Max-Planck-Institut für Mathematik
Bonn

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Preprint Series 2021 (22)

Date of submission: May 27, 2021

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MPIM 21-22

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April-May, 2021
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Abstract

Algebraic differential operators are described acting on automorphic forms $\boldsymbol{\varphi}$ on unitary groups $U(n,n)$ over an imaginary quadratic field $\mathcal{K} = \mathbb{Q}(\sqrt{-D_{\mathcal{K}}}) \subset \mathbb{C}$.

Applications are given to Shimura's zeta functions $L(s, \mathbf{f})$ [Shi00] attached special L-values $L(s, \boldsymbol{\varphi})$ attached to $\boldsymbol{\varphi}$. and normalized in accordance with Deligne's Gamma factors rule [De79]. An explicit description of Shimura's Γ-factors is used.

Figure 1: Workshop "Automorphic forms and integrable systems" February 26, 2020, Sochi, Russia https://trv-science.ru/2020/04/sirius/

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0.1 Algebraic differential operators in the simplest case of modular forms for $\Gamma = \mathrm{SL}_2(\mathbb{Z})$

Action of the derivative $D = \frac{1}{2\pi i}\frac{d}{dz} = q\frac{d}{dq}$(where $q = e^{2\pi i z}$) on a modular form $g = \sum_{n=0}^{\infty} b_n q^n$ is not a modular form, but it is quasi-modular ([DZ], p.59, [MaRo], p.67): the function $f = D^r g = \sum_{n=0}^{\infty} n^r b_n q^n$ satisfies the following transformation law:

$$(cz+d)^{-\ell-2r} D^r g(\gamma z) = \sum_{t=0}^{r} \binom{r}{t} \frac{\Gamma(r+\ell)}{\Gamma(t+\ell)} \left(\frac{1}{2\pi i}\frac{c}{cz+d}\right)^{r-t} D^t g(z)$$

for a modular form $g \in \mathcal{M}_\ell(\Gamma)$ of weight ℓ, $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma$.
In order to adjust it to the weight $\ell + 2r$, let us use $S = \frac{1}{4\pi y}$, $\mathrm{Im}\, z = \frac{z-\bar{z}}{2i}$, and $\frac{1}{\mathrm{Im}\,\gamma z} = \frac{|cz+d|^2}{\mathrm{Im}\, z} = (cz+d)\left(-2ic + \frac{cz+d}{y}\right)$:

0.2 Maass-Shimura differential operator

If $f = D^r g$ where $g \in \mathcal{M}_\ell(\Gamma)$ is a modular form of weight ℓ, then the transformation law produces also the Maass-Shimura differential operator δ_ℓ to the space of *nearly holomorphic forms* of weight $\ell + 2r$:

$$\delta_\ell^r g(z) = \sum_{t=0}^{r} \binom{r}{t} \frac{\Gamma(r+\ell)}{\Gamma(r-t+\ell)} (-S)^t D^{r-t} g(z), \text{ where } S = \frac{1}{4\pi y},$$

which preserves the rationality of the coefficients of S and q. It comes again from the above transformation law of $D^r g$. Notice:
$\delta_\ell(g) = \frac{1}{2\pi i} y^{-\ell} \frac{\partial}{\partial z}(y^\ell g) = \frac{1}{2\pi i}\left(\frac{\partial g(z)}{\partial z} + \frac{\ell}{2iy} g(z)\right) = (D - \ell S)(g)$, which is of weight $\ell + 2$ and its *degree of near holomorphy* (in the variable S) is increased by one.
For an integer $r \geq 0$, $\delta_\ell^r := \delta_{\ell+2r-2} \circ \cdots \circ \delta_\ell$ (see also [U14]).

A conceptual explanation of the *algebraicity* comes from the Gauss-Manin connection (due to Grothendieck in higher dimensions see [Gr66], [KaOd68]).

0.3 Algebraic differential operators on symplectic groups

On scalar-valued Siegel modular forms: Let $Z = (z_{ij}) \in \mathrm{GL}_n(\mathbb{C})$, $Z = {}^t Z$, $\partial_{ij} = \frac{1}{2\pi\sqrt{-1}} \begin{cases} \frac{\partial}{\partial z_{ij}} & i = j \\ \frac{1}{2}\frac{\partial}{\partial z_{ij}} & i \neq j \end{cases}$, *Maass operator* $\Delta = \det(\partial_{ij})$ acts by $\Delta q^T = \det(T) q^T$

on $q^T = \exp(2\pi i \mathrm{tr}(TZ))$.
The *Maass-Shimura operator* $\delta_k f(Z) = (-4\pi)^{-n}\det(Z-\bar{Z})^{\frac{1+n}{2}-k}\Delta(\det(Z-\bar{Z})^{k-\frac{1+n}{2}+1}f(Z))$
acts on q^T via the polynomial representations $\rho_r : \mathrm{GL}_n(\mathbb{C}) \to \mathrm{GL}(\wedge^r\mathbb{C}^n)$ and its adjoint ρ_r^* (see [CourPa]) $\delta_k(q^T) = \sum_{\ell=0}^{n}(-1)^{n-\ell}c_{n-\ell}(k+1-\frac{1+n}{2})\mathrm{tr}({}^t\rho_{n-\ell}(S)\rho_\ell^*(T))q^T$,
where $c_{n-\ell}(s) = s(s-\frac{1}{2})\cdots(s-\frac{n-\ell-1}{2})$, $S = (2\pi i(\bar{z}-z))^{-1}$.
For a $\mathbb{C}^d$-valued Siegel modular form f this *algebraic operator* extends to a $\mathbb{C}^d$-valued smooth function of $Z = (z_{ij}) = X + \sqrt{-1}Y$.
Let $S_e(\mathrm{Sym}^2(R^n), R^d)$ be the R-module of all polynomial maps of $\mathrm{Sym}^2(R^n)$ into R^d *homogeneous of degree* e. Define inductively $S_1(\mathrm{Sym}^2(\mathbb{C}^n), \mathbb{C}^d)$-valued smooth functions:
$(Df)(u) = \sum_{1\le i\le j\le n} u_{ij}\frac{\partial f}{\partial(2\pi\sqrt{-1}z_{ij})}$, $(Cf)(u) = (Df)((Z-\bar{Z})u(Z-\bar{Z}))$, $(C^e(f)(u) = C(C^{e-1}(f)(u)$
$D_\rho^e(f) := (\rho\otimes\tau^e)(Z-\bar{Z})^{-1}C^e(\rho(Z-\bar{Z})f)$, where
$[(\rho\otimes\tau)(\alpha)(h)](u) := \rho(\alpha)h({}^t\alpha\cdot u\cdot\alpha)$.
Then D_ρ^e equals $(2\sqrt{-1}\pi)^{-e}$ times the (vector-valued) *Maass-Shimura differential operator*.

0.4 From symplectic case (Type C) to unitary case (Type A)

Siegel modular forms of degree n are holomorphic (vector-valued) functions on $\mathbb{H}_n = \{Z = {}^tZ \in \mathbb{C}_n^n, \mathrm{Im}(Z) > 0\}$ (the Siegel space, (Type C) [Shi00]).

Automorphic forms on unitary groups (Type A) in [Shi00]
$U(a,b)$ (of degree $n = a+b$) $\rightsquigarrow$ the double group $U(n,n)$,
and the corresponding hermitian space of degree n: $\mathcal{H}_n = \{z \in \mathbb{C}_n^n \mid i(z^*-z) > 0\}$ where $z^* = {}^t\bar{z}$, $x := (z+z^*)/2$ the hermitian part of z, and $y := (z-z^*)/2$ the anti-hermitian part, such that $i(z^*-z)/2 = iy$ is a positive hermitian matrix.

