Max-Planck-Institut für Mathematik
Bonn

n-cluster tilting subcategories for radical square zero algebras

by

Laertis Vaso

Max-Planck-Institut für Mathematik
Preprint Series 2021 (20)

Date of submission: May 20, 2021

n-cluster tilting subcategories for radical square zero algebras

by

Laertis Vaso

Max-Planck-Institut für Mathematik
Vivatsgasse 7
53111 Bonn
Germany

MPIM 21-20

n-CLUSTER TILTING SUBCATEGORIES FOR RADICAL SQUARE ZERO ALGEBRAS

LAERTIS VASO

ABSTRACT. We give a characterization of radical square zero bound quiver algebras $\mathbf{k}Q/\mathcal{J}^2$ that admit n-cluster tilting subcategories and $n\mathbb{Z}$-cluster tilting subcategories in terms of Q. We also show that if Q is not of cyclically oriented extended Dynkin type $\tilde{A}$, then the poset of n-cluster tilting subcategories of $\mathbf{k}Q/\mathcal{J}^2$ with relation given by inclusion forms a lattice isomorphic to the opposite of the lattice of divisors of an integer which depends on Q.

INTRODUCTION

Representation theory of algebras can be described as the study of the category $\operatorname{mod}\Lambda$ of finite-dimensional (right) modules over an algebra Λ. One of the most helpful tools in that study has been Auslander–Reiten theory. In recent years a higher-dimensional analogue of Auslander–Reiten theory has been introduced by Iyama [Iya07b, Iya07a]; see also [Iya08]. In this theory, instead of focusing on $\operatorname{mod}\Lambda$, one restricts to a suitable subcategory $\mathcal{C}$ of $\operatorname{mod}\Lambda$ called an *n-cluster tilting subcategory* for some positive n, while if $\mathcal{C}$ has an additive generator M, then M is called an *n-cluster tilting module.* In this setting one may describe $\mathcal{C}$ using an n-dimensional version of Auslander–Reiten theory.

Every algebra Λ admits a unique 1-cluster tilting subcategory, namely $\operatorname{mod}\Lambda$ itself. On the other hand, if $n \geq 2$, then an n-cluster tilting subcategory may not exist. Generally, it is not easy to find algebras which admit n-cluster tilting subcategories. Recently there has been a lot of research in trying to find or construct n-cluster tilting subcategories, see for example [IO11, HI11, IO13, CIM19, JKPK19, CDIM20].

For simplicity, we assume that all quivers in this article are connected; the results of this paper can be straightforwardly generalised for quivers which are not connected. For an integer $m \in \mathbb{Z}_{\geq 1}$ we denote by A_m the quiver $1 \longrightarrow 2 \longrightarrow \cdots \longrightarrow m$ and by $\tilde{A}_m$ the quiver

$$\begin{tikzcd} & m \arrow[r, bend left] & 1 \arrow[dr, bend left] & \\ m-1 \arrow[ur, bend left] & & & 2. \arrow[dl, bend left] \\ & \cdot \arrow[ul, bend left] & \cdots & \cdot \end{tikzcd}$$

For a quiver Q we denote by $\mathcal{J}$ the ideal of the path algebra $\mathbf{k}Q$ generated by the arrows of Q. One may then ask for which integers $l \geq 2$ does the bound quiver algebra $\Lambda = \mathbf{k}Q/\mathcal{J}^l$ admit an n-cluster tilting subcategory. Several results are known in that direction. The case $Q = A_m$ has been studied in [Vas19], while the case $Q = \tilde{A}_m$ has been studied in [DI20]. The case where n is the global dimension of Λ was studied in [ST21]. In this article we consider the case where $l = 2$. As a first result we have the following theorem.

2020 *Mathematics Subject Classification.* 16G20 (Primary), 16G70 (Secondary).

Key words and phrases. Auslander–Reiten theory, n-cluster tilting subcategory, radical square zero algebra, string algebra, representation-finite algebra.

Theorem A (Proposition 2.9 and Theorem 2.10). *Let $\Lambda = \mathbf{k}Q/\mathcal{J}^2$ and $n \geq 2$. If Λ admits an n-cluster tilting subcategory, then Λ is a representation-finite string algebra. Moreover, if X is an indecomposable Λ-module and X is not simple, then X is projective or injective.*

Theorem A shows that a radical square zero bound quiver algebra Λ which admits an n-cluster tilting subcategory is well-understood from the point of view of representation theory. In particular, since Λ is representation-finite, every n-cluster tilting subcategory $\mathcal{C}$ of $\operatorname{mod}\Lambda$ is of the form $\mathcal{C} = \operatorname{add}(M)$ for an n-cluster tilting module $M \in \operatorname{mod}\Lambda$. We then give the following characterization, which is the main result of this paper.

Theorem B (Theorem 4.1). *Let $\Lambda = \mathbf{k}Q/J^2$ and $n \geq 2$. Then Λ admits an n-cluster tilting subcategory $\mathcal{C}$ if and only if Q is an n-admissible quiver. If moreover $Q \neq \tilde{A}_m$, then $\mathcal{C}$ is unique and $\mathcal{C} = \operatorname{add}\left(\bigoplus_{j\geq 0} \tau_n^{-j}(\Lambda)\right)$ where $\tau_n^- = \tau^-\Omega^{-(n-1)}$.*

For the definition of n-admissible quivers we refer to Definition 2.6 and Definition 3.1; we refer to Remark 4.4 for an easy way to construct n-admissible quivers. Given Theorem B, it is not hard to classify radical square zero bound quiver algebras which admit $n\mathbb{Z}$-cluster tilting subcategories in the sense of [IJ17].

Theorem C (Theorem 4.7). *Let $\Lambda = \mathbf{k}Q/\mathcal{J}^2$ and $n \geq 2$. Then Λ admits an $n\mathbb{Z}$-cluster tilting subcategory if and only if $Q = A_m$ and $n \mid (m-1)$ or $Q = \tilde{A}_m$ and $n \mid m$.*

Finally, we show that if $Q \neq \tilde{A}_m$, then the set of n-cluster tilting subcategories of $\mathbf{k}Q/\mathcal{J}^2$ forms a lattice isomorphic to the lattice of a certain integer which depends only on Q.

Theorem D (Theorem 4.12). *Let $\Lambda = \mathbf{k}Q/\mathcal{J}^2$. Assume that $Q \neq \tilde{A}_m$ and that Q has admissible degree N. Set*

$$\mathbf{CT}(\Lambda) := \{\mathcal{C} \subseteq \operatorname{mod}\Lambda \mid \text{there exists } n \in \mathbb{Z}_{\geq 1} \text{ such that } \mathcal{C} \text{ is } n\text{-cluster tilting}\}.$$

Then $(\mathbf{CT}(\Lambda), \subseteq)$ is a complete lattice isomorphic to the opposite of the lattice of divisors of N.

For the definition of the admissible degree of a quiver we refer to Definition 4.11.

This paper is organized as follows. In Section 1 we establish notation and include some general results about n-cluster tilting subcategories and radical square zero algebras. In Section 2 we find some necessary conditions for a radical square zero bound quiver algebra to admit an n-cluster tilting subcategories. In Section 3 we show that these necessary conditions are also sufficient. In Section 4 we state our main result and a few applications.

1. Preliminaries and notation

Let $\mathbf{k}$ be a field. By an algebra we mean a finite-dimensional associative $\mathbf{k}$-algebra with a unit and by a module we mean a finite-dimensional right module.

Let Λ be an algebra. The *(Jacobson) radical* $\operatorname{rad}(\Lambda)$ of Λ is the intersection of all the maximal right ideals of Λ. The algebra Λ is a *radical square zero* algebra if $\operatorname{rad}^2(\Lambda) = 0$. We denote by $\operatorname{mod}\Lambda$ the category of Λ-modules. A Λ-module $M \in \operatorname{mod}\Lambda$ is called *basic* if all indecomposable direct summands of M are pairwise non-isomorphic. For $M \in \operatorname{mod}\Lambda$ we denote by $\Omega(M)$ the *syzygy* of M, that is the kernel of $P(M) \twoheadrightarrow M$, where $P(M)$ is the *projective cover* of M and by $\Omega^-(M)$ the *cosyzygy* of M, that is the cokernel of $M \hookrightarrow I(M)$ where $I(M)$ is the *injective hull* of M. Note that $\Omega(M)$ and $\Omega^-(M)$ are unique up to isomorphism.

We denote by D the duality $\operatorname{Hom}(-, K)$ between $\operatorname{mod}\Lambda$ and $\operatorname{mod}\Lambda^{\mathrm{op}}$. We denote by τ and τ^- the *Auslander–Reiten translations* and we recall the *Auslander–Reiten duality*

$$\operatorname{Ext}^1_\Lambda(M, N) \cong D\underline{\operatorname{Hom}}_\Lambda(\tau^-(N), M),$$

for all $M, N \in \operatorname{mod}\Lambda$, where $\underline{\operatorname{Hom}}_\Lambda(-,-)$ denotes morphisms in the *projectively stable category* $\underline{\operatorname{mod}}\Lambda$. For more details about the representation theory of finite-dimensional algebras and Auslander–Reiten theory we refer to [ARS95, ASS06].

Throughout this article n denotes a positive integer. A subcategory $\mathcal{C} \subseteq \operatorname{mod}\Lambda$ is called *n-rigid* if $\operatorname{Ext}^i_\Lambda(\mathcal{C},\mathcal{C}) = 0$ for all $i \in \{1,\dots,n-1\}$. A functorially finite subcategory $\mathcal{C} \subseteq \operatorname{mod}\Lambda$ is called *n-cluster tilting* if

$$\begin{aligned}\mathcal{C} &= \{X \in \operatorname{mod}\Lambda \mid \operatorname{Ext}^i_\Lambda(X,\mathcal{C}) = 0 \text{ for all } 0 < i < n\}\\ &= \{X \in \operatorname{mod}\Lambda \mid \operatorname{Ext}^i_\Lambda(\mathcal{C},X) = 0 \text{ for all } 0 < i < n\}.\end{aligned}$$

If moreover $\mathcal{C} = \operatorname{add}(M)$ for a module $M \in \operatorname{mod}\Lambda$, then M is called an *n-cluster tilting module.* Notice that any category of the form $\operatorname{add}(M)$ for some $M \in \operatorname{mod}\Lambda$ is functorially finite. In particular, if Λ is representation-finite, then any subcategory of $\operatorname{mod}\Lambda$ is functorially finite. Clearly if $\mathcal{C} \subseteq \operatorname{mod}\Lambda$ is n-cluster tilting, then $\Lambda, D(\Lambda) \in \mathcal{C}$; we use this fact throughout. We denote by τ_n and τ_n^- the *n-Auslander–Reiten translations* defined by $\tau_n = \tau\Omega^{n-1}$ and $\tau_n^- = \tau^-\Omega^{-(n-1)}$. For more details about higher dimensional Auslander–Reiten theory we refer to [Iya08].

Notice that there exists a unique 1-cluster tilting subcategory of $\operatorname{mod}\Lambda$, namely $\operatorname{mod}\Lambda$ itself. In the rest of this paper we assume that $n \geq 2$, unless otherwise stated. We also need the following observations.

Proposition 1.1. Let Λ be a finite-dimensional algebra and let $\mathcal{C} \subseteq \operatorname{mod}\Lambda$ be an n-cluster tilting subcategory.

(a) The functors $\tau_n : \mathcal{C}_\mathcal{P} \to \mathcal{C}_\mathcal{I}$ and $\tau_n^- : \mathcal{C}_\mathcal{I} \to \mathcal{C}_\mathcal{P}$ induce mutually inverse bijections, between the set $\mathcal{C}_\mathcal{P}$ of isomorphism classes of indecomposable nonprojective Λ-modules and the set $\mathcal{C}_\mathcal{I}$ of isomorphism classes of indecomposable noninjective Λ-modules.
(b) If $\mathcal{D} \subseteq \operatorname{mod}\Lambda$ is an n-cluster tilting subcategory such that $\mathcal{D} \subseteq \mathcal{C}$, then $\mathcal{C} = \mathcal{D}$.
(c) Let $M = \bigoplus_{j\geq 0} \tau_n^{-j}(\Lambda)$. Then $M \in \mathcal{C}$. If moreover M is an n-cluster tilting module, then $\mathcal{C} = \operatorname{add}(M)$.

Proof. (a) See [Iya08, Theorem 2.8].
(b) Follows directly from the definition of n-cluster tilting subcategories.
(c) Since $\Lambda \in \mathcal{C}$, we have that $M \in \mathcal{C}$ by (a). In particular we have $\operatorname{add}(M) \subseteq \mathcal{C}$. Hence if M is an n-cluster tilting module, then by (b) we conclude that $\mathcal{C} = \operatorname{add}(M)$. □

Lemma 1.2. Let Λ be a finite-dimensional algebra and $M, N \in \operatorname{mod}\Lambda$ with $M \neq 0$. Assume that $\tau_x^-(N) \cong M$ for some $x \geq 1$. Then $\operatorname{Ext}^x_\Lambda(M,N) \neq 0$.

Proof. We first consider the case $x = 1$. By additivity of τ^- and $\operatorname{Ext}^x_\Lambda(-,-)$ we may assume that M and N are indecomposable. Since $\tau^-(N) \cong M$ and M is nonzero, it follows that N is noninjective. Then there exists an almost split sequence $0 \longrightarrow N \longrightarrow F \longrightarrow \tau^-(N) \longrightarrow 0$ in $\operatorname{mod}\Lambda$ and the result follows. For $x \geq 2$ we have using dimension shift that

$$\operatorname{Ext}^x_\Lambda(M,N) \cong \operatorname{Ext}^1_\Lambda(\tau_x^-(N), \Omega^{-(x-1)}(N)) = \operatorname{Ext}^1_\Lambda(\tau^-(\Omega^{-(x-1)}(N)), \Omega^{-(x-1)}(N)) \neq 0,$$

where the last inequality follows from the case $x = 1$. □

Next we recall some background on bound quiver algebras. A *quiver* $Q = (Q_0, Q_1, s, t)$ is a quadruple consisting of a set Q_0 of *vertices*, a set Q_1 of *arrows* and two maps $s, t : Q_1 \to Q_0$ called *source map* and *target map*. All quivers in this article are *finite*, that is both Q_0 and Q_1 are finite sets. Moreover, for simplicity, we assume that all quivers in this article are *connected*, that is the underlying unoriented graph of Q is connected. For a vertex $v \in Q_0$, the *ingoing degree* of v, denoted by $\delta^-(v)$, is the number of arrows ending at v and the *outgoing degree* of v, denoted by $\delta^+(v)$, is the number of arrows starting at v. The *degree* of v is the tuple $(\delta^-(v), \delta^+(v))$. For a quiver Q and $k \geq 1$, a *path* $\mathbf{p}$ *of length* k in Q is a sequence of k consecutive arrows

$$\mathbf{p} = \; v_1 \xrightarrow{\alpha_1} v_2 \xrightarrow{\alpha_2} \cdots \longrightarrow v_k \xrightarrow{\alpha_k} v_{k+1},$$

in Q. We also assign a trivial path ϵ_v of length 0 to each vertex $v \in Q_0$.

Let Q be a quiver. We denote by $\mathbf{k}Q$ the *path algebra* of Q and we denote by $\mathcal{J} \subseteq \mathbf{k}Q$ the *arrow ideal* of Q, that is the ideal of $\mathbf{k}Q$ generated by the arrows of Q. An ideal $\mathcal{I} \subseteq \mathbf{k}Q$ is called *admissible* if there exists $k \geq 2$ such that $\mathcal{J}^k \subseteq \mathcal{I} \subseteq \mathcal{J}^2$. If $\mathcal{I}$ is an admissible ideal, then the *bound quiver algebra* $\Lambda = \mathbf{k}Q/\mathcal{I}$ is a finite-dimensional algebra. Throughout we identify Λ-modules and representations of Q bound by $\mathcal{I}$. For a vertex $v \in Q_0$, we denote by $P(v)$, $I(v)$ and $S(v)$ the indecomposable projective, injective and simple Λ-modules corresponding to v. When clear from context, we use composition series to denote Λ-modules. For more details on bound quiver algebras and their representation theory we refer to [ARS95, ASS06].

For radical square zero algebras we have the following easy observations.

Lemma 1.3. Let Λ be a radical square zero algebra and let M be a nonprojective Λ-module. Then $\Omega(M)$ is semisimple.

Proof. Since M is nonprojective, it follows that $\Omega(M) \neq 0$. Let $P(M)$ be the projective cover of M. Then $\operatorname{rad}^2(P(M)) = P(M)\operatorname{rad}^2(\Lambda) = 0$ and so $\operatorname{rad}(P(M))$ is semisimple. Since $\Omega(M)$ is a submodule of $\operatorname{rad}(P(M))$ and $\Omega(M) \neq 0$, we conclude that $\Omega(M)$ is semisimple. $\square$

Lemma 1.4. Let Λ be a radical square zero algebra and assume that $\mathcal{C} \subseteq \operatorname{mod}\Lambda$ is an n-cluster tilting subcategory. Let I be an indecomposable injective Λ-module. Then $\dim(\Omega(I)) \leq 1$.

