Reconstruction of schemes

Alexander L. Rosenberg

Max-Planck-Institut
für Mathematik
Gottfried-Claren-Str. 26
53225 Bonn

Germany

MPI 96-108

Reconstruction of schemes.

Alexander L. Rosenberg

Table of contents:

Introduction

One of the central results of the famous work of P. Gabriel [Gab] is the following theorem:

Theorem ([Gab], Ch.VI). *Any noetherian scheme can be reconstructed uniquely up to isomorphism from the category of quasi-coherent sheaves on this scheme.*

Actually Gabriel produced a construction which assigns to any locally noetherian abelian category $\mathcal{A}$ a ringed space $(X_{\mathcal{A}}, \mathcal{O}_{\mathcal{A}})$ which in the case when $\mathcal{A}$ is the category $Qcoh_{\mathbf{X}}$ of quasi-coherent sheaves on a noetherian scheme $\mathbf{X}$ happens to be isomorphic to the scheme $\mathbf{X}$. Recall that the space $X_{\mathcal{A}}$ – the Gabriel spectrum of $\mathcal{A}$ – consists of isomorphism classes of indecomposable injectives of the category $\mathcal{A}$ with a natural topology: the base of closed sets are supports of noetherian objects. Here the *support* of an object M consists of equivalence classes of those indecomposable injectives I for which there exists a nonzero arrow from M to I.

One of the main purposes of this work is to prove the following Theorem:

Theorem. *Any scheme can be reconstructed uniquely up to isomorphism from the category of quasi-coherent sheaves on this scheme.*

We present here a construction which assigns to any abelian category $\mathcal{A}$ a ringed space $\mathbf{X}_{\mathcal{A}}$ such that if $\mathcal{A}$ and $\mathcal{A}'$ are equivalent categories, $\mathbf{X}_{\mathcal{A}}$ is naturally isomorphic to $\mathbf{X}_{\mathcal{A}'}$. And if $\mathcal{A}$ is the category of quasi-coherent sheaves of a scheme $\mathbf{X}$, then the scheme $\mathbf{X}$ is canonically isomorphic to $\mathbf{X}_{\mathcal{A}}$. The underlying topological space of $\mathbf{X}_{\mathcal{A}}$ is $(\mathbf{Spec}\mathcal{A}, \mathfrak{T})$. Here $\mathbf{Spec}\mathcal{A}$ is the spectrum of the category $\mathcal{A}$ introduced in [R1] (cf. also [R], Chapter III); $\mathfrak{T}$ is *the strong Zariski topology* which appears here for the first time. The structure sheaf is a sheaf of commutative rings analogous to the one used in the theorem of P. Gabriel. It is worth to mention that the construction of this work is not an extension of the Gabriel construction to a wider class of categories. On the contrary, for almost all locally noetherian categories (in particular, for almost all categories of left modules over noncommutative left noetherian rings) the underlying space, $(\mathbf{Spec}\mathcal{A}, \mathfrak{T})$, is much smaller than the Gabriel spectrum of $\mathcal{A}$. And it could be described, or at least estimated, in a quite a few cases of interest (see [R], Chapters II, IV, and V, and [R3]). The meaning of this remark is that the construction of this paper can be used for studying some 'noncommutative spaces' of interest which are introduced as abelian categories thought as categories of quasi-coherent sheaves on 'would-be spaces' (cf. [M], [A], [AZ], [R]). The paper is written with such applications in mind. In particular, we study the geometrical structure of an abelian category in more detail than is strictly necessary to prove the main theorem of the work.

The article is organized as follows.

Section 1 contains some basic facts of spectral theory of abelian categories. Main references on the subject are [R1] or [R], Chapter III.

In Section 2 subschemes of an abelian category are introduced.

In Section 3 we study Zariski closed subschemes and Zariski topology on the spectrum.

In Section 4 we define reduced and Zariski reduced subschemes and study relations between them in some cases of interest. We establish the stability of subschemes and reduced subschemes with respect to flat localizations.

In Section 5 we introduce the *prime spectrum* and *Levitzki spectum* of an abelian category which are naturally related (especially the Levitzki spectrum) with the Zariski topology on the spectrum.

The Zariski closed subschemes and Zariski topology are not, in general, compatible with localizations. We introduce in Section 6 *the strongly closed subschemes* and *the strong Zariski topology* which are stable with respect to flat localizations.

In Section 7 we define a ringed space associated to any choice of a topology on the spectrum of an abelian category and prove that any scheme can be canonically reconstructed, uniquely up to isomorphism, from the category of quasi-coherent sheaves on that scheme.

I would like to thank Max-Planck Institut für Mathematik for hospitality and excellent working conditions.

1. Preliminaries on the spectrum of abelian categories.

Fix an abelian category $\mathcal{A}$. Recall that, for any two objects X, Y of $\mathcal{A}$, we write $X \succ Y$ if Y is a subquotient of a finite direct sum of copies of X. For any $X \in Ob\mathcal{A}$, denote by $\langle X \rangle$ the full subcategory of $\mathcal{A}$ such that $Ob\langle X \rangle = Ob\mathcal{A} - \{Y \in Ob\mathcal{A} \mid Y \succ X\}$. It is easy to check that $X \succ Y$ iff $\langle Y \rangle \subseteq \langle X \rangle$. This observation provides a convenient realization of the quotient of $(Ob\mathcal{A}, \succ)$ with respect to the equivalence relation induced by $\succ$: $X \approx Y$ if $X \succ Y \succ X$. Namely, $(Ob\mathcal{A}, \succ)/\approx$ is isomorphic to $(\{\langle X \rangle \mid X \in Ob\mathcal{A}\}, \supseteq)$.

Set $Spec\mathcal{A} = \{P \in Ob\mathcal{A} \mid P \neq 0$, and for any nonzero subobject X of $P, X \succ P\}$. *The spectrum*, $\mathbf{Spec}\mathcal{A}$, *of* the category $\mathcal{A}$ is the preordered set of equivalence (with respect to $\succ$) classes of objects of $Spec\mathcal{A}$. The canonical realization of $(Ob\mathcal{A}, \succ)/\approx$ induces a canonical realization of $\mathbf{Spec}\mathcal{A}$: $(\mathbf{Spec}\mathcal{A} = \{\langle P \rangle \mid P \in Spec\mathcal{A}\}, \supseteq)$.

An abelian category $\mathcal{A}$ has the property (sup) if, for any ascending chain Ω of subobjects of an object M, the supremum of Ω exists and for any subobject L of M, the natural morphism $\sup\{X \cap L | X \in \Omega\} \longrightarrow (\sup \Omega) \cap L$ is an isomorphism.

The categories with the property (sup) are sometimes called *the categories with exact direct limits.*

1.1. Proposition. *For any $P \in Spec\mathcal{A}$, the subcategory $\langle P\rangle$ is a Serre subcategory of $\mathcal{A}$. If $\mathcal{A}$ is a category with the property (sup), then the converse is true: if X is an object of $\mathcal{A}$ such that $\langle X\rangle$ is a Serre subcategory of $\mathcal{A}$, then X is equivalent (in the sense of $\succ$) to* a $P \in Spec\mathcal{A}$*; i.e.* $\langle X\rangle = \langle P\rangle$.

Proof. See Proposition 2.3.3 and 2.4.7 in [R]. ■

A nonzero object X of a category $\mathcal{A}$ is called *quasifinal* if, for any nonzero object Y of $\mathcal{A}, Y \succ X$. The category $\mathcal{A}$ having a quasifinal object is called *local.*

One can check that all simple objects of a local category (if any) are isomorphic to each other. In particular, the category of left modules over a commutative ring R is local iff the ring R is local.

1.2. Proposition. *The quotient category $\mathcal{A}/\langle P\rangle$ is local.*

Proof. See Proposition 3.3.1 and Corollary 3.3.2 in [R]. ■

1.3. Proposition. *(a) For any topologizing (i.e. full and closed with respect to taking direct sums and subquotients) subcategory $\mathbb{T}$ of $\mathcal{A}$, the inclusion functor $\mathbb{T} \longrightarrow \mathcal{A}$ induces an embedding* $\mathbf{Spec}\mathbb{T} \longrightarrow \mathbf{Spec}\mathcal{A}$.

(b) For any exact localization $Q : \mathcal{A} \longrightarrow \mathcal{A}/\mathbf{S}$ and for any $P \in Spec\mathcal{A}$, either $P \in Ob\mathbb{S}$, or $Q(P) \in Spec\mathcal{A}/\mathbb{S}$; hence Q induces an injective map from $\mathbf{Spec}\mathcal{A} - \mathbf{Spec}\mathbb{S}$ *to* $\mathbf{Spec}\mathcal{A}/\mathbb{S}$.

1.4. The support of an object. For any $M \in Ob\mathcal{A}$, *the support of M*, $\mathbf{Supp}(M)$, consists of all $\langle P\rangle \in \mathbf{Spec}\mathcal{A}$ such that $M \notin Ob\langle P\rangle$.

1.5. Localizations at subsets of the spectrum. For any subset U of $\mathbf{Spec}\mathcal{A}$, denote by $\langle U\rangle$ the intersection $\bigcap_{\langle P\rangle \in U}\langle P\rangle$. Being the intersection of a set of Serre subcategories, $\langle U\rangle$ is a Serre subcategory. *A localization at U* is a localization at the Serre subcategory $\langle U\rangle$.

1.6. Some of the canonical topologies on the spectrum.

1.6.1. The topology τ. We denote this way the strongest topology compatible with the preorder $\supseteq$ (recall that $\mathbf{P} \supseteq \mathbf{P}'$ means that $\mathbf{P}'$ is a specialization of $\mathbf{P}$). Its explicit description: the closure of a subset W of $\mathbf{Spec}\mathcal{A}$ consists of all specializations of all points of W.

Recall that a topological space X is called a *Kolmogorov's space*, if it satisfies the following property:

(T_0) If x and y are two distinct points of X, then there exists an open set containing only one of these points.

In other words, if $x \in \{y\}^-$ and $y \in \{x\}^-$, then $x = y$.

Since the closure of a point in the topology τ consists of the set of its specializations and two points which are specializations of each other coincide by definition of $\mathbf{Spec}\mathcal{A}$, $(\mathbf{Spec}\mathcal{A}, \tau)$ is a Kolmogorov's space. In particular, if an irreducible subset of $(\mathbf{Spec}\mathcal{A}, \tau)$ has a generic point, this generic point is unique. The irreducible closed subsets of $(\mathbf{Spec}\mathcal{A}, \tau)$ having generic points (i.e. the closures of points) are sets of specializations of points of $\mathbf{Spec}\mathcal{A}$.

