FOLIATIONS ON 3-MANIFOLDS

A report on D.Gabai's work

by

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It is well-known that every compact 3-manifold, M, (under certain conditions on ∂M) admits a foliation of codimension one. The proof of this result (in the closed orientable case) relies heavily on the fact that every such manifold can be obtained by Dehn surgery on a braid in the 3-sphere. This in turn implies that all foliations constructed in this way necessarily do admit a Reeb component. Indeed, by the work of S.P.Novikov, Reeb components cannot be avoided in any foliation on M when the fundamental group of M is finite.

Consequently, it is an interesting problem to characterize in some way all those compact 3-manifolds which admit a foliation of codimension one without Reeb components. In its generality this problem is still unsolved although meanwhile there are many contributions to it, mainly in the direction to generalize the condition on the fundamental group of M for the necessary existence of a Reeb component; see for instance.....

D.Gabai's work, however, must be considered as a very important step towards a complete solution of the above stated problem. In [GaI] he proves the following theorem:

Let M be a compact irreducible oriented 3-manifold whose boundary is a (possibly empty) union of tori. Let S be any norm minimizing surface in $(M,\partial M)$ such that $[S] \in H_2(M,\partial M)$ is non-trivial. Then there exists a codimension-one foliation, F, on M without Reeb components and such that F is transverse to ∂M and contains S as a leaf. Moreover, if S is not a torus then F is smooth.

A more detailed version of this theorem, again due to Gabai, and some of the most striking consequences following from it will be given in chapter III.

Note that, by work of Thurston, any compact leaf in a foliation without Reeb components is norm minimizing, and, by results of Alexander and Rosenberg, the underlying manifold must be irreducible; [Al], [Ro]. Also a standard argument using the Euler characteristic shows that the condition on the non-triviality of $[S]$ cannot be dropped.

These notes refer mainly to the paper [GaI]. They are based on lectures I gave at the University of Bielefeld in 1985 and are organized as follows:

In chapter I we study the basic theory of coloured 3-manifolds. (Gabai calls them "sutured" 3-manifolds, but the sutures do not play any role in our context.) The central result here will be the proof of the existence of a suitable splitting surface in any taut coloured 3-manifold such that the decomposed manifold is again taut. (For unexplained definitions see chp.I, § 4 and 5.)

In chapter II we adapt the classical concept of Haken hierarchies to coloured manifolds and show the existence of coloured manifold hierarchies. This will be done in an inductive way of proof using the notion of complexity for taut coloured manifolds.

Coloured manifold hierarchies are used in chapter III to construct the desired foliations. After having performed these constructions we shall discuss some corollaries following from the existence of such foliations.

CHAPTER I

Coloured 3-manifolds-basic theory and decomposition theorem

1. Preliminaries

2. Incompressibility

3. The Thurston norm

4. Coloured 3-manifolds

5. Coloured manifold decompositions

6. The coloured manifold decomposition theorem

1. Preliminaries

The goal of this first chapter is the proof of the decomposition theorem for taut coloured 3-manifolds (see 6.1). This provides the first of three main steps by which the principal result of these notes is composed.

Although most often the 3-manifold M under consideration will be connected we also have to deal with non-connected compact 3-manifolds. Whenever this occurs and it is essential for the argument that M is not necessarily connected we will point out this fact by speaking of a 3-manifold system.

A surface is always connected. If we have to do with a not necessarily connected compact 2-manifold then we refer to it as a surface system. ‡

1.1. - Orientations. In what follows all 3-manifolds M and surfaces are smooth and oriented. Then ∂M supports an induced orientation which is determined by the requirement that the normal to ∂M be pointing outwards of M.

Also when S is a properly embedded orientable surface in M then fixing an orientation of S amounts to choosing a normal direction to S. We then can speak of the right

‡ For definitions concerning 3-manifolds and surfaces related to them we refer the reader to the books [He], [Ja] and [JS] of Hempel, Jaco and Shalen.

(resp. left) hand side of S .

Note that ∂S , if non-empty, also carries an orientation which is determined by that of S in just the same way as that of ∂M above.

1.2. - <u>Gluing</u>. Suppose we are given 3-manifolds M_0 and M_1 and a diffeomorphism

$$\varphi : R_0 \longrightarrow R_1$$

where R_i is a system of compact surfaces in ∂M_i, $i = 0,1$, possibly with non-empty boundary. Then

$$N = M_0 \underset{\varphi}{\cup} \bar{M}_1$$

denotes the manifold obtained by gluing together M_0 and M_1 by means of φ .

If M_0 and M_1 are oriented and φ is orientation preserving then we obtain an orientation of N by changing the orientation of M_1 but not that of ∂M_1 .

Similarly, when S_i denotes a properly embedded surface system in M_i meeting R_i transversely and such that φ maps $S_0 \cap R_0$ diffeomorphically onto $S_1 \cap R_1$ then $S = S_0 \underset{\varphi}{\cup} S_1$ is a properly embedded surface system in M .

Moreover, smoothing the corners possibly arising in this process all of N, $R = R_0 = R_1$ and S become smooth.

The case most interesting for us is when $M_0 = M_1$, and $\varphi = \mathrm{id}$. The resulting manifold is then referred to as the <u>double of</u> M <u>along</u> $R (= R_0 = R_1)$.

1.3. - <u>Transversality</u>. In what follows we always require that surfaces in M occuring in the argument intersect one another transversely. In particular, any proper surface S meets the boundary of M transversely. Moreover, if R is a compact subsurface of ∂M arising from context then S is required to be transverse to R and to ∂R .

Note that transversality can always be established by an arbitrarily small isotopy of one of the involved surfaces. Actually, all we do in the sequel is independent of such small isotopies.

1.4. - <u>Modifying transverse surfaces</u>. Let M be a compact oriented 3-manifold, possibly with $\partial M \neq \phi$ and let K be a surface system in ∂M , possibly with $\partial K \neq \phi$. Suppose that S and T are properly embedded oriented surface systems such that $\partial S \cup \partial T \subset \operatorname{int} K$. We modify S and T in a neighbourhood of $S \cap T$ so as to obtain a new oriented surface system, denoted $S \supset\!\subset T$, with the following properties:

(1) $\quad [S \supset\!\subset T] = [S] + [T] \in H_2(M,K)$,

(2) $\quad \chi(S \supset\!\subset T) = \chi(S) + \chi(T)$.

By transversality, $S \cap T$ consists of finitely many arcs and circles. Let c be any such component. Then in a neighbourhood of c, $S \cup T$ looks like $X \times c$, where

$$X = \{(x,y) \in \mathbb{R}^2 \mid -1 \leq x,y \leq 1 \ , \ x = 0 \ \text{ or } \ y = 0\} \ .$$

We now modify S and T by replacing $X \times c$ by $X' \times c$, where

$$X' = \{(x,y) \in \mathbb{R}^2 \mid 0 \leq xy \leq 1, (x-1)^2+(y-1)^2=1 \ \text{ or } \ (x+1)^2+(y+1)^2=1\}$$

or by $X'' \times c$, where

$$X'' = \{(x,y) \in \mathbb{R}^2 \mid -1 \leq xy \leq 0, (x+1)^2+(y-1)^2=1 \ \text{ or } \ (x-1)^2+(y+1)^2=1\} \ ,$$

according as S and T are (transversely) oriented as indicated in fig.1 a) or b).

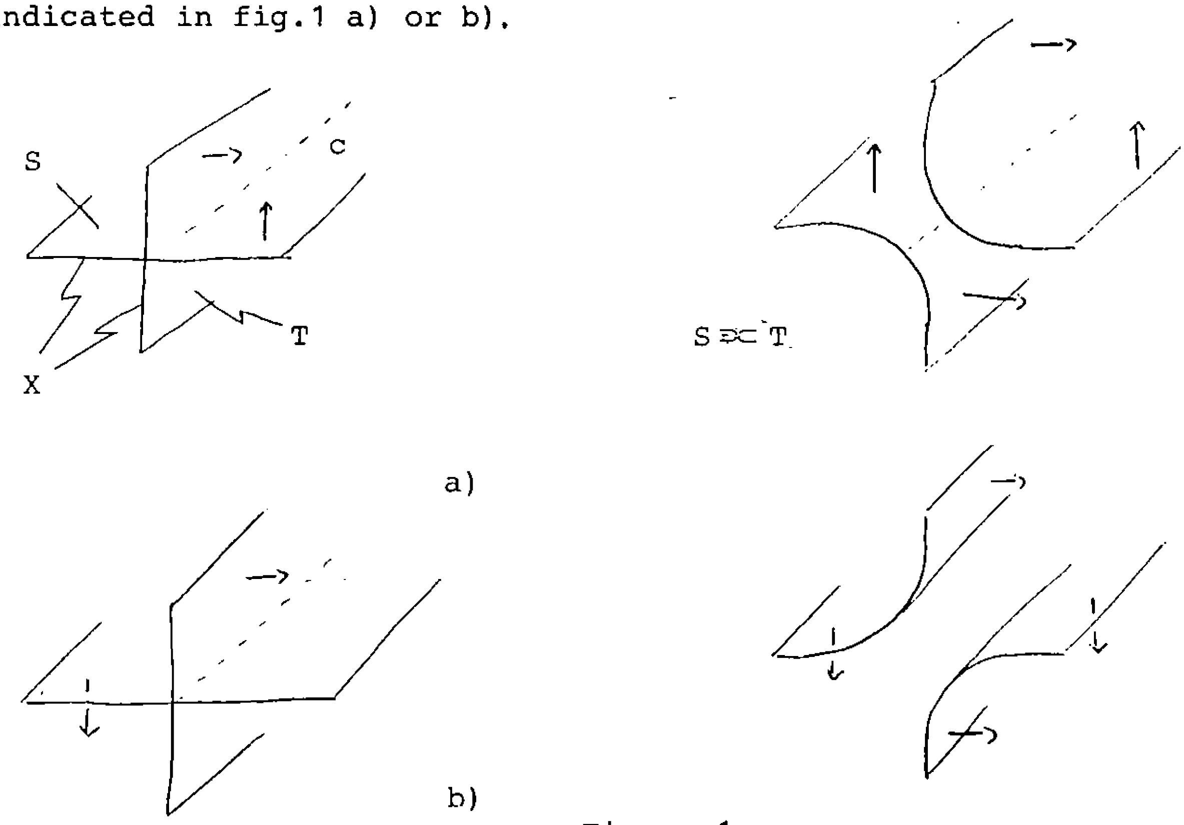


Figure 1

It is clear from this construction that conditions (1) and (2) hold.

1.5. - <u>The Pontryagin construction for proper surfaces</u>

By our discussion above, an orientation of a surface system S in an oriented 3-manifold is the same as choosing a normal field or, what is the same, a framing of S. If S is proper and $\partial S \neq \emptyset$ then, by our transversality convention, a framing may be always found so that its restriction on ∂S is a framing of $\partial S \subset \partial M$.

Pontryagin gave a construction which he used to prove the following result. You can find this construction and a proof of the theorem in Milnor's book [Mi]. There you also find the corresponding definitions. Although Milnor considers only the case that S and M are closed the theorem holds for arbitrary surface systems which are transverse to the boundary of M. Indeed, this generalization is routine work.

<u>Theorem</u> (Pontryagin [Po1], [Po2]) <u>Let</u> M <u>be an orientable compact</u> 3-<u>manifold. Then there exists a one to one correspondence between homotopy classes of smooth maps of</u> M <u>to</u> S^1 <u>and framed cobordism classes of framed surface systems in</u> M.

<u>Furthermore, every properly embedded orientable surface system</u> S <u>in</u> M <u>is of the form</u> $S = f^{-1}(t)$ <u>for some smooth map</u> $f : M \longrightarrow S^1$ <u>and regular value</u> $t \in S^1$.

1.6. - Simple curves on surfaces. Later we shall frequently use the following result on the homology of curves in surfaces.

Proposition. Let V be a compact oriented surface, possibly with $\partial V \neq \emptyset$, and let C be a system of pairwise disjoint proper curves in V such that one of the following two conditions holds:

(1) $0 \neq [C] \in H_1(V,\partial V)$ and $\langle C,c\rangle = 0$ for every component c of ∂V.

(2) V is planar and $\langle C,c\rangle \neq 0$ for at most two components c of ∂V.

Then there exists a sequence of systems of pairwise disjoint proper curves $C_0,\ldots\ldots,C_p$ such that

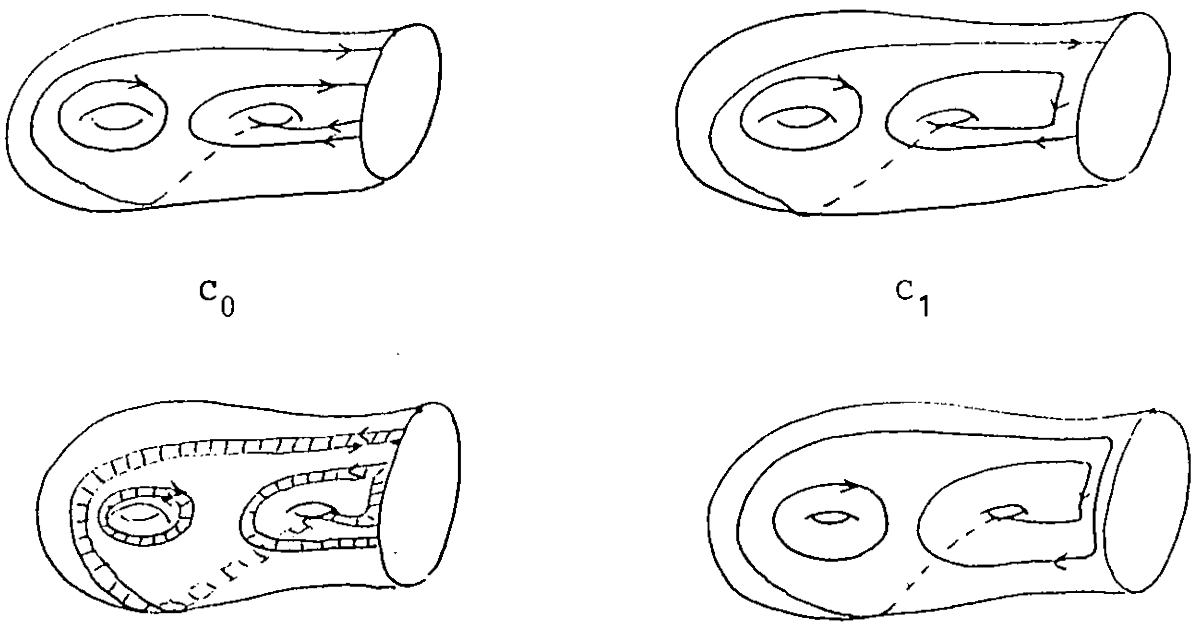
$$C_0 = C\ ,\ C_{i+1} \cup (-C_i) = \partial W_i \pmod{\partial V}$$

for some compact sub-surface W_i of V (with orientation inherited from V), and for some $k \in \mathbb{Z}$ we have according as (1) or (2) holds:

(1') C_p is a system of k parallel oriented simple closed curves.

(2') C_p is a system of k parallel oriented proper arcs.

Proof. We first consider the case that (1) holds. Then C_1 is obtained from $C_0 = C$ by reducing the number of points in $C \cap \partial V$ two by two as indicated in figure 2. If necessary we iterate this process until we have a system of curves, C_j, consisting entirely of circles in $\mathring{V}$.

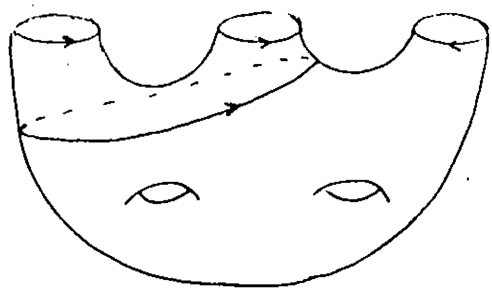


C_0 C_1

$C_1 \cup (-C_0) = \partial W_0 \mod \partial V$ $C_2 = C_j$

Figure 2

Now let W denote the closure of some component of $V - C_j$. If more than two components of ∂W come from different components of C_j then two of these are cobordant in W to a simple closed curve; see figure 3.



W

Figure 3

This shows that by a series of reductions of the number of components of C_j we obtain a system C_k such that the boundary of every component of $V-C_k$ has exactly two curves belonging to C_k . This shows that (1') holds.

For the proof of (2') we can assume (possibly after a series of reductions as above) that C is a system of proper arcs such that

$$|< C,c >| = \# (C \cap c)$$

for every boundary curve c of V . Then, since V is planar, any two arcs belonging to C are homologous mod ∂V . We conclude as in the proof of (1') to see that (2') holds.

$\square$

1.7. - Existence of suitable homology classes.

In the decomposition theorem (6.1) we have to find a splitting surface S in M whose intersection with ∂M is non-trivial in homology. The existence of such a surface is based on 3.2, iii) together with the following general fact on the homology of the pair $(M, \partial M)$.

Proposition. *Let* $M \neq D^3$ *be a compact oriented irreducible 3-manifold with* $\partial M \neq \phi$ *and let* R *be a compact sub-manifold of* ∂M *such that* $\Sigma = \partial M - \overset{\circ}{R}$ *consists of annuli and tori or is empty. Then there exists* $\alpha \in H_2(M, \partial M)$ *such that*

(1) $0 \neq \partial\alpha \in H_1(\partial M)$,

(2) *for each non-planar component,* V , *of* R *and each component,* c , *of* ∂V *we have* $\langle\alpha, [c]\rangle = 0$,

(3) *for each planar component* V *of* R *there exist at most two components,* c_1 *and* c_2 , *of* ∂V *such that* $\langle\alpha, [c_i]\rangle \neq 0$, $i = 1,2$.

Proof. Let $k = \frac{1}{2} b_1(\partial M) > 0$ be the number of handles of ∂M and let

$$\partial : H_2(M, \partial M) \longrightarrow H_1(\partial M)$$

be the boundary homomorphism. From the exact homology sequence of $(M, \partial M)$ we deduce

$$b_3(M,\partial M)-b_2(\partial M)+b_2(M)-b_2(M,\partial M)+\operatorname{rank}(\operatorname{im}\partial) = 0 ,$$

and thus (as $b_0(M,\partial M)=0$ and $b_2(M) = b_1(M,\partial M)$)

$$\begin{aligned}\operatorname{rank}(\operatorname{im}\partial) &= \chi(M,\partial M) + b_2(\partial M)\\ &= b_2(\partial M) - \tfrac{1}{2}\chi(\partial M)\\ &= \tfrac{1}{2} b_1(\partial M)\\ &= k .\end{aligned}$$

Now we choose generators $\alpha_1,\dots,\alpha_k$, $\alpha_{k+1},\dots$ for $H_2(M,\partial M)$ in such a way that $\partial\alpha_1,\dots,\partial\alpha_k$ are linearly independent in $H_1(\partial M) \otimes \mathbb{R}$. We also select a maximal family of annuli $A_1,\dots,A_j$ in Σ such that no component of $\partial M - \cup \mathring{A}_i$ is planar. Clearly, $0 \leq j \leq k-1$, and $j = 0$ only if $\Sigma = \phi$ or all annulus components of Σ lie on tori. In these cases conditions (2) and (3) are, however, trivially fulfilled.

It follows that there exists an integer linear combination α of the α_i such that

$$\langle\partial\alpha, c_i\rangle = 0 \quad \text{for} \quad i = 1,\dots,j ,$$

where c_i is any boundary curve of A_i .

Let S be a proper surface system in M representing α . We may assume that

$$S \cap \cup A_i = \phi .$$

When $S \cap A_i$ contains circles or proper arcs each of which separates A_i, this can be achieved simply by pushing S off A_i. If $S \cap A_i$ consists of pairs of oppositely oriented parallel arcs none of which is separating then we remove two innermost of these arcs by attaching a square Q in A_i to S and then pushing the surface so obtained along Q slightly into $\mathring{M}$; see figure 4. This process can be repeated.

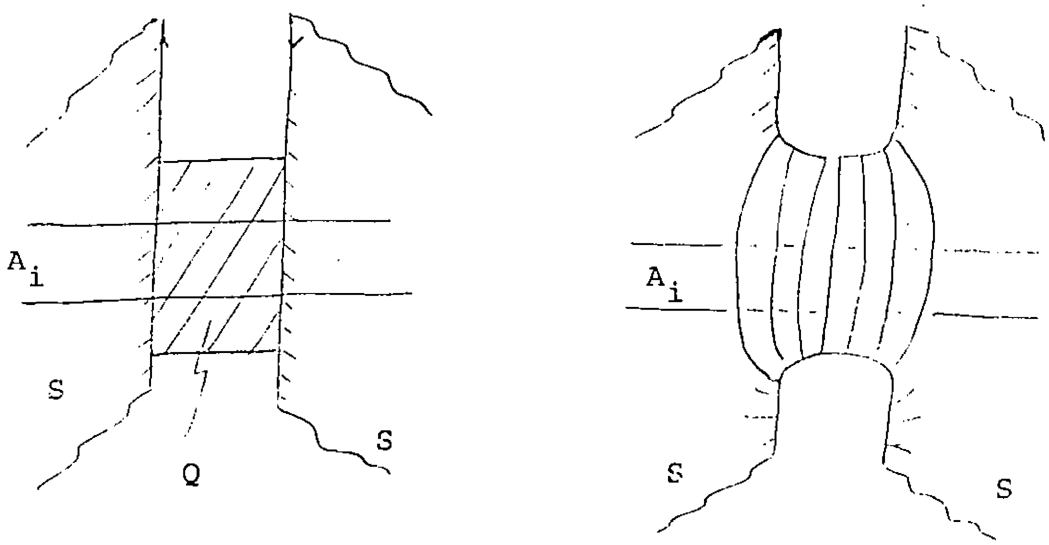
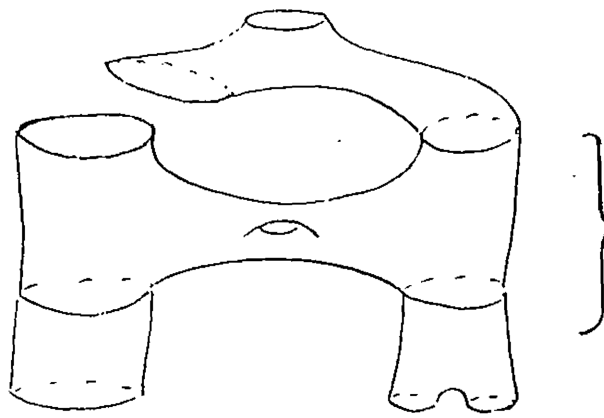


Figure 4: Pushing the square Q into $\mathring{M}$.

Let V be a component of R and let W be the component of $\partial M - \cup A_i$ which contains V.

If V is non-planar then, by the maximality of $\cup A_i$, V and W have the same genus. It follows that every boundary circle of V is homologous in W to a union of boundary circles of W; see fig.5. Since $\partial W \cap S = \phi$ this implies (2).



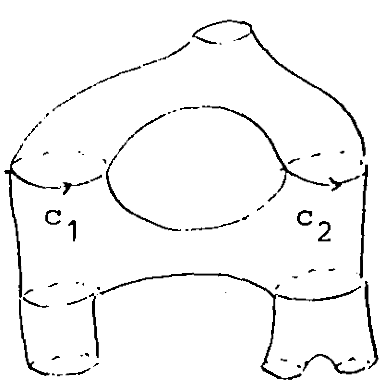
$V \subset W$

Figure 5

If V is planar then let c_1 be a boundary circle of V such that $\langle S,c_1\rangle \neq 0$. Then, since $S \cap \partial W = \phi$, c_1 is contained in $\mathring{W}$ and is not homologous in W to a set of boundary curves of W . We conclude that $W-c_1$ is connected and, a second time by the maximality of $\cup A_i$, the component of $W-\mathring{V}$ containing c_1 in its boundary must be planar, with exactly one further boundary circle, c_2 , belonging to V (see fig.6). It follows that $\langle S,c_2\rangle \neq 0$.

Again by the maximality of $\cup A_i$, all other boundary curves of V are null-homologous mod ∂W and thus have trivial algebraic intersection with S . This proves (3).

□



$V \subset W$

Figure 6

2. Incompressibility

According to Jaco [Ja] we use the following definition of incompressibility.

2.1. - Definition. Let S be a surface system in M whose components are either sub-surfaces of boundary components or properly embedded. We say that S is compressible in M if either S contains a 2-sphere bounding a ball or there exists a disk D in M such that $D \cap S = \partial D$ and $0 \neq [\partial D] \in \pi_1(S_0)$, where $S_0 \subset S$ is the component containing ∂D.

Otherwise the system S is referred to as being incompressible in M.

Note that, by this definition, any system of properly embedded disks is incompressible in M.