Proof. If I is projective, then $\dim(\Omega(I)) = 0$. Otherwise, assume that I is nonprojective. By Lemma 1.3 we have that $\Omega(I)$ is semisimple. By [Vas19, Corollary 3.3] and since $I \in \mathcal{C}_{\mathcal{P}}$, we have that $\Omega(I)$ is indecomposable. Since $\Omega(I)$ is semisimple and indecomposable, it follows that $\dim(\Omega(I)) \leq 1$. $\square$

In this paper we study radical square zero bound quiver algebras. These can be easily described as in the following lemma.

Lemma 1.5. A bound quiver algebra $\mathbf{k}Q/\mathcal{I}$ is a radical square zero algebra if and only if $\mathcal{I} = \mathcal{J}^2$.

Proof. Since $\mathcal{I}$ is admissible, we have that $\mathcal{I} = \mathcal{J}^2$ if and only if $\mathcal{J}^2 \subseteq \mathcal{I}$, which is equivalent to the ideal $(\mathcal{J}/\mathcal{I})^2$ being equal to the zero ideal. Since $(\mathcal{J}/\mathcal{I})^2 = \operatorname{rad}^2(\mathbf{k}Q/\mathcal{I})$, the result follows. $\square$

As a corollary, any radical square zero algebra over an algebraically closed field is Morita equivalent to a bound quiver algebra of the form $\mathbf{k}Q/\mathcal{J}^2$.

Proposition 1.6. Let Λ be a basic and connected finite-dimensional $\mathbf{k}$-algebra and assume that $\mathbf{k}$ is algebraically closed. Then Λ is a radical square zero algebra if and only if $\Lambda \cong \mathbf{k}Q/\mathcal{J}^2$ for some quiver Q.

Proof. Since Λ is basic and $\mathbf{k}$ is algebraically closed, there exists a quiver Q and an admissible ideal $\mathcal{I} \subseteq \mathbf{k}Q$ such that $\Lambda \cong \mathbf{k}Q/\mathcal{I}$. The result follows from Lemma 1.5. $\square$

We also need to recall the following notion.

Definition 1.7. A bound quiver algebra $\mathbf{k}Q/\mathcal{I}$ is a *string algebra* if the following conditions hold:

(S1) For every vertex $v \in Q_0$ we have that $\delta^-(v) \leq 2$ and $\delta^+(v) \leq 2$.
(S2) For every arrow $\alpha \in Q_1$ there exists at most one arrow $\beta \in Q_1$ such that $\beta\alpha \notin \mathcal{I}$ and at most one arrow $\gamma \in Q_1$ such that $\alpha\gamma \notin \mathcal{I}$.
(S3) The ideal $\mathcal{I}$ can be generated by paths.

Indecomposable modules over string algebras are classified in [BR87] using the combinatorics of strings and bands. We briefly recall these combinatorics.

Let $\mathbf{k}Q/\mathcal{I}$ be a string algebra. For every arrow $\alpha \in Q_1$ we define a formal inverse α^- such that $s(\alpha^-) = t(\alpha)$ and $t(\alpha^-) = s(\alpha)$. We define $Q_1^- = \{\alpha^- \mid \alpha \in Q_1\}$ and we set $(\alpha^-)^- = \alpha$. We call elements of Q_1 *direct arrows* and elements of Q_1^- *inverse arrows*. A *formal path of length* $k \geq 1$ is a sequence $\boldsymbol{\ell} = \ell_k \dots \ell_1$ such that $\ell_i \in Q_1 \cup Q_1^-$ and such that for all $i \in \{1, \dots, k-1\}$ we have $t(\ell_i) = s(\ell_{i+1})$ and $\ell_i \neq \ell_{i+1}^-$. We also set $\boldsymbol{\ell}^- = \ell_1^- \dots \ell_k^-$. We say that $\boldsymbol{\ell}$ is a *string of length* k if no formal path of the form $\ell_{i+r} \dots \ell_i$ or $\ell_i^- \dots \ell_{i+r}^-$ is in $\mathcal{I}$ for $1 \leq i \leq i+r \leq k$. To each vertex $v \in Q_0$ we also associate a string $\boldsymbol{e_v}$ of length 0. We say that a string $\boldsymbol{\ell}$ is a *band* if $s(\ell_1) = t(\ell_k)$ and $\boldsymbol{\ell}^q$ is a string for every $q \geq 1$, and moreover there is no string $\boldsymbol{\ell'} \neq \boldsymbol{\ell}$ such that $\boldsymbol{\ell'} \dots \boldsymbol{\ell'} = \boldsymbol{\ell}$. For each string or band

$\boldsymbol{\ell}$ we can define a corresponding string or band module $M(\boldsymbol{\ell})$ which is indecomposable. Furthermore, every indecomposable $\mathbf{k}Q/\mathcal{I}$-module is isomorphic to a string or band module. For more details on the definition of $M(\boldsymbol{\ell})$ and on other facts about the representation theory of string algebras we refer to [BR87, Section 3].

2. Necessary conditions

In this section we investigate the existence of an n-cluster tilting subcategory $\mathcal{C} \subseteq \operatorname{mod}\Lambda$ where $\Lambda = \mathbf{k}Q/\mathcal{J}^2$ and $n \geq 2$. Our aim is to show that these assumptions impose some important restrictions on Q and Λ.

2.1. **n-pre-admissible quivers.** Recall that if $\mathcal{C} \subseteq \operatorname{mod}\Lambda$ is an n-cluster tilting subcategory, then $\Lambda, D(\Lambda) \in \mathcal{C}$. We start with showing that the degree of a vertex in Q is bounded.

Lemma 2.1. Let $\Lambda = \mathbf{k}Q/\mathcal{J}^2$ and assume that $\mathcal{C} \subseteq \operatorname{mod}\Lambda$ is an n-cluster tilting subcategory. Let $v \in Q_0$ be a vertex. Then $\delta^-(v) \leq 2$ and $\delta^+(v) \leq 2$.

Proof. We only show that $\delta^-(v) \leq 2$; the inequality $\delta^+(v) \leq 2$ follows dually. Consider the short exact sequence $0 \longrightarrow \Omega(I(v)) \longrightarrow P(I(v)) \longrightarrow I(v) \longrightarrow 0$. Then

$$\dim(I(v)) = \dim(P(I(v))) - \dim(\Omega(I(v))) \geq \dim(P(I(v))) - 1, \tag{2.1}$$

where the last inequality follows from Lemma 1.4. Moreover, since Λ is a radical square zero bound quiver algebra, it immediately follows that $\dim(I(v)) = \delta^-(v) + 1$ and $\dim(P(I(v))) \geq 2\delta^-(v)$. Hence (2.1) gives

$$\delta^-(v) + 1 \geq 2\delta^-(v) - 1,$$

or equivalently $2 \geq \delta^-(v)$. □

We continue with showing that there are no multiple arrows between two vertices of Q.

Lemma 2.2. Let $\Lambda = \mathbf{k}Q/\mathcal{J}^2$ and assume that $\mathcal{C} \subseteq \operatorname{mod}\Lambda$ is an n-cluster tilting subcategory. Let $v, u \in Q_0$ be vertices. Then $|\{\alpha \in Q_1 \mid s(\alpha) = v \text{ and } t(\alpha) = u\}| \leq 1$.

Proof. By Lemma 2.1 we have that $|\{\alpha \in Q_1 \mid s(\alpha) = v, t(\alpha) = u\}| \leq 2$. Assume towards a contradiction that there exist two arrows $\alpha_1 : v \longrightarrow u$ and $\alpha_2 : v \longrightarrow u$. Then, by Lemma 2.1 we have that the composition series of $I(u)$ is $\begin{smallmatrix} v & & v \\ & u & \end{smallmatrix}$ while the composition series of $P(v)$ is $\begin{smallmatrix} & v & \\ u & & u \end{smallmatrix}$. Hence the projective cover of $I(u)$ is $P(I(u)) \cong P(v) \oplus P(v)$ and

$$\dim(\Omega(I(u))) = \dim(P(I(u))) - \dim(I(u)) = 2\dim(P(v)) - \dim(I(u)) = 2 \cdot 3 - 3 = 3,$$

which contradicts Lemma 1.4. □

Next we show that no vertex can have degree $(0, 2)$ or $(2, 0)$ and, moreover, that if $n > 2$, then no vertex can have degree $(2, 2)$ either.

Lemma 2.3. Let $\Lambda = \mathbf{k}Q/\mathcal{J}^2$ and assume that $\mathcal{C} \subseteq \operatorname{mod}\Lambda$ is an n-cluster tilting subcategory. Let $v \in Q_0$ be a vertex.

(a) If $\delta^+(v) = 2$, then $\delta^-(v) \geq 1$.
(b) If $\delta^-(v) = 2$, then $\delta^+(v) \geq 1$.
(c) If $\delta^-(v) = \delta^+(v) = 2$, then $n = 2$.

Proof. (a) Since $\Lambda = \mathbf{k}Q/\mathcal{J}^2$ and $\delta^+(v) = 2$, it follows that $\dim(P(v)) = 3$. Assume towards a contradiction that $\delta^-(v) = 0$. Then $I(v) = S(v)$ and $P(I(v)) = P(v)$. Hence

$$\dim(\Omega(I(v))) = \dim(P(I(v))) - \dim(I(v)) = 3 - 1 = 2,$$

which contradicts Lemma 1.4.

(b) Dual to (a).

(c) Let $\alpha_1 : v \longrightarrow u_1$, $\alpha_2 : v \longrightarrow u_2$, $\beta_1 : w_1 \longrightarrow v$ and $\beta_2 : w_2 \longrightarrow v$ be the arrows starting and ending at v. Then the composition series of $P(v)$ is ${}_{u_1}{}^{v}{}_{u_2}$ while the composition series of $I(v)$ is ${}^{w_1}{}_{v}{}^{w_2}$. For $i = 1, 2$, let $\pi_i : P(v) \longrightarrow \begin{smallmatrix} v \\ u_i \end{smallmatrix}$ be the projective cover of $\begin{smallmatrix} v \\ u_i \end{smallmatrix}$ and $\iota_i : \begin{smallmatrix} w_i \\ v \end{smallmatrix} \longrightarrow I(v)$ be the injective envelope of $\begin{smallmatrix} w_i \\ v \end{smallmatrix}$. Then it follows that

$$\operatorname{coker}\left(\begin{bmatrix} \pi_1 \\ \pi_2 \end{bmatrix} : P(v) \longrightarrow \begin{smallmatrix} v \\ u_1 \end{smallmatrix} \oplus \begin{smallmatrix} v \\ u_2 \end{smallmatrix} \right) \cong S(v) \cong \ker\left(\begin{bmatrix} \iota_1 & \iota_2 \end{bmatrix} : \begin{smallmatrix} w_1 \\ v \end{smallmatrix} \oplus \begin{smallmatrix} w_2 \\ v \end{smallmatrix} \longrightarrow I(v) \right).$$

Hence the sequence

$$\begin{tikzcd}
0 \arrow[r] & P(v) \arrow[r, "\begin{bmatrix} \pi_1 \\ \pi_2 \end{bmatrix}"] & \begin{smallmatrix} v \\ u_1 \end{smallmatrix} \oplus \begin{smallmatrix} v \\ u_2 \end{smallmatrix} \arrow[rr] \arrow[dr, two heads] & & \begin{smallmatrix} w_1 \\ v \end{smallmatrix} \oplus \begin{smallmatrix} w_2 \\ v \end{smallmatrix} \arrow[r, "\begin{bmatrix} \iota_1 & \iota_2 \end{bmatrix}"] & I(v) \arrow[r] & 0 \\
& & & S(v) \arrow[ur, hook] & & &
\end{tikzcd}$$

gives a nonzero element of $\operatorname{Ext}^2_\Lambda(I(v), P(v))$. Since $I(v), P(v) \in \mathcal{C}$ and $\operatorname{Ext}^2_\Lambda(I(v), P(v)) \neq 0$, it follows that $n \leq 2$. Since by assumption we have that $n \geq 2$, we conclude that $n = 2$. $\square$

Finally we examine how an arrow between two vertices affects the degree of the two vertices.

Lemma 2.4. Let $\Lambda = \mathbf{k}Q/\mathcal{J}^2$ and assume that $\mathcal{C} \subseteq \operatorname{mod}\Lambda$ is an n-cluster tilting subcategory.

(a) Let $w_1 \longrightarrow v \longleftarrow w_2$ be a subquiver of Q. Then $\delta^+(w_1) = \delta^+(w_2) = 1$.
(b) Let $u_1 \longleftarrow v \longrightarrow u_2$ be a subquiver of Q. Then $\delta^-(u_1) = \delta^-(u_2) = 1$.

Proof. We only prove (a); (b) follows dually. By symmetry it is enough to show that $\delta^+(w_1) = 1$. Since by Lemma 2.1 we have that $\delta^+(w_1) \leq 2$ and since by assumption we have that $\delta^+(w_1) \geq 1$, it is enough to show that $\delta^+(w_1) \neq 2$. Assume towards a contradiction that $\delta^+(w_1) = 2$. Since $\Lambda = \mathbf{k}Q/\mathcal{J}^2$, it follows from Lemma 2.1 that $\dim(I(v)) = 3$, $\dim(P(w_1)) = 3$, $\dim(P(w_2)) \geq 2$ and $P(I(v)) \cong P(w_1) \oplus P(w_2)$. Then

$$\dim(\Omega(I(v))) = \dim(P(I(v))) - \dim(I(v)) = \dim(P(w_1)) + \dim(P(w_2)) - \dim(I(v)) \geq 3 + 2 - 3 = 2,$$

which contradicts Lemma 1.4. $\square$

Corollary 2.5. Let $\Lambda = \mathbf{k}Q/\mathcal{J}^2$ and assume that $\mathcal{C} \subseteq \operatorname{mod}\Lambda$ is an n-cluster tilting subcategory. Let $v \longrightarrow u$ be an arrow in Q. Then $\delta^+(v) + \delta^-(u) \leq 3$.

Proof. Follows immediately by Lemma 2.1 and Lemma 2.4. $\square$

The results of this section motivate the following definition.

Definition 2.6. A quiver Q is called *n-pre-admissible* if the following conditions are satisfied.

(i) For all vertices $v \in Q_0$ we have $\delta(v) \in \{(0,0), (0,1), (1,0), (1,1), (1,2), (2,1)\} \cup E$, where
$$E = \begin{cases} \{(2,2)\}, & \text{if } n = 2, \\ \varnothing, & \text{otherwise.} \end{cases}$$
(ii) There exist no multiple arrows between two vertices.
(iii) For all arrows $v \longrightarrow u$ in Q we have $\delta^+(v) + \delta^-(u) \leq 3$.

Remark 2.7. It follows immediately by the definition of n-pre-admissible quivers that an n-pre-admissible quiver which has no vertex of degree $(2,2)$ is n-pre-admissible for any $n \geq 2$.

Example 2.8. (a) The quivers A_m and $\tilde{A}_m$ are n-pre-admissible for any $n \geq 2$.
(b) The quiver

$$\begin{tikzcd} 1 \arrow[r] & 2 \arrow[r] \arrow[loop above] & 3 \end{tikzcd}$$

is not n-pre-admissible for any $n \geq 2$ since there exists an arrow $2 \longrightarrow 2$, but $\delta^+(2) + \delta^-(2) = 4$.

(c) The quiver

$$\begin{tikzcd}
1 \arrow[loop above] \arrow[r] & 2 \arrow[r] & 3 \arrow[r] \arrow[dr] & 4 \arrow[r] & 5 \arrow[loop above] \\
& 8 \arrow[ur] & & 6 \arrow[dl] & \\
& & 7 \arrow[ul] & &
\end{tikzcd}$$

is 2-pre-admissible but not n-pre-admissible for $n \geq 3$ since $\delta(3) = (2, 2)$.

(d) The quiver

$$\begin{tikzcd}
1 \arrow[r] \arrow[dr] & 2 \arrow[r] & 3 \arrow[r] & 4 \arrow[ddd] \\
& 9 \arrow[d] & 12 \arrow[ur] & \\
& 10 \arrow[dl] & 11 \arrow[u] & \\
8 \arrow[uuu] & 7 \arrow[l] & 6 \arrow[l] & 5 \arrow[l] \arrow[ul]
\end{tikzcd}$$

is n-pre-admissible for any $n \geq 2$.

We have the following immediate result.

Proposition 2.9. Let $\Lambda = \mathbf{k}Q/\mathcal{J}^2$ and assume that $\mathcal{C} \subseteq \operatorname{mod}\Lambda$ is an n-cluster tilting subcategory. Then Q is n-pre-admissible.

Proof. Follows immediately by Lemma 2.1, Lemma 2.3, Lemma 2.2 and Corollary 2.5. □

Radical square zero bound quiver algebras with n-pre-admissible quivers are especially easy to study from the point of view of representation theory. Indeed, we have the following result.

Theorem 2.10. *Let Q be an n-pre-admissible quiver and let $\Lambda = \mathbf{k}Q/\mathcal{J}^2$.*

(a) Λ is a string algebra.
(b) If $\ell_k \dots \ell_1$ is a string in Λ, then $k \leq 2$. In particular, there are no bands in Λ.
(c) Λ is representation-finite.
(d) If M is an indecomposable Λ-module and M is not simple, then M is projective or injective.

