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COMBINED COUNT OF REAL RATIONAL CURVES OF CANONICAL DEGREE 2 ON REAL DEL PEZZO SURFACES WITH $K^2 = 1$

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Abstract. For del Pezzo surfaces of this type, we propose two ways of counting real rational curves of canonical degree 2. The both have exceptionally strong invariance property: the result does not depend on the choice of a surface. The first one includes all divisor classes of canonical degree 2 and gives in total 30. The other one excludes the class $-2K$, but adds up the results of counting for Bertini pair of real structures. This count gives 96.

Vor dem Gesetz steht ein Trhter.
Zu diesem Trhter kommt ein Mann vom Lande und bittet um Eintritt in das Gesetz. Aber der Trhter sagt, da er ihm jetzt den Eintritt nicht gewhren knne. Der Mann berlegt und fragt dann, ob er also spter werde eintreten drfen.
"Es ist mglich", sagt der Trhter, "jetzt aber nicht."

F. Kafka, Die Parabel "Vor dem Gesetz"

1. Introduction

This work is based on our previous paper [FK]. So we start with recalling its setup and principal ingredients.

1.1. **Short review of** [FK]**.** By definition, a compact complex surface X is a *del Pezzo surface of degree* 1, if X is non-singular and irreducible, its anticanonical class $-K_X$ is ample, and $K_X^2 = 1$. The image of X by a bi-anticanonical map $X \to \mathbb{P}^3$ is then a non-degenerate quadratic cone $Q \subset \mathbb{P}^3$, with $X \to Q$ being a double covering branched at the vertex of the cone and along a non-singular sextic curve $C \subset Q$ (a transversal intersection of Q with a cubic surface). In particular, each del Pezzo surface of degree 1 carries a non-trivial automorphism, known as the *Bertini involution*, that is the deck transformation τ_X of the covering.

Any real structure, $\mathrm{conj} : X \to X$, has to commute with τ_X, and this gives another real structure $\tau_X \circ \mathrm{conj} = \mathrm{conj} \circ \tau_X$ which we call *Bertini dual* to conj. It is such a pair of real structures, $\{\mathrm{conj},\ \mathrm{conj} \circ \tau_X\}$, that we call a *Bertini pair*. We generally use notation $\mathrm{conj}^{\pm}$ for Bertini pairs of real structures and write $X^{\pm}$ for the corresponding pairs of real del Pezzo surfaces to simplify a more formal notation $(X, \mathrm{conj}^{\pm})$.

The bi-anticanonical map projects the real loci $X^{\pm}_{\mathbb{R}}$ to two complementary domains $Q^{\pm}_{\mathbb{R}} \subset Q_{\mathbb{R}}$ on $Q_{\mathbb{R}}$, where the latter is a cone over a real non-singular conic *with non-empty real locus.* The branching curve C is real too, and its real locus $C_{\mathbb{R}}$ together with the vertex of the cone form the common boundary of $Q^{\pm}_{\mathbb{R}}$. Conversely, for any real non-singular curve $C \subset Q$ which is a transversal intersection of Q with a real cubic surface, the surface X which is the double covering of Q branched at the vertex of Q and along C is a del Pezzo surface of degree 1 inheriting from Q a pair of Bertini dual real structures $\mathrm{conj}^{\pm}$.

Recall also an intrinsic description of the basic Pin^--structure introduced in [FK].

1.1.1. **Theorem.** *There is a unique way to supply each real del Pezzo surface X of degree 1 with a Pin^--structure θ_X on $X_{\mathbb{R}}$, so that the following properties hold:*

1. *θ_X is invariant under real automorphisms and real deformations of X. In particular, the associated quadratic function $q_X : H_1(X_{\mathbb{R}}; \mathbb{Z}/2) \to \mathbb{Z}/4$ is preserved by the Bertini involution.*
2. *q_X vanishes on each real vanishing cycle in $H_1(X_{\mathbb{R}}; \mathbb{Z}/2)$ and takes value 1 on the class dual to $w_1(X_{\mathbb{R}})$.*
3. *If $X^{\pm}$ is a Bertini pair of real del Pezzo surfaces of degree 1, then the corresponding quadratic functions $q_{X^{\pm}}$ take equal values on the elements represented in $H_1(X^{\pm}_{\mathbb{R}}; \mathbb{Z}/2)$ by the connected components of $C_{\mathbb{R}}$.*

The Picard group of a del Pezzo surface (as well as that of any rational surface) is naturally isomorphic to the second homology group with integer coefficients, $\operatorname{Pic} X = H_2(X)$. It has a natural grading by *canonical degree*, $\alpha \mapsto -\alpha K_X$. In the case of del Pezzo surfaces of degree 1, the subgroup of minimal, zero, grading is the lattice $K_X^{\perp}$ which is isomorphic to E_8 and generated by geometric vanishing cycles. Over $\mathbb{C}$, the set of geometric vanishing cycles coincides with the set of roots. Neither of divisor classes of degree 0 is effective. If X is equipped with a real structure conj, then the Picard group of real divisor classes is a subgroup of $\operatorname{Pic} X$ naturally isomorphic to $H_2(X) \cap \ker(1 + \mathrm{conj}_*)$. The latter splits in an orthogonal direct sum $\mathbb{Z}K_X \oplus (K_X^{\perp} \cap \ker(1 + \mathrm{conj}_*))$. For the list of isomorphism classes of the lattices $K_X^{\perp} \cap \ker(1 + \mathrm{conj}_*)$, see for example Table 1 in Section 2.

The only effective divisor classes in the coset $-K_X + E_8$ of degree 1 classes are $-K_X$ and $-K_X - e$ where e is any root of E_8. The linear system $|-K_X|$ is of projective dimension 1 and consists of pull-backs to X of line generators of Q. The divisors representing the classes $-K_X - e$ are rigid, they are (-1)-curves, called by definition *lines*. Over $\mathbb{C}$ they are in one-to-one correspondence with the roots e in $K_X^{\perp}$, and over $\mathbb{R}$ with the roots in $K_X^{\perp} \cap \ker(1 + \mathrm{conj}_*)$.

It is for a signed count of real lines that the above Pin^--structure was employed in [FK]. Namely, a real line $L \subset X$ was called *hyperbolic* (resp. *elliptic*) if $q_X(L_{\mathbb{R}}) = 1 \in \mathbb{Z}/4$ (resp. $q_X(L_{\mathbb{R}}) = -1 \in \mathbb{Z}/4$), and an integer weight $s(L) = 1$ (resp. $s(L) = -1$) was attributed to hyperbolic (resp. elliptic) lines. As was shown in [FK], for counting real lines with these weights the following fundamental relations hold:

$$\sum_{\text{real lines } L \subset X} s(L) = 2\,\mathrm{rk}(K_X^{\perp} \cap \ker(1 + \mathrm{conj}_*)) \tag{1.1.1}$$

and, for each Bertini pair $X^{\pm}$,

$$\sum_{\text{real lines } L \subset X^+} s(L) + \sum_{\text{real lines } L \subset X^-} s(L) = 16. \tag{1.1.2}$$

The divisor classes of lines together with $-K_X$ constitute the *first layer* in the semigroup of effective divisor classes, that is the set of effective divisor classes α of canonical degree $-\alpha K_X = 1$. The curves in the divisor class $-K_X$ form a pencil of curves of arithmetic genus 1, so that a generic curve in this pencil is a nonsingular curve of geometric genus 1 while a finite number of curves in this pencil is rational. Therefore, it is reasonable to combine the count of real lines with a count of real rational curves A belonging to the divisor class $-K_X$, attributing (similar to the above) to each of them the weight $s(A) = i^{q_X(A_{\mathbb{R}})-1}w(A)$ where $w(A)$ stands for a *modified Welschinger weight* of A, that is $(-1)^{c_A}$ where c_A stands for the number of non-solitary real nodes of A. Then, as it can be easily deduced from (1.1.1), an additional relation holds when we are performing such a weighted count of real rational curves over the whole first layer.

1.1.2. **Theorem.** *For any real del Pezzo surface X of degree* 1*,*

$$\sum_{\text{real lines } L\subset X} s(L) \quad + \sum_{\text{real rational curves } A\in|-K_X|} s(A) = 8. \tag{1.1.3}$$
□

1.2. **Next step.** In this paper, we carry out a similar study of the *second layer* of the semigroup of effective divisor classes, that is the set of effective divisor classes α of canonical degree $-\alpha K_X = 2$. As is known, this semigroup is generated by the elements of its first layer (namely, by $-K$ and the divisor classes of lines), and thus the second layer is formed by sums of two elements in the first one.

To extend the universal counting relations (1.1.2) and (1.1.3) to the second layer, we exclude from the consideration the divisor classes of type $-2K_X - 2e$ where e is a root in $K_X^{\perp}$, and the classes of type $-2K_X - e_1 - e_2$ where e_1, e_2 are roots in $K_X^{\perp}$ with $e_1 \cdot e_2 = -1$. Their exclusion from consideration is motivated mainly by conventions in Gromov-Witten theory that impose to count such classes for 0 (neither of these classes α allows to trace a rational curve of divisor class α through a $1 = -\alpha K_X - 1$ fixed generic point).