2.2. - Remarks. i) If every component of a surface system S is incompressible then clearly so is S. It is not hard to see that the converse of this statement is also true. We leave this as an exercise to the reader.

ii) (Cf. [He; 6.1 and 6.2]) Suppose as always that M and S are orientable. Then S incompressible implies that for each component S_0 of S the inclusion of S_0 in M induces an injection $\pi_1 S_0 \longrightarrow \pi_1 M$ of fundamental groups.

Conversely, if $\ker(\pi_1 S_0 \longrightarrow \pi_1 M) = 1$ for some non-spherical component S_0 of S then S_0 is incompressible.

iii) Suppose that the 3-manifold N is obtained by gluing together (M,K) and (M',K') by means of a homeomorphism between K and K', where K (and similarly K') is a surface system in ∂M, possibly with $\partial K \neq \phi$. Then $K = K'$ is incompressible in N if and only if K and K' are incompressible in M resp. M'.

iv) Moreover, if K is incompressible in N then N is irreducible if and only if both M and M' are irreducible.

Statements i), iii) and iv) are proved by standard arguments and are left as exercises to the reader. A proof of ii) involves the Loop theorem. Note also that N need not be irreducible when K or K' is compressible in M resp. M'.

2.3. - Boundary incompressibility. Let $K \subset \partial M$ be as above and let S be a properly embedded surface system in M with $S \cap K = \partial S$.

We say that S is K-compressible if either there exists a disk component of S which is parallel to a disk in K or there exist a component $S_0 \neq D^2$ of S and a disk D in M such that

$$D \cap S = D \cap S_0 = c$$

is an arc in ∂D and

$$\partial D = c \cup d, \; c \cap d = \partial c = \partial d ,$$

where d is an arc in K and either S_0 is separated by c, in which case none of the resulting surfaces is a disk, or c does not separate S_0; see fig. 7.

Otherwise we call S K-<u>incompressible</u>.

When $K = \partial M$, we simply write ∂-compressible resp. ∂-incompressible and say that S is <u>boundary compressible</u> resp. <u>boundary incompressible</u>.

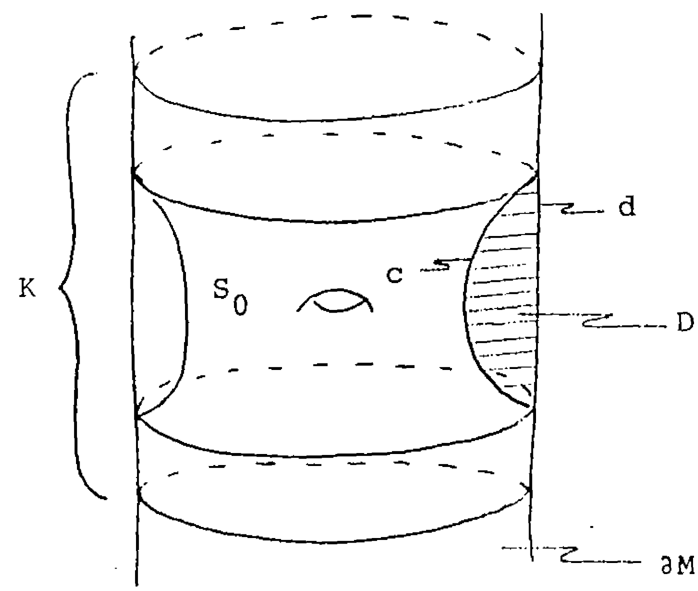


Figure 7

Again it is not hard to see that in case S is incompressible the K-incompressibility of S is equivalent to the K-incompressibility of each of its components.

2.4. - <u>Compressions</u>. i) Let (M,K) be as above and suppose we are given a surface system S with $S \cap K = \partial S$

and such that every component of S is either contained in ∂M or is properly embedded.

If $S_0 \neq S^2$ is a compressible component of S then we can "compress" S_0 by means of a spanning disk in the usual way so as to obtain from S_0 a new surface system S_0' (consisting of one or two components) and leaving $S - S_0$ unchanged. This process is described in detail for instance in [ ]; compare also 2.2, i).

The system S_0' and similarly the system $(S - S_0) \cup S_0'$ enjoys the following properties:

(1) S_0' is homologous to S_0 rel K .

(2) $\chi(S_0') = \chi(S_0) + 2$.

(3) If S_0 is a torus then S_0' is a sphere. Thus $[S_0] = 0 \in H_2(M,K)$ when M is irreducible.

(4) If S_0 is an annulus then S_0' consists of two disks. Therefore if K is incompressible and M is irreducible then again $[S_0] = 0$.

(5) If S_0 is a punctured torus then S_0' is a single disk or consists of a disk and a torus. If K is incompressible and M is irreducible then it follows that $[S_0] = 0$ or $[S_0] \neq 0$ and is represented by a torus in M .

ii) In a similar way we proceed when $S_0 \neq D^2$ is K-compressible. We have that

(1) S_0' is homologous to S_0 rel K ,

(2) $\chi(S_0') = \chi(S_0) + 1$,

(3) if S_0 is an annulus then S_0' is a disk. So if M is irreducible and K is incompressible then $[S_0] = 0 \in H_2(M,K)$.

3. The Thurston norm

3.1. - Definitions and remark. i) Let S be a properly embedded surface in M . We define the Thurston seminorm of S (or the norm of S for short) by

$$\| S \| \begin{cases} 0 & \text{if } S = D^2 \text{ or } S = S^2 \\ |\chi(S)| & \text{otherwise.} \end{cases}$$

When $S = \cup S_i$, where the S_i are connected, we define

$$\| S \| = \sum_i \| S_i \| \ .$$

ii) Now let (as always in these notes) K be a

2-dimensional submanifold of ∂M, possibly with $\partial K \neq \phi$.
For $\alpha \in H_2(M,K;\mathbf{Z})$ we define

$$\|\alpha\| = \min\{\|S\|;\ (S,\partial S) \text{ properly embedded in } (M,K), [S]=\alpha\}\ .$$

Note that a system S with $[S] = \alpha$ always exists.

iii) The surface system S is called <u>norm minimizing</u> if the following conditions are satisfied:

(1) S is incompressible,

(2) no proper sub-system of S is null-homologous rel K,

(3) $\|S\| = \|[S]\|$.

When S is a surface system satisfying condition (3) then a sub-system of S which is null-homologous consists only of components with non-negative Euler characteristic.

In general, (1) is not a consequence of (2) and (3). However, when M is irreducible, K is incompressible and S is not a null-homologous torus, annulus or sphere then (1) can be deduced from (2) and (3); cf. 2.4.

3.2. - <u>Examples and observations</u>. i) Let $M = T \times [-1,1]$ where T is a compact (oriented) surface with $\partial T \neq \phi$ and $\chi(T) < 0$. Then $S = T \times \{0\}$ is incompressible. If $K = \partial M$

or $K = T \times \{-1,1\}$ then S is not norm minimizing. However, if $K = \partial T \times [-1,1]$ then S is norm minimizing in (M,K) .

ii) If S is norm minimizing then obviously so is any sub-system of S . On the other hand, if S and T are (disjoint and) norm minimizing then evidently $T \cup S$ need not be norm minimizing.

iii) Let $\alpha \in H_2(M,K;\mathbb{Z})$. If $K = \phi$ suppose that $\alpha \neq 0$. Then α is representable by a norm minimizing surface system.

Indeed, when $\alpha = 0$ any properly embedded K-compressible disk is a norm minimizing representative of α . Otherwise, among all possible surface systems T with $[T] = \alpha \neq 0$ choose one satisfying conditions (2) and (3) of 3.1, iii). Then either this T is automatically incompressible or we can perform on T the necessary compressions in order to make it incompressible, without changing the norm of T . Thus in both cases we obtain a norm minimizing representative of α .

iv) If S is norm minimizing and $[S] \neq 0$ then every component S_0 of S with $\chi(S_0) \neq 0$ is K-incompressible. Indeed, when S_0 is a disk this follows from 3.1, iii), condition (2), and when $\chi(S_0) < 0$ it follows from 2.4, ii), (3). Recall also 2.4, ii), (3).

v) It should be clear that the Thurston norm is not a genuine norm on $H_2(M,K)$. Indeed, every incompressible torus or annulus gives rise to an element $\alpha \in H_2(M,K)$ with $\|\alpha\| = 0$, but in general $\alpha \neq 0$.

On the other hand, when M is irreducible, closed and atoroidal (i.e. does not contain any incompressible torus) then $\|\alpha\| = 0$ implies $\alpha = 0$.

3.3. - Disks and spheres in modified norm minimizing systems

When $(S,\partial S)$ and $(T,\partial T)$ are (properly embedded, oriented, transversely intersecting) surface systems in (M,K) , we denote by $|S \cap T|$ the number of components of $S \cap T$ (recall 1.3). By $\hat{S}$ ($\hat{T}$ etc.) we denote that part of S consisting only of those components which are neither disks nor spheres.

We are interested in the behaviour of disks and spheres of S and T under modification.

Lemma. - Let S and T be norm minimizing such that $|S \cap T|$ is minimal under homology rel K . Then the disk (sphere) components of $S \asymp T$ were already disk (resp. sphere) components of $S \cup T$. In particular, we have that

$$\chi(S \,\hat{\asymp}\, T) \geq \chi(\hat{S}) + \chi(\hat{T}) .$$

Proof. To begin with, we show that in $S \asymp T$ no new disks are created. For that let us assume to the contrary that D is a disk component of $S \asymp T$ but not of S or T . Then D stems from a disk D' that is embedded in

$S \cup T$; the situation is schematized in fig. 8. In D' we have proper arcs and circles

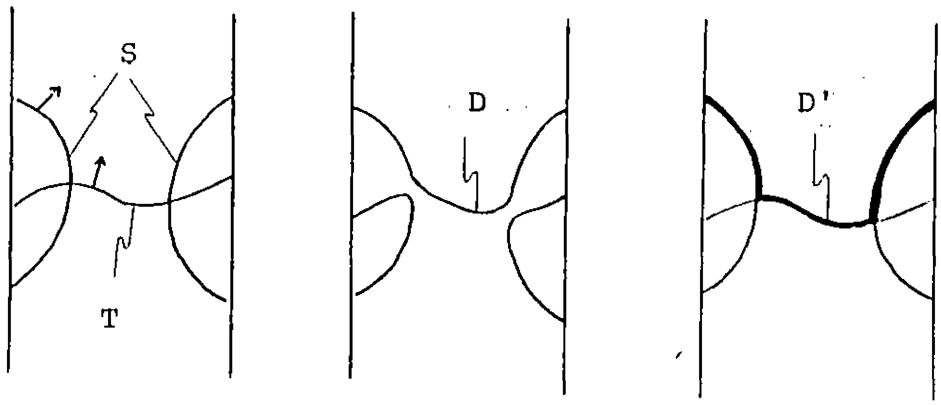


Figure 8

which are components of $S \cap T$. We distinguish between two cases.

First assume that D' contains no circles of $S \cap T$ in its interior. Then let $d \subset S \cap T$ be an outermost arc in D' . Then d dissects D' into two disks one of which, D_0 , does not contain any further arc component of $S \cap T$. Let us say without loss of generality that $D_0 \subset S$.

Clearly, d belongs also to a component, T_0 , of T . We can therefore compress T_0 along D and thus obtain a system T' such that

$$[T'] = [T] \in H_2(M,K;\mathbb{Z}) \quad \text{and} \quad |S \cap T'| < |S \cap T| \ .$$

But, by 3.2, iv), T_0 is either an annulus or $\chi(T_0) \neq 0$ and T_0 is K-incompressible.

In the first case, T' is again norm minimizing. In the second case, the arc d must split off a disk from T_0 whence it also follows that T' is norm minimizing. This provides the desired contradiction to the minimality of $|S \cap T|$.

Secondly, we assume that $\operatorname{int} D'$ contains at least one circle component of $S \cap T$. Let c be an innermost of these circles. Then c bounds a disk $D_0 \subset D'$ which lies entirely, say, in S. On the other hand, ∂D_0 also lies in some component, T_0, of T and thus we can compress T_0 along D_0. It follows from the incompressibility of T_0 that the system T' arising from T in this way is again norm minimizing. Since obviously

$$|S \cap T'| < |S \cap T|$$

we again have a contradiction.

The investigation of a sphere component of $S \sqsupset\!\sqsubset T$ goes similarly.

Finally, the relation between the Euler characteristics is now easily deduced.

$\square$

Corollary. If S and T are norm minimizing and diskless (resp. sphereless) and $|S \cap T|$ is minimal then

$S \asymp T$ *is diskless (sphereless)*.

$\square$

3.4. - *Homogeneity and subadditivity*. In the next proposition we denote for $n \in \mathbb{N}$ by nS an oriented surface system consisting of n parallel copies of the originally given oriented surface system S in (M,K) .

Proposition. i) $\|n\alpha\| = n\|\alpha\|$ *for all* $n \in \mathbb{Z}$ *and every* $\alpha \in H_2(M,K;\mathbb{Z})$.

ii) $\|\alpha+\beta\| \leq \|\alpha\| + \|\beta\|$ *for all* $\alpha,\beta \in H_2(M,K;\mathbb{Z})$.

Proof of i). Clearly, we may assume that $\alpha \neq 0$ and $n \in \mathbb{N}$. Let S and T be norm minimizing surface systems such that

$$[S] = \alpha \quad \text{and} \quad [T] = n\alpha \ .$$

By 1.5, there are smooth maps $f,g : M \longrightarrow S^1$ such that

$$S = f^{-1}(1) \quad \text{and} \quad T = g^{-1}(1) \ .$$

Furthermore, we may assume that the n-th roots of unity $\zeta_1 = 1,\zeta_2,\ldots,\zeta_n$ are all regular values of f .

Denote by $q : S^1 \longrightarrow S^1$ the covering map given by $q(z) = z^n$ for $z \in S^1 \subset \mathbb{C}$. Since

$$[T] = [nS] = [(q\circ f)^{-1}(1)] \in H_2(M,K) \ ,$$

the systems T and $(q\circ f)^{-1}(1) = \bigcup_i f^{-1}(\zeta_i)$ are framed cobordant. Therefore the maps $\tilde{f} = q\circ f$ and g are homotopic. It follows by homotopy lifting that there exists a map $\tilde{g} : M \longrightarrow S^1$ homotopic to f and with $q\circ\tilde{g} = g$.

$$\begin{tikzcd}
 & S^1 \arrow[d, "q"] \\
M \arrow[ur, "{f,\tilde{g}}"] \arrow[r, "{\tilde{f},g}"'] & S^1
\end{tikzcd}$$

Now with $T_i = \tilde{g}^{-1}(\zeta_i)$, $i=1,\ldots,n$, we have that

$$T = g^{-1}(1) = \bigcup_i T_i \ .$$

Since f and $\tilde{g}$ are homotopic, we know from 1.5 that

$$[S] = [T_i] \quad \text{for} \quad i=1,\ldots,n \ .$$

Consequently,

$$\begin{aligned}
\|n\alpha\| = \|[T]\| = \|T\| = \sum_i \|T_i\| \geq \| \sum_i \|[T_i]\| \\
= \sum_i \|[S]\| \\
= n\,\|\alpha\| \ .
\end{aligned}$$

On the other hand, since T and nS are homologous rel K and S and T are norm minimizing, we see that

$$\|n\alpha\| = \|T\| \leq \|nS\| = n\|S\| = n\|\alpha\| .$$

This establishes i).

<u>Proof of ii)</u>. We assume that $\alpha,\beta \neq 0$ and take norm minimizing representatives S and T of α and β, respectively. Then by 3.3, we obtain

$$\begin{aligned} \|\alpha+\beta\| \leq \|S \supset\!\subset T\| &= -\chi(\widehat{S \supset\!\subset T}) \\ &\leq -\chi(\hat{S})-\chi(\hat{T}) \\ &= \|S\| + \|T\| \\ &= \|\alpha\| + \|\beta\| \quad . \end{aligned}$$

□

<u>Corollary</u>. <u>When</u> S <u>is norm minimizing then so is</u> nS <u>for any</u> $n \in \mathbb{Z}-\{0\}$.

□

3.5. - Extending the Thurston norm. We recall two important facts on seminorms; for a proof see Thurston [Th] and Fried [Fr].

Proposition. i) *A seminorm on $\mathbb{Z}^n$ with values in $\mathbb{Z}_+$ extends uniquely to a seminorm on $\mathbb{R}^n$ with values in $\mathbf{R}_+$.*

ii) *A seminorm* $\| \ \| : \mathbb{R}^n \longrightarrow \mathbf{R}_+$ *takes integer values on* $\mathbb{Z}^n$ *if and only if there is a finite set* $\Gamma \subset \mathrm{Hom}(\mathbf{Z}^n, \mathbb{Z})$ *such that*

$$\| x \| = \max_{\gamma \in \Gamma} |\gamma(x)| .$$

□

Consequence. For the extended Thurston norm on $H_2(M,K;\mathbf{R})$ the unit ball B is a finite convex polyhedron and the domains where equality holds in the triangle inequality are precisely the cones over the faces of ∂B with the origin as vertex; see fig. 9.

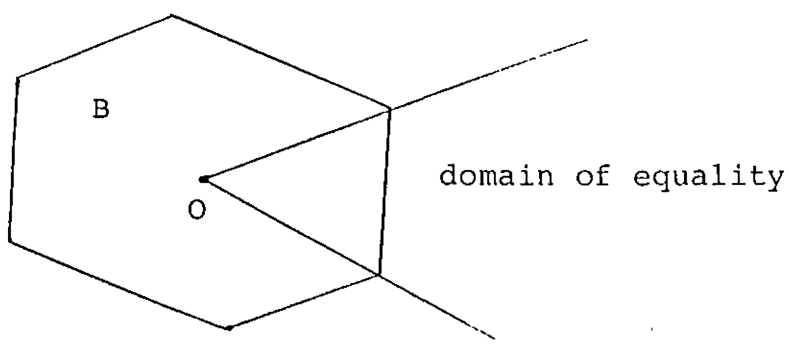


Figure 9

3.6. - Exercises. i) Find (M,K) and a properly embedded surface S with $\partial S \subset K$ and $[S] \neq 0$ which is norm minimizing but not K-incompressible.

ii) Find (M,K) such that the Thurston norm is not a norm on $H_2(M,K)$.

iii) Let S be a surface system in (M,K) such that no sub-system of S is null-homologous rel K . Then S is norm minimizing if and only if $\hat{S}$ is norm minimizing (cf. 3.3).

iv) Let S be a surface system in (M,K) such that $\| S \| = \| [S] \|$. If S is incompressible then any sub-system of S that is null-homologous rel K consists only of components with non-negative Euler characteristic.

4. Coloured 3-manifolds

Let us keep in mind that our final goal consists of the construction of foliations on a compact manifold M which are transverse to the boundary. This will be done by means of a hierarchy of M . Clearly, a foliation of M can be transverse to the boundary only when ∂M consists of tori. However, the manifolds occuring in the hierarchy of M may have boundary components with non-zero Euler chararcteristic. Therefore we have to consider these 3-manifolds as manifolds with corners where the foliation is transverse to one part of the boundary, consisting of tori and annuli, and is tangent to the rest of the boundary.

4.1. - Definition. i) Let M be a compact oriented 3-manifold with $\partial M \neq \phi$. By a colouring of ∂M we understand a partition of ∂M ,

$$\partial M = \Sigma \cup R_+ \cup R_- ,$$

into compact sub-manifolds which only intersect in boundary circles and such that the following holds:

(1) Σ is a union of pairwise disjoint annuli and tori.

(2) $R_+ \cap R_- = \phi$ and R_+ (resp. R_-) is oriented so that its normal points out of (into) M .

(3) If A is an annulus component of Σ then one boundary circle of A belongs to R_+ and the other to R_- .

The examples following below are to illustrate this definition. Note that Σ or $R_+(R_-)$ may be empty.

ii) By a coloured 3-manifold we mean a compact 3-manifold together with a colouring of ∂M .

iii) Two coloured manifolds $(M,\Sigma,R_\pm)$ and $(M',\Sigma', R'_\pm)$ are considered as being the same if there exists an orientation preserving diffeomorphism between M and M' taking R_+ to R'_+ and R_- to R'_- .

Instead of $(M,\Sigma,R_\pm)$ we often simply write (M,Σ) . The part of Σ consisting of the annuli components is denoted by $A(\Sigma)$, and R stands for $R_+ \cup R_-$. In pictures we indicate R_+

(resp. R_-) simply by a +sign (-sign).

4.2. - <u>Examples</u>. i) The <u>taut</u> coloured 3-ball. Here $M = D^2 \times [0,1]$, $\Sigma = \partial D^2 \times I$, $R_+ = D^2 \times \{i\}$, $R_- = D^2 \times \{i+1\}$, $i = 0,1 \pmod 2$.

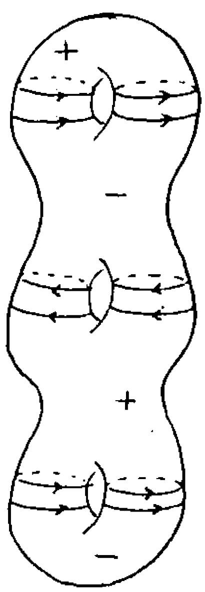
More generally, when P is any compact orientable surface with boundary, we obtain a coloured manifold by

$$M = P \times [0,1],\ \Sigma = \partial P \times [0,1],\ R_+ = P \times \{0\},\ R_- = P \times \{1\}\ .$$

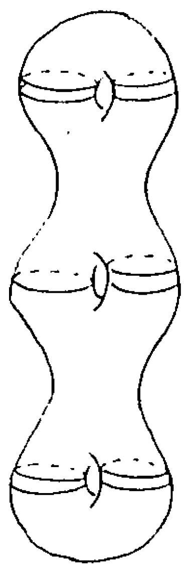
ii) $M = D^2 \times S^1$, $\Sigma = \partial D^2 \times I$ where $I \subset S^1$ is an interval, $R_+ = \partial M - \mathring{\Sigma}$, $R_- = \phi$, does not constitute a coloured manifold because condition (3) is violated.

iii) Any compact oriented 3-manifold whose boundary is a union of tori is coloured whether a specification of orientations for some of these tori is given or not.

iv) A typical colouring of the closed orientable surface of genus three is depicted in fig. 10a). The decomposition in fig. 10b), however, does not constitute a colouring.



a)



b)

Figure 10

4.3. - Taut coloured 3-manifolds. We call a coloured 3-manifold $(M,\Sigma,R_{\pm})$ taut if M is irreducible and R_+ and R_- are norm minimizing in $H_2(M,\Sigma)$.

Here it is understood that R_+ (resp. R_-) is norm minimizing if it is empty. For example, $(D^2\times S^1, \partial D^2\times S^1, \phi)$ is taut. On the other hand, the handle body of genus three with the colouring presented in fig. 7a) is not taut, for both R_+ and R_- are compressible.

The next result provides one of the key ingredients of the splitting theorem 6.1.

4.4. - Doubling taut coloured 3-manifolds. When $(M,\Sigma,R_{\pm})$ is a coloured 3-manifold, we wish to perform modifications using the manifold $R = R_+ \cup R_-$. Since R is not proper in M , we first double M along R and then do the desired modifications.

The next result will be needed in the proof of the decomposition theorem. It relies on 2.2 iii) and iv) and is easily verified.

Lemma. Let (M,Σ) be taut and let $(N,\partial N,\phi)$ be the coloured manifold obtained by doubling (M,Σ) along R . Then $(N,\partial N)$ is taut, unless $N = D^2 \times S^1$ and (M,Σ) is the taut 3-ball.

□

4.5. - Proposition. Let $(M,\Sigma,R_{\pm})$ be a taut coloured manifold and let N be the manifold obtained by doubling M

along $R = R_+ \cup R_-$. *Then for any* $\alpha \in H_2(N,\partial N; \mathbf{Z})$, $\alpha \neq 0$, *there exists an integer* $n \geq 0$ *and a properly embedded oriented surface* T *in* N *such that*

(1) $[T] = n[R] + \alpha \in H_2(N,\partial N)$.

(2) T *is norm minimizing*.

(3) *If* V *is a component of* R *then* T *meets* V *transversely and no union of components of* $V \cap T$ *represents the trivial element in* $H_1(V,\partial V)$.

Furthermore, the following holds:

(a) *If* $\langle\alpha,[c]\rangle = 0$ *for every component* c *of* ∂V *then* $T \cap V$ *is a union of* $k \geq 0$ *parallel oriented homologically non-trivial simple closed curves*.

(b) *If* V *is planar such that* $\langle\alpha,[c_i]\rangle \neq 0$ *for exactly two components* c_1, c_2 *of* ∂V *then* $T \cap V$ *is a union of* $|\langle\alpha,[c_i]\rangle|$ *parallel oriented proper arcs*.

Proof. First of all we observe that the proposition is true when (M,Σ) is the taut 3-ball. So for the rest of the proof we suppose that (M,Σ) is not the taut 3-ball.

