ be a $\mathbb{Z}_2$-grading of outer type and $\mathfrak{g} = \mathfrak{g}_{\bar 0} \oplus \mathfrak{g}_{\bar 1}$ the associated diagram $\mathbb{Z}_2$-grading. Let W' be the set of representatives of minimal length for $W_{\bar 0}/W_0$. Then*

$$Spin_0(\mathfrak{g}_1) = \textstyle\bigoplus_{w\in W'} \mathbb{V}_{\lambda_w} \ .$$

Proof. First, note that if Θ is a diagram involutory automorphism, then $W_{\bar 0} = W_0$. Here the theorem claims that $Spin_0(\mathfrak{g}_1)$ is irreducible, with highest weight $(\rho_{\bar 0})_s$. This was already demonstrated in Theorem 2.9 and Prop. 3.8. In the general case, we proceed as follows.

By Prop. 6.4, $\bigoplus_{w\in W'} \mathbb{V}_{\lambda_w} \subset Spin_0(\mathfrak{g}_1)$, and the equality will follow from the fact that $\dim(Spin_0(\mathfrak{g}_1)^{\otimes 2})^{\mathfrak{g}_0} = \#W'$. As in the proof of Theorem 5.2, a crucial step in the next argument is of "cohomological" nature. Since $\mathfrak{t}_0$ contains regular elements, $\dim(\mathfrak{g}_1)^{\mathfrak{t}_0} = \dim\mathfrak{t} - \dim\mathfrak{t}_0$, i.e., the multiplicity of the zero weight in $\mathfrak{g}_1$ is equal to $\mathrm{rk}\,\mathfrak{g} - \mathrm{rk}\,\mathfrak{g}_0$. By Prop. 2.4(ii),

$$\wedge^\bullet\mathfrak{g}_1 \simeq 2^{\mathrm{rk}\mathfrak{g}-\mathrm{rk}\mathfrak{g}_0}{\cdot}Spin_0(\mathfrak{g}_1)^{\otimes 2}$$

and hence

$$\dim(\wedge^\bullet \mathfrak{g}_1)^{\mathfrak{g}_0} = 2^{\mathrm{rk}\mathfrak{g}-\mathrm{rk}\mathfrak{g}_0}\cdot \dim(Spin_0(\mathfrak{g}_1)^{\otimes 2})^{\mathfrak{g}_0}\ .$$

At the rest of the proof, $\Bbbk = \mathbb{C}$. Inspecting the list of the symmetric spaces of outer type and their cohomology rings over $\mathbb{C}$ (see e.g. [Ta62, § 4]) yields the equality

$$\dim H^*(G/G_0) = 2^{\mathrm{rk}\mathfrak{g}-\mathrm{rk}\mathfrak{g}_0}\cdot\#(W_{\bar 0}/W_0)\ .$$

Since $(\wedge^\bullet\mathfrak{g}_1)^{\mathfrak{g}_0} \simeq H^*(G/G_0)$, we are done. □

The following proof, although also being not free of case-by-case arguments, does not appeal to $\mathbb{C}$.

(6.6) Another proof of Theorem 6.5. Arguing as in (5.6) and using Prop. 6.4(3), we obtain

(6.7)
$$\mathrm{ch}\Bigl(\bigoplus_{w\in W'} \mathbb{V}_{\lambda_w}\Bigr) = \sum_{w\in W'} \frac{\sum_{\tilde w\in W_0} \varepsilon_0(\tilde w) e^{\tilde w(\rho_0+\lambda_w)}}{\prod_{\alpha\in\Delta_0^+}(e^{\alpha/2}-e^{-\alpha/2})} =$$
$$= \frac{\sum_{w\in W'}\sum_{\tilde w\in W_0}\varepsilon_0(\tilde w)e^{\tilde w w^{-1}\rho}}{\prod_{\alpha\in\Delta_0^+}(e^{\alpha/2}-e^{-\alpha/2})} = \frac{\sum_{w\in W_{\bar 0}}\tau(w)e^{w\rho}}{\prod_{\alpha\in\Delta_0^+}(e^{\alpha/2}-e^{-\alpha/2})}.$$