Proof. (a) Since Q is n-pre-admissible, we have that $\delta^+(v) \leq 2$ and $\delta^-(v) \leq 2$ for every vertex $v \in Q_0$. Since $\mathcal{J}^2$ is generated by all paths of length 2, it immediately follows that Λ is a string algebra.

(b) Let $\ell_k \dots \ell_1$ be a string in Λ and assume towards a contradiction that $k \geq 3$. Consider the string $\ell_3\ell_2\ell_1$. Since every path of length two is in $\mathcal{J}^2$, it follows that ℓ_1 and ℓ_2 cannot be both direct or both inverse letters. Similarly ℓ_3 and ℓ_2 cannot be both direct or both inverse letters. Hence $\ell_3\ell_2\ell_1$ is either of the form $\alpha\beta^-\gamma$ or of the form $\alpha^-\beta\gamma^-$ for some arrows $\alpha, \beta, \gamma \in Q_1$ with $\alpha \neq \beta$ and $\gamma \neq \beta$. If $\alpha = \gamma$, then we readily get that $s(\alpha) = s(\beta)$ and $t(\alpha) = t(\beta)$, which contradicts Definition 2.6(ii). Otherwise, if $\alpha \neq \gamma$, then we readily get that

$$\delta^+(s(\beta)) + \delta^-(t(\beta)) \geq 4,$$

which contradicts Definition 2.6(iii). Hence $k \leq 2$. Since the length of a string is bounded by 2, it follows that there are no bands in Λ.

(c) Follows immediately from (b) since indecomposable Λ-modules are classified by string and band modules, see [BR87, Section 3].

(d) Let M be an indecomposable Λ-module and assume that M is not simple. From (b) it follows that M is isomorphic to a string module $M(\boldsymbol{\ell})$ where $\boldsymbol{\ell}$ has length at most 2 (for the definition of $M(\boldsymbol{\ell})$ we refer to [BR87, Section 3]). Since M is not simple, it follows that $\boldsymbol{\ell}$ has length different than 0 and so $\boldsymbol{\ell}$ has length 1 or 2. If the length of $\boldsymbol{\ell}$ is 1, then $\boldsymbol{\ell} = \alpha$ for some arrow $\alpha \in Q_1$ (the modules $M(\alpha)$ and $M(\alpha^-)$ are isomorphic). Let $\alpha : v \longrightarrow u$. Then $\delta^+(v) \in \{1, 2\}$

and using Definition 2.6(iii) and the fact that Λ is a radical square zero algebra it is easy to see that

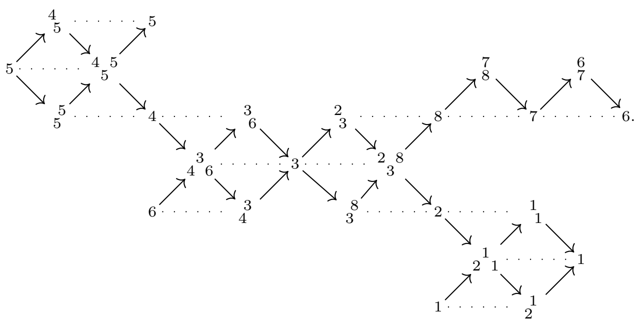
$$M(\alpha) \cong \begin{cases} P(v), & \text{if } \delta^+(v) = 1, \\ I(u), & \text{if } \delta^+(v) = 2. \end{cases}$$

If the length of $\boldsymbol{\ell}$ is 2, and since Λ is a radical square zero algebra, then $\boldsymbol{\ell} = \alpha\beta^-$ or $\boldsymbol{\ell} = \alpha^-\beta$ for some arrows $\alpha, \beta \in Q_1$. Let $\alpha : v \longrightarrow u$. Similarly to the case of length 1, it is easy to see that

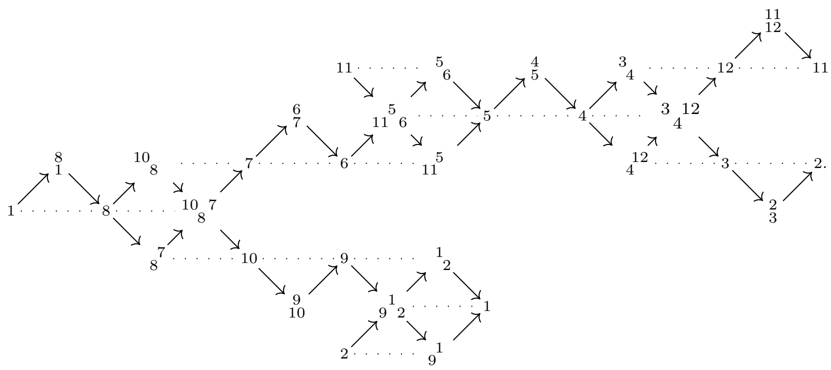
$$M(\boldsymbol{\ell}) \cong \begin{cases} P(v), & \text{if } \boldsymbol{\ell} = \alpha\beta^-, \\ I(u), & \text{if } \boldsymbol{\ell} = \alpha^-\beta, \end{cases}$$

and so in both cases M is projective or injective. $\square$

Example 2.11. (a) Let Q be as in Example 2.8(c). The Auslander–Reiten quiver of $\mathbf{k}Q/\mathcal{J}^2$ is



(b) Let Q be as in Example 2.8(d). The Auslander–Reiten quiver of $\mathbf{k}Q/\mathcal{J}^2$ is



2.2. **Flow paths.** We have seen that if $\Lambda = \mathbf{k}Q/\mathcal{J}^2$ and $\mathcal{C} \subseteq \operatorname{mod}\Lambda$ is an n-cluster tilting subcategory, then Q is n-pre-admissible. The opposite is not true in general. It turns out that there are additional properties that Q must satisfy. To describe these properties we need to consider certain paths in Q.

Definition 2.12. Let Q be an n-pre-admissible quiver and let $k \geq 2$. A *$(k$-$)$flow path* $\mathbf{v}$ *in* Q is a path

$$\mathbf{v} = \; v_1 \longrightarrow v_2 \longrightarrow \cdots \longrightarrow v_{k-1} \longrightarrow v_k, \tag{2.2}$$

such that $\delta(v_s) = (1,1)$ if and only if $1 < s < k$.

Notice that since Q is n-pre-admissible, there are no multiple arrows between two vertices. Hence a flow path is defined uniquely by its vertices and we do not need to label arrows in a flow path. For a flow path $\mathbf{v}$ in Q we use v_i to denote its vertices as in (2.2). Moreover, in what follows we write "k-flow path" when the length k of the flow path is important and "flow path" otherwise. Many of the results presented in this section have a dual version which although we omit for brevity, we sometimes use. We first study the case where there are no flow paths in Q.

Lemma 2.13. Let Q be an n-pre-admissible quiver. Then there exists a flow path in Q if and only if $Q \neq A_1$ and $Q \neq \tilde{A}_m$ for some $m \geq 1$.

Proof. It is clear by the definition of a flow path that if there exists a flow path in Q, then $Q \neq A_1$ and $Q \neq \tilde{A}_m$. For the other direction, assume that $Q \neq A_1$ and $Q \neq \tilde{A}_m$ and we show that there exists a flow path in Q. Since Q is connected and $Q \neq \tilde{A}_m$, there exists a vertex v_1 in Q with degree $\delta(v_1) \neq (1,1)$. Since $Q \neq A_1$, we have $\delta(v_1) \neq (0,0)$. Since Q is finite, any path starting or ending at v_1 eventually passes through a vertex v_k with $\delta(v_k) \neq (1,1)$; let $\mathbf{v}$ be a minimal such path. Then $\mathbf{v}$ is a flow path by definition. □

Proposition 2.14. Let Q be an n-pre-admissible quiver and let $\Lambda = \mathbf{k}Q/\mathcal{J}^2$.

(a) If $Q = A_1$, then $\mathcal{C}$ is an n-cluster tilting subcategory of $\operatorname{mod}\Lambda$ if and only if $\mathcal{C} = \operatorname{mod}\Lambda = \operatorname{add}(\Lambda)$.

(b) If $Q = \tilde{A}_m$ for some $m \geq 1$, then $\mathcal{C}$ is an n-cluster tilting subcategory of $\operatorname{mod}\Lambda$ if and only if $n \mid m$ and $\mathcal{C} = \operatorname{add}\left(\Lambda \oplus \left(\bigoplus_{j=0}^{\frac{m}{n}-1} \tau_n^{-j}(S)\right)\right)$ for some simple module $S \in \operatorname{mod}\Lambda$.

Proof. (a) In this case $\Lambda = \mathbf{k}$ and the result is clear.

(b) Follows from [DI20, Theorem 5.1]. □

By Lemma 2.13 we have that the only n-pre-admissible quivers that do not have flow paths are the quivers A_1 and $\tilde{A}_m$ for $m \geq 1$. Proposition 2.14 classifies radical square zero bound quiver algebras with such quivers that admit n-cluster tilting subcategories. Hence it remains to study n-pre-admissible quivers that have flow paths. For the rest of this section we fix an n-pre-admissible quiver Q such that $Q \neq A_1$ and $Q \neq \tilde{A}_m$ for any $m \geq 1$. It then follows that there exists a flow path in Q. We further set $\Lambda \coloneqq \mathbf{k}Q/\mathcal{J}^2$. We start with some simple but important observations about flow paths.

Lemma 2.15. Let $\mathbf{v}$ be a k-flow path in Q. Let $1 \leq s \leq t \leq k$.

(a) If $1 < s$ and $t < k$, then $v_s = v_t$ if and only if $s = t$.

(b) If $s < t$ and $v_s = v_t$, then $s = 1$ and $t = k$. In particular, in this case $v_1 = v_k$.

Proof. (a) If $s = t$, then clearly $v_s = v_t$. Assume towards a contradiction that $v_s = v_t$ but $s < t$. Without loss of generality, we may assume that $s < t$ are minimal among $\{2, \dots, k-1\}$ with these properties. By the definition of a k-flow path and since $\delta(v_s) = \delta(v_t) = (1,1)$, it follows that $v_{s-1} = v_{t-1}$. By minimality of s and t we conclude that $s - 1 = 1$. Moreover, we have $1 < s \leq t-1 < t < k$ and so $\delta(v_{t-1}) = (1,1)$. Then

$$(1,1) \neq \delta(v_1) = \delta(v_{s-1}) = \delta(v_{t-1}) = (1,1),$$

which is a contradiction.

(b) Since $s < t$ and $v_s = v_t$, it follows from (a) that $s = 1$ or $t = k$. In both cases we get that $\delta(v_s) = \delta(v_t) \neq (1,1)$. It follows from the definition of a k-flow path that $s = 1$ and $t = k$. □

Lemma 2.16. Let $\mathbf{v}$ be a k-flow path in Q and let $\mathbf{u}$ be a k'-flow path in Q.

(a) Let v_s be a vertex in $\mathbf{v}$ with $\delta(v_s) = (1,1)$ and assume that $v_s = u_t$ for some vertex u_t in $\mathbf{u}$. Then $\mathbf{v} = \mathbf{u}$.

(b) Assume that $v_k = u_{k'}$ and that $v_{k-1} = u_{k'-1}$. Then $\mathbf{v} = \mathbf{u}$.

(c) Assume that $v_k = u_{k'}$ and that $\delta^-(v_k) = 1$. Then $\mathbf{v} = \mathbf{u}$.

Proof. (a) Since $\delta(v_s) = (1,1)$, it follows from the definition of a flow path that $1 < s < k$. Since $v_s = u_t$ it follows that $\delta(u_t) = (1,1)$. Without loss of generality we may assume that $s \leq t$. By the definition of a k-flow path it follows that $v_{s-1} = u_{t-1}$. Continuing inductively we see that $v_2 = u_{t-(s-2)}$ and this vertex has degree $(1,1)$. Hence the only arrow ending at $v_2 = u_{t-(s-2)}$ is the arrow coming from v_1. Since $\delta(v_1) \neq (1,1)$ and $\delta(u_{t-(s-2)}) = (1,1)$, and since there exists an arrow $v_1 \longrightarrow u_{t-(s-2)}$, it follows that $u_1 = v_1$ and $u_2 = u_{t-(s-2)}$. By Lemma 2.15(a) it follows that $2 = t - (s-2)$ and so $s = t$. A dual argument shows that $v_k = u_{k'}$. Since $v_1 = u_1$, $v_k = u_k$ and $v_s = u_s$ for some s with $1 < s < k$, it readily follows that $\mathbf{v} = \mathbf{u}$.

(b) Since $\mathbf{v}$ and $\mathbf{u}$ are flow paths, there exist arrows $\alpha : v_{k-1} \longrightarrow v_k$ and $\beta : u_{k'-1} \longrightarrow u_{k'}$. Since $v_{k-1} = u_{k'-1}$ and $v_k = u_{k'}$, and since there exist no multiple arrows between two vertices of Q, it follows that $\alpha = \beta$. If $\delta(v_{k-1}) \neq (1,1)$, then $\mathbf{v} = v_{k-1} \longrightarrow v_k = u_{k'-1} \longrightarrow u_{k'} = \mathbf{u}$, as required. Otherwise, if $\delta(v_{k-1}) = (1,1)$, then the result follows from (a).

(c) Since $\delta^-(v_k) = \delta^-(u_{k'}) = 1$, there exists a unique arrow ending at $v_k = u_{k'}$. Since there exist arrows $v_{k-1} \longrightarrow v_k$ and $u_{k'-1} \longrightarrow u_{k'}$, we conclude that $v_{k-1} = u_{k'-1}$. The result follows from (b). □

Corollary 2.17. Let $\alpha : w \longrightarrow v$ be an arrow in Q.

(a) If $\delta(v) = (1,1)$, then there exists a unique flow path $\mathbf{v}$ in Q through v.

(b) If $\delta(v) \neq (1,1)$, then there exists a unique flow path $\mathbf{v}$ in Q ending at v such that α is the last arrow of $\mathbf{v}$.

In both cases we have that $v = v_j$ for some $j > 1$.

Proof. The existence of the flow path is clear since $Q \neq A_1$ and $Q \neq \tilde{A}_m$. The uniqueness follows from Lemma 2.16. □

If $\mathbf{v}$ is a k-flow path in Q, then the arrows ending and starting at the vertices v_1 and v_k play an important role in our investigation. Hence we also label the following vertices

$$\begin{tikzcd}
v^{-2} \arrow[dr, dotted] & & v^{-1} & & v^{+1} \arrow[dr, dotted] & & v^{+2} \\
& v_1 \arrow[ur, dotted] \arrow[r] & v_2 \arrow[r] & \cdots \arrow[r] & v_{k-1} \arrow[r] & v_k \arrow[ur, dotted] \arrow[dr, dotted] & \\
v^{-3} \arrow[ur, dotted] & & & & & & v^{+3},
\end{tikzcd} \tag{2.3}$$

where a dotted arrow means that such an arrow may or may not exist. When $\delta^-(v_1) = 1$, we assume that the arrow $v^{-2} \longrightarrow v_1$ is the one that exists and when $\delta^+(v_k) = 1$ we assume that the arrow $v_k \longrightarrow v^{+2}$ is the one that exists. Notice that by Definition 2.6(ii) we have that $v^{-1} \neq v_2$ and $v^{+1} \neq v_{k-1}$, if the vertices v^{-1} and v^{+1} exist.

We also set

$$I(\mathbf{v}) := \begin{cases} I(v_1), & \text{if } \delta^+(v_1) = 1, \\ I(v^{-1}), & \text{if } \delta^+(v_1) = 2, \end{cases} \quad \text{and} \quad P(\mathbf{v}) := \begin{cases} P(v_k), & \text{if } \delta^-(v_k) = 1, \\ P(v^{+1}), & \text{if } \delta^-(v_k) = 2. \end{cases}$$

With this notation, we have the following technical results.

Lemma 2.18. Let $\mathbf{v}$ be a k-flow path in Q and let $\mathbf{u}$ be a k'-flow path in Q. Then $P(\mathbf{v})$ is not injective and $P(\mathbf{v}) \cong P(\mathbf{u})$ if and only if $\mathbf{v} = \mathbf{u}$.

Proof. That $P(\mathbf{v})$ is not injective follows immediately by the definition of $P(\mathbf{v})$ and since $\delta(v_k) \in \{(1,0),(1,2),(2,1),(2,2)\}$. That $\mathbf{v} = \mathbf{u}$ implies $P(\mathbf{v}) \cong P(\mathbf{u})$ is clear. Now assume that $P(\mathbf{v}) \cong P(\mathbf{u})$ and we show that $\mathbf{v} = \mathbf{u}$. We first claim that $\delta^-(v_k) = \delta^-(u_{k'})$. Indeed, assume towards a contradiction that $\delta^-(v_k) \neq \delta^-(u_{k'})$. Without loss of generality we may assume that $\delta^-(v_k) = 1$ and $\delta^-(u_{k'}) = 2$. Then $P(\mathbf{v}) = P(v_k)$ and $P(\mathbf{u}) = P(u^{+1})$. Hence $v_k = u^{+1}$. By Definition 2.6(iii), it follows that $\delta^+(u^{+1}) = 1$. Therefore $\delta(v_k) = (\delta^-(v_k), \delta^+(v_k)) = (1, \delta^+(u^{+1})) = (1,1)$, which contradicts the definition of a k-flow path.

