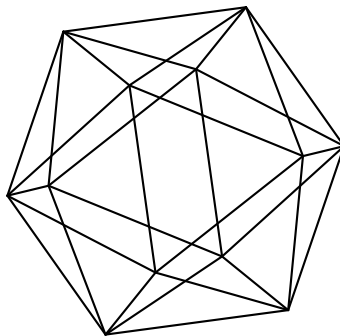# Max-Planck-Institut für Mathematik Bonn

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zeta functions via profinite completions

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PÓLYA–CARLSON DICHOTOMY FOR COINCIDENCE REIDEMEISTER ZETA FUNCTIONS VIA PROFINITE COMPLETIONS

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ABSTRACT. We consider coincidence Reidemeister zeta functions for tame endomorphism pairs of nilpotent groups of finite rank, shedding new light on the subject by means of profinite completion techniques.

In particular, we provide a closed formula for coincidence Reidemeister numbers for iterations of endomorphism pairs of torsion-free nilpotent groups of finite rank, based on a weak commutativity condition, which derives from simultaneous triangularisability on abelian sections. Furthermore, we present results in support of a Pólya–Carlson dichotomy between rationality and a natural boundary for the analytic behaviour of the zeta functions in question.

1. INTRODUCTION

In classical topological fixed point theory, Reidemeister numbers arise as homotopy invariants associated to iterates of a continuous self-map of a connected compact polyhedron. Passing to the fundamental group of the polyhedron, the Reidemeister numbers admit an algebraic treatment in terms of twisted conjugacy classes; compare [7]. In this paper we take a group-theoretic point of view, inspired by, but otherwise largely independent of the topological origins of the subject.

Let G be a group and let $\varphi, \psi \colon G \to G$ be endomorphisms of G. Elements $x, y \in G$ are said to be (φ, ψ)-*twisted conjugate* to one another if there exists $g \in G$ such that $x = (g\varphi)^{-1}y(g\psi)$. We observe that this sets up an equivalence relation on G; the corresponding equivalence classes are called (φ, ψ)-*twisted conjugacy classes* or (φ, ψ)-*coincidence Reidemeister classes*. We denote by $\mathcal{R}(\varphi, \psi)$ the set of all (φ, ψ)-coincidence Reidemeister classes of G, and $R(\varphi, \psi) = |\mathcal{R}(\varphi, \psi)|$ is called the (φ, ψ)-*coincidence Reidemeister number*.

We call the pair (φ, ψ) of endomorphisms *tame* if the Reidemeister numbers $R(\varphi^n, \psi^n)$ are finite for all $n \in \mathbb{N}$. For such a tame pair of endomorphisms we define the (φ, ψ)-*coincidence Reidemeister zeta function*

$$Z_{\varphi,\psi}(s) = \exp\left(\sum_{n=1}^{\infty} \frac{R(\varphi^n, \psi^n)}{n} s^n\right),$$

where s denotes a complex variable. This zeta function can be regarded as an analogue of the Hasse–Weil zeta function of an algebraic variety over a finite field or the Artin–Mazur zeta function of a continuous self-map of a topological space. For $\psi = \mathrm{id}_G$, we recover ordinary φ-twisted conjugacy classes (or φ-Reidemeister classes) and the ordinary φ-Reidemeister zeta function that were studied, for instance in [6, 7, 19, 2, 9, 3, 11, 10].

For short, we set $\mathcal{R}(\varphi) = \mathcal{R}(\varphi, \mathrm{id}_G)$ and $R(\varphi) = R(\varphi, \mathrm{id}_G)$; if (φ, id_G) is tame, we say that φ is tame and write $Z_\varphi(s) = Z_{\varphi, \mathrm{id}_G}(s)$.

Remark 1.1. (1) The anti-isomorphism $G \to G$, $x \mapsto x^{-1}$ sets up a one-to-one correspondence between (φ, ψ)-twisted and (ψ, φ)-twisted conjugacy classes. Consequently, (φ, ψ) is tame if and only if (ψ, φ) is tame, and $Z_{\varphi,\psi}(s) = Z_{\psi,\varphi}(s)$ for tame pairs.

(2) If $\psi \colon G \to G$ is an automorphism, it is easy to check that $\mathcal{R}(\varphi, \psi) = \mathcal{R}(\psi^{-1}\varphi)$ and consequently $R(\varphi, \psi) = R(\psi^{-1}\varphi)$. Moreover, if (φ, ψ) is tame and if $\varphi\psi = \psi\varphi$ (in some situations a weak concept of commutativity suffices, as we will see), then $\psi^{-1}\varphi$ is tame and $Z_{\varphi,\psi}(s) = Z_{\psi^{-1}\varphi}(s)$.

In order to illustrate these concepts by means of a simple example, namely the infinite cyclic group, and for later use we record the following easy fact.

Lemma 1.2. *Let $\varphi, \psi \colon G \to G$ be endomorphisms of a group G such that the image of ψ is central in G, i.e. $G\psi \subseteq \mathrm{Z}(G)$. Then $H_{\varphi,\psi} = \{(g\varphi)^{-1}(g\psi) \mid g \in G\}$ is a subgroup of G and $\mathcal{R}(\varphi, \psi) = H_{\varphi,\psi}\backslash G$; consequently, $R(\varphi, \psi) = |G : H_{\varphi,\psi}|$.*

Example 1.3. Let $G = \mathbb{Z}$ be the infinite cyclic group, written additively, and let
$$\varphi \colon G \to G, \quad x \mapsto d_\varphi x \qquad \text{and} \qquad \psi \colon G \to G, \quad x \mapsto d_\psi x$$
for $d_\varphi, d_\psi \in \mathbb{Z}$. According to the lemma, for every $n \in \mathbb{N}$, we have
$$R(\varphi^n, \psi^n) = \begin{cases} |d_\psi^n - d_\varphi^n| & \text{if } d_\varphi^n \neq d_\psi^n, \\ \infty & \text{otherwise.} \end{cases}$$
Consequently, (φ, ψ) is tame precisely when $|d_\varphi| \neq |d_\psi|$ and, in this case,
$$Z_{\varphi,\psi}(s) = \frac{1 - d_2 s}{1 - d_1 s} \qquad \text{where } d_1 = \max\{|d_\varphi|, |d_\psi|\} \text{ and } d_2 = \frac{d_\varphi d_\psi}{d_1}.$$

This simple example (or at least special cases of it) are known. The aim of the current paper is to generalise this example – as far as possible – to torsion-free nilpotent groups of finite rank. Our approach via profinite completions sheds new light on the subject.