Therefore, the remaining part $\mathcal{B}(X)$ of the second layer and its *real part* $\mathcal{B}_{\mathbb{R}}(X) = \mathcal{B}(X) \cap \ker(1 + \mathrm{conj}_*)$ split as

$$\begin{aligned}\mathcal{B}(X) =&\mathcal{B}^0(X) \cup \mathcal{B}^2(X) \cup \mathcal{B}^4(X), \quad \mathcal{B}^{2k}(X) = \{-2K_X - v \,|\, v \in K_X^{\perp}, v^2 = -2k\},\\ \mathcal{B}_{\mathbb{R}}(X) =&\mathcal{B}^0_{\mathbb{R}}(X) \cup \mathcal{B}^2_{\mathbb{R}}(X) \cup \mathcal{B}^4_{\mathbb{R}}(X), \quad \mathcal{B}^{2k}_{\mathbb{R}}(X) = \mathcal{B}^{2k}(X) \cap \ker(1 + \mathrm{conj}_*).\end{aligned}$$

The curves $A \subset X$ in each of the divisor classes $\alpha \in \mathcal{B}^{2k}$ have arithmetic genus $2 - k$ and form a linear system of projective dimension $3 - k$. Thus, in our counts we pick a point $x \in X_{\mathbb{R}}$ and for each $\alpha \in \mathcal{B}^{2k}$ introduce into consideration the following sets of curves

$$\begin{aligned}\mathcal{C}^{2k}(\alpha, x, X) =&\{A \subset X \,|\, A \text{ is rational}, [A] = \alpha, x \in A\},\\ \mathcal{C}^{2k}_{\mathbb{R}}(\alpha, x, X) =&\{A \in \mathcal{C}^{2k}(\alpha, x, X) \,|\, A \text{ is real}\}\end{aligned}$$

and put

$$\mathcal{C}^{2k}(x, X) = \bigcup_{\alpha\in\mathcal{B}^{2k}(X)} \mathcal{C}^{2k}(\alpha, x, X), \quad \mathcal{C}^{2k}_{\mathbb{R}}(x, X) = \bigcup_{\alpha\in\mathcal{B}^{2k}_{\mathbb{R}}(X)} \mathcal{C}^{2k}_{\mathbb{R}}(\alpha, x, X),$$
$$\begin{aligned}\mathcal{C}(x, X) =&\mathcal{C}^0(x, X) \cup \mathcal{C}^2(x, X) \cup \mathcal{C}^4(x, X),\\ \mathcal{C}_{\mathbb{R}}(x, X) =&\mathcal{C}^0_{\mathbb{R}}(x, X) \cup \mathcal{C}^2_{\mathbb{R}}(x, X) \cup \mathcal{C}^4_{\mathbb{R}}(x, X).\end{aligned}$$

For a generic choice of $x \in X_{\mathbb{R}}$, each of these sets is finite.

The main results of this paper are the following two theorems, which provide an extension of the strong invariance properties (1.1.2) and (1.1.3) form the first layer to the second.

1.2.1. **Theorem.** *For any real del Pezzo surface X of degree 1 and any generic point $x \in X_{\mathbb{R}}$, we have*

$$\sum_{A \in \mathcal{C}_{\mathbb{R}}(x,X)} \mathfrak{s}(A) = 30 \tag{1.2.1}$$

with

$$\mathfrak{s}(A) = i^{\widehat{\mathrm{q}}_X([A])} w(A), \quad w(A) = (-1)^{c_A} \tag{1.2.2}$$

where $\widehat{\mathrm{q}}_X([A]) = q_X(A_{\mathbb{R}})$ and c_A stands for the number of non-solitary real nodes of A.

1.2.2. **Theorem.** *For any Bertini pair of real del Pezzo surfaces $X^{\pm}$ of degree 1 and any pair of real generic points $x^{\pm} \in X^{\pm}_{\mathbb{R}}$, we have*

$$\sum_{A \in \mathcal{C}^2_{\mathbb{R}}(x^+,X^+) \cup \mathcal{C}^4_{\mathbb{R}}(x^+,X^+)} \widetilde{\mathfrak{s}}(A) + \sum_{A \in \mathcal{C}^2_{\mathbb{R}}(x^-,X^-) \cup \mathcal{C}^4_{\mathbb{R}}(x^-,X^-)} \widetilde{\mathfrak{s}}(A) = 96 \tag{1.2.3}$$

where

$$\widetilde{\mathfrak{s}}(A) = \begin{cases} \mathfrak{s}(A), & \text{if } A \in \mathcal{C}^2_{\mathbb{R}}(x^{\pm}, X^{\pm}), \\ 2\mathfrak{s}(A), & \text{if } A \in \mathcal{C}^4_{\mathbb{R}}(x^{\pm}, X^{\pm}). \end{cases} \tag{1.2.4}$$

It may be worth to mention that our initial motivation was to study quadric sections of a real quadric cone $Q \subset \mathbb{P}^3$ that are 6-tangent to a fixed real sextic curve $C \subset \mathbb{P}^3$ and to elaborate for them a count which would have as strong invariance properties as the count of real 3-tangent hyperplane sections established in [FK] as one of the main applications of the relation (1.1.2) (see Section 5). It is from analysis of the underlying wall-crossing phenomena that we came to an idea to combine together $\mathcal{B}^2(X)$ and $\mathcal{B}^4(X)$ and developed the corresponding system of weights. It is in this way that we arrived to Theorem 1.2.2 and proved it initially. Later on we elaborated another, more arithmetic oriented, proof. Below, we give the both proofs with a hope that either of them may help to reveal a general law. At least, it is due to this arithmetic proof that we noticed another system of weights that led us to Theorem 1.2.1, where by such a change of weights and introducing into consideration $\mathcal{B}^0(X)$ we turned to achieve the strong invariance statement for each of the real structures in a Bertini pair separately.

The paper is organized as follows. The arithmetic proof is presented in Sections 2 and 3. Namely, in Section 2 we treat separately a bit more tricky case of maximal and submaximal surfaces, while the other cases are carried out in Section 3. Another proof, via wall-crossing, is discussed in Section 4. In Section 5 we present an application of Theorems 1.2.1-1.2.2 to counting real quadric sections 6-tangent to a real sextic curve on a real quadric cone.

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2. Preliminary count for surfaces with a connected maximal or submaximal real locus

2.1. **Real forms of del Pezzo surfaces of degree 1.** Recall that the real deformation class of any real del Pezzo surface X of degree 1 is determined by the topology of $X_{\mathbb{R}}$. There are 11 deformation classes. The corresponding topological types are shown in the first line of Table 1, where $\mathbb{T}^2$ stands for a 2-torus and $\mathbb{K}$ for a Klein bottle.

The lattice $\Lambda(X) = K_X^{\perp} \cap \ker(1+\mathrm{conj}_*)$ is one of the main deformation invariants which plays a crucial role in the further proofs. These lattices are enumerated in the bottom lines of Table 1.

This table is organized according to the so-called *Smith type* of surfaces, with a code $(M-k)$, which means that in the Smith inequality $\dim H_*(X_{\mathbb{R}};\mathbb{Z}/2) \leqslant \dim H_*(X;\mathbb{Z}/2)$ the right-hand side is greater by $2k$ than the left-hand side. The $(M-2)$-case includes four deformation classes and two of them, encoded with $(M-2)_I$ are *of type I*, which means that the fundamental class of $X_{\mathbb{R}}$ is realizing $w_2(X) = K_X(\mathrm{mod} 2)$ in $H_2(X;\mathbb{Z}/2)$.

Surfaces belonging to the same real Bertini pair have the same Smith type. The real Bertini pairs form 7 pairs of real deformation types. In 3 pairs the deformation types (indicated in the last 3 columns of Table 1) are dual to itself: X^+ is deformation equivalent to X^-. The other 4 pairs are shown in the 4 columns marked M, (M-1), (M-2), and (M-3).

Note also that the lattices $\Lambda(X^+)$ and $\Lambda(X^-)$ are orthogonal complements to each other in $E_8 = K_X^{\perp}$.

Table 1. The root lattices $\Lambda(X) = K_X^{\perp} \cap \ker(1+\mathrm{conj}_*)$

Smith type of $X_{\mathbb{R}}$	M	$(M-1)$	$(M-2)$	$(M-3)$	$(M-4)$	$(M-2)_I$	$(M-2)_I$
Topology of $X_{\mathbb{R}}$	$\mathbb{RP}^2\#4\mathbb{T}^2$	$\mathbb{RP}^2\#3\mathbb{T}^2$	$\mathbb{RP}^2\#2\mathbb{T}^2$	$\mathbb{RP}^2\#\mathbb{T}^2$	$\mathbb{RP}^2$	$\mathbb{RP}^2\perp\mathbb{K}^2$	$(\mathbb{RP}^2\#\mathbb{T}^2)\perp\Sigma^2$
$\Lambda(X)$	E_8	E_7	D_6	D_4+A_1	$4A_1$	D_4	D_4

Smith type of $X_{\mathbb{R}}$	M	$(M-1)$	$(M-2)$	$(M-3)$
Topology of $X_{\mathbb{R}}$	$\mathbb{RP}^2\perp4\Sigma^2$	$\mathbb{RP}^2\perp3\Sigma^2$	$\mathbb{RP}^2\perp2\Sigma^2$	$\mathbb{RP}^2\perp\Sigma^2$
$\Lambda(X)$	0	A_1	$2A_1$	$3A_1$

2.2. **Cremona transformation of Pin-codes.** By a *code of a real blowup model* $X \to \mathbb{P}^2$ of a real del Pezzo $(M-r)$-surface X of degree 1 with r pairs of complex conjugate imaginary exceptional classes $e_{8-2k} = -\mathrm{conj}_* e_{8-2k-1}, 0 \leqslant k \leqslant r-1$, and $8-2r$ real exceptional classes $e_1, \dots, e_{8-2r}$ we mean the sequence $(a_0, \dots, a_{8-2r})$ of residues $\pm 1 \mod 4$, where $a_i = \widehat{\mathrm{q}}_X(e_i)$ for $i \geqslant 1$ and $a_0 = \widehat{\mathrm{q}}_X(h)$ with h staying for the class realized by the pull-back of straight lines. The condition $\widehat{\mathrm{q}}(h) = q_X(w_1(X_{\mathbb{R}})) = 1$ imposes the relation

$$a_0 + \dots + a_{8-2r} = 1 \mod 4.$$

2.2.1. **Lemma.** *An elementary Cremona transformation based on a triple e_i, e_j, e_k with $1 \leqslant i < j < k \leqslant 8-2r$ changes each of the residues a_0, a_i, a_j, a_k by the sum of three others, and does not change a_l for $l \neq 0, i, j, k$. In particular: a sequence a_0, a_i, a_j, a_k formed by $1, 1, 1, 1$ is replaced by $-1, -1, -1, -1$ and vice versa; a sequence $1, 1, -1, -1$ is replaced by $-1, -1, 1, 1$ and vice versa; sequences $1, 1, 1, -1$ and $-1, 1, 1, 1$ are not modified.*

If $r>0$ and we choose a triple e_i, e_7, e_8 (where e_7, e_8 are conjugate imaginary), then the pair a_0, a_i is changed to a_i, a_0 while the other elements of the code are not modified.