Note that $z = x+iy$, but x, y are not real: for a hermitian matrix h, the real matrices $\dot{h} = \dfrac{\omega\,{}^th - \bar{\omega}h}{\omega-\bar{\omega}}$, $\ddot{h} = \dfrac{h - {}^th}{\omega-\bar{\omega}}$ are used for $\omega = \dfrac{1}{2}(\delta+\delta^{\frac{1}{2}})$, δ the discriminant of $\mathcal{K}$, so that $h = \dot{h} + \omega\ddot{h}$ (notation in [Bra51]).

Automorphic L functions on unitary groups and related geometric objects where discussed by M. Harris (ICM 2014), *Automorphic Galois representations and the cohomology of Shimura varieties.*, [Ha14], and by P.Scholze (ICM 2018), Applications of p-adic geometry to automorphic Galois representations on unitary groups in [Scho18].

1 Unitary groups and forms, [Ha97],[EE], [Shi00]

1.1 Unitary groups $U(a,b)$ $(a+b=n)$ and $U(n,n)$ (the double group)

Let V be an n-dimensional space over an imaginary quadratic field $\mathcal{K}=\mathbb{Q}(\sqrt{-D_{\mathcal{K}}})$, and let $\langle\cdot,\cdot\rangle$ be a *non degenerate hermitian pairing* of signature (a,b) on V relative to $\mathcal{K}\subset\mathbb{C}$.
Let us write $-V$ for the vector space V with the pairing $\langle\cdot,\cdot\rangle_{-V}=-\langle\cdot,\cdot\rangle_V$ (of signature (b,a)).

Let $2V$ denote the *double vector space* $V\oplus V$ with the pairing $\langle\cdot,\cdot\rangle_{2V}$ defined for all vectors $v_1,v_2,w_1,w_2\in V$ by $\langle(v_1,v_2),(w_1,w_2)\rangle_{2V}:=\langle(v_1,w_1)\rangle_V+\langle(v_2,w_2)\rangle_{-V}$
(of signature $(b+a,a+b)=(n,n)$).

For a vector space W with hermitian pairing $\langle\cdot,\cdot\rangle_W$, and a $\mathbb{Q}$-algebra R, the *unitary groups are defined by* $U(W)(R)=\{g\in \mathrm{GL}(W\otimes R)|\forall v,v',\langle gv,gv'\rangle=\langle v,v'\rangle\}$
$GU(W)(R)=\{g\in \mathrm{GL}(W\otimes R)|\forall v,v',\exists\nu(g)\in R^*,\langle gv,gv'\rangle=\nu(g)\langle v,v'\rangle\}$. Then

$$U(2V)(R)\cong U(n,n)(R)=\left\{M=\begin{pmatrix}A&B\\C&D\end{pmatrix}\in \mathrm{GL}_{2n}(\mathcal{K}\otimes R)|M\eta_nM^*=\eta_n\right\},$$

where $\eta_n=\begin{pmatrix}0_n&-I_n\\I_n&0_n\end{pmatrix}$. The group $U(n,n)$ acts on the *hermitian space*

$$\mathcal{H}_n=\{z\in\mathbb{C}_n^n\mid i(z^*-z)>0\},\ \text{with } z^*:={}^t\bar z.$$

1.2 Algebraic geometric approach: families of abelian varieties of CM-type and unitary groups

Main arithmetical applications of unitary groups $U(n,n)$ use Shimura's analytic families of abelian varieties A of CM-type of $\dim_{\mathbb{C}}A=2n$, that is, with fixed imbedding $\iota:\mathcal{K}\hookrightarrow \mathrm{End}(A)\otimes\mathbb{Q}$, and other PEL-structures ("polzarization, endomorphisms, level", following [EE], §2).

Recall that elliptic curves E with complex multiplication by $\mathcal{K}$ correspond to certain CM-points on the upper half plaine $\mathbb{H}$, that is $E\xrightarrow{\sim}\mathbb{C}/L$, where $L=\langle 1,\alpha\rangle\subset\mathcal{K}=\mathbb{Q}(\alpha)$ is a lattice in $\mathbb{C}$ and $\mathrm{Im}\,(\alpha)>0$ (only *special CM-points, not analytic families*).

Families of $2n$*-dimensional CM-abelian varieties* A use the analytic parameter $z\in\mathcal{H}_n$. Any row vector $x\in\mathcal{K}^1_{2n}$ defines a z-holomorphic $\mathbb{C}^{2n}$-valued function $p_z(x)$ by

$$p_z(x)=\left([z,1_n]\cdot x^*,[{}^tz,1_n]\cdot{}^tx\right)$$

For a fixed lattice $L\subset\mathcal{K}^{2n}\subset\mathbb{C}^{2n}$, denote by $L_z=p_z(L)$ a $4n$-dimensional CM-lattice of analytic parameter z.

1.3 Explicit matrix description by the complex torus $\mathbb{C}^{2n}/L_z$

Any $2n$-dimensional abelian variety of CM-type is isomorphic to A_z, with the action of $\mathcal{K}$ given by $\iota_z(a)\cdot v = \mathrm{diag}[\bar{a}\cdot 1_n, a\cdot 1_n]\cdot v$.

Universal analytic family $\mathcal{A}_{univ}$ over $\mathcal{H}_n$: taking L the lattice in $\mathcal{K}^1_{2n}$ generated by the standard basis vectors $e_1,\cdots,e_{2n}$, and the vectors $\alpha\cdot e_1,\cdots,\alpha\cdot e_{2n}$ with α a generator of $\mathcal{K}$ over $\mathbb{Q}$. Then the fiber A_z over each point $z=(z_{ij})\in\mathcal{H}_n$ is the abelian variety $A_z\cong\mathbb{C}^{2n}/L$, where L_z the $\mathbb{Z}$-lattice generated by $4n$ rows:

$$\begin{aligned}
z_j &= (z_{1j},\cdots,z_{nj},z_{j1},\cdots,z_{jn})\\
e_j &= \text{ vector with 1 in the } j\text{-th and } j+n\text{-th positions}\\
&\quad\ \text{and zeroes everywhere else,}\\
z'_j &= (\bar{\alpha}z_{1j},\cdots,\bar{\alpha}z_{nj},\alpha z_{j1},\cdots,\alpha z_{jn})\\
e'_j &= \text{ vector with } \bar{\alpha} \text{ in the } j\text{-th, and } \alpha \text{ in the } j+n\text{-th positions}\\
&\quad\ \text{and zeroes everywhere else.}
\end{aligned}$$

1.4 Vector-valued automorphic forms on unitary groups, [EE], p.18

Weight ρ of an automorphic form on G is a representation of the maximal compact subgroup $K\subset G$. Weights are constructed via the following polynomial representations $\rho_{\boldsymbol{\kappa}}:\mathrm{GL}_n\to\mathrm{GL}(V_{\boldsymbol{\kappa}})$.

For each set $\boldsymbol{\kappa}$ of orderered integers $\kappa_1\geq\cdots\geq\kappa_n$ there is a representation $(\rho_{\boldsymbol{\kappa}},\mathrm{GL}_n)$ *of highest weight* $\boldsymbol{\kappa}$, constructed as
$V_{\boldsymbol{\kappa}} = \mathrm{Sym}^{\kappa_1-\kappa_2}(R^n)\otimes\mathrm{Sym}^{\kappa_2-\kappa_3}(\wedge^2(R^n))\otimes\cdots\otimes\mathrm{Sym}^{\kappa_n}(\wedge^n(R^n))$ with the standard GL_n-action, over any $\mathbb{Q}$-algebra R.

Vector valued modular forms $\mathcal{M}_{\boldsymbol{\kappa}}$ (symplectic case) and $\mathcal{M}_{\boldsymbol{\kappa},\boldsymbol{\kappa}'}$ (unitary case) can be attached to the representations with highest weight $\rho=\rho_{\boldsymbol{\kappa}}$ and $\rho^+_{\boldsymbol{\kappa}}\otimes\rho^-_{\boldsymbol{\kappa}'}$ of the maximal compact subgroups $K\cong U(n)\subset Sp_{2n}(\mathbb{R})$ and $K\cong U(a)\times U(b)\subset U(a,b)$.
These modular forms take values in $V_{\boldsymbol{\kappa}}$ and $V_{\boldsymbol{\kappa},\boldsymbol{\kappa}'}$, and defined on the symmetric spaces G/K, $G=Sp(\mathbb{R})$ or $G=U(a,b)$.