According to 3.5 there exists $m \geq 0$ such that for all $k \geq 0$ we have that

(*) $\|(m+k)[R] + \alpha\| = \|m[R] + \alpha\| + k\|[R]\|$;

see figure 11. Note that then (*) also holds for all $m + \ell$,

$\ell \in \mathbb{N}$.

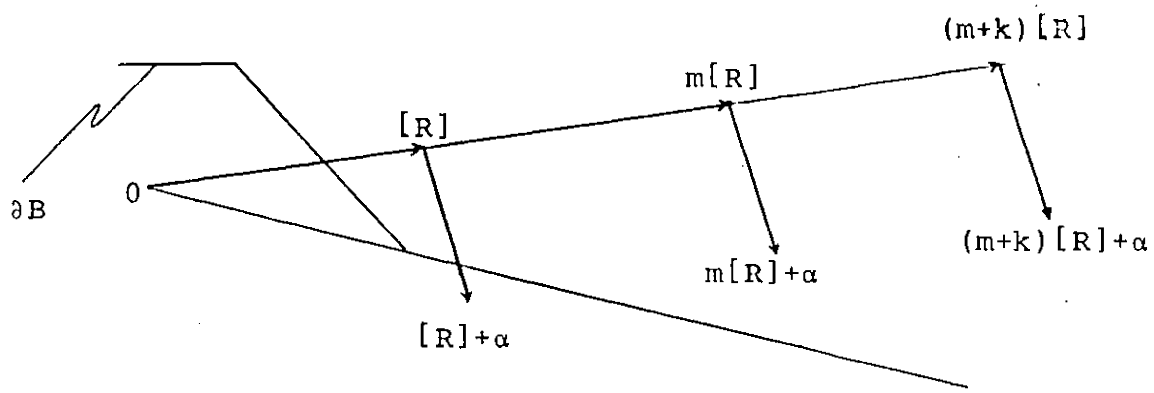


Figure 11

Now let T_0 be a norm minimizing surface in $(N,\partial N)$ representing $m[R] + \alpha$ (see 3.2, iii)). T_0 may be chosen such that for any boundary component B of N , $T_0 \cap B$ is a union of parallel oriented simple closed curves. This can be seen by capping off pairs of oppositely oriented curves of $T_0 \cap B$ by annuli within $\overset{\circ}{N}$. Moreover, we may assume that for any component d of ∂R the following holds.

Either $\langle d,T_0\rangle \neq 0$ then

$$|\langle d,T_0\rangle| = \#\,(d \cap T_0)$$

or $\langle d,T_0\rangle = 0$, in which case $d \cap T_0 = \phi$ and

$$[T_0 \cap B] = k[d] \in H_1(B) \quad \text{with} \quad k \geq 0 \ .$$

(In order to achieve that $k \in \mathbb{Z}$ indeed is non-negative, we only have to take $m \in \mathbb{N}$ sufficiently large.)

Next we consider $T_1 = T_0 \asymp R$ and recall 1.6. Let W be a subsurface of R such that $\partial W = C \cup (-D) \pmod{\partial R}$ and $\operatorname{int} W \cap T_0 = \phi$, where D is a union of components of $T_0 \cap R$ and $C \cap T_0 = \phi$. Then we can isotope T_1 slightly near W so that

$$T_1 \cap R = (T_0 \cap R - D) \cup C .$$

This shows that modifying stepwise T_0 and r parallel copies of R and performing the necessary isotopies we obtain a surface system T_2 with

$$[T_2] = [T_0] + r[R] \in H_2(N, \partial N)$$

and such that for any component V of R no union of components of $V \cap T_2$ is null-homologous in $V \pmod{\partial V}$. More precisely, since the subsurfaces W above are chosen as in 1.6, we see that conditions a) and b) of (3) can be satisfied additionally. So it remains to show that T_2 contains a norm minimizing homologous sub-system.

By 2.2, iii) and iv), N is irreducible and thus any sphere possibly contained in T_2 may be omitted. So we assume that T_2 is sphereless. Moreover, the existence of a disk component, D, of T_2 or T_0 would imply that either $0 \neq [D] \in H_2(N, \partial N)$, in which case D can be omitted, or that $N = D^2 \times S^1$ whence it would follow that (M, Σ) is the taut 3-ball. As this is excluded here, we now have that

$$\|T_2\| = -\chi(T_2) = -\chi(T_0) - r\,\chi(R)$$
$$= \|T_0\| + r\|R\|$$
$$= \|m[R] + \alpha\| + r\|R\|$$
$$= \|(m+r)[R] + \alpha\|\,,\ \text{by}\ (*)\ ,$$
$$= \|[T_2]\|\ .$$

This also shows that every component of T_2 with negative Euler characteristic is incompressible. Consequently, if Q is a maximal system of components of T_2 (necessarily tori) such that $[Q] = 0$ in $H_2(N,\partial N)$ then

$$T = T_2 - Q$$

is as required.

□

4.6. - Observation. The proof of 4.5 also showed the following. If $S = (T \asymp R) \cap M$, where (M,Σ) is viewed as being embedded in $(N,\partial N)$, then, if necessary after a slight isotopy of S , each component, c , of $S \cap \Sigma$ satisfies one of the following conditions:

(1) c is a properly embedded non-separating arc in $A(\Sigma)$.

(2) c is a simple closed curve in an annular component A of Σ and, with the orientation induced by that of S, is homologous to any of the boundary curves of A (cf. 4.1) .

(3) c is a homologically non-trivial curve in a torus component B of Σ, and if c' is another component of $B \cap S$ then c and c', with their induced orientations, are homologous in B .

4.7. - Exercises. i) Let $(M,\Sigma,R_{\pm})$ be taut. If one component of R is a disk then (M,Σ) is the taut 3-ball.

If $(N,\partial N)$ denotes the double of M along R then $N = D^2 \times S^1$ if and only if (M,Σ) is the taut 3-ball.

ii) Find all taut coloured 3-manifolds with $D^2 \times S^1$ as underlying manifold.

iii) If (M,Σ) is taut then Σ is incompressible unless (M,Σ) is the taut 3-ball or $(M,\Sigma) = (D^2 \times S^1, \partial D^2 \times S^1)$.

5. Coloured manifold decompositions

5.1. - Definition. Let $(M,\Sigma,R_{\pm})$ be a coloured 3-manifold and suppose we are given a properly embedded surface system S in M such that either $S \cap \Sigma = \phi$ or for every component of $\Sigma \cap S$ one of the three conditions of 4.6 is satisfied.

Now we construct a new coloured manifold $(M',\Sigma',R'_{\pm})$ by cutting M along S. To be more precise we do the following. First we choose a regular neighbourhood $N(S)$ of $(S,\partial S)$ in $(M,\partial M)$ and set

$$M' = M - \text{int } N(S) .$$

Next, letting S'_+ (resp. S'_-) be that component of $\partial N(S) \cap M'$ whose normal points out of (into) M', we create R'_+ (resp. R'_-) by adding S'_+ (S'_-) to what is left of R_+ (R_-). Finally, we separate R'_+ from R'_- by introducing an annulus resp. disk for each component of $R'_+ \cap R'_-$; see figures 12 and 13.

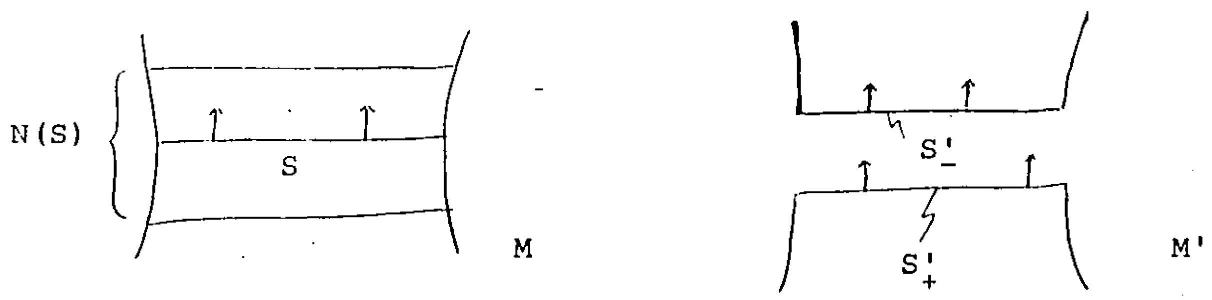


Figure 12

In formulae this reads as follows:

$$\Sigma' = (\Sigma \cap M) \cup N(S'_+ \cap R_-) \cup N(S'_- \cap R_+)$$

$$R'_+ = ((R_+ \cap M') \cup S'_+) - \operatorname{int} \Sigma'$$

$$R'_- = ((R_- \cap M') \cup S'_-) - \operatorname{int} \Sigma' \ .$$

This process is referred to as a <u>coloured manifold decomposition</u> and is denoted by

$$(M,\Sigma) \overset{S}{\rightsquigarrow} (M',\Sigma') \ .$$

We point out that the manifold M' obtained by splitting M along S need not be connected even if S is connected.

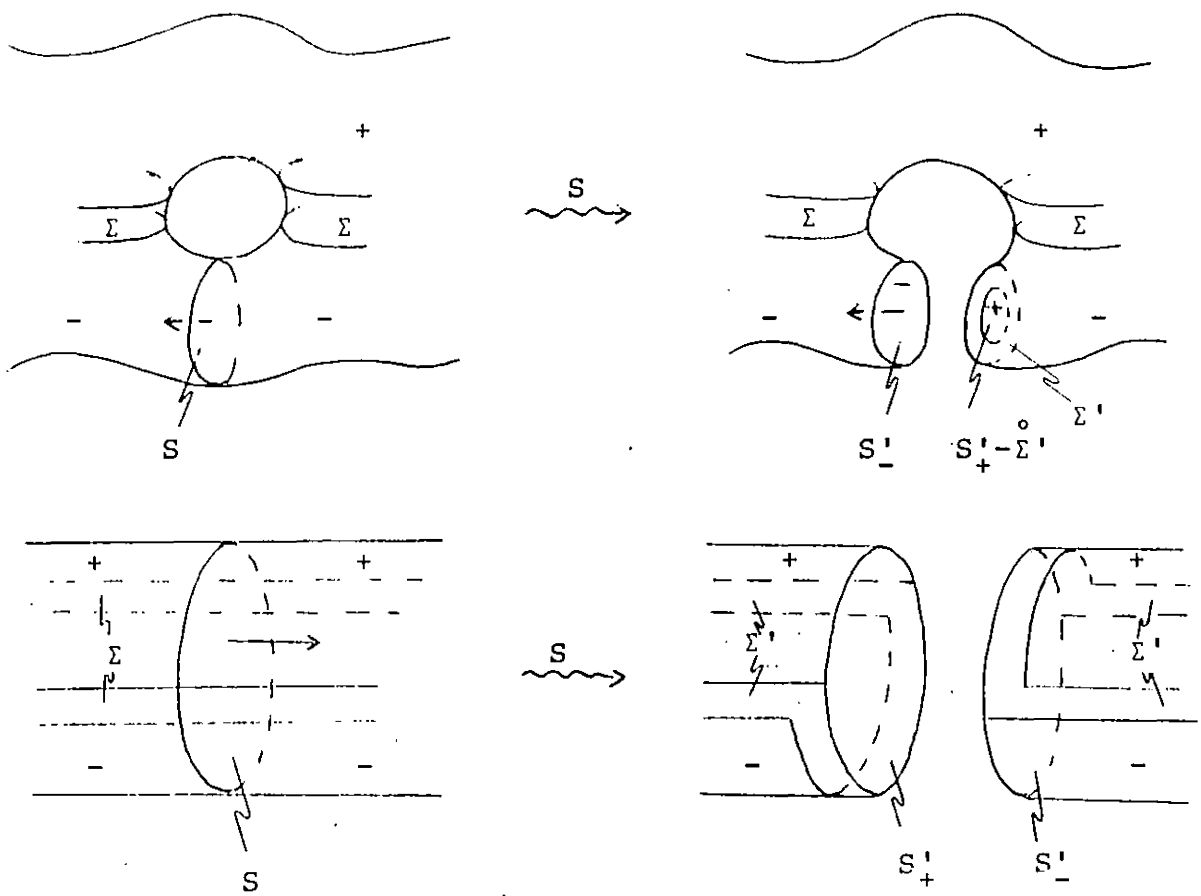


Figure 13

5.2. - Remarks. i) If $S = S_0 \cup S_1$ then the coloured manifolds obtained by decomposing (M,Σ) along S is the same as that obtained from (M,Σ) by first decomposing along S_0 and then along S_1 .

ii) Suppose that $(M,\Sigma) \overset{S}{\rightsquigarrow} (M',\Sigma')$ is a coloured manifold decomposition. Then it follows as in 2.2, iii) that S is incompressible in M if and only if S'_+ and S'_+ are incompressible in M' .

Similarly, if S is incompressible then M' is irreducible if and only if M is irreducible.

On the other hand, if (M',Σ') is taut then standard arguments show that M is irreducible independent of whether S is incompressible or not.

5.3. - A necessary condition for tautness

We want to know under what circumstances in a decomposition $(M,\Sigma) \overset{S}{\rightsquigarrow} (M',\Sigma')$ the tautness of one of the two involved manifolds implies that of the other. The next exercises are to show that there is no general result in this direction.

Since, in general, a decomposition does not yield a connected manifold M' , even if S is connected, we agree that a system of coloured manifolds is taut if each of its components is taut.

At first let us prove the following necessity criterion and a preliminary lemma.

Lemma. Let $(D^3,\Sigma) \overset{S}{\rightsquigarrow} (M',\Sigma')$ be a coloured manifold decomposition with $\Sigma \neq \phi$ and S a system of disks such that

(M', Σ') <u>is taut. Then</u> (D^3, Σ) <u>is the taut</u> 3-<u>ball</u>.

<u>Proof</u>. We induct on the number of components of S. At first let us consider the case that S is a single disk. Then, by hypothesis, (M', Σ') consists of two taut 3-balls (D_0^3, Σ_0) and (D_1^3, Σ_1).

When $S \cap \Sigma = \phi$, we easily see that (D^3, Σ) is taut. So let us assume that $S \cap \Sigma \neq \phi$. Assuming furthermore without loss of generality that the copy of S belonging to ∂D_0^3 is S'_+, we have the following picture (fig. 14a)).

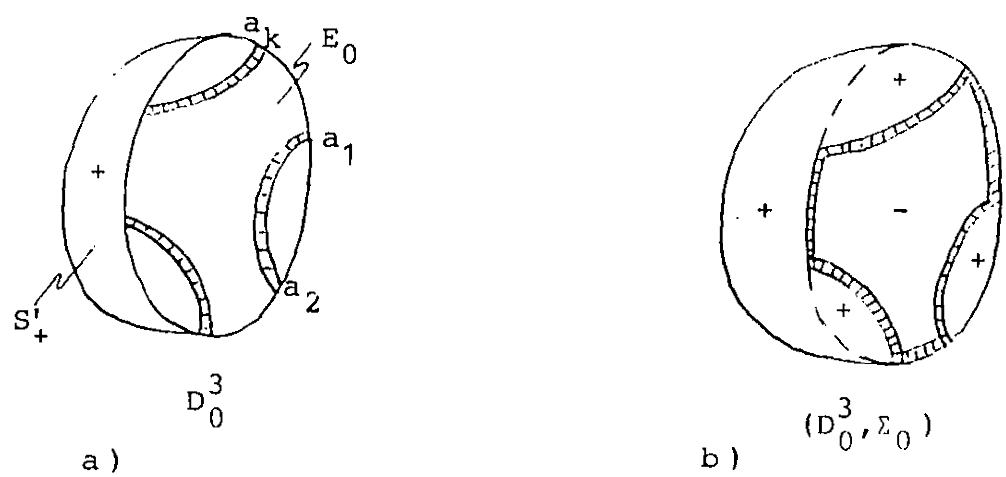


Figure 14

The disk $E_0 = \partial D_0^3 - \mathring{S}'_+$ contains bands coming from Σ. Each of these bands is outermost in that it splits off a disk from E_0 that does not contain any further band.

Now the only possibility to connect these bands in order to obtain the annulus Σ_0 is as indicated in fig. 14b).

A similar argument holds for (D_1^3, Σ_1). Let $a_1, \ldots, a_k$ be the cyclically ordered arcs of $\Sigma \cap \partial S$. Then it follows from the fact that (D_0^3,Σ_0) and (D_1^3,Σ_1) are taut that if a_i and a_{i+1} belong to the same band in E_0 then a_i and a_{i-1} belong to the same band in $E_1 = \partial D_1^3 - \mathring{S}^1_-$. This implies that Σ is connected whence it follows that (D^3,Σ) is taut.

Finally, when S is not connected, we take a component, D , of S and consider the diagram of coloured manifold decompositions

$$\begin{tikzcd}
(D^3,\Sigma) \arrow[rr, squiggly, "S"] \arrow[dr, squiggly, "S-D"'] & & (M',\Sigma') \\
& (M_1,\Sigma_1) \arrow[ur, squiggly, "D"'] &
\end{tikzcd}$$

Since S consists of disks, each component of M_1 is a 3-ball. Therefore, by what was just proved, (M_1,Σ_1) is taut. The induction hypothesis thus shows that (D^3,Σ) is taut.

$\square$

Proposition. *Let* $(M,\Sigma,R_\pm) \overset{S}{\rightsquigarrow} (M',\Sigma',R'_\pm)$ *be a coloured manifold decomposition such that* (M',Σ') *is taut and* S *is incompressible. If* $M = D^2 \times S^1$ *and* $\Sigma = \phi$ *suppose that* S *is not a system of* $k \geq 1$ *parallel oriented meridional disks. Then* (M,Σ) *is taut*.

Proof. The case $M = D^3$ was already proved in the preceding lemma. Consequently, we may now assume that $M \neq D^3$.

By 5.2, ii), M is irreducible, so it remains to show that R_+ and R_- are norm minimizing.

Assuming that R_+ is not norm minimizing means that

(a) either there exists a properly embedded surface system $(T,\partial T)$ in (M,Σ) such that

$$[T]=[R_+]\in H_2(M,\Sigma) \quad \text{and} \quad \| T \| < \| R_+ \|$$

(b) or R_+ has minimal norm within its homology class but at least one component of R_+ is compressible.

To begin with, let us assume that b) holds but not a). Then clearly $\chi(R_0) = 0$ for any compressible component R_0 of R_+. Moreover, if the component B of ∂M containing R_0 is a torus then $M = D^2 \times S^1$. The only incompressible proper surfaces in $D^2 \times S^1$ are disks or annuli whose fundamental groups inject into that of $D^2 \times S^1$. It is not hard to check that in this situation either the exceptional case in the statement of the proposition holds or the tautness of (M',Σ') implies that of (M,Σ).

If B is not a torus then R_- has a compressible component with negative Euler characteristic and we henceforth could argue with R_- instead of R_+. It suffices therefore to consider the case a).

To this end let T be as in a). Moreover, since

$$\| T\| <\| R_+\| \quad \text{if and only if} \quad \| \hat{T}\| < \| \hat{R}_+\|$$

(see the beginning of 3.3 and 3.6, iii)) we may choose T such that if

$$T = \hat{T} \cup (\ell \text{ disks}) \quad \text{and} \quad R_+ = \hat{R}_+ \cup (n \text{ disks})$$

then

$$(*) \qquad n \leq \ell \ .$$

Next, let

$$T_1 = T \asymp S$$

and let T' be the surface system in M' resulting from T_1 after cutting M along S ; see fig.15 . Then we have

$$[T'] = [R'_+] \in H_2(M',\Sigma') \ .$$

Moreover, since (M',Σ') is supposed to be taut, it follows, possibly after suppressing pairs of oppositely oriented parallel disk components of T' that

$$(**) \qquad m \leq p \ ,$$

where m and p are the number of disk components of T' and R'_+ , respectively. All together this yields a contradiction as follows.

$$
\begin{aligned}
\| T' \| &= -\chi(T')+m \\
&= -\chi(T)-\chi(S)+m \\
&= \| T \| -\chi(S)+m \\
&< \| R_+ \| -\chi(S)+m-\ell \\
&= -\chi(R_+)-\chi(S)-m-\ell+n \\
&= -\chi(R_+')+m-\ell+n \\
&= \| R_+' \| +m-\ell+n-p \\
&\le \| R_+' \| \qquad\qquad , \text{ by } (*) \text{ and } (**).
\end{aligned}
$$

□

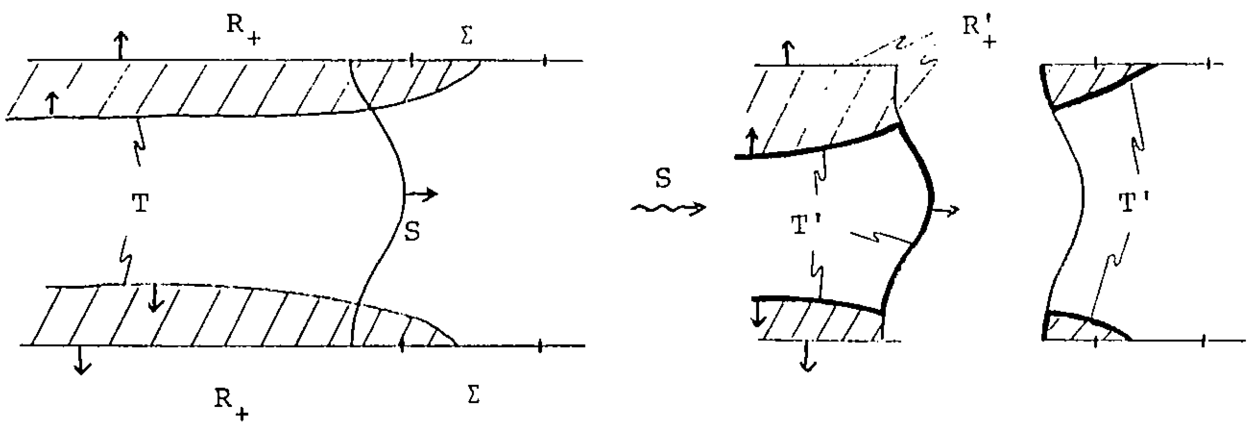


Figure 15

5.4. - <u>A criterion for tautness</u>.

When the decomposing surface is of a special kind, we even have the following necessary and sufficient condition for tautness.

<u>Proposition</u>. <u>Let</u> $(M,\Sigma,R_{\pm}) \overset{S}{\rightsquigarrow} (M',\Sigma',R'_{\pm})$ <u>be a coloured manifold decomposition where</u> S <u>is either a disk and</u> $|S \cap \Sigma| = 2$ <u>or an incompressible annulus such that one component of</u> ∂S <u>lies in</u> R_+ <u>and the other in</u> R_- . <u>Then</u> (M,Σ) <u>is taut if and only if</u> (M',Σ') <u>is taut</u>.

<u>Proof</u>. We only have to show that (M,Σ) being taut implies that (M',Σ') is taut, for the converse is an immediate consequence of 5.3.

If (M,Σ) is taut then obviously M' is irreducible. So it remains to prove that R'_+ and R'_- are norm minimizing, i.e. incompressible and with minimal norm within their homology classes.

Thus assuming that, say, R'_+ is compressible means that the component of R'_+ containing S'_+ is compressible. However, by the special choice of S , a compressing disk may then be chosen so that it does not meet S'_+ at all. This yields no contradiction to the tautness of R_+ only when S is an annulus and the component of ∂S lying in $R_0 \subset R_+$ separates R_0 in such a way that one of the resulting pieces is a disk. But this is impossible because S is incompressible. Hence R_+ , and similarly R_- , is incompressible.

Finally, we have to show that $\| R'_+ \|$ is minimal. Otherwise there exists a proper surface system T' in (M',Σ') such that

$$[T'] = [R'_+] \in H_2(M',\Sigma') \quad \text{and} \quad \| T' \| < \| R'_+ \| \ .$$

Then, similar to the proof of proposition 5.3, we construct from T' and S a surface system T in M such that

$$[T] = [R_+] \in H_2(M,\Sigma) \quad \text{and} \quad \| T \| < \| R_+ \| \ .$$

□

5.5. - Exercises. i) Find an example of a coloured manifold decomposition $(M,\Sigma) \overset{S}{\rightsquigarrow} (M',\Sigma')$ where (M',Σ') is taut but the splitting surface S is compressible (resp. Σ-compressible).

ii) Find a decomposition $(M,\Sigma) \overset{S}{\rightsquigarrow} (M',\Sigma')$ where (M,Σ) is taut, S is incompressible but (M',Σ') is not taut.

iii) Let $(M,\Sigma) \overset{S}{\rightsquigarrow} (M',\Sigma')$ be a decomposition such that $\partial S \subset \Sigma$, S is incompressible and has minimal norm in (M,Σ) . Then (M,Σ) is taut if and only if (M',Σ') is taut.

iv) Prove 5.2, ii).

6. The coloured manifold decomposition theorem

We are now ready to prove the decomposition theorem for taut coloured manifolds which is the chief result of this chapter.