We prove a number of useful results, some of which we summarise already here, after fixing some basic notation. We write $\mathbb{P}$ the set of all rational primes; for $p \in \mathbb{P}$, the field of p-adic numbers is denoted by $\mathbb{Q}_p$, the ring of p-adic integers by $\mathbb{Z}_p$, and the p-adic absolute value (as well as its unique extension to the algebraic closure $\overline{\mathbb{Q}}_p$) by $|\cdot|_p$. The absolute value on $\mathbb{C}$ is denoted by $|\cdot|_\infty$.

Theorem 1.4. *Let $\varphi, \psi \colon G \to G$ be a tame pair of endomorphisms of a torsion-free nilpotent group G of finite Prüfer rank. Let c denote the nilpotency class of G and, for $1 \leq k \leq c$, let $\varphi_k, \psi_k \colon G_k \to G_k$ denote the induced endomorphisms of the torsion-free abelian factor groups $G_k = \overline{\gamma}_k(G)/\overline{\gamma}_{k+1}(G)$ of finite rank, $d_k \geq 1$ say, that arise from the isolated lower central series (2.1) of G. Then the following hold.*

(1) For each $n \in \mathbb{N}$, there is a bijection between the set $\mathcal{R}(\varphi^n, \psi^n)$ of (φ, ψ)-coincidence Reidemeister classes and the cartesian product $\prod_{k=1}^c \mathcal{R}(\varphi_k^n, \psi_k^n)$; consequently,
$$R(\varphi^n, \psi^n) = \prod_{k=1}^c R(\varphi_k^n, \psi_k^n) \qquad \text{for } n \in \mathbb{N}.$$

(2) For $1 \leq k \leq c$, let
$$\varphi_{k,\mathbb{Q}}, \psi_{k,\mathbb{Q}} \colon G_{k,\mathbb{Q}} \to G_{k,\mathbb{Q}}$$
denote the extensions of φ_k, ψ_k to the divisible hull $G_{k,\mathbb{Q}} = \mathbb{Q} \otimes_\mathbb{Z} G_k \cong \mathbb{Q}^{d_k}$ of G_k. Suppose that each pair of endomorphisms $\varphi_{k,\mathbb{Q}}, \psi_{k,\mathbb{Q}}$ is simultaneously triangularisable. Let $\xi_{k,1}, \ldots, \xi_{k,d_k}$ and $\eta_{k,1}, \ldots, \eta_{k,d_k}$ be the eigenvalues of $\varphi_{k,\mathbb{Q}}$ and $\psi_{k,\mathbb{Q}}$ in a fixed algebraic closure of the field $\mathbb{Q}$, including multiplicities, ordered so that, for $n \in \mathbb{N}$, the eigenvalues

of $\varphi_{k,\mathbb{Q}}^n - \psi_{k,\mathbb{Q}}^n$ are $\xi_{k,1}^n - \eta_{k,1}^n, \ldots, \xi_{k,d_k}^n - \eta_{k,d_k}^n$. Set $L_k = \mathbb{Q}(\xi_{1,k}, \ldots, \xi_{d_k,k}, \eta_{1,k}, \ldots, \eta_{d_k,k})$; for each $p \in \mathbb{P}$ fix an embedding $L_k \hookrightarrow \overline{\mathbb{Q}}_p$ and choose an embedding $L_k \hookrightarrow \mathbb{C}$.

Then there exist subsets $I_k(p) \subseteq \{1, \ldots, d_k\}$, for $p \in \mathbb{P}$, such that the following hold.

(i) *For each $p \in \mathbb{P}$, the polynomials $\prod_{i \in I_k(p)}(X - \xi_{k,i})$ and $\prod_{i \in I_k(p)}(X - \eta_{k,i})$ have coefficients in $\mathbb{Z}_p$; in particular, $|\xi_{k,i}|_p, |\eta_{k,i}|_p \leq 1$ for $i \in I_k(p)$.*

(ii) *For each $n \in \mathbb{N}$,*

$$(1.1) \quad R(\varphi_k^n, \psi_k^n) = \prod_{p \in \mathbb{P}} \prod_{i \in I_k(p)} |\xi_{k,i}^n - \eta_{k,i}^n|_p^{-1} = \prod_{i=1}^{d_k} |\xi_{k,i}^n - \eta_{k,i}^n|_\infty \cdot \prod_{p \in \mathbb{P}} \prod_{i \notin I_k(p)} |\xi_{k,i}^n - \eta_{k,i}^n|_p;$$

as this number is a positive integer, $|\xi_{k,i}^n - \eta_{k,i}^n|_p = 1$ for $1 \leq i \leq d_k$ for almost all $p \in \mathbb{P}$.

(3) *Suppose that, for each $k \in \{1, \ldots, c\}$, the primes p that contribute non-trivial factors $\prod_{i \notin I_k(p)} |\xi_{k,i}^n - \eta_{k,i}^n|_p \neq 1$ to the product on the far right-hand side of (1.1) form a finte subset $\mathbf{P}_k \subseteq \mathbb{P}$ and that $|\xi_{k,i}|_\infty \neq |\eta_{k,i}|_\infty$ for $1 \leq i \leq d_k$.*

Then the coincidence Reidemeister zeta function $Z_{\varphi,\psi}(s)$ is either a rational function or it has a natural boundary at its radius of convergence. Furthermore, the latter occurs if and only if $|\xi_{k,i}|_p = |\eta_{k,i}|_p$ for some $k \in \{1, \ldots, c\}$, $p \in \mathbf{P}_k$ and $i \notin I_k(p)$.

For a full discussion and several intermediate results, we refer to the main text. The paper is organised as follows.

Section 2.1 puts results of Roman'kov [16] on Reidemeister classes of finitely generated torsion-free nilpotent groups in the more general perspective of coincidence Reidemeister classes of torsion-free nilpotent groups of finite rank. Of particular interest is Proposition 2.1. In Section 2.2 we establish a natural bijection between the set of coincidence Reidemeister classes of a pair (φ, ψ) of endomorphisms of an almost abelian group G and the corresponding set for the induced pair $(\widehat{\varphi}, \widehat{\psi})$ of endomorphisms of the profinite completion $\widehat{G}$; see Proposition 2.8. As explained and illustrated there, this closes some small gaps in the literature.