Hence we have shown that $\delta^-(v_k) = \delta^-(u_{k'})$. Next we consider the cases $\delta^-(v_k) = 1$ and $\delta^-(v_k) = 2$ separately.

Case $\delta^-(v_k) = 1$. In this case $\delta^-(u_{k'}) = 1$ and so $P(v_k) \cong P(u_{k'})$. It follows that $v_k = u_{k'}$. Therefore we have that $\mathbf{v} = \mathbf{u}$ by Lemma 2.16(c).

Case $\delta^-(v_k) = 2$. In this case $\delta^-(u_{k'}) = 2$ and so $P(v^{+1}) \cong P(u^{+1})$. It follows that $v^{+1} = u^{+1}$. Since $\delta^+(v^{+1}) = 1$ and there exist an arrow $v^{+1} \longrightarrow v_k$ and an arrow $v^{+1} = u^{+1} \longrightarrow u_{k'}$, it follows that $v_k = u_{k'}$. Since $v^{+1} = u^{+1} \neq u_{k'-1}$, it follows that $u_{k'-1} = v_{k-1}$. Therefore we have that $\mathbf{v} = \mathbf{u}$ by Lemma 2.16(b). □

Lemma 2.19. Let $v \in Q_0$ be a vertex. Then exactly one of the following three conditions hold:

(i) $P(v)$ is injective.
(ii) $\delta(v) = (2, 2)$.
(iii) $P(v) = P(\mathbf{v})$ for some flow path $\mathbf{v}$.

Proof. Notice that conditions (i) and (iii) cannot hold simultaneously since by Lemma 2.18 we have that $P(\mathbf{v})$ is not injective. Moreover, by the definition of $P(\mathbf{v})$, conditions (ii) and (iii) also cannot hold simultaneously. It is also clear that conditions (i) and (ii) cannot hold simultaneously, since if $\delta(v) = (2, 2)$, then $P(v)$ does not have simple socle. Hence it is enough to show that one of the conditions (i),(ii) or (iii) holds. We consider the cases $\delta^+(v) = 0$, $\delta^+(v) = 1$ and $\delta^+(v) = 2$ separately.

Case $\delta^+(v) = 0$. In this case $\delta(v) = (1, 0)$ and by Corollary 2.17(b) there exists a unique flow path $\mathbf{v}$ ending at v. It follows from the definition of $P(\mathbf{v})$ that $P(\mathbf{v}) = P(v)$ and so condition (iii) holds.

Case $\delta^+(v) = 1$. Let $\alpha : v \longrightarrow u$ be the unique arrow starting at v. We consider the subcases $\delta^-(u) = 1$ and $\delta^-(u) = 2$ separately.

- Subcase $\delta^-(u) = 1$. In this case $P(v) = I(u)$ is injective and so condition (i) holds.
- Subcase $\delta^-(u) = 2$. Let $\beta : w \longrightarrow u$ be the other arrow ending at u. By Corollary 2.17(b) and since $\delta(u) \neq (1, 1)$, it follows that there exists a unique flow path $\mathbf{v}$ such that the last arrow of $\mathbf{v}$ is β. It follows from the definition of $P(\mathbf{v})$ that $P(\mathbf{v}) = P(v)$ and so condition (iii) holds.

Case $\delta^+(v) = 2$. We consider the subcases $\delta(v) = (1, 2)$ and $\delta(v) = (2, 2)$ separately.

- Subcase $\delta(v) = (1, 2)$. By Corollary 2.17(b) there exists a unique flow path $\mathbf{v}$ ending at v. It follows from the definition of $P(\mathbf{v})$ that $P(\mathbf{v}) = P(v)$ and so condition (iii) holds.
- Subcase $\delta(v) = (2, 2)$. In this case condition (ii) holds. □

Let $v \in Q_0$ be a vertex. If there exists an n-cluster titling subcategory $\mathcal{C} \subseteq \operatorname{mod}\Lambda$, then we have that $P(v) \in \mathcal{C}$. By Proposition 1.1(a) we then have that $\tau_n^{-j}(P(v)) \in \mathcal{C}$ for all $j \geq 0$. By Lemma 2.19 there are three different cases for $P(v)$. If $P(v)$ belongs to the first case, that is if $P(v)$ is injective, then $\tau_n^{-j}(P(v)) = 0$ for $j \geq 1$. Our aim now is to compute $\tau_n^{-j}(P(v))$ for the two remaining cases. To this end we need the following lemma.

Lemma 2.20. Let $v \in Q_0$ be a vertex.

(a) If $\delta^-(v) = 0$, then $\Omega^-(S(v)) = \tau^-(S(v)) = 0$.
(b) If $\delta^-(v) = 1$, let $w \longrightarrow v$ be the unique arrow ending at v. Then $\Omega^-(S(v)) \cong S(w)$ and $\tau^-(S(v)) \cong \operatorname{coker}(S(v) \hookrightarrow P(w))$.
(c) If $\delta^-(v) = 2$, let $w_1 \longrightarrow v$ and $w_2 \longrightarrow v$ be the two arrows ending at v. Then $\Omega^-(S(v)) \cong S(w_1) \oplus S(w_2)$ and $\tau^-(S(v)) \cong I(v)$.

Proof. By Theorem 2.10 the algebra Λ is a string algebra. The Auslander–Reiten translations for modules over string algebras are computed in [BR87]. We include here a simple proof in this special case.

(a) If $\delta^-(v) = 0$, then $S(v) = I(v)$ is injective and so $\Omega^-(S(v)) = \tau^-(S(v)) = 0$.
(b) Since Λ is a radical square zero algebra and $\delta^-(v) = 1$, we have that $I(v) = \begin{smallmatrix} w \\ v \end{smallmatrix}$. Hence there exists a minimal injective presentation of $S(v)$ of the form

$$\begin{tikzcd} 0 \arrow[r] & S(v) \arrow[r, "i_0"] & I(v) \arrow[rr, "i_1"] \arrow[dr, two heads] & & I(w), \\ & & & S(w) \arrow[ur, hook] & \end{tikzcd}$$

from which it follows that $\Omega^-(S(v)) \cong S(w)$. Furthermore, by applying the inverse Nakayama functor ν^- to the above presentation we obtain an exact sequence

$$\begin{tikzcd} 0 \arrow[r] & \nu^-(S(v)) \arrow[r, "\nu^-(i_0)"] & P(v) \arrow[rr, "\nu^-(i_1)"] \arrow[dr, two heads] & & P(w) \arrow[r] & \tau^-(S(v)), \\ & & & S(v) \arrow[ur, hook] & & \end{tikzcd}$$

from which it follows that $\tau^{-}(S(v)) \cong \operatorname{coker}(S(v) \hookrightarrow P(w))$.

(c) Since Λ is a radical square zero algebra and $\delta^{-}(v) = 2$, we have that $I(v) = \begin{smallmatrix} w_1 & & w_2 \\ & v & \end{smallmatrix}$. Hence there exists a minimal injective presentation of $S(v)$ of the form

$$\begin{tikzcd}
0 \arrow[r] & S(v) \arrow[r, "i_0"] & I(v) \arrow[rr, "i_1"] \arrow[dr, two heads] & & I(w_1) \oplus I(w_2), \\
& & & S(w_1) \oplus S(w_2) \arrow[ur, hook] &
\end{tikzcd}$$

from which it follows that $\Omega^{-}(S(v)) \cong S(w_1) \oplus S(w_2)$. By applying the inverse Nakayama functor ν^{-} to the above presentation we obtain an exact sequence

$$\begin{tikzcd}
0 \arrow[r] & \nu^{-}(S(v)) \arrow[r, "\nu^{-}(i_0)"] & P(v) \arrow[rr, "\nu^{-}(i_1)"] \arrow[dr, two heads] & & P(w_1) \oplus P(w_2) \arrow[r] & \tau^{-}(S(v)). \\
& & & S(v) \arrow[ur, hook] & &
\end{tikzcd}$$

By Definition 2.6(iii) we have that $P(w_1) = \begin{smallmatrix} w_1 \\ v \end{smallmatrix}$ and $P(w_2) = \begin{smallmatrix} w_2 \\ v \end{smallmatrix}$. Then $\operatorname{coker}(S(v) \hookrightarrow P(w_1) \oplus P(w_2)) \cong I(v)$ and the result follows. $\square$

We can now compute $\tau_n^{-j}(P(v))$ in the second case of Lemma 2.19, that is when $\delta(v) = (2,2)$. Notice that in this case we have $n = 2$ by Definition 2.6(i).

Corollary 2.21. Let $v \in Q_0$ be a vertex with $\delta(v) = (2,2)$. Then $\tau_2^{-}(P(v)) \cong I(v)$.

Proof. Let $v \longrightarrow u_1$ and $v \longrightarrow u_2$ be the arrows starting at v. By Definition 2.6(iii) we have that $\delta^{-}(u_1) = \delta^{-}(u_2) = 1$. It follows that $\Omega^{-}(P(v)) \cong S(v)$. By Lemma 2.20(c) we have that $\tau^{-}(S(v)) \cong I(v)$. Hence

$$\tau_2^{-}(P(v)) = \tau^{-}\Omega^{-}(P(v)) \cong \tau^{-}(S(v)) \cong I(v),$$

as required. $\square$

Before continuing with the computation of $\tau_n^{-j}(P(v))$ in the last case, that is when $P(v) = P(\mathbf{v})$ for a flow path $\mathbf{v}$ in Q, let us introduce one more piece of notation.

Definition 2.22. Let $\mathbf{v} = v_1 \longrightarrow v_2 \longrightarrow \cdots \longrightarrow v_k$ be a k-flow path. We define

$$q_1 = q_1(\mathbf{v}) \coloneqq \begin{cases} 1, & \text{if } \delta(v_1) = (2,1), \\ 0, & \text{if } \delta(v_1) \neq (2,1), \end{cases} \text{ and } q_k = q_k(\mathbf{v}) \coloneqq \begin{cases} 1, & \text{if } \delta(v_k) = (1,2), \\ 0, & \text{if } \delta(v_k) \neq (1,2). \end{cases}$$

We also define

$$q(\mathbf{v}) \coloneqq -1 + q_1 + q_k = \begin{cases} 1, & \text{if } \delta(v_1) = (2,1) \text{ and } \delta(v_k) = (1,2), \\ 0, & \text{if either } \delta(v_1) = (2,1) \text{ or } \delta(v_k) = (1,2), \\ -1, & \text{if } \delta(v_1) \neq (2,1) \text{ and } \delta(v_k) \neq (1,2). \end{cases}$$

With this definition we can write some of the next results in a more compact way. First we have the following statement.

Lemma 2.23. Let $\mathbf{v}$ be a k-flow path in Q. Let $s \in \mathbb{Z}$ and assume that $2 \leq s \leq k - 1 + q_k$. Then $\delta^{-}(v_s) = 1$.

Proof. We have $s \leq k - 1 + q_k \leq k$. We consider the cases $s \leq k-1$ and $s = k$ separately. If $s \leq k-1$, then $\delta(v_s) = (1,1)$ by the definition of flow paths and so the result holds. If $s = k$, then $k - 1 + q_k = k$ and so $q_k = 1$. Then by the definition of q_k we have $\delta(v_s) = \delta(v_k) = (1,2)$, and so the result holds again. $\square$

With this we are ready to make the following computations.

Lemma 2.24. Let $\mathbf{v}$ be a k-flow path in Q. Let $s, x \in \mathbb{Z}_{\geq 0}$ and assume that $1 \leq s \leq k - 1 + q_k$.

(a) If $s - x \geq 1$, then $\Omega^{-x}(S(v_s)) \cong S(v_{s-x})$.

(b) If $1 \leq x \leq s-1+q_1$, then $\tau_x^-(S(v_s)) \cong \begin{cases} S(v_{s-x}), & \text{if } 1 \leq x < s-1+q_1, \\ I(\mathbf{v}), & \text{if } x = s-1+q_1. \end{cases}$

Proof. (a) We use induction on x. If $x=0$, then the result holds trivially. Assume now that the result holds for $x-1 \geq 0$ and we show that it holds for x. Since $s-x \geq 1$, we have that $s-(x-1) \geq 1$. Hence by induction hypothesis we have that $\Omega^{-(x-1)}(S(v_s)) \cong S(v_{s-(x-1)})$. Then

$$2 = 1+1 \leq (s-x)+1 = s-(x-1) \leq s \leq k-1+q_k,$$

and so $\delta^-(v_{s-(x-1)}) = 1$ by Lemma 2.23. Then by the definition of flow paths and Lemma 2.20(b) applied on $v_{s-(x-1)}$ it follows that

$$\Omega^{-x}(S(v_s)) \cong \Omega^-(S(v_{s-(x-1)})) \cong S(v_{s-x}),$$

as required.

(b) Since $x \leq s-1+q_1$, we have that $s-(x-1) \geq 2-q_1 \geq 1$. Therefore, by (a) we have that

$$\tau_x^-(S(v_s)) = \tau^-\Omega^{-(x-1)}(S(v_s)) \cong \tau^-(S(v_{s-(x-1)})).$$

Hence it is enough to show that

$$\tau^-(S(v_{s-(x-1)})) \cong \begin{cases} S(v_{s-x}), & \text{if } 1 \leq x < s-1+q_1, \\ I(\mathbf{v}), & \text{if } x = s-1+q_1. \end{cases}$$

We consider the cases $1 \leq x < s-1+q_1$ and $x = s-1+q_1$ separately.

Case $1 \leq x < s-1+q_1$. In this case we want to show that $\tau^-(S(v_{s-(x-1)})) \cong S(v_{s-x})$. We have

$$2-q_1 < s-(x-1) \leq s \leq k-1+q_k \leq k. \tag{2.4}$$

Hence $2 \leq s-(x-1) \leq k-1+q_k$ and so by Lemma 2.23 we have that $\delta^-(v_{s-(x-1)}) = 1$. It follows from Lemma 2.20(b) that it is enough to show that $\delta^+(v_{s-x}) = 1$. We consider the subcases $q_1 = 0$ and $q_1 = 1$ separately.

- Subcase $q_1 = 0$. Then by (2.4) we conclude that $2 \leq s-x \leq k-1$ and so $\delta^+(v_{s-x}) = 1$.
- Subcase $q_1 = 1$. Then by (2.4) we conclude that $1 \leq s-x \leq k-1$. Since in this case we have $\delta(v_1) = (2,1)$, it follows that $\delta^+(v_{s-x}) = 1$.

Case $x = s-1+q_1$. In this case we have $s-(x-1) = 2-q_1$ and we want to show that $\tau^-(S(v_{2-q_1})) \cong I(\mathbf{v})$. We consider the cases $q_1 = 0$ and $q_1 = 1$ separately.

- Subcase $q_1 = 0$. Then the result follows immediately by Lemma 2.20(b) and by considering the possibilities $\delta(v_1) = (0,1)$, $\delta(v_1) = (1,2)$ and $\delta(v_1) = (2,2)$ separately.
- Subcase $q_1 = 1$. Then $\delta(v_1) = (2,1)$ and by Lemma 2.20(c) we have $\tau^-(S(v_1)) \cong I(v_1) = I(\mathbf{v})$. ☐

Lemma 2.25. Let $\mathbf{v}$ be a k-flow path in Q. Let $x \in \mathbb{Z}_{\geq 1}$.

(a) If $k-x+q_k \geq 1$, then $\Omega^{-x}(P(\mathbf{v})) \cong S(v_{k-x+q_k})$.

(b) If $1 \leq x \leq k+q(\mathbf{v})$, then $\tau_x^-(P(\mathbf{v})) \cong \begin{cases} S(v_{k-x+q_k}), & \text{if } 1 \leq x < k+q(\mathbf{v}), \\ I(\mathbf{v}), & \text{if } x = k+q(\mathbf{v}). \end{cases}$

Proof. (a) If $x=1$, then the result follows immediately by considering the cases $\delta(v_k) = (1,0)$, $\delta(v_k) = (1,2)$, $\delta(v_k) = (2,1)$ and $\delta(v_k) = (2,2)$ separately (recall that if $\delta(v_k) = (1,2)$, then $\delta^-(v^{+2}) = \delta^-(v^{+3}) = 1$ by Definition 2.6(iii)). For $x \geq 2$ notice that $1 \leq k-x+q_k$ implies that $k-1+q_k-(x-1) \geq 1$. Hence we can apply Lemma 2.24(a) to obtain

$$\Omega^{-x}(P(\mathbf{v})) = \Omega^{-(x-1)}\Omega^-(P(\mathbf{v})) \cong \Omega^{-(x-1)}(S(v_{k-1+q_k})) \cong S(v_{k-x+q_k}),$$

as required.

(b) We first show the result for $x=1$. We consider the cases $1 = x < k+q(\mathbf{v})$ and $1 = x = k+q(\mathbf{v})$ separately.

Case $1 = x < k+q(\mathbf{v})$. In this case we want to show that $\tau^-(P(\mathbf{v})) \cong S(v_{k-1+q_k})$. We consider the subcases $\delta(v_k) = (1,0)$, $\delta(v_k) = (1,2)$ and $\delta(v_k) \in \{(2,1),(2,2)\}$ separately.