1.6.2. The topology τ_*. A base of closed sets of τ_* consists of $Supp(V)$, where V runs through finite direct sums of objects of $Spec\mathcal{A}$. In other words, the base of closed sets of τ_* consists of finite unions of the sets of specializations of points of the spectrum. The topology τ_* is the weakest topology on $\mathbf{Spec}\mathcal{A}$ having the property: the closure of any point $\mathbf{P}$ of $(\mathbf{Spec}\mathcal{A}, \tau_*)$ is the set $\{\mathbf{P}' \in \mathbf{Spec}\mathcal{A} | \mathbf{P}' \subseteq \mathbf{P}\}$ of specializations of $\mathbf{P}$. In particular, $(\mathbf{Spec}\mathcal{A}, \tau_*)$ is a Kolmogorov's space with the same irreducible closed subsets having a generic point as $(\mathbf{Spec}\mathcal{A}, \tau)$.

1.6.3. The topology τ^*. Recall that an object M of a category C is *of finite* type if for any directed set Ω of subobjects of M such that $\sup \Omega = M$, there exists a subobject $M' \in \Omega$ such that $M' = M$. We define the topology τ^* on $\mathbf{Spec}\mathcal{A}$ by declaring the sets $Supp(V)$, where V runs through the class of all objects of $\mathcal{A}$ of finite type, a base of closed subsets. Note that this base is closed with respect to taking finite unions, since $\bigcup_{i \in J} Supp(V_i) = Supp(\oplus_{i \in J} V_i)$ and a finite direct sum of objects of finite type is an object of finite type. A similar statement about finite intersections is not true.

1.6.3.1. Lemma. *Suppose that the category $\mathcal{A}$ has the property: any nonzero object of $\mathcal{A}$ has a nonzero subobject of finite type. Then the topology τ_* is weaker than the topology τ^*.*

Proof. In fact, under the conditions of the lemma, every element of $\mathbf{Spec}\mathcal{A}$ is of the form $\langle P \rangle$, where P is of finite type. ∎

1.6.3.2. Corollary. *Under the conditions of Lemma 1.6.3.1,* $(\mathbf{Spec}\mathcal{A}, \tau^*)$ *is a Kolmogorov's space.*

1.6.4. The case of a locally noetherian category. The *flat spectrum* of an abelian category $\mathcal{A}$ is the set of all Serre subcategories $\mathbf{P}$ of $\mathcal{A}$ such that $\mathcal{A}/\mathbf{P}$ is a local category (cf. [R], Ch. VI for a detailed study of $\mathbf{Spec}^-\mathcal{A}$). The following proposition has a well known commutative (i.e. when $\mathcal{A} = R - mod$ and R is commutative) prototype.

1.6.4.1. Proposition. *Let M be a noetherian object of $\mathcal{A}$ such that every nonzero subquotient of M has an associated point. Then there exists an increasing filtration $0 = M_0 \subset M_1 \subset M_2 \subset \ldots \subset M_n = M$ such that $M_i/M_{i-1} \in Spec\mathcal{A}$ for $1 \leq i \leq n$.*

Proof. Note that any noetherian object M of $\mathcal{A}$ has a finite Gabriel-Krull dimension which implies the existence of a finite filtration $0 = M_0 \subset M_1 \subset M_2 \subset \ldots \subset M_n = M$ such that, for all $1 \leq i \leq n$, $M_i/M_{i-1} \in Spec^-\mathcal{A}$. By the assumption, $Ass(M_i/M_{i-1}) \neq \emptyset$. So that $M_i/M_{i-1} \in Spec\mathcal{A}$. ∎

1.6.4.2. Corollary. *Suppose that $\mathcal{A}$ is a locally noetherian category such that any nonzero object of $\mathcal{A}$ has an associated point (in* $\mathbf{Spec}\mathcal{A}$*). Then* $\mathbf{Spec}^-\mathcal{A} = \mathbf{Spec}\mathcal{A}$.

Proof. Since $\mathcal{A}$ is locally noetherian, every $\mathbf{P} \in \mathbf{Spec}^-\mathcal{A}$ can be represented by a noetherian ($\mathbf{P}$-torsion free) object P. It follows from Proposition 1.6.4.1 that $P \in Spec\mathcal{A}$. ∎

1.6.4.3. Corollary. *Suppose that $\mathcal{A}$ is a locally noetherian category such that any nonzero object of $\mathcal{A}$ has an associated point (in* **Spec**$\mathcal{A}$*). Then the topologies τ^* and τ_* on* **Spec**$\mathcal{A}$ *coincide.*

Proof. The category $\mathcal{A}$ being locally noetherian, objects of finite type are exactly noetherian objects; and all elements of **Spec**$\mathcal{A}$ are of the form $\langle P \rangle$ with a noetherian P. It follows from Proposition 1.6.4.1 that the support of any object M of finite type is a finite union of the sets of specializations of points of **Spec**$\mathcal{A}$: $Supp(M) = \bigcup_{1 \leq i \leq n} Supp(M_i/M_{i-1})$ (cf. 1.6.4.1). So that the topologies τ^* and τ_* coincide. ∎

2. Subschemes.

2.1. Subschemes of an abelian category. Recall that a subcategory $\mathbb{T}$ of $\mathcal{A}$ is coreflective if the inclusion functor $\mathbb{T} \longrightarrow \mathcal{A}$ has a right adjoint. This property means that any object M has the biggest subobject, $t_{\mathbb{T}}M$ (called the '$\mathbb{T}$-torsion of M'), from $\mathbb{T}$.

If $\mathcal{A}$ has supremums of subobjects (which is the case if $\mathcal{A}$ has small direct sums), then being coreflective means exactly that supremum of any set of subobjects of an object which belong to $\mathbb{T}$ is also in $\mathbb{T}$.

A full subcategory $\mathbb{T}$ of an abelian category $\mathcal{A}$ is called *topologizing* if it is closed with respect to finite direct sums and contains all subquotions of any of its objects.

We call a coreflective topologizing subcategory of $\mathcal{A}$ a *subscheme of* $\mathcal{A}$.

2.2. The Gabriel multiplication. The *Gabriel product* of two subcategories $\mathbb{T}$ and $\mathbb{S}$ of an abelian category $\mathcal{A}$ is the full subcategory $\mathbb{T} \bullet \mathbb{S}$ of $\mathcal{A}$ generated by all $X \in Ob\mathcal{A}$ for which there exists an exact sequence

$$0 \longrightarrow X' \longrightarrow X \longrightarrow X'' \longrightarrow 0$$

with $X' \in Ob\mathbb{S}$ and $X'' \in Ob\mathbb{T}$. If $\mathbb{T}$ and $\mathbb{S}$ are topologizing subcategories, then such is $\mathbb{T} \bullet \mathbb{S}$. This multiplication is associative and has an identity element – the subcategory $\mathbf{0}$. Note that a topologizing subcategory $\mathbb{T}$ of $\mathcal{A}$ is *thick* iff $\mathbb{T} \bullet \mathbb{T} = \mathbb{T}$.

By Lemma III.6.2.1 in [R], if $\mathbb{T}$, $\mathbb{S}$ are subschemes of $\mathcal{A}$, their Gabriel product $\mathbb{T} \bullet \mathbb{S}$ is a subscheme.

2.2.1. An application: the n-th neighborhood of a topologizing subcategory. Given a subscheme $\mathbb{T}$ of $\mathcal{A}$, define the n-th neighborhood of $\mathbb{T}$ as the n-th power of $\mathbb{T}$; i.e. $\mathbb{T}^{(n)} := \mathbb{T} \bullet \ldots \bullet \mathbb{T}$ (n times). All $\mathbb{T}^{(n)}$ are subschemes of $\mathcal{A}$ and $\mathbb{T}^{(n)} \subseteq \mathbb{T}^{(n+1)}$ for all n. One can check that $\mathbb{T}^{(\infty)} := \bigcup_{n \geq 1} \mathbb{T}^{(n)}$ is a thick subcategory of $\mathcal{A}$ which coincides with the intersection of all thick subcategories containing $\mathbb{T}$. Note that $\mathbb{T}^{(\infty)}$ is not, in general, a subscheme of $\mathcal{A}$. It is a subscheme if $\mathbb{T}$ is locally noetherian. ∎

2.3. Subschemes and the topology τ. Fix an abelian category $\mathcal{A}$. Note that, for any subscheme $\mathbb{T}$ of $\mathcal{A}$, **Spec**$\mathbb{T}$ is a closed subset of the topological space (**Spec**$\mathcal{A}, \tau$). Conversely, any closed subset W of (**Spec**$\mathcal{A}, \tau$) coincides with **Spec**$\mathbb{T}$ for some subscheme $\mathbb{T}$. The biggest subscheme of $\mathcal{A}$ having such property is the Serre subcategory $\mathcal{A}_W$ of $\mathcal{A}$ generated by all objects M such that $Supp(M) \subseteq W$. Thus arbitrary subschemes could be interpreted as closed subschemes of the space (**Spec**$\mathcal{A}, \tau$).

2.4. Subschemes of the category of modules. Let $\mathcal{A}$ be the category $R-mod$ of left modules over an associative ring R. And let $\mathbb{T}$ be any topologizing subcategory of $\mathcal{A}$. Denote by $F_{\mathbb{T}}$ the set of all left ideals m in R such that $R/m \in Ob\mathbb{T}$.

Conversely, for any set F of left ideals in R, denote by $\mathbb{T}_F$ the full subcategory of $R-mod$ generated by all modules M such that, for any $z \in M$, $Ann(z) \in F$.

2.4.1. Lemma. *1) For any topologizing subcategory $\mathbb{T}$ of $R-mod$, the set $F = F_{\mathbb{T}}$ has the following properties:*

(a) $m, n \in F$ implies that $m \cap n \in F$;

(b) if $m \in F$, then any left ideal n containing m belongs to F;

(c) for any $m \in F$ and any finite subset x of elements of R, $(m : x) \in F$.

2) If F is a subset of the set I_lR of left ideals of R having the properties (a), (b), (c), then the subcategory $\mathbb{T}_F$ is topologizing and coreflective.

Proof. 1) *(a)* is a consequence of the fact that the quotient module $R/m \cap n$ is a submodule of the direct sum $R/m \oplus R/n$.

(b) The module R/n is a quotient of R/m; hence $R/n \in Ob\mathbb{T}$ together with R/m.

(c) Let u denote the image of the identity element in R/m. The left ideal $(m : x)$ is the annihilator of the element $\oplus_{r\in x} ru$ of the direct sum of $| x |$ copies of R/m; hence $R/(m : x)$, being a submodule of a module from $\mathbb{T}$, belongs to $\mathbb{T}$.

2) For any module M, the set $M_F := \{z \in M \mid Ann(z) \in F\}$ is a submodule.

In fact, for any $z, z' \in M$ and any $r \in R$, we have:

$Ann(z + z') \supseteq Ann(z) \cap Ann(z')$, and $Ann(rz) = (Ann(z) : r)$.