In Section 3 we derive a closed formula for the sequence of coincidence Reidemeister numbers for iterations of tame endomorphism pairs of abelian groups of finite rank. Our approach is via profinite completions and sheds new light on a similar formula of Miles [13], which was established in a different way, namely based upon techniques from commutative algebra. Corollary 3.2 provides a reduction to torsion-free groups, which are then duly dealt with in Proposition 3.4.

In Section 4 we present, in analogy to work of Bell, Miles and Ward [1], results in support of a Pólya–Carlson dichotomy between rationality and a natural boundary for the analytic behaviour of the coincidence Reidemeister zeta functions of torsion-free nilpotent groups of finite rank. The underlying ideas are already present in the abelian case which is covered in Theorem 4.3. The simple, but important Example 4.5 illustrates a new phenomenon that occurs for coincidence Reidemeister zeta functions associated to non-commuting pairs of endomorphisms. The paper concludes with the proof of Theorem 1.4 already stated above.

2. Nilpotent groups and profinite completions

2.1. **Nilpotent groups.** A group G has *finite (Prüfer) rank* if there exists an integer r such that every finitely generated subgroup of G can be generated by r elements; the least such r is called the *(Prüfer) rank* of G. It is known that G is a torsion-free nilpotent group of finite rank if and only if there is some integer n such that G is isomorphic to a

subgroup of $\mathrm{Tr}_1(n, \mathbb{Q})$, the group of all upper uni-triangular matrices over $\mathbb{Q}$; compare [18, Cor. to Thm. 2.5].

Let G be a torsion-free nilpotent group of class $c = c(G)$. As usual, let $G = \gamma_1(G) \geq \gamma_2(G) \geq \ldots \geq \gamma_c(G) \geq \gamma_{c+1}(G) = 1$ denote the lower central series. The elements of finite order in a nilpotent group form a subgroup, the *torsion subgroup* of G. For $1 \leq i \leq c + 1$, define $\overline{\gamma}_i(G)$ to be subgroup of G such that $\overline{\gamma}_i(G)/\gamma_i(G)$ is the torsion subgroup of $G/\gamma_i(G)$. Then it is routine to check that

$$(2.1) \qquad G = \overline{\gamma}_1(G) \geq \overline{\gamma}_2(G) \geq \ldots \geq \overline{\gamma}_c(G) \geq \overline{\gamma}_{c+1}(G) = 1$$

is a descending central series of fully invariant subgroups of G such that each factor group $\overline{\gamma}_i(G)/\overline{\gamma}_{i+1}(G)$ is torsion-free, for $1 \leq i \leq c$.

For completeness, we sketch the argument. It is straightforward that each $\overline{\gamma}_i(G)$ is fully invariant in G and that $G = \overline{\gamma}_1(G) \supseteq \ldots \supseteq \overline{\gamma}_{c+1}(G) = 1$. Clearly, each factor group $\overline{\gamma}_i(G)/\overline{\gamma}_{i+1}(G)$ is torsion-free. It remains to show that $\overline{\gamma}_i(G)/\overline{\gamma}_{i+1}(G)$ is abelian and, in fact, central in $G/\overline{\gamma}_{i+1}(G)$. In both cases it is enough to check the assertion for finitely generated subgroups of G; hence without loss of generality we take G to be finitely generated. By construction, $\overline{\gamma}_i(G)/\overline{\gamma}_{i+1}(G)$ is central-by-finite; thus Schur's theorem [15, §10.1.4] implies that $\overline{\gamma}_i(G)/\overline{\gamma}_{i+1}(G)$ is finite-by-abelian, hence abelian. Now $\gamma_i(G)\overline{\gamma}_{i+1}(G)/\overline{\gamma}_{i+1}(G)$ is central in $G/\overline{\gamma}_{i+1}(G)$ and of finite index in the finitely generated torsion-free abelian group $\overline{\gamma}_i(G)/\overline{\gamma}_{i+1}(G)$. This implies that $\overline{\gamma}_i(G)/\overline{\gamma}_{i+1}(G)$ is central in $G/\overline{\gamma}_{i+1}(G)$.

This establishes all claimed properties of the series (2.1); we refer to it as the *isolated lower central series* of G. If, in addition, G has finite rank then each factor group $\overline{\gamma}_i(G)/\overline{\gamma}_{i+1}(G)$ is a torsion-free abelian group of rank d_i, say, and hence isomorphic to a subgroup of the additive vector group $\mathbb{Q}^{d_i}$ that contains a $\mathbb{Q}$-basis.

The following proposition puts results of Roman'kov [16] for twisted conjugacy classes in nilpotent groups in a more general perspective.

Proposition 2.1. *Let G be a torsion-free nilpotent group, of class c and with isolated lower central series (2.1). Let $\varphi, \psi\colon G \to G$ be endomorphisms and, for $1 \leq i \leq c$, let $\varphi_i, \psi_i\colon \overline{\gamma}_i(G)/\overline{\gamma}_{i+1}(G) \to \overline{\gamma}_i(G)/\overline{\gamma}_{i+1}(G)$ denote the induced endomorphisms of the torsion-free abelian factor groups arising from (2.1).*

(1) Suppose that G has finite rank and that $R(\varphi, \psi) < \infty$. Then, for $1 \leq i \leq c$, the coincidence set

$$\mathrm{Coin}(\varphi_i, \psi_i) = \left\{ x.\overline{\gamma}_{i+1}(G) \in \overline{\gamma}_i(G)/\overline{\gamma}_{i+1}(G) \mid \left(x.\overline{\gamma}_{i+1}(G)\right)\varphi_i = \left(x.\overline{\gamma}_{i+1}(G)\right)\psi_i \right\}$$

is a singleton, i.e. equal to $\{1.\overline{\gamma}_{i+1}(G)\}$.

(2) Suppose that, for $1 \leq i \leq c$, the coincidence set $\mathrm{Coin}(\varphi_i, \psi_i)$ is a singleton. Then there is a bijection between $\mathcal{R}(\varphi, \psi)$ and $\prod_{i=1}^{c} \mathcal{R}(\varphi_i, \psi_i)$. In particular, $R(\varphi, \psi) < \infty$ if and only if $R(\varphi_i, \psi_i) < \infty$ for $1 \leq i \leq c$, and in this case $R(\varphi, \psi) = \prod_{i=1}^{c} R(\varphi_i, \psi_i)$.

Proof. It suffices to check that the arguments developed in [16] can be adapted so that they work in the more general situation considered here. By induction, we may assume that $c \geq 1$ and that the analogous assertions hold for the endomorphisms of $\overline{\varphi}, \overline{\psi}\colon G/\overline{\gamma}_c(G) \to G/\overline{\gamma}_c(G)$ that are induced by φ and ψ.