Here $\tau(w)$ is the cunning parity for $w\in W_{\bar 0}$, relative to the subgroup W_0. To get another expression for the numerator, we exploit the following observation concerning the pairs $(\mathfrak{g}_0,\mathfrak{g}_{\bar 0})$ in Table 2. Although Δ_0 is not closed in $\Delta_{\bar 0}$, the dual root system $\widetilde\Delta_0$ is closed in $\widetilde\Delta_{\bar 0}$ and, moreover, it is a "symmetric" subset. That is, the partition $\widetilde\Delta_{\bar 0} = \widetilde\Delta_0 \sqcup (\widetilde\Delta_{\bar 0}\setminus\widetilde\Delta_0)$ arises from an *inner* involutory automorphism of the "dual" Lie algebra. (E.g. the pair $(\mathbf{C}_4,\mathbf{F}_4)$ inverts in $(\mathbf{B}_4,\mathbf{F}_4)$.) Here $\widetilde\Delta_{\bar 0} = (\Delta_{\bar 0})_l \sqcup 2(\Delta_{\bar 0})_s$. Recall from (6.2) that $\Delta_{\bar 1} = (\Delta_{\bar 0})_s = (\Delta_0)_s$. This means in particular that $\Delta_{\bar 0}\setminus\Delta_0$ consists of long roots and these are exactly the roots constituting $(\Delta_1)_l$. Hence $\widetilde\Delta_{\bar 0}\setminus\widetilde\Delta_0 = (\Delta_1)_l$. After these preparations, write out the identity from Theorem 5.4 for the partition $\widetilde\Delta_{\bar 0} = \widetilde\Delta_0\sqcup(\Delta_1)_l$:

(6.8)
$$\sum_{w\in W_{\bar 0}}\tau(w)e^{w\tilde\rho} = \prod_{\alpha\in\widetilde\Delta_0^+}(e^{\alpha/2}-e^{-\alpha/2})\prod_{\mu\in(\Delta_1^+)_l}(e^{\mu/2}+e^{-\mu/2}).$$

Here $\tilde\rho := \frac12|\widetilde\Delta_{\bar 0}^+| = \frac12|(\Delta_{\bar 0}^+)_l| + |(\Delta_{\bar 0}^+)_s| = (\rho_{\bar 0})_l + 2(\rho_{\bar 0})_s = \rho_{\bar 0} + (\rho_{\bar 0})_s = \rho$ (see Eq. 6.3). Transforming the first factor on the right hand side of Eq. (6.8) yields

$$\prod_{\alpha\in\widetilde\Delta_0^+}(e^{\alpha/2}-e^{-\alpha/2}) = \prod_{\alpha\in(\Delta_0^+)_s}(e^{\alpha}-e^{-\alpha})\prod_{\beta\in(\Delta_0^+)_l}(e^{\beta/2}-e^{-\beta/2}) =$$
$$= \prod_{\alpha\in(\Delta_0^+)_s}(e^{\alpha/2}-e^{-\alpha/2})(e^{\alpha/2}+e^{-\alpha/2})\prod_{\beta\in(\Delta_0^+)_l}(e^{\beta/2}-e^{-\beta/2}) =$$
$$= \prod_{\alpha\in\Delta_0^+}(e^{\alpha/2}-e^{-\alpha/2})\prod_{\mu\in(\Delta_0^+)_s}(e^{\mu/2}+e^{-\mu/2}).$$

Since $(\Delta_0^+)_s = (\Delta_1^+)_s$, the whole expression on the right hand side of Eq. (6.8) is equal to

$$\prod_{\alpha\in\Delta_0^+}(e^{\alpha/2}-e^{-\alpha/2})\prod_{\alpha\in\Delta_1^+}(e^{\mu/2}+e^{-\mu/2}).$$

Substituting this expression for $\sum_{w\in W_{\overline{0}}}\tau(w)e^{w\rho}$ in Eq. (6.7), we obtain

$$\mathrm{ch}\Big(\bigoplus_{w\in W'}\mathbb{V}_{\lambda_w}\Big)=\prod_{\alpha\in\Delta_1^+}(e^{\mu/2}+e^{-\mu/2})=\mathrm{ch}\,Spin(\mathfrak{g}_1). \qquad \text{q.e.d.}$$

(6.9) Connexion with cohomology of symmetric spaces. Using classical structure results on $H^*(G/G_0)$ (see e.g. [On95, ch.3]), one may notice some interesting coincidences.