Clearly M_F is the largest submodule of M which belongs to $\mathbb{T}_F$. This means that the subcategory $\mathbb{T}_F$ is coreflective.

If $M, M' \in Ob\mathbb{T}_F$, then $M \oplus M' \in Ob\mathbb{T}_F$, since for any $z \in M$, $z' \in M'$, $Ann(z \oplus z')$ equals to the intersection of $Ann(z)$ and $Ann(z')$. Clearly any subobject of an object of $\mathbb{T}_F$ belongs to $\mathbb{T}_F$. Finally, a quotient of any object of $\mathbb{T}_F$ belongs to $\mathbb{T}_F$. So that the subcategory $\mathbb{T}_F$ is topologizing. ■

The sets F of left ideals satisfying the conditions of Lemma 2.4.1 are called *topologizing filters.*

2.4.2. Note. For any topologizing subcategory $\mathbb{T}$, the subcategory $\mathbb{T}_F$, where $F = F_{\mathbb{T}}$ is the set $\{m \in I_lR \mid R/m \in Ob\mathbb{T}\}$ is the intersection of all coreflective topologizing subcategories of $R-mod$ containing $\mathbb{T}$. ■

2.4.3. Example. Let m be any left ideal in R. Denote by $[R/m]$ the full subcategory of $\mathcal{A}$ generated by all modules M such that, for any $z \in M, m \leq$Ann(z). One can check that the subcategory $[R/m]$ is topologizing and coreflective. Moreover, $[R/m]$ is the smallest subscheme of $\mathcal{A}$ containing the module R/m. One can see that $[R/m] = \mathbb{T}_{[m]}$, where $[m] := \{n \in I_lR \mid m \leq n\}$.

The topologizing subcategories $\mathbb{T}_{[m]}$ are minimal in the following sense: for any topologizing filter F of left ideals in R, $\mathbb{T}_F = \bigcup_{m\in F} \mathbb{T}_{[m]}$. ■

Example 2.4.3 is extended to any abelian category $\mathcal{A}$ with the property (sup) as follows. For any object V of $\mathcal{A}$, denote by $V_{\succ}$ the full subcategory of $\mathcal{A}$ generated by all objects X such that $V \succ X$. Note that $V_{\succ}$ is topologizing, since it is closed under finite direct sums; and if $X \in \mathrm{ObV}_{\succ}$ and $X \succ Y$, then $Y \in \mathrm{ObV}_{\succ}$. But, in general, the subcategory $V_{\succ}$ is not coreflective. The full subcategory $[V]$ of $\mathcal{A}$ generated by all $X \in Ob\mathcal{A}$ which are supremums of their subobjects from $V_{\succ}$ is both topologizing and coreflective.

Note that any subscheme $\mathbb{T}$ of $\mathcal{A}$ is of the form $\bigcup_{V \in \mathfrak{X}}[V]$, where $\mathfrak{X}$ is a class of objects of $\mathbb{T}$ having the property: for any $Y \in Ob\mathbb{T}$, there exists $X \in \mathfrak{X}$ such that $X \succ Y$.

Clearly $V \succ W$ if and only if $W_{\succ} \subseteq V_{\succ}$. In particular, the subcategories $V_{\succ}$ and $[V]$ depend only on the equivalence class $\langle V \rangle$ of the object V. One can see that $Spec[V] = SpecV_{\succ} = Supp(V)$. So that if $V \in Spec\mathcal{A}$, then $\mathbf{Spec}V_{\succ}$ is the set of all specializations of $\langle V \rangle$.

Assume that the category $\mathcal{A}$ has no nonzero objects with empty support. In this case, if $V \in Spec\mathcal{A}$ and is a closed point, then all nonzero objects of $V_{\succ}$ are equivalent to V.

If $\mathcal{A} = R - mod$ and $V = R/m$ for some left ideal m in R, the subcategory $[V]$ coincides with the subcategory $[R/m]$ of Example 2.4.3.

2.4.4. Residue field of a point of the spectrum. If $\mathcal{A}$ is a local category and V is a quasifinal object, then $[V]$ is the *residue category of* $\mathcal{A}$: $[V] = \mathcal{K}(\mathcal{A})$. It does not depend on the choice of a quasifinal object. If $\mathcal{A}$ has a simple object M, then $[V] = [M]$ is equivalent to the category of vector spaces over skew residue field of $K(\mathcal{A}) = End(M)$ (cf. [R], III.5.4.1). Since all simple objects of a local category are isomorphic to each other, the residue field $K(\mathcal{A})$ is defined uniquely up to isomorphism.

In particular, for any abelian category $\mathcal{A}$ and any element $\mathbf{P}$ of $\mathbf{Spec}\mathcal{A}$ we have a well defined residue category $\mathcal{K}_{\mathbf{P}} := \mathcal{K}(\mathcal{A}/\mathbf{P})$ of $\mathbf{P}$. And if the category $\mathcal{A}/\mathbf{P}$ has simple objects, we have a defined uniquely up to isomorphism skew residue field $K_{\mathbf{P}} := K(\mathcal{A}/\mathbf{P})$ of $\mathbf{P}$.

3. Zariski closed subschemes.

A subcategory $\mathbb{T}$ of $\mathcal{A}$ is called *reflective* if the inclusion functor $\mathbb{T} \longrightarrow \mathcal{A}$ has a left adjoint. We say that a subscheme $\mathbb{T}$ of $\mathcal{A}$ is *Zariski closed* (or simply *closed*) if $\mathbb{T}$ is a reflective subcategory of $\mathcal{A}$.

By Lemma III.6.2.1 in [R], if $\mathbb{T}, \mathbb{S}$ are closed subschemes of $\mathcal{A}$, then their Gabriel product $\mathbb{T} \bullet \mathbb{S}$ is a closed subscheme.

3.1. Example: closed subschemes of $R - mod$. If $\mathcal{A} = R - mod$, reflective topologizing subcategory of $\mathcal{A}$ are in one-to-one correspondence with two-sided ideals of the ring R: to any two-sided ideal α there corresponds the full subcategory $[R/\alpha]$ generated by all modules annihilated by α (cf. [R], Proposition III.6.4.1). In particular, any reflective topologizing subcategory of $R - mod$ is coreflective. ∎

3.2. Note. It follows from Example 3.1 that if the ring R is simple, there are only trivial Zariski closed subschemes of $R - mod$ and lots of subschemes. ∎

3.3. Operations with subschemes. Fix an abelian category $\mathcal{A}$ having the property (sup).

3.3.1. Lemma. *(a) The intersection of any set of subschemes of $\mathcal{A}$ is a subscheme.*

(b) The intersection of any set of Zariski closed subschemes of $\mathcal{A}$ is a Zariski closed subscheme.

Proof. *(a)* Clearly the intersection of any set of topologizing subcategories is a-topologizing subcategory. Similarly, the intersection of any family $\mathfrak{X}$ of coreflective subcategories is a coreflective subcategory.

In fact, let Ω be a family of subobjects of an object Y which belong to the intersection $\bigcap_{\mathbb{S}\in\mathfrak{X}}\mathbb{S}$. Since each of the subcategories $\mathbb{S}\in\mathfrak{X}$ is coreflective, $\sup\Omega$ belongs to this intersection too. This implies the coreflectivity of $\bigcap_{\mathbb{S}\in\mathfrak{X}}\mathbb{S}$.

(b) Let now $\mathfrak{F}$ be a family of Zariski closed subschemes. And let, for any $\mathbb{T}$ in $\mathfrak{F}$, $\hat{\ }J_{\mathbb{T}}$ denote a left adjoint to the inclusion $J_{\mathbb{T}} : \mathbb{T} \longrightarrow \mathcal{A}$, and $\eta_{\mathbb{T}}$ the adjunction arrow $\mathrm{Id}_{\mathcal{A}} \longrightarrow J_{\mathbb{T}}\circ\hat{\ }J_{\mathbb{T}}$. Let $K_{\mathbb{T}}$ denote the kernel of $\eta_{\mathbb{T}}$. Note that $\eta_{\mathbb{T}}$ is an epimorphism; so that $J_{\mathbb{T}}\circ\hat{\ }J_{\mathbb{T}} \simeq \mathrm{Cok}(\eta_{\mathbb{T}})$. Set $K\mathfrak{F} := \sup\{K_{\mathbb{T}} \mid \mathbb{T}\in\mathfrak{F}\}$. For any $M\in Ob\mathcal{A}$, $M/K\mathfrak{F}(M)$ is a quotient of $M/K_{\mathbb{T}}(M)$ for any $\mathbb{T}\in\mathfrak{F}$; hence it is an object of $\bigcap_{\mathbb{T}\in\mathfrak{F}}\mathbb{T}$. Conversely, if Y is an object of $\bigcap_{\mathbb{T}\in\mathfrak{F}}\mathbb{T}$, then an arbitrary morphism $f : M \longrightarrow Y$ factors by $M \longrightarrow M/K_{\mathbb{T}}(M)$. So that Ker$f$ 'contains' $K\mathfrak{F}(M)$. All together shows that the map $M \mapsto M/K\mathfrak{F}(M)$ extends to a left adjoint to the inclusion functor $\bigcap_{\mathbb{T}\in\mathfrak{F}}\mathbb{T} \longrightarrow \mathcal{A}$; i.e. $\bigcap_{\mathbb{T}\in\mathfrak{F}}\mathbb{T}$ is a reflective subcategory of $\mathcal{A}$. ∎

3.3.2. The supremum of subschemes. The *supremum*, $\sup\mathfrak{F}$, of a family $\mathfrak{F} = \{\mathbb{S}_i \mid i\in J\}$ of subschemes is the smallest subscheme of $\mathcal{A}$ containing all the subschemes of the family $\mathfrak{F}$.

Let $\{\mathbb{S}_i \mid i\in J\}$ be any family of topologizing subcategories of $\mathcal{A}$. Then the smallest topologizing subcategory containing all the subcategories $\mathbb{S}_i$ equals to the union of the subcategories $X_{\succ}$, where X runs through $\oplus_{i\in J}X_i$ in which $X_i\in Ob\mathbb{S}_i$ for all $i\in J$, and only finite number of X_i are nonzero. If all the subcategories $\mathbb{S}_i$ are coreflective and arbitrary direct sums $\oplus_{i\in J}X_i, X_i\in Ob\mathbb{S}_i$, exist, then we have an analogous description of the smallest subscheme $\mathbb{S}$ containing all $\mathbb{S}_i$: the subcategory $\mathbb{S}$ is the union of the subcategories $[X]$, where X runs through all sums $\oplus_{i\in J}X_i$ with $X_i\in Ob\mathbb{S}_i$.

Note that 'all sums' in this description can be replaced by the requirement $X_i\in\Xi_i$, where Ξ_i is a set of objects of $\mathbb{S}_i$ such that $\mathbb{S}_i = \bigcup_{Y\in\Xi_i}[Y]$.