Clearly, two elements $x, y \in G$ can be (φ, ψ)-twisted conjugate only if their images in $G/\overline{\gamma}_c(G)$ are $(\overline{\varphi}, \overline{\psi})$-twisted conjugate to one another. Furthermore, we observe that, for any central subgroup A of G,

$$L_{G,\varphi,\psi}(A) = \{a \in A \mid \exists g \in G : (g\varphi)a = g\psi\} = \{a \in A \mid \exists g \in G : a = (g\varphi)^{-1}(g\psi)\}$$

is a subgroup of A. If, in addition, A is φ- and ψ-invariant, then

$$A/L_{G,\varphi,\psi}(A) \to \{[a] \in \mathcal{R}(\varphi, \psi) \mid a \in A\}, \quad a L_{G,\varphi,\psi}(A) \mapsto [a]$$

is bijective, and hence $|\{[a] \in \mathcal{R}(\varphi, \psi) \mid a \in A\}| = |A : L_{G,\varphi,\psi}(A)|$.

(1) Suppose that G has finite rank and that $R(\varphi, \psi) < \infty$. By induction, the coincidence sets $\mathrm{Coin}(\varphi_i, \psi_i)$ are singletons for $1 \leq i \leq c - 1$, and consequently

$$L_{G,\varphi,\psi}(A) = L_{A,\varphi_c,\psi_c}(A) \qquad \text{for } A = \overline{\gamma}_c(G).$$

For a contradiction, assume that $\mathrm{Coin}(\varphi_c, \psi_c)$ is not a singleton. Then the kernel of the endomorphism $A \to A$, $b \mapsto (b\varphi)^{-1}(b\psi)$ is non-trivial, hence its image, viz. $L_{A,\varphi_c,\psi_c}(A)$, has strictly smaller rank than A. (After tensoring the $\mathbb{Z}$-module A with $\mathbb{Q}$, we are reduced to considering a linear endomorphism of the finite dimensional $\mathbb{Q}$-vector space $\mathbb{Q} \otimes_{\mathbb{Z}} A$.) This implies that

$$R(\varphi, \psi) \geq |\{[a] \in \mathcal{R}(\varphi, \psi) \mid a \in A\}| = |A : L_{A,\varphi_c,\psi_c}(A)| = \infty.$$

(2) Suppose that, for $1 \leq i \leq c$, the coincidence set $\mathrm{Coin}(\varphi_i, \psi_i)$ is a singleton. We need to investigate how the pre-image of a twisted conjugacy class in $G/\overline{\gamma}_c(G)$ splits into twisted conjugacy classes in G. Fix $x \in G$ and consider $z, \tilde{z} \in \overline{\gamma}_c(G) \subseteq \mathrm{Z}(G)$. The elements xz and $x\tilde{z}$ are (φ, ψ)-twisted conjugate if and only if there exists $g \in G$ such that $\tilde{z}^{-1}z = (g\vartheta)^{-1}(g\psi)$, where $\vartheta = \varphi\gamma_x$ and $\gamma_x \colon G \to G$, $h \mapsto x^{-1}hx$ denotes the inner automorphism that corresponds to conjugation by x.

Again we write $A = \overline{\gamma}_c(G)$, and we note that, for $1 \leq i \leq c$, the endomorphism $\vartheta_i \colon \overline{\gamma}_i(G)/\overline{\gamma}_{i+1}(G) \to \overline{\gamma}_i(G)/\overline{\gamma}_{i+1}(G)$ induced by ϑ is equal to φ_i. In particular, the coincidence sets $\mathrm{Coin}(\vartheta_i, \psi_i)$, for $1 \leq i \leq c-1$, are singletons and $L_{G,\vartheta,\psi}(A) = L_{A,\vartheta_c,\psi_c}(A)$. Thus we conclude: the collection of twisted conjugacy classes in G which make up the pre-image of the twisted conjugacy class of the image of x in $G/\overline{\gamma}_c(G)$ is in bijection to

$$A/L_{G,\vartheta,\psi}(A) = A/L_{A,\vartheta_c,\psi_c}(A) = A/L_{A,\varphi_c,\psi_c}(A) = \mathcal{R}(\varphi_c, \psi_c). \qquad \square$$

Example 2.2. It is easy to produce a torsion-free abelian group A, not of finite rank, and an endomorphism $\varphi \colon A \to A$ such that $R(\varphi) < \infty$, but with infinite coincidence set $\mathrm{Coin}(\varphi, \mathrm{id}_A) = \mathrm{Fix}(\varphi)$. For instance, consider the Cartesian product $A = \prod_{i \in \mathbb{N}} \mathbb{Z}$ and the shift-map $\varphi \colon A \to A$, $(a_i)_{i \in \mathbb{N}} \mapsto (a_{i+1})_{i \in \mathbb{N}}$. In this case, $R(\varphi) = 1$, but $\mathrm{Coin}(\varphi, \mathrm{id}_A)$ consists of all constant sequences and is thus infinite.

2.2. **Profinite completions.** Let $\varphi, \psi \colon G \to G$ be endomorphisms of a group G. This induces continuous endomorphisms $\widehat{\varphi}, \widehat{\psi} \colon \widehat{G} \to \widehat{G}$ of the profinite completion $\iota \colon G \to \widehat{G}$, and a natural map

$$\mathcal{R}(\varphi, \psi) \to \mathcal{R}(\widehat{\varphi}, \widehat{\psi}), \quad [x]_{\varphi,\psi} \mapsto [x\iota]_{\widehat{\varphi},\widehat{\psi}}.$$

This constellation was already considered in [7, Sec. 5.3.2] and [10].

Lemma 2.3. *In the situation described above, the following hold.*
   (1) *Each $(\widehat{\varphi}, \widehat{\psi})$-twisted conjugacy class in $\widehat{G}$ is compact and hence closed in $\widehat{G}$.*
   (2) *For $x \in G$, the twisted conjugacy class $[x\iota]_{\widehat{\varphi},\widehat{\psi}}$ is the closure of $[x]_{\varphi,\psi}\iota$ in $\widehat{G}$.*
   (3) *If $R(\varphi, \psi) < \infty$, then the natural map $\mathcal{R}(\varphi, \psi) \to \mathcal{R}(\widehat{\varphi}, \widehat{\psi})$ is surjective and each $(\widehat{\varphi}, \widehat{\psi})$-twisted conjugacy class is open in $\widehat{G}$.*
   (4) *If G is abelian and $R(\varphi, \psi) < \infty$ then the natural map $\mathcal{R}(\varphi, \psi) \to \mathcal{R}(\widehat{\varphi}, \widehat{\psi})$ is bijective.*

Proof. (1) For $x \in \widehat{G}$, the image of $\widehat{G}$ under the continuous map $\widehat{G} \to \widehat{G}$, $g \mapsto (g\widehat{\varphi})^{-1}x(g\widehat{\psi})$ is compact.