1. Recall that the image of the canonical map $\eta: H^*(G/G_0)\to H^*(G)$ is generated by primitive elements. Actually, there exists a subalgebra of $H^*(G/G_0)$ that is mapped isomorphically onto $\mathrm{Im}\,(\eta)$. It is called *Samelson* and denoted by $Sam(G/G_0)$. There exists also another subalgebra of $H^*(G/G_0)$, which is called *characteristic* and denoted by ${}_0H^*(G/G_0)$. It is the zero component for a natural grading in $H^*(G/G_0)$. Then $H^*(G/G_0)\simeq Sam(G/G_0)\otimes {}_0H^*(G/G_0)$. This holds not only for the symmetric spaces but for a wider class of *formal* homogeneous spaces, see [loc. cit., § 12, Th.2]. A general fact for the formal homogeneous spaces is that $\dim Sam(G/G_0)=2^{\mathrm{rk}\mathfrak{g}-\mathrm{rk}\mathfrak{g}_0}$. Hence $\dim Spin_0(\mathfrak{g}_1)=\dim {}_0H^*(G/G_0)$, which suggests that $Spin_0(\mathfrak{g}_1)$ might somehow be related to the characteristic subalgebra of $H^*(G/G_0)$.

2. Note that $H^*(G/G_0)={}_0H^*(G/G_0)$ if and only if Θ is inner, and $H^*(G/G_0)=Sam(G/G_0)$ if and only if Θ is a diagram involutory automorphism. In the mixed case, the associated diagram involutory automorphism $\overline{\Theta}$ seems to yield a splitting for $H^*(G/G_0)$. Namely, one has $\dim Sam(G/G_0)=\dim H^*(G/G_{\overline{0}})$ and $\dim {}_0H^*(G/G_0)=\dim H^*(\widetilde{G}_{\overline{0}}/\widetilde{G}_0)$, where $\widetilde{G}_{\overline{0}}$ and $\widetilde{G}_0$ are the groups corresponding to the dual root systems $\widetilde{\Delta}_{\overline{0}}$ and $\widetilde{\Delta}_0$. Moreover, these two pairs of *graded* algebras have equal Poincaré polynomials and it is likely that they are naturally isomorphic. I think that a better understanding of this situation as well as elimination of the case-by-case arguments in section 6 can be achieved through application of the theory of twisted affine Kac–Moody algebras.

3. In the above exposition, $\overline{\Theta}$ has appeared as *deus ex machina*. But the Kac-Moody theory provides some explanation for this. Namely, the outer automorphism Θ determines the twisted affine Kac-Moody algebra $\hat{\mathcal{L}}(\mathfrak{g},\Theta,2)=\hat{\mathcal{L}}(\mathfrak{g})$ [Ka90, ch. 8]. Next, $\hat{\mathcal{L}}(\mathfrak{g})$ has the standard $\mathbb{Z}$-grading associated with the special vertex of the Dynkin diagram of $\hat{\mathcal{L}}(\mathfrak{g})$. If $\mathfrak{g}\ne\mathfrak{so}_{2n+1}$, this $\mathbb{Z}$-grading determines another outer automorphism of $\mathfrak{g}$, which is just $\overline{\Theta}$.

6.10 Examples. 1. $\mathfrak{g}=\mathfrak{sl}_{2n}$, $\mathfrak{g}_0=\mathfrak{so}_{2n}$. Then $\mathfrak{g}_1\simeq\mathbb{V}_{2\varphi_1}$. As indicated in Table 2, $\mathfrak{g}_{\overline{0}}=\mathfrak{sp}_{2n}$ and therefore $\#(W_{\overline{0}}/W_0)=2$. Here $\Delta_0=\{\pm\varepsilon_i\pm\varepsilon_j\mid 1\le i,j\le n, i\ne j\}$ and $\Delta_{\overline{0}}=\{\pm\varepsilon_i\pm\varepsilon_j,\pm2\varepsilon_i\}$. This example is a kind of outer version of Example 5.7(1). Indeed, taking the "dual" Lie algebras for $(\mathfrak{g}_0,\mathfrak{g}_{\overline{0}})$ yields the symmetric pair considered there. In our case, $\Delta_1=\Delta_{\overline{0}}$ and $W'=\{id,w_n\}$, where $w_n(\varepsilon_i)=\varepsilon_i$ $(i\le n-1)$ and $w_n(\varepsilon_n)=-\varepsilon_n$. Therefore $(\Delta_1)^+_{id}=\Delta^+_{\overline{0}}=\Delta^+_0\cup\{2\varepsilon_1,\dots,2\varepsilon_n\}$ and $(\Delta_1)^+_{w_n}=\Delta^+_0\cup\{2\varepsilon_1,\dots,2\varepsilon_{n-1},-2\varepsilon_n\}$. Hence the extreme weights are $\rho_0+2\varphi_n$ and $\rho_0+2\varphi_{n-1}$. Thus, $Spin_0(\mathbb{V}_{2\varphi_1})=\mathbb{V}_{\rho_0+2\varphi_{n-1}}\oplus\mathbb{V}_{\rho+2\varphi_n}$ and, because $m_{2\varphi_1}(0)=n-1$, $\wedge^\bullet\mathbb{V}_{2\varphi_1}=2^{n-1}\cdot(\mathbb{V}_{\rho+2\varphi_{n-1}}\oplus\mathbb{V}_{\rho+2\varphi_n})^{\otimes2}$.