- Subcase $\delta(v_k) = (1, 0)$. In this case $q_k = 0$ and $P(\mathbf{v}) = S(v_k)$ and so we want to show that $\tau^-(S(v_k)) \cong S(v_{k-1})$. We claim that $\delta^+(v_{k-1}) = 1$. Indeed, assume towards a contradiction that $\delta^+(v_{k-1}) = 2$. Then $v_{k-1} = v_1$ and so $k = 2$ and $q_1 = 0$. Hence $1 = x < 2 + q(\mathbf{v}) = 2 - 1 = 1$, which is a contradiction. Hence by Lemma 2.20(b) and since $\delta^+(v_{k-1}) = 1$, it follows that $\tau^-(S(v_k)) \cong S(v_{k-1})$.
- Subcase $\delta(v_k) = (1, 2)$. In this case $q_k = 1$ and $P(\mathbf{v}) = P(v_k)$ and so we want to show that $\tau^-(P(v_k)) \cong S(v_k)$. By the dual of Lemma 2.20(c) we have that $\tau(S(v_k)) \cong P(v_k)$. By applying τ^- we obtain $\tau^-(P(v_k)) \cong \tau^-\tau(S(v_k)) \cong S(v_k)$.
- Subcase $\delta(v_k) \in \{(2, 1), (2, 2)\}$. In this case $q_k = 0$ and $P(\mathbf{v}) = P(v^{+1})$ and so we want to show that $\tau^-(P(v^{+1})) \cong S(v_{k-1})$. By Definition 2.6(iii) we have that $\delta^+(v_{k-1}) = 1$. By the dual of Lemma 2.20(b) we then have that $\tau(S(v_{k-1})) \cong P(v^{+1})$. By applying τ^- we obtain $\tau^-(P(v^{+1})) \cong \tau^-\tau(S(v_{k-1})) \cong S(v_{k-1})$.

Case $1 = x = k + q(\mathbf{v})$. In this case we have that $k = 2$ and $q(\mathbf{v}) = -1$ and we want to show that $\tau^-(P(\mathbf{v})) \cong I(\mathbf{v})$. Since $q(\mathbf{v}) = -1$, we have $\delta(v_1) \neq (2, 1)$ and $\delta(v_2) \neq (1, 2)$. We consider the subcases $\delta(v_2) = (1, 0)$ and $\delta(v_2) \in \{(2, 1), (2, 2)\}$ separately.

- Subcase $\delta(v_2) = (1, 0)$. In this case we have $P(\mathbf{v}) = S(v_2)$ and so we want to show that $\tau^-(S(v_2)) \cong I(\mathbf{v})$. If $\delta^+(v_1) = 1$, since $\delta(v_1) \neq (2, 1)$ and since by the definition of a flow path we have $\delta(v_1) \neq (1, 1)$, we conclude that $\delta(v_1) = (0, 1)$. Hence by Lemma 2.20(b) we have $\tau^-(S(v_2)) \cong S(v_1) = I(\mathbf{v})$, where the last equality follows from the definition of $I(\mathbf{v})$. If $\delta^+(v_1) = 2$, then by Lemma 2.20(b) and Definition 2.6(iii) we have $\tau^-(S(v_2)) \cong I(v^{-1}) = I(\mathbf{v})$, where the last equality again follows from the definition of $I(\mathbf{v})$.
- Subcase $\delta(v_2) \in \{(2, 1), (2, 2)\}$. In this case we have $P(\mathbf{v}) = P(v^{+1})$ and so we want to show that $\tau^-(P(v^{+1})) \cong I(\mathbf{v})$. Since $k = 2$ and $\delta^-(v_2) = 2$, by Definition 2.6(iii) we have that $\delta^+(v_1) = 1$. Since $\delta(v_1) \neq (2, 1)$ and since by the definition of a flow path we have $\delta(v_1) \neq (1, 1)$, we conclude that $\delta(v_1) = (0, 1)$. It follows that $I(\mathbf{v}) = S(v_1)$. By the dual of Lemma 2.20(b) we then have that $\tau(S(v_1)) \cong P(v^{+1})$. By applying τ^- we obtain $\tau^-(P(v^{+1})) \cong \tau^-\tau(S(v_1)) \cong S(v_1) = I(\mathbf{v})$.

Now let $x \geq 2$. Then $2 \leq k + q(\mathbf{v})$ gives $k - 1 + q_k \geq 1$. Hence by (a) we have that

$$\tau_x^-(P(\mathbf{v})) = \tau_{x-1}^-\Omega^-(P(\mathbf{v})) \cong \tau_{x-1}^-(S(v_{k-1+q_k})).$$

Moreover, since $1 \leq k - 1 + q_k$ and

$$(k - 1 + q_k) - 1 + q_1 = k + q(\mathbf{v}) - 1 \geq x - 1 \geq 1,$$

we can apply Lemma 2.24(b) to obtain

$$\tau_{x-1}^-(S(v_{k-1+q_k})) \cong \begin{cases} S(v_{k-1+q_k-(x-1)}), & \text{if } 1 \leq x - 1 < (k - 1 + q_k) - 1 + q_1, \\ I(\mathbf{v}), & \text{if } x - 1 = (k - 1 + q_k) - 1 + q_1. \end{cases}$$

After simplifying the above expression, we get

$$\tau_x^-(P(\mathbf{v})) \cong \tau_{x-1}^-(S(v_{k-1+q_k})) \cong \begin{cases} S(v_{k-x+q_k}), & \text{if } 2 \leq x < k + q(\mathbf{v}), \\ I(\mathbf{v}), & \text{if } x = k + q(\mathbf{v}), \end{cases}$$

which proves the case $x \geq 2$. □

With the above computation we can show the following important results about flow paths in Q.

Proposition 2.26. Let $\mathbf{v}$ be a k-flow path in Q and assume that $\mathcal{C} \subseteq \operatorname{mod}\Lambda$ is an n-cluster tilting subcategory. Then $n \mid (k + q(\mathbf{v}))$.

Proof. We write $k + q(\mathbf{v}) = pn + r$ where $p \in \mathbb{Z}_{\geq 0}$ and $0 \leq r \leq n - 1$. We first claim that $p \geq 1$. Indeed, assume towards a contradiction that $p = 0$. Then $1 \leq k + q(\mathbf{v}) = r$. Hence by Lemma 2.25(b) we have that $\tau_r^-(P(\mathbf{v})) \cong I(\mathbf{v})$. By Lemma 1.2 we obtain $\operatorname{Ext}_\Lambda^r(I(\mathbf{v}), P(\mathbf{v})) \neq 0$. But this contradicts the fact that $\mathcal{C}$ is an n-cluster tilting subcategory, since $I(\mathbf{v}), P(\mathbf{v}) \in \mathcal{C}$ and $1 \leq r \leq n - 1$.

Hence $p \geq 1$ and it remains to show that $r = 0$. Assume towards a contradiction that $r \geq 1$. Then

$$1 \leq n \leq pn = k + q(\mathbf{v}) - r < k + q(\mathbf{v}).$$

Hence we can apply Lemma 2.25(b) to obtain that $\tau_n^-(P(\mathbf{v})) \cong S(v_{k-n+q_k})$. Then we can apply Lemma 2.24(b) repeatedly $p-1$ more times to obtain

$$\tau_n^{-p}(P(\mathbf{v})) \cong \tau_n^{-(p-1)}(S(v_{k-n+q_k})) \cong \tau_n^{-(p-2)}(S(v_{k-2n+q_k})) \cong \cdots \cong S(v_{k-pn+q_k}) = S(v_{r+1-q_1}).$$

By Proposition 1.1(a) and since $P(\mathbf{v}) \in \mathcal{C}$, it follows that $S(v_{r+1-q_1}) \in \mathcal{C}$. By Lemma 2.24(b) we have $\tau_r^-(S(v_{r+1-q_1})) \cong I(\mathbf{v})$. By Lemma 1.2 we obtain $\mathrm{Ext}_\Lambda^r(I(\mathbf{v}), S(v_{r+1-q_1})) \neq 0$. But this contradicts the fact that $\mathcal{C}$ is an n-cluster tilting subcategory, since $I(\mathbf{v}), S(v_{r+1-q_1}) \in \mathcal{C}$ and $1 \leq r \leq n-1$. ☐

Corollary 2.27. Let $\mathbf{v}$ be a k-flow path in Q. Assume that $k + q(\mathbf{v}) = pn$ for some $p \geq 1$ and let $j \in \mathbb{Z}$ with $0 \leq j \leq p$. Then

$$\tau_n^{-j}(P(\mathbf{v})) \cong \begin{cases} P(\mathbf{v}), & \text{if } j = 0, \\ S(v_{k-jn+q_k}), & \text{if } 1 \leq j \leq p-1, \\ I(\mathbf{v}), & \text{if } j = p. \end{cases} \tag{2.5}$$

Moreover, if $1 \leq j \leq p-1$, then $\delta(v_{k-jn+q_k}) = (1,1)$. In particular, the module $\tau_n^{-j}(P(\mathbf{v}))$ is indecomposable and not projective-injective.

Proof. We first prove (2.5). For $j = 0$ the result is clear. For $1 \leq j \leq p-1$ we use induction on j, where the base case $j = 1$ follows from Lemma 2.25(b), while the induction step follows from Lemma 2.24(b).

Next, if $1 \leq j \leq p-1$, then

$$2 = 2 + 1 - 1 \leq n + 1 - q_1 = k - (p-1)n + q_k \leq k - jn + q_k \leq k - n + q_k \leq k - 2 + 1 = k - 1,$$

from which it follows that $\delta(v_{k-jn+q_k}) = (1,1)$ and so $S(v_{k-jn+q_k})$ is neither projective nor injective.

Finally, if $j = 0$, then $\tau_n^{-j}(P(\mathbf{v})) \cong P(\mathbf{v})$ is not injective by Lemma 2.18, while if $j = p$, then $\tau_n^{-j}(P(\mathbf{v})) \cong I(\mathbf{v})$ is not projective by the dual of Lemma 2.18. ☐

3. Sufficient conditions

Motivated by Proposition 2.14 and Proposition 2.26 we give the following definition.

Definition 3.1. Let Q be an n-pre-admissible quiver. We say that Q is *n-admissible* if one of the following conditions hold:

(a) $Q = \tilde{A}_m$ and $n \mid m$, or
(b) $Q \neq \tilde{A}_m$ and for every k-flow path $\mathbf{v}$ in Q we have that $n \mid (k + q(\mathbf{v}))$.

Example 3.2. (a) The quiver A_m is n-admissible if and only if $n \mid (m-1)$. In particular, the quiver A_1 is n-admissible for all $n \geq 2$.
(b) The quiver of Example 2.8(c) is 2-admissible.
(c) The quiver of Example 2.8(d) is 3-admissible but not n-admissible for any $n \neq 3$.

Remark 3.3. (a) When studying n-admissible quivers, the cases $Q = A_1$ and $Q = \tilde{A}_m$ for $m \geq 1$ usually behave differently from the rest of the cases; the reason for this is that the quivers A_1 and $\tilde{A}_m$ are the only n-pre-admissible quivers that do not have flow paths as Lemma 2.13 shows. Hence many times in the rest of this paper we will exclude one or both of the cases $Q = A_1$ and $Q = \tilde{A}_m$ from our statements. We remind the reader that this does not present a problem in our aim of classification of n-cluster tilting subcategories for radical square zero bound quiver algebras since such a classification in these exceptional cases is given in Proposition 2.14.
(b) If Q is an n-admissible quiver and n' is an integer such that $n' \geq 2$ and $n' \mid n$, then it follows directly from Remark 2.7 and Definition 3.1 that Q is also an n'-admissible quiver.

By Proposition 2.14 and Proposition 2.26 it follows that if Q is a quiver and there exists an n-cluster tilting subcategory $\mathcal{C} \subseteq \mathrm{mod}(\mathbf{k}Q/\mathcal{J}^2)$, then Q is n-admissible. The aim of this section is to show that the opposite is also true. We also want to show that if $Q \neq \tilde{A}_m$, then $\mathcal{C}$ is unique and give a description of $\mathcal{C}$.

For the rest of this section we fix an n-admissible quiver Q with $Q \neq A_1$ and $Q \neq \tilde{A}_m$ and we set $\Lambda \coloneqq \mathbf{k}Q/\mathcal{J}^2$. We denote by $\mathbf{V}$ the set of all flow paths in Q. Note that by Lemma 2.13 we have that $\mathbf{V} \neq \varnothing$. For a k-flow path $\mathbf{v} \in \mathbf{V}$ we set $p(\mathbf{v}) = \frac{k+q(\mathbf{v})}{n}$; since Q is n-admissible, it follows that $p(\mathbf{v})$ is an integer. We define

$$M(\mathbf{v}) \coloneqq \bigoplus_{j=0}^{p(\mathbf{v})} \tau_n^{-j}(P(\mathbf{v})) \cong P(\mathbf{v}) \oplus \left(\bigoplus_{j=1}^{p(\mathbf{v})-1} S(v_{k-jn+q_k}) \right) \oplus I(\mathbf{v}),$$

where the last isomorphism follows from Corollary 2.27. We also set $M(\mathbf{V}) \coloneqq \bigoplus_{\mathbf{v}\in\mathbf{V}} M(\mathbf{v})$. With this notation we have the following lemmas.

Lemma 3.4. (a) The module $M(\mathbf{v})$ is basic and has no projective-injective direct summand.
(b) The module $M(\mathbf{V})$ is basic and has no projective-injective direct summand.

Proof. (a) Follows immediately by Corollary 2.27 and Lemma 2.15(a).
(b) By (a) we have that $M(\mathbf{V})$ has no projective-injective direct summand. It remains to show that $M(\mathbf{V})$ is basic. Since the module $M(\mathbf{v})$ for $\mathbf{v} \in \mathbf{V}$ is basic by (a), it is enough to show that if $\mathbf{v}$ and $\mathbf{u}$ are two flow paths in Q with $\mathbf{v} \neq \mathbf{u}$, then $M(\mathbf{v})$ and $M(\mathbf{u})$ have no isomorphic direct summands. Assume towards a contradiction that there exist indecomposable direct summands V of $M(\mathbf{v})$ and U of $M(\mathbf{u})$ such that $V \cong U$ but $\mathbf{v} \neq \mathbf{u}$. Then $V \cong \tau_n^{-j_v}(P(\mathbf{v}))$ and $U \cong \tau_n^{-j_u}(P(\mathbf{u}))$ for some $j_v, j_u \in \mathbb{Z}_{\geq 0}$ with $j_v \leq p(\mathbf{v})$ and $j_u \leq p(\mathbf{u})$. Without loss of generality we assume that $j_u \geq j_v$. It follows that

$$\tau_n^{-(p(\mathbf{v})-j_v+j_u)}(P(\mathbf{u})) = \tau_n^{-(p(\mathbf{v})-j_v)}\tau_n^{-j_u}(P(\mathbf{u})) \cong \tau_n^{-(p(\mathbf{v})-j_v)}\tau_n^{-j_v}(P(\mathbf{v})) = \tau_n^{-p(\mathbf{v})}(P(\mathbf{v})) \cong I(\mathbf{v}),$$

where the last isomorphism follows from Corollary 2.27. In particular, we have that the module $\tau_n^{-(p(\mathbf{v})-j_v+j_u)}(P(\mathbf{u}))$ is injective and nonzero. By Corollary 2.27 we have that $\tau_n^{-j'}(P(\mathbf{u})) = 0$ for $j' > p(\mathbf{u})$ and $\tau_n^{-j'}(P(\mathbf{u}))$ is not injective for $j' < p(\mathbf{u})$. We conclude that $p(\mathbf{v}) - j_v + j_u = p(\mathbf{u})$ and so $I(\mathbf{u}) \cong \tau_n^{-p(\mathbf{u})}(P(\mathbf{u})) = \tau_n^{-(p(\mathbf{v})-j_v+j_u)}(P(\mathbf{u})) \cong I(\mathbf{v})$. Then by the dual of Lemma 2.18 it follows that $\mathbf{v} = \mathbf{u}$, which contradicts our assumption $\mathbf{v} \neq \mathbf{u}$. ☐

Lemma 3.5. Let $i \in \{1, \dots, n-1\}$. Then $\mathrm{Ext}^i_\Lambda(M(\mathbf{V}), M(\mathbf{V})) = 0$.

Proof. Let $\mathbf{v}$ be a k-flow path in Q and let $\mathbf{u}$ be a k'-flow path in Q. By the definition of $M(\mathbf{V})$ and additivity of $\mathrm{Ext}^i_\Lambda(-,-)$ it is enough to show that $\mathrm{Ext}^i_\Lambda(M(\mathbf{u}), M(\mathbf{v})) = 0$. By the definition of $M(\mathbf{u})$ and $M(\mathbf{v})$ and additivity of $\mathrm{Ext}^i_\Lambda(-,-)$ it is enough to show that

$$\mathrm{Ext}^i_\Lambda(\tau_n^{-x}(P(\mathbf{u})), \tau_n^{-y}(P(\mathbf{v}))) = 0 \tag{3.1}$$

for any $x \in \{0, 1, \dots, p(\mathbf{u})\}$ and $y \in \{0, 1, \dots, p(\mathbf{v})\}$. If $x = 0$, then $\tau_n^{-x}(P(\mathbf{u})) = P(\mathbf{u})$ is projective and so (3.1) holds. If $y = p(\mathbf{v})$, then by Corollary 2.27 we have that $\tau_n^{-p(\mathbf{v})}(P(\mathbf{v})) \cong I(\mathbf{v})$ is injective and so (3.1) holds again. Hence we may assume that $x > 0$ and $y < p(\mathbf{v})$.