(2) From (1) we conclude that $\overline{[x]_{\varphi,\psi}\iota} \subseteq [x\iota]_{\widehat{\varphi},\widehat{\psi}}$. The reverse inclusion holds, because $((g_\lambda\varphi)^{-1}x(g_\lambda\psi))\iota$ converges to $(g\widehat{\varphi})^{-1}(x\iota)(g\widehat{\psi})$, for any $g \in \widehat{G}$ and any net $(g_\lambda)_{\lambda \in \Lambda}$ in G such that $(g_\lambda\iota)_{\lambda \in \Lambda}$ converges to g in $\widehat{G}$.

(3) Suppose that $R(\varphi, \psi) < \infty$. Then every $y \in \widehat{G}$ is the limit of a net $(x_\lambda\iota)_{\lambda \in \Lambda}$, where the elements $x_\lambda \in G$ all belong to the same (φ, ψ)-twisted conjugacy class in G. The claim follows from (2).

(4) Suppose that G is abelian and $R(\varphi, \psi) < \infty$. Then $\mathcal{R}(\varphi, \psi) = H_{\varphi,\psi} \backslash G$, where $H_{\varphi,\psi} = \{(g\varphi)^{-1}(g\psi) \mid g \in G\} \leq G$ is a finite-index subgroup. Also $\widehat{G}$ is abelian and $\mathcal{R}(\widehat{\varphi}, \widehat{\psi}) = H_{\widehat{\varphi},\widehat{\psi}} \backslash G$, where $H_{\widehat{\varphi},\widehat{\psi}} = \{(g\widehat{\varphi})^{-1}(g\widehat{\psi}) \mid g \in \widehat{G}\} \leq \widehat{G}$ equals the closure of $H_{\varphi,\psi}\iota$ in $\widehat{G}$ and $|\widehat{G} : H_{\widehat{\varphi},\widehat{\psi}}| = |G : H_{\varphi,\psi}|$. The claim follows from (3). $\qquad\square$

Example 2.4. For $G = \mathbb{Z}$ and $\varphi = \mathrm{id}_G$, the natural map $\mathcal{R}(\varphi) \to \mathcal{R}(\widehat{\varphi})$ is not surjective. For $G = \mathbb{Q}$ and $\varphi = \mathrm{id}_G$, the natural map $\mathcal{R}(\varphi) \to \mathcal{R}(\widehat{\varphi})$ is not injective.

The following examples illustrate that an argument given in [7, §5.3.2], intended to show that every (finitely generated) almost-abelian group contains a fully invariant, abelian finite-index subgroup, is not quite complete.

Example 2.5. Consider $G = \langle z \rangle \times (\langle a \rangle \ltimes \langle b \rangle) \cong C_2 \times \mathrm{Sym}(3)$, where $\mathrm{ord}(z) = \mathrm{ord}(a) = 2$, $\mathrm{ord}(b) = 3$ and $b^a = a^{-1}$. The 2-Sylow subgroup $A = \langle z, a \rangle \cong C_2 \times C_2$ is abelian and of index $|G : A| = 3$. There are two other 2-Sylow subgroups: $A^b = \langle z, ab^2 \rangle$ and $A^{b^2} = \langle z, ab \rangle$. Thus the intersection of all subgroups of index 3 in G, namely $Z = \langle z \rangle$, is not fully invariant in G; for instance, the endomorphism $\varphi \colon G \to G$ given by $z \mapsto a$ and $a, b \mapsto 1$ does not map Z to itself.

Example 2.6. Consider $G = \langle a, b_0 \rangle \times \langle b_1, b_2, \ldots \rangle \cong \mathrm{Alt}(3) \times \coprod_{i \in \mathbb{N}} C_2$, where $\mathrm{ord}(a) = 3$ and $\mathrm{ord}(b_0) = \mathrm{ord}(b_1) = \ldots = 2$, equipped with the endomorphism $\varphi \colon G \to G$ given by $a\varphi = b_0\varphi = 1$ and $b_i\varphi = b_{i-1}$ for $i \in \mathbb{N}$. Every φ-invariant finite-index subgroup of G contains b_0. Consequently, the almost abelian group G has no φ-invariant, abelian finite-index normal subgroup.

The following lemma partly fixes this small gap, in a slightly more general context.

Lemma 2.7. *Let G be finitely generated and almost-$\mathcal{P}$, where $\mathcal{P}$ is a group-theoretic property that is inherited by finite-index subgroups. Then G admits a fully invariant finite-index subgroup satisfying $\mathcal{P}$.*

Proof. Put $d = \mathrm{d}(G)$, the minimal number of generators of G, and let $A \leq G$ be a subgroup that has property $\mathcal{P}$ and finite index $|G : A| < \infty$. Replacing A by its core in G, we may assume that $A \trianglelefteq G$. Denote by W the set of group words $w = w(x_1, \ldots, x_d)$, i.e. elements in the free group on free generators $x_1, \ldots, x_d$, such that w is a law in the finite group G/A. Then the verbal subgroup $B = \langle w(g_1, \ldots, g_d) \mid w \in W, \ g_1, \ldots, g_d \in G \rangle$ is fully invariant in G and has finte index $|G : B| < \infty$; see [14, Thm. 15.4]. As $B \leq A$, we conclude that B satisfies $\mathcal{P}$. $\qquad\square$

The following proposition generalises Lemma 2.3(4) and altogether by-passes some of the apparent difficulties in [7, §5.3.2].