Some notation $\alpha(z)=(az+b)(cz+d)^{-1}$, $\lambda(z)=\bar{c}\cdot{}^t\bar{z}+\bar{d}$, $\mu(z)=c\cdot z+d$ (used for the automorphy factors of weight ρ, and for the Eisenstein series).

1.5 C^∞-differential operators via Shimura's approach

For each $z\in\mathcal{H}_n$, let $\Xi(z)=(\xi(z),\eta(z))=(i(\bar{z}-{}^tz),i(z^*-z))$, so that ${}^t\xi(z)=\eta(z)=i(z^*-z))$. *The tangent space* $T=\mathbb{C}^n_n$ over $\mathbb{C}$ has a $\mathbb{R}$-rational basis $\{e_\nu\}$, $u:=\sum_\nu u_\nu\varepsilon_\nu$, $z:=\sum_\nu z_\nu\varepsilon_\nu$.

Let $(\rho, V) = (\rho_- \otimes \rho_+, V_- \otimes V_+)$ be a finite dimensional representation of $\mathrm{GL}_n(\mathbb{C}) \times \mathrm{GL}_n(\mathbb{C})$, and e be a positive integer. For vector spaces X and Y, define $S_e(Y, X)$ the vector space of *degree e homogeneous polynomial maps* of Y into X, i.e. the space of maps h from Y to X such that $h(a \cdot y) = a^e h(y)$, $S_e(Y) = S_e(Y, \mathbb{C})$.

For $f \in C^\infty(\mathcal{H}_n, V)$, put $\Xi = (\xi, \eta) \in S_1(T, \mathbb{C})$, and define operators $C, D : C^\infty(\mathcal{H}_n, V) \to C^\infty(\mathcal{H}_n, S_1(T, V))$ by

$$(Df)(u) = \sum_\nu u_\nu \frac{\partial f}{\partial z_\nu}$$
$$(Cf)(u) = (\tau^1(\Xi) Df)(u) := Df({}^t\xi u \eta).$$

For $e > 1$ write $D^e(f)$ and $C^e(f)$ for $D(D^{e-1}f)$ and $C(C^{e-1}f)$, viewed as $C^\infty(\mathcal{H}_n, S_e(T, V))$ - valued.

1.5.1 Action on vector-values automorphic forms

Given $g = (a, b) \in \mathrm{GL}_n(\mathbb{C}) \times \mathrm{GL}_n(\mathbb{C})$, (ρ, X) a polyomial representation, and $h \in M\ell_e(T, X) = M\ell_e(T, \mathbb{C}) \otimes X$ (symmetric $\mathbb{R}$-multilinear map viewed also as element $S_e(T, X)$), define $[\tau^e(a, b)h](u_1, \cdots, u_e) = ({}^t a u_1 b, \cdots, {}^t a u_e b)$, and a representation $\rho \otimes \tau^e$ of $\mathrm{GL}_n(\mathbb{C}) \times \mathrm{GL}_n(\mathbb{C})$ on $M\ell_e(T, \mathbb{C}) \otimes X$

$$[(\rho \otimes \tau^e)(g)](h(u) \otimes x) = \tau^e(g)h \otimes \rho(g)x$$

for each $g \in \mathrm{GL}_n(\mathbb{C}) \times \mathrm{GL}_n(\mathbb{C})$, $h \in M\ell_e(T, \mathbb{C})$, and $x \in X$. For $e > 1$ write $D^e(f) = D(D^{e-1}(f))$ and $C^e(f) = C(C^{e-1})(f)$.

Such operators *take automorphic forms of weight ρ to automorphic forms of weight $\rho \otimes \tau^e$* as follows: define

$$(D_\rho f)(u) = \rho(\Xi)^{-1} D[\rho(\Xi) f](u) = (\rho \otimes \tau)(\Xi)^{-1} C[\rho(\Xi) f](u).$$

and $(D_\rho^e f)(u) = (\rho \otimes \tau^e)(\Xi)^{-1} C^e[\rho(\Xi) f]$ for $e > 1$.
Then D_ρ^e maps automorphic forms of weight ρ to automorphic forms of weight $\rho \otimes \tau^e$.

1.5.2 General Shimura's differential operators D_ρ^Z via φ_Z

The classification of the irreducible subspaces of polynomial representations of $\mathrm{GL}_n(\mathbb{C})$ and of irreducible subspaces of τ^e is studied in [Shi00], Theorem 12.7, in terms of highest weights. Given a matrix $a \in \mathbb{C}_n^n$, let $\det_j(a)$ denote the determinant of the upper left $j \times j$ submatrix of a. *If ρ and σ are irreducible representations* of $\mathrm{GL}_n(\mathbb{C})$, $\rho \otimes \sigma$ occurs in τ^e if and only if ρ and σ are representations of the same highest weights $\kappa_1 \geq \cdots \geq \kappa_n$ as each other $\kappa_1 + \cdots + \kappa_n = e$, and the corresponding irreducible subspace of $S_e(T)$ contains a polynomial $p(x)$ defined by

$$\prod_{j=1}^{n} \det_j(x)^{e_j} \ (x \in T = \mathbb{C}_n^n, e_j = \kappa_j - \kappa_{j+1}, 1 \le j \le n-1, e_n = \kappa_n)$$

If ρ is the representation of $\mathrm{GL}_n(\mathbb{C}) \times \mathrm{GL}_n(\mathbb{C})$, there is a differential operator D^Z defined for a stable quotient of $S_e(T)$ with the projection φ_Z of $S_r(T) \otimes X$ onto $Z \otimes X$. Then the operator $D^Z_\rho = \varphi_Z D^e_\rho$ is a map from the space of automorphic forms of weight ρ to automorphic forms of weight $\rho \otimes \tau_Z$, where τ_Z denotes the restriction of τ to Z. *There is a formula for the action of the algebraic differential operators* θ^Z_ρ on formal q-expansions on the double group G at a at a cusp (which is a certain formal object) $f = \sum_{L_m \ni \boldsymbol{\beta} > 0} a(\boldsymbol{\beta}) q^{\boldsymbol{\beta}}$, where L_m is the lattice in $Herm_{\mathcal{K}}$ determined by m. If ζ is a highest-weight vector in Z, then it follows from the formulas in [EE], §9, that $\theta(\zeta)(f) = \sum_{\boldsymbol{\beta}} a(\boldsymbol{\beta})\zeta(\boldsymbol{\beta})q^{\boldsymbol{\beta}}$.

1.6 Holomorphic discrete series of $U(a, b)$

following P.Garrett,[Ga05] let us recall the structure of *holomorphic discrete series representations* of unitary groups $U(a, b)$ for sufficiently high highest weight. For $U(a, b)$, the maximal compact is $U(a)xU(b)$, and for ρ with highest weight $(\kappa_1, \dots, \kappa_a) \times (\kappa'_1, \dots, \kappa'_b)$ it is sufficient to assume that

$$\kappa_1 \ge \cdots \ge \kappa_a \ge \frac{a+b-1}{2}, \quad \kappa'_1 \ge \cdots \ge \kappa'_b \ge \frac{a+b-1}{2}$$

Let $\mathfrak{g}$ be the Lie algebra of $G = U(a, b)$ where the latter is the isometry group of the standard hermitian form given by $(a+b) \times (a+b)$-matrix $H = \begin{pmatrix} 1_a & 0 \\ 0 & -1_b \end{pmatrix}$. The copy K of $U(a) \times U(b)$ in G is $K = \{\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \mid A \in U(a), B \in U(b)\}$, the center of K is $Z = \begin{pmatrix} \lambda 1_a & 0 \\ 0 & \mu 1_b \end{pmatrix}$, $\lambda, \mu \in U(1)$, $\mathfrak{p}^+ = \{\begin{pmatrix} 0 & S \\ 0 & 0 \end{pmatrix}\}$, with S a-by-b, $\mathfrak{p}^- = \{\begin{pmatrix} 0 & 0 \\ S & 0 \end{pmatrix}\}$, with S b-by-a, and the Lie algebra of K denoted by $\mathfrak{k}$. The elements of $\mathfrak{p}^+$ are the *raising* operators, the elements of $\mathfrak{p}^-$ are the *lowering* operators, and $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}^+ \oplus \mathfrak{p}^-$ is the Harish-Chandra decomposition.