Proof. Such transformation changes e_i to $h-e_j-e_k$ and h to $2h-e_i-e_j-e_k$, and the result follows from quadraticity of $\widehat{\mathrm{q}}_X$ and its additivity on pairwise orthogonal elements. □

2.3. **Signed count for connected M-surfaces.**

2.3.1. **Proposition.** *If $X_{\mathbb{R}} = \mathbb{RP}^2\#4\mathbb{T}^2$, then X admits a real blowup model with 8 real blown up points and code $(1,1,\dots,1)$.*

Proof. Let us blow up P^2 first at 4 generic points $p_1,\dots,p_4$ and next make a generic infinitely near blowup over each of the points p_i. The result is a singular del Pezzo surface of degree 1, with 4 nodes. A non-singular del Pezzo surface X obtained by its perturbation can be interpreted as replacing of the 4 infinitely near blow ups by blowing up at points $p_{i+4}\in P^2$ located somewhere in close proximity to p_i, $i=1,\dots,4$. Let $e_i\in H_2(X)$ denote the exceptional classes of blowing up at p_i. In the real setting, our assumption $X_{\mathbb{R}} = \mathbb{RP}^2\#4\mathbb{T}^2$ means that all points $p_i, 1\leqslant i\leqslant 8$, are real.

Note moreover, that $\widehat{\mathrm{q}}_X(e_i)$ and $\widehat{\mathrm{q}}_X(e_{i+4})$ are of opposite signs, since e_i-e_{i+4} is a vanishing class. This implies $\widehat{\mathrm{q}}_X(h)=1$. Lemma 2.2.1 implies that an elementary Cremona transformation based at two negative and one positive classes e_i leads to a real blowup model with totally 3 negative classes e_i. After another transformation based at these three, we obtain a real blowup model with code $(1,1,\dots,1)$, as required.

To extend the result from a particular surface X (constructed above) to any other real del Pezzo surface X' of degree 1 with the real locus $X'_{\mathbb{R}} = \mathbb{RP}^2\#4\mathbb{T}^2$, it is sufficient to use their real deformation equivalence, the invariance of the quadratic function $\widehat{\mathrm{q}}$ under real deformation, and the natural bijection between the set of real (-1)-curves and the set of divisor classes α with $\alpha K=\alpha^2=-1$. □

Starting from here, given a surface X, we use the following notation:

$$R^2(X)=\{e\in K_X^\perp \,|\, e^2=-2\},\quad R^4(X)=\{v\in K_X^\perp \,|\, v^2=-4\}.$$

If X is real then we consider the real part of the above families of divisor classes and put

$$R^a_{\mathbb{R}}(X)=\{x\in R^a(X) \,|\, \mathrm{conj}_*(x)=-x\}.$$

As is well known (see, for example, [FK]), if (X,conj) is a maximal real del Pezzo surface of degree 1 and $X_{\mathbb{R}}$ is connected, then $X_{\mathbb{R}} = \mathbb{RP}^2\#4\mathbb{T}^2$ and $K_X^\perp = \ker(1+\mathrm{conj}_*)$, so that in this case $R^2_{\mathbb{R}}(X)=R^2(X)$ is nothing but the set of roots in $\ker(1+\mathrm{conj}_*)=K_X^\perp\cong E_8$. To enumerate the elements of this set and to determine their $\widehat{\mathrm{q}}$-values, we use the special blowup model given by Proposition 2.3.1, which we call a *positive blowup model.*

A straightforward calculation shows that with respect to a positive blowup model the 240 roots that constitute $R^2_{\mathbb{R}}(X)$ in the case $X_{\mathbb{R}} = \mathbb{RP}^2\#4\mathbb{T}^2$ split into 4 types with corresponding values of $\widehat{\mathrm{q}}$ as shown in Table 2. Each type is characterized there by its level, that is equal up to sign to the coefficient at h in the basic coordinate expansion.

Table 2. Real roots in the case $X_{\mathbb{R}} = \mathbb{RP}^2\#4\mathbb{T}^2$

level	type of roots $e \in R^2_{\mathbb{R}}(X) \cong E_8$	number	$\widehat{\mathrm{q}}(e)$
0	$e_{i_1} - e_{i_2}$	56	2
1	$\pm(h - e_{i_1} - e_{i_2} - e_{i_3})$	$2\binom{8}{3} = 112$	0
2	$\pm(2h - e_{i_1} - e_{i_2} - e_{i_3} - e_{i_4} - e_{i_5} - e_{i_6})$	$2\binom{8}{6} = 56$	2
3	$\pm(3h - 2e_{i_1} - e_{i_2} - e_{i_3} - e_{i_4} - e_{i_5} - e_{i_6} - e_{i_7} - e_{i_8})$	16	0

As a consequence the 240 elements in $\mathcal{B}^2_{\mathbb{R}}(X)$ split into 7 types with corresponding values of $\widehat{\mathrm{q}}$ as shown in Table 3.

Table 3. $\mathcal{B}^2_{\mathbb{R}}$-classes in the case $X_{\mathbb{R}} = \mathbb{RP}^2\#4\mathbb{T}^2$

level	type of classes $e \in \mathcal{B}^2_{\mathbb{R}}(X)$	number	$\widehat{\mathrm{q}}(e)$
3	$3h - e_{i_1} - e_{i_2} - e_{i_3} - e_{i_4} - e_{i_5} - e_{i_6} - e_{i_7}$	8	0
4	$4h - 2e_{i_1} - 2e_{i_2} - e_{i_3} - e_{i_4} - e_{i_5} - e_{i_6} - e_{i_7} - e_{i_8}$	28	2
5	$5h - 2e_{i_1} - 2e_{i_2} - 2e_{i_3} - 2e_{i_4} - 2e_{i_5} - e_{i_6} - e_{i_7} - e_{i_8}$	56	0
6	$6h - 3e_{i_1} - 2e_{i_2} - 2e_{i_3} - 2e_{i_4} - 2e_{i_5} - 2e_{i_6} - 2e_{i_7} - e_{i_8}$	56	2
7	$7h - 3e_{i_1} - 3e_{i_2} - 3e_{i_3} - 2e_{i_4} - 2e_{i_5} - 2e_{i_6} - 2e_{i_7} - 2e_{i_8}$	56	0
8	$8h - 3e_{i_1} - 3e_{i_2} - 3e_{i_3} - 3e_{i_4} - 3e_{i_5} - 3e_{i_6} - e_{i_7} - e_{i_8}$	28	2
9	$9h - 4e_{i_1} - 3e_{i_2} - 3e_{i_3} - 3e_{i_4} - 3e_{i_5} - 3e_{i_6} - 3e_{i_7} - 3e_{i_8}$	8	0

Similarly, the 2160 elements of $\mathcal{B}^4_{\mathbb{R}}(X)$ split into the following 11 subsets with corresponding values of $\widehat{\mathrm{q}}$ as shown in Table 4.

Table 4. $\mathcal{B}^4_{\mathbb{R}}$-classes in the case $X_{\mathbb{R}} = \mathbb{RP}^2\#4\mathbb{T}^2$

level	type of classes $\beta \in \mathcal{B}^4(X)$	number	$\widehat{\mathrm{q}}(\beta)$
1	$h - e_i$	8	2
2	$2h - e_{i_1} - e_{i_2} - e_{i_3} - e_{i_4}$	70	0
3	$3h - 2e_{i_1} - e_{i_2} - e_{i_3} - e_{i_4} - e_{i_5} - e_{i_6}$	168	2
4	$4h - 2e_{i_1} - 2e_{i_2} - 2e_{i_3} - e_{i_4} - e_{i_5} - e_{i_6} - e_{i_7}$	280	0
4	$4h - 3e_{i_1} - e_{i_2} - e_{i_3} - e_{i_4} - e_{i_5} - e_{i_6} - e_{i_7} - e_{i_8}$	8	0
5	$5h - 2e_{i_1} - 2e_{i_2} - 2e_{i_3} - 2e_{i_4} - 2e_{i_5} - 2e_{i_6} - e_{i_7}$	56	2
5	$5h - 3e_{i_1} - 2e_{i_2} - 2e_{i_3} - 2e_{i_4} - e_{i_5} - e_{i_6} - e_{i_7} - e_{i_8}$	280	2
6	$6h - 3e_{i_1} - 3e_{i_2} - 2e_{i_3} - 2e_{i_4} - 2e_{i_5} - 2e_{i_6} - e_{i_7} - e_{i_8}$	420	0
7	$7h - 3e_{i_1} - 3e_{i_2} - 3e_{i_3} - 3e_{i_4} - 2e_{i_5} - 2e_{i_6} - 2e_{i_7} - e_{i_8}$	280	2
7	$7h - 4e_{i_1} - 3e_{i_2} - 2e_{i_3} - 2e_{i_4} - 2e_{i_5} - 2e_{i_6} - 2e_{i_7} - 2e_{i_8}$	56	2
8	$8h - 3e_{i_1} - 3e_{i_2} - 3e_{i_3} - 3e_{i_4} - 3e_{i_5} - 3e_{i_6} - 3e_{i_7} - e_{i_8}$	8	0
8	$8h - 4e_{i_1} - 3e_{i_2} - 3e_{i_3} - 3e_{i_4} - 3e_{i_5} - 2e_{i_6} - 2e_{i_7} - 2e_{i_8}$	280	0
9	$9h - 4e_{i_1} - 4e_{i_2} - 3e_{i_3} - 3e_{i_4} - 3e_{i_5} - 3e_{i_6} - 3e_{i_7} - 2e_{i_8}$	168	2
10	$10h - 4e_{i_1} - 4e_{i_2} - 4e_{i_3} - 4e_{i_4} - 3e_{i_5} - 3e_{i_6} - 3e_{i_7} - 3e_{i_8}$	70	0
11	$11h - 4e_{i_1} - 4e_{i_2} - 4e_{i_3} - 4e_{i_4} - 4e_{i_5} - 4e_{i_6} - 4e_{i_7} - 3e_{i_8}$	8	2