6.1. - Theorem. Suppose that $(M,\Sigma,R\pm)$ is taut and $H_2(M,\partial M) \neq 0$. Then there exists a proper surface S in $(M,\partial M)$ such that

(1) S is incompressible.

(2) $0 \neq [\partial M] \in H_1(\partial M)$ provided that $\partial M \neq \phi$,

(3) S is a splitting surface for (M,Σ) ,

(4) the coloured manifold (M',Σ') obtained by decomposing (M,Σ) by means of S is taut.

Furthermore, S can be chosen specifically so that it meets every component, V , of $R_+(R_-)$ in a system of $k \geq 0$ parallel oriented (each) non-separating simple closed curves if V is non-planar or proper arcs if V is planar.

For the proof of this theorem we need another observation that follows. Its proof is easy and therefore omitted. Note that in this observation we do not require that S be norm minimizing in that condition (2) of 3.1, iii) may fail. Cf. also 5.5, iii).

6.2. - Lemma. Let $(N,\partial N)$ be the double of M along $R = R_+ \cup R_-$ (where (M,Σ) is as in 6.1) and let $(N,\partial N) \overset{T'}{\rightsquigarrow} (N',\Phi')$ be a coloured manifold decomposition where T' is incompressible and has minimal norm. Then (N',Φ') is taut.

□

Proof of 6.1. When M is closed, let S be any norm minimizing surface in M . When $\partial M \neq \phi$, let $\alpha \in H_2(M,\partial M)$ be as provided by proposition 1.7 and let P be a proper surface system such that $[P] = \alpha$. Doubling P along $\partial P - \mathring{\Sigma}$ yields a proper surface system P' in $(N,\partial N)$. (Note that if $\partial P \subset \Sigma$ then we are in the easy case of exercise 5.5, iii).)

Next we apply proposition 4.5 to $\alpha' = [P'] \in H_2(N,\partial N)$ and let T be the resulting surface system. Then, by the special choice of P , T meets each non-planar component of R in a system of $k \geq 0$ parallel oriented simple closed homologically non-trivial curves and T meets each planar component of R in a system of $k \geq 0$ parallel oriented proper arcs. Moreover, $k > 0$ for at least one component.

Now put

$$S' = T \cap M$$

and let S be a component of S' such that $0 \neq [\partial S] \in H_1(\partial M)$. Clearly S' and thus S is incompressible. We consider the commutative diagram of coloured manifold decompositions and inclusions

$$\begin{tikzcd}
(M,\Sigma) \arrow[r, hook] \arrow[d, rightsquigarrow, "S"'] \arrow[dr, rightsquigarrow, "S'"] & (N,\partial N) \arrow[r, rightsquigarrow, "T'"] & (N',\Phi') \arrow[d, rightsquigarrow, "J"] \\
(M',\Sigma') \arrow[r, rightsquigarrow, "S'-S"] & (M'',\Sigma'') \arrow[r, hook] & (N'',\Phi'')
\end{tikzcd} ,$$

where $T' = T \sqsupset\!\sqsubset R$ and J is a system of annuli and disks of the form $J_0 = c \times [0,1]$ where c is a component of $T \cap R$; see 1.4 and figure 16.

As in proposition 4.5 we see that T' is incompressible and has minimal norm. Therefore, by lemma 6.1, (N',Φ') is taut.

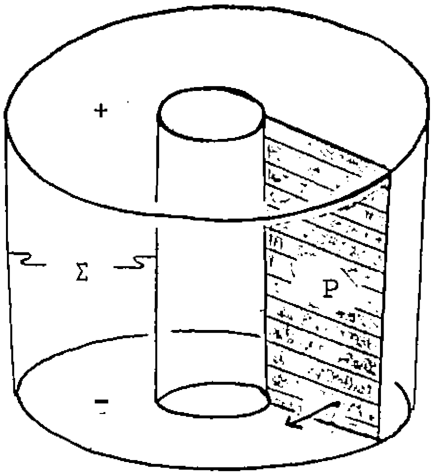
Furthermore, each component of the system J satisfies the hypothesis of proposition 5.4. Indeed, the only point here that is not quite obvious is the incompressibility of the annular components of J . However, any such annulus, A , comes from a circle component, c , of $T \cap R$. If A is compressible then c bounds a disk in N and thus , by the incompressibility of R , also a disk in R . But this contradicts the fact that the circles of $T \cap R$ are homologically non-trivial. Now, by 5.4, (N'',Φ'') and thus (M'',Σ'') as a component of it are taut.

Finally, since S' is incompressible, it follows from proposition 5.3 that (M',Σ') is taut. □

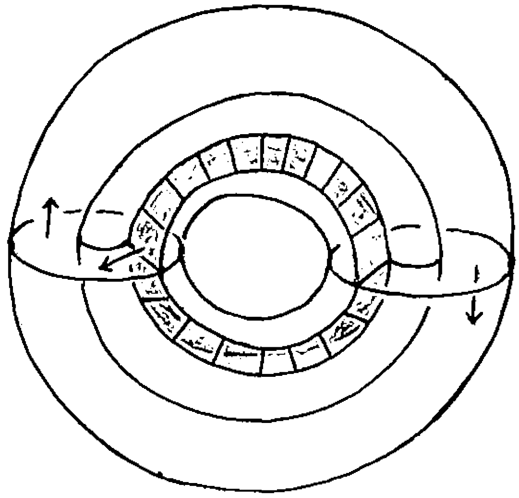
Remarks. i) We do not claim and cannot prove that the surface S obtained by the theorem is norm minimizing or at least ∂-incompressible. This will be one reason for the difficulties we have to encounter in chapter II.

ii) M. Scharlemann (sutured manifolds and generalized Thurston norms) improved the proof of theorem 6.1 considerably by working directly in (M,Σ) instead of its double.

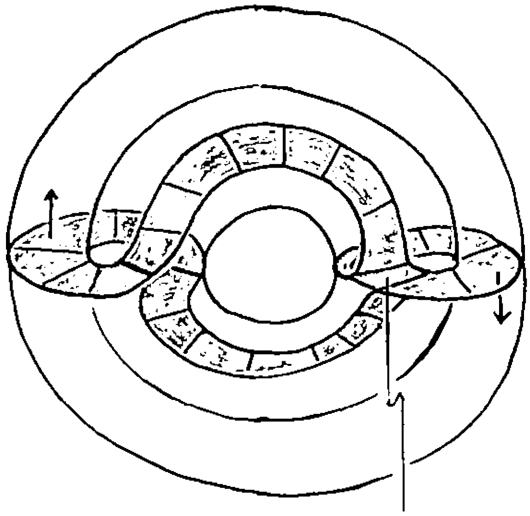
□



P in (M,Σ)



$P' = T$ in $(N,\partial N)$



$T' = T \natural R$ in $(N,\partial N)$

$J_0 \subset J$

Figure 16

Chapter II

EXISTENCE OF COLOURED MANIFOLD HIERARCHIES

We recall that our aim is to construct foliations without Reeb components on compact 3-manifolds. It is well known that such a foliation can exist only when the underlying manifold is irreducible (see [Al] and [Ro]).

In our construction we want to use hierarchies. This concept from the general theory of compact 3-manifolds permits it to decompose a given manifold M by a finite number of splittings at properly embedded surfaces into balls. We adapt this concept to our purposes. In particular, we do not insist that the final pieces be balls.

Hierarchies exist always when M is of Haken type, for instance, when M is irreducible and $\partial M \neq \phi$ (see I; 1.5, 1.7, and 3.2, as well as [Ha; p.101], [He; p.62f], [Ja], and [Wa; p.60]). Therefore, in what follows we restrict our interest to Haken manifolds. Moreover, for reasons which will become evident later we have to require in most cases that coloured manifolds are taut.

Of course, our transversality and orientability assumptions on 3-manifolds and surfaces lying in them remain valid also in this chapter.

1. Coloured manifold hierarchies

To begin with let us make precise what a hierarchy for a coloured manifold is to be.

1.1. - Definition. - i) A coloured manifold hierarchy is a finite sequence of coloured manifold decompositions

$$(M_0,\Sigma_0) \overset{S_0}{\rightsquigarrow} (M_1,\Sigma_1) \overset{S_1}{\rightsquigarrow} \cdots \overset{S_{m-1}}{\rightsquigarrow} (M_m,\Sigma_m) ,$$

where (M_m,Σ_m) is a system of coloured products, i.e.

$$(M_m,\Sigma_m) = (R \times I,\ \partial R \times I),\ (R_m)_+ = R \times 1 ,$$

for some compact surface system R .

ii) A coloured manifold hierarchy for a coloured manifold (M,Σ) is one with $(M_0,\Sigma_0) = (M,\Sigma)$.

The goal of this chapter is to show that coloured manifold hierarchies exist for a big class of coloured manifolds. Following Gabai [Ga 1] we shall use for the construction of such hierarchies the notion of complexity of a coloured 3-manifold. Roughly the complexity measures how far a coloured manifold is from being a coloured product. The existence of a coloured manifold hierarchy is then established by induction on the complexity.

1.2. - The length of a Haken manifold

Let us briefly recall the notion of length of a Haken manifold. This notion enters in the definition of complexity and is based on the following observations which are proved, for instance, in [Ja, p.42f and p.57-61].

Proposition. Let M be a Haken manifold, then we have:

i) There exists a minimal integer $h(M)$ (the so-called closed Haken number of M) such that if $S_1,\ldots,S_n$ is any system of pairwise disjoint incompressible, ∂-incompressible, closed surfaces in M then either $n < h(M)$ or for some $i \neq j$, S_i is parallel to S_j in M.

ii) Suppose

$$(*)\quad M = M_0 \overset{S_0}{\rightsquigarrow} M_1 \rightsquigarrow \ldots \overset{S_{m-1}}{\rightsquigarrow} M_m$$

is a sequence of decompositions where each S_i is an incompressible, ∂-incompressible surface in M_i which is not ∂-parallel and not a disk. Then $m \leq 3h(M)$.

□

These statements permit us to define the length of M to be the maximal number of decompositions occuring in any sequence (*) for M.

If the manifold M is not connected then the length of M is understood to be the sum of the lengths of its components.

Remarks. - i) If $M \overset{S}{\rightsquigarrow} M'$ is a decomposition between Haken manifolds where the surface $S \neq D^2$ is incompressible,

∂-incompressible and not ∂-parallel then

$$\text{length } M' < \text{length } M$$

ii) Since

$$\text{length } M \leq 3h(M) ,$$

and $h(M) = 0$ if and only if M is a handlebody, it follows that any sequence (*) with $m = \text{length } M$ terminates with a system of handlebodies.

These remarks make it plausible that the notion of length is useful in an inductive construction of hierarchies for coloured manifolds.

The length of a Haken manifold behaves well with respect to splitting at disks. This will be made precise in the next section.

1.3. - Complexity disks. - Supposing that M is a Haken manifold one may induct on the Euler characteristic of ∂M in order to see that there exists a system of proper disks $\mathcal{D}$ in M such that each component of M split along $\mathcal{D}$ is ∂-irreducible or a 3-ball.

We call $\mathcal{D}$ a *system of complexity disks* for M.

Lemma. Let $\mathcal{D}$ *be any proper disk system in* M *and let* M' *be obtained by splitting* M *at* $\mathcal{D}$. *Then we have*

$$\text{length } M' = \text{length } M .$$

Proof. It suffices to give a proof for $\mathcal{D}$ being a single disk D . Let

$$M = M_0 \overset{S_0}{\rightsquigarrow} M_1 \rightsquigarrow \dots \overset{S_{m-1}}{\rightsquigarrow} M_m$$

be a length defining sequence of splittings for M , i.e length $M = m$, and each $S_i \neq D^2$ is incompressible, ∂-incompressible and not ∂-parallel in M_i . In particular, the components of M_m are handlebodies.

Now, as M is irreducible, we can isotope S_0 so that

$$D \cap S_0 = \phi \ .$$

Furthermore, if D' and D'' denote the two copies of D in $\partial M'$ then we may assume that

$$S_1 \cap D' = S_1 \cap D'' = \phi \ .$$

Procceding inductively, we thus obtain a commutative diagram of decompositions

$$(*) \qquad \begin{array}{ccccccc} M_0 & \overset{S_0}{\rightsquigarrow} & M_1 & \rightsquigarrow & \dots & \overset{S_{m-1}}{\rightsquigarrow} & M_m \\ D \downarrow & & D \downarrow & & & & \downarrow D \\ M'_0 & \overset{S_0}{\rightsquigarrow} & M'_1 & \rightsquigarrow & \dots & \overset{S_{m-1}}{\rightsquigarrow} & M'_m \end{array}$$

Note that M'_m consists of handlebodies if and only if M_m consists of handlebodies. As each S_i is incompressible and ∂-incompressible in M'_i it follows that

$$\text{length } M' \leq \text{length } M \; .$$

Conversely, if the bottom line of (*) is length defining for M' then we may clearly assume that

$$S_0 \cap D' = S_0 \cap D'' = \phi \; .$$

Therefore we again obtain a diagram (*) . We know that each S_i is incompressible also in M_i (cf.I, 2.2) . Moreover, S_i is ∂-incompressible in M_i , for any boundary compressing disk for S_i in M_i , even if its intersection with D is non-empty, would lead to a boundary compressing disk for S_i in M'_i . We conclude

$$\text{length } M \leq \text{length } M' \; .$$

□

1.4. - <u>Special complexity disks</u>

In the definition of the complexity of a coloured manifold we need a special sort of complexity disks. The existence of such a special disk system is established by the next lemma.

Lemma. *Let* M *be a Haken manifold. Then there exists a system* $\mathcal{D}$ *of complexity disks for* M *such that the following conditions hold.*

(1) *If* M_κ *denotes any* ∂*-irreducible component of* $M - \text{int}\, N(\mathcal{D})$ *and if* V *is a component of* ∂M_κ *then* $V \cap N(\mathcal{D})$ *has at most one component.*

(2) *If* B_λ *is any ball component of* $M - \text{int}\, N(\mathcal{D})$ *and* $V = \partial B_\lambda$ *then* $V \cap N(\mathcal{D})$ *has exactly three components or is empty, unless* V *intersects a unique* $N(D_\nu)$ (*where* D_ν is a component of $\mathcal{D}$) *and* $|V \cap N(D_\nu)| = 2$.

Proof. The existence of such a system is fairly obvious. We simply start with an arbitrary system of complexity disks $\mathcal{E}$ for M so that no two components of $\mathcal{E}$ are parallel and no component is ∂-compressible. Then, if M_0 is a component of $M - \text{int}\, N(\mathcal{E})$ which does not satisfy conditions (1) and (2), we split M_0 along an additional proper disk lying in M_0 so as to diminish the number of components of $N(\mathcal{E}) \cap M_0$. This creates a new ball component satisfying (2).

The system $\mathcal{D}$ then arises from $\mathcal{E}$ by adding these new disks.

□

2. The complexity of a coloured 3-manifold

It is convenient to define the complexity for coloured manifolds which need not be connected.

2.1. - Definition and remarks. - Let (M,Σ) be a system of taut coloured 3-manifolds.

i) A system of complexity disks $\mathcal{D} = D_1 \cup \ldots \cup D_n$ for (M,Σ) satisfies by definition the following

(1) $\mathcal{D}$ is a system of complexity disks for M .

(2) If V is a component of ∂M_κ (where M_κ is a ∂-irreducible component of $M - \operatorname{int} N(\mathcal{D})$) then $V \cap N(\mathcal{D})$ is connected (possibly empty).

(3) If $V = \partial B_\lambda$ (where B_λ is a ball component of $M - \operatorname{int} N(\mathcal{D})$) then $V \cap N(\mathcal{D})$ is empty or has exactly three components, unless V intersects a unique $N(D_\nu)$ and $|V \cap N(D_\nu)| = 2$.

(4) $\mathcal{D}$ is a splitting disk system for (M,Σ) . In particular, $\mathcal{D} \cap A(\Sigma)$ consists of proper (each) non-separating arcs.

ii) As condition (4) can always be arranged, the existence of a system of complexity disks for any taut coloured 3-manifold is established by the preceding lemma.

iii) If $\partial D_\nu \subset T(\Sigma)$ for some component D_ν of $\mathcal{D}$ then the component of (M,Σ) containing D_ν is $(D^2 \times S^1, \partial D^2 \times S^1)$. This is the last one of the possibilities listed in (3).

iv) If $D_\nu \cap A(\Sigma) \neq \phi$ then each component of $D_\nu \cap A(\Sigma)$ connects R_+ with R_- . It follows that $|D_\nu \cap A(\Sigma)|$ is even.

v) Condition (3) of the definition is put in order to distinguish between taut coloured handlebodies.

vi) If V is a component of ∂M_κ with $V \cap N(\mathcal{D}) = \phi$ then V is an incompressible component of ∂M .

2.2. - Minimal complexity disks

Suppose we are given a system of complexity disks $\mathcal{D} = D_1 \cup \ldots \cup D_n$ for (M,Σ) such that M-int $N(\mathcal{D})$ decomposes into ∂-irreducible components $M_1,\ldots,M_k$ and 3-balls $B_1,\ldots,B_\ell$. Then, if necessary after re-indexing the M_κ we may assume that for some $r \leq k$ M_κ is diffeomorphic to $P \times I$ for some closed surface P if and only if $\kappa \leq r$. Observe that then P is necessarily incompressible in M .

Next, let

$$\hat{\mathcal{D}} = \hat{D}_1 \cup \ldots \cup \hat{D}_s$$

be the sub-system of $\hat{\mathcal{D}}$ consisting of those D_ν such that

$$N(D_\nu) \cap \partial M_\kappa \neq \phi \quad \text{for some} \quad \kappa \leq r \ .$$

We set

$$a_i = |\hat{D}_i \cap A(\Sigma)| \ , \ i = 1,\ldots,s \ ,$$

and order the a_i so that $a_{i_1} \geq \ldots \geq a_{i_s}$. Finally, we set

$$\zeta_3^{\mathcal{D}} = (a_{i_1},\ldots,a_{i_s}) \quad .$$

If all a_i are zero or $\hat{\mathcal{D}} = \phi$ then we simply write $\zeta_3^{\mathcal{D}} = 0$. Similarly, we define

$$b_\nu = \begin{cases} |D_\nu \cap A(\Sigma)| & \text{if} \quad |D_\nu \cap A(\Sigma)| > 2 \\ 0 & \text{if} \quad |D_\nu \cap A(\Sigma)| \leq 2 \end{cases}$$

and order the b_ν so that

$$b_{\nu_1} \geq \ldots \geq b_{\nu_{n_0}} > b_{\nu_{n_0+1}} = \ldots = b_{\nu_n} = 0 \ .$$

Now we set

$$\zeta_4^{\mathcal{D}} = (b_{\nu_1}, \ldots, b_{\nu_{n_0}}) \ ,$$

and $\zeta_4^{\mathcal{D}} = 0$ when all $b_\nu = 0$ or $\mathcal{D} \cap A(\Sigma) = \phi$.

We say that $\mathcal{D}$ is a system of <u>minimal</u> complexity disks if

$$(\zeta_3^{\mathcal{D}}, \zeta_4^{\mathcal{D}}) = \min_{\mathcal{E}} \ (\zeta_3^{\mathcal{E}}, \zeta_4^{\mathcal{E}}) \ ,$$

where $\mathcal{E}$ ranges over all systems of complexity disks for (M,Σ) , and the pairs is given the dictionary ordering. To be more precise, we define $(\zeta_3,\zeta_4) < (\eta_3,\eta_4)$ if either $\zeta_3 < \eta_3$ or $\zeta_3 = \eta_3$ and $\zeta_4 < \eta_4$. Here we have $\zeta < \eta$ for tuples $\zeta = (a_1,\ldots,a_n)$ and $\eta = (b_1,\ldots,b_m)$ with $a_i \geq a_{i+1}$, $1 \leq i \leq n$, $b_j \geq b_{j+1}$, $1 \leq j \leq m$, if for some j , $a_i = b_i$ for $i < j$ and either $a_j < b_j$ or $n = j < m$ holds.

2.3. - Two basic properties of complexity disks

Systems of complexity disks are unique in the following sense.

Lemma 1. Let $\mathcal{D}$ and $\mathcal{D}'$ be systems of complexity disks for (M,Σ) and denote by M_κ , $\kappa = 1,\dots,k$, resp. $M'_{\kappa'}$, $\kappa' = 1,\dots,k'$, the ∂-irreducible components of $M - \operatorname{int} N(\mathcal{D})$ and $M - \operatorname{int} N(\mathcal{D}')$. Then $k = k'$ and, if necessary after a permutation of indices, M_κ is homeomorphic to M'_κ by an isotopy of M . In particular we have

(1) $r = r'$,

(2) $|\hat{D}| = |\hat{D}'|$,

where r' and $\hat{D}'$ are defined similarly to r resp. $\hat{D}$ in 2.2.

(3) If V denotes a component of ∂M_κ and V' is the corresponding component of $\partial M'_\kappa$ then $V \cap N(\mathcal{D}) = \emptyset$ if and only if $V' \cap N(\mathcal{D}') = \emptyset$.

Proof. We show that the M_κ for $\kappa \leq r$ are uniquely determined, independent of the choice of $\mathcal{D}$. For this let

$$M_\kappa = P_\kappa \times I \ , \ 1 \leq \kappa \leq r \ ,$$

where P_κ is a closed surface which is, moreover, incompressible in M . Then, if necessary after an isotopy of M_κ , we may assume that

$$P_\kappa \times I \cap \mathcal{D}' = \phi .$$

Now let M_0 be the component of $M\text{-int } N(\mathcal{D}')$ that contains $P_\kappa \times I$. As P_κ is incompressible M_0 is not a ball. Therefore M_0 is ∂-irreducible and we can isotope $\mathcal{D}$ so that

$$M_0 \cap \mathcal{D} = \phi .$$

It follows that

$$M_0 \subset P_\kappa \times I \quad \text{and} \quad P_\kappa \times I \subset M_0 ,$$

up to isotopy.

Thus to each component M_κ , $\kappa \leq r$, corresponds a component of $M\text{-int } N(\mathcal{D}')$ which is homeomorphic to it. Interchanging the rôles of $\mathcal{D}$ and $\mathcal{D}'$ we see that this correspondence is bijective. In particular we have condition (1).

In a similar way one shows that the components M_κ , $\kappa \geq r+1$, are uniquely determined up to isotopy.

The above analysis also shows that the component M_κ is glued together with M_μ if and only if M'_κ is glued together with M'_μ . This finally shows that conditions (2) and (3) hold.

□

If $\mathcal{D}$ is a system of complexity disks and S is a splitting surface in M such that $\mathcal{D} \cap S = \phi$ then $\mathcal{D}$ need not be a system of complexity disks for M-int $N(S)$, even if S is incompressible and ∂-incompressible. However, we have the following special result which will be used later.

Lemma 2. <u>Let $\mathcal{D}$ be a system of minimal complexity disks for the taut coloured manifold</u> (M,Σ) . <u>Denote by</u> (M',Σ') <u>the coloured manifold obtained by decomposing</u> (M,Σ) <u>along the sub-system</u> $\mathcal{D}_0$ <u>of</u> $\mathcal{D}$. <u>Suppose, moreover, that</u> $|D_\nu \cap A(\Sigma)| = 2$ <u>for every</u> $D_\nu \subset \mathcal{D}_0$. <u>Then</u> $\mathcal{D}' = \mathcal{D} - \mathcal{D}_0$ <u>is a system of minimal complexity disks for</u> (M',Σ') <u>and</u>

$$(\zeta_3^{\mathcal{D}'},\zeta_4^{\mathcal{D}'}) \leq (\zeta_3^{\mathcal{D}},\zeta_4^{\mathcal{D}}) .$$

<u>Proof</u>. By I; 5.4, (M',Σ') is taut, and it suffices to consider the case that $\mathcal{D}_0$ is a single disk.

Certainly $\mathcal{D}'$ is a system of complexity disks for (M',Σ') . Also it is easily seen that

$$\zeta_3^{\mathcal{D}'} \leq \zeta_3^{\mathcal{D}} \quad \text{and} \quad \zeta_4^{\mathcal{D}'} = \zeta_4^{\mathcal{D}} ,$$

for $|\mathcal{D}_0 \cap A(\Sigma)|=2$ and thus either $\zeta_3^{\mathcal{D}'} = \zeta_3^{\mathcal{D}}$ or $(\zeta_3^{\mathcal{D}'},2) = \zeta_3^{\mathcal{D}}$.

Now, if $\mathcal{D}'$ was not minimal then there would exist a system of complexity disks E for (M',Σ') with

$$(\zeta_3^{E},\zeta_4^{E}) < (\zeta_3^{\mathcal{D}'},\zeta_4^{\mathcal{D}'}) .$$

We want to run into a contradiction by showing that then $E_0 = E \cup \mathcal{D}_0$ is a system of complexity disks for (M,Σ) with smaller complexity than that of $\mathcal{D}$. That E_0 indeed is a system of complexity disks for (M,Σ) follows from lemma 1.