2 Algebraic differential operators

on automorphic forms on unitary groups. Fix a $\mathcal{O}_{\mathcal{K}}$-algebra $\mathcal{R}$ with inclusion $\iota : \mathcal{R} \to \mathbb{C}$ and a weight representation $\rho = (\rho^+, \rho^-)$ of the maximal compact subgroup $K = U(n) \times U(n)$ of $U = U(n, n)$. Following §8 and 9 of [EE], write an automorphic form in $\mathcal{M}_\rho(\mathcal{R})$ with values in an $\mathcal{R}$-module $V = V^\rho(\mathcal{R}^d)$ on the hermitian space $\mathcal{H}_n = U/K$ as a formal q-expansion $f(q) = \sum_{\boldsymbol{\beta}\in H_{\geq 0}} c_{\boldsymbol{\beta}}(\Xi)q^{\boldsymbol{\beta}}$ with vector-valued polynomial coefficients $c_{\boldsymbol{\beta}}(\Xi) \in V^\rho$ of $q^{\boldsymbol{\beta}}$=exp($2\pi i$tr($\boldsymbol{\beta}z$)), $z \in \mathcal{H}_n$, where $\Xi(z) = (i(\bar{z} - {}^t z), i(z^* - z)) = (\xi, \eta)$ (Shimura's notation), $T = \mathbb{C}^n_n$, and $\{e_\nu\}$ a $\mathbb{R}$-rational basis of T over $\mathbb{C}$, $H_{\geq 0}$ is a lattice of hermitian semi-integral non-negative matrices.

Then a general algebraic operator $\theta(f)$ is defined as above via $\theta(\zeta)(f)$, using $\boldsymbol{\beta}$ and Ξ as formal variables over a cusp: $\theta(\zeta)(f)(q) = \sum_{\beta\in H_{\geq 0}} \zeta(\boldsymbol{\beta})c_{\boldsymbol{\beta}}(\Xi)q^{\boldsymbol{\beta}}$

more general formal q-expansions: $f(q) = \sum_{\beta\in H_{\geq 0}} c_\beta(\Xi; T_1, \ldots, T_n)q^\beta$ with additional polynomial variables $T_1, \ldots, T_n$, and define $\theta(f) = \sum_{\beta\in H_{\geq 0}} d_\beta(\Xi; T_1, \ldots, T_n)q^\beta$, where $T_1, \cdots, T_n \in T^{\cdot}(\mathcal{R}^n)$ in the tensor algebra of n letters, $d_\beta = \sum_{\beta_{i,j}\in H_{\geq 0}} \beta_{i,j}c(\beta)\cdot (T_i \otimes T_j)$. This construction allows to treat *vector-valued modular forms as polynomial-valued*, and to prove congruences between them monomial-by-monomial.

2.1 Classical setting: arithmetic differential operators

In the Unitary case such operators were studied in [EE]; we may write $\boldsymbol{\beta} = \begin{pmatrix} \beta_1 & \beta_2 \\ \beta_3 & \beta_4 \end{pmatrix}$ in the q expansion on the double group, with hermitian matrices β_1, β_4 , and $\beta_2^* = \beta_3$. In the Sp-case such operator studied in [BS00] and [Do17] are compositions Shimura-type operators, described then via its action on the q-expansions.

For $\nu \in \mathbb{N}$, we put

$$\mathfrak{D}^\nu_{n,\alpha} = \mathfrak{D}_{n,\alpha+\nu-1} \circ \ldots \circ \mathfrak{D}_{n,\alpha}$$
$$\mathring{\mathfrak{D}}^\nu_{n,\alpha} = (\mathfrak{D}^\nu_{n,\alpha}) \mid_{z_2=0} .$$

The arithmetic applications of this differential operator is due to its explicit action on the exponentials in the Fourier expansion as follows: for $\mathcal{T} \in \mathbb{C}^{2n,2n}_{\text{sym}}$, we recall a polynomial $\mathfrak{P}^\nu_{n,\alpha}(\mathcal{T})$ defined by S. Böcherer in the entries $t_{ij} (1 \leq i \leq j \leq 2n)$ of $\mathcal{T}$ by

$$\mathring{\mathfrak{D}}^\nu_{n,\alpha}(e^{\text{tr}(\mathcal{T}Z)}) = \mathfrak{P}^\nu_{n,\alpha}(\mathcal{T})e^{\text{tr}(\mathcal{T}_1 z_1 + \mathcal{T}_4 z_4)}, \mathcal{T} = \begin{pmatrix} \mathcal{T}_1 & \mathcal{T}_2 \\ {}^t\mathcal{T}_2 & \mathcal{T}_4 \end{pmatrix}, \mathfrak{Z} = \begin{pmatrix} z_1 & z_2 \\ z_3 & z_4 \end{pmatrix}$$

that is, it represents "action of differential operator on exponential function". The $\mathfrak{P}^\nu_{n,\alpha}$ are homogenous polynomials of degree $n\nu$.

2.2 Applications to critical values

of the *standard zeta function* $L(\boldsymbol{\varphi}, \chi, s)$ of vector-valued automorphic forms $\boldsymbol{\varphi}$ on unitary groups, see [Ha97], [EHLS].

More generaly, take a unitary group U of a n-dimensional $\mathcal{K}$-vector space with a non-degenerate hermitian form $\langle\cdot,\cdot\rangle_V : V\times V \to \mathcal{K}$of signature (a,b), $a+b=n$. Then a vector-valued automorphic (Hecke eigenform) $\boldsymbol{\varphi}$ on U generates a cuspidal automorphic representation $\pi=\pi_{\boldsymbol{\varphi}}$ of the adelic group $U(\mathbb{A})$.
The *standard zeta function* $L(\boldsymbol{\varphi}, \chi, s) = L(\pi_{\boldsymbol{\varphi}}, \chi, s)$ with a Hecke character $\chi : \mathbb{A}_{\mathcal{K}}^\times \to \mathbb{C}^\times$ of allowed type χ_∞ is a certain Euler product $L(\boldsymbol{\varphi}, \chi, s) = \prod_{\mathfrak{q}} L_{\mathfrak{q}}(\boldsymbol{\varphi}, \chi, s)$, where $L_{\mathfrak{q}}(\boldsymbol{\varphi}, \chi, s)^{-1} = L_{\mathfrak{q}}(\boldsymbol{\varphi}, X)$ is a polynomial of deg $=2n$ of $X=N(\mathfrak{q})^{-s}\chi(\mathfrak{q})$ given by the Satake parameters $t_{\mathfrak{q},i}$ $(i=1,\dots,n)$ of $\pi_{\mathfrak{q},\boldsymbol{\varphi}}$ (for $\mathfrak{q}$ outside a finite set S). The signature (a,b) is such that $n=a+b$ and $s=\frac{n-1}{2}$ is critical for the L-function $L(\pi,\chi,s)=L(\pi_{\boldsymbol{\varphi}},\chi,s)$.