Using dimension shift and the Auslander–Reiten duality we compute

$$\begin{aligned}
\mathrm{Ext}^i_\Lambda(\tau_n^{-x}(P(\mathbf{u})), \tau_n^{-y}(P(\mathbf{v}))) &\cong \mathrm{Ext}^1_\Lambda(\tau_n^{-x}(P(\mathbf{u})), \Omega^{-(i-1)}\tau_n^{-y}(P(\mathbf{v}))) \\
&\cong D\underline{\mathrm{Hom}}_\Lambda(\tau^{-}\Omega^{-(i-1)}\tau_n^{-y}(P(\mathbf{v})), \tau_n^{-x}(P(\mathbf{u}))) \\
&\cong D\underline{\mathrm{Hom}}_\Lambda(\tau_i^{-}\tau_n^{-y}(P(\mathbf{v})), \tau_n^{-x}(P(\mathbf{u}))) \\
&\cong D\underline{\mathrm{Hom}}_\Lambda(S(v_{k-yn-i+q_k(\mathbf{v})}), \tau_n^{-x}(P(\mathbf{u}))),
\end{aligned}$$

where the last isomorphism follows from Lemma 2.25(b) if $y = 0$ and by Corollary 2.27 and Lemma 2.24(b) if $y > 0$. Hence it is enough to show that

$$D\underline{\mathrm{Hom}}_\Lambda(S(v_{k-yn-i+q_k(\mathbf{v})}), \tau_n^{-x}(P(\mathbf{u}))) = 0. \tag{3.2}$$

Assume towards a contradiction that (3.2) does not hold. We consider the cases $0 < x < p(\mathbf{u})$ and $x = p(\mathbf{u})$ separately and reach a contradiction in each case.

Case $0 < x < p(\mathbf{u})$. In this case by Corollary 2.27 we have that $\tau_n^{-x}(P(\mathbf{u})) \cong S(u_{k'-xn+q_{k'}(\mathbf{u})})$. Then it follows that $\mathrm{Hom}_\Lambda(S(v_{k-yn-i+q_k(\mathbf{v})}), S(u_{k'-xn+q_{k'}(\mathbf{u})})) \neq 0$. Since both modules are simple, we conclude that $v_{k-yn-i+q_k(\mathbf{v})} = u_{k'-xn+q_{k'}(\mathbf{u})}$. By Corollary 2.27 and since $0 < x < p(\mathbf{u})$, it follows that $\delta(u_{k'-xn+q_{k'}(\mathbf{u})}) = (1,1)$. Thus by Lemma 2.16(a) we obtain $\mathbf{v} = \mathbf{u}$. In particular, we have that $k = k'$ and $q_k(\mathbf{v}) = q_{k'}(\mathbf{u})$ and so $v_{k-yn-i+q_k(\mathbf{v})} = v_{k-xn+q_k(\mathbf{v})}$. Hence by Lemma 2.15(a) it follows that $k - yn - i + q_k(\mathbf{v}) = k - xn + q_k(\mathbf{v})$. Equivalently we get $(x-y)n = i$, which contradicts $1 \leq i \leq n-1$.

Case $x = p(\mathbf{u})$. In this case by Corollary 2.27 we have that $\tau_n^{-x}(P(\mathbf{u})) \cong I(\mathbf{u})$. Since we assume that (3.2) does not hold, and since $I(\mathbf{u})$ is indecomposable and injective, it follows that $S(v_{k-yn-i+q_k(\mathbf{v})}) \cong \mathrm{soc}(I(\mathbf{u}))$. We consider the subcases $\delta^+(u_1) = 1$ and $\delta^+(u_1) = 2$ separately.

- Subcase $\delta^+(u_1) = 1$. In this case we have $I(\mathbf{u}) = I(u_1)$ by definition. Hence $v_{k-yn-i+q_k(\mathbf{v})} = u_1$ and so $\delta(v_{k-yn-i+q_k(\mathbf{v})}) \neq (1,1)$. By the definition of a k-flow path we obtain that $k - yn - i + q_k(\mathbf{v}) \in \{1, k\}$. We claim that $k - yn - i + q_k(\mathbf{v}) = 1$. Indeed, assume towards a contradiction that $k - yn - i + q_k(\mathbf{v}) = k$. Since $0 \leq y \leq p(\mathbf{v}) - 1$, $1 \leq i \leq n-1$ and $0 \leq q_k(\mathbf{v}) \leq 1$, it follows that $y = 0$, $i = 1$ and $q_k(\mathbf{v}) = 1$. But then $(1,2) = \delta(v_k) = \delta(v_{k-yn-i+q_k(\mathbf{v})}) = \delta(u_1)$ contradicts the fact that $\delta^+(u_1) = 1$.

  Hence we have $k - yn - i + q_k(\mathbf{v}) = 1$. Using this equality together with $k + q(\mathbf{v}) = p(\mathbf{v})n$, we obtain that $(p(\mathbf{v}) - y)n = i + q_1(\mathbf{v})$. Since $y < p(\mathbf{v})$ and $1 \leq i \leq n-1$, it follows that $q_1(\mathbf{v}) = 1$. Hence we have $v_1 = v_{k-yn-i+q_k(\mathbf{v})} = u_1$ and $\delta(v_1) = (2,1)$. Then any morphism from $S(v_{k-yn-i+q_k(\mathbf{v})}) = S(v_1) = {\scriptstyle v_1}$ to $\tau_n^{-x}(P(\mathbf{u})) \cong I(u_1) = I(v_1) = \begin{smallmatrix} v^{-2} & & v^{-3} \\ & v_1 & \end{smallmatrix}$ clearly factors through $P(v^{-2}) = \begin{smallmatrix} v^{-2} \\ v_1 \end{smallmatrix}$. But this shows that (3.2) holds, which is a contradiction.
- Subcase $\delta^+(u_1) = 2$. In this case we have $I(\mathbf{u}) = I(u^{-1})$ by definition. Hence $v_{k-yn-i+q_k(\mathbf{v})} = u^{-1}$. Then any morphism from $S(v_{k-yn-i+q_k(\mathbf{v})}) = S(u^{-1}) = {\scriptstyle u^{-1}}$ to $\tau_n^{-x}(P(\mathbf{u})) \cong I(u^{-1}) = \begin{smallmatrix} u_1 \\ u^{-1} \end{smallmatrix}$ clearly factors through $P(u_1) = \begin{smallmatrix} & u_1 & \\ u^{-1} & & u_2 \end{smallmatrix}$. But this shows that (3.2) holds, which is a contradiction. $\square$

Lemma 3.6. Let $v, u \in Q_0$ be such that $\delta(v) = \delta(u) = (2,2)$.

(a) We have $\mathrm{Ext}^1_\Lambda(M(\mathbf{V}), P(v)) = 0$ and $\mathrm{Ext}^1_\Lambda(I(v), M(\mathbf{V})) = 0$.

(b) We have $\mathrm{Ext}^1_\Lambda(I(u), P(v)) = 0$.

Proof. (a) We only show that $\mathrm{Ext}^1_\Lambda(M(\mathbf{V}), P(v)) = 0$; the other equality follows dually. Let $\mathbf{w}$ be a k-flow path in Q. By additivity of $\mathrm{Ext}^1_\Lambda(-,-)$ it is enough to show that

$$\mathrm{Ext}^1_\Lambda(\tau_n^{-x}(P(\mathbf{w})), P(v)) = 0$$

for any $x \in \{0, 1, \ldots, p(\mathbf{w})\}$. If $x = 0$, then $\tau_n^{-x}(P(\mathbf{w})) = P(\mathbf{w})$ is projective and so the result follows. Otherwise, assume that $1 \leq x \leq p(\mathbf{w})$. By the dual of Lemma 2.20(c) we have that $\tau^-(P(v)) \cong S(v)$. Then by the Auslander–Reiten duality, it is enough to show that

$$D\underline{\mathrm{Hom}}_\Lambda(S(v), \tau_n^{-x}(P(\mathbf{w}))) = 0. \tag{3.3}$$

We consider the cases $1 \leq x \leq p(\mathbf{w}) - 1$ and $x = p(\mathbf{w})$ separately.

Case $1 \leq x \leq p(\mathbf{w}) - 1$. In this case by Corollary 2.27 we have that $\tau_n^{-x}(P(\mathbf{w})) \cong S(w_{k-xn+q_k})$. Assume towards a contradiction that (3.3) does not hold. Then $S(v) \cong S(w_{k-xn+q_k})$ from which it follows that $v = w_{k-xn+q_k}$. By Corollary 2.27 we have that $\delta(w_{k-xn+q_k}) = (1,1)$, which contradicts $\delta(v) = (2,2)$.

Case $x = p(\mathbf{w})$. In this case by Corollary 2.27 we have that $\tau_n^{-x}(P(\mathbf{w})) \cong I(\mathbf{w})$. Assume towards a contradiction that (3.3) does not hold. Then $S(v) \cong \mathrm{soc}(I(\mathbf{w}))$ from which it follows that $I(v) \cong I(\mathbf{w})$. But this contradicts the dual of Lemma 2.19 since $\delta(v) = (2,2)$.

(b) By the dual of Lemma 2.20(c) we have that $\tau^-(P(v)) \cong S(v)$. Then by the Auslander–Reiten duality it is enough to show that

$$D\underline{\mathrm{Hom}}_\Lambda(S(v), I(u)) = 0.$$

If $v \neq u$, then $\mathrm{Hom}_\Lambda(S(v), I(u)) = 0$ and the result follows. Otherwise, assume that $v = u$. Let $w_1 \longrightarrow v$ and $w_2 \longrightarrow v$ be the arrows ending at v. Then any morphism from $S(v) = {\scriptstyle v}$ to $I(u) = I(v) = {\scriptstyle w_1}{}_{\scriptstyle v}{\scriptstyle w_2}$ clearly factors through $P(w_1) = {\scriptstyle \begin{smallmatrix} w_1 \\ v \end{smallmatrix}}$, which shows that $D\underline{\mathrm{Hom}}_\Lambda(S(v), I(u)) = 0$. □

Next, let $\{R_t\}_{t=1}^{f}$ be a complete collection of representatives of pairwise non-isomorphic projective-injective Λ-modules. Set

$$M \coloneqq M(\mathbf{V}) \oplus \left(\bigoplus_{t=1}^{f} R_t \right) \oplus \left(\bigoplus_{\substack{v \in Q_0 \\ \delta(v) = (2,2)}} (P(v) \oplus I(v)) \right). \tag{3.4}$$

The main aim of this section is to show that M is the unique n-cluster tilting module of Λ. We start by giving an alternate description of M.

Corollary 3.7. The module M is basic and $M \cong \bigoplus_{j \geq 0} \tau_n^{-j}(\Lambda)$. In particular, we have that $D(\Lambda) \in M$.

Proof. We set

$$R \coloneqq \bigoplus_{t=1}^{f} R_t, \text{ and } M_{(2,2)} \coloneqq \bigoplus_{\substack{v \in Q_0 \\ \delta(v) = (2,2)}} (P(v) \oplus I(v)).$$

By Lemma 2.19 we have that

$$\Lambda \cong \left(\bigoplus_{\mathbf{v} \in \mathbf{V}} P(\mathbf{v}) \right) \oplus R \oplus \left(\bigoplus_{\substack{v \in Q_0 \\ \delta(v) = (2,2)}} P(v) \right).$$

Then by (3.4) and Corollary 2.21 it follows that $M \cong \bigoplus_{j \geq 0} \tau_n^{-j}(\Lambda)$.

To see that M is basic, we have that $M(\mathbf{V})$ is basic by Lemma 3.4(b), that R is basic by definition and that $M_{(2,2)}$ is basic since $P(v)$ is never injective if $\delta^+(v) = 2$. By Corollary 2.27 and by Lemma 2.19 and its dual and by comparing direct summands of $M(\mathbf{V})$, R and $M_{(2,2)}$, it easily follows that M is basic.

Finally, we show that $D(\Lambda) \in \mathrm{add}(M)$. It is enough to show that for every vertex $v \in Q_0$, the indecomposable injective Λ-module $I(v)$ corresponding to the vertex $v \in Q_0$ belongs to $\mathrm{add}(M)$. If $\delta(v) = (2,2)$ or $I(v)$ is projective, then clearly $I(v) \in \mathrm{add}(M)$ by the definition of M. Otherwise, by the dual of Lemma 2.19 it follows that $I(v) \cong I(\mathbf{v})$ for some flow path $\mathbf{v}$ in Q. Then by Corollary 2.21 and Proposition 1.1(a) we have

$$I(v) \cong \tau_n^{-p(\mathbf{v})}(P(\mathbf{v})) \in \mathrm{add}(M),$$

as required. □

Next we want to show that M is n-rigid.

Proposition 3.8. Let $i \in \{1, \ldots, n-1\}$. Then $\mathrm{Ext}^i_\Lambda(M, M) = 0$.

Proof. By Lemma 3.5 and since R_t is projective-injective for every $t \in \{1, \ldots, f\}$, it follows that the module $M(\mathbf{V}) \oplus \left(\bigoplus_{t=1}^{f} R_t \right)$ is n-rigid. Hence if there exists no vertex $v \in Q_0$ with degree $\delta(v) = (2,2)$, the result follows immediately, while if there exists a vertex $v \in Q_0$ with degree $\delta(v) = (2,2)$, the result follows from Lemma 3.6. □

We are now ready to show that M is n-cluster tilting.

Proposition 3.9. The module M is an n-cluster tilting Λ-module and any basic n-cluster tilting Λ-module is isomorphic to M.

Proof. To show that M is an n-cluster tilting module we need to show that

$$\begin{aligned}\operatorname{add}(M) &= \{X \in \operatorname{mod}\Lambda \mid \operatorname{Ext}^i_\Lambda(M,X)=0 \text{ for all } 0<i<n\} \\ &= \{X \in \operatorname{mod}\Lambda \mid \operatorname{Ext}^i_\Lambda(X,M)=0 \text{ for all } 0<i<n\}.\end{aligned}$$

We only show the first equality; the other follows dually. Since by Proposition 3.8 the module M is n-rigid, the inclusion

$$\operatorname{add}(M) \subseteq \{X \in \operatorname{mod}\Lambda \mid \operatorname{Ext}^i_\Lambda(M,X)=0 \text{ for all } 0<i<n\}$$

holds. It remains to show the opposite inclusion, that is that if $\operatorname{Ext}^i_\Lambda(M,X)=0$ for all $0<i<n$, then $X \in \operatorname{add}(M)$. We show the contrapositive statement that if $X \notin \operatorname{add}(M)$, then $\operatorname{Ext}^i_\Lambda(M,X) \neq 0$ for some $i \in \{1,\dots,n-1\}$. By additivity of $\operatorname{Ext}^i_\Lambda(-,-)$ we may assume that X is indecomposable. Since by Corollary 3.7 we have that $\Lambda \in \operatorname{add}(M)$ and $D(\Lambda) \in \operatorname{add}(M)$, it follows that X is neither projective nor injective. Since X is neither projective nor injective, it follows from Theorem 2.10(d) that X is simple. Then $X \cong S(v)$ for some vertex $v \in Q_0$. Clearly $\delta(v) \neq (0,1)$ and $\delta(v) \neq (1,0)$ because in the first case we have that $S(v)$ is injective while in the second case we have that $S(v)$ is projective. We consider the cases $\delta^-(v)=2$ and $\delta^-(v)=1$ separately.

Case $\delta^-(v)=2$. In this case we have by Lemma 2.20(c) that $\tau^-(S(v)) \cong I(v)$. By Lemma 1.2 it follows that $\operatorname{Ext}^1_\Lambda(I(v),S(v)) \neq 0$. Since by Corollary 3.7 we have that $I(v) \in \operatorname{add}(M)$, we conclude that $\operatorname{Ext}^1_\Lambda(M,S(v)) \neq 0$, as required.

Case $\delta^-(v)=1$. In this case we have that $\delta(v)=(1,1)$ or $\delta(v)=(1,2)$. By Corollary 2.17 there exists a unique k-flow path $\mathbf{v}$ in Q such that $v=v_j$ for some $j>1$. Notice that $j<k+q_k$ also holds. We first claim that n does not divide $k-j+q_k$.

To show this, assume towards a contradiction that $k-j+q_k=mn$ for some $m \in \mathbb{Z}_{\geq 0}$. Then $j=k-mn+q_k$. Since we have $1<j<k+q_k$, we obtain

$$1 < k-mn+q_k < k+q_k.$$

Using $k+q(\mathbf{v})=p(\mathbf{v})n$ and $q(\mathbf{v})=-1+q_1+q_k$, we obtain that

$$0 < m < p(\mathbf{v}) - \frac{q_1}{n},$$

which implies $0<m<p(\mathbf{v})$. But then by Corollary 2.27 we have

$$X \cong S(v) = S(v_j) = S(v_{k-mn+q_k}) \cong \tau_n^{-m}(P(\mathbf{v})) \in \operatorname{add}(M),$$

which contradicts $X \notin \operatorname{add}(M)$.