2. $\mathfrak{g} = \mathfrak{e}_6$, $\mathfrak{g}_0 = \mathfrak{sp}_8$. Then $\mathfrak{g}_1 \simeq \mathbb{V}_{\varphi_4}$. As indicated in Table 2, $\mathfrak{g}_{\bar{0}} = \mathfrak{f}_4$ and hence $\#(W_{\bar{0}}/W_0) = 3$. This is the outer version of Example 5.7(2). Here $\Delta_{\bar{0}} = \{\pm\varepsilon_i \pm \varepsilon_j, \pm\varepsilon_i, (\pm\varepsilon_1 \pm\varepsilon_2 \pm\varepsilon_3 \pm\varepsilon_4)/2 \mid 1 \le i,j \le 4, i \ne j\}$ and $\Delta_0 = (\Delta_{\bar{0}})_s \cup \{\pm\varepsilon_1 \pm\varepsilon_2, \pm\varepsilon_3 \pm\varepsilon_4\}$. The standard set of simple roots for $\Delta_{\bar{0}}$ is $\alpha'_1 = \frac{1}{2}(\varepsilon_1 - \varepsilon_2 - \varepsilon_3 - \varepsilon_4)$, $\alpha'_2 = \varepsilon_4$, $\alpha'_3 = \varepsilon_3 - \varepsilon_4$, $\alpha'_4 = \varepsilon_2 - \varepsilon_4$. The roots for $\mathfrak{sp}_8$ have non standard presentation, but it is not hard to find that the simple roots in $\Delta^+_{\bar{0}} \cap \Delta_0$ are $\alpha_1 = \varepsilon_2$, $\alpha_2 = \frac{1}{2}(\varepsilon_1 - \varepsilon_2 - \varepsilon_3 - \varepsilon_4)$, $\alpha_3 = \varepsilon_4$, $\alpha_4 = \varepsilon_3 - \varepsilon_4$. Therefore the fundamental weights for $\mathfrak{sp}_8$ are $\varphi_1 = \frac{1}{2}(\varepsilon_1 + \varepsilon_2)$, $\varphi_2 = \varepsilon_1$, $\varphi_3 = \varepsilon_1 + \frac{1}{2}(\varepsilon_3 + \varepsilon_4)$, $\varphi_4 = \varepsilon_1 + \varepsilon_3$.
Then an explicit verification shows that $W' = \{id, w', w''\}$, where w', w'' are permutations of $\{\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4\}$ determined by the cycles (23) and (432), respectively. (E.g. $w''(\varepsilon_4) = \varepsilon_3$.)
The direct computation of the extreme weights gives:

$$\lambda_{id} = \rho_{\bar{0}} + \rho_{\bar{1}} - \rho_0 = \tfrac{1}{2}(9\varepsilon_1 + 5\varepsilon_2 + \varepsilon_3 + \varepsilon_4) = 5\varphi_1 + \varphi_2 + \varphi_3,$$
$$\lambda_{w'} = (w')^{-1}(\rho_{\bar{0}} + \rho_{\bar{1}}) - \rho_0 = \tfrac{1}{2}(9\varepsilon_1 + 3\varepsilon_2 + 3\varepsilon_3 + \varepsilon_4) = 3\varphi_1 + \varphi_2 + \varphi_3 + \varphi_4 = \rho_0 + 2\varphi_1,$$
$$\lambda_{w''} = (w'')^{-1}(\rho_{\bar{0}} + \rho_{\bar{1}}) - \rho_0 = \tfrac{1}{2}(9\varepsilon_1 + \varepsilon_2 + 3\varepsilon_3 + 3\varepsilon_4) = \varphi_1 + \varphi_2 + 3\varphi_3.$$

Whence

$$Spin_0(\mathbb{V}_{\varphi_4}) = \mathbb{V}_{5\varphi_1+\varphi_2+\varphi_3} \oplus \mathbb{V}_{\varphi_1+\varphi_2+3\varphi_3} \oplus \mathbb{V}_{\rho_0+2\varphi_1}$$

and

$$\wedge^\bullet \mathbb{V}_{\varphi_4} = 4\,(\mathbb{V}_{5\varphi_1+\varphi_2+\varphi_3} \oplus \mathbb{V}_{\varphi_1+\varphi_2+3\varphi_3} \oplus \mathbb{V}_{\rho_0+2\varphi_1})^{\otimes 2}\ .$$

7 Decomposably-generated ‘Spin’ modules and a Casimir element

Recall that we have given a geometric description of the extreme weights of Spin-representations in (3.4).