Let $\hat{\mathcal{D}}$ (and similarly $\hat{E}$, $\hat{\mathcal{D}}'$, $\hat{E}_0$) be the sub-system of $\mathcal{D}$ as above in the definition. We know from lemma 1 that

$$(*) \qquad |\hat{\mathcal{D}}'| = |\hat{E}| \quad \text{and} \quad |\hat{\mathcal{D}}| = |\hat{E}_0| \ .$$

Therefore, assuming that

$$(**) \qquad \zeta_3^{E} < \zeta_3^{\mathcal{D}'}$$

we have

$$\zeta_3^{E} = (a_1,\ldots,a_s) \ , \ \zeta_3^{\mathcal{D}'} = (b_1,\ldots,b_s) \ ,$$

i.e. both tuples have the same length, and there exists $j \leq s$ such that $a_i = b_i$ for $i < j$ and $a_j < b_j$. By lemma 1, we have to discuss two possibilities.

If $\mathcal{D}_0$ is a component of $\hat{\mathcal{D}}$ and of $\hat{E}_0$ then $(**)$ implies that

$$\zeta_3^{E_0} = (a_1,\ldots,a_s,2) < (b_1,\ldots,b_s,2) = \zeta_3^{\mathcal{D}} \ .$$

This contradicts the minimality of $\mathcal{D}$.

If $\mathcal{D}_0$ is neither a component of $\hat{\mathcal{D}}$ nor of $\hat{E}_0$ then we have

$$\zeta_3^{E_0} = \zeta_3^{E} \quad \text{and} \quad \zeta_3^{\mathcal{D}} = \zeta_3^{\mathcal{D}'} \ ,$$

and thus obtain a contradiction to $(**)$.

Finally, if

$$(***) \qquad \zeta_3^{E} = \zeta_3^{\mathcal{D}'} \quad \text{and} \quad \zeta_4^{E} < \zeta_4^{\mathcal{D}'} ,$$

then the above discussion shows that $\zeta_3^{E_0} = \zeta_3^{\mathcal{D}}$ and, since $|\mathcal{D}_0 \cap A(\Sigma)| = 2$, we conclude by (***)

$$\zeta_4^{E_0} = \zeta_4^{E} < \zeta_4^{\mathcal{D}} = \zeta_4^{\mathcal{D}}$$

This again is incompatible with the minimality of $\mathcal{D}$.

□

2.4. - <u>Definition of complexity</u>

We now complete the definition of the complexity of a taut coloured 3-manifold as follows.

As before let $\mathcal{D} = D_1 \cup \ldots \cup D_n$ be a system of minimal complexity disks for (M,Σ) and let $M_1,\ldots,M_k$ and $B_1,\ldots,B_\ell$ be the ∂-irreducible resp. ball components of M-int $N(\mathcal{D})$. Again we fix $r \leq k$ so that M_κ is diffeomorphic to $P_\kappa \times I$ if and only if $\kappa \leq r$.

We define the <u>complexity</u> of (M,Σ) to be the 4-tuple

$$C(M,\Sigma) = (C_1,C_2,C_3,C_4) ,$$

where

$$C_1 = C_1(M) = \sum_{\kappa=r+1}^{k} \text{length } M_\kappa \ ,$$

$$C_\alpha = C_\alpha(M,\Sigma) = \zeta_\alpha^{\mathcal{D}} \ , \ \alpha = 3,4 \ ,$$

and C_2 is the 6-tuple of non-negative integers

$$C_2 = C_2(M,\Sigma) = (a_1,\ldots,a_6) \ ,$$

with

$$a_1 = \left|\left(\bigcup_{\kappa=r+1}^{k} \partial M_\kappa\right) \cap N(\mathcal{D})\right| \ ,$$

$$a_2 = \#\{\kappa \mid \kappa \leq r \ \text{ and } \ |M_\kappa \cap N(\mathcal{D})| = 2\} \ ,$$

$$a_3 = \#\{\kappa \mid \kappa \leq r \ \text{ and } \ |M_\kappa \cap N(\mathcal{D})| = 1\} \ ,$$

$$a_4 = \#\{V \mid V \subset \partial M_\kappa \ \text{ for } \ \kappa > r \ \text{ and } \ V \cap A(\Sigma) \neq \phi\} \ ,$$

$$a_5 = \#\{\kappa \mid \kappa \leq r \ \text{ and } \ V \cap A(\Sigma) \neq \phi \ \text{ for every } \ V \subset \partial M_\kappa\} \ ,$$

$$a_6 = \#\{\kappa \mid \kappa \leq r \ \text{ and } \ V \cap A(\Sigma) \neq \phi \ \text{ for exactly one component } \ V \subset \partial M_\kappa\} \ .$$

(Here as always $N(\mathcal{D})$ denotes a regular neighbourhood of $\mathcal{D}$ in M .)

In words, $a_1(a_4)$ is the number of components of $\partial M_{r+1} \cup \ldots \cup \partial M_k$ which (non-trivially) intersect $N(\mathcal{D})$

(resp. $A(\Sigma)$); cf.2.1; (2). Further, $a_2(a_3)$ is the number of product components M_κ , $\kappa \leq r$, such that both (resp. exactly one) components of ∂M_κ intersect $N(\mathcal{D})$, and $a_5(a_6)$ is the number of product components M_κ , $\kappa \leq r$, such that both (resp. exactly one) components of ∂M_κ meet $A(\Sigma)$.

Clearly, the colouring of (M,Σ) comes in only in C_3, C_4 and the last three components of C_2 , whereas C_1 and the first three components of C_2 depend only on the topology of M .

Example. - If (M,Σ) is a taut 3-ball or is homeomorphic to $(D^2 \times S^1, \partial D^2 \times S^1)$ then $C(M,\Sigma) = (0,0,0,0)$.

2.5. - Invariance of the complexity

The following alternative characterization of complexity will turn out to be very useful in applications. It also shows that the complexity of a taut coloured manifold is independent of the special choice of the system used in its definition.

For $\mathcal{D}$ being any system of complexity disks for (M,Σ) define the $\mathcal{D}$-complexity of (M,Σ) , denoted $C^{\mathcal{D}}(M,\Sigma)$, in just the same way as $C(M,\Sigma)$.

Proposition. For a taut coloured 3-manifold (M,Σ) we have

$$C(M,\Sigma) = \operatorname*{Min}_{\mathcal{D}} C^{\mathcal{D}}(M,\Sigma)$$

where $\mathcal{D}$ runs through all systems of complexity disks for (M,Σ) .

Moreover, $C_1(M,\Sigma)$ and the first three components of $C_2(M,\Sigma)$ can be computed using any system of complexity disks for (M,Σ) .

Proof. Without loss of generality we may assume that no component of (M,Σ) is a taut 3-ball or is homeomorphic to $(D^2 \times S^1, \partial D^2 \times S^1)$.

Let $\mathcal{D}$ and $\mathcal{D}'$ be systems of complexity disks for (M,Σ) . By 2.3, lemma 1, we have (using the same notation as there)

$$\begin{aligned} c_1^{\mathcal{D}}(M,\Sigma) &= \sum_{\kappa = r + 1}^{k} \text{length } M_\kappa \\ &= \sum_{\kappa = r + 1}^{k} \text{length } M'_\kappa \\ &= c_1^{\mathcal{D}'}(M,\Sigma) \ . \end{aligned}$$

Next, coming to the invariance of

$$c_2^{\mathcal{D}}(M,\Sigma) = (a_1,\dots,a_6) \ ,$$

condition (3) of the same lemma tells us that

$$a_i = a'_i \quad \text{for} \quad i = 1,2,3 \ .$$

Moreover, an incompressible component V of ∂M with $V \cap A(\Sigma) \neq \phi$ counts in the same way for both a_i and a'_i , where $i = 4,5$ or 6 . Furthermore, as (M,Σ) is taut and no component of it is a taut 3-ball or is homeomorphic to $(D^2 \times S^1, \partial D^2 \times S^1)$, every component of $\mathcal{D}$ or $\mathcal{D}'$ meets $A(\Sigma)$ non-trivially.

Therefore, if $\mathcal{D}$ and $\mathcal{D}'$ are minimal then

$$C(M,\Sigma) = C^{\mathcal{D}}(M,\Sigma) = C^{\mathcal{D}'}(M,\Sigma) .$$

Finally, taking a system $\mathcal{E}$ so that

$$C^{\mathcal{E}}(M,\Sigma) = \operatorname*{Min}_{\mathcal{D}} C^{\mathcal{D}}(M,\Sigma)$$

it clearly follows by what we have already proved that $\mathcal{E}$ is minimal and thus

$$C^{\mathcal{E}}(M,\Sigma) = C(M,\Sigma).$$

$\square$

2.6. - Trivial complexity and a first reduction step

Obviously, every strictly decreasing sequence of complexities is finite. This enables us to use the complexity as an inductive method for the construction of coloured manifold hierarchies. We can see this more clearly once we know what the taut coloured manifolds with trivial (zero) complexity are.

Proposition. - A connected taut coloured 3-manifold $(M,\Sigma,R_{\pm})$ has trivial complexity if and only if it belongs to the following list:

a) $(M,\Sigma) = (D^2 \times S^1, \partial D^2 \times S^1)$,

b) $(M,\Sigma) = (T^2 \times I, T^2 \times 0)$, and $R = R_+ = T^2 \times 1$,

c) $(M,\Sigma) = (T^2 \times I, \phi)$, and $R = \partial M = R_+$,

d) $(M,\Sigma) = (T^2 \times I, T^2 \times \partial I)$,

e) $(M,\Sigma) = (P_g \times I, \phi)$, where P_g is a closed orientable surface of genus $g \geq 1$, and $R_+ = P_g \times 1$, $R_- = P_g \times 0$,

f) $(M,\Sigma) = (P \times I, \partial P \times I)$, where P is a compact orientable surface with $\partial P \neq \phi$, and $R_+ = P \times 1$, $R_- = P \times 0$.

Proof. At first let us consider the case that M is ∂-irreducible, i.e. $\mathcal{D} = \phi$. Then $C_1 = 0$ implies that $M = P_g \times I$ for some closed surface P_g of genus $g \geq 1$. Here we use the fact that a 3-manifold has length zero if and only if it is a handlebody and that handlebodies are ∂-reducible.

Further, since $a_5 = a_6 = 0$, we see that

$$\partial M \cap A(\Sigma) = \phi .$$

Thus either $g > 1$ and $R = \partial M$, a situation that is covered by e), or $g = 1$ and (M,Σ) is one of those coloured manifolds listed in b), c), d), or constitutes the remaining case of e).

Now let us investigate what is going on when M is ∂-reducible, so that the underlying system $\mathcal{D}$ of minimal complexity disks for (M,Σ) is non-empty. As before we denote by $M_1,\dots,M_r,\dots,M_k$, $B_1,\dots,B_\ell$ the components of $M - \text{int } N(\mathcal{D})$ where the M_κ are ∂-irreducible and $M_\kappa = P_\kappa \times I$ if and only if $\kappa \leq r$. As no M_κ can be a handlebody, we conclude that

$$r = k .$$

Moreover, the hypothesis $a_2 = a_3 = 0$ implies that

$$k = 0 ,$$

i.e. $M - \text{int } N(\mathcal{D})$ consists entirely of balls. This shows that M is a handlebody.

Next, since $C_4 = 0$, every component D_ν of $\mathcal{D}$ intersects $A(\Sigma)$ in at most two components. However if $|D_\nu \cap A(\Sigma)| = 0$ then, by norm-minimality of R_+ and R_- , necessarily $\partial D_\nu \subset T(\Sigma)$ which means that a) holds. It therefore remains to consider the case that

$$|D_\nu \cap A(\Sigma)| = 2 \quad \text{for every} \quad D_\nu \subset \mathcal{D} .$$

We want to show that then (M,Σ) is of type f).

Proceeding by induction on the genus of M we first assume that M is a solid torus. Then $\mathcal{D}$ is a meridional disk, and our assertion is easily verified (especially by those having mastered exercise 4.7, ii) of chapter I).

To establish the induction claim we take a disk D_ν of $\mathcal{D}$ which is non-separating in M. Then, by I; 5.4, decomposing (M,Σ) along D_ν provides a taut coloured handlebody (M',Σ'). By 2.3, lemma 2, (M',Σ') has trivial complexity whence it follows by the induction hypothesis that

$$(M',\Sigma') = (P' \times I,\ \partial P' \times I) .$$

Now we look at P' as a bouquet of bands. Then (M',Σ') can be visualized as indicated in fig. 3a).

To reconstruct (M,Σ) from (M',Σ) we have to identify two disks E_+ and E_- in $\partial M'$, where E_+ (and similarly E_-) is a union of disks E_+' and E_+'' with

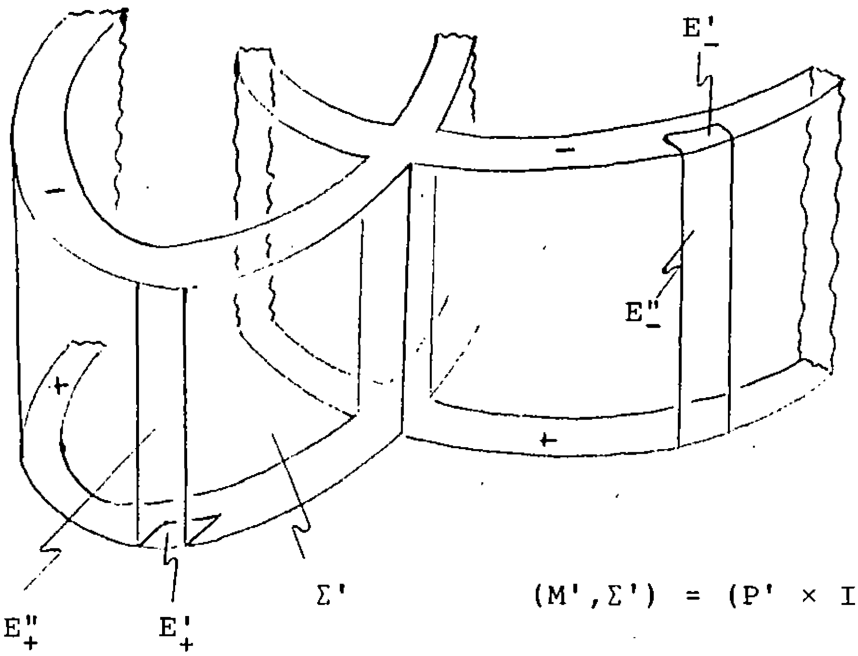
$$E_+' \subset R_+' \qquad \text{and} \qquad E_+'' \subset A(\Sigma') ;$$

cf. fig. 3a).

We now modify (M',Σ') homeomorphically so that E_+ and E_- come to lie in the vertical part of ∂M ; see fig. 3b). It is then evident that identification of E_+ and E_- again gives a product of type f).

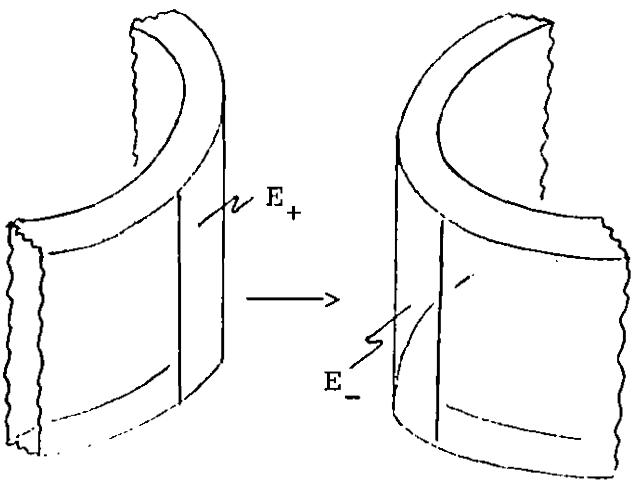
To complete the proof we observe that each member of our list indeed has zero complexity.

□



a)

$(M',\Sigma') = (P' \times I, \partial P' \times I)$



b)

Figure 3

As another application of our new concept let us prove the following generalization of lemma 2 of section 2.3 which will be used later in an essential manner.

Lemma. Let $\mathcal{D}$ be a system of minimal complexity disks for the taut coloured manifold (M,Σ) and let $\mathcal{D}_0 \subset \mathcal{D}$ be a sub-system such that $|D_\nu \cap A(\Sigma)| = 2$ for every component D_ν of $\mathcal{D}_0$. Then

$$C(M',\Sigma') \leq C(M,\Sigma) ,$$

where (M',Σ') is obtained by splitting (M,Σ) along $\mathcal{D}_0$.

Proof. By I; 5.4, (M',Σ') is taut, so its complexity is defined. Also we already know from lemma 2 of section 2.3 that $\mathcal{D}' = \mathcal{D} - \mathcal{D}_0$ is a system of minimal complexity disks for (M',Σ') and

$$(C_3(M,\Sigma),\ C_4(M,\Sigma)) \leq (C_3(M',\Sigma'),\ C_4(M',\Sigma')) .$$

Furthermore

$$C_1(M,\Sigma) = C_1(M',\Sigma')$$

holds trivially. Also inspecting the components $a_1,\ldots,a_6$ of $C_2(M,\Sigma)$ we convince ourselves immediately that

$$a_i' \leq a_i \ , \ i = 1,2 \ ,$$

and

$$a_3' > a_3 \ \text{ only if } \ a_2' < a_2 \ .$$

Finally, as $V \cap A(\Sigma) \neq \phi$ if and only if $V \cap A(\Sigma') \neq \phi$ for any component $V \subset \partial M_K$ we conclude that

$$a_i' = a_i \ \text{ for } \ i = 4,5,6 \ .$$

□

2.7. - Exercises. - i) Find a Haken manifold M, a system $\mathcal{D}$ of complexity disks and a proper surface $S \neq D^2$ in M which is incompressible (∂-incompressible, not ∂-parallel) such that $\mathcal{D} \cap S = \phi$ and $\mathcal{D}$ is not a system of complexity disks for $M\text{-int } N(S)$.

ii) Show that the coloured manifolds listed in the proposition of section 2.6 indeed have trivial complexity.

iii) For any $k \geq 1$ find a taut coloured manifold (M,Σ) where M is the handlebody of genus two such that $C(M,\Sigma)$ is non-trivial and Σ consists of k components.

iv) Find a taut coloured manifold (M,Σ) and a system $\mathcal{D}$ of complexity disks for M such that $|D \cap A(\Sigma)| = 2$ for every component D of $\mathcal{D}$ but (M,Σ) is not a coloured product.

3. Splitting surfaces and complexity disks

Suppose we are given a taut coloured manifold (M,Σ) with $C(M,\Sigma) \neq 0$. We are looking for a (non-separating) incompressible splitting surface S for (M,Σ) such that the coloured manifold (M',Σ') obtained by splitting (M,Σ) along S is taut and has complexity smaller than that of (M,Σ) . When $H_2(M,\Sigma) \neq 0$ such a surface is provided by theorem I; 6.1. However, to decide that $C(M',\Sigma')$ is indeed smaller than $C(M,\Sigma)$ we first have to put S in a special position with respect to a system of complexity disks $\mathcal{D}$ for (M,Σ) . We shall do this in two steps.

To simplify language we call a splitting system S taut if the coloured manifold obtained by splitting along S is taut.

Now among all non-separating incompressible taut splitting surfaces for (M,Σ) we choose one so that $|\mathcal{D} \cap S|$ is minimal. It follows by the incompressibility of S and the irreducibility of M that then $\mathcal{D} \cap S$ does not contain any circle.

3.1 Deforming the splitting surface in a nice position

To begin with we deform the splitting surface S so that it becomes nice in the complement of $\mathcal{D}$.

Lemma 1. - Let (M,Σ) be a taut coloured manifold and let $\mathcal{D}$ be a system of complexity disks for (M,Σ) .

Suppose S is a splitting surface for (M,Σ) as above. Then S can be deformed into a taut splitting surface, again denoted S, such that $|\mathcal{D} \cap S|$ is also minimal and, furthermore, the following holds.

(1) If A is a component of $A(\Sigma)$ then each component of $S \cap A$ either intersects each component of $\mathcal{D} \cap A$ in exactly one point or $S \cap A \cap \mathcal{D} = \emptyset$.

(2) There exists a tubular neighbourhood $N(\mathcal{D}) = \mathcal{D} \times I$, $I = [-1,1]$ of $\mathcal{D}$ such that we have:
If M_0 is a component of $cl(M-N(\mathcal{D}))$ and S_0 denotes a component of $S \cap M_0$ then either

a) $S = S_0$ and is ∂-parallel in M_0, or

b) S_0 is not ∂-parallel in M_0, or

c) S_0 is parallel into $cl(\partial M_0 - N(\mathcal{D}))$ rel $N(\mathcal{D})$, i.e. there exists an embedding

$$\varphi:(S_0 \times [0,1],\ S_0 \cap N(\mathcal{D}) \times [0,1]) \to (M_0, M_0 \cap N(\mathcal{D}))$$

such that $\varphi | S_0 \times 0 = id_{S_0}$ and $\varphi(S_0 \times 1) \subset cl(\partial M_0 - N(\mathcal{D}))$.

Proof. At first we deform S so that condition (1) holds. Then we choose a small tubular neighbourhood

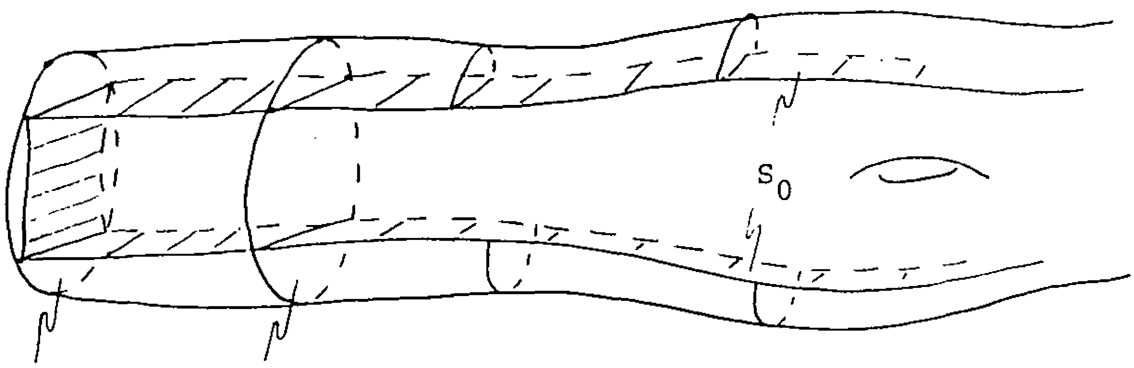
$N(\mathcal{D}) = \mathcal{D} \times I$ of $\mathcal{D}$ in M such that

$$N(\mathcal{D}) \cap \Sigma = (\mathcal{D} \cap \Sigma) \times I$$

and

$$N(\mathcal{D}) \cap S = (\mathcal{D} \cap S) \times I .$$

Now for a component M_0 of $cl(M-N(\mathcal{D}))$ we consider an arbitrary component S_0 of $S \cap M_0$. If S_0 is ∂-parallel and neither a) nor c) holds for S_0 then clearly S_0 is ∂-parallel in the component M_0' of $M-(\mathcal{D} \times (-\frac{1}{2}, \frac{1}{2}))$ containing M_0 . We choose a corresponding isotopy whose restriction to ∂S_0 may be assumed to stay within an arbitrarily small neighbourhood of ∂S_0 in M_0' , and first deform S_0 by means of this isotopy to lie in a small collar to $\partial M_0'$ in M_0' ; see fig.4. Then the components of the deformed surface S_0 lying in M_0 satisfy condition c). Note also that there is no problem in extending such an isotopy to one on $M \supset M_0'$ which is constant outside a small neighbourhood of M_0' in M . So condition (1) is still satisfied.



$D \times \frac{1}{2}$ $D \times 1$ $M_0 \subset M_0'$ $(D \subset \mathcal{D})$

Figure 4

Now for a component T of S_0 in M_0 satisfying condition (2), c) means that T separates M_0 and the closure of one of the resulting components is of the form $T \times [0,1]$ where $T \times 0 = T$ and $T \times 1 \subset \partial M_0$. Therefore, if T_1 is another component of $S \cap M_0$ that lies in $T \times [0,1]$ then T_1 is parallel to some sub-surface of $\partial(T \times [0,1])-T$; see [Wa; p.65]. Here we use that S is connected and that $B \cap N(\mathcal{D})$ is connected for every boundary component B of M_0 , provided $M_0 \neq D^3$. It follows that the above deformation process can be repeated with other components of $S \cap M_0$ which are ∂-parallel. This observation completes the proof of the lemma.

$\square$

Note that the minimality of $|\mathcal{D} \cap S|$ was used so far only in that $\mathcal{D} \cap S$ does not contain circle components. Observe also that the new surface S evidently also satisfies the minimality condition.