Proposition 2.8. *Let $\varphi, \psi \colon G \to G$ be endomorphisms of an almost abelian group G such that $R(\varphi, \psi) < \infty$. Then the natural map $\mathcal{R}(\varphi, \psi) \to \mathcal{R}(\widehat{\varphi}, \widehat{\psi})$ is a bijection.*

Proof. Fix $x \in G$. In view of Lemma 2.3, it suffices to show that the map $\mathcal{R}(\varphi, \psi) \to \mathcal{R}(\widehat{\varphi}, \widehat{\psi})$ is injective. For this it suffices to prove that, for every $x \in G$, there exists a finite-index subgroup $H \leq G$ such that $[x]_{\varphi,\psi} \supseteq xH$; indeed, if $y \in G$ is such that $\overline{[x]_{\varphi,\psi}\iota} = \overline{[y]_{\varphi,\psi}\iota}$ then y is (φ, ψ)-twisted conjugate to an element of the open neighbourhood xH of $x \in G$ (in the profinite topology on G, which is not necessarily Hausdorff), hence y is (φ, ψ)-twisted conjugate to x.

Fix $x \in G$, and let $A \leq G$ be an abelian finite-index subgroup. We observe that, for $a, \tilde{a} \in A$, the elements xa and $x\tilde{a}$ are (φ, ψ)-twisted conjugate if and only if there exists $g \in G$ such that $a = (g\vartheta)^{-1}\tilde{a}(g\psi)$, where $\vartheta = \varphi\gamma_x$ and $\gamma_x \colon G \to G$, $h \mapsto x^{-1}hx$ denotes the inner automorphism that corresponds to conjugation by x. Put $A_0 = A\vartheta^{-1} \cap A\psi^{-1} \cap A$, a finite-index subgroup of A, and let Y be a right-transversal for A_0 in G with $1 \in Y$. Writing $g = by$, with $b \in A_0$ and $y \in Y$, we obtain

(2.2) $(g\vartheta)^{-1}\tilde{a}(g\psi) = (y\vartheta)^{-1} \cdot \tilde{a}(b\vartheta)^{-1}(b\psi) \cdot (y\psi)$.

We observe that $B = \{(b\vartheta)^{-1}(b\psi) \mid b \in A_0\}$ is a subgroup of A. Furthermore, (2.2) shows that $x\tilde{a}$ is (φ, ψ)-twisted conjugate to one of finitely many elements $xa_1, \dots, xa_m$, where $a_1, \dots, a_m \in A$, if and only if the coset $\tilde{a}B$ contains one of finitely many elements $(y\vartheta)a_i(y\psi)^{-1}$, with $y \in Y$ and $1 \le i \le m$. Thus $R(\varphi, \psi) < \infty$ implies that $|G : B| = |G : A||A : B| < \infty$. Taking $y = 1$ in (2.2), we conclude that $[x]_{\varphi, \psi} \supseteq [x]_{\varphi, \psi} \cap xA \supseteq xB$. $\quad\square$

3. An explicit formula for abelian groups of finite rank

Using techniques from commutative algebra, Miles derived in [13] a closed formula for periodic point counts for ergodic finite-entropy endomorphisms of finite-dimensional compact abelian groups. This approach was then used in [1] to conjecture a Pólya–Carlson dichotomy for the analytic behaviour of dynamical zeta functions of compact group automorphisms; supportive evidence comes from a certain class of automorphisms of solenoids (connected finite-dimensional compact abelian groups). Under the Pontryagin duality, finite-dimensional compact abelian groups correspond to abelian groups of finite rank and, in the more restricted setting, solenoids correspond to torsion-free abelian groups of finite rank.

Our aim in this section is to consider somewhat more general situations and to arrive at a formula similar to Miles' by an entirely different route, namely via profinite completions. We start with a reduction to torsion-free groups.

Lemma 3.1. *Let G be an abelian group of finite rank, and let $\varphi, \psi \colon G \to G$ be endomorphisms such that $R(\varphi, \psi) < \infty$. Let $T \le G$ denote the torsion subgroup of G, and write $\tilde{\varphi}, \tilde{\psi} \colon G/T \to G/T$ and $\varphi_0, \psi_0 \colon T \to T$ for the induced endomorphisms.*
   Then $R(\varphi, \psi) = R(\tilde{\varphi}, \tilde{\psi})R(\varphi_0, \psi_0)$ and, in particular, $R(\tilde{\varphi}, \tilde{\psi}), R(\varphi_0, \psi_0) < \infty$.

Proof. We follow the same ideas as in the proof of Proposition 2.1. Two elements $x, y \in G$ can be (φ, ψ)-twisted conjugate only if their images in G/T are $(\tilde{\varphi}, \tilde{\psi})$-twisted conjugate to one another. Thus $R(\tilde{\varphi}, \tilde{\psi}) < \infty$, and $\mathrm{Coin}(\tilde{\varphi}, \tilde{\psi})$ is a singleton by Proposition 2.1. We need to investigate how the pre-image of a twisted conjugacy class in G/T splits into twisted conjugacy classes in G. Fix $x \in G$ and consider $z, \tilde{z} \in T$. The elements xz and $x\tilde{z}$ are (φ, ψ)-twisted conjugate if and only if there exists $g \in G$ such that $\tilde{z}^{-1}z = (g\varphi)^{-1}(g\psi)$, because in contrast to the proof of Proposition 2.1 we can ignore inner automorphisms altogether. Since $\mathrm{Coin}(\tilde{\varphi}, \tilde{\psi})$ is trivial, we conclude that xz and $x\tilde{z}$ are (φ, ψ)-twisted conjugate if and only if there exists $g \in T$ such that $\tilde{z}^{-1}z = (g\varphi)^{-1}(g\psi)$, which is to say that z and $\tilde{z}$ are (φ_0, ψ_0)-twisted equivalent. $\quad\square$

Corollary 3.2. *Let G be an abelian group of finite rank, and let $\varphi, \psi \colon G \to G$ be endomorphisms such that (φ, ψ) is tame. Let $T \le G$ denote the torsion subgroup of G, and write $\tilde{\varphi}, \tilde{\psi} \colon G/T \to G/T$ and $\varphi_0, \psi_0 \colon T \to T$ for the induced endomorphisms.*
   Then $(\tilde{\varphi}, \tilde{\psi})$ and (φ_0, ψ_0) are tame and

$$Z_{\varphi, \psi}(s) = Z_{\tilde{\varphi}, \tilde{\psi}}(s) * Z_{\varphi_0, \psi_0}(s)$$

is an additive convolution, in the sense of [8, §1].