Definition. Given an orthogonal $\mathfrak{g}$-module $\mathbb{V}$, the $\mathfrak{g}$-submodule of $Spin_0(\mathbb{V})$ generated by the extreme weight vectors is denoted by $Spin_0^{dg}(\mathbb{V})$; $Spin_0(\mathbb{V})$ is called *decomposably-generated*, if it is equal to $Spin_0^{dg}(\mathbb{V})$, i.e., if all its highest weights are extreme.

Since the extreme weights are of multiplicity 1, “decomposably-generated” implies “multiplicity free”. As a consequence of previous development, we have

7.1 Proposition. *Let $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ be a $\mathbb{Z}_2$-graded semisimple Lie algebra. Then the $\mathfrak{g}_0$-module $Spin_0(\mathfrak{g}_1)$ is decomposably-generated (and multiplicity free).*

Proof. The problem immediately reduces to the case in which $\mathfrak{g}$ is an irreducible $\mathbb{Z}_2$-graded algebra. Then either $\mathfrak{g}$ is simple or $\mathfrak{g} \simeq \mathfrak{h} \oplus \mathfrak{h}$, where $\mathfrak{h}$ is simple and $\Theta(h_1, h_2) = (h_2, h_1)$. In the second case, $\mathfrak{g}_0 \simeq \mathfrak{h}$ is the diagonal in $\mathfrak{g}$, and $\mathfrak{g}_1 \simeq \mathfrak{g}_0$ as $\mathfrak{g}_0$-module. Here the conclusion follows by Kostant's result, see Example 2.5(1). In the first case, for $\mathfrak{g}_0$ of inner type, use Prop. 5.1(3) and Theorem 5.2; for $\mathfrak{g}_0$ of outer type, use Prop. 6.4(3) and Theorem 6.5. □

An explanation of the term “decomposably-generated” comes from Example 2.5(3). If $\mathbb{V} = \mathbb{W} \oplus \mathbb{W}^*$, then $Spin(\mathbb{V}) \simeq \Bbbk_{-\nu} \otimes \wedge^\bullet \mathbb{W}$ and each extreme weight vector is represented

by a decomposable vector in the exterior algebra.
I think that the property of being "decomposably-generated" characterizes the representations of the form $Spin_0(\mathfrak{g}_1)$, i.e.,

7.2 Conjecture. *Let $\mathbb{V}$ be an orthogonal $\mathfrak{g}$-module. Then $Spin_0(\mathbb{V})$ is decomposably-generated if and only if $\tilde{\mathfrak{g}} := \mathfrak{g} \oplus \mathbb{V}$ is a $\mathbb{Z}_2$-graded semisimple Lie algebra.*

The conjecture will be proved in a particular case. Until the end of the section, the following situation is being considered: $\mathfrak{g}$ is semisimple, $\mathfrak{h}$ is a reductive subalgebra of $\mathfrak{g}$, and $\mathfrak{m} := \mathfrak{h}^\perp \subset \mathfrak{g}$. Then $\mathfrak{m}$ is an orthogonal $\mathfrak{h}$-module and $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ is a vector space sum. Clearly, this decomposition is a $\mathbb{Z}_2$-grading if and only if $[\mathfrak{m}, \mathfrak{m}] \subset \mathfrak{h}$. The representation $\mathfrak{h} \to \mathfrak{so}(\mathfrak{m})$ is the isotropy representation of the affine homogeneous space G/H. Our aim is to study $Spin_0(\mathfrak{m})$ and $Spin_0^{dg}(\mathfrak{m})$ in the equal rank case. That is, it is assumed from now on that $\mathrm{rk}\,\mathfrak{h} = \mathrm{rk}\,\mathfrak{g}$ and, more precisely, $\mathfrak{t} \subset \mathfrak{h}$. Then $\Delta^+ = \Delta^+_{\mathfrak{h}} \sqcup \Delta^+_{\mathfrak{m}}$, $\mathfrak{m}$ has no zero weight and $\wedge^\bullet\mathfrak{m} \simeq (Spin_0(\mathfrak{m}))^{\otimes 2}$. Denoting by $W_{\mathfrak{h}}$ the Weyl group of $(\mathfrak{h}, \mathfrak{t})$, one may consider the minimal length "section" $W^{\mathfrak{h}}$ for $W \to W/W_{\mathfrak{h}}$ and the cunning parity $\tau : W \to \{1, -1\}$, determined by $\Delta^+_{\mathfrak{h}}$. The proof of Prop. 5.1 applies in the present situation as well. This yields exactly $\#W^{\mathfrak{h}}$ extreme weights of $Spin_0(\mathfrak{m})$. Hence