For instance, if $\mathbb{S}_i = [X_i]$ for some $X_i\in Ob\mathcal{A}$, $i\in J$, then $\sup\{\mathbb{S}_i \mid i\in J\} = [\oplus_{i\in J}X_i]$.

3.3.2.1. Lemma. *The supremum of a finite number of Zariski closed subschemes is a Zariski closed subscheme.*

Proof. We shall use the notations of the argument of Lemma 3.3.1.

Let $\mathfrak{F}$ be a finite family of Zariski closed subschemes of $\mathcal{A}$. Denote by $K_{\mathfrak{F}}$ the functor which assigns to any $M\in Ob\mathcal{A}$ the intersection $\bigcap_{\mathbb{T}\in\mathfrak{F}}K_{\mathbb{T}}(M)$. Since $\mathfrak{F}$ is finite, $M/K_{\mathfrak{F}}(M)$ is a subobject of ${}_{\mathbb{T}}\mathfrak{F}_{\mathfrak{F}}M/K_{\mathbb{T}}(M) =_{\mathbb{T}} \mathfrak{F}_{\mathfrak{F}}\hat{\ }J_{\mathbb{T}}(M)$. Denote by $\Psi_{\mathfrak{F}}$ the (uniquely defined) extension of the map $M\mapsto M/K_{\mathfrak{F}}(M)$ to a functor from $\mathcal{A}$ to $\mathcal{A}$. Since the direct sum ${}_{\mathbb{T}}\mathfrak{F}_{\mathfrak{F}}M/K_{\mathbb{T}}(M)$ is an object of $\sup\mathfrak{F}$, the functor $\Psi_{\mathfrak{F}}$ takes values in the subcategory $\sup\mathfrak{F}$. On the other hand, if $M\in Ob\sup\mathfrak{F}$, then $K_{\mathfrak{F}}(M) = 0$; i.e. the natural epimorphism $M \longrightarrow \Psi_{\mathfrak{F}}(M)$ is an isomorphism. This shows that $\Psi_{\mathfrak{F}}$ is left adjoint to the inclusion $\sup\mathfrak{F} \longrightarrow \mathcal{A}$. ∎

4. Irreducible and reduced subschemes.

4.1. Irreducible subschemes and points of the spectrum. Call a subscheme $\mathbb{X}$ of $\mathcal{A}$ *irreducible* if, for any subschemes $\mathbb{T}$ and $\mathbb{S}$, we have the implication:

$$\mathbb{X} \subseteq \mathbb{S} \cup \mathbb{T} \Leftrightarrow \text{ either } \mathbb{X} \subseteq \mathbb{T}, \text{ or } \mathbb{X} \subseteq \mathbb{S}.$$

4.1.1. Lemma. *For any* $\mathbf{P} \in \mathbf{Spec}\mathcal{A}$*, the subscheme* $[\mathbf{P}]$ *has the property: if* $[\mathbf{P}] \subseteq \mathbb{S} \bullet \mathbb{T}$ *for some subschemes* $\mathbb{S}$ *and* $\mathbb{T}$*, then either* $[\mathbf{P}] \subseteq \mathbb{S}$*, or* $[\mathbf{P}] \subseteq \mathbb{T}$*. In particular,* $[\mathbf{P}]$ *is an irreducible subscheme.*

Proof. Let $\mathbf{P} = \langle P \rangle$ for some $P \in Spec\mathcal{A}$; so that $[\mathbf{P}] = [P]$. The inclusion $[P] \subseteq \mathbb{S} \bullet \mathbb{T}$ means that $P \in Ob\mathbb{S} \bullet \mathbb{T}$; i.e. there exists an exact sequence $0 \longrightarrow X \longrightarrow P \longrightarrow Y \longrightarrow 0$ such that $X \in Ob\mathbb{T}$ and $Y \in Ob\mathbb{S}$. If $X \neq 0$, then $X \succ P$; hence $P \in Ob\mathbb{T}$. If $X = 0$, then $Y \simeq P$; so $P \in Ob\mathbb{S}$.

If $[\mathbf{P}] \subseteq \mathbb{S} \cup \mathbb{T}$, then $[\mathbf{P}] \subseteq \mathbb{S} \bullet \mathbb{T}$. Therefore either $[\mathbf{P}] \subseteq \mathbb{S}$, or $[\mathbf{P}] \subseteq \mathbb{T}$. ■

4.2. Reduced subschemes. We call $\mathcal{A}$ *reduced* if any subscheme $\mathbb{T}$ of $\mathcal{A}$ such that the the natural embedding $\mathbf{Spec}\mathbb{T} \longrightarrow \mathbf{Spec}\mathcal{A}$ is a bijection coincides with $\mathcal{A}$.

4.2.1. Proposition. *The intersection* $\mathbb{T}_{\mathfrak{red}}$ *of all subschemes* $\mathbb{T}'$ *of* $\mathcal{A}$ *such that* $\mathbf{Spec}\mathbb{T}' = \mathbf{Spec}\mathbb{T}$ *is a unique reduced subscheme of* $\mathcal{A}$ *having the same spectrum as* $\mathbb{T}$*. The subscheme* $\mathbb{T}_{\mathfrak{red}}$ *coincides with* $\sup\{[\mathbf{P}] \mid \mathbf{P} \in \mathbf{Spec}\mathbb{T}\}$.

Proof. We can assume without loss of generality that $\mathbb{T}$ coincides with $\mathcal{A}$. The equality $\mathbf{Spec}\mathbb{T}' = \mathbf{Spec}\mathcal{A}$ is equivalent, for any subscheme (more generally, for any topologizing subcategory)$\mathbb{T}'$ of $\mathcal{A}$, to the equality $Spec\mathbb{T}' = Spec\mathcal{A}$. Therefore $Spec\mathcal{A}_{\mathfrak{red}}$ is equal to $Spec\mathcal{A}$. Since a subscheme of a subscheme of $\mathcal{A}$ is a subscheme of $\mathcal{A}, \mathcal{A}_{\mathfrak{red}}$ is reduced.

Clearly the spectrum of the subscheme $\sup\{[\mathbf{P}] \mid \mathbf{P} \in \mathbf{Spec}\mathcal{A}\}$ coincides with $\mathbf{Spec}\mathcal{A}$. And any subscheme $\mathbb{T}$ of $\mathcal{A}$ such that $Spec\mathbb{T} = Spec\mathcal{A}$ contains all subschemes $[\mathbf{P}]$; hence it contains $\sup\{[\mathbf{P}] \mid \mathbf{P} \in \mathbf{Spec}\mathcal{A}\}$. Therefore $\mathcal{A}_{\mathfrak{red}} = \sup\{[\mathbf{P}] \mid \mathbf{P} \in \mathbf{Spec}\mathcal{A}\}$. ■

4.2.2. Remark. Clearly $\mathcal{A}_{\mathfrak{red}} = \sup\{[\mathbf{P}] \mid \mathbf{P} \in \mathfrak{X}\}$, where $\mathfrak{X}$ is any subset of $\mathbf{Spec}\mathcal{A}$ such that any point of $\mathbf{Spec}\mathcal{A}$ is a specialization of some point of $\mathfrak{X}$. ■

4.3. Closed subsets of the spectrum and reduced subschemes. For any subset W of $\mathbf{Spec}\mathcal{A}$, let $[W]$ denote the reduced subcategory $\sup\{[P] \mid \langle P \rangle \in W\}$.

4.3.1. Proposition. *The correspondence* $W \mapsto [W]$ *establishes an isomorphism from the category (preorder) of closed subsets of the space* $(\mathbf{Spec}\mathcal{A}, \tau)$ *onto the category of reduced subschemes of* $\mathcal{A}$.

Proof. In fact, a subscheme $\mathbb{T}$ is reduced iff $\mathbb{T} = \sup\{[P] \mid \langle P \rangle \in \mathbf{Spec}\mathbb{T}\}$. ■

4.4. Zariski reduced subschemes. We call a closed subscheme $\mathbb{T}$ of $\mathcal{A}$ *Zariski reduced* if any Zariski closed subscheme $\mathbb{T}'$ of $\mathcal{A}$ such that $\mathbb{T}' \subseteq \mathbb{T}$ and $Spec\mathbb{T}' = Spec\mathbb{T}$ coincides with $\mathbb{T}$. For any closed subscheme $\mathbb{T}$, the intersection $\mathbb{T}_{\mathrm{red}}$ of all closed subschemes $\mathbb{T}'$ of $\mathcal{A}$ such that $\mathbf{Spec}\mathbb{T}' = \mathbf{Spec}\mathbb{T}$ is the smallest Zariski reduced subscheme of $\mathcal{A}$ having the same spectrum as $\mathbb{T}$. We call $\mathbb{T}_{\mathrm{red}}$ *the Zariski reduced subscheme associated to* $\mathbb{T}$. In particular, there exists *the Zariski reduced subscheme* $\mathcal{A}_{\mathrm{red}}$ *associated to* $\mathcal{A}$.

4.4.3. Example: the reduced subscheme associated to the category of modules over a Goldie ring. Let $\mathcal{A} = R-mod$, where R is a left Goldie ring. Then there is a finite set $\mathfrak{X}$ of minimal primes in R. These minimal primes are left annihilators in R; therefore, by Proposition I.6.4.5 in [R], they belong to $Spec_l$R. Since every ideal $p \in Spec_l R$ contains a prime ideal, $(p : R)$, any ideal of $Spec_l R$ contains an ideal from $\mathfrak{X}$. Therefore $\mathcal{A}_{\mathfrak{red}}$ is the supremum of Zariski closed irreducible subschemes $[R/p] \simeq R/p - mod$, where p runs through (the finite set) $\mathfrak{X}$.

Note that $\sup\{[R/p] \mid p \in \mathfrak{X}\} = [\oplus_{p\in\mathfrak{X}} R/p] = [R/\bigcap_{p\in\mathfrak{X}} p] = [R/\mathcal{L}(R)]$, where $\mathcal{L}(R) = \bigcap_{p\in\mathfrak{X}} p$ is the Levitzki radical (which is by definition the biggest locally nilpotent ideal) of R. The last equality follows from the following facts:

a) $R/\mathcal{L}(R) \succ R/p$ for each $p \in \mathfrak{X}$ (actually, for each $p \in Spec_l R$) which implies the inclusion $\sup\{[R/p] \mid p \in \mathfrak{X}\} \subseteq [R/\mathcal{L}(R)]$.

b) On the other hand, $R/\mathcal{L}(R)$ is a subobject (thanks to the finiteness of $\mathfrak{X}$) of $\oplus_{p\in\mathfrak{X}} R/p$ which implies the inverse inclusion $[R/\mathcal{L}(R)] \subseteq [\oplus_{p\in\mathfrak{X}} R/p]$.