Hence n does not divide $k-j+q_k$. Let m be the unique integer such that $m < \frac{k-j+q_k}{n} < m+1$. Using $1<j<k+q_k$, we obtain that $0 \leq m \leq p(\mathbf{v})-1$. Then by Lemma 2.25, Corollary 2.27 and Lemma 2.24 it follows that

$$\Omega^{-(k-mn+q_k-j)}\tau_n^{-m}(P(\mathbf{v})) \cong S(v_j). \tag{3.5}$$

Set $i \coloneqq (m+1)n-k-q_k+j$. Since $m < \frac{k-j+q_k}{n} < m+1$, we obtain that $0<i<n$. Then, using (3.5), we compute

$$\begin{aligned}\tau_i^-(S(v_j)) &\cong \tau_i^-\Omega^{-(k-mn+q_k-j)}\tau_n^{-m}(P(\mathbf{v})) \\ &= \tau^-\Omega^{-(i-1+k-mn+q_k-j)}\tau_n^{-m}(P(\mathbf{v})) \\ &= \tau^-\Omega^{-((m+1)n-k-q_k+j-1+k-mn+q_k-j)}\tau_n^{-m}(P(\mathbf{v})) \\ &= \tau^-\Omega^{-(n-1)}\tau_n^{-m}(P(\mathbf{v})) \\ &= \tau_n^{-(m+1)}(P(\mathbf{v})).\end{aligned}$$

By Corollary 2.27 and since $0 \leq m \leq p(\mathbf{v})-1$, it follows that $\tau_n^{-(m+1)}(P(\mathbf{v})) \neq 0$. Then by Lemma 1.2 we have that $\operatorname{Ext}^i_\Lambda(\tau_n^{-(m+1)}(P(\mathbf{v})),S(v_j)) \neq 0$, which shows that $\operatorname{Ext}^i_\Lambda(M,S(v)) \neq 0$ since $\tau_n^{-(m+1)}(P(\mathbf{v})) \in \operatorname{add}(M)$.

Finally, the fact that M is the unique basic n-cluster tilting module up to isomorphism follows from Proposition 1.1(c). $\square$

4. Main result and applications

We are now ready to state our main result.

Theorem 4.1. *Let Q be a quiver, let $\Lambda = \mathbf{k}Q/\mathcal{J}^2$ and let $n \in \mathbb{Z}_{\geq 2}$. Then the algebra Λ admits an n-cluster tilting subcategory $\mathcal{C} \subseteq \operatorname{mod}\Lambda$ if and only if Q is an n-admissible quiver. If moreover $Q \neq \tilde{A}_m$ for any $m \geq 1$, then $\mathcal{C}$ is unique and $\mathcal{C} = \operatorname{add}\left(\bigoplus_{j\geq 0} \tau_n^{-j}(\Lambda)\right)$.*

Proof. The statement that if Λ admits an n-cluster tilting subcategory, then Q is an n-admissible quiver follows from Proposition 2.9, Proposition 2.14 and Proposition 2.26. The statement that if Q is an n-admissible quiver, then Λ admits an n-cluster tilting subcategory follows from Proposition 2.14 and Proposition 3.9. The description of $\mathcal{C}$ in the case $Q \neq \tilde{A}_m$ follows from Proposition 3.9. $\square$

Remark 4.2. In Theorem 4.1 we classify n-cluster tilting subcategories for bound quiver algebras of the form $\mathbf{k}Q/\mathcal{J}^2$ when $n \geq 2$. We also find that all of them are of the form $\operatorname{add}(M)$ for an n-cluster tilting module M. If $n = 1$, then the algebra $\Lambda = \mathbf{k}Q/\mathcal{J}^2$ admits a unique 1-cluster tilting subcategory, namely the whole module category $\operatorname{mod}\Lambda$. Moreover, the module category $\operatorname{mod}\Lambda$ is of the form $\operatorname{add}(M)$ if and only if Λ is a representation-finite algebra. A result of Gabriel [Gab72] classifies representation-finite algebras with radical square zero in terms of their *separated quiver*; see also [ARS95, Section X.2].

Using Theorem 4.1 we can construct many examples of algebras that admit n-cluster tilting modules and have many interesting properties. As an example, an answer to a question of Erdmann and Holm from [EH08] is given in [MV21] using radical square zero bound quiver algebras.

Example 4.3.
(a) Let $Q = A_m$, let $\Lambda = \mathbf{k}Q/\mathcal{J}^2$ and let $n \geq 2$ be such that $n \mid (m-1)$. Then
$$\Lambda \oplus \left(\bigoplus_{j=1}^{\frac{m-1}{n}} S(m - jn) \right)$$
is the unique basic n-cluster tilting Λ-module.

(b) Let Q be as in Example 2.8(c) and let $\Lambda = \mathbf{k}Q/\mathcal{J}^2$. Then the module
$$\begin{aligned} M &= \Lambda \oplus \tau_2^{-}(\Lambda) \oplus \tau_2^{-2}(\Lambda) \\ &\cong \Lambda \oplus \left(\begin{smallmatrix}1\\2\end{smallmatrix} \oplus 7 \oplus \begin{smallmatrix}2&&8\\&3&\end{smallmatrix} \oplus \begin{smallmatrix}4&&5\\&5&\end{smallmatrix} \oplus \begin{smallmatrix}3\\6\end{smallmatrix} \oplus \begin{smallmatrix}1\\1\end{smallmatrix} \right) \oplus \begin{smallmatrix}3\\4\end{smallmatrix} \end{aligned}$$
is the unique basic 2-cluster tilting Λ-module.

(c) Let Q be as in Example 2.8(d) and let $\Lambda = \mathbf{k}Q/\mathcal{J}^2$. Then the module
$$\begin{aligned} M &= \Lambda \oplus \tau_3^{-}(\Lambda) \\ &\cong \Lambda \oplus \left(\begin{smallmatrix}10&&7\\&8&\end{smallmatrix} \oplus \begin{smallmatrix}5\\6\end{smallmatrix} \oplus \begin{smallmatrix}3&&12\\&4&\end{smallmatrix} \oplus \begin{smallmatrix}1\\2\end{smallmatrix} \oplus \begin{smallmatrix}&5\\11&\end{smallmatrix} \oplus \begin{smallmatrix}&1\\9&\end{smallmatrix} \right) \end{aligned}$$
is the unique basic 3-cluster tilting Λ-module.

In the rest of this section we further investigate some properties of radical square zero bound quiver algebras which admit n-cluster tilting subcategories. We start with describing a method to construct n-admissible quivers.

Remark 4.4. Starting from any n-pre-admissible quiver Q, it is not difficult to construct an n-admissible quiver by adjusting the lengths of flow paths in Q appropriately. For example, if Q is the quiver
$$\begin{tikzcd} 1 \arrow[loop above] \arrow[r] & 2, \end{tikzcd}$$
then Q is n-pre-admissible for any $n \geq 2$ and there are two flow paths in Q, namely
$$\begin{aligned} \mathbf{v} &: 1 \longrightarrow 1, \\ \mathbf{u} &: 1 \longrightarrow 2. \end{aligned}$$

In particular, we have $q(\mathbf{v}) = 0$ and $q(\mathbf{u}) = -1$. Now let us fix an $n \geq 2$ and construct an n-admissible quiver. We pick $k_{\mathbf{v}}, k_{\mathbf{u}} \geq 2$ such that $n \mid k_{\mathbf{v}}$ and $n \mid (k_{\mathbf{u}} - 1)$. Then the quiver

$$\begin{tikzcd}
 & \cdots \arrow[dr] & & & & \\
v_2 \arrow[ur] & & v_{k_{\mathbf{v}}-1} \arrow[dl] & & & \\
 & 1 \arrow[ul] \arrow[r] & u_2 \arrow[r] & \cdots \arrow[r] & u_{k_{\mathbf{u}}-1} \arrow[r] & 2
\end{tikzcd}$$

is n-admissible.

4.1. **$n\mathbb{Z}$-cluster tilting subcategories.** We recall the definition of $n\mathbb{Z}$-cluster tilting subcategories from [IJ17].

Definition 4.5. [IJ17, Definition-Proposition 2.15] Let Λ be an algebra and let $\mathcal{C} \subseteq \operatorname{mod}\Lambda$ be an n-cluster tilting subcategory. We say that $\mathcal{C}$ is an *$n\mathbb{Z}$-cluster tilting subcategory* if one of the two equivalent conditions

(a) $\Omega^n(\mathcal{C}) \subseteq \mathcal{C}$, and
(b) $\Omega^{-n}(\mathcal{C}) \subseteq \mathcal{C}$

holds.

In this subsection we classify radical square zero bound quiver algebras which admit $n\mathbb{Z}$-cluster tilting subcategories. We start with the following proposition.

Proposition 4.6. Let $\Lambda = \mathbf{k}Q/\mathcal{J}^2$ and assume that there exists an $n\mathbb{Z}$-cluster tilting subcategory $\mathcal{C} \subseteq \operatorname{mod}\Lambda$. Let $v \in Q_0$ be a vertex of Q. Then $\delta(v) \in \{(0,0),(0,1),(1,0),(1,1)\}$.

Proof. Since $\mathcal{C}$ is an $n\mathbb{Z}$-cluster tilting subcategory, it follows that Q is an n-admissible quiver. Hence $\delta^+(v) \leq 2$ and $\delta^-(v) \leq 2$ and it is enough to show that $\delta^+(v) \neq 2$ and $\delta^-(v) \neq 2$. We show that $\delta^+(v) \neq 2$; the fact that $\delta^-(v) \neq 2$ follows dually.

Assume towards a contradiction that $\delta^+(v) = 2$ and let $v \longrightarrow u_1$ and $v \longrightarrow u_2$ be the two arrows starting at v. Then $P(v) \in \mathcal{C}$ and $P(v)$ is not injective. It follows from Proposition 1.1(a) that $P(v) \cong \tau_n(X)$ for some nonprojective indecomposable module $X \in \mathcal{C}$. In particular, the module $\Omega^{n-1}(X)$ is nonprojective and so we have

$$\Omega^{n-1}(X) \cong \tau^-\tau\Omega^{n-1}(X) = \tau^-\tau_n(X) \cong \tau^-(P(v)) \cong S(v),$$

where the last isomorphism follows from the dual of Lemma 2.20(c). Since by the dual of Lemma 2.20(c) we have $\Omega(S(v)) \cong S(u_1) \oplus S(u_2)$, we obtain that

$$\Omega^n(X) = \Omega\Omega^{n-1}(X) \cong \Omega(S(v)) \cong S(u_1) \oplus S(u_2).$$

Since $\mathcal{C}$ is an $n\mathbb{Z}$-cluster tilting subcategory, it follows that $S(u_1) \oplus S(u_2) \in \mathcal{C}$. But then a direct computation shows that $\Omega(I(u_2)) \cong S(u_1)$, from which we conclude that $\operatorname{Ext}^1_\Lambda(I(u_2), S(u_1) \oplus S(u_2)) \neq 0$. This contradicts the fact that $\mathcal{C}$ is an n-cluster tilting subcategory since $I(u_2), S(u_1) \oplus S(u_2) \in \mathcal{C}$. $\square$

We can now give the classification of $n\mathbb{Z}$-cluster tilting subcategories for radical square zero bound quiver algebras.

Theorem 4.7. *Let Q be a quiver, let $\Lambda = \mathbf{k}Q/\mathcal{J}^2$ and let $n \in \mathbb{Z}_{\geq 2}$. Then the algebra Λ admits an $n\mathbb{Z}$-cluster tilting subcategory $\mathcal{C} \subseteq \operatorname{mod}\Lambda$ if and only if $Q = A_m$ and $n \mid (m-1)$ or $Q = \tilde{A}_m$ and $n \mid m$.*

Proof. If $Q = A_m$ and $n \mid (m-1)$ or $Q = \tilde{A}_m$ and $n \mid m$, then Λ admits an n-cluster tilting subcategory $\mathcal{C} \subseteq \operatorname{mod}\Lambda$ by Theorem 4.1. Moreover, in this case, it is easy to see that $\tau(M) \cong \Omega(M)$ for any $M \in \operatorname{mod}\Lambda$ and hence $\mathcal{C}$ is also an $n\mathbb{Z}$-cluster tilting subcategory by Proposition 1.1(a).

For the other direction, assume that Λ admits an $n\mathbb{Z}$-cluster tilting subcategory $\mathcal{C}$. Then by Proposition 4.6 we have that if $v \in Q_0$, then $\delta(v) \in \{(0,0),(1,0),(0,1),(1,1)\}$. Since Q is connected, we conclude that there exists some $m \in \mathbb{Z}_{\geq 1}$ such that $Q = A_m$ or $Q = \tilde{A}_m$. Since $\mathcal{C}$ is n-cluster tilting, it follows that Q is n-admissible from Theorem 4.1. Hence we conclude that if $Q = A_m$, then $n \mid (m-1)$, while if $Q = \tilde{A}_m$, then $n \mid m$, as required. $\square$

In particular we see that the only radical square zero bound quiver algebras which admit $n\mathbb{Z}$-cluster tilting subcategories are Nakayama algebras.

4.2. **A lattice of n-cluster tilting subcategories.** Before giving our next result, let us recall the following classical definition.

Definition 4.8. A *poset* is a partially ordered set. A *lattice* is a partially ordered set in which every two elements have a *meet*, that is a greatest lower bound and a *join*, that is a least upper bound. A *complete lattice* is a lattice in which any subset has a greatest lower bound and a least upper bound.

Example 4.9. Let N be a positive integer. Then the set $D(N) = \{x \in \mathbb{Z} \mid x \geq 1 \text{ and } x \mid N\}$ forms a complete lattice called the *lattice of divisors of* N under the relation $x \leq y$ if $x \mid y$. If $x, y \in D(N)$, then their meet corresponds to their greatest common divisor $\gcd(x, y)$ and their join corresponds to their least common multiple $\operatorname{lcm}(x, y)$.

For the rest of this article, we drop our assumption that we consider n-cluster tilting subcategories for $n \geq 2$ and we assume that $n \geq 1$ instead. Let $Q \neq \tilde{A}_m$ be a quiver and let $\Lambda = \mathbf{k}Q/\mathcal{J}^2$. Our aim is to show that the collection of n-cluster tilting subcategories (for varying n) of $\operatorname{mod}\Lambda$ forms a lattice with respect to inclusion of subcategories. We start with the following result.

Proposition 4.10. Let $\Lambda = \mathbf{k}Q/\mathcal{J}^2$ be a radical zero bound quiver algebra and assume that $Q \neq A_1$ and $Q \neq \tilde{A}_m$ for any $m \geq 1$. Let $\mathcal{C}_n \subseteq \operatorname{mod}\Lambda$ be an n-cluster tilting subcategory and $\mathcal{C}_{n'} \subseteq \operatorname{mod}\Lambda$ be an n'-cluster tilting subcategory. Then $\mathcal{C}_n \subseteq \mathcal{C}_{n'}$ if and only if $n' \mid n$.

Proof. If $n' = 1$, then the result is clear since $\mathcal{C}_{n'} = \mathcal{C}_1 = \operatorname{mod}\Lambda$. If $n = 1$, then the result is also clear since $\operatorname{mod}\Lambda$ is an n-cluster tilting subcategory if and only if $n = 1$ (since $Q \neq A_1$).

Hence we may assume that $n > 1$ and $n' > 1$. Then Q is n-admissible and n'-admissible by Theorem 4.1. Moreover, we have that $\mathcal{C}_n = \operatorname{add}(M_n)$ and $\mathcal{C}_{n'} = \operatorname{add}(M_{n'})$ where

$$M_n = \bigoplus_{j\geq 0} \tau_n^{-j}(\Lambda) \text{ and } M_{n'} = \bigoplus_{j \geq 0} \tau_{n'}^{-j}(\Lambda).$$

Assume first that $n' \mid n$. Then $n = hn'$ for some $h \geq 1$. Let $X \in \mathcal{C}_n$ and we show that $X \in \mathcal{C}_{n'}$. Since $\mathcal{C}_n$ and $\mathcal{C}_{n'}$ are closed under direct sums and summands, we may assume that X is indecomposable. If X is projective or injective, then $X \in \mathcal{C}_{n'}$ since $\mathcal{C}_{n'}$ is an n'-cluster tilting subcategory. Otherwise we have by (3.4) and Corollary 3.7 that $X \cong \tau_n^{-j}(P(\mathbf{v}))$ for some k-flow path $\mathbf{v}$ in Q and some $j \geq 1$. Since Q is n-admissible and n'-admissible, we have that $k + q(\mathbf{v}) = pn$ and $k + q(\mathbf{v}) = p'n'$ for some $p, p' \in \mathbb{Z}_{\geq 1}$. In particular, we have $p = \frac{p'n'}{n}$. Moreover, by Corollary 2.27 and since X is not injective, we have that $1 \leq j \leq p - 1$. Hence we obtain

$$1 \leq jh \leq (p-1)h = ph - h = \frac{p'n'}{n}\frac{n}{n'} - h = p' - h \leq p' - 1,$$

and so $1 \leq jh \leq p' - 1$. Hence by Corollary 2.27 we have

$$X \cong \tau_n^{-j}(P(\mathbf{v})) \cong S(v_{k-jn+q_k}) = S(v_{k-jhn'+q_k}) \cong \tau_{n'}^{-jh}(P(\mathbf{v})) \in \operatorname{add}(M_{n'}) = \mathcal{C}_{n'},$$

as required.