2.3.2. **Proposition.** *If $X_{\mathbb{R}} = \mathbb{RP}^2\#4\mathbb{T}^2$ then*

$$\sum_{\beta \in \mathcal{B}^4_{\mathbb{R}}(X)} i^{\widehat{\mathrm{q}}_X(\beta)} = 2[-8 + 70 - 168 + (280 + 8) - (280 + 56)] + 420$$
$$= -308 + 420 = 112. \qquad \square$$

2.4. **Signed count for connected $(M-1)$-surfaces.**

2.4.1. **Proposition.** *If $X_{\mathbb{R}} = \mathbb{RP}^2\#3\mathbb{T}^2$, then X admits a real blowup model with 6 real blown up points and code $(-1, -1, \ldots, -1)$.*

Proof. Like in the proof of Proposition 2.3.1 we construct a real del Pezzo surface X by blowing up P^2 at three pairs of real points p_i and p_{i+3}, $i = 1, 2, 3$, located sufficiently close to each other in each pair. Then we additionally blow up at a a pair of imaginary complex-conjugate points p_7 and p_8, assuming that the whole configuration of 8 points is generic. For a similar reason, $\widehat{\mathrm{q}}(e_i)$ and $\widehat{\mathrm{q}}(e_{i+3})$ are of opposite signs for each $i = 1, 2, 3$. This implies $\widehat{\mathrm{q}}(h) = 1$, and performing a Cremona transformation based at those 3 points p_i for which $\widehat{\mathrm{q}}(e_i)$ is positive, we obtain a blowup model with code $(-1, -1, \dots, -1)$, as required.

The same deformation arguments as at the end of the proof of Proposition 2.3.1 apply and extend the result from the surface X constructed to any real del Pezzo surface of degree 1 with real locus of the same topological type. $\square$

TABLE 5. $\mathcal{B}^4_{\mathbb{R}}$-classes in the case $X_{\mathbb{R}} = \mathbb{RP}^2 \# 3\mathbb{T}^2$

bi-level	type of classes $\beta \in \mathcal{B}^4(X)$	number	$\widehat{\mathrm{q}}(\beta)$
1,1	$h - e_i$	6	2
0,4	$2h - e_{i_1} - e_{i_2} - e_{i_3} - e_{i_4}$	15	0
0,2	$2h - e_{i_1} - e_{i_2} - e_7 - e_8$	15	2
1,5	$3h - 2e_{i_1} - e_{i_2} - e_{i_3} - e_{i_4} - e_{i_5} - e_{i_6}$	6	2
1,3	$3h - 2e_{i_1} - e_{i_2} - e_{i_3} - e_{i_4} - e_7 - e_8$	60	0
0,2	$4h - 2e_{i_1} - 2e_{i_2} - 2e_{i_3} - e_{i_4} - e_{i_5} - e_7 - e_8$	60	2
0,4	$4h - 2e_{i_1} - e_{i_2} - e_{i_3} - e_{i_4} - e_{i_5} - 2e_7 - 2e_8$	30	0
0,6	$4h - 3e_{i_1} - e_{i_2} - e_{i_3} - e_{i_4} - e_{i_5} - e_{i_6} - e_7 - e_8$	6	2
1,1	$5h - 2e_{i_1} - 2e_{i_2} - 2e_{i_3} - 2e_{i_4} - e_{i_5} - 2e_7 - 2e_8$	30	2
1,3	$5h - 3e_{i_1} - 2e_{i_2} - 2e_{i_3} - 2e_{i_4} - e_{i_5} - e_{i_6} - e_7 - e_8$	60	0
1,5	$5h - 3e_{i_1} - 2e_{i_2} - e_{i_3} - e_{i_4} - e_{i_5} - e_{i_6} - 2e_7 - 2e_8$	30	2
0,2	$6h - 3e_{i_1} - 3e_{i_2} - 2e_{i_3} - 2e_{i_4} - 2e_{i_5} - 2e_{i_6} - e_7 - e_8$	15	2
0,4	$6h - 3e_{i_1} - 3e_{i_2} - 2e_{i_3} - 2e_{i_4} - e_{i_5} - e_{i_6} - 2e_7 - 2e_8$	90	0
0,2	$6h - 2e_{i_1} - 2e_{i_2} - 2e_{i_3} - 2e_{i_4} - e_{i_5} - e_{i_6} - 3e_7 - 3e_8$	15	2
1,5	$7h - 3e_{i_1} - 3e_{i_2} - 3e_{i_3} - 3e_{i_4} - 2e_{i_5} - e_{i_6} - 2e_7 - 2e_8$	30	2
1,3	$7h - 3e_{i_1} - 3e_{i_2} - 2e_{i_3} - 2e_{i_4} - 2e_{i_5} - e_{i_6} - 3e_7 - 3e_8$	60	0
1,1	$7h - 4e_{i_1} - 3e_{i_2} - 2e_{i_3} - 2e_{i_4} - 2e_{i_5} - 2e_{i_6} - 2e_7 - 2e_8$	30	2
0,6	$8h - 3e_{i_1} - 3e_{i_2} - 3e_{i_3} - 3e_{i_4} - 3e_{i_5} - e_{i_6} - 3e_7 - 3e_8$	6	2
0,4	$8h - 4e_{i_1} - 3e_{i_2} - 3e_{i_3} - 3e_{i_4} - 3e_{i_5} - 2e_{i_6} - 2e_7 - 2e_8$	30	0
0,2	$8h - 4e_{i_1} - 3e_{i_2} - 3e_{i_3} - 2e_{i_4} - 2e_{i_5} - 2e_{i_6} - 3e_7 - 3e_8$	60	2
1,3	$9h - 4e_{i_1} - 4e_{i_2} - 3e_{i_3} - 3e_{i_4} - 3e_{i_5} - 2e_{i_6} - 3e_7 - 3e_8$	60	0
1,5	$9h - 3e_{i_1} - 3e_{i_2} - 3e_{i_3} - 3e_{i_4} - 3e_{i_5} - 2e_{i_6} - 4e_7 - 4e_8$	6	2
0,2	$10h - 4e_{i_1} - 4e_{i_2} - 4e_{i_3} - 4e_{i_4} - 3e_{i_5} - 3e_{i_6} - 3e_7 - 3e_8$	15	2
0,4	$10h - 4e_{i_1} - 4e_{i_2} - 3e_{i_3} - 3e_{i_4} - 3e_{i_5} - 3e_{i_6} - 4e_7 - 4e_8$	15	0
1,1	$11h - 4e_{i_1} - 4e_{i_2} - 4e_{i_3} - 4e_{i_4} - 4e_{i_5} - 3e_{i_6} - 4e_7 - 4e_8$	6	2

If (X, conj) is an $(M-1)$-surface and $X_{\mathbb{R}}$ is connected, then $X_{\mathbb{R}} = \mathbb{RP}^2 \# 3\mathbb{T}^2$ and $K_X^{\perp} \cap \ker(1 + \mathrm{conj}_*) \cong E_7$ (see [FK]). To enumerate the elements of $\mathcal{B}^4_{\mathbb{R}}$ and to determine their $\widehat{\mathrm{q}}_X$-values, we use the special blowup model given by Proposition 2.4.1, which we call a *submaximal negative blowup model*.

First of all, we observe that among 2160 classes $v \in K^{\perp} \cong E_8$ with $v^2 = -4$ precisely $\frac{126 \times 60}{10} = 756$ are real, i.e. belong to $R^4_{\mathbb{R}}(X)$. Same calculation as in M-case above shows that the corresponding 756 elements of $\mathcal{B}^4_{\mathbb{R}}(X)$ split into 11 subsets shown in Table 5. In accordance with notation in Proposition 2.4.1, by e_i (with unspecified value of index) there meant the classes of the 6 real exceptional divisors, while e_7, e_8 specify the pair of complex conjugate imaginary ones. Furthermore, each type is accompanied by an indication of its *bi-level*, that is a pair (a, b) where a is the $\mathbb{Z}/2$-residue of the coefficient at h and b the number of classes $e_1, \dots, e_6$

which enter with odd coefficients in the expansion of the element. Due to the choice of the negative blowup model, for $\beta \in \mathcal{B}^4(X)$ of level a, b the value $\widehat{\mathrm{q}}(\beta)$ (shown in Table 5) is equal to $a + b \mod 4$.