$$\dim(\wedge^\bullet\mathfrak{m})^{\mathfrak{h}} = \dim(Spin_0(\mathfrak{m})^{\otimes 2})^{\mathfrak{h}} \geq \#W^{\mathfrak{h}} \ .$$

For $w \in W^{\mathfrak{h}}$, the corresponding extreme weight is $\lambda_w = w^{-1}\rho - \rho_{\mathfrak{h}}$, where $\rho_{\mathfrak{h}} = \frac{1}{2}|\Delta^+_{\mathfrak{h}}|$. Therefore, arguing as in (5.6), we obtain

$$\mathrm{ch}\,(Spin_0^{dg}(\mathfrak{m})) = \mathrm{ch}\,\Big(\bigoplus_{w \in W^{\mathfrak{h}}} \mathbb{V}_{\lambda_w}\Big) = \frac{\sum_{w\in W} \tau(w) e^{w\rho}}{\prod_{\alpha \in \Delta^+_{\mathfrak{h}}} (e^{\alpha/2} - e^{-\alpha/2})} \ .$$

Recall that $\mathrm{ch}\, Spin_0(\mathfrak{m}) = \prod_{\mu \in \Delta^+_{\mathfrak{m}}} (e^{\mu/2} + e^{-\mu/2})$. On the other hand, $\dim H^*(G/H) = \#W^{\mathfrak{h}}$ [On95, § 13, Th. 2] and $H^*(G/H)$ can be computed via the complex of G-invariant exterior forms on G/H, i.e., the complex $((\wedge^\bullet(\mathfrak{g}/\mathfrak{h})^*)^{\mathfrak{h}}, d)$, where d is the usual Lie algebra coboundary operator. We shall identify the $\mathfrak{h}$-modules $(\mathfrak{g}/\mathfrak{h})^*$ and $\mathfrak{m}$. Having compared the previous expressions, we obtain

7.3 Proposition. *Let $\mathfrak{h} \subset \mathfrak{g}$ be a reductive subalgebra of maximal rank and $\mathfrak{h} \to \mathfrak{so}(\mathfrak{m})$ the isotropy representation. Then the following conditions are equivalent:*

(i) *d is trivial on $(\wedge^\bullet\mathfrak{m})^{\mathfrak{h}}$;*
(ii) *$\dim(\wedge^\bullet\mathfrak{m})^{\mathfrak{h}} = \#W^{\mathfrak{h}}$;*
(iii) *$Spin_0(\mathfrak{m}) = Spin_0^{dg}(\mathfrak{m})$;*
(iv) *$\sum_{w\in W} \tau(w) e^{w\rho} = \prod_{\alpha \in \Delta^+_{\mathfrak{h}}} (e^{\alpha/2} - e^{-\alpha/2}) \prod_{\mu \in \Delta^+_{\mathfrak{m}}} (e^{\mu/2} + e^{-\mu/2})$.* □

Thus, conjecture 7.2 claims that neither of these conditions holds unless G/H is symmetric.

7.4 Theorem. *Let $\mathfrak{h}$ be a reductive subalgebra of $\mathfrak{g}$, with $\mathrm{rk}\,\mathfrak{h} = \mathrm{rk}\,\mathfrak{g}$, and $\mathfrak{m} := \mathfrak{h}^\perp$. Suppose $[\mathfrak{m}, \mathfrak{m}] \not\subset \mathfrak{h}$; then d is non-trivial on $(\wedge^\bullet\mathfrak{m})^{\mathfrak{h}}$. More precisely, $d((\wedge^3\mathfrak{m})^{\mathfrak{h}}) \neq 0$.*