Assume now that $\mathcal{C}_n \subseteq \mathcal{C}_{n'}$. Then by Lemma 2.13 there exists a k-flow path $\mathbf{v}$ in Q. Since Q is n-admissible and n'-admissible, we have that $k + q(\mathbf{v}) = pn$ and $k + q(\mathbf{v}) = p'n'$ for some $p, p' \in \mathbb{Z}_{\geq 1}$. If $p = 1$, then $n = p'n'$ and $n' \mid n$ as required. Otherwise, assume that $p > 1$. Then by Corollary 2.27 and Proposition 1.1(a) we have that

$$\tau_n^{-}(P(\mathbf{v})) \cong S(v_{k-n+q_k}) \in \mathcal{C}_n.$$

Since by assumption we have $\mathcal{C}_n \subseteq \mathcal{C}_{n'}$, we conclude that $S(v_{k-n+q_k}) \in \mathcal{C}_{n'}$. Write $n = hn' + r$ with $h \in \mathbb{Z}_{\geq 0}$ and $0 \leq r \leq n' - 1$. We first claim that $1 \leq h \leq p' - 1$.

First assume towards a contradiction that $h = 0$. Then $n = r$ and

$$k - n + q_k = (p-1)n + 1 - q_1 \geq (2-1)\cdot 1 + 1 - 1 = 1.$$

Hence by Lemma 2.25(a) we have that $\Omega^{-n}(P(\mathbf{v})) \cong S(v_{k-n+q_k})$. Since $0 < n = r < n'$, we have $0 < n' - n < n'$. Hence by Proposition 1.1(a) we obtain

$$\tau_{n'}^{-}(P(\mathbf{v})) = \tau_{n'-n}^{-}\Omega^{-n}(P(\mathbf{v})) \cong \tau_{n'-n}^{-}(S(v_{k-n+q_k})) \in \mathcal{C}_{n'}.$$

It follows from Lemma 1.2 that $\operatorname{Ext}_{\Lambda}^{n'-n}(\tau_{n'}^{-}(P(\mathbf{v})), S(v_{k-n+q_k})) \neq 0$. This contradicts the fact that $\mathcal{C}_{n'}$ is n'-cluster tilting since $\tau_{n'}^{-}(P(\mathbf{v})), S(v_{k-n+q_k}) \in \mathcal{C}_{n'}$.

Next assume towards a contradiction that $h \geq p'$. Then we have

$$n < pn = k + q(\mathbf{v}) = p'n' \leq hn' \leq hn' + r = n,$$

which is a contradiction.

We conclude that $n = hn' + r$ with $1 \leq h \leq p' - 1$ and we now claim that $r = 0$. Assume towards a contradiction that $r > 0$. By Corollary 2.27 and Proposition 1.1(a) we have that

$$\tau_{n'}^{-h}(P(\mathbf{v})) \cong S(v_{k-hn'+q_k}) \in \mathcal{C}_{n'}.$$

Then $1 \leq k - hn' + q_k \leq k - 1 + q_k$ and $(k - hn' + q_k) - r = k - (hn' + r) + q_k = k - n + q_k \geq 1$, and so by Lemma 2.24(a) we have that

$$\Omega^{-r}(S(v_{k-hn'+q_k})) \cong S(v_{k-hn'-r+q_k}) = S(v_{k-n+q_k}).$$

But then we have that

$$\operatorname{Ext}_{\Lambda}^{r}(S(v_{k-n+q_k}), S(v_{k-hn'+q_k})) \cong \operatorname{Ext}_{\Lambda}^{r}(\Omega^{-r}(S(v_{k-hn'+q_k})), S(v_{k-hn'+q_k})) \neq 0.$$

This contradicts the fact that $\mathcal{C}_{n'}$ is n'-cluster tilting since $S(v_{k-n+q_k}), S(v_{k-hn'+q_k}) \in \mathcal{C}_{n'}$ and $1 \leq r \leq n' - 1$. We conclude that $r = 0$ and so $n = hn'$, as required. $\square$

We also need the following definition.

Definition 4.11. Let Q be a quiver. We define the *admissible degree* of Q to be

$$N(Q) := \begin{cases} \max\left(\{n \in \mathbb{Z}_{\geq 2} \mid Q \text{ is } n\text{-admissible}\} \cup \{1\}\right), & \text{if } Q \neq A_1, \\ 1, & \text{if } Q = A_1. \end{cases}$$

Since Q is finite, it follows that $N(Q)$ is well-defined. We now give the main result for this section.

Theorem 4.12. *Let Q be a quiver with admissible degree $N = N(Q)$ and let $D(N) = \{n \in \mathbb{Z} \mid n \geq 1$ and $n \mid N\}$. Let $\Lambda = \mathbf{k}Q/\mathcal{J}^2$.*

(a) If $Q \neq A_1$, then there exists an n-cluster tilting subcategory $\mathcal{C}_n \subseteq \operatorname{mod}\Lambda$ if and only if $n \in D(N)$.

(b) If $Q \neq \tilde{A}_m$, set

$$\mathbf{CT}(\Lambda) := \{\mathcal{C} \subseteq \operatorname{mod}\Lambda \mid \text{there exists } n \in \mathbb{Z}_{\geq 1} \text{ such that } \mathcal{C} \text{ is } n\text{-cluster tilting}\}.$$

Then for every $n \in D(N)$ there exists a unique n-cluster tilting subcategory $\mathcal{C}_n$. Moreover, the pair $(\mathbf{CT}(\Lambda), \subseteq)$ is a poset isomorphic to the opposite of the poset of divisors of N. In particular, $(\mathbf{CT}(\Lambda), \subseteq)$ forms a complete lattice where the meet of $\mathcal{C}_n$ and $\mathcal{C}_{n'}$ is given by $\mathcal{C}_{\operatorname{lcm}(n,n')}$ and the join of $\mathcal{C}_n$ and $\mathcal{C}_{n'}$ is given by $\mathcal{C}_{\gcd(n,n')}$.

Proof. (a) For $n = 1$ the result is clear since $\operatorname{mod}\Lambda$ is a 1-cluster tilting subcategory. Assume now that $n > 1$. If $n \in D(N)$, then it follows from Remark 3.3(b) and Theorem 4.1 that Λ admits an n-cluster tilting subcategory, which proves one direction.

For the other direction assume that there exists an n-cluster tilting subcategory $\mathcal{C}_n \subseteq \operatorname{mod}\Lambda$ and we show that $n \in D(N)$. It follows from Theorem 4.1 that Q is n-admissible and so $1 < n \leq N$. Hence by Definition 4.11 it follows that Q is N-admissible too. We consider the cases $Q = \tilde{A}_m$ and $Q \neq \tilde{A}_m$ separately.

If $Q = \tilde{A}_m$ for some $m \geq 1$, then we have by Definition 3.1 that $N \mid m$ and so $N \leq m$. Moreover, the quiver $\tilde{A}_m$ is always m-admissible and so $m \leq N$. It follows that $m = N$. Since Q is also n-admissible, we have that $n \mid m = N$ and so $n \in D(N)$.

If $Q \neq \tilde{A}_m$, then there exists a flow path $\mathbf{v}$ in Q by Lemma 2.13. Moreover, for every $k_{\mathbf{v}}$-flow path $\mathbf{v}$ we have that $n \mid (k_{\mathbf{v}} + q(\mathbf{v}))$ and $N \mid (k_{\mathbf{v}} + q(\mathbf{v}))$. It follows that $\operatorname{lcm}(n,N) \mid (k_{\mathbf{v}} + q(\mathbf{v}))$ for every flow path $\mathbf{v}$ in Q. Hence Q is $\operatorname{lcm}(n,N)$-admissible and so $\operatorname{lcm}(n,N) \leq N$. We conclude that $n \mid N$ and so $n \in D(N)$.

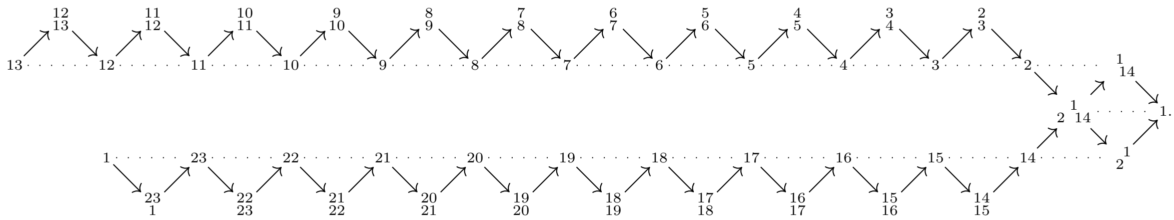
(b) If $N(Q) = 1$, then $\mathbf{CT}(\Lambda) = \{\operatorname{mod}\Lambda\}$ and the result is clear. If $N(Q) > 1$, then existence of $\mathcal{C}_n$ follows from (a) and uniqueness by Theorem 4.1. Then $(\mathbf{CT}(\Lambda), \subseteq)$ is a poset isomorphic to the opposite of the poset of divisors of N by Proposition 4.10. That $\mathbf{CT}(\Lambda)$ forms a complete lattice with the given meet and join follows from Example 4.9. $\square$

We finish with an example which illustrates Theorem 4.12.

Example 4.13. Let Q be the quiver

$$\begin{tikzcd}
 & 23 \arrow[dl] & 22 \arrow[l] & 21 \arrow[l] & 20 \arrow[l] & 19 \arrow[l] \\
1 \arrow[r] \arrow[d] & 14 \arrow[r] & 15 \arrow[r] & 16 \arrow[r] & 17 \arrow[r] & 18 \arrow[u] \\
2 \arrow[r] & 3 \arrow[r] & 4 \arrow[r] & 5 \arrow[r] & 6 \arrow[r] & 7 \arrow[r] & 8 \arrow[r] & 9 \arrow[r] & 10 \arrow[r] & 11 \arrow[r] & 12 \arrow[r] & 13.
\end{tikzcd}$$

Then $N(Q) = 12$. The Auslander–Reiten quiver of $\Lambda = \mathbf{k}Q/\mathcal{J}^2$ is



The divisors of 12 are $D(12) = \{1,2,3,4,6,12\}$. For $n \in D(12)$ we set $M_n = \bigoplus_{j \geq 0} \tau_n^{-j}(\Lambda)$ and $\mathcal{C}_n = \operatorname{add}(M_n)$. Then we have

$\mathcal{C}_1 = \operatorname{mod}\Lambda,$

$\mathcal{C}_2 = \operatorname{add}\{\Lambda,\ 11,\ 9,\ 7,\ 5,\ 3,\ \begin{smallmatrix}1\\14\end{smallmatrix},\ 23,\ 21,\ 19,\ 17,\ 15,\ \begin{smallmatrix}1\\2\end{smallmatrix}\},$

$\mathcal{C}_3 = \operatorname{add}\{\Lambda,\ 10,\ 7,\ 4,\ \begin{smallmatrix}1\\14\end{smallmatrix},\ 22,\ 19,\ 16,\ \begin{smallmatrix}1\\2\end{smallmatrix}\},$

$\mathcal{C}_4 = \operatorname{add}\{\Lambda,\ 9,\ 5,\ \begin{smallmatrix}1\\14\end{smallmatrix},\ 21,\ 17,\ \begin{smallmatrix}1\\2\end{smallmatrix}\},$

$\mathcal{C}_6 = \operatorname{add}\{\Lambda,\ 7,\ \begin{smallmatrix}1\\14\end{smallmatrix},\ 19,\ \begin{smallmatrix}1\\2\end{smallmatrix}\},$

$\mathcal{C}_{12} = \operatorname{add}\{\Lambda,\ \begin{smallmatrix}1\\14\end{smallmatrix},\ \begin{smallmatrix}1\\2\end{smallmatrix}\},$

and $\mathcal{C}_n$ is an n-cluster tilting subcategory of $\operatorname{mod}\Lambda$ by Theorem 4.1. Then the lattice

$$\begin{tikzcd}
 & 12 \arrow[dl, dash] \arrow[dr, dash] & & \\
4 \arrow[dr, dash] & & 6 \arrow[dl, dash] \arrow[dr, dash] & \\
 & 2 \arrow[dr, dash] & & 3 \arrow[dl, dash] \\
 & & 1 &
\end{tikzcd}$$

of divisors of 12 corresponds to the lattice

$$\begin{tikzcd}
 & \mathcal{C}_{12} \arrow[dl, phantom, "\supset" sloped] \arrow[dr, phantom, "\subset" sloped] & & \\
\mathcal{C}_4 \arrow[dr, phantom, "\subset" sloped] & & \mathcal{C}_6 \arrow[dl, phantom, "\supset" sloped] \arrow[dr, phantom, "\subset" sloped] & \\
 & \mathcal{C}_2 \arrow[dr, phantom, "\subset" sloped] & & \mathcal{C}_3 \arrow[dl, phantom, "\supset" sloped] \\
 & & \mathcal{C}_1 &
\end{tikzcd}$$

of inclusions of n-cluster tilting subcategories of $\operatorname{mod}\Lambda$.

Acknowledgements

The author wishes to thank Hipolito Treffinger for valuable discussions on string algebras and helpful suggestions about the article. The author also thanks René Marczinzik for bringing to his attention the question in paragraph 5.4 of [EH08]. The author is grateful to Max Planck Institute for Mathematics in Bonn for its hospitality and financial support.

References

[ARS95] Maurice Auslander, Idun Reiten, and Svere Olaf Smalø. *Representation theory of Artin algebras*, volume 36 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1995.

[ASS06] Ibrahim Assem, Daniel Simson, and Andrzej Skowroński. *Elements of the Representation Theory of Associative Algebras: Volume 1: Techniques of Representation Theory*. Elements of the Representation Theory of Associative Algebras. Cambridge University Press, 2006.

[BR87] Michael C.R. Butler and Claus Michael Ringel. Auslander–Reiten sequences with few middle terms and applications to string algebras. *Communications in Algebra*, 15(1–2):145–179, 1987.

[CDIM20] Aaron Chan, Erik Darpö, Osamu Iyama, and René Marczinzik. Periodic trivial extension algebras and fractionally Calabi–Yau algebras. *arXiv e-prints*, 2020, 2012.11927.

[CIM19] Aaron Chan, Osamu Iyama, and René Marczinzik. Auslander–Gorenstein algebras from Serre-formal algebras via replication. *Advances in Mathematics*, 345:222–262, 2019.

[DI20] Erik Darpö and Osamu Iyama. d-Representation-finite self-injective algebras. *Advances in Mathematics*, 362:106932, 2020.

[EH08] Karin Erdmann and Thorsten Holm. Maximal n-orthogonal modules for selfinjective algebras. *Proceedings of the American Mathematical Society*, 136(9):3069–3078, 2008.

[Gab72] Peter Gabriel. Unzerlegbare Darstellungen I. *Manuscripta Mathematica*, 6(1):71–103, March 1972.

[HI11] Martin Herschend and Osamu Iyama. Selfinjective quivers with potential and 2-representation-finite algebras. *Compositio Mathematica*, 147(6):1885–1920, 2011.

[IJ17] Osamu Iyama and Gustavo Jasso. Higher Auslander Correspondence for Dualizing R-Varieties. *Algebras and Representation Theory*, 20(2):335–354, Apr 2017.

[IO11] Osamu Iyama and Steffen Oppermann. n-representation-finite algebras and n-APR tilting. *Transactions of the American Mathematical Society*, 363(12):6575–6614, 2011.

[IO13] Osamu Iyama and Steffen Oppermann. Stable categories of higher preprojective algebras. *Advances in Mathematics*, 244:23–68, 2013.

[Iya07a] Osamu Iyama. Auslander correspondence. *Advances in Mathematics*, 210(1):51–82, 2007.

[Iya07b] Osamu Iyama. Higher-dimensional Auslander–Reiten theory on maximal orthogonal subcategories. *Advances in Mathematics*, 210(1):22–50, 2007.

[Iya08] Osamu Iyama. Auslander–Reiten theory revisited. In *Trends in representation theory of algebras and related topics*, EMS Series of Congress Reports, pages 349–397. European Mathematical Society, Zürich, 2008.

[JKPK19] Gustavo Jasso, Julian Külshammer, Chrysostomos Psaroudakis, and Sondre Kvamme. Higher Nakayama algebras I: Construction. *Advances in Mathematics*, 351:1139–1200, 2019.

[MV21] René Marczinzik and Laertis Vaso. Existence of a 2-cluster tilting module does not imply finite complexity. *arXiv e-prints*, 2021, 2101.05671.

[ST21] Mads Hustad Sandøy and Louis-Philippe Thibault. Classification results for n-hereditary monomial algebras. *arXiv e-prints*, 2021, 2101.12746.

[Vas19] Laertis Vaso. n-Cluster tilting subcategories of representation-directed algebras. *Journal of Pure and Applied Algebra*, 223(5):2101–2122, 2019.