FACTORIZATION OF BIRATIONAL MAPS
(after J. Włodarczyk)

Lecture given on the Mathematische Arbeitstagung
by Dan Abramovich
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This lecture is centered around the following theorem:

Theorem 1 (Weak Factorization). *Let $\phi : X_1 \dashrightarrow X_2$ be a birational map between two nonsingular complex projective varieties X_1 and X_2. Then ϕ can be factorized into a sequence of blowings up and blowings down with smooth centers. In other words, there exist nonsingular complex projective varieties V_i and a diagram*

$$\begin{tikzcd}
& W_1 \arrow[dl] \arrow[dr] & & \cdots \arrow[dl] \arrow[dr] & & W_{n-1} \arrow[dl] \arrow[dr] & \\
V_1 \arrow[d, equal] & & V_2 & & V_{n-1} & & V_n \arrow[d, equal] \\
X_1 & & & & & & X_2
\end{tikzcd}$$

where $W_i \to V_i$ and $W_i \to V_{i+1}$ are blowings up with nonsingular centers.

Much of the work leading to this theorem is due to Jarosław Włodarczyk, who also completed one of two currently known proofs of the theorems. The other proof is given in a joint paper of Kalle Karu, Kenji Matsuki, Jarosław Włodarczyk, and myself.

A more general version of this theorem is true - the reader is referred to the latter paper for a statement.

After giving a general overview of the results leading to the theorem, the lecture focuses on Włodarczyk's *birational cobordisms*, which are the crucial tools behind the two proofs.

Given a birational map $\phi : X_1 \dashrightarrow X_2$, a *birational cobordism* $B_\phi(X_1, X_2)$ is a variety of dimension $\dim(X_1) + 1$ with an action of the multiplicative group $\mathbb{C}^*$. It is analogous to the usual cobordism $B(M_1, M_2)$ between differentiable manifolds M_1 and M_2 given by a Morse function f. In the differential setting one can construct an action of the additive real group $\mathbb{R}$, where the "time" $t \in \mathbb{R}$ acts as a diffeomorphism induced by integrating the vector field $grad(f)$; hence the multiplicative group $(\mathbb{R}_{>0}, \times) = \exp(\mathbb{R}, +)$ acts as well. The critical points of f are precisely the fixed points of the action of the multiplicative group, and the homotopy type of fibers of f changes when we pass through these critical points. Analogously, in the algebraic setting "passing through" the fixed points of the $\mathbb{C}^*$-action induces a birational transformation. Looking at the action on the tangent space at each fixed point, we obtain a locally toric description of the transformation. This already gives a factorization of ϕ into *locally toric birational maps* among varieties with locally toric structures.

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