At this point we dispose of a tubular neighbourhood $N(\mathcal{D})$ of our fixed system of complexity disks for (M,Σ) so that, in the complement of $N(\mathcal{D})$, the splitting surface S is in a nice position. We now achieve more niceness by also improving the position of S in $N(\mathcal{D})$.

In the next lemma we have the same hypotheses as in lemma 1.

Lemma 2. - Let S be the surface in M obtained by lemma 1. Then S can be isotopically deformed in $N(\mathcal{D}) = \mathcal{D} \times I$ such that it still suffices conditions (1) and (2) provided by lemma 1 and so that, moreover, $|\mathcal{D} \cap S|$ is still minimal and the restriction of S to $N(\mathcal{D})$ can be described as follows.

For each component D of $\mathcal{D}$ we have

$$(*) \qquad N(D) \cap \partial S = (D \cap \partial S) \times I \,,$$

and if (*) is not an equality between empty sets then there exist numbers $-1 = t_1 < \ldots\ldots < t_m = 1$ such that, for $1 \leq j \leq m$, $S \cap (D \times t_j)$ is a system of properly embedded arcs $a_{j_1}, \ldots, a_{j_k}$ and, for $1 \leq j \leq m-1$, each component of $S \cap (D \times [t_j, t_{j+1}])$ is either of the form $a_{j_i} \times [t_j, t_{j+1}]$ or is a saddle as indicated in fig.5.

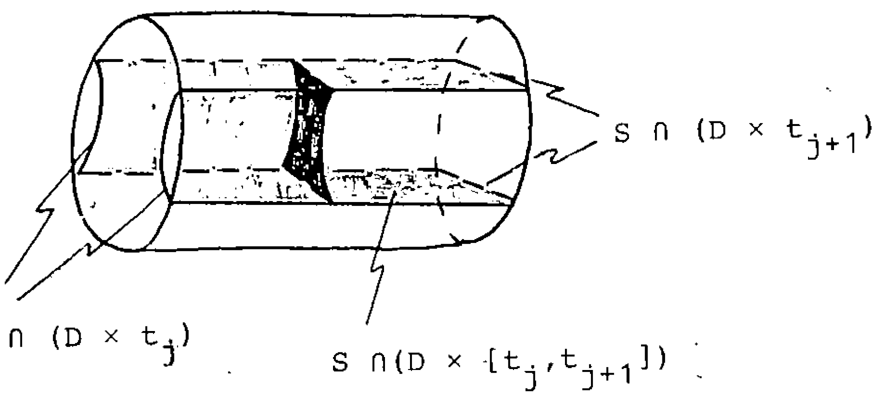


Figure 5

Proof. Initially, every component of $N(D) \cap S$ was a disk of the form $a \times [-1,1]$ where a is an arc in D. By lemma 1, this situation is possibly changed in so far as (possibly multi-pronged) saddles were pushed from $cl(M-N(\mathcal{D}))$ into $N(D)$. This shows that the components of $N(D) \cap S$ are still disks and that (*) holds.

Now when E denotes such a disk containing a saddle that is multi-pronged we can isotope S so that (*) still holds and the multi-pronged saddle is replaced by saddles as in fig.5 which are on different levels. As this isotopy can be chosen to be constant off $N(\mathcal{D})$ the results of lemma 1 remain valid.

□

3.2 Special boundary compressions

Let us suppose that S and $N(\mathcal{D})$ are as provided by lemma 2. Then, with the notation of lemma 2, a component c of $(D \times t_j) \cap S$ may occur a priori as in one of the seven cases depicted in figure 6.

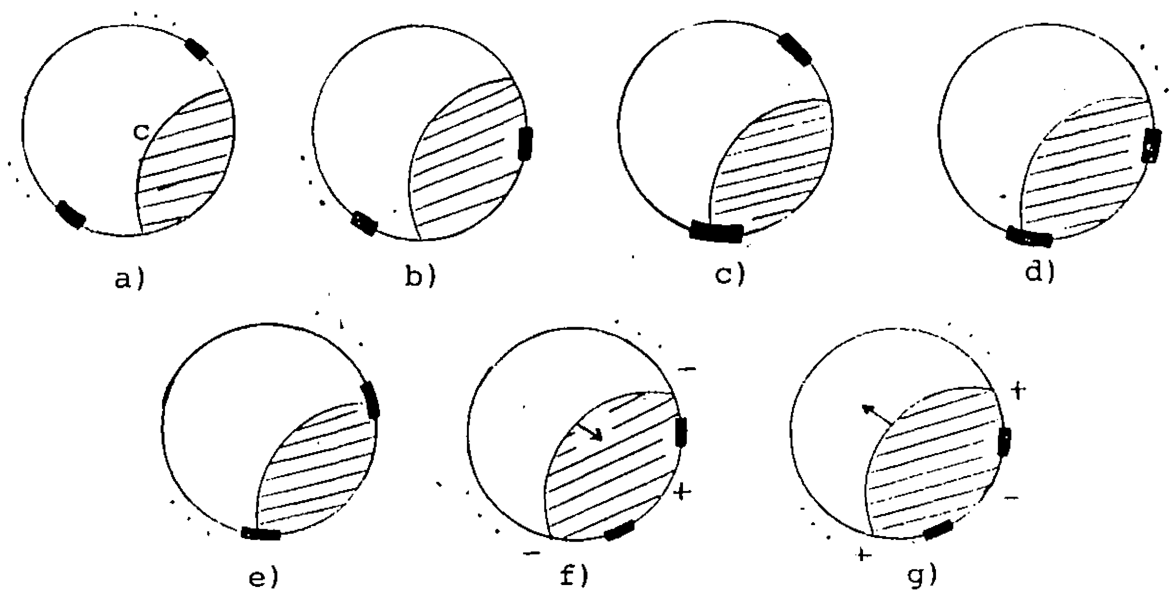


Figure 6

Here the thickened arcs are components of $(D \times t_j) \cap A(\Sigma)$, the shaded area is a disk E in $D \times t_j$, the arrows as usual denote normal orientation and the $\pm$-sign indicates that the corresponding arc belongs to $R_{\pm}$, respectively.

The essential point is that each such disk E corresponds to a splitting disk for (M',Σ') with $|E \cap A(\Sigma')| \leq 2$.

We are now going to show that in the special situation at hand neither of these possibilities can occur, for the appearance of any of these possibilities would contradict the minimality of $|\mathcal{D} \cap S|$. Let us first focus our interest on case a) of figure 6.

<u>Lemma 1</u>. - <u>In case a) of figure 6 the situation specifically cannot occur as shown in figure 7</u>.

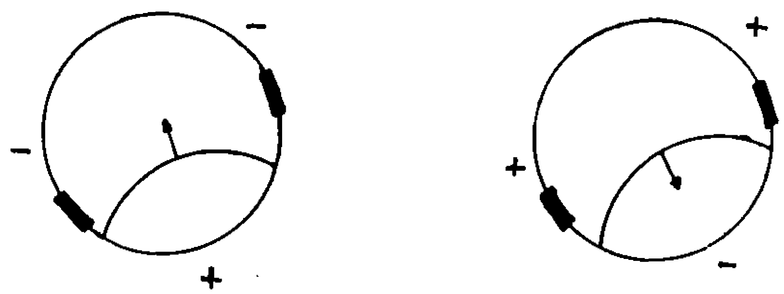


Figure 7

<u>Proof</u>. Both cases would yield a splitting disk E for (M',Σ') such that

$$|E \cap A(\Sigma')| = 0 .$$

As R' is incompressible, ∂E would bound a disk E' in R'. By the construction of (M',Σ') from (M,Σ), E' itself would be the union of two disks F and G the first one of which belonging to S and the other one to ∂M ; see figure 8.

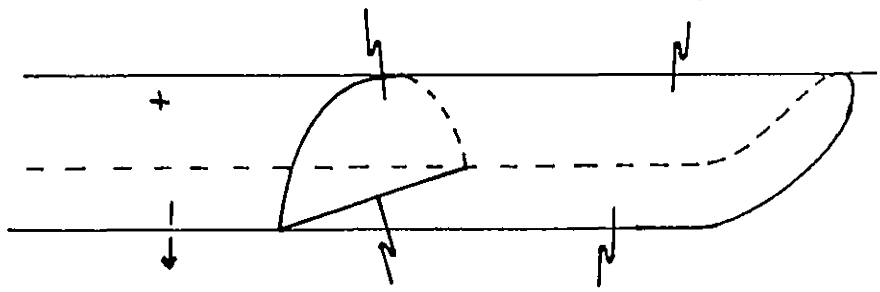


Figure 8

Now $E \cup F \cup G$ is a sphere in M and thus bounds a ball. Therefore, if $G \cap A(\Sigma) = \emptyset$ then we can deform S so that c disappears. Obviously, this would contradict the minimality of $|\mathcal{D} \cap S|$.

On the other hand, if $G \cap A(\Sigma) \neq \emptyset$ then somewhere a constellation as indicated in fig.9 a) must occur. However, as (M',Σ') is required to be taut, we are allowed to deform S as indicated in fig.9 b). The important point is that if S is oriented as in a) then the colouring of ∂M must be necessarily as in a). Otherwise (M',Σ') would not be taut. We can repeat deforming S in this way until we eventually again have $G \cap A(\Sigma) = \emptyset$. As this is impossible the lemma is proved. $\square$

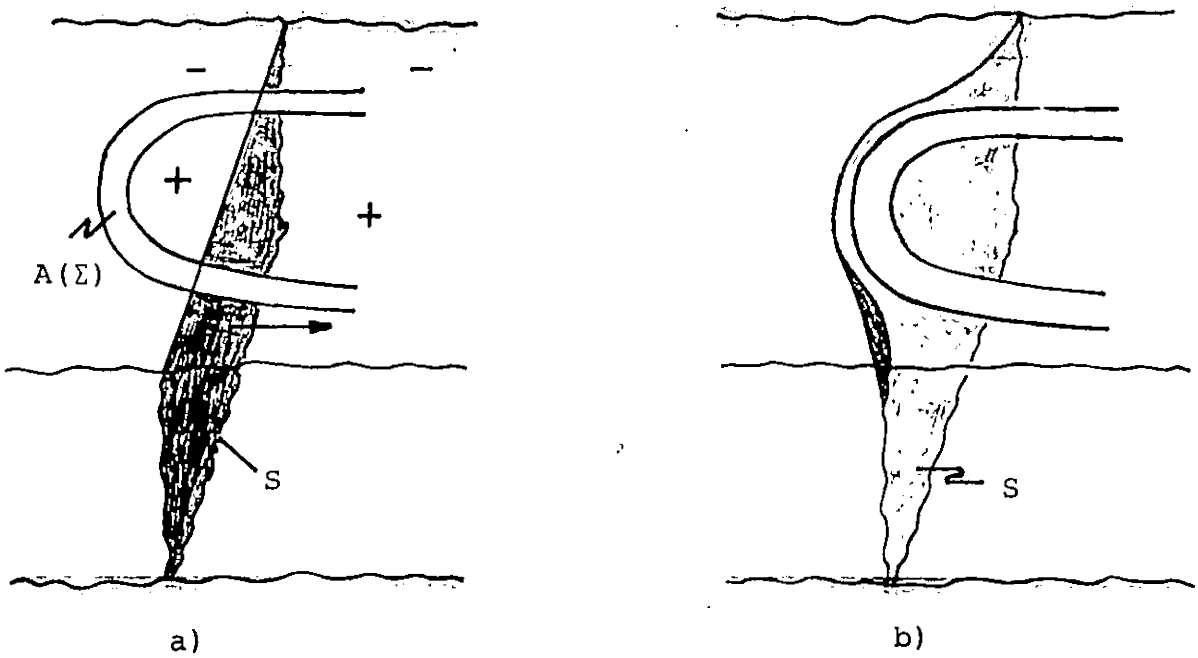


Figure 9

We shall now treat the remaining cases of fig.6. These are the cases where $|E \cap A(\Sigma')| = 2$.

Lemma 2. - <u>Under the given assumptions none of the cases illustrated by figure</u> 6 <u>can appear</u>.

<u>Proof</u>. Assume E is a disk in $D \times t_j$ satisfying one of the remaining cases of fig.6. If necessary after an isotopy of S we may assume that one of the cases a), b), f) or g) holds.

Now we perform a boundary compression of S along E and thus obtain a splitting surface system T for (M,Σ) . As (M',Σ') is taut, it follows from I ; 5.4 that T is taut. Moreover, by I ; proposition 5.3, we may assume that T is connected and non-separating. Therefore, in order to produce a contradiction it suffices to show that T can be deformed isotopically rel $A(\Sigma)$ so that

$$(*) \qquad |\mathcal{D} \cap T| < |\mathcal{D} \cap S| .$$

Clearly we have

$$|D \times t_j \cap T| = |D \times t_j \cap S| - 1$$

and as $D \times t_j$ is isotopic to D rel $A(\Sigma)$ we can find an isotopy of T rel $A(\Sigma)$ so that (*) holds.

$\square$

3.4. Complexity disks for the decomposed manifold

Given a coloured manifold decomposition $(M,\Sigma) \overset{S}{\rightsquigarrow} (M',\Sigma')$ and a system of complexity disks $\mathcal{D}$ for (M,Σ), there is no general recipe how to obtain from $\mathcal{D}$ and S a system of complexity disks for M' or even for (M',Σ'). However, in order to decide that $C(M',\Sigma') < C(M,\Sigma)$ we always have to refer to such a system.

The following situations will arise in the proof of the existence theorem in the next paragraph. So let us specify in either of these a system of complexity disks for M' or (M',Σ').

As before we suppose that (M,Σ) and (M',Σ') are taut and that S and $N(\mathcal{D})$ are provided by lemma 3.2. In particular, S is connected, incompressible and non-separating. Recall also from paragraph 2 the decomposition of $cl(M-N(\mathcal{D}))$ into balls B_λ, and other pieces $M_1,\dots,M_k$ where $M_\kappa = P_\kappa \times I$ (P_κ a closed surface) if and only if $\kappa \leq r$.

1. case: S is closed.

Then S is contained in some component M_0 of $cl(M-N(\mathcal{D}))$. Furthermore, as S is incompressible, $cl(M_0-N(S))$ is ∂-irreducible. We conclude that $\mathcal{D}' = \mathcal{D}$ is a system of complexity disks for (M',Σ').

2. case: $\partial S \neq \phi$ and $S \cap \mathcal{D} = \phi$.

Also in this case S is contained in some M_0 as above, but this time $cl(M-N(S))$ need not be ∂-irreducible.

i) If $M_0 \neq B_\lambda$ then let $\mathcal{E}$ be any system of complexity disks for $cl(M_0-N(S))$. Then by $\mathcal{D}' = \mathcal{D} \cup \mathcal{E}$ we clearly obtain a system of complexity disks for M'.

ii) If $M_0 = B_\lambda$ then S is a disk. As

$$M_0 \cap \partial N(\mathcal{D}) \neq \phi$$

and consists of at most three disks, S is parallel to some component D of $\mathcal{D}$. Then it is easily seen that some subsystem $\mathcal{D}'$ of $\mathcal{D}-D$ is a system of complexity disks for (M',Σ'). (The choice of $\mathcal{D}'$ depends on the special situation. However in the argument later we only need that $\mathcal{D}' \subset \mathcal{D}-D$, so that further specification of $\mathcal{D}'$ is unnecessary.)

3. case: $S \cap \mathcal{D} \neq \phi$.

As $\mathcal{D} \cap S$ consists of arcs, the components of $\mathcal{D} \cap M'$ are disks. The system of these disks can be completed to a system $\mathcal{E}$ of complexity disks for M'.

i) If there exists a component S_0 of $S \cap cl(M-N(\mathcal{D}))$ which is contained in some M_κ, $\kappa \geq r+1$, and is not ∂-parallel in M_κ, then we take $\mathcal{D}' = \mathcal{E}$ (where

$\mathcal{E}$ is as above) as a system of complexity disks for M' .

ii) If i) does not hold then we shall have to refer to a system of complexity disks $\mathcal{D}'$ for (M',Σ') , not just one for M' . However the system $\mathcal{E}$ as above in general does not suffice condition (3) of definition 2.1, i) and thus cannot serve as $\mathcal{D}'$. To overcome this difficulty we start with the system

$$\mathcal{E}' = \bigcup_{\nu,j} D_\nu \times t_j \cap M' \text{ , } \nu = 1,\dots,n \text{ ; } j=j(\nu)=1,\dots,m(\nu)$$

and consider a ball

$$B = D \times [t_j, t_{j+1}] \subset \mathcal{D} \times I = N(\mathcal{D}) \text{ ;}$$

cf. 3.1, lemma 2. If $B \cap S$ contains a saddle then according to 3.1., lemma 2, $cl(B-(N(S) \cup N(D \times \{t_j,t_{j+1}\})))$ consists of $p+2$ balls $B_1,\dots\dots,B_{p+2}$ where

$$|B_\pi \cap N(\mathcal{E})| = \begin{cases} 3 & \text{if } \pi \leqq 2 \\ 2 & \text{if } 3 \leqq \pi \leqq p+2 \text{ .} \end{cases}$$

Next we observe that if T is an incompressible surface in a product $P \times I$ (where P is a closed surface) then any component of $cl(P \times I - N(T))$ is either a handlebody or is homeomorphic to $P \times I$.

It follows that a sub-system of $\mathcal{E}'$ may be completed to a system $\mathcal{D}'$ for (M',Σ') where the additional disks are needed to cut down the handlebody components of $cl(M-(N(\mathcal{D}) \cup N(S)))$ to balls. (These additional disks are completely irrelevant as in the present case we do not need $\zeta_4^{\mathcal{D}'}$ in comparing $C(M,\Sigma)$ with $C(M',\Sigma')$.)

4. Coloured manifold hierarchies exist

4.1. - The existence theorem (statement).

In this paragraph we show that a coloured manifold hierarchy exists for most taut coloured 3-manifolds. More precisely, we shall prove:

Theorem. - *Every connected taut coloured manifold* (M,Σ) , *where* M *is not a rational homology sphere containing no incompressible torus, has a coloured manifold hierarchy*

$$(M,\Sigma) = (M_0,\Sigma_0) \overset{S_0}{\rightsquigarrow} (M_1,\Sigma_1) \rightsquigarrow \ldots\ldots \overset{S_{m-1}}{\rightsquigarrow} (M_m,\Sigma_m) \ .$$

Moreover, each S_i *is connected and if* $\partial M_i \neq \phi$ *then* $S_i \cap \partial M_i \neq \phi$.

As already mentioned, the proof of this result is by induction on the complexity of (M_i,Σ_i) . It is carried out by several steps according to the several positions the splitting surface may have. Before we start with the actual induction process, let us briefly recapitulate what we already have achieved.

We are given a (connected) taut coloured manifold (M,Σ) whose complexity is supposed to be non-trivial and a system of minimal complexity disks $\mathcal{D}$ for (M,Σ). $\mathcal{D}$ is empty whenever the manifold M is ∂-irreducible. Furthermore, if $A(\Sigma) \neq \phi$ then, by lemma 2.6, we may assume that

$$|D_\nu \cap A(\Sigma)| \geq 4 \quad \text{for all} \quad 1 \leq \nu \leq n .$$

Now in case $H_2(M,\partial M) \neq 0$ we have found a non-separating, incompressible taut splitting surface for (M,Σ) whose position in M with respect to a regular neighbouhood $N(\mathcal{D})$ of $\mathcal{D}$ and to the components $M_1,\dots,M_k$, $B_1,\dots,B_\ell$ of $cl(M-N(\mathcal{D}))$ is nice; see 3.1.

4.2. - The induction step

We turn to the proof of theorem 4.1. First of all let us perform the induction step under the special assumption that there exists a nice splitting surface. So in the next two lemmas S denotes an incompressible, non-separating splitting surface for (M,Σ) which is in a nice position as provided by section 3.1. Furthermore, the coloured manifold (M',Σ') obtained by splitting along S is taut and as a system of complexity disks $\mathcal{D}'$ for M' resp. (M',Σ') we use the one supplied by 3.4.

<u>Lemma 1</u>. - <u>If</u> $\mathcal{D} \cap S = \phi$ <u>then we have</u>

$$C(M',\Sigma') < C(M,\Sigma) .$$

<u>Proof</u>. 1. case: S is closed.

Then S is contained in some ∂-irreducible component M_0 of $cl(M-N(\mathcal{D}))$.

If S is not ∂-parallel in M_0 then, by [Wa; prop.3.1], $M \neq P \times I$. Therefore, by remark 1.2, i), we conclude

$$C_1(M',\Sigma') < C_1(M,\Sigma) .$$

If S is ∂-parallel in M_0 then clearly

$$C_1(M',\Sigma') = C_1(M,\Sigma) .$$

But in this case our hypothesis implies that $N(\mathcal{D}) \cap B$ is non-empty, where B is the component of ∂M_0 into which S is parallel. Therefore, if $M_0 \neq P \times I$ then, with the notation

$$C_2(M,\Sigma) = (a_1,\ldots,a_6) \quad \text{and} \quad C_2(M',\Sigma') = (a_1',\ldots,a_6')$$

we conclude that

$$a_1' < a_1 .$$

If $M = P \times I$ then both boundary components of M_0 necessarily meet $N(\mathcal{D})$. Hence it follows that in this case we have that

$$a_2' < a_2 .$$

Here we referred to proposition 2.5. Our claim is proved when S is closed.

2. case: $\partial S \neq \phi$.

Clearly in this case S is also contained in some component M_0 of $cl(M-N(\mathcal{D}))$ which now, however, may be a 3-ball. We thus distinguish between two possibilities.

i) M_0 is not a 3-ball.

If S is ∂-parallel in M_0 , and $\mathcal{D}' = \mathcal{D} \cup \mathcal{E}$ is as in the corresponding case of 3.4, then $\mathcal{E}$ is a system of complexity disks for the handlebody split off from M_0 by S . As S is not ∂-parallel in M we conclude as in the 1. case that

$$C_1(M',\Sigma') = C_1(M,\Sigma)$$

and

$$C_2(M',\Sigma') < C_2(M,\Sigma) .$$

A similar argument holds if S is not ∂-parallel in M_0 and $M_0 = P \times I$.

Next we assume that S is not ∂-parallel in M_0 and $M_0 \neq P \times I$. If S is ∂-compressible, we perform the necessary compressions on S in order to obtain a ∂-incompressible surface system S' (cf. [Ja; p.44]). As S is irreducible and not ∂-parallel in M_0 the same holds for S' . Moreover, S' is not a disk because M_0 is ∂-irreducible. We thus obtain a commutative diagramm of decompositions

$$\begin{tikzcd} M_0 \arrow[r, rightsquigarrow, "S"] \arrow[dr, rightsquigarrow, "S'"'] & M_0' \arrow[d, rightsquigarrow, "\mathcal{E}'"] \\ & M_0'' \end{tikzcd} ,$$

where $\mathcal{E}'$ denotes the disk system used to create S' .

Now it follows from remark i) of section 1.2 and lemma 1.3 that

$$\text{length } M_0' = \text{length } M_0'' < \text{length } M_0 \ .$$

Applying lemma 1.3 a second time we eventually see that

$$C_1(M',\Sigma') < C_1(M,\Sigma) \ .$$

ii) M_0 is a ball.

In this case S must be a disk. Furthermore, as $\partial M_0 \cap N(\mathcal{D})$ is non-empty and consists of at most three disks, S is parallel to some component D of $\mathcal{D}$. Furthermore, since by assumption

$$|D \cap A(\Sigma)| \geq 4 ,$$

D counts in $\zeta_4^{\mathcal{D}}$ (cf. 2.2). Here we have to observe that our hypotheses imply that $A(\Sigma) \neq \phi$. Now, since D does not count in $\zeta_4^{\mathcal{D}'}$, we conclude that

$$\begin{aligned}(C_3(M',\Sigma'), C_4(M',\Sigma')) &\leq (\zeta_3^{\mathcal{D}'}, \zeta_4^{\mathcal{D}'}) \\ &< (\zeta_3^{\mathcal{D}}, \zeta_4^{\mathcal{D}}) \\ &= (C_3(M,\Sigma), C_4(M,\Sigma)) .\end{aligned}$$

Finally, as

$$C_1(M',\Sigma') = C_1(M,\Sigma)$$

and

$$C_2(M',\Sigma') \leq C_2(M,\Sigma) ,$$

we see that the lemma holds also in this case.

□

<u>Lemma 2</u>. - <u>If</u> $\mathcal{D} \cap S \neq \phi$ <u>then we also have</u>

$$C(M',\Sigma') < C(M,\Sigma) .$$

<u>Proof</u>. i) At first we assume that there exists a component S_0 of $S \cap cl(M-N(\mathcal{D}))$ which is contained in some M_κ , $\kappa \geq r+1$, and is not ∂-parallel in M_κ .