3 The integral representation for the L-function $L(\boldsymbol{\varphi}, \chi, s)$

is on the double group $G=U(a+b,a+b)\supset U\times U$ of type

$$\int\limits_{U\times U} E((g_1,g_2),f)\chi^{-1}(\det g_2)\varphi_1(g_1)\varphi_2(g_2)dg_1dg_2$$

$= Z_S(s)L^S(\pi_{\boldsymbol{\varphi}},\chi,s+\dfrac{n-1}{2})\langle\varphi_1,\varphi_2\rangle$
where $E((g_1,g_2),f_{s,\chi})$ denotes the restriction to $(g_1,g_2)$of an *Eisenstein series on the double adelic group* $G=U(a+b,a+b)$, the series defined from a suitably chosen section $f=f_{s,\chi}\in \mathrm{Ind}_{P_{Siegel}}^G$, $\varphi_1\in\pi, \varphi_2\in\tilde{\pi}$, with $P_{Siegel} = \begin{pmatrix} * & * \\ 0_{a+b} & * \end{pmatrix}$ is the Siegel parabolic in G, $E(g,f) = \sum\limits_{\gamma\in P(\mathcal{K})\backslash G(\mathcal{K})} f(\gamma g)$, $f_{k,\chi} = \chi(\det(c))\det(cz+d)^{-k}$, $\langle\varphi_1,\varphi_2\rangle = \int\limits_{U(a,b)} \varphi_1(g)\varphi_2(g)dg.$

The section f is an automorphic form on $U(n,n)$ has a weight, which is a representation ρ of $\mathrm{GL}_n\times\mathrm{GL}_n$. In the special case where this representation is of the form $\rho(a,b)=\det(a)^{k+\nu}\det(b)^{-\nu}$ f is said an automorphic form of weight k,ν. *For the critical values* $s=s_*,\dots,s^*$ *we use certain algebraic operators* θ_{s^*-s} to move the Eisenstein series from s^* to s by acting on the section $f_{s^*,\chi}$ to get $f_{s,\chi}$. This allows to compare their q-expansions and get congruences for the critical values.

3.1 Classical setting: pull-back identity

This integral representation takes the form of a double Petersson product.

In the Sp case (see [BS00]) it becomes a *double integral representation* (pull-back identity) for the normalized L-function $\mathcal{D}(\boldsymbol{f}, s, \chi)$ and its critical values at t with $k + t = \ell$,

$$\mathcal{F}(g) = \frac{\left\langle \left\langle \boldsymbol{f}_1^0(w), g(*,*) \right\rangle^w, \boldsymbol{f}_2^0(z) \right\rangle^z}{\langle \boldsymbol{f}_1^0, \boldsymbol{f}_2^0 \rangle}$$

From test functions $g = g_{\chi_i, s_i}(*,*)$ to normalized critical L-values $\mathcal{D}(\boldsymbol{f}, t_i, \chi_i) = \mathcal{F}(g_{\chi_i, s_i}) = L^*_{geom}(\boldsymbol{\pi}, s_i, \chi_i)$ at t_i with $k_i + t_i = \ell$

Here $g(z, w) = \mathcal{H}_{t,\chi}(-\bar{z}, w)$ is a function in the tensor product of certain spaces of automorphic forms

$$\mathcal{H}_{t,\chi} \in C^\infty M_n^\ell(\Gamma_0(M), \varphi)|_z \otimes_{\mathbb{C}} C^\infty M_n^\ell(\Gamma_0(M), \varphi)|_w,$$

obtained from a double Eisenstein series E_{k_i,χ_i} on $U(n, n)$ of the above type, with $\boldsymbol{f}_1^0$, $\boldsymbol{f}_2^0$ suitably chosen eigenfunctions of Atkin's type operator $U_p : \sum_H A_H q^H \mapsto \sum_H A_{pH} q^H$ (the Hermitian Fourier expansion): .

This analytic properties of the L-function indicate that the representation π_∞ eventually produces a geometric object of a certain Hodge type, described in [EHLS], (4.4.19) at p.66 in terms of its Hodge polygon. The existence of such objects was proved by P.Scholze via geometric p-adic Galois representations of Fontaine-Mazur type ([Scho15]).

3.2 Eisenstein series and congruences (Unitary case)

The (Siegel-Hermite) Eisenstein series $E_{2\ell,n,K}(Z)$ of weight 2ℓ, character $\det^{-\ell}$, is defined in [Ike08] by $E_{2\ell,n,K}(Z) = \sum\limits_{g \in \Gamma_{n,K,\infty} \backslash \Gamma_{n,K}} (\det g)^\ell j(g, Z)^{-2\ell}$ (converges for $\ell > n$). *The normalized Eisenstein series* is given by
$\mathcal{E}_{2\ell,n,K}(Z) = 2^{-n} \prod_{i=1}^n L(i - 2\ell, \theta^{i-1}) \cdot E_{2\ell,n,K}(Z)$.

If $H \in \Lambda_n(\mathcal{O})^+$, then the H-th Fourier coefficient of $\mathcal{E}_{2\ell}^{(n)}(Z)$ is *polynomial over* $\mathbb{Z}$ in variables $\{p^{\ell-(n/2)}\}_p$, and equals

$$|\gamma(H)|^{\ell-(n/2)} \prod_{p|\gamma(H)} \tilde{F}_p(H, p^{-\ell+(n/2)}), \gamma(H) = (-D_K)^{[n/2]} \det H.$$

Here, $\tilde{F}_p(H, X)$ is a certain Laurent polynomial in the variables $\{X_p = p^{-s}, X_p^{-1}\}_p$ over $\mathbb{Z}$. This polynomial is a *key point in proving congruences* for the modular forms in both the *pull-back double integral representation* and *Rankin-Selberg integral.*

3.3 Strategy of the construction of p-adic L-functions

It slightly differs from that on [EHLS] and uses our *method of automorphic distributions* on the p-adic weight space $X_{\boldsymbol{\pi}}$ in [PaTV], [Pa05]. This method allows to treat a general non-ordinary case.

- The integral representation for the normalized critical values $L^*(\boldsymbol{\pi}, \chi_i, s_i,)$ via the doubling method: $Z_S(s_i)L^S(\boldsymbol{\pi}_{\boldsymbol{\varphi}}, \chi_i, s_i + \frac{n-1}{2}) \times \langle \boldsymbol{\varphi}_{i,1}, \boldsymbol{\varphi}_{i,2}\rangle = \int\limits_{U\times U_-} E((g_1, g_2), f_{s_i,\chi_i})\chi_i^{-1}(\det g_2)\boldsymbol{\varphi}_{i,1}(g_1)\boldsymbol{\varphi}_{i,2}(g_2)dg_2$
  where $\boldsymbol{\varphi}_{i,1} \in \boldsymbol{\pi}, \boldsymbol{\varphi}_{i,1} \in \tilde{\boldsymbol{\pi}}$ are chosen functions in dual spaces (factorizable adelic Schwartz functions on the group $U(n)(\mathbb{A})$), $E((g_1, g_2), f_{s_i,\chi_i})$ the pull-back of the Eisenstein series on $U(n, n)$, $f = f_{s_i,\chi_i}$ its Siegel section $f \in I_P^U = \mathrm{Ind}_{P_{Siegel}}^{U(n,n)}$, $E(g, f) = \sum\limits_{\gamma\in P(\mathcal{K})\backslash G(\mathcal{K})} f(\gamma g)$.

- From Siegel sections f_{χ_i,s_i} to critical values $L^*_{geom}(\boldsymbol{\pi}, s_i, \chi_i)$.

  Families of automorphic distributions $\{\boldsymbol{\mu}_r\}$, $0 \le r \le s^* - s_*$ on the weight space X attached to $U(a, b)$. They produce $\bar{\mathbb{Q}}$-valued distributions $\boldsymbol{\mu}_i$ on X such that $\int_X \chi_i(x_p)d\boldsymbol{\mu}_{s^*-s_i} = L^*_{geom}(\boldsymbol{\pi}, s_i, \chi_i)$, where $X_\pi \to \mathbb{Z}_p^*$ is a p-part projection. Fixing embeddings $\bar{\mathbb{Q}} \overset{i_\infty}{\hookrightarrow} \mathbb{C}$, $\bar{\mathbb{Q}} \overset{i_p}{\hookrightarrow} \mathbb{C}_p = \hat{\bar{\mathbb{Q}}}_p$ produces p-adic-valued distributions.