2.4.2. **Proposition.** *If $X_{\mathbb{R}} = \mathbb{RP}^2 \# 3\mathbb{T}^2$ then*

$$\sum_{\beta \in \mathcal{B}^4_{\mathbb{R}}(X^+)} i^{\widehat{\mathrm{q}}_{X^+}(\beta)} = 2[-6 + 15 - 15 - 6 + 60 - 60 + 30 - 6 - 30 + 60 - 30]$$
$$-15 + 90 - 15 = 2 \cdot 12 + 60 = 84. \qquad \square$$

3. Proof of Theorems 1.2.1 and 1.2.2

3.1. Signed count of curves in $\mathcal{C}^2_{\mathbb{R}}(x, X)$.

3.1.1. **Proposition.** *For every Bertini pair of real del Pezzo surfaces $X^{\pm}$ of degree 1, and any pair of real generic points $x^{\pm} \in X^{\pm}_{\mathbb{R}}$, we have*

$$\sum_{A \in \mathcal{C}^2(x^+, X^+)} \mathfrak{s}(A) = 2(r^- - r^+)r^+, \qquad \sum_{A \in \mathcal{C}^2(x^-, X^-)} \mathfrak{s}(A) = 2(r^+ - r^-)r^-,$$
$$\sum_{A \in \mathcal{C}^2(x^+, X^+) \cup \mathcal{C}^2(x^-, X^-)} \mathfrak{s}(A) = -\,2(r^+ - r^-)^2.$$

where $r^{\pm} = \mathrm{rk}(K^{\perp}_{X^{\pm}} \cap \ker(1 + \mathrm{conj}^{\pm}_*))$.

Proof. Any class $\alpha = -2K - e \in \mathcal{B}^2_{\mathbb{R}}(X^{\pm})$ gives a real net of elliptic curves. By fixing a generic base point $x \in X^{\pm}_{\mathbb{R}}$, we obtain a real pencil, whose other base point $x' \neq x$ has to be real. Singular curves from this pencil are irreducible (and thus, rational, with one node), except one curve which has to be real and gives a splitting $\alpha = (-K) + (-K - e)$. The first, *anticanonical*, component is real elliptic, passing through x, and the second component is a real line. After blowing up the points x, x' we obtain a real fibration $X^{\pm}_{\mathbb{R}} \# 2\mathbb{RP}^2 \to \mathbb{RP}^1$, so that counting the Euler characteristic of $X^{\pm}_{\mathbb{R}} \# 2\mathbb{RP}^2$ by means of this fibration we get the relation

$$-1 + \sum_{A \in \mathcal{C}^2_{\mathbb{R}}(\alpha, x, X^{\pm})} w(A) = \chi(X^{\pm}_{\mathbb{R}}) - 2 = r^{\mp} - r^{\pm} \tag{3.1.1}$$

where -1 is the Euler characteristic of the reducible fiber and $w(A)$ stands for $\chi(A_{\mathbb{R}}) = s_A - c_A$.

On the other hand, $\sum_{\alpha \in \mathcal{B}^2_{\mathbb{R}}(X^{\pm})} i^{\widehat{\mathrm{q}}_{X^{\pm}}(\alpha)} = 2r^{\pm}$ due to Proposition 3.4.5 in [FK]. Since $\mathfrak{s}(A) = i^{\widehat{\mathrm{q}}_{X^{\pm}}(\alpha)} w(A)$, we conclude that

$$\sum_{A \in \mathcal{C}^2(x, X^{\pm})} \mathfrak{s}(A) = \sum_{\alpha \in \mathcal{B}^2_{\mathbb{R}}(X^{\pm})} i^{\widehat{\mathrm{q}}_{X^{\pm}}(\alpha)} \sum_{A \in \mathcal{C}^2_{\mathbb{R}}(\alpha, x, X^{\pm})} w(A) = 2r^{\pm}(r^{\mp} - r^{\pm}).$$

The third identity in the statement is nothing but the sum of the first two. $\square$

3.2. **On deformation invariance of partial counts.** As it could be already observed in the proof of Proposition 3.1.1, the value of a sum

$$\sum_{A \in \mathcal{C}^2_{\mathbb{R}}(\alpha, x, X^{\pm})} i^{\widehat{\mathrm{q}}_{X^{\pm}}(\alpha)} w(A)$$

does not depend on a choice of the point x, and moreover is invariant under real deformations of $X^{\pm}$. The same property holds for $\sum_{A \in \mathcal{C}^{2k}_{\mathbb{R}}(\alpha, x, X^{\pm})} i^{\widehat{\mathrm{q}}_{X^{\pm}}(\alpha)} w(A)$

with $k=0$ and 2. In fact, such an invariance property holds in much more general situation.

3.2.1. **Proposition.** *For any $\beta\in H_2(X)$ each sum of the form*

$$\sum_A f([\beta]^2, K_X\beta, \widehat{\mathrm{q}}(\beta))w(A)$$

where f is any function $\mathbb{Z}\times\mathbb{Z}\times\mathbb{Z}/4\to\mathbb{R}$ and the sum is taken over all real rational curves A belonging to the divisor class β and passing through a fixed real collection of $-K_X\beta-1$ points $\mathbf{x}\subset\mathbf{X}$ depends only on β, f and the number of real points in $\mathbf{x}$. Such a sum is also invariant under real deformations of X.

Proof. Follows from the real deformation invariance of $\widehat{\mathrm{q}}$, see Theorem 1.1.1, and the real deformation invariance, as well as the stated in Proposition independence on $\mathbf{x}$, of the sum $\sum_A w(A)$ taken, as in the statement, over all real rational curves A belonging to the divisor class β and passing through a fixed real collection of $-K_X\beta-1$ points (for a proof of the latter invariance, see [Br-2]). □

3.3. **Signed count of curves in $\mathcal{C}^4_{\mathbb{R}}(x,X)$ for $X_{\mathbb{R}}$ other than $\mathbb{RP}^2\#4\mathbb{T}^2$ or $\mathbb{RP}^2\#3\mathbb{T}^2$.** In the case-by-case analysis of each of the lattices

$$\Lambda(X^{\pm})=K_X^{\perp}\cap\ker(1+\mathrm{conj}_*)$$

from Table 1 we use such a root basis on which $\widehat{\mathrm{q}}_X$ is vanishing identically. For existence of such a basis, see [FK, Lemma 3.1.2].

3.3.1. **Cases $X_{\mathbb{R}}=\mathbb{RP}^2\perp kS^2$.** If $X_{\mathbb{R}}=\mathbb{RP}^2\perp kS^2$, $0\leqslant k\leqslant 4$, then $\Lambda(X)$ is isomorphic to $(4-k)A_1$. Since each (-4)-vector in $(4-k)A_1$ splits into a sum of generators of a pair of A_1-summands, the number of such vectors is $4\binom{4-k}{2}$ which is 0 for k equal 3 and 4. Each (-4)-vector yields a unique curve $A\in\mathcal{C}^4_{\mathbb{R}}(x,X)$ and for them the value $\mathfrak{s}(A)=i^{\widehat{\mathrm{q}}_X(v)}$ is 1 since $\widehat{\mathrm{q}}_X=0$ on each A_1-summand. Thus, we just count the number $\mathrm{Card}(R^4_{\mathbb{R}})$ of (-4)-vectors in $\Lambda(X)$:

$$\sum_{A\in\mathcal{C}^4_{\mathbb{R}}(x,X)}\mathfrak{s}(A)=\mathrm{Card}(R^4_{\mathbb{R}})=4\binom{4-k}{2}.$$

3.3.2. **Cases $X_{\mathbb{R}}=\mathbb{RP}^2\perp\mathbb{K}^2$ and $X_{\mathbb{R}}=(\mathbb{RP}^2\#\mathbb{T}^2)\perp S^2$.** According to Table 1, if $X^+_{\mathbb{R}}=\mathbb{RP}^2\perp\mathbb{K}^2$ (resp. $X^+_{\mathbb{R}}=(\mathbb{RP}^2\#\mathbb{T}^2)\perp S^2$) then $X^-_{\mathbb{R}}=\mathbb{RP}^2\perp\mathbb{K}^2$ (resp. $X^-_{\mathbb{R}}=(\mathbb{RP}^2\#\mathbb{T}^2)\perp S^2$) too, and in all the cases both $\Lambda(X^{\pm})$ are isomorphic to D_4. Note that D_4 can be seen as a sublattice of $4\langle-1\rangle$ generated by the roots $e_0=(1,1,0,0), e_1=(1,-1,0,0), e_2=(0,1,-1,0)$ and $e_3=(0,0,1,-1)$. With respect to this presentation, the (-4)-vectors split into 2 kinds: sixteen vectors $(\pm1,\pm1,\pm1,\pm1)$ and eight vectors with one coordinate ±2 and the others 0. Thus, the vanishing of $\widehat{\mathrm{q}}_X$ on $e_0,\dots,e_3$ implies its vanishing on all the (-4)-vectors in D_4. Indeed, it has then to vanish on e_1+e_3 and thus on all the sixteen first-kind (-4)-vectors (as they are congruent modulo $2D_4$ to each other), while vanishing of $\widehat{\mathrm{q}}_X$ on the eight second-kind (-4)-vectors follows from their presentation as mod 2-orthogonal sum of the first-kind vectors. For each of $X=X^{\pm}$, this gives

$$\sum_{A\in\mathcal{C}^4_{\mathbb{R}}(x,X)}\mathfrak{s}(A)=\sum_{v\in R^4_{\mathbb{R}}}i^{\widehat{\mathrm{q}}_{X^{\pm}}(v)}=\mathrm{Card}(R^4_{\mathbb{R}})=16+8=24.$$