Thus, we have established the following fact:

4.4.3.1. Proposition. *For any left Goldie ring $R, R - mod_{\mathfrak{red}} = [R/\mathcal{L}(R)] \simeq R/\mathcal{L}(R) - mod$.*

In particular, $R - mod_{\mathfrak{red}}$ is a Zariski closed subscheme of $R - mod$; therefore $R - mod_{\mathfrak{red}} = R - mod_{\mathrm{red}}$.

4.4.3.2. Corollary. *Let $\mathcal{A}$ be the category of left modules over a left noetherian ring* R. *Then, for any closed subscheme $\mathbb{T}$ of $\mathcal{A}, \mathbb{T}_{\mathrm{red}} = \mathbb{T}_{\mathfrak{red}}$.*

Proof. Any closed subscheme $\mathbb{T}$ of $R - mod$ (for an arbitrary associative ring R) is naturally isomorphic to the category $R/\alpha - mod$ for a (uniquely defined) two-sided ideal α (cf. Example 3.1). The isomorphism $\mathbb{T} \simeq R/\alpha - mod$ induces isomorphisms between $\mathbb{T}_{\mathrm{red}}$ and $R/\alpha - mod_{\mathrm{red}}$ and between $\mathbb{T}_{\mathfrak{red}}$ and $R/\alpha - mod_{\mathfrak{red}}$. Since the ring R is left noetherian such is the ring R/α; in particular, R/α is a left Goldie ring. By Proposition 4.4.3.1, the inclusion $R/\alpha - mod_{\mathfrak{red}} \subseteq R/\alpha - mod_{\mathrm{red}}$ turns out to be the equality. Hence $\mathbb{T}_{\mathfrak{red}} = \mathbb{T}_{\mathrm{red}}$. ∎

4.4.4. The reduced subschemes associated to the category of modules: the general case. Consider now the general affine case: $\mathcal{A} = R - mod$, where R is an arbitrary associative ring. We have:

$$\mathcal{A}_{\mathfrak{red}} = \sup\{[R/p] \mid p \in Spec_l R\} = [\oplus_{p\in Spec_l R} R/p] \subseteq [R/\mathcal{L}(R)] \simeq R/\mathcal{L}(R) - mod$$

where $\mathcal{L}(R)$ is the Levitzki radical of R. It follows from Theorem I.4.10.2 in [R] that the subcategory $[R/\mathcal{L}(R)]$ coincides with $\mathcal{A}_{\mathrm{red}}$.

If $Spec_l R$ has a finite subset of minimal points (as in the case when R is a left Goldie ring; cf. Example 4.4.3), then, repeating the argument of Example 4.4.3, one can see that $\mathcal{A}_{\mathfrak{red}} = [R/\mathcal{L}(R)] = \mathcal{A}_{\mathrm{red}}$.

In the general (even commutative) case, the reduced subscheme $\mathcal{A}_{\mathfrak{red}}$ is, usually, strictly smaller than $\mathcal{A}_{\mathrm{red}}$.

4.5. Subschemes and localizations. The following assertion shows that subschemes are stable with respect to flat localizations.

4.5.1. Proposition. *(a) Let $Q : \mathcal{A} \longrightarrow \mathcal{A}/\mathbb{S}$ be a flat localization. For any subscheme $\mathbb{T}$ of $\mathcal{A}$, $\mathbb{T} \cap \mathbb{S}$ is a Serre subcategory of $\mathbb{T}$ and, given a localization $Q' : \mathbb{T} \longrightarrow \mathbb{T}/\mathbf{S} \cap \mathbb{T}$, there exists a unique functor $J : \mathbb{T}/\mathbb{T} \cap \mathbb{S} \longrightarrow \mathcal{A}/\mathbb{S}$ such that the diagram*

$$\begin{array}{ccccc} \mathbb{T}\cap\mathbb{S} & \longrightarrow & \mathbb{T} & \xrightarrow{Q'} & \mathbb{T}/\mathbb{T}\cap\mathbb{S} \\ \downarrow & & \downarrow J_{\mathbf{T}} & & \downarrow J \\ \mathbb{S} & \longrightarrow & \mathcal{A} & \xrightarrow{Q} & \mathcal{A}/\mathbb{S} \end{array} \qquad (1)$$

is commutative. The functor J establishes an equivalence between $\mathbb{T}/\mathbb{T}\cap\mathbb{S}$ and the minimal subscheme $[Q(\mathbb{T})]$ of the category $\mathcal{A}/\mathbb{S}$ containing $Q(\mathbb{T})$.

(b) If $\mathbb{T}$ is a reduced subscheme of $\mathcal{A}$, then $[Q(\mathbb{T})]$ is a reduced subscheme of the category $\mathcal{A}/\mathbb{S}$.

Proof. *(a)* The assertion *(a)* follows from Lemma VI.1.4.1 and (the argument of) Proposition VI.2.5.1 in [R].

(b) Suppose now that the subscheme $\mathbb{T}$ is reduced; i.e. $\mathbb{T} = \sup\{[\mathbf{P}] | \mathbf{P} \in \mathbf{Spec}\mathbb{T}\}$. Clearly $[Q(\mathbb{T})] = \sup\{[Q(P)] | \langle P\rangle \in \mathbf{Spec}\mathcal{A}\}$. Since $Q(P)$ is either zero, or belongs to $Spec\mathcal{A}/\mathbf{S}$, the subscheme $[Q(\mathbb{T})]$ is reduced. ∎

5. The Prime and Levitzki spectra of an abelian category and Zariski reduced subschemes.

Let $\mathbf{I}(\mathcal{A})$ denote the set of Zariski closed subschemes of an abelian category $\mathcal{A}$. Denote by $\mathbf{Prime}\mathcal{A}$ the set of all $\mathbb{P} \in \mathbf{I}(\mathcal{A})$ such that, for any pair $\mathbf{S}, \mathbb{T} \in \mathbf{I}(\mathcal{A})$, the inclusion $\mathbb{P} \subseteq \mathbb{S} \cup \mathbb{T}$ implies that either $\mathbb{P} \subseteq \mathbb{S}$, or $\mathbb{P} \subseteq \mathbb{T}$.

We consider $\mathbf{Prime}\mathcal{A}$ together with the 'specialization' preorder $\supseteq$.

The Zariski topology on $\mathbf{Prime}\mathcal{A}$ is defined in a usual way: closed subsets are all sets $\mathbf{V}(\mathbb{T}) := \{\mathbb{P} \in \mathbf{Prime}\mathcal{A} \mid \mathbb{P} \subseteq \mathbb{T}\}$, where $\mathbb{T}$ runs though $\mathbf{I}(\mathcal{A})$.

It follows from the definition of $\mathbf{Prime}\mathcal{A}$ that

$$\mathbf{V}(\mathbb{T} \cup \mathbf{S}) = \mathbf{V}(\mathbb{T}) \bigcup \mathbf{V}(\mathbf{S}) \qquad (1)$$

for any $\mathbb{T}, \mathbb{S} \in \mathbf{I}(\mathcal{A})$. And, for any family $\Xi \subseteq \mathbf{I}(\mathcal{A})$, we have

$$\mathbf{V}(\bigcap_{\mathbf{T}\in\Xi} \mathbb{T}) = \bigcap_{\mathbb{T}\in\Xi} \mathbf{V}(\mathbb{T}). \qquad (2)$$

For any object M in $\mathcal{A}$, define *the annihilator of M* as the intersection of all $\mathbb{T} \in \mathbf{I}(\mathcal{A})$ which contain M. The notation: $Ann(M)$.

5.1. Lemma. *(a) If $M \succ M'$, then $Ann(M) \subseteq Ann(M')$.*
In particular, $Ann(M)$ is well defined for any $M \in Ob\mathcal{A}$.
(b) $\bigcap_{M\in\Xi} Ann(M) = Ann(\oplus_{M\in\Xi} M)$.
(c) For any $\mathbf{P} \in \mathbf{Spec}\mathcal{A}$, $Ann(\mathbf{P}) \in \mathbf{Prime}\mathcal{A}$.

Proof. *(a)* The first assertion is evident.

(b) Any reflective subcategory $\mathbb{T}$ contains coproducts (taken in $\mathcal{A}$) of any set of its objects (provided this coproduct exists).

(c) Fix $P \in Spec\mathcal{A}$. Let $\mathbb{S}, \mathbb{T} \in \mathbf{I}(\mathcal{A})$ be such that $P \in Ob\mathbb{S} \cup \mathbb{T}$. Since $\mathbb{S} \cup \mathbb{T} \subseteq \mathbb{S} \bullet \mathbb{T}$, $P \in Ob\mathbb{S} \bullet \mathbb{T}$ which implies that either $P \in \mathbb{S}$, or $P \in \mathbb{T}$; i.e. either $Ann(\langle P \rangle) \subseteq \mathbb{S}$, or $Ann(\langle P \rangle) \subseteq \mathbb{T}$. ∎

5.2. Corollary. *If $\mathcal{A}$ is non-degenerate (i.e. $Supp(M) = \emptyset$ only if $M = 0$), then* $\mathbf{V}(\mathbb{T})$ *is non-empty for any* $\mathbb{T} \neq \mathbf{0}$.

Proof. In fact, for any $M \in Ob\mathbb{T}$, $\{P \in Spec\mathcal{A} | \langle P \rangle \in \mathrm{Supp(M)}\} \subseteq Ob\mathbb{T}$. Therefore, if $\mathbb{T}$ is nonzero, it contains an object P of $Spec\mathcal{A}$; hence it contains $Ann(P)$ which is prime by Lemma 5.1. ∎

5.3. Lemma. *Suppose that $\mathcal{A}$ is a local category. And let P be a quasifinal object in $\mathcal{A}$. Then $Ann(P)$ is the unique minimal element of* $\mathbf{I}(\mathcal{A}) - \mathbf{0}$.

Proof. In fact, any nonzero topologizing subcategory $\mathbb{T}$ contains all quasifinal objects of $\mathcal{A}$. ∎

5.4. Levitzki spectrum of an abelian category. Recall that a topological space X is called *sober* if every irreducible closed subset of X has a unique generic point. The inclusion functor $J_{\mathfrak{s}}$ from the category $\mathfrak{STop}$ of sober topological spaces into the category $\mathfrak{Top}$ of topological spaces has a left adjoint, $^{\wedge}J_{\mathfrak{s}}$, which assigns to any topological space X the set $X_{\mathfrak{s}}$ of all irreducible closed subsets of X with the strongest topology such that the map $\varphi_X : X \longrightarrow X_{\mathfrak{s}}, x \mapsto \{x\}^-$, is continuous. The map φ_X is a quasi-homeomorphism (i.e. it induces a bijection of the sets of open subsets of the spaces). The sober spaces are exactly topological spaces Y having the property: every quasi-homeomorphism with the domain Y is a homeomorphism. In particular, any quasi-homeomorphism from X to a sober space Y induces a homeomorphism from $X_{\mathfrak{s}}$ to Y.