Remark 3.3. A similar result for finitely generated abelian groups and additional facts were established in [8, Thm. 2]. The point here is that we may effectively replace the data (G, φ, ψ) by the similar data $(G/T \times T, \tilde{\varphi} \times \varphi_0, \tilde{\psi} \times \psi_0)$ without any change to the coincidence Reidemeister zeta function. Furthermore, by Proposition 2.8, the group T can be replaced by $T_1 = T/T_{\mathrm{div}}$, where T_{div} denotes the maximal divisible subgroup of T, equipped with the induced endomorphisms φ_1, ψ_1 of T_1: the group T_1 is of the form $\bigoplus_{p \in \mathbb{P}} F_p$, for finite abelian p-groups of uniformly bounded rank, and the profinite completion of T is $\widehat{T} \cong \widehat{T_1} \cong \prod_{p \in \mathbb{P}} F_p$. In the setting of [13], the group F_p is trivial for almost all primes p; thus T_1 is finite and its (φ_1, ψ_1)-coincidence Reidemeister zeta function is easily understood; compare [8].

We now focus on the torsion-free case and derive a formula similar to the one of Miles [13], but perhaps more directly accessible by group-theoretic means. For this we introduce the following notion of 'weak commutativity': We say that two endomorphisms $\alpha, \beta \colon V \to V$ of a finite-dimensional $\mathbb{Q}$-vector space V are (absolutely) *simultaneously triangularisable*, if there is a finite algebraic extension L of $\mathbb{Q}$ such that the induced endomorphisms of the L-vector space $L \otimes_{\mathbb{Q}} V$ simultaneously preserve a complete flag of subspaces. For this to happen, it suffices that α and β commute; for a more precise characterisation of the property we refer to [4].

Proposition 3.4. *Let G be a torsion-free abelian group of finite rank $d \geq 1$, and let (φ, ψ) be a tame pair of endomorphisms for G. Let $\varphi_{\mathbb{Q}}, \psi_{\mathbb{Q}} \colon G_{\mathbb{Q}} \to G_{\mathbb{Q}}$ denote the extensions of φ, ψ to the divisible hull $G_{\mathbb{Q}} = \mathbb{Q} \otimes_{\mathbb{Z}} G \cong \mathbb{Q}^d$ of G, and suppose that the endomorphisms $\varphi_{\mathbb{Q}}, \psi_{\mathbb{Q}}$ are simultaneously triangularisable. Let $\xi_1, \ldots, \xi_d$ and $\eta_1, \ldots, \eta_d$ denote the eigenvalues of $\varphi_{\mathbb{Q}}$ and $\psi_{\mathbb{Q}}$ in a fixed algebraic closure of the field $\mathbb{Q}$, including multiplicities, ordered so that, for $n \in \mathbb{N}$, the eigenvalues of $\varphi_{\mathbb{Q}}^n - \psi_{\mathbb{Q}}^n$ are $\xi_1^n - \eta_1^n, \ldots, \xi_d^n - \eta_d^n$. Set $L = \mathbb{Q}(\xi_1, \ldots, \xi_d, \eta_1, \ldots, \eta_d)$.*

For each $p \in \mathbb{P}$, we fix an embedding $L \hookrightarrow \overline{\mathbb{Q}}_p$, where $\overline{\mathbb{Q}}_p$ denotes the algebraic closure of $\mathbb{Q}_p$, and we write $|\cdot|_p$ for the unique prolongation of the p-adic absolute value to $\overline{\mathbb{Q}}_p$; this is a way of choosing a particular extension of the p-adic absolute value to L. Likewise, we choose an embedding $L \hookrightarrow \mathbb{C}$ and denote by $|\cdot|_\infty$ the usual absolute value on $\mathbb{C}$.

Then there exist subsets $I(p) \subseteq \{1, \ldots, d\}$, for $p \in \mathbb{P}$, such that the following hold.

(1) *For each $p \in \mathbb{P}$, the polynomials $\prod_{i \in I(p)} (X - \xi_i)$ and $\prod_{i \in I(p)} (X - \eta_i)$ have coefficients in $\mathbb{Z}_p$; in particular, $|\xi_i|_p, |\eta_i|_p \leq 1$ for $i \in I(p)$.*

(2) *For each $n \in \mathbb{N}$,*

$$(3.1) \qquad R(\varphi^n, \psi^n) = \prod_{p \in \mathbb{P}} \prod_{i \in I(p)} |\xi_i^n - \eta_i^n|_p^{-1} = \prod_{i=1}^d |\xi_i^n - \eta_i^n|_\infty \cdot \prod_{p \in \mathbb{P}} \prod_{i \notin I(p)} |\xi_i^n - \eta_i^n|_p;$$

as this number is a positive integer, $|\xi_i^n - \eta_i^n|_p = 1$ for $1 \leq i \leq d$ for almost all $p \in \mathbb{P}$.

Proof. Without loss of generality, $\mathbb{Z}^d \leq G \leq \mathbb{Q}^d$ and we use additive notation. Using Lemma 2.3(4), we may pass to the profinite completion.

The profinite completions of the abelian group G and its endomorphisms decompose as direct products

$$\iota \colon G \to \widehat{G} = \prod_{p \in \mathbb{P}} \widehat{G}_p \qquad \text{and} \qquad \widehat{\varphi} = \prod_{p \in \mathbb{P}} \widehat{\varphi}_p, \quad \widehat{\psi} = \prod_{p \in \mathbb{P}} \widehat{\psi}_p,$$

where, for each prime p, the Sylow pro-p subgroup of $\widehat{G}$ constitutes the pro-p completion $\widehat{G}_p$ of G, equipped with endomorphisms $\widehat{\varphi}_p, \widehat{\psi}_p \colon \widehat{G}_p \to \widehat{G}_p$. Proposition 2.8 shows that

$$R(\varphi^n, \psi^n) = \prod_{p \in \mathbb{P}} R(\widehat{\varphi}_p^n, \widehat{\psi}_p^n), \qquad \text{for } n \in \mathbb{N};$$

in particular, $R(\varphi^n, \psi^n) < \infty$ implies that $R(\widehat{\varphi}_p^n, \widehat{\psi}_p^n) = 1$ for almost all $p \in \mathbb{P}$ so that the product is only formally infinite.