3.3.3. **Cases $X_{\mathbb R}=\mathbb{RP}^2\#\mathbb T^2$ and $X_{\mathbb R}=\mathbb{RP}^2\perp S^2$.** They form a Bertini pair: if $X^+=\mathbb{RP}^2\#\mathbb T^2$ then $X^-=\mathbb{RP}^2\perp S^2$, and vice versa. According to Table 1, for such a pair, $\Lambda(X^+)$ is $D_4\oplus A_1$ and $\Lambda(X^-)$ is $3A_1$. A (-4)-vector in $D_4\oplus A_1$ is either one of the (-4)-vectors of D_4 (24 choices), or a sum of one root in D_4 (24 choices) with one root in A_1 (2 choices). On the (-4)-vectors of the first-kind the form $\widehat{\mathrm q}_X$ vanishes, like in the previous case. On the latter sums we have $\widehat{\mathrm q}_X(v_1+v_2)=\widehat{\mathrm q}_X(v_1)$, and in accordance with [FK] the signed count of the 2-roots v_1 gives $2\operatorname{rk}|D_4|=8$, which after that should be multiplied by 2 because of two choices of v_2 in A_1. Thus,

$$\sum_{A\in\mathcal C^4_{\mathbb R}(x,X)}\mathfrak s(A)=\sum_{v\in R^4_{\mathbb R}} i^{\widehat{\mathrm q}_X(v)}=\begin{cases}24+16=40, & \text{for } \Lambda(X)=D_4\oplus A_1,\\ 4\binom{3}{2}=12, & \text{for } \Lambda(X)=A_3.\end{cases}$$

3.3.4. **Cases $X_{\mathbb R}=\mathbb{RP}^2\#2\mathbb T^2$ and $\mathbb{RP}^2\perp 2S^2$.** They also form a Bertini pair: if $X^+=\mathbb{RP}^2\#2\mathbb T^2$ then $X^-=\mathbb{RP}^2\perp 2S^2$, and vice versa. According to Table 1, for such a pair, $\Lambda(X^+)$ is isomorphic to D_6 and $\Lambda(X^-)$ to $2A_1$. The lattice D_6 can be seen as a sublattice of $6\langle -1\rangle$ generated by the following roots $\boxed{\begin{matrix}-1&-1&0&0&0&0\\1&-1&0&0&0&0\\0&1&-1&0&0&0\\0&0&1&-1&0&0\\0&0&0&1&-1&0\\0&0&0&0&1&-1\end{matrix}}$

The vanishing of $\widehat{\mathrm q}_{X^+}$ on these basic roots implies immediately its vanishing on the twelve (-4)-vectors that contain ±2 as one coordinate and 0 as others. The rest (-4)-vectors are obtained from $(\pm1,\pm1,\pm1,\pm1,0,0)$ by permutation of coordinates (totally $2^4\binom{6}{2}$). The $\binom{6}{2}$ permutations interpreted as partitions $n_1+n_2+n_3=4$ have either all n_i even (6 cases), in which case $\widehat{\mathrm q}_X=0$, of give two odd summands n_i (9 cases), in which case $\widehat{\mathrm q}_X=2$.

For $\Lambda(X)=D_6$, this yields

$$\sum_{A\in\mathcal C^4_{\mathbb R}(x,X)}\mathfrak s(A)=\sum_{v\in R^4_{\mathbb R}} i^{\widehat{\mathrm q}_X(v)}=12+16(9-6)=60.$$

For $\Lambda(X)=2A_1$, we have $\sum_{A\in\mathcal C^4_{\mathbb R}(x,X)}\mathfrak s(A)=\sum_{v\in R^4_{\mathbb R}} i^{\widehat{\mathrm q}_X(v)}=4$.

3.4. **Proof of Theorems 1.2.2 and 1.2.1.** The results obtained above are summarized in Table 6 which is organized by columns according to Smith types of Bertini pairs and where the rows show the result of the signed count over $A\in\mathcal C^{2k}(x,X)$ for $X=X^\pm$ in each Bertini pair. For the first 4 columns our convention is that X^+ refers to surfaces with connected real locus (see Table 1).

For $k=1$ calculations are given by Proposition 3.1.1, for $k=2$ they are taken from Propositions 2.3.2, 2.4.2 and previous Subsection. For $k=0$ calculations are due to [Br-2] (note that the original Welschinger weight used in [Br-2] coincides with our $\mathfrak s$-weight in the case of curves $A\in\mathcal C^0(x,X)$, since their arithmetic genus $g_A=2$ is even, and thus c_A and s_A have the same pairity).

Adding the 3 terms $\sum_{A\in\mathcal C^0(x,X)}\mathfrak s(A)+\sum_{A\in\mathcal C^2(x,X)}\mathfrak s(A)+\sum_{A\in\mathcal C^4(x,X)}\mathfrak s(A)$ for each type of $X=X^\pm$ we obtain Theorem 1.2.1.

As we take the sum $\sum_{A\in\mathcal C^2(x,X)}\mathfrak s(A)+2\sum_{A\in\mathcal C^4(x,X)}\mathfrak s(A)$ for $X=X^+$ and add it with the same sum for $X=X^-$, we obtain Theorem 1.2.2.

TABLE 6. $\sum_{A\in\mathcal{C}^2(x,X)}\mathfrak{s}(A)$ for each type of Bertini pairs $X^{\pm}$.

	M	$M-1$	$M-2$	$M-3$	$M-4$	$(M-2)_I$
$A\in\mathcal{C}^2(x^+,X^+)$	-128	-84	-48	-20	0	0
$A\in\mathcal{C}^2(x^-,X^-)$	0	12	16	12	0	0
$A\in\mathcal{C}^4(x^+,X^+)$	112	84	60	40	24	24
$A\in\mathcal{C}^4(x^-,X^-)$	0	0	4	12	24	24
$A\in\mathcal{C}^0(x^+,X^+)$	46	30	18	10	6	6
$A\in\mathcal{C}^0(x^-,X^-)$	30	18	10	6	6	6

4. Wall crossing

Richtiges Auffassen einer Sache und Miverstehen der gleichen Sache schlieen einander nicht vollstndig aus.

F. Kafka, "Der Proze. Kapitel 9: Im Dom"

Here, we present a proof of Theorems 1.2.1 and 1.2.2 based on a real version of Abramovich-Bertram-Vakil wall-crossing formula [Va, Theorem 4.2]. More precisely, it is based on the underlying it gluing procedure as it is presented in [IKS, Proposition 4.1] and [Br-P, Theorem 2.5].

4.1. **A special choice of walls.** As is known, the bi-anticanonical map establishes an isomorphism between the moduli space of real del Pezzo surfaces X of degree 1 and that of real non-singular sextics C on the real quadric cone Q based on a non-empty real conic. In particular, such an isomorphism allows not only to identify the real deformation classes of these objects but also to visualize nodal degenerations of the former with nodal degenerations of the latter.

To be precise, let us define by a (simple) *nodal degeneration* of sextics on Q a complex analytic family of curves $C(t), t\in\mathbb{C}, |t|<1$ such that $C(t)$ with $t\neq 0$ are non-singular sextics while $C(0)$ is uninodal. We call it *Morse-Lefschetz* if the total space of the family is smooth (in other words, "wall-intersection" is transverse), and *real* if it is equipped with an antiholomorphic involution mapping $C(t)$ to $C(\bar{t})$ for each t. Starting from a nodal degeneration of sextics, $\{C(t), t\in\mathbb{C}, |t|<1\}$, and taking the double coverings $X(t)\to Q$ branched in $C(t)$ we obtain a complex analytic family of surfaces $X(t), t\in\mathbb{C}, |t|<1$ such that $X(t)$ with $t\neq 0$ are del Pezzo surfaces of degree 1 while $X(0)$ is a uninodal surface. It is this kind of families that we call *nodal degenerations of del Pezzo surfaces of degree 1.* Such a degeneration will be called Morse-Lefschetz, if the family $C(t)$ is Morse-Lefschetz. Note also that if the family $C(t)$ is real, then the real structure $\mathrm{conj}: Q\to Q$ lifts to two, Bertini dual, real structures on the total space of the family $X(t)$ so that, in particular, for each real t the surface $X(t)$ acquires a pair of Bertini dual real structures. If $X(t)$ is a real Morse-Lefschetz family, then for each of the Bertini dual structures the real loci of $X(t)$ (as well as the real loci of $C(t)$) with t real experience a Morse transformation when t is crossing 0. We say that such a degeneration *avoids contracting a real spherical component* if for both directions of crossing 0 and for each of the Bertini dual structures the Morse transformation $X_{\mathbb{R}}(t)\to X_{\mathbb{R}}(-t)$ is not contracting a real spherical component of $X_{\mathbb{R}}(t)$.

4.1.1. **Proposition.** *Any pair of real del Pezzo surfaces of degree 1 can be connected by a finite sequence of real Morse-Lefschetz nodal degenerations that avoid contracting of a real spherical component.*

Proof. As it follows from definitions, a real Morse-Lefschetz family $X(t)$ is not contracting a real spherical component if and only if the underlying real Morse-Lefschetz family $C(t)$ is not contracting a real oval. Thus, it remains to check that any two real sextics $C(t)$ can be connected by a finite sequence of real Morse-Lefschetz nodal degenerations that avoid contracting a real oval. Such degenerations can be found, for example, on Figure 1 in [FK]. $\square$

4.2. **Enumerating of limit splittings.** Consider an arbitrary Morse-Lefschetz family $X(t), t \in \mathbb{C}, |t| < 1$ of del Pezzo surfaces of degree 1. To kill the monodromy, introduce, in addition, an *associated untwisted family*, $X'(\tau), \tau \in \mathbb{C}, |\tau| < 1$, induced from the given one by degree 2 base change $t = \tau^2$. The total space $X' = \cup X'(\tau)$ of the untwisted family acquires a node at the nodal point of $X'(0) = X(0)$. We make X' non-singular by blowing up this node and get a new family $\tilde{X}(\tau), \tau \in \mathbb{C}, |\tau| < 1$ with $\tilde{X}(\tau) = X'(\tau)$ for $\tau \neq 0$ while $\tilde{X}(0)$ is reduced and consists of two irreducible components with normal crossing: one component, which we denote $\tilde{X}^1(0)$, is the minimal nonsingular model of $X'(0)$, the other one, denoted $\tilde{X}^0(0)$, is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$, and they intersect along a nonsingular rational curve E which is the blown in (-2)-curve in $\tilde{X}^1(0)$ and which can be seen in $\tilde{X}^0(0) = \mathbb{P}^1 \times \mathbb{P}^1$ as the diagonal.