Then, again using the fact that splitting a product $P \times I$ along an incompressible surface system never leads to any non-product component (which possibly could give rise to an increase of $C_1(M',\Sigma')$) , we conclude, as in the proof of lemma 1 (2. case, i)), that

$$C_1(M',\Sigma') < C_1(M,\Sigma) \ .$$

ii) Now let us investigate the situation when i) does not hold. Then a system $\mathcal{D}'$ of complexity disks for (M',Σ') which is adapted to this case is provided by 3.4 (3. case, ii)). It has the following helpful properties:

(1) For every ∂-irreducible component M_0' of $cl(M'-N(\mathcal{D}'))$ there is exactly one component M_0 of $cl(M-N(\mathcal{D}))$ which is homeomorphic to M_0' by an isotopy of M and contains M_0' .

(2) A component V' of $\partial M_0'$ satisfies $V' \cap N(\mathcal{D}') \neq \phi$ if and only if the corresponding component V of ∂M_0 satisfies $V \cap N(\mathcal{D}) \neq \phi$. Moreover we have that

$$\hat{\mathcal{D}}' \subset \hat{E} \cap M' ,$$

where $\mathcal{E} = \bigcup_{\nu,j} D_\nu \times t_j$; see 2.2 for definition of $\hat{\mathcal{D}}$ (resp. $\hat{\mathcal{D}}'$, $\hat{\mathcal{E}}$) .

(3) For every component D' of $\mathcal{D}'$ which is properly contained in some $D_\nu \times t_j$ we have

$$|D' \cap A(\Sigma')| < |D_\nu \cap A(\Sigma)| .$$

Observations (1) and (2) are fairly clear, but (3) needs an explanation. To see that it holds we let s be the number of components of $\partial D' - \partial(D_\nu \times t_j)$. Then at least s disks, $E_1,\ldots,E_s$, in the complement of $D' \cup N(S)$ in $D_\nu \times t_j$ are outermost. In fig. 10 we have $s = 3$.

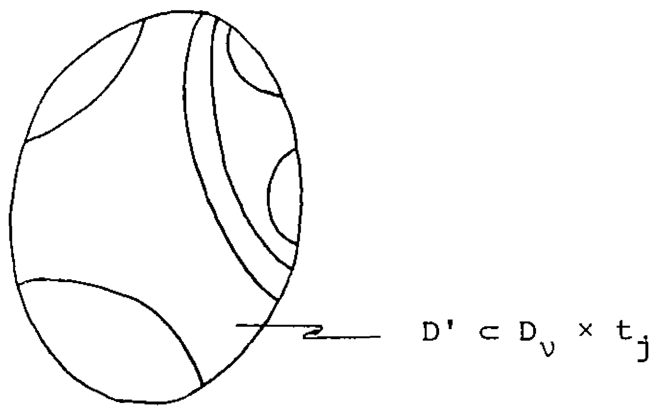


Figure 10

Since none of them is of the type excluded by figure 6 of section 3.2, it follows that

(*) $\quad |E_\sigma \cap A(\Sigma')| \geq 4 \quad \text{for} \quad \sigma = 1,\ldots,s$.

Furthermore, as $C(M,\Sigma)$ is supposed to be non-trivial we see that

(**) $\quad |E \cap A(\Sigma')| \geq 2$

for every component E of $cl(D_\nu \times t_j - N(S))$. Now, observing that each arc of $D_\nu \times t_j \cap S$ contributes two new components of $\mathcal{D}' \cap A(\Sigma')$, we deduce from (*) and (**)

$$|D_\nu \times t_j \cap A(\Sigma)| + 2s \geq |D' \cap A(\Sigma')| + \sum_{\sigma=1}^{s} |E_\sigma \cap A(\Sigma')|$$

$$\geq |D' \cap A(\Sigma')| + 4s \ .$$

Finally, as

$$|D_\nu \times t_j \cap A(\Sigma)| = |D_\nu \cap A(\Sigma)|$$

we see that (3) is indeed true.

To complete the proof of the lemma we note that (1) to (3) imply that

$$C_i^{\mathcal{D}'}(M',\Sigma') \leq C_i(M,\Sigma) \quad \text{for} \quad i = 1,2,3 \ .$$

Moreover, if $S \cap \hat{\mathcal{D}} \neq \phi$ then, by (3), we have that

$$\zeta_3^{\mathcal{D}'} < \zeta_3^{\mathcal{D}} \ ,$$

and if $S \cap \hat{\mathcal{D}} = \phi$ then (3) shows that

$$\zeta_4^{\mathcal{D}'} < \zeta_4^{\mathcal{D}} .$$

□

4.3. - Proof of the existence theorem (end)

If M is closed then our hypotheses guarantee the existence of a norm minimizing splitting surface S in M. When $H_1(M;\mathbb{Q}) = 0$ this surface is a torus, otherwise it may be chosen non-separating. We decompose (M,Σ) along S and thus obtain a taut coloured 3-manifold (system) (M_1,Σ_1) with $\Sigma_1 = \phi$, and $(R_1)_+ = S'_+$, $(R_1)_- = S'_-$; cf. I; 5.1, and I; 5.2. Therefore, no component of M_1 is a 3-ball and thus has non-trivial relative second homology.

Consequently, we are reduced to finding a coloured manifold hierarchy for a taut coloured manifold (M,Σ) with $\partial M \neq \phi$.

To begin with let us find a hierarchy when (M,Σ) has trivial complexity. According to proposition 2.6, we only have to investigate cases a) to d), whereas in cases e) and f) of that proposition the desired hierarchy consists only of (M,Σ) itself.

In case a) we simply take as splitting surface any meridional disk of $M = D^2 \times S^1$, and in cases b), c), and d) we chose as splitting surface an annulus of the form $C \times I$ where $C = S^1 \times \{point\} \subset S^1 \times S^1 = T^2$. In either of these cases the result of the splitting is a taut coloured product as required.

So for the rest of the proof we suppose that the complexity of (M,Σ) is non-trivial.

At first let us assume that $A(\Sigma) = \phi$. This means that either $\Sigma = \phi$ or Σ consists only of tori. In both cases we have $\mathcal{D} = \phi$ showing that

$$C(M,\Sigma) = (C_1,0,0,0) .$$

As C_1 = length $M \neq 0$, there is, by I; 6.1, a non-separating, taut splitting surface $S \neq D^2$ in M so that

$$\text{length } cl(M-N(S)) < \text{length } M .$$

It follows that the taut coloured manifold (M_1,Σ_1) obtained by decomposing (M,Σ) along S satisfies $A(\Sigma_1) \neq \phi$ and

$$C(M_1,\Sigma_1) < C(M,\Sigma) .$$

Now it follows from lemma 2.6 and section 4.2 that there is a splitting

$$(M_1,\Sigma_1) \overset{S_1}{\rightsquigarrow} (M_2,\Sigma_2)$$

such that (M_2,Σ_2) is taut and

$$C(M_2,\Sigma_2) < C(M_1,\Sigma_1) .$$

As $A(\Sigma_2)$ is again non-empty, we conclude that after a finite number of splittings along taut surfaces we arrive at a taut coloured product.

□

<u>Remark</u>. Modifying the proof of the existence theorem slightly it can be shown with not too much additional work that all splitting surfaces S_i can be found so that for every component V of R_i , $S_i \cap V$ is a system of k (≥ 0) parallel oriented non-separating simple closed curves or arcs.

CHAPTER III

Construction of foliations from coloured manifold hierarchies

1. Generalities on Reeb components
2. Norm minimality of compact leaves
3. Construction of foliations with corners
4. The Main theorem
5. Applications to links

CHAPTER III

CONSTRUCTION OF FOLIATIONS FROM COLOURED MANIFOLD HIERARCHIES

The objective of this chapter is the proof of Gabai's existence theorem of foliations without Reeb components announced in the introduction (see also 4.1). The proof, of course, will make explicit use of a coloured manifold hierarchy for the underlying manifold. The existence of such a hierarchy was established in chapter II.

Before giving the proof of Gabai's central result in § 4 we recollect in § 1 some general facts on Reeb components, and in § 2 we illustrate Thurston's result telling us that compact leaves are always norm minimizing, as long as the foliation contains no Reeb component; cf [Th2]. This all is to give the reader a better impression of the importance of Gabai's theorem.

Again all 3-manifolds and surfaces embedded in them are supposed to be compact and oriented unless otherwise stated. We presuppose that the reader is familiar with some basic concepts of geometric foliation theory. All we need can be found for instance in [HH], except the notion of depth which will, however, be explained in the text.

1. Generalities on Reeb components

Suppose specifically that the 3-manifold is closed and fibres over S^1. Then every fibre is met by a closed transversal. Even more, when the fibre is connected, we can find a global transversal, that is an embedded S^1 intersecting every fibre non-trivially and transversely.

In contrast to that the torus leaf of a Reeb component embedded as part of a foliation in any 3-manifold does not admit any closed transversal; see [No]. Further, the manifold M above can be endowed with a riemannian metric so that all fibres become minimal 2-manifolds, i.e. have mean curvature zero. On the other hand, according to Sullivan [Su], a foliation $\mathcal{F}$ on a 3-manifold can be equipped with a riemannian metric so that all leaves become minimal if and only if $\mathcal{F}$ does not contain a Reeb component.

These two phenomena are to show that in order to discover properties of (well-understood) surface bundles over S^1 which carry over to foliations of codimension-one the existence of Reeb components is an essential obstruction.

1.1 - Reeb components cannot be always avoided

We here have to recall the following striking results due to S.P. Novikov [No].

Theorem. - Let M be a closed orientable 3-manifold.

(i) If M has finite fundamental group $\pi_1 M$ then every codimension-one foliation on M has a Reeb component.

(ii) If $\mathcal{F}$ is a codimension-one foliation on M such that for some leaf L of $\mathcal{F}$ the map $\pi_1 L \longrightarrow \pi_1 M$ induced by inclusion is not injective then $\mathcal{F}$ has a Reeb component.

□

As already mentioned, the boundary leaf of a Reeb component does not admit any closed transversal. Thus, in order to show that a given foliation F has no Reeb component it suffices to verify that each leaf of F is met by a closed transversal. This criterion will be applied in the proof of the main theorem.

1.2 - Reeb components and irreducibility

Novikov's investigations go even further in proving that the universal cover of a transversely orientable foliation without Reeb components is contractible, and the leaves of the lifted foliation are all planes. Moreover, H. Rosenberg has shown the following result which, in connection with theorem 1.1, implies that a closed 3-manifold admitting a transversely orientable 2-dimendional foliation without Reeb components must be irreducible.

Theorem. - ([Ro; theorem 6]) *Let N be a 3-manifold, not necessarily compact. If N admits a foliation by planes then every 2-sphere in N bounds an embedded ball.*

□

These comments motivate our restriction to irreducible 3-manifolds in most parts of chapters I and II, and also in the remainder of this chapter.

1.3 - Making surfaces transverse

Another crucial property of Reeb components is the fact that they are an obstruction to making surfaces embedded in a foliated manifold transverse to the foliation. We shall make use of the following result due to Roussarie, Thurston, and Gabai; see [Rou], [Th1], and in particular [Ga1] and [Th2].

Theorem. - Let M be a compact oriented 3-manifold and $\mathcal{F}$ a transversely orientable codimension-one foliation on M without Reeb components. If S is a properly embedded incompressible surface in M such that each component of ∂S is either contained in a leaf of $\mathcal{F}$ or is transverse to $\mathcal{F}$ then S is isotopic to a properly embedded surface which is either a leaf of $\mathcal{F}$, or has only saddle singularities for the induced foliation with singularities on S. Moreover, every boundary component of the deformed S is either a leaf of $\mathcal{F}|\partial M$ or is transverse to $\mathcal{F}|\partial M$.

□

2. Norm minimality of compact leaves

2.1 - More on the Thurston norm

Using Poincaré-Lefschetz duality, the Thurston norm $\| \ \|$ (see I; 3) on $H_2(M)$ or $H_2(M,\partial M)$ gives rise to a dual map $\| \ \|^*$ on $H^2(M)$ resp. $H^2(M,\partial M)$ (real coefficients) defined by

$$\|u\|^* = \sup_{\|\alpha\| \leq 1} \langle u,\alpha\rangle ,$$

where $\langle \ , \ \rangle$ denotes cup product. If $\| \ \|$ is not a norm then $\| \ \|^*$ may become infinite. So we understand $\| \ \|^*$

as the restriction to the subspace where it is a norm.

2.2 - Theorem. - (Thurston [Th2]) *Let M be a compact oriented 3-manifold and $\mathcal{F}$ a transversely orientable foliation on M without Reeb components. When $\partial M \neq \phi$, suppose further that each component B of ∂M is either a leaf of $\mathcal{F}$ or $\mathcal{F}$ is transverse to B and $\mathcal{F}|B$ has no (2-dimensional) Reeb component. Then every compact leaf of $\mathcal{F}$ is norm minimizing.*

The proof is carried out in [Th2]. It uses theorem 1.3 to show that for the Euler class $e(T\mathcal{F}) \in H^2(M)$ resp. $H^2(M,\partial M)$ $\mathcal{F}$ we have the inequality

$$\| e(T\mathcal{F}) \|^* \leq 1 \ .$$

Therefore, if L is any compact leaf of $\mathcal{F}$ with negative Euler characteristic then we obtain

$$\| L \| = |\chi(L)| = |\langle e(T\mathcal{F}),[L]\rangle| \leq \| e(T\mathcal{F}) \|^* \| [L] \| \leq \| [L] \| \ .$$

Showing that

$$\| L \| = \| [L] \| \ .$$

If $\chi(L) \geq 0$ then the result follows from Reeb stability and the hypothesis that there are no Reeb components in $\mathcal{F}$.

□

3. Construction of foliations with corners

Given a coloured manifold hierarchy

$$(M,\Sigma) \overset{S}{\rightsquigarrow} (M_1,\Sigma_1) \overset{S_1}{\rightsquigarrow} \ldots \rightsquigarrow (M_k,\Sigma_k) = (P\times I, \partial P\times I)$$

we want to construct by means of this hierarchy a foliation $\mathcal{F}$ on M which is transverse to Σ and tangent to $R = \partial M - \overset{\circ}{\Sigma}$. If the hierarchy is good enough then $\mathcal{F}$ will not contain a Reeb component.

The foliation $\mathcal{F}$ will be constructed stepwise by starting with the product foliation on $P\times I$ with leaves $P\times\{t\}$, $t \in I$, and then going backwards along the hierarchy.

Strictly speaking, the "foliations" we have to deal with at this stage are not foliations in the usual sense because they have corners.

3.1 - Foliations with corners

i) A <u>foliation with corners</u> $\mathcal{F}$ on a compact 3-manifold is by definition a partition of M into injectively immersed surfaces locally modelled on the space $D^2 \times I$. Thus the points of ∂M correspond either to points of $\partial D^2 \times I$ or of $D^2 \times \partial I$. The former constitute the set Φ where $\mathcal{F}$ is transverse to ∂M, the latter the subset R of ∂M where $\mathcal{F}$ is tangent. R and Φ meet in a union of circles, the "corners".

As $F|\Phi$ is a genuine foliation, it follows that Φ is a union of annuli A and tori (possibly empty or all of ∂M).

Note that the double of a foliation with corners on M along A is a genuine foliation on the resulting manifold.

ii) A foliation with corners is <u>transversely orientable</u> if its double A is transversely orientable.

iii) By a <u>foliation on a coloured manifold</u> (M,Σ,R) we mean a foliation with corners F on M such that F is transverse to Σ and tangent to R .

iv) A foliation F on (M,Σ) is <u>transversely oriented</u>, if a transverse orientation can be chosen so that on $R_\pm$ the two normal orientations agree, respectively.

<u>Examples</u> - i) $D^2 \times I$ with leaves $D^2 \times \{t\}$ is a foliation on the taut coloured 3-ball. More generally, when P is any (orientable) compact surface, we obtain a foliation on $(P\times I,\ \partial P\times I)$ in the obvious way. Clearly this foliation is transversely oriented once we have chosen an orientation for P .

ii) Assume F is a foliation on the manifold M tangent to ∂M and A is an annulus, properly embedded and transverse to F . Then cutting M along A yields a foliation with corners on $M' = M - \mathrm{int}\, N(A)$ which is transverse precisely to the two copies of A in $\partial M'$.

In the next section we will see how a coloured manifold decomposition

$$(M,\Sigma) \overset{S}{\rightsquigarrow} (M',\Sigma') ,$$

together with a foliation on (M',Σ') yields a foliation on (M,Σ) .

3.2 - Recursive construction of foliations

Suppose that we are given a coloured manifold decomposition

$$(M,\Sigma,R) \overset{S}{\rightsquigarrow} (M',\Sigma',R')$$

and a foliation F' on (M',Σ') . We are going to show how to obtain from F' a foliation F on (M,Σ) . The construction of F , of course, will also depend on S . We have to discuss three possibilities. In doing so we may restrict ourselves to the special situation where S is connected, and for every component V of R, $V \cap S$ is a system of $k \geq 0$ parallel homologically non-trivial simple closed curves (if V is non-planar) or arcs (if V is planar). Moreover, we assume that $F'|\Sigma'$ has no 2-dimensional Reeb component.

Case 1. $\partial S \cap R = \phi$. This is the easiest case. Our hypothesis guarantees that the two copies S'_+ and S'_- of S in $\partial M'$ are components of R'_+ resp. R'_- , and hence homeomorphic leaves of F' . We can therefore reglue S'_+ and S'_- and obtain by this from F' a foliation F on (M,Σ) as required.

Case 2. $\partial S \subset R$. By our assumption on S , a component V of R with $S \cap V \neq \phi$ is non-planar, and $V \cap S$ consists of parallel, in particular coherently oriented, circles. Then, to each of these circles corresponds an annular component of Σ' ; see fig. 1.

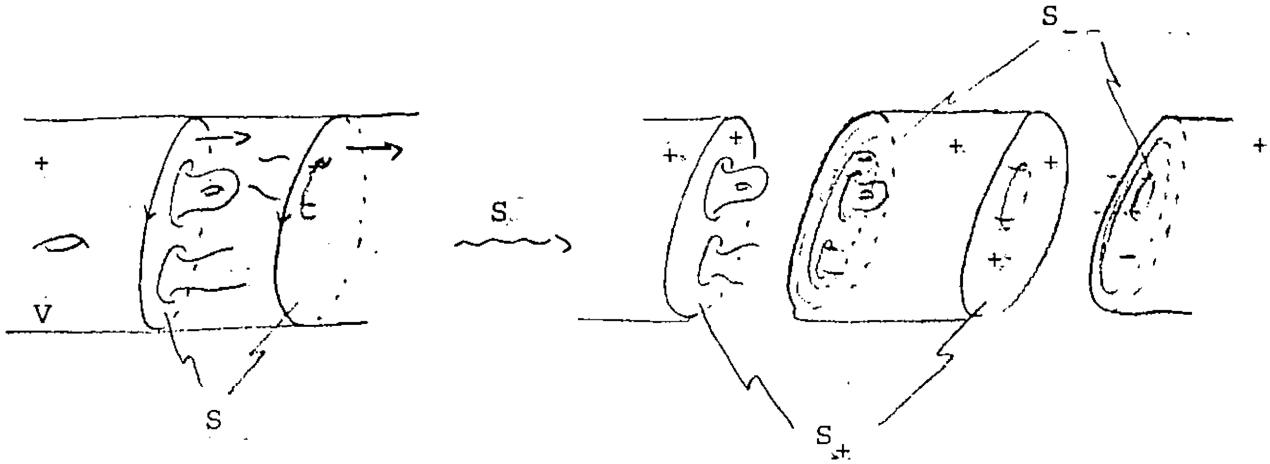
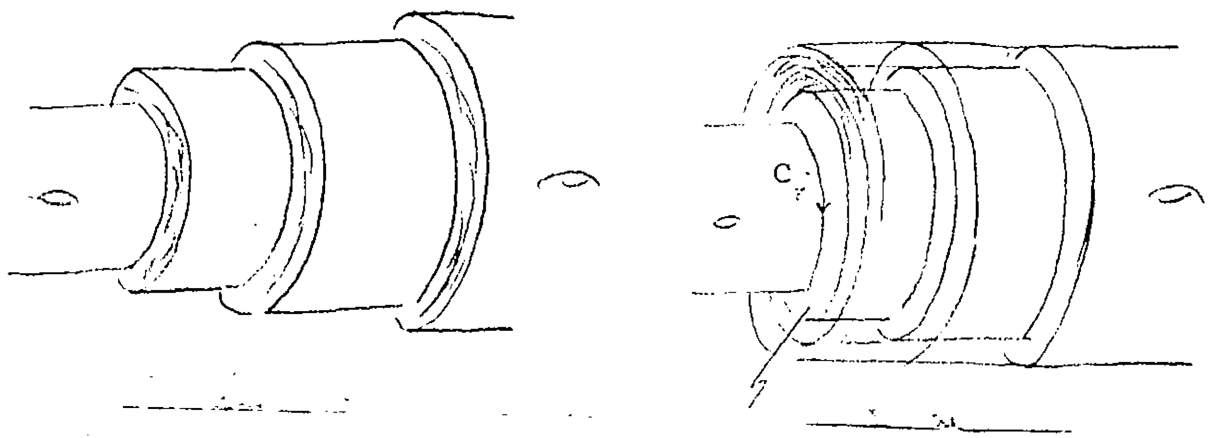


Figure 1

Now again we glue S'_+ to S'_- and thus this time obtain a manifold M_0 homeomorphic to M and an induced foliation with corners and singularities F_0 on M_0 . Note that F_0 is transverse to Σ and to $k \geq 0$ annuli corresponding to the circles of $V \cap S$.

Next, we thicken (M_0, F_0) in the obvious way so that only the outermost of these annuli, denoted A , remains in the boundary, whereas all others disappear from the boundary. See fig. 2. This is possible since the circles of $V \cap S$ are coherently oriented. Call the new foliation with corners again (M_0, F_0) .



(M_0, F_0) old   A (M_0, F_0) new

Figure 2

In order to obtain a foliation on (M,Σ) we have to spiral the leaves of F_0 towards V. This process of spiraling is analogous to the process of extending a foliated pseudo bundle of rank one over S^1 (in the sense of [HH]) to a foliated bundle on $S^1 \times I$. To be more precise, we view M_0 as embedded in M so that $N = \overline{M-M_0}$ is homeomorphic to $V \times I$ and is endowed with a foliation with corners in the following way.

Let $C = A \cap S$ (see fig. 2) and let C' be a simple closed curve on V whose geometric intersection with C is one. We cut V open along $C \cup C'$ and give $(V-N(C \cup C')) \times I$ the product foliation. Denote by $f : I \to I$ the holonomy map of $F_0|A$, and let $g : I \to I$ be defined by $g(0) = 0$ and

$$g(t) = \frac{1}{2^\nu} f(2^\nu(t-\frac{1}{2^\nu})) + \frac{1}{2^\nu}, \text{ for } t \in (\frac{1}{2^\nu}, \frac{1}{2^{\nu-1}}].$$

So g maps each interval $[\frac{1}{2^\nu}, \frac{1}{2^{\nu-1}}]$ homeomorphically onto itself and $g|[\frac{1}{2^\nu}, \frac{1}{2^{\nu-1}}]$ is conjugate to f, $\nu = 1,2,\ldots$. This permits us to identify the two copies of $C'\times I$ on $\partial(V-N(C \cup C') \times I)$ by means of

$$(x,t) \longmapsto (x,g(t)) .$$

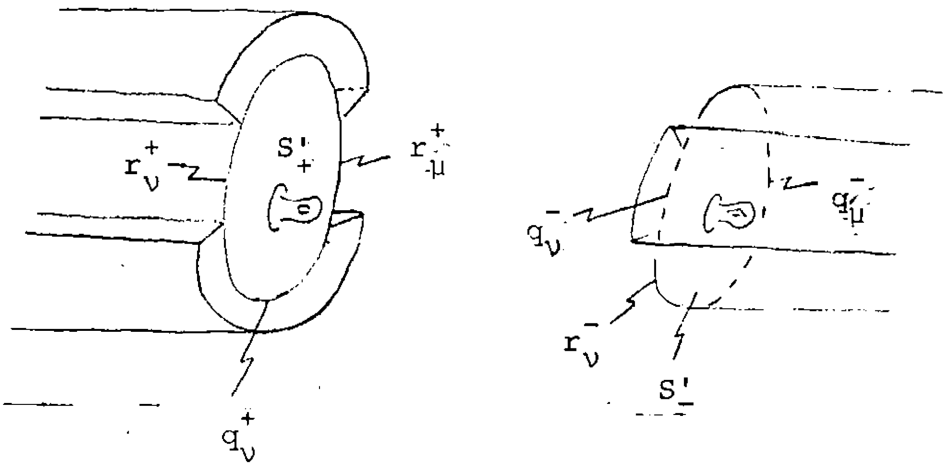
This gluing clearly respects the product foliation. What we have produced is a foliation on $(V-N(C)) \times I$ such that the two induced foliations on the two copies A_0 and A_1 of $C \times I$ are the same and have holonomy equal to g . Now the construction of the desired foliation on N is accomplished by gluing $A_0 = C \times [0,1]$ to $C \times [0,\frac{1}{2}]$ by means of the map

$$A_0 \longrightarrow A_1 , \quad (x,t) \longmapsto (x,\tfrac{t}{2}) .$$

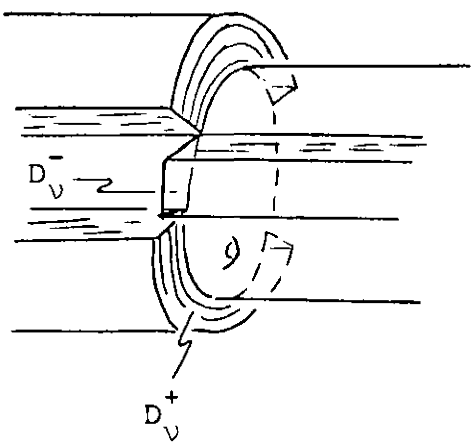
Finally we glue N to M_0 so as to obtain the foliation F on (M,Σ) as required. As the preceding construction can be carried out independently for each component V of ∂M with $S \cap V \neq \phi$ we have completed our discussion of case 2.