3.4 Constructing p-adic measures via congruences

- Proving Kummer type congruences in the form
  Definition. Let M be a $\mathcal{O}$-module of finite rank where $\mathcal{O} \subset \mathbb{C}_p$. For $h \ge 1$, consider the following $\mathbb{C}_p$-vector spaces of functions on $\mathbb{Z}_p^* : \mathcal{C}^h \subset \mathcal{C}^{loc-an} \subset \mathcal{C}$. Then *a continuous homomorphism* $\mu : \mathcal{C} \to M$ is called a *(bounded) M-valued measure* on $\mathbb{Z}_p^*$. Let us define a measure with given integrals.

  Take a dense family of continuous functions $\{\boldsymbol{\varphi}_i = \boldsymbol{\varphi}_{s_i,\chi_i}\}$ in $\mathcal{C}(X_\pi, \mathbb{C}_p)$ on the p-adic space X_π. Then Kummer says:
  $\sum\limits_i \beta_i\boldsymbol{\varphi}_i \equiv 0 \bmod p^N \Longrightarrow \sum\limits_i \beta_i L^*_{geom}(\boldsymbol{\pi}, s_i, \chi_i) \equiv 0 \bmod p^N.$

  Each $\boldsymbol{\varphi} \in \mathcal{C}(X_\pi, \mathbb{C}_p)$ can be approximated by $\{\boldsymbol{\varphi}_i\}_i$, and a measure $\mu_{\boldsymbol{\pi}}(\boldsymbol{\varphi})$ with given $\mu_{\boldsymbol{\pi}}(\varphi_i) = L^*_{geom}(\boldsymbol{\pi}, s_i, \chi_i)$ is a well-defined limit over all approximations of $\boldsymbol{\varphi}$.

- From bounded measures on X to admissible measures using $h_{\boldsymbol{\pi},p} = P_{Newton,p}(d/2) - P_{Hodge}(d/2) \ge 0$.

  Computing critical values at $s = s_*, \cdots, s^*$ and prove admissibility congruences for them as follows

A $\mathbb{C}_p$-*linear mapping* $\mu : \mathcal{C}^h \to M$ is called an *h admissible M-valued measure* on $\mathbb{Z}_p^*$ if the following growth condition is satisfied

$$\left| \int_{a+(p^v)} (x-a)^j d\mu \right|_p \le p^{-v(h-j)}$$

for $j = 0, 1, ..., h-1$. Such μ extends to $\mathcal{C}^{loc-an}$ (and to $\mathcal{Y}_p = Hom_{cont}(\mathbb{Z}_p^*, \mathbb{C}_p^*)$, the space of definition of *p-adic Mellin transform*)

3.5 Perspectives and applications

1. The case $U(n,n)$: a striking analogue of Manin-Mazur's result on p-adic analytic interpolation of critical values, [Ma73], [MTT], to any imaginary quadraic $\mathcal{K}$, a hermitian Hecke-eigenform of weight $\ell > 2n$, $s_* = n$, $s^* = \ell - n$.

2. Using the Hodge and Newton polygons of an Euler product with a functional equation, for its geometric recognition

3. Link to a new revolutionary tool – Prisms and Prismatic cohomology (by P.Scholze-B.Bhatt [BhSc19], via Kisin-Fargue-Wach-modules and Iwasawa cohomology, using the obtained Iwasawa series,.

   Given a formally smooth $\mathbb{Z}_p$-scheme X, this cohomology yields a universal q-deformation of the de Rham cohomology of $X/\mathbb{Z}_p$ across the map $\mathbb{Z}_p[[q-1]] \xrightarrow{q \to 1} \mathbb{Z}_p$, and the Iwasawa algebra $\mathbb{Z}_p[[q-1]]$ provides a description.

4. Special hypergeometric motives and their L-functions: Asai recognition, see [DPVZ] The generalized hypergeometric functions are often used in arithmetic and algebraic geometry. They come as periods of certain algebraic varieties, and consequently they encode important information about the invariants of these varieties. Euler factors, Newton and Hodge polygons attached to them, provide a tool for their geometric recognition.

4 The case $U(n,n)$. Hermitian modular group $\Gamma_{n,K}$ and the standard zeta function $\mathcal{Z}(s, \mathbf{f})$ (definitions)

The followng function $\mathcal{Z}(s, \mathbf{f})$ is a special case of Euler products constructed by G. Shimura. Let $\theta = \theta_K$ be the quadratic character attached to $K = \mathbb{Q}(\sqrt{-D_K})$,

$n' = \left[\frac{n}{2}\right]$.

$$\Gamma_{n,K} = \left\{ M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{GL}_{2n}(\mathcal{O}_K) | M\eta_n M^* = \eta_n \right\},\ \eta_n = \begin{pmatrix} 0_n & -I_n \\ I_n & 0_n \end{pmatrix},$$

$$\mathcal{Z}(s, \mathbf{f}) = \left(\prod_{i=1}^{2n} L(2s - i + 1, \theta^{i-1}) \right) \sum_{\mathfrak{a}} \lambda(\mathfrak{a}) N(\mathfrak{a})^{-s},$$

(defined via Hecke's eigenvalues: $\mathbf{f}|T(\mathfrak{a}) = \lambda(\mathfrak{a})\mathbf{f}, \mathfrak{a} \subset \mathcal{O}_K$)

$= \prod_{\mathfrak{q}} \mathcal{Z}_{\mathfrak{q}}(N(\mathfrak{q})^{-s})^{-1}$(an Euler product over primes $\mathfrak{q} \subset \mathcal{O}_K$,

with $\deg \mathcal{Z}_{\mathfrak{q}}(X) = 2n,$ the Satake parameters $t_{i,\mathfrak{q}}, i = 1, \cdots, n$),

$\boxed{\mathcal{D}(s, \mathbf{f}) = \mathcal{Z}(s - \frac{\ell}{2} + \frac{1}{2}, \mathbf{f})}$ (Geometrically normalized standard zeta function

with a functional equation $s \mapsto \ell - s$; $\mathrm{rk} = 4n,$ and geometric weight $\ell - 1$),

$$\Gamma_{\mathcal{D}}(s) = \prod_{i=0}^{n-1} \Gamma_{\mathbb{C}}(s - i)^2.$$

Main result in the lifted case: Assuming $\ell > 2n$, a p-adic interpolation is constructed of all critical values $\mathcal{D}(s, \mathbf{f}, \chi)$ normalized by $\times\Gamma_{\mathcal{D}}(s)/\Omega_{\mathbf{f}}$, in the critical strip $n \le s \le \ell - n$ for all χ mod p^r in both *bounded or unbounded case*, i.e. when the product $\alpha_{\mathbf{f}} = \left(\prod_{\mathfrak{q}|p} \prod_{i=1}^{n} t_{\mathfrak{q},i}\right) p^{-n(n+1)}$ is *not a p-adic unit.*

4.1 The Hodge and Newton polygons of $\mathcal{D}(s)$

are used in order to state our Main result. The Hodge polygon $P_H(t) : [0, d] \to \mathbb{R}$ of the function $\mathcal{D}(s)$ and
the Newton polygon $P_{N,p}(t) : [0, d] \to \mathbb{R}$ at p are piecewise linear:

The Hodge polygon of (weak) pure weight w has the slopes j of $length_j = h^{j,w-j}$ given by Serre's Gamma factors of the functional equation of the form $s \mapsto w + 1 - s$, relating $\Lambda_{\mathcal{D}}(s, \chi) = \Gamma_{\mathcal{D}}(s)\mathcal{D}(s, \chi)$ and $\Lambda_{\mathcal{D}^\rho}(w + 1 - s, \bar{\chi})$, where ρ is the complex conjugation of a_n, and $\Gamma_{\mathcal{D}}(s) = \Gamma_{\mathcal{D}^\rho}(s)$ equals to the product $\Gamma_{\mathcal{D}}(s) = \prod_{j \le \frac{w}{2}} \Gamma_{j,w-j}(s)$, where