Contracting $\tilde{X}^0(0)$ to E along the lines of one of the rulings, we get a smooth family of smooth surfaces with $\tilde{X}^1(0)$ as central fiber (see [A] for details). A choice of such a contraction provides then natural isomorphisms

$$\mathrm{Pic}(X(t)) = H_2(X(t)) \xleftarrow{\simeq} H_2(\tilde{X}(\tau)) \xrightarrow{\simeq} H_2(\tilde{X}^1(0)) = \mathrm{Pic}(\tilde{X}^1(0)), \quad t = \tau^2, \tau \neq 0$$

preserving the intersection form. This allows us to use, for the sake of brevity, the same symbol for all corresponding divisor and homology classes (with a precaution that the composed map $H_2(X(t)) \xleftarrow{\simeq} H_2(\tilde{X}(\tau)) \xrightarrow{\simeq} H_2(\tilde{X}^1(0)) \xleftarrow{\simeq} H_2(\tilde{X}(-\tau)) \xrightarrow{\simeq} H_2(X(t))$ is not identity but the Dehn twist $x \mapsto x + ([E] \circ x)[E]$).

In accordance with the Abramovich-Bertram-Vakil approach, we pick a generic section $t \in \mathbb{C}, |t| < 1 \mapsto x(t) \in X(t)$ and lift each of the involved into our count family of curves $A(t) \in \mathcal{C}^2(x(t), X(t)) \cup \mathcal{C}^4(x(t), X(t))$ in $X(t)$ with $t \neq 0$ up to a family $\tilde{A}(\tau)$ in $\tilde{X}(\tau)$ with $\tau \neq 0$ where we put $\tilde{A}(\tau) = A(\tau^2)$ using $\tilde{X}(\tau) \equiv X(\tau^2)$. As a first step in the proof of Theorem 1.2.2, we enumerate below the possible types for splittings $\tilde{A}(0) = D_0 + rE$ with $r > 0$ that can appear in the limit $\tilde{A}(0) = \lim \tilde{A}(\tau) \subset \tilde{X}^1(0)$.

4.2.1. **Proposition.** *For $A(t) \in \mathcal{C}^2(x(t), X(t))$ (so that $[A(t)] = -2K - e$ with $e^2 = -2, eK = 0$) there are no splittings with $r > 2$. If $r = 1$ then either $e \cdot [E] = 1$ or $e \cdot [E] = 0$. Furthermore, the following holds:*

(1) *If $r = 1$ and $e \cdot [E] = 1$, then $[D_0] \in -2K - e_0, e_0^2 = -2, e_0K = 0$, $[D_0]^2 = 2$ and $[D_0] \cdot [E] = 1$.*
(2) *If $r = 1$ and $e \cdot [E] = 0$, then $[D_0] \in -2K - v$, $v^2 = -4$, $vK = 0$, $v = e + [E]$, $[D_0]^2 = 0$ and $D_0 \cdot E = 2$.*
(3) *If $r = 2$ then $e = -[E]$, $[D_0] \in -2K - [E]$, $[D_0]^2 = 2$, and $D_0 \cdot E = 2$.*

Proof. The bound $r \leqslant 2$ follows from $e \cdot [E] \leqslant 2$ (general property of roots in a root lattice) combined with $2 + e \cdot [E] - 2r = [D_0] \cdot (-K - [E]) \geqslant 0$ (this positivity of intersection holds, since $-K - [E]$ is represented by an effective divisor lying over a generator of the cone through the node, and by this reason is not contained in D_0 which pass, by definition, through a fixed generic point). Furthermore, by the same argument, if $r = 2$ then $e \cdot [E] = 2$ and thus $e = -[E]$, while if $r = 1$ then $e \cdot [E]$ equals 1 or 0. The rest of the statement is a straightforward lattice arithmetic. □

4.2.2. **Proposition.** *For $A(t) \in \mathcal{C}^4(x(t), X(t))$ (so that $[A(t)] = -2K - v$, with $v^2 = -4, vK = 0$), there are no splittings with $r > 2$. Furthermore, the following holds:*

1. *If $r = 1$ then $v \cdot [E] = 1$, $[D_0] \in -2K - v_0$, $v_0 = v + [E]$, $v_0^2 = -4$, $[D_0]^2 = 0$, and $[D_0] \cdot [E] = 1$.*
2. *If $r = 2$ then $v \cdot [E] = 2$, $[D_0] \in -2K - v_0$, $v_0^2 = -4$, $v_0 = v + 2[E]$, $[D_0]^2 = 0$, and $[D_0] \cdot [E] = 2$.*

Proof. As is known (and easy to show), each vector $v \in E_8$ with $v^2 = -4$ can be represented as a sum of two orthogonal roots. This implies that $v \cdot [E] \leqslant 2$. So, by the same arguments as in the proof of Proposition 4.2.1 we conclude that $r \leqslant 2$ and that $r = 2$ may hold only if $v \cdot [E] = 2$. The rest of the statement is a straightforward lattice arithmetic. □

4.3. **Alternative proof of Theorem 1.2.2.** Due to Proposition 3.2.1, to prove Theorem 1.2.2 it is sufficient to prove the invariance of our count under the *wall-crossing*, *i.e.* in real Morse-Lefschetz families of real del Pezzo surfaces of degree 1, $X(t), t \in \mathbb{C}, |t| < 1$ with $\operatorname{conj}(t) : X(t) \to X(\bar{t})$. To shorten case by case considerations we consider only nodal degenerations with $E_{\mathbb{R}} \neq \varnothing$ on both $X^+(0)$ and $X^-(0)$, which is sufficient due to Proposition 4.1.1.

The wall-crossings are enumerated in Propositions 4.2.1 and 4.2.2, and in each case the groups of curves $A(t)$ that are involved into the count share in the limit $A(0) = D_0 + rE$ with $r \geqslant 0$ the same divisor D_0. For each group we calculate "a loss" and "a gain" happening under crossing a wall not only for the surfaces $X^+(t) = X(t), t \in \mathbb{R}$, but also for their Bertini duals $X^-(t)$.

To be able to apply the Abramovich-Bertram-Vakil gluing procedure and to be in accordance with the setting in Section 4.2, we always choose the real coordinate t in the Morse-Lefschetz family under consideration so, that the Euler charateristic of $X^+_{\mathbb{R}}(t) = X_{\mathbb{R}}(t)$ is smaller for $t > 0$ than for $t < 0$. With this choice the both rulings of $\tilde{X}^0(0)$ are real, and by this reason the real structure descends from the family $\tilde{X}(\tau)$ to the family obtained by contraction of any of the two rulings. When passing from $X^+(t)$ to the Bertini dual family $X^-(t)$, we do the same, only the direction is changing: due to $\chi(X^+_{\mathbb{R}}(t)) + \chi(X^-_{\mathbb{R}}(t)) = 2$, the Euler characteristic of $X^-_{\mathbb{R}}(t)$ is smaller for $t < 0$ than for $t > 0$.

4.3.1. **Wall-crossing 4.2.1(1) and 4.2.2(1).** These are the cases with $D_0 \cdot E = 1$. Here, we apply the Abramovich-Bertram-Vakil gluing procedure and observe, for $t > 0$, two real curves $A'(t,)A''(t)$ with $[A''_{\mathbb{R}}(t)] = [A'_{\mathbb{R}}(t)] + [E_{\mathbb{R}}]$ that merge together in the limit. They are both of type $-2K - e \in \mathcal{B}^2_{\mathbb{R}}(X)$ in the case 4.2.1(1), and of type $-2K - v \in \mathcal{B}^4_{\mathbb{R}}(X)$ in the case 4.2.2(1). Furthermore, these curves have the same number of non-solitary nodes and $q_X([A''_{\mathbb{R}}(t)]) = q_X([A'_{\mathbb{R}}(t)] + [E]) = q_X([A'_{\mathbb{R}}(t)]) + q_X([E]) + 2 = q_X([A'_{\mathbb{R}}(t)]) + 2$. Hence, their common input is zero.

Since neither of $[A'(t)], [A''(t)]$ is orthogonal to $[E]$, it follows that on the side $t < 0$ there are no real curves in these classes at all. Thus, in these cases there is no "loss" or "gain" when comparing the counts for $X(t)$ and $X(-t)$. Besides, the same arguments apply to the Bertini duals of $X(-t)$ and $X(t)$.