Proof. For any $x \in \mathfrak{g}$, let $x_{\mathfrak{h}}$ and $x_{\mathfrak{m}}$ denote its components in $\mathfrak{h}$ and $\mathfrak{m}$, respectively. Given $x, y \in \mathfrak{m}$, consider the decomposition $[x,y] = [x,y]_{\mathfrak{h}} + [x,y]_{\mathfrak{m}}$. We regard $[\ ,\ ]_{\mathfrak{m}}$ as mapping from $\mathfrak{m} \times \mathfrak{m}$ to $\mathfrak{m}$, and likewise for $[\ ,\ ]_{\mathfrak{h}}$. By assumption, $[\ ,\ ]_{\mathfrak{m}} \not\equiv 0$. On the other hand, applying construction from section 1 (see (1.4) and around) to $\mathfrak{h}$ and $\mathfrak{m}$ in place of $\mathfrak{g}$ and $\mathbb{V}$, we see that $[x,y]_{\mathfrak{h}} = \bar\mu(x,y)$ for any $x, y \in \mathfrak{m}$.

Define the 3-form $\Psi : \wedge^3\mathfrak{m} \to \Bbbk$ by $\Psi(x,y,z) = \Phi([x,y],z)$. Obviously, Ψ is $\mathfrak{h}$-invariant. The assumption $[\mathfrak{m},\mathfrak{m}] \not\subset \mathfrak{h}$ precisely means that $\Psi \not\equiv 0$. We shall prove $d\Psi \neq 0$. To compute $d\Psi$, we regard Ψ as $\mathfrak{h}$-invariant 3-form on $\mathfrak{g}$, orthogonal to $\mathfrak{h}$, and use the standard formula for d. The resulting expression is

$$d\Psi(x,y,z,u) = 2\Big(\Phi([x,y]_{\mathfrak{m}},[z,u]_{\mathfrak{m}}) + \Phi([y,z]_{\mathfrak{m}},[x,u]_{\mathfrak{m}}) + \Phi([z,x]_{\mathfrak{m}},[y,u]_{\mathfrak{m}})\Big)\ .$$

Since $[x,y] = \bar\mu(x,y) + [x,y]_{\mathfrak{m}}$, $\mathfrak{h}$ is orthogonal to $\mathfrak{m}$, and $([x,y],[z,u]) + ([y,z],[x,u]) + ([z,x],[y,u]) = 0$ (because of the Jacobi identity), we have

$$d\Psi(x,y,z,u) = -2\Big(\Phi(\bar\mu(x,y),\bar\mu(z,u)) + \Phi(\bar\mu(y,z),\bar\mu(x,u)) + \Phi(\bar\mu(z,x),\bar\mu(y,u))\Big)\ .$$

In the notation of Prop. 1.5, for $\mathfrak{h}$ and $\mathfrak{m}$ in place of $\mathfrak{g}$ and $\mathbb{V}$, this means that $d\Psi = -2\kappa$. Assume that $\kappa = 0$. Then $\mathfrak{h}\oplus\mathfrak{m}$ equipped with the modified multiplication $[\ ,\ ]^{\sim}$ becomes a $\mathbb{Z}_2$-graded Lie algebra (see Prop. 1.5). Clearly, the multiplication does change only for pairs of elements in $\mathfrak{m}$: $[m_1,m_2]^{\sim} := [m_1,m_2]_{\mathfrak{h}}$, whereas the structure of Lie algebra on $\mathfrak{h}$ and the $\mathfrak{h}$-module structure on $\mathfrak{m}$ remain undisturbed. Let $\tilde{\mathfrak{g}}$ denote the Lie algebra with modified multiplication and Θ the corresponding involutory automorphism of $\tilde{\mathfrak{g}}$. It is easily seen that $\tilde{\mathfrak{g}}$ is semisimple (use the proof of Theorem 1.7) and Θ is inner (because $\mathfrak{t} \subset \mathfrak{h}$ remains a Cartan subalgebra in $\tilde{\mathfrak{g}}$). For the symmetric space $\tilde G/H$, we have $H^3(\tilde G/H) = (\wedge^3\mathfrak{m})^{\mathfrak{h}} \neq 0$. But $H^{odd}(\cdot) = 0$ for the symmetric spaces of inner type [On95, § 13, n.3]. This contradiction proves $\kappa = d\Psi \neq 0$. $\square$

7.5 Corollary. *Conjecture 7.2 is true for the isotropy representations of affine homogeneous spaces G/H with $\operatorname{rk}\mathfrak{g} = \operatorname{rk}\mathfrak{h}$.* $\square$

Remark. Theorem 7.4 is true even if $\operatorname{rk}\mathfrak{h} < \operatorname{rk}\mathfrak{g}$ and some mild conditions are satisfied (e.g. $\mathfrak{g}$ is simple). However this has no immediate relation to Conjecture 7.2.