5.4.1. Definition. *The Levitzki spectrum* of an abelian category $\mathcal{A}$ is the subspace $\mathbf{LSpec}\mathcal{A}$ of $\mathbf{Prime}\mathcal{A}$ formed by all $\mathbf{P} \in \mathbf{Prime}\mathcal{A}$ which are Zariski reduced.

5.4.2. Note. Clearly $\mathrm{Ann}(\mathbf{P}) \in \mathbf{LSpec}\mathcal{A}$ for any $\mathbf{P} \in \mathbf{Spec}\mathcal{A}$. If follows from the argument of Corollary 5.2 that if $\mathbb{T}$ is a nonzero closed subscheme, the corresponding closed subset of $\mathbf{LSpec}\mathcal{A}$, $\mathbf{VL}(\mathbb{T}) := \mathbf{V}(\mathbb{T}) \cap \mathbf{LSpec}\mathcal{A}$ is nonempty. ∎

5.4.3. Proposition. *Let $\mathcal{A}$ be an abelian category. The Levitzki spectrum* $\mathbf{LSpec}\mathcal{A}$ *of $\mathcal{A}$ is a sober topological space. The map φ :* $\mathbf{Spec}\mathcal{A} \longrightarrow \mathbf{LSpec}\mathcal{A}$, $\mathbf{P} \mapsto Ann(\mathbf{P})$, *is a quasi-homeomorphism of* $(\mathbf{Spec}\mathcal{A}, \tau_3)$ to $\mathbf{LSpec}\mathcal{A}$.

Proof. The map φ is continuous because

$$\varphi^{-1}(\mathbf{VL}(\mathbb{T})) := \{\langle P \rangle \in \mathbf{Spec}\mathcal{A} |\ Ann(P) \subseteq \mathbb{T}\} = \{\langle P \rangle \in \mathbf{Spec}\mathcal{A} | P \in Ob\mathbb{T}\} := V(\mathbb{T}). \quad (1)$$

Moreover, it follows from (1) that φ^{-1} defines a bijective map from the set of closed subsets of $\mathbf{LSpec}\mathcal{A}$ onto the set of closed subsets of $(\mathbf{Spec}\mathcal{A}, \tau_3)$; i.e. φ is a quasi-homeomorphism. It remains to show that $\mathbf{LSpec}\mathcal{A}$ is a sober space.

Note that a closed subset W of $\mathbf{LSpec}\mathcal{A}$ is irreducible iff $W = \mathbf{V}(\mathbb{P})$ for a prime subscheme $\mathbb{P}$. In fact, let $\mathbb{P}$ be the intersection of all closed subschemes of $\mathcal{A}$ containing elements of W. We claim that it is prime. Suppose $\mathbb{P} \subseteq \mathbb{S} \cup \mathbb{T}$ for some closed subschemes $\mathbb{S}$ and $\mathbb{T}$. Then $W \subseteq \mathbf{V}(\mathbb{P}) \subseteq \mathbf{V}(\mathbb{S} \cup \mathbb{T}) = \mathbf{V}(\mathbb{S}) \cup \mathbf{V}(\mathbb{T})$. Since W is irreducible, either $W \subseteq \mathbf{V}(\mathbb{S})$, or $W \subseteq \mathbf{V}(\mathbb{T})$. If $W \subseteq \mathbf{V}(\mathbb{S})$, then $\mathbb{P} \subseteq \mathbb{S}$ by definition of $\mathbb{P}$ and $\mathbf{V}(\mathbb{S})$.

Since all elements of W are reduced closed subschemes, $\mathbb{P}$ is a reduced closed subscheme: $\mathbb{P} = \mathbb{P}_{\mathrm{red}}$. Therefore $\mathbb{P} \in \mathbf{LSpec}\mathcal{A}$. Since $W = \mathbf{V}(\mathbb{P}) \cap \mathbf{LSpec}\mathcal{A}, \mathbb{P} \in$W. Thus every irreducible closed subset of $\mathbf{LSpec}\mathcal{A}$ has a generic point. The last observation to finish the argument: $\mathbf{LSpec}\mathcal{A}$ is a Kolmogorov's space, hence an irreducible closed set of $\mathbf{LSpec}\mathcal{A}$ cannot have more than one generic point. This follows directly from definitions: the relation $\mathbb{P} \in \mathbf{V}(\mathbb{P}')$ means that $\mathbb{P} \subseteq \mathbb{P}'$. So that if, in addition, $\mathbb{P}' \in \mathbf{V}(\mathbb{P})$, then $\mathbb{P} = \mathbb{P}'$. ∎

5.4.4. Proposition. *The map $\mathbb{T} \mapsto \mathbf{VL}(\mathbb{T})$ establishes a one-to-one correspondence between Zariski reduced closed subschemes of $\mathcal{A}$ and closed subsets of* $\mathbf{LSpec}\mathcal{A}$.

Proof. We have a map which assigns to any closed subset W of $\mathbf{LSpec}\mathcal{A}$ the intersection $\mathbb{T}_W$ of all closed subschemes of $\mathcal{A}$ containing all elements of W. Since W consists of Zariski reduced subschemes, $\mathbb{T}_W$ is also Zariski reduced. And $\mathbf{VL}(\mathbb{T}_W)$ coincides with W. This shows that the composition $\psi \circ \phi$ of the map $\phi : W \mapsto \mathbb{T}_W$ with the map $\psi : \mathbb{T} \mapsto \mathbf{VL}(\mathbb{T})$ is identity. On the other hand, if $\mathbb{T} = \mathbb{T}_{\mathrm{red}}$, the set $W = \mathbf{VL}(\mathbb{T})$ contains all $Ann(\mathbf{P})$, $\mathbf{P} \in \mathbf{Spec}\mathbb{T}$. This implies that $\mathbb{T}_W$ contains all $P \in Spec\mathbb{T}$. Therefore, since $\mathbb{T}$ is reduced, $\mathbb{T} \subseteq \mathbb{T}_W$. It follows from the definition of $\mathbb{T}_W$ that $\mathbb{T}_W \subseteq \mathbb{T}$. This shows that $\phi \circ \psi = id$. ∎

5.4.5. The prime spectrum and the Levitzki spectrum of the category of modules. Let $\mathcal{A} = R - mod$ for an associative ring R. Then set $\mathbf{I}(\mathcal{A})$ of closed subschemes of $\mathcal{A}$ are in bijective correspondence with the set $I(R)$ of two-sided ideals in R: to any two-sided ideal α there corresponds the full subcategory of $R - mod$ generated by all R-modules M such that $\alpha \subseteq Ann(M)$ (cf. Example 3.1). This correspondence induces a homeomorphism from $\mathbf{Prime}\mathcal{A}$ to the prime spectrum $SpecR$ of the ring R with Zariski topology. It follows from Theorem I.4.10.2 (and Lemma I.5.2) in [R] that the homeomorphism $\mathbf{Prime}\mathcal{A} \longrightarrow SpecR$ induces a homeomorphism from $\mathbf{LSpec}\mathcal{A}$ onto the *Levitzki spectrum LSpecR* of R which by definition consists of all prime ideals p in R such that the quotient ring R/p has no locally nilpotent ideals.

6. Strong Zariski topology.

Let $\mathbb{T}$ be a Zariski closed subscheme, $J_{\mathbf{T}}$ the inclusion functor $\mathbb{T} \longrightarrow \mathcal{A}$, $\mathcal{O}_{\mathbf{T}} := J_{\mathbf{T}} \circ {}^{\wedge} J_{\mathbf{T}} : \mathcal{A} \longrightarrow \mathcal{A}$. We say that $\mathbb{T}$ is compatible with localization $Q : \mathcal{A} \longrightarrow \mathcal{A}/\mathbb{S}$ if $\mathcal{O}_{\mathbb{T}}$ is compatible with Q. The latter means that $Q \circ \mathcal{O}_{\mathbf{T}}(s)$ is invertible for any $s \in \mathrm{Hom}\mathcal{A}$ such that Qs is invertible.

6.1. Lemma. *Suppose $\mathbb{T}$ is compatible with a flat localization $Q : \mathcal{A} \longrightarrow \mathcal{B}$. Then the minimal subscheme of $\mathcal{B}$ generated by $Q(\mathbb{T})$ is closed.*

Proof. In fact, the compatibility of $\mathcal{O}_{\mathbb{T}}$ with Q means that there exists a unique functor $\mathcal{O}'_{\mathbb{T}}$ such that $Q \circ \mathcal{O}_{\mathbf{T}} = \mathcal{O}'_{\mathbb{T}} \circ$Q. The latter equality and the isomorphism $Q \circ Q^{\wedge} \longrightarrow \mathrm{Id}_{\mathcal{B}}$ imply that there is a canonical isomorphism $\mathcal{O}'_{\mathbf{T}} \simeq Q \circ \mathcal{O}_{\mathbf{T}} \circ Q^{\wedge}$. Replacing $\mathcal{O}'_{\mathbf{T}}$ in the equality $Q \circ \mathcal{O}_{\mathbb{T}} = \mathcal{O}'_{\mathbf{T}} \circ Q$ by $Q \circ \mathcal{O}_{\mathbf{T}} \circ Q^{\wedge}$, we obtain the following criteria of compatibility:

$\mathcal{O}_{\mathbb{T}}$ is compatible with the localization Q iff the canonical morphism $Q \circ \mathcal{O}_{\mathbb{T}} \longrightarrow Q \circ \mathcal{O}_{\mathbf{T}} \circ Q^{\hat{}} \circ Q$ is an isomorphism. The adjunction epimorphism δ:$\mathrm{Id}_{\mathcal{A}} \longrightarrow \mathcal{O}_{\mathbf{T}}$ induces a morphism $\delta' : Id_{\mathcal{B}} \longrightarrow \mathcal{O}'_{\mathbf{T}}$ which is the composition of the inverse to the adjunction isomorphism $Id_{\mathcal{B}} \simeq Q \circ Q^{\hat{}}, Q\delta Q^{\hat{}}$, and the isomorphism $Q \circ \mathcal{O}_{\mathbb{T}} \circ Q^{\hat{}} \longrightarrow \mathcal{O}'_{\mathbf{T}}$. We have canonical isomorphisms:

$$\mathcal{O}'_{\mathbf{T}} \circ \mathcal{O}'_{\mathbf{T}} \simeq (Q \circ \mathcal{O}_{\mathbf{T}} \circ Q^{\hat{}}) \circ (Q \circ \mathcal{O}_{\mathbf{T}} \circ Q^{\hat{}}) \simeq Q \circ \mathcal{O}_{\mathbf{T}} \circ \mathcal{O}_{\mathbf{T}} \circ Q^{\hat{}} \simeq Q \circ \mathcal{O}_{\mathbf{T}} \circ Q^{\hat{}} \simeq \mathcal{O}'_{\mathbf{T}}$$

showing that the functor $\mathcal{O}'_{\mathbf{T}}$ induces a functor $\mathcal{B} \longrightarrow \mathbb{T}'$, where $\mathbb{T}'$ is the full subcategory of $\mathcal{B}$ generated by all objects M of $\mathcal{B}$ such that $\delta'(M)$ is an isomorphism.