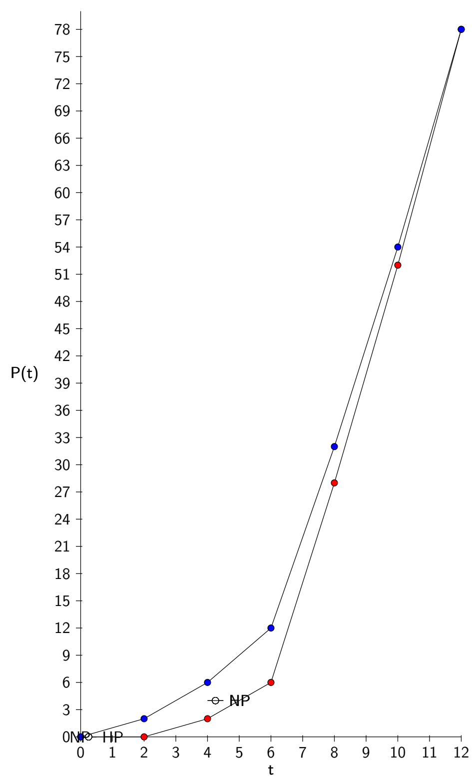
$$\Gamma_{j,w-j}(s) = \begin{cases} \Gamma_{\mathbb{C}}(s - j)^{h^{j,w-j}}, & \text{if } j < w, \\ \Gamma_{\mathbb{R}}(s - j)^{h_+^{j,j}} \Gamma_{\mathbb{R}}(s - j + 1)^{h_-^{j,j}}, & \text{if } 2j = w, \text{ where} \end{cases}$$

$\Gamma_{\mathbb{R}}(s) = \pi^{-\frac{s}{2}} \Gamma\left(\frac{s}{2}\right), \Gamma_{\mathbb{C}}(s) = \Gamma_{\mathbb{R}}(s)\Gamma_{\mathbb{R}}(s + 1) = 2(2\pi)^{-s}\Gamma(s)$, $h^{j,j} = h_+^{j,j} + h_-^{j,j}$, $\sum_j h^{j,w-j} = d$, see [Coh15] for the various examples with Gamma factors.

The Newton polygon at p is the convex hull of points $(i, \mathrm{ord}_p(a_i))$ $(i = 0, \ldots, d)$; its slopes λ are the p-adic valuations $\mathrm{ord}_p(\alpha_i)$ of the inverse roots α_i of $\mathcal{D}_p(X) \in \bar{\mathbb{Q}}[X] \subset \mathbb{C}_p[X]$: $length_\lambda = \sharp\{i \mid \mathrm{ord}_p(\alpha_i) = \lambda\}$. According to [BeOg78], Th8.36, $P_{Newton,p}(t) \geq P_{Hodge}(t)$ on $[0, d]$, see also [BrCo].

4.2 Hodge/Newton polygons for f $= Lift(\Delta)$, $n = 3$, $U(3,3)$

Let us draw $P_{Hodge}(t)$ (slopes 0,1,2,11,12,13), and $P_{Newton,p}(t)$ (slopes 1,2,3,10,11,12), symmetry for slopes: $j \mapsto 13 - j$, for $p = 7$, $\boldsymbol{f} = Lift(\Delta)$, $k = 12$, $n' = 1$, $\ell = 14 = k + 2n'$, $d = 4n = 12$, $\Gamma_{\mathcal{D}}(s) = \Gamma_{\mathbb{C}}(s)^2\Gamma_{\mathbb{C}}(s-1)^2\Gamma_{\mathbb{C}}(s-2)^2$, symmetry $s \mapsto 14 - s$. $P_{Newton,p}(6) = 12$, $P_{Hodge}(6) = 6$, $h = 6$ ("the Hasse invariant")



4.3 Description of the Main theorem

Let $\Omega_{\mathbf{f}}$ be a period attached to an Hermitian cusp eigenform $\mathbf{f}$, $\mathcal{D}(s,\mathbf{f}) = \mathcal{Z}(s - \frac{\ell}{2} + \frac{1}{2}, \mathbf{f})$ the standard zeta function, and

$$\alpha_{\mathbf{f}} = \alpha_{\mathbf{f},p} = \left(\prod_{\mathfrak{q}|p}\prod_{i=1}^{n} t_{\mathfrak{q},i}\right) p^{-n(n+1)}, \quad h = \mathrm{ord}_p(\alpha_{\mathbf{f},p}),$$

The number $\alpha_{\mathbf{f}}$ turns out to be an eigenvalue of Atkin's type operator $U_p : \sum_H A_H q^H \mapsto \sum_H A_{pH} q^H$ (the Hermitian Fourier expansion) on some $\mathbf{f_0}$, and $h = P_N(\frac{d}{2}) - P_H(\frac{d}{2})$, $d = 4n$, $\dfrac{d}{2} = 2n$.

Definition. Let M be a $\mathcal{O}$-module of finite rank where $\mathcal{O} \subset \mathbb{C}_p$. For $h \geq 1$, consider the following $\mathbb{C}_p$-vector spaces of functions on $\mathbb{Z}_p^* : \mathcal{C}^h \subset \mathcal{C}^{loc-an} \subset \mathcal{C}$. Then

- a continuous homomorphism $\mu : \mathcal{C} \to M$ is called a *(bounded) measure* M-valued measure on $\mathbb{Z}_p^*$.

- $\mu : \mathcal{C}^h \to M$ is called an h *admissible measure* M-valued measure on $\mathbb{Z}_p^*$ measure if the following growth condition is satisfied

$$\left| \int_{a+(p^v)} (x-a)^j d\mu \right|_p \leq p^{-v(h-j)}$$

for $j = 0, 1, ..., h-1$, and et $\mathcal{Y}_p = Hom_{cont}(\mathbb{Z}_p^*, \mathbb{C}_p^*)$ be the space of definition of *p-adic Mellin transform*

Theorem ([Am-V], [MTT]) For an h-admissible measure μ, the Mellin transform $\mathcal{L}_\mu : \mathcal{Y}_p \to \mathbb{C}_p$ exists and has growth $o(\log^h)$ (with infinitely many zeros).

4.4 Main Theorem.

Let $\mathbf{f}$ be a Hermitian cusp eigenform of degree $n \geq 2$ and of weight $\ell > 2n$. There exist distributions $\mu_{\mathcal{D},s}$ for $s = n, \cdots, \ell - n$ with the properties:

i) for all pairs (s, χ) such that $s \in \mathbb{Z}$ with $n \leq s \leq \ell - n$,

$$\int_{\mathbb{Z}_p^*} \chi d\mu_{\mathcal{D},s} = A_p(s,\chi) \frac{\mathcal{D}^*(s, \mathbf{f}, \overline{\chi})}{\Omega_{\mathbf{f}}}$$

(under the inclusion i_p), with elementary factors $A_p(s,\chi) = \prod_{\mathfrak{q}|p} A_{\mathfrak{q}}(s,\chi)$ including a finite Euler product, Satake parameters $t_{\mathfrak{q},i}$, gaussian sums, the conductor of χ; the integral is a finite sum.

(ii) if $\mathrm{ord}_p \left((\prod_{\mathfrak{q}|p} \prod_{i=1}^n t_{\mathfrak{q},i}) p^{-n(n+1)} \right) = 0$ then the above distributions $\mu_{\mathcal{D},s}$ are bounded measures, we set $\mu_{\mathcal{D}} = \mu_{\mathcal{D},s^*}$ and the integral is defined for all continuous characters $y \in \mathrm{Hom}(\mathbb{Z}_p^*, \mathbb{C}_p^*) =: \mathcal{Y}_p$.