For a reductive Lie algebra $\mathfrak{h}$, the Casimir element in $U(\mathfrak{h})$ is determined by the choice of an invariant bilinear form on $\mathfrak{h}$. If $\mathfrak{h}$ is not simple, then the choice is essentially non unique. But for the isotropy representations one has a preferred choice of the bilinear form. In the above setting, let $\Phi(\ ,\ )_{\mathfrak{h}}$ be the restriction of $\Phi(\ ,\ )$ to $\mathfrak{h}$. Notice that even if $\mathfrak{h}$ is semisimple and we begin with the Killing form on $\mathfrak{g}$, then $\Phi(\ ,\ )_{\mathfrak{h}}$ is not necessarily proportional to the Killing form on $\mathfrak{h}$. Let $c_{\mathfrak{h}} \in U(\mathfrak{h})$ be the Casimir element with respect to $\Phi(\ ,\ )_{\mathfrak{h}}$. Recall that the W-invariant scalar product on $\mathcal{P}_{\mathbb{Q}}$ is determined by $\Phi(\ ,\ )$.

7.6 Proposition. *Suppose* $\operatorname{rk}\mathfrak{h} = \operatorname{rk}\mathfrak{g}$. *Then the Casimir element $c_{\mathfrak{h}}$ acts scalarly on $Spin_0^{dg}(\mathfrak{m})$. Its eigenvalue is equal to $(\rho,\rho) - (\rho_{\mathfrak{h}},\rho_{\mathfrak{h}})$.*

Proof. As is indicated above, $Spin_0^{dg}(\mathfrak{m}) = \oplus_{w\in W^\mathfrak{h}} \mathbb{V}_{\lambda_w}$ and $\lambda_w = w^{-1}\rho - \rho_\mathfrak{h}$. Therefore the eigenvalue of $c_\mathfrak{h}$ on $\mathbb{V}_{\lambda_w}$ is $(\lambda_w + 2\rho_\mathfrak{h}, \lambda_w) = (w^{-1}\rho, w^{-1}\rho) - (\rho_\mathfrak{h}, \rho_h) = (\rho, \rho) - (\rho_\mathfrak{h}, \rho_\mathfrak{h})$. $\square$

7.7 Theorem. *Let $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ be a $\mathbb{Z}_2$-graded semisimple Lie algebra. Define the Casimir element c_0 for $\mathfrak{g}_0$ using the restriction of $\Phi(\ ,\ )$ to $\mathfrak{g}_0$. Then c_0 acts on $Spin_0(\mathfrak{g}_1)$ scalarly, with value $(\rho, \rho) - (\rho_0, \rho_0)$.*

Proof. 1. If Θ is inner, then $\operatorname{rk}\mathfrak{g}_0 = \operatorname{rk}\mathfrak{g}$; we conclude by Propositions 7.1, 7.6.
2. If Θ is outer, some accuracy is needed, since $\operatorname{rk}\mathfrak{g}_0 < \operatorname{rk}\mathfrak{g}$. We use notation and information from section 6. Since $Spin_0(\mathfrak{g}_1) = Spin_0^{dg}(\mathfrak{g}_1) = \oplus_{w\in W'} \mathbb{V}_{\lambda_w}$ and $\lambda_w = w^{-1}\rho - \rho_0$, the value of c_0 on $\mathbb{V}_{\lambda_w}$ is equal to $(w^{-1}\rho, w^{-1}\rho) - (\rho_0, \rho_0)$. Here $W' \subset W_{\overline{0}}$, where $W_{\overline{0}}$ is the Weyl group of $\mathfrak{g}_{\overline{0}}$. Recall that $\mathfrak{t}_0 = \mathfrak{t}^\Theta$ is a Cartan subalgebra for both $\mathfrak{g}_{\overline{0}}$ and $\mathfrak{g}_0$. As $\mathfrak{t}_0$ contains regular elements of $\mathfrak{t}$, we have $N_{G_{\overline{0}}}(\mathfrak{t}_0) \subset N_G(\mathfrak{t})$. Furthermore, since $G_{\overline{0}}$ is connected and $\mathfrak{t}_0$ is Cartan, we have $G_{\overline{0}} \cap T = T_0$. Therefore $W_{\overline{0}} = N_{G_{\overline{0}}}(\mathfrak{t}_0)/T_0$ can be identified with a subgroup of $W = N_G(\mathfrak{t})/T$. Hence $(w^{-1}\rho, w^{-1}\rho) = (\rho, \rho)$. $\square$

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