(a) It follows from the definition of $\mathbb{T}'$ that $\mathbb{T}'$ is a reflective subcategory of $\mathcal{B}$: the functor $\mathcal{O}'_{\mathbf{T}}$ takes values in $\mathbb{T}'$ and induces a left adjoint to the inclusion functor $\mathbb{T}' \longrightarrow \mathcal{B}$.

(b) Note that $Q(\mathbb{T}) \subseteq \mathbb{T}'$. Indeed, since $Ob\mathbb{T} = \{X \in Ob\mathcal{A} | \mathcal{O}_{\mathbb{T}}(X)\}$, for any $X \in Ob\mathbb{T}$, we have isomorphisms

$$\mathcal{O}'_{\mathbf{T}} \circ Q(X) \simeq Q \circ \mathcal{O}_{\mathbf{T}} \circ Q^{\hat{}} \circ Q(X) \simeq Q \circ \mathcal{O}_{\mathbf{T}}(X) \simeq Q(X).$$

On the other hand, an object $Q(Y)$ belongs to $\mathbb{T}'$ iff the adjunction morphism $Q(Y) \longrightarrow \mathcal{O}'_{\mathbf{T}} \circ Q(Y)$ is an isomorphism. Thus we have canonical isomorphisms

$$Q(Y) \simeq \mathcal{O}'_{\mathbf{T}} \circ Q(Y) \simeq Q \circ \mathcal{O}_{\mathbf{T}} \circ Q^{\hat{}} \circ Q(Y) \simeq Q \circ \mathcal{O}_{\mathbb{T}}(Y)$$

showing that the object $Q(Y)$ is isomorphic to an object of $Q(\mathbb{T})$.

(c) We claim that $\mathbb{T}'$ is a subscheme of $\mathcal{B}$. In fact, since $\mathbb{T}'$ it reflective, contains all quotients of each of its objects. Let $M \in Ob\mathbb{T}'$; and let $L \longrightarrow M$ be a monomorphism in $\mathcal{B}$. By a standard argument, there exists a commutative diagram

$$\begin{array}{ccc} L & \longrightarrow & M \\ \uparrow & & \uparrow \\ Q(L') & \xrightarrow{Q\iota} & Q(M') \end{array} \tag{1}$$

in which ι is a monomorphism, $M' \in Ob\mathbb{T}$, and both vertical arrows are isomorphisms. Since $\mathbb{T}$ is a topologizing subcategory, $L' \in Ob\mathbb{T}$. By (c), $Q(L') \in Ob\mathbb{T}'$. Hence $L \in Ob\mathbb{T}'$. Thus $\mathbb{T}'$ is a topologizing subcategory of $\mathcal{B}$. Being a reflective subcategory of $\mathcal{B}$ (cf. (a)), $\mathbb{T}'$ is closed with respect to colimits (taken in $\mathcal{B}$). Hence $\mathbb{T}'$ is a closed subscheme of $\mathcal{B}$. ∎

6.2. Note. A closed subscheme $\mathbb{T}$ is compatible with a flat localization Q iff the defining ideal $K_{\mathbf{T}}$:=Ker($\mathrm{Id}_{\mathcal{A}} \longrightarrow \mathcal{O}_{\mathbf{T}}$) of $\mathbb{T}$ is compatible with Q. In fact, we have a commutative diagram

$$\begin{array}{ccccccccc} 0 \longrightarrow & Q \circ K_{\mathbf{T}} \circ Q^{\hat{}} \circ Q & \longrightarrow & Q \circ Q^{\hat{}} \circ Q & \longrightarrow & Q \circ \mathcal{O}_{\mathbb{T}} \circ Q^{\hat{}} \circ Q & \longrightarrow 0 \\ & \uparrow & & \uparrow Q^{\hat{}} & & \uparrow & \\ 0 \longrightarrow & Q \circ K_{\mathbf{T}} & \longrightarrow & Q^{\hat{}} & \longrightarrow & Q \circ \mathcal{O}_{\mathbb{T}} & \longrightarrow 0 \end{array}$$

with exact rows and isomorphic central vertical arrow. Therefore the subscheme $\mathbb{T}$ is compatible with the localization Q, i.e. the right vertical arrow is an isomorphism iff the left vertical arrow is an isomorphism. The latter means that the defining ideal $K_{\mathbf{T}}$ is compatible with Q. ∎

6.3. Lemma. *Let $Q : \mathcal{A} \longrightarrow \mathcal{B}$ be a flat localization. The class of $\mathfrak{Z}_{\{Q\}}$ of closed subschemes of $\mathcal{A}$ compatible with Q is stable with respect to taking finite supremums and any intersections.*

Proof. Let $\mathfrak{F}$ be a family of closed subschemes.

(a) The defining ideal of $\cap\mathfrak{F}$ is $K_{\cap\mathfrak{F}} = \sup\{K_{\mathbb{T}} | \mathbb{T} \in \mathfrak{F}\}$ (cf. Lemma 2.7.1). Since the functor Q preserves colimits and $\mathcal{A}$ is a category with the property (sup), the fact that the natural arrow $Q \circ K_{\mathbf{T}} \longrightarrow Q \circ K_{\mathbf{T}} \circ Q^{\hat{}} \circ Q$ is an isomorphism for any $\mathbb{T} \in \mathfrak{F}$ implies the isomorphness of $Q \circ K_{\cap\mathfrak{F}} \longrightarrow Q \circ K_{\cap\mathfrak{F}} \circ Q^{\hat{}} \circ \mathrm{Q}$.

(b) Suppose the family $\mathfrak{F}$ is finite. The defining ideal of $\sup\mathfrak{F}$ is $\bigcap_{\mathbb{T}\in\mathfrak{F}} K_{\mathbf{T}}$ – the kernel of the canonical morphism $Id_{\mathcal{A}} \longrightarrow \oplus_{\mathbf{T}\in\mathfrak{F}}\mathcal{O}_{\mathbf{T}}$. Thus we have a commutative diagram

$$\begin{array}{ccccccccc}
0 \longrightarrow & Q \circ K_{\sup\mathfrak{F}} \circ Q^{\hat{}} \circ Q & \longrightarrow & Q \circ Q^{\hat{}} \circ Q & \longrightarrow & Q \circ (\oplus_{\mathbf{T}\in\mathfrak{F}}\mathcal{O}_{\mathbf{T}}) \circ Q^{\hat{}} \circ Q & \longrightarrow 0 \\
& \alpha \uparrow & & \uparrow Q\eta & & \uparrow & \\
0 \longrightarrow & Q \circ K_{\sup\mathfrak{F}} & \longrightarrow & Q & \longrightarrow & Q \circ (\oplus_{\mathbf{T}\in\mathfrak{F}}\mathcal{O}_{\mathbf{T}}) & \longrightarrow 0
\end{array} \qquad (1)$$

with exact rows. The central vertical arrow is an isomorphism. Since functors $\mathcal{O}_{\mathbf{T}}$ are compatible with the localization Q, the arrow α is also an isomorphism. Therefore the third vertical arrow is an isomorphism. Hence $\sup\mathfrak{F}$ is compatible with the localization Q. ∎

6.3.1. Corollary. *For any family Ξ of localizating subcategories (i.e. kernels of flat localizations) of an abelian category $\mathcal{A}$, the class $\mathfrak{Z}_{\Xi}$ of closed subschemes of $\mathcal{A}$ compatible with localizations at all $\mathbf{S} \in \Xi$ is stable with respect to taking finite supremums and any intersections.*

6.3.2. Corollary. *For any family Ξ of localizing subcategories of an abelian category $\mathcal{A}$ containing $\mathbf{0}$, the class the family of sets $\{\mathbf{Spec}\mathbb{T} | \mathbb{T} \in \mathfrak{Z}_{\Xi}\}$ (cf. Corollary 6.3.1) is the family of all closed sets of a topology $\tau_{\mathfrak{Z}_{\Xi}}$ on $\mathbf{Spec}\mathcal{A}$.*

6.4. The strong Zariski topology. Assume that every $\mathbf{P} \in \mathbf{Spec}\mathcal{A}$ is localizable. We denote by $\mathfrak{T}_{\mathfrak{Z}}$ the topology $\tau_{\mathfrak{Z}_{\Xi}}$ of Corollary 6.3.2 in the case when Ξ consists of $\mathbf{0}$ and all $\mathbf{P} \in \mathbf{Spec}\mathcal{A}$. We call $\mathfrak{T}_{\mathfrak{Z}}$ *the strong Zariski topology.*

6.4.1. Lemma. *Let Ξ be a set of thick subcategories of an abelian category $\mathcal{A}$; and let $F : \mathcal{A} \longrightarrow \mathcal{A}$ be a functor compatible with localizations at each $\mathbb{S} \in \Xi$. Then F is compatible with localizations at the thick subcategory $\bigcap_{\mathbb{S}\in\Xi} \mathbb{S}$.*

Proof. The compatibility of F with localization at $\mathbb{S}$ means that, for any arrow s such that $Ker(s)$ and $Cok(s)$ are objects of $\mathbb{S}$, the morphism Fs enjoys the same property. Therefore $[Ker(s), Cok(s) \in Ob\bigcap_{\mathbb{S}\in\Xi}\mathbb{S}] \Leftrightarrow [KerF(s), CokF(s) \in Ob\bigcap_{\mathbb{S}\in\Xi}\mathbb{S}]$. ∎

6.4.2. Corollary. *Suppose a functor $F : \mathcal{A} \longrightarrow \mathcal{A}$ is compatible with the localizations at all points of* $\mathbf{Spec}\mathcal{A}$. *Then F is compatible with the localizations at any subset of* $\mathbf{Spec}\mathcal{A}$.