Their *Mellin transforms* $\mathcal{L}_{\mu_{\mathcal{D},s}}(y) = \int_{\mathbb{Z}_p^*} y d\mu_{\mathcal{D},s}$, $\mathcal{L}_{\mu_{\mathcal{D}}} : \mathcal{Y}_p \to \mathbb{C}_p$, give bounded p-adic analytic interpolation of the above L-values to on the $\mathbb{C}_p$-analytic group $\mathcal{Y}_p$; and these distributions are *related by*: $\int_X \chi d\mu_{\mathcal{D},s} = \int_X \chi x^{s^*-s} \mu_{\mathcal{D},s^*}$, $X = \mathbb{Z}_p^*$, where $s^* = \ell - n$, $s_* = n$.

Main theorem (continued)

(iii) in the *admissible* case assume that $0 < h \le s^* - s_* + 1 = \ell + 1 - 2n$, where $h = \mathrm{ord}_p\left((\prod_{\mathfrak{q}|p} \prod_{i=1}^n t_{\mathfrak{q},i}) p^{-n(n+1)}\right) > 0$, Then there exists an h–admissible measure $\mu_{\mathcal{D}}$ whose integrals $\int_{\mathbb{Z}_p^*} \chi x_p^s d\mu_{\mathcal{D}}$ are given by $i_p\left(A_p(s,\chi)\dfrac{\mathcal{D}^*(s, \mathbf{f}, \overline{\chi})}{\Omega_{\mathbf{f}}}\right) \in \mathbb{C}_p$ with $A_p(s,\chi)$ as in (i); their Mellin transforms $\mathcal{L}_{\mathcal{D}}(y) = \int_{\mathbb{Z}_p^*} y d\mu_{\mathcal{D}}$, belong to the type $o(\log x_p^h)$. (iv) the functions $\mathcal{L}_{\mathcal{D}}$ are determined by (i)-(iii).

Remarks. (a) Interpretation of s^*: the smallest of the "big slopes" of P_H
(b) Interpretation of $s_* - 1$: the biggest of the "small slopes" of P_H.

A Appendix . Recovering geometric objects from automorphic forms and special functions

For an irreducible automorphic representation $\pi = \pi_{\boldsymbol{\varphi}}$ of a $\mathbb{Q}$-algebraic group $G(\mathbb{A})$, the eventual geometric type of π is determined by the component π_∞, where $\pi = \otimes_v \pi_v$, v the set of valuations.

- (Wiles) Elliptic curves $E/\mathbb{Q} \leftrightarrow$ Hecke cusp eigenforms $f = \sum_{n=1}^\infty a_n q^n$ of weight $w = 2$ and $a_n \in \mathbb{Q}$ (where $q = e^{2\pi i z}$).
- (Deligne,Serre, Scholl, Carayol) Holomorphic modular forms of higher weight $w \ge 2 \rightsquigarrow X_f$, certain $(w-1)$-dimensional parts X_f (called "motives") of a Kuga-Sato variety E_{univ}^{w-2}, such that $L_f(s) = \sum_{n=1}^\infty a_n n^{-s} = L(H^{w-1}(X_f), s)$
- (Manin-Shimura-Mazur) Periods and modular symbols $\int_x^{i\infty} f(z) z^r dz \rightsquigarrow$ Normalized special values $L_f^*(r+1,\chi)$, where $L_f^*(s,\chi) := \Gamma(s) L_f(s,\chi)$, for any Dirichlet character χ, $0 \le r \le w-2, x \in \mathbb{Q}$). That is, the integrals on the left give linear forms on homology classes of geodesics $\{x, i\infty\}$, i.e. elements of certain cohomology groups $H^{w-1}(X_f)$, producing X_f and $L(X_f), s)$.

- The use of the Iwasawa algebra $\Lambda = \mathbb{Z}_p[[T]] = Dist(\mathbb{Z}_p, \mathbb{Z}_p)$, $\Lambda \ni \mu \longleftrightarrow A_\mu(T) = \sum_{k\geq 0} A_k T^k$, where $A_k = \int_{\mathbb{Z}_p} \binom{x}{k} d\mu$.
  The integral $I = \int\limits_{\mathbb{Z}_p} \varphi(x) d\mu(x)$ *of any continuous function* $\varphi = \sum_{k\geq 0} a_k \binom{x}{k} \in \mathcal{C}(\mathbb{Z}_p, \mathbb{Z}_p)$ becomes $I = \sum_{k\geq 0} a_k A_k$.

B Appendix.Prisms and Prismatic cohomology [BhSc19]

This new tool in the theory of geometric p-adic Galois representations appeared since [Scho15], [Scho18] and can be used for the study of q-universal deformation the De Rham cohomology of locally-symmetric hermitian spaces (or Shimura varieties of PEL-type). The above example of unitary groups $U_{\mathcal{K}}(n, n)$ describes analytic families of abelian varieties A with imbedding $\iota : \mathcal{K} \hookrightarrow \mathrm{End}_{\mathcal{K}}(A)$. Thus obtained p-adic schemes $X_{\pi,p}$ produce de Rham cohomology groups as above, and their universal deformations can be described using prisms [BhSc19] as cerain Iwasawa-type modules, notably, $\mathbb{Z}_p[[q-1]]$-modules, where $T = q-1$ is the Iwasawa variable attached to the quantum variable q.

According to [BhSc19], the notion of a prism substitutes in applications the notion of a perfectoid ring. Using prisms, one may attach a ringed site - the prismatic site - to a formal $\mathbb{Z}_p$-scheme. The resulting cohomology theory specializes to most known integral p-adic cohomology theories (étale, crystalline, de Rham). As application, a co-ordinate free description of q-de Rham cohomology is given.

Given a formally smooth $\mathbb{Z}_p$-scheme X, this cohomology yields a deformation of the de Rham cohomology of $X/\mathbb{Z}_p$ across the map $\mathbb{Z}_p[[q-1]] \overset{q\to 1}{\longrightarrow} \mathbb{Z}_p$.

C Appendix . Ikeda's lifting $f \rightsquigarrow \mathbf{f} = Lift(f)$

Its L-function gives a crucial motivation for both complex and p-adic theory of L-functions on unitary groups, and extends to a general (not necessarily lifted) case. Recall that in [Ike08]

$S_{2k+1}(\Gamma_0(D),\theta) \ni f \rightsquigarrow \mathbf{f} = Lift(f) \in \mathcal{S}_{2k+2n'}(\Gamma_{K,n}),$ $_{\text{if}}\ n=2n'$ $_{\text{is even}}\ (E)$

$S_{2k}(\mathrm{SL}(\mathbb{Z})) \ni f \rightsquigarrow \mathbf{f} = Lift(f) \in \mathcal{S}_{2k+2n'}(\Gamma_{K,n}),$ $_{\text{if}}\ n=2n'+1$ $_{\text{is odd}}\ (O)$

the standard L-function of $\mathbf{f} = Lift^{(n)}(f)$ is a nice product: $\mathcal{Z}(s,\mathbf{f}) =$

$$\prod_{i=1}^{n} L(s+k+n'-i+(1/2),f)L(s+k+n'-i+(1/2),f,\theta) \quad \text{[Ike08]}$$

$$= \prod_{i=0}^{n-1} L(s+\ell/2-i-(1/2),f)L(s+\ell/2-i-(1/2),f,\theta).$$

Notice $k+n'=\ell/2$, then the Gamma factor of the standard zeta function with the symmetry $s\mapsto 1-s$ becomes $\Gamma_{\mathcal{Z}}(s) = \prod_{i=0}^{n-1}\Gamma_{\mathbb{C}}(s+\ell/2-i-(1/2))^2$.

D Appendix . Special hypergeometric motives and their L-functions: Asai recognition, [DPVZ]

The generalized hypergeometric functions are a familiar player in arithmetic and algebraic geometry. They come quite naturally as periods of certain algebraic varieties, and consequently they encode important information about the invariants of these varieties.

Euler factors, Newton and Hodge polygons attached to them, provide a tool for their geometric recognition.

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