Case 3. There is a planar component $V \neq D^2$ of R meeting S non-trivially. Thus $S \cap V$ is a system of parallel properly embedded arcs between different boundary components of V .

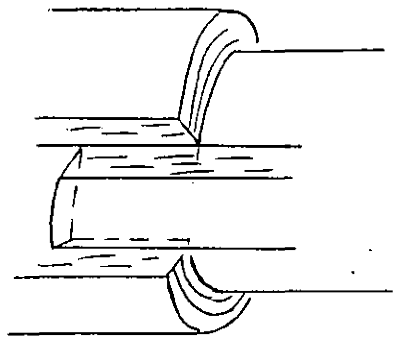
Recalling how (M',Σ') was obtained from (M,Σ) by means of S , we see that $\partial S'_+$ is the union of two systems of arcs $q_1^+,\ldots,q_n^+$ and $r_1^+,\ldots,r_n^+$ where $q_\nu \subset \partial\Sigma'$ and $r_\nu \subset R'_+$, $\nu = 1,\ldots,n$, and similarly for $\partial S'_-$. Now we reidentify S'_+ with S'_- in such a way that q_ν^+ is glued to r_ν^- and r_ν^+ is glued to q_ν^-, $\nu = 1,\ldots,n$; see fig. 3,a).



a)



b)



c)

Figure 3

The manifold M_0 obtained by this gluing process has a "singular foliation with corners" C' which is induced by F'. Each of the arcs q_ν^+ and q_ν^- is contained in the boundary of a disk D_ν^+ resp. D_ν^- where all these disks belong to Σ', see fig. 3b). Clearly, we may assume that F' induces on each $D_\nu^\pm$ a product foliation. Thus we can extend C' by gluing a product $D_\nu^\pm \times I$ to M_0 where the identification is along

$D_\nu^\pm \times \{0\} \cup q_\nu^\pm \times I$; see fig. 3c). Denote the resulting singular foliation on M_0 by $\mathcal{C}$.

Now let us have a closer look at $\mathcal{C}$. As in case 2 we consider M_0 as being embedded in M so that $M-M_0$ is contained in a neighbourhood of R . Then, if V is a component of R as before, we have established a situation as in fig. 4. Here we (again can and do) assume that $S \cap V$ is connected.

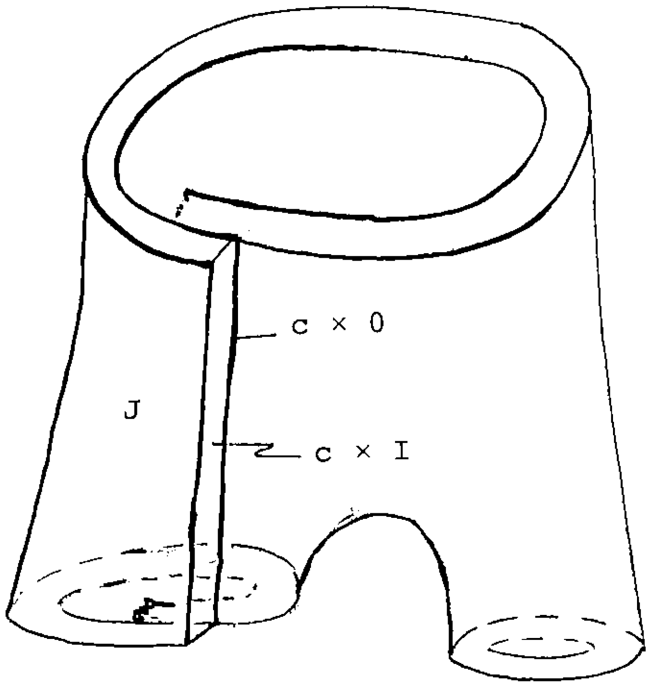


Figure 4

More precisely, fig. 4 shows $N(V) \cap M_0$ which is homeomorphic to $V \times I$ where

$$V \times 1 = J \cup (c \times I) ,$$

with J contained in a leaf L of $\mathcal{C}$, $c \times 0$ a properly embedded arc in both $V \times 1$ and L , and $c \times 1 \subset \partial L$ properly embedded in $V \times 1$. Moreover, $\mathcal{C}|(c \times I)$ is the product foliation with leaves $c \times t$, $t \in I$.

Next, let

$$M_1 = M_0 \underset{J' \times 0}{\cup} J' \times I$$

where

$$J' = cl(J - N(c \times 0))$$

with $N(c \times 0)$ a collar neighbourhood of $c \times 0$ in J . Clearly, M_1 is homeomorphic to M_0 . We now have to fill out the ditch formed by

$$N(c \times 0) \cup (c \times [0,2]) \cup (c_1 \times [0,1]) \subset \partial M_1$$

where c_1 is the copy of $c = c \times 0$ in $\partial N(c \times 0) \subset J$; see fig. 5. This is done by taking the product $N(c \times 0) \times I$, gluing $N(c \times 0) \times 0$ onto $N(c \times 0)$, and identifying $c \times I$ and $c_1 \times I$ leaf preservingly with the corresponding walls of the ditch. Note that this gluing proceduce necessarily produces holonomy in $\mathcal{F}|\Sigma$.

We observe that spiraling $\mathcal{F}'$ in a neighbourhood of V in order to eliminate transversality near $V \cap S$ can be exhibited successively for every planar component V of R with $V \cap S \neq \phi$. This completes the discussion of case 3.

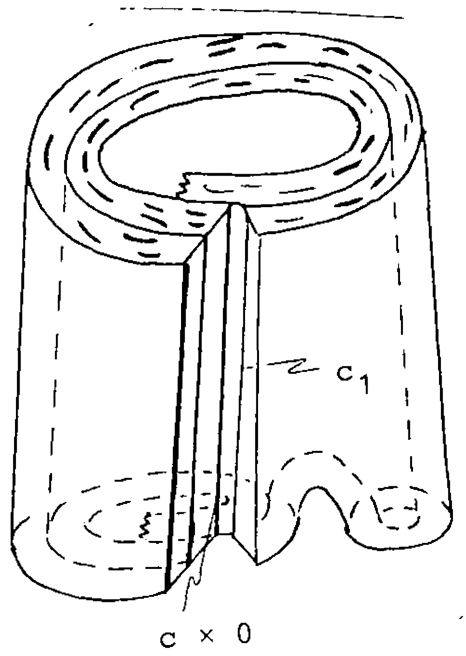


Figure 5

Note that if F' is transversely orientable then so is F .

Summarizing we obtain:

Proposition. - Let $(M,\Sigma,R) \overset{S}{\rightsquigarrow} (M',\Sigma',R')$ be a coloured manifold decomposition where S is connected and such that for every component V of R, $V \cap S$ is a family of $k \geq 0$ parallel homologically non-trivial simple closed curves (if V is non-planar) or arcs (if V is planar). Then any foliation F' on (M',Σ') where $F'|\Sigma'$ has no Reeb component induces in a natural way a foliation F on (M,Σ) such that $F|\Sigma$ has no Reeb component. Moreover, if F' is transversely orientable then so is F .

Proof. We glue S'_+ to S'_- . If case 1 holds then we are done. Otherwise in each of the cases 2 and 3 we perform the constructions described above. This gives us the foliation F on (M,Σ) .

□

In the next three sections we discuss some relevant properties of the foliation F .

3.3 - Differentiability of F

A foliation on an arbitrary coloured 3-manifold (M,Σ) is said to be of class C^r if its double along Σ is a C^r foliation in the usual sense. We call the foliation smooth if it is of class C^∞ .

Now let F' and F be as in the previous section. Unfortunately, F need not be smooth even if F' is. This happens necessarily to be the case when the gluing somewhere is as in case 2 and the corresponding holonomy map $f : I \to I$ of $F_0|A$ (cf. 3.2, case 2) is not the identity. For in this case the map $g : I \to I$ defined by means of f is only of class C^0 , at least at 0. This, however, implies that the leaf V of F has only C^0 holonomy.

Nevertheless, we shall see in a minute how Gabai proceeds in order to obtain a smooth foliation even if $f \neq Id$, at least when $V \neq T^2$.

On the other hand, when F' is smooth and the gluing is as in case 1 then a sufficient condition for F to be smooth is that the holonomy of the two leaves S'_+ and S'_- of F' which are going to be identified is everywhere C^∞ flat , i.e. if

$$f : [0,\varepsilon) \longrightarrow [0,\delta)$$

denotes an element of the holonomy pseudogroup of such a leaf then

$$\frac{d^r f}{dt^r}(0) = \begin{cases} 1 & \text{if} \quad r=1 \\ 0 & \text{if} \quad r>1 \ ; \end{cases}$$

cf. [HH; IV, 4.1.3].

A similar remark applies in case 3 provided that the filling out of the ditch is made by means of a diffeomorphism of the interval which is C^∞ flat at both end points.

Let us now see how Gabai proceeds in order to obtain a smooth foliation even in case 2, provided $V \neq T^2$. Of course, we need that F' is smooth. Furthermore, we have to require that F' be C^∞ flat near R' and that (with the notation of 3.2, case 2) the holonomy map $f : I \longrightarrow I$ of $F_0|A$ be C^∞ flat at both end points. All these conditions will turn out to be satisfied later in the applications.

To begin with let us assume that $\partial V \neq \phi$. We choose a collar $C \times [0,1]$ of an arbitrary boundary component C of V . Then we give $(C \times [0,1]) \times I$ the structure of a foliated I-bundle with foliation transverse to the third factor and with holonomy f^{-1} . Next we connect $C \times 1 \times I$ and the annulus A by a thickened band $(I \times I) \times I$. This is done by gluing $0 \times I \times I$ to $C \times 1 \times I$, $1 \times I \times I$ to A and $I \times I \times 0$ to ∂M_0 ; see fig.6. Moreover, when $(I \times I) \times I$ is endowed with the product foliation transverse to the third factor, we do these identifications so that we create a new foliation with corners which is transverse to the annulus A' composed by sub-disks of A and $C \times 1 \times I$, together with $I \times \partial I \times I$. By construction, the holonomy along A' now has become trivial. We thus have reduced to the case $f = Id$.

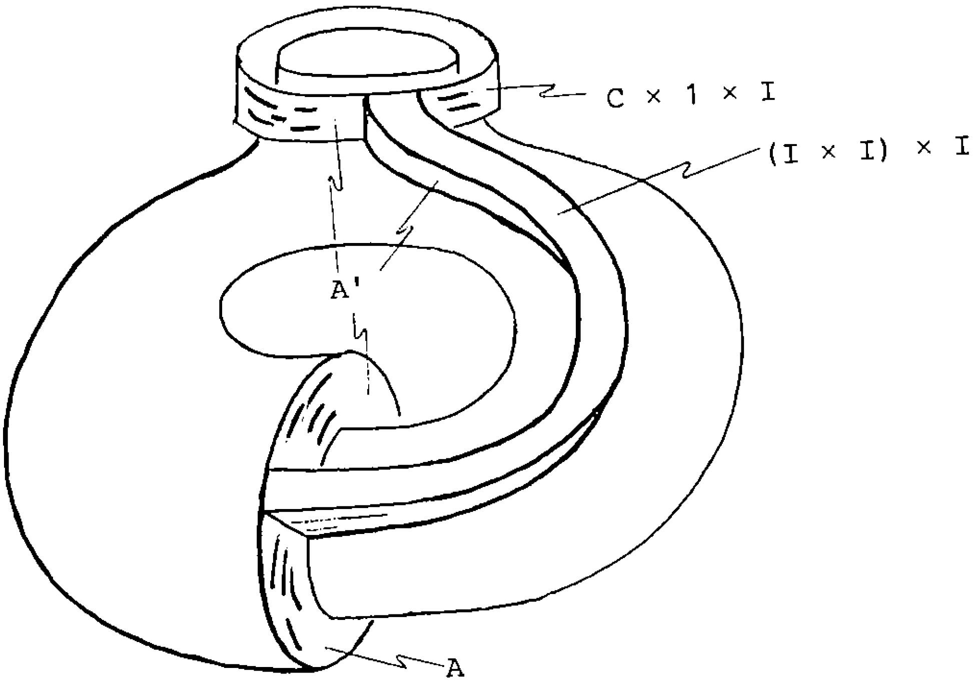


Figure 6

When V is closed and has genus greater than one we have to refer to the following result ascribed by Gabai to Mather, Sergeraert, and Thurston (cf. [Se; théorème 6.6]).

<u>Proposition</u>. - <u>Let</u> $f : I \longrightarrow I$ <u>be a</u> C^∞ <u>diffeomorphism which is</u> C^∞ <u>flat at both end points. Then there exist</u> C^∞ <u>diffeomorphisms</u> $g_\nu, h_\nu : I \longrightarrow I$, $\nu = 1,\dots,n$, <u>which are all</u> C^∞ <u>flat at both end points of</u> I <u>such that</u>

$$f \circ (g_1 \circ h_1 \circ g_1^{-1} \circ h_1^{-1}) \circ \dots \circ (g_n \circ h_n \circ g_n^{-1} \circ h_n^{-1}) = Id .$$

□

In order to apply this result to our situation we glue, similarly to the previous case, thickened bands G_1 and H_1, both homeomorphic to $S^1 \times I \times I$, on the component of $\partial M_0 - \Sigma'$ that contains the annulus A. This gluing is done along $S^1 \times I \times 0$ and so that A is not met. Moreover, the intersection of G_1 and H_1 should consist of a single cube; see fig.7.

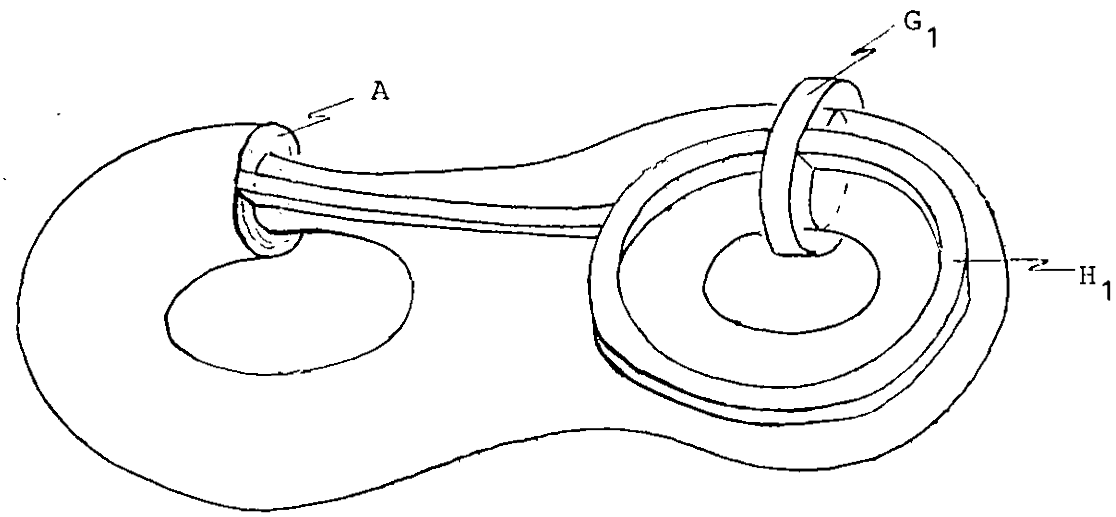


Figure 7

We endow G_1 and H_1 both with the structure of a foliated bundle where the holonomy on G_1 is g_1 and on H_1 is h_1. We may assume that on the cube $G_1 \cap H_1$ both foliations agree.

Next, we connect one of the bands G_1 and H_1, say H_1, by a third thickened band, this time homeomorphic to $I \times I \times I$, with the annulus A. This is done as indicated in fig.7. If the gluing is performed correctly, we obtain by this operation an extension of F_0 with a transverse annulus A_1 such that the holonomy along A_1 is given by $f \circ g_1 \circ h_1 \circ g_1^{-1} \circ h_1^{-1}$. Exhibiting this operation n times, we therefore finally get an extended foliation of F_0 with a transverse annulus A_n such that the holonomy along A_n is trivial.

We have shown:

Proposition. - Suppose we are given a decomposition

$$(M,\Sigma,R) \overset{S}{\rightsquigarrow} (M',\Sigma',R')$$

and a foliation F' on (M',Σ') such that $F'|\Sigma'$ has no Reeb components, as before. Suppose further that no component of R is a torus and F' is smooth and C^∞ flat at R' . Then the foliation F on (M,Σ) may be constructed to be smooth and C^∞ flat at R .

□

3.4 - The closed transversal property

Definition. - A foliation on a coloured manifold has the closed transversal property (c.t.p. in short) if every leaf it met by a closed transversal or by a properly embedded transverse arc.

Note that a Reeb component whether embedded or not does not have the c.t.p.

Given a coloured manifold decomposition $(M,\Sigma) \overset{S}{\rightsquigarrow} (M',\Sigma')$ as before, and a foliation F' on (M',Σ') we want to see to what extend the c.t.p. of F' carries over to the foliation F on (M,Σ) obtained from F' by the constructions of the preceding paragraphs.

Proposition. *Let* $(M,\Sigma,R) \overset{S}{\rightsquigarrow} (M',\Sigma',R')$ *be as in* 3.2. *Suppose that no component of* R *is closed. On* (M',Σ') *let a transversely orientable foliation* F' *be given where* $F'|\Sigma'$ *has no* 2-*dimensional Reeb component*.

If F' *has the c.t.p. then so does the foliation* F *on* (M,Σ) *obtained by the recipe of sections* 3.2 *and* 3.3.

Proof. By section 3.2, F is obtained from F' by gluing S'_+ to S'_- and a series of extensions of the foliation with corners (and singularities) obtained by this gluing procedure. Now the proof of our assertion is based on the following three observations.

(1) If W_+ and W_- are components of R' such that $W_+ \cap A \neq \phi$ and $W_- \cap A \neq \phi$ for some annular component A of Σ' then for any two points $x_+ \in W_+$ and $x_- \in W_-$ there is a proper transversal t of F' with end points x_+ and x_-.

(2) If t is a proper transversal of F' connecting $x_+ \in S'_+$ with $x_- \in S'_-$ then we may assume that x_+ and x_- are identified under the gluing of S'_+ to S'_-.

(3) If there is a proper transversal t of F' connecting the point x_+ of R'_+ to S'_- and (2) does not apply then after gluing S'_+ to S'_- the arc t can be extended to a properly embedded transversal through x_+.

Clearly, (3) follows from (1). Note also that if (2) applies then the transversal t gives rise to a closed transversal through the leaf of $\mathcal{F}$ containing S . This situation holds for instance when $(M',\mathcal{F}')$ is a coloured product and $S'_\pm = R'_\pm$.

Now, in case 1 of section 3.2, our claim manifestly follows from (1), (2), (3). The same holds in case 2 whenever no component of R is closed. For then $\partial V \neq \phi$ and the existence of a proper transversal also for the boundary leaf V of the spiraled foliation follows from (1) and the fact that the foliated pseudobundle that is attached has a global transversal, i.e. one intersecting every leaf. Observe, however, that the exceptional case actually occurs, for instance when $(M,\Sigma,R) = (D^2 \times S^1,\phi,\partial D^2 \times S^1)$ and S is a meridian disk. In this case, when $\mathcal{F}'$ is a product we obtain as $\mathcal{F}$ a Reeb component.

Finally, in case 3, it suffices to observe that a proper transversal t of $\mathcal{F}'$ with one end point in the leaf of $\mathcal{F}'$ containing S'_+ (resp. S'_-) can be isotopically deformed through proper transversals meeting the same leaves as t but not S'_+ (S'_-) .

The proof is completed by noticing that the modifications performed in 3.3 in order to make $\mathcal{F}$ differentiable do not have any influence on the c.t.p. .

□

3.5 - <u>Boundary triviality</u>

Let us say that a foliation $\mathcal{F}$ on the coloured manifold (M,Σ) is <u>boundary trivial</u> (∂-trivial) if $\mathcal{F}|\Sigma$ is trivial on each of its components.

Boundary triviality will play a rôle in chapter IV when Gabai's machinery of coloured manifolds is applied to the surgery problem for knots in S^3. There we shall be interested in foliations without Reeb components on knot complements so that the foliation induced on the boundary torus is by circles each of which represents a prescribed homology class.

<u>Proposition</u>. - <u>Let the decomposition</u>

$$(M,\Sigma) \overset{S}{\rightsquigarrow} (M',\Sigma')$$

<u>be as in 3.2. Suppose further that the foliation</u> $\mathcal{F}'$ <u>on</u> (M',Σ') <u>is</u> ∂-<u>trivial. Then the foliation</u> $\mathcal{F}$ <u>on</u> (M,Σ) <u>obtained from</u> $\mathcal{F}'$ <u>by the procedure of section 3.2 also is</u> ∂-<u>trivial, provided that for the construction of</u> $\mathcal{F}$ <u>only gluing and extensions as in case 1 and 2 of 3.2 are used</u>.

<u>Proof</u>. The assertion is obviously true when the gluing of S'_+ and S'_- is as in case 1, for then $\mathcal{F}|\Sigma$ is obtained from $\mathcal{F}'|\Sigma'$ by identifying certain components of Σ' along boundary curves.

In case 2, the attached pseudobundle over $V \subset \partial M$ by construction is trivial over ∂V. This implies our claim.

□

4. The Main Theorem

In this paragraph we present the main result of these notes (Gabai's theorem [Ga I; 5.5]) and draw a first consequence.

4.1 - Main theorem. - *Let* M *be a compact (connected) irreducible oriented 3-manifold whose boundary is a (possibly empty) union of tori. Let* S *be a norm minimizing properly embedded surface system in* M *representing a non-trivial element of* $H_2(M,\partial M)$ *. Then there exists a transversely oriented foliation* $\mathcal{F}$ *on* M *such that:*

(1) $\mathcal{F}$ *is transverse to* ∂M *and* $\mathcal{F}|\partial M$ *has no Reeb component.*

(2) *Every leaf of* $\mathcal{F}$ *is met by a closed transversal.*

(3) *The components of* S *are leaves of* $\mathcal{F}$ *.*

(4) $\mathcal{F}$ *is smooth except possibly along torus components of* S *.*

Proof. Consider the decomposition

$$(*) \qquad (M,\partial M) \overset{S}{\rightsquigarrow} (M_1,\Sigma_1) \ .$$

Since S is norm minimizing, (M_1,Σ_1) is taut. As $\partial M_1 \neq 0$ we have that $H_2(M_1,\partial M_1) \neq 0$. We can therefore apply theorem 4.1 of chapter II to (M_1,Σ_1) and extend the decomposition (*) to a coloured manifold hierarchy

$$(M,\Sigma) \overset{S}{\rightsquigarrow} (M_1,\Sigma_1) \overset{S_1}{\rightsquigarrow} \dots \overset{S_{m-1}}{\rightsquigarrow} (M_m,\Sigma_m)$$

where each S_μ is connected and has non-empty boundary, and $\Sigma = \partial M'$. We can moreover assume (see remark at the end of proof of II; 4.1) that for every S_μ and every component V of $R_\mu = \partial M_\mu - \mathring{\Sigma}_\mu$, $V \cap S_\mu$ is a system of $k (\geq 0)$ parallel homologically non-trivial simple closed curves (if V is non-planar) or arcs (if V is planar).

Now, starting with the product foliation on (M_m, Σ_m) we construct recursively foliations F_μ on (M_μ, Σ_μ). As no S_μ is closed, it follows by section 3.4 that F_1 has the closed transversal property and satisfies the hypotheses of proposition 3.3. We conclude that the foliation F on $(M, \partial M)$ which is constructed by means of F_1 as in § 3 satisfies conditions (1), (3) and (4). Finally, as $\partial M = \Sigma$ we deduce from proposition 3.4 that F satisfies also condition (2).

$\square$