6.4.3. Corollary. *For any subset V of* $\mathbf{Spec}\mathcal{A}$ *the map* $\varphi : \mathbf{Spec}\mathcal{A} \longrightarrow \mathbf{Spec}\mathcal{A}/\langle V\rangle$ *sends closed subset of* $(\mathbf{Spec}\mathcal{A}, \mathfrak{T}_3)$ *into closed subsets of* $\mathrm{im}(\varphi)$ *regarded as a subspace of* $(\mathbf{Spec}\mathcal{A}/\langle V\rangle, \tau_3)$*, where* τ_3 *is the Zariski topology.*

Proof. Let $Q = Q_{\langle V\rangle}$ be a localization at $\langle V\rangle$. Let W be a closed subset of $(\mathbf{Spec}\mathcal{A}, \mathfrak{T}_3)$; i.e. $W = \mathbf{Spec}\mathbb{T}$ for some strongly closed subscheme $\mathbb{T}$. By 6.4.2, $\mathbb{T}$ is compatible with the localization Q. Hence the minimal subscheme $\mathbb{T}_V$ of $\mathcal{A}/\langle V\rangle$ generated by $\mathbb{T}$ is closed (Lemma 6.1). Clearly this miminal subscheme is compatible with localizations at every point of the image of $\mathbf{Spec}\mathcal{A}$. ∎

7. Reconstruction of schemes.

7.1. Ringed spaces associated to a category. Recall that the center of a category $\mathcal{A}$ is the ring $\mathfrak{C}\,(\mathcal{A})$ of endomorphisms of the identical functor $Id_{\mathcal{A}}$. A localization $Q : \mathcal{A} \longrightarrow \mathcal{B}$ maps the center of $\mathcal{A}$ into the center of $\mathcal{B}$. In particular, given a topology $\mathcal{T}$ of $\mathbf{Spec}\mathcal{A}$, there is a presheaf $\mathfrak{O} = \mathfrak{O}_{\mathcal{T}}$ of commutative rings on the space $(\mathbf{Spec}\mathcal{A}, \mathcal{T})$ which assigns to any open set U the center of the quotient category $\mathcal{A}/\langle U\rangle$. Denote the sheaf associated to $\mathfrak{O}_{\mathcal{T}}$ by $\mathcal{O}_{\mathcal{T}}$.

In the following theorem $\mathcal{T} = \mathfrak{T}_3$ – the strong Zariski topology.

7.2. Theorem. *Suppose that $\mathcal{A}$ is the category of quasi-coherent sheaves on an arbitrary scheme* $\mathbf{X}$. *Then the ringed space* $((\mathbf{Spec}\mathcal{A}, \mathfrak{T}_3), \mathcal{O}_{\mathcal{A}})$ *is isomorphic to* $\mathbf{X}$.

Proof. (a) Let $\mathcal{A}$ denote the category $Qcoh_{\mathbf{X}}$ of quasi-coherent sheaves on the scheme $\mathbf{X} = (X, \mathcal{O})$. We claim that the underlying space X is isomorphic to $(\mathbf{Spec}\mathcal{A}, \tau_3)$.

(a1) A map $\phi : X \longrightarrow \mathbf{Spec}\mathcal{A}$. Fix a point $x \in X$. Let p_x be the corresponding prime ideal in the local ring $\mathcal{O}_x$. We define a function P' on affine open sets of X as follows. To any affine open set U containing x, we assign the $\mathcal{O}(U)$-module $\mathcal{O}(U)/p_U$, where p_U is the preimage of p_x. In other words, p_U is the kernel of the canonical $\mathcal{O}(U)$-module morphism $\mathcal{O}(U) \longrightarrow (j_{U,x})_*(\mathcal{O}_x/p_x)$, where $j_{U,x}$ is the embedding $(Spec\mathcal{O}_x, \mathcal{O}_x) \longrightarrow \mathbf{X}$. We assign zero to any affine set which does not contain x. A standard argument shows that there exists a sheaf, unique up to isomorphism, P_x on X such that $P_x(U) = P'(U)$ for all affine open sets. Clearly P_x is quasi-coherent. We claim that $P_x \in Spec\mathcal{A}$.

In fact, let $\mathcal{M} \longrightarrow P_x$ be a monomorphism of quasi-coherent sheaves, and $\mathcal{M} \neq 0$. The latter implies that $\mathcal{M}(U) \neq 0$ for some affine neighborhood of x. Since $\mathcal{M}(U)$ is a submodule of $P_x(U) = \mathcal{O}(U)/p_U$ and p_U is a prime ideal in $\mathcal{O}(U)$, any choice of a nonzero element in $\mathcal{M}(U)$ provides a monomorphism $P_x(U) \longrightarrow \mathcal{M}(U)$. But then, by [Gr], Proposition I.9.4.2, $j_{U*}(P_x\,|_U)$ is a subsheaf of $\mathcal{M}$. Here j_U denotes the canonical embedding $(U, \mathcal{O}\,|_U) \longrightarrow \mathbf{X}$. Note now that P_x is a subsheaf of $j_{U*}(P_x\,|_U)$; hence P_x is a subsheaf of $\mathcal{M}$. This shows that $P_x \in Spec\mathcal{A}$. Since the sheaf P_x is defined uniquely up to isomorphism, P_x defines an element of $\mathbf{Spec}\mathcal{A}$ which does not depend on the choices made in the construction of P_x.

(a2) A map $\psi : \mathbf{Spec}\mathcal{A} \longrightarrow X$. Note that, for any $P \in Spec\mathcal{A}$, $Supp(P)$ is an irreducible closed subset of X. In fact, let U be any affine subset of X such that $P\,|_U$ is nonzero. Then $P(U) \in Spec(\mathcal{O}(U) - mod)$. This implies that $P(U)$ is equivalent to the module $\mathcal{O}(U)/p_U$, where p_U is the annihilator of $P(U)$, and p_U is a prime ideal. Therefore $SuppP(U) = V(p_U) := \{x \in U \mid p_U \subseteq p_x\}$ is an irreducible closed subset. Note that

$\mathcal{O}(U)/p_U$ is a submodule of $P(U)$. And $\mathcal{O}(U)/p_U$ can be identified with $P_x(U)$, where x is the generic point of $SuppP(U)$ (cf. (a1)). Since $P_x \mid_U$ is a subsheaf of $P \mid_U$, P_x is a subsheaf of P (cf. the argument in (a1)). Since $P \in Spec\mathcal{A}$ and P_x is nonzero, P_x is equivalent to P. This implies that Supp(P)=Supp(P_x) $= \{x\}^-$. The map ψ assigns to the point $\langle P \rangle$ of $\mathbf{Spec}\mathcal{A}$ the generic point x of the support of P. It remains to notice that the map ψ is well defined, i.e. it does not depend on the choice of P inside of the equivalence class. In fact, if $\mathcal{M}$ is a quasi-coherent sheaf such that $\mathcal{M} \succ P$, then, for any $y \in X$, $\mathcal{M}_y \succ P_y$. This follows from the fact that the localization at a point, $\mathcal{M} \mapsto \mathcal{M}_y$, is an exact functor, hence it preserves the preorder $\succ$. In particular $\mathcal{M}_y = 0$ implies that $P_y = 0$; i.e. $Supp(P) \subseteq Supp(\mathcal{M})$. Thus if $\mathcal{M} \succ P \succ \mathcal{M}$, then $Supp(P) = Supp(\mathcal{M})$.

It is clear from the argument above that $\phi \circ \psi = id_{\mathbf{Spec}\mathcal{A}}$ and $\psi \circ \phi = Id_X$. One can see also that the map ψ is a morphism of ordered sets with the preorder on X given by specialization. Therefore ψ and ϕ are mutually inverse isomorphisms of the preordered sets.

(a3) For any closed subset V of X, denote by $\mathcal{J}_V$ the defining ideal of the reduced subscheme of $\mathbf{X}$ with the underlying space V. Thus we have a map

$$V \mapsto \mathbf{Spec}[\mathcal{O}_{\mathbf{X}}/\mathcal{J}_V] = \{\langle P \rangle \in \mathbf{Spec}\mathcal{A} | P \text{ is annihilated by } \mathcal{J}_V\}$$

from the set of Zariski closed subsets of X to the set of Zariski closed subsets of $\mathbf{Spec}\mathcal{A}$. This map is nothing else but $\phi = \psi^{-1} : V \mapsto \phi(V)$. Hence the map ψ is continuous.

Conversely, let W be a closed subset of $(\mathbf{Spec}\mathcal{A}, \mathfrak{T}_3)$. Let $\mathbb{T} = \mathbb{T}_W$ denote a strongly closed subscheme of $\mathcal{A}$ having the spectrum W. It follows from Lemma 6.4.3 that $\psi(W) \cap U$ is a Zariski closed subset of U for any affine open set U. Therefore $\psi(W) = \phi^{-1}(W)$ is Zariski closed which proves that ϕ is continuous.

(b) For any open affine subset U of $X, \mathcal{O}(U)$ is isomorphic to the center of the the category $\mathcal{A}/\langle U \rangle$. This is due to the equivalence of categories $\mathcal{A}/\langle U \rangle$ and $\mathcal{O}(U) - mod$ (cf. (a4)) and to the fact that, since the ring $\mathcal{O}(U)$ is commutative, the center of $\mathcal{O}(U) - mod$ is naturally isomorphic to $\mathcal{O}(U)$. ∎

7.3. Remark. We could use other canonical topologies in the construction of Theorem 7.2 to reconstruct schemes which belong to a certain class. For instance, we could use the topology τ_* (cf. 1.6.2) to reconstruct noetherian schemes and the topology τ^* of Subsection 1.6.3 to reconstruct quasi-compact quasi-separated schemes.

It is worth to mention that the reconstruction procedure presented here works in a much wider category than the category of schemes and provides a base for reconstruction theorems in other categories of spaces. For instance it can be used (as a principal step) for the reconstruction of certain classes of analytic spaces. ∎

References.

[A] M. Artin, Geometry of quantum planes, Contemporary Mathematics, v.124, 1992

[AZ] M. Artin, J.J. Zhang, Noncommutative projective schemes, preprint, 1994

[Gab] P.Gabriel, Des catégories abéliennes, Bull. Soc. Math. France, 90 (1962), 323-449

[Gr] A. Grothendieck, EGA I, 1961

[GZ] P. Gabriel and M. Zisman, Calculus of fractions and homotopy theory, Springer Verlag, Berlin-Heidelberg-New York, 1967
[M] Yu.I.Manin, Quantum Groups and Noncommutative Geometry, Publ. du C.R.M.; Univ. de Montreal, 1988
[R] A.L. Rosenberg, Noncommutative algebraic geometry and representations of quantized algebras, Kluwer Academic Publishers, Mathematics and Its Applications, v.330 (1995), 328 pages.
[R1] A.L. Rosenberg, Noncommutative local algebra, Geometric and Functional Analysis (GAFA), v.4, no.5 (1994), 545-585
[R2] A.L.Rosenberg, Left Spectrum, Levitzki Radical and Noncommutative Schemes, Proc. of Natnl. Acad., v.87, 1990
[R3] A. L. Rosenberg, The Spectrum of the Algebra of Skew Differential Operators and the Irreducible Representations of the Quantum Heisenberg Algebra, Commun. Math. Phys. 142, 567-588, 1991
[S] J.-P. Serre, Faisceaux algébriques cohérents, Annals of Math.62, 1955