4.3.2. **Wall-crossing 4.2.1(2) and 4.2.2(2).** These are the cases with $D_0 \cdot E = 2$ and $D_0^2 = 0$. Being combined together, they provide 3 groups of families of curves sharing the same divisor D_0 in the limit; they correspond to divisor classes of type $-2K - e - [E], -2K - e, -2K - e + [E]$. Here, using the assumption $E_{\mathbb{R}} \neq \varnothing$, we take the base point $x(0) \in X_{\mathbb{R}}(0)$o close to a generic point of $E_{\mathbb{R}}$, which insures that $D_0 \in -2K - e - [E]$ intersects E at 2 real points, and observe, for $t > 0$, two real curves $A'(t), A''(t)$ of type $-2K - e$ that merge together and split both into $D_0 \sqcup E$ while D_0 being a non-multiple limit of real curves $D(t)$ of type $-2K - e - [E]$. Here, $[A'(t)] = [A''(t)] = [D_0] + [E]$ while the number of cross-points in $A'_{\mathbb{R}}(t,)A''_{\mathbb{R}}(t)$ is by 1 greater that the number of cross-points in $[(D_0)_{\mathbb{R}}]$. Therefore, the common input of $A'_{\mathbb{R}}(t), A''_{\mathbb{R}}(t)$ is opposite to the input of $D(t)$, as it follows from the definitions of weights, (1.2.2) and (1.2.4). The term corresponding to $-2K - e + [E]$ gives an equal (also non-multiple) input as that of $D(t)$. Thus, the total input of the 3 groups under consideration is equal to $2 \sum i^{\widehat{\mathrm{q}}(e)}$ where the sum is taken over the roots $e \in \Lambda(X^+(t))$. This gives a "loss" equal to $4\big(\operatorname{rk}(K^\perp \cap \ker(1 + \operatorname{conj}_*(t))) - 1\big)$ taken for $\operatorname{conj}(t) : X(t) \to X(t)$ with $t > 0$ (*cf.* [FK, Proposition 3.4.5]). There are no real curves in divisor classes $-2K - e - [E], -2K - e, -2K - e + [E]$ on $X(-t)$ (for $-2K - e$ it holds, since with our choice of $x(0)$ on $X_{\mathbb{R}}(0)$ all the counted rational curves in $X_{\mathbb{R}}(0)$ of class $-2K - e$ intersect $E_{\mathbb{R}}$ at real points, *cf.* [Br-P, Theorem 2.5]). The same arguments applied to the Bertini duals $X^-(-t)$ and $X^-(t)$ of $X^+(-t) = X(-t)$ and $X^+(t) = X(t)$ show that from the Bertini duals we have a "gain" equal to $4\big(\operatorname{rk}(K^\perp \cap \ker(1 + \operatorname{conj}^-_*(-t))) - 1\big) = 4 \operatorname{rk}(K^\perp \cap \ker(1 - \operatorname{conj}_*(t))) = 4 \operatorname{rk} \ker(1 - \operatorname{conj}_*(t))$ where $\operatorname{conj}^-(-t) : X^-(-t) \to X^-(-t)$ stands for the Bertini dual real structure $\operatorname{conj}^-(-t) = \operatorname{conj}(-t) \circ \tau_X$. Therefrom, and in accordance with Lefschetz trace formula applied to $\operatorname{conj}(t)$, the deficiency (loss minus gain) is equal to $-4\chi(X_{\mathbb{R}}(t))$ with $t > 0$.

4.3.3. **Wall-crossing 4.2.1(3).** In this case there are, on $X(t)$ with $t > 0$, some number of curves of type $-2K + [E]$, and hence all with value 0 of $\widehat{\mathrm{q}}$, that merge to the same curve of type $-2K - [E]$. The input of Welschinger factors for all these curves together can be counted in a similar way as in (3.1.1) in the proof of Proposition 3.1.1, which gives a signed "loss" equal to $2 \times (\chi(X_{\mathbb{R}}(t)) - 1)$. A signed "gain" from the side $t < 0$ is, similarly, $2 \times (\chi(X^-_{\mathbb{R}}(-t)) - 1)$ where $X^-(-t)$ is the Bertini pair for $X^+(-t) = X(-t)$. Therefrom, in this case the deficiency of inputs is equal to $2 \times (\chi(X_{\mathbb{R}}(t)) - 1) - 2 \times (\chi(X^-_{\mathbb{R}}(-t)) - 1) = 4\chi(X_{\mathbb{R}}(t))$ with $t > 0$.

Thus the total, summarizing all the cases, deficiency is zero, and we conclude that the total count is preserved. □

4.4. **Alternative proof of Theorem 1.2.1.** We proceed as above in "Alternative proof" of Theorem 1.2.2 (now, without appealing to the Bertini dual surfaces). In the cases 4.2.1(1) and 4.2.2(1) the arguments remain literally the same. The cases 4.2.1(2) and 4.2.2(2), being combined together and treated as in that proof, give now the total input equal to 0 from $t > 0$ side just because the two families of curves of types $-2K - e - [E], -2K - e + [E]$ give an input of the opposite sign and the same absolute value (equal to 2) as the two families of curves of (intermediate) type

$-2K - e$. As in the above proof, in these cases there are no input from the opposite side. It is in the, only remaining, case 4.2.1(3) that curves of type $-2K$ show them up. Here, in accordance with Abramovich-Bertram-Vakil formula we need to treat curve families sharing in the limit a common divisor D_0 of type $-2K - E$. If the both points where D_0 meets E are real, then due to Abramovich-Vakil-Bertram gluing procedure we have here four families of curves, one of type $-2K - E$, two of type $-2K$, and two of type $-2K + E$ that have D_0 as the limit. The first and the last one are of opposite sign and the same absolute value (equal to 1) as each of the two of type $-2K$. Hence, their common input is 0. And, since there is no input on the opposite ($t < 0$) side, we are done in the case of real points. If the points where D_0 meets E are imaginary (complex conjugate), then there are no real curves of type $-2K$ converging to D_0 (this time there are instead two imaginary complex conjugate families converging to D_0 from $t > 0$ side) but by contrary, due to Brugallé-Puignau surgery description, there are two real families of curves of type $-2K$ that converge to D_0 from $t < 0$ side). Furthermore, these curves acquire an additional, with respect to D_0, node and this node is solitary. Hence, they of the same sign as the two curves merging to D_0 from $t > 0$ (that we have treated before). Thus, here, we have the same input (equal to ± 2) from both sides, which finishes the proof of invariance. $\square$

5. Count of quartics 6-tangent to a sextic on a quadratic cone

Here, by a *quartic 6-tangent to a sextic* $C \subset Q$ we mean a transverse intersection $A = Z \cap Q$ of Q with a quadric Z such that the intersection divisor $Z \cdot C = A \cdot C$ contains each point with even multiplicity. Let us denote by $T(Q, C, z)$ the set of rational irreducible reduced 6-tangent quartics that pass through a fixed point $z \in Q \setminus C$. For a generic $z \in Q \setminus C$, it contains, as is well-known, 2400 elements, and for $z = \pi(x), x \in X$ (where as usual X stands for the del Pezzo surface associated with C), the projection $\mathcal{C}(x, X) \to T(Q, C, z)$ induced by the double covering $\pi : X \to Q$ is a bijection.

Over $\mathbb{R}$, with each real point $z^\pm \in Q^\pm_\mathbb{R}$ we associate a set of real 6-tangent quartics, $T_\mathbb{R}(Q^\pm_\mathbb{R}, C, z^\pm) = \{A \text{ is real}, A \in T(Q, C, z^\pm)\}$. For each $A \in T_\mathbb{R}(Q^\pm_\mathbb{R}, C, z^\pm)$, the real locus $A_\mathbb{R}$ of A lies entirely in the same half $Q^\pm_\mathbb{R}$ of $Q_\mathbb{R} \setminus C_\mathbb{R}$ as $z^\pm$, and thus the curve A lifts to a pair of real rational curves $A_k \in \mathcal{C}^2_\mathbb{R}(x_k, X^\pm) \cup \mathcal{C}^4_\mathbb{R}(x_k, X^\pm)$, $k = 1, 2$, with $\pi(x_k) = z \in Q^\pm$. For each $k = 1, 2$, this induces a bijection

$$\mathcal{C}^2_\mathbb{R}(x_k, X^\pm) \cup \mathcal{C}^4_\mathbb{R}(x_k, X^\pm) \to T_\mathbb{R}(Q^\pm, C, z^\pm).$$

By Theorem 1.1.1(1), for each $A \in T_\mathbb{R}(Q^\pm_\mathbb{R}, C, z^\pm)$, the both real curves A_k ($k = 1, 2$) that form $\pi^{-1}(A)$ are of the same type. This allows us to split real 6-tangent quartics into two species: *hyperbolic* if $\mathfrak{s}(A_k) > 0$, and *elliptic* if $\mathfrak{s}(A_k) < 0$.

Following the setting of Theorem 1.2.1, we consider also a bigger set of sections, $\tilde{T}_\mathbb{R}(Q^\pm, C, z^\pm) \supset T_\mathbb{R}(Q^\pm, C, z^\pm)$, the set which contains in addition the real conics in $Q^\pm_\mathbb{R}$ which are 2-tangent to C and pass through $z^\pm$. Each of these conics A lifts to a curve $\tilde{A}$ from $\mathcal{C}^0_\mathbb{R}(x_k, X^\pm)$ both for $k = 1$ and $k = 2$, which provides us, for each $k = 1, 2$, with a bijection

$$\mathcal{C}^0_\mathbb{R}(x_k, X^\pm) \cup \mathcal{C}^2_\mathbb{R}(x_k, X^\pm) \cup \mathcal{C}^4_\mathbb{R}(x_k, X^\pm) \to \tilde{T}_\mathbb{R}(Q^\pm, C, z^\pm).$$

Similar to above, we call a real conic $A \in \tilde{T}_\mathbb{R}(Q^\pm, C, z^\pm)$ hyperbolic if $\mathfrak{s}(\tilde{A}) > 0$ and elliptic otherwise.

Theorems 1.2.1 and 1.2.2 imply then the following result.

5.0.1. **Theorem.** *Assume that a real sextic curve $C \subset Q$ is a transversal intersection of a real quadratic cone $Q \subset \mathbb{P}^3$ (whose base is non-singular and has non-empty real locus) with a real cubic surface. Then, for any generic pair of points $z^{\pm} \in Q^{\pm}_{\mathbb{R}}$, the following holds.*

(1) *The number of hyperbolic minus the number of elliptic elements in the set $\tilde{T}_{\mathbb{R}}(Q^+, C, z^+) \cup \tilde{T}_{\mathbb{R}}(Q^-, C, z^-)$ is 30.*
(2) *The number of real quartics A that are 6-tangent to C (i.e. of elements in $T_{\mathbb{R}}(Q^+, C, z^+) \cup T_{\mathbb{R}}(Q^-, C, z^-)$) counted with weight $\widetilde{\mathfrak{s}}(A_1) = \widetilde{\mathfrak{s}}(A_2)$ is 96.*

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