Fix $p \in \mathbb{P}$. The pro-p group $\widehat{G}_p$ is torsion-free, abelian and of rank at most d, hence $\widehat{G}_p \cong \mathbb{Z}_p^{d(p)}$, where $d(p) = \mathrm{d}(\widehat{G}_p) \leq d$. Indeed, we can arrive at $\widehat{G}_p$ in the following way, which is somewhat roundabout but useful for our purposes. We observe that the group G has the same pro-p completion as $G_{\mathbb{Z}_p} = \mathbb{Z}_p \otimes_{\mathbb{Z}} G$, where $\mathbb{Z}_p$ denotes the ring of p-adic integers. The $\mathbb{Z}_p$-module $G_{\mathbb{Z}_p}$ decomposes as a direct sum of its maximal divisible submodule $D \cong \mathbb{Q}_p^{d-d(p)}$ and a free $\mathbb{Z}_p$-module $A \cong \mathbb{Z}_p^{d(p)}$; compare [12, Thm. 20]. Furthermore, the submodule D is invariant under the induced $\mathbb{Z}_p$-module endomorphisms

$\varphi_{\mathbb{Z}_p}, \psi_{\mathbb{Z}_p} \colon G_{\mathbb{Z}_p} \to G_{\mathbb{Z}_p}$; thus we obtain endomorphisms $\overline{\varphi}_{\mathbb{Z}_p}, \overline{\psi}_{\mathbb{Z}_p} \colon G_{\mathbb{Z}_p}/D \to G_{\mathbb{Z}_p}/D$. Consequently, there is an isomorphism $\sigma \colon \widehat{G}_p \to G_{\mathbb{Z}_p}/D$ of pro-p groups that is compatible with the endomorphism pairs $\widehat{\varphi}_p, \widehat{\psi}_p$ and $\overline{\varphi}_{\mathbb{Z}_p}, \overline{\psi}_{\mathbb{Z}_p}$, i.e.

$$\widehat{\varphi}_p \sigma = \sigma \overline{\varphi}_{\mathbb{Z}_p} \qquad \text{and} \qquad \widehat{\psi}_p \sigma = \sigma \overline{\psi}_{\mathbb{Z}_p}.$$

We conclude that there exist a subset $I(p) \subseteq \{1, \ldots, d\}$ such that the endomorphisms $\widehat{\varphi}_p$ and $\widehat{\psi}_p$ have eigenvalues ξ_i, $i \in I(p)$, and η_i, $i \in I(p)$. In particular, the polynomials $\prod_{i \in I(p)} (X - \xi_i)$ and $\prod_{i \in I(p)} (X - \eta_i)$ have coefficients in $\mathbb{Z}_p$.

Finally, Lemma 1.2 yields

$$R(\widehat{\varphi}_p^n, \widehat{\psi}_p^n) = |\det(-\widehat{\varphi}_p^n + \widehat{\psi}_p^n)|_p = \prod_{i \in I(p)} |\xi_i^n - \eta_i^n|_p^{-1}.$$

Taking the product over all primes p, we arrive at the first formula in (2). Using the adelic formula $|a|_\infty \prod_{p \in \mathbb{P}} |a|_p = 1$ for $a \in \mathbb{Q} \setminus \{0\}$, we obtain the second formula in (2). $\qquad \square$

4. Pólya–Carlson dichotomy for coincidence Reidemeister zeta functions

In this section we present, in analogy to [1], results in support of a Pólya–Carlson dichotomy between rationality and a natural boundary for the analytic behaviour of the coincidence Reidemeister zeta functions for tame pairs of commuting endomorphisms of a torsion-free nilpotent group of finite rank. The classical Pólya–Carlson theorem, as discussed in [17, §6.5], provides the following connection between the arithmetic properties of the coefficients of a complex power series and its analytic behaviour.

Pólya–Carlson Theorem. *A power series with integer coefficients and radius of convergence 1 is either a rational function or has the unit circle as a natural boundary.*

Translated to our set-up, Bell, Miles and Ward [1] conjectured that if φ is a tame automorphism of an abelian group G of finite rank then $Z_\varphi(s)$ is either rational or admits a natural boundary at its radius of convergence. They collected substantial evidence for this conjecture from a certain class of automorphisms of torsion-free abelian groups of finite rank.

Let G be a group with tame endomorphism pair (φ, ψ). It is convenient to introduce a notation

$$Z_{\varphi,\psi}^*(s) = \sum_{n=1}^\infty R(\varphi^n, \psi^n) s^n = s \cdot Z_{\varphi,\psi}'(s)/Z_{\varphi,\psi}(s)$$

for the Dirichlet series that enumerates directly the numbers of coincidence Reidemeister classses. The following lemma is basic: it shows in particular that, if $Z_{\varphi,\psi}^*(s)$ has a natural boundary at its radius of convergence, then so does $Z_{\varphi,\psi}(s)$; compare [1].

Lemma 4.1. *If $Z_{\varphi,\psi}(s)$ is rational then $Z_{\varphi,\psi}^*(s)$ is rational. If $Z_{\varphi,\psi}(s)$ admits analytic continuation beyond its radius of convergence, then so does $Z_{\varphi,\psi}^*(s)$.*

For the proofs of the main theorems in this section we rely on the following key result of Bell, Miles and Ward [1, Lem. 17]; one of the ingredients in its proof is the Hadamard quotient theorem.

Lemma 4.2. *Let S be a finite list of places of algebraic number fields and, for each $v \in S$, let ξ_v be an element of the appropriate number field that is not a unit root and such that $|\xi_v|_v = 1$. Then the complex function*

$$F(s) = \sum_{n=1}^\infty f(n) s^n, \qquad \text{where } f(n) = \prod_{v \in S} |\xi_v^n - 1|_v \text{ for } n \geq 1,$$

has the unit circle as a natural boundary.

First we state and prove our result in the case of abelian groups, in order to illustrate some of the underlying ideas.

Theorem 4.3. *Let $\varphi, \psi \colon G \to G$ be a tame pair of endomorphisms of a torsion-free abelian group G of finite rank $d \geq 1$. Using the notation from Proposition 3.4, suppose that $\varphi_{\mathbb{Q}}$ and $\psi_{\mathbb{Q}}$ are simultaneously triangularisable and, in addition, that the primes p that contribute non-trivial factors $\prod_{i \notin I(p)} |\xi_i^n - \eta_i^n|_p$ to the product on the far right-hand side of (3.1) form a finte subset $\mathbf{P} \subseteq \mathbb{P}$. For instance, this is the case, when $I(p) = \{1, \ldots, d\}$ for almost all $p \in \mathbb{P}$, equivalently when G has no elements of infinite p-height for almost all $p \in \mathbb{P}$. Suppose further that $|\xi_i|_\infty \neq |\eta_i|_\infty$ for $1 \leq i \leq d$.*

Then the coincidence Reidemeister zeta function $Z_{\varphi,\psi}(s)$ is either a rational function or it has a natural boundary at its radius of convergence. Furthermore, the latter occurs if and only if $|\xi_i|_p = |\eta_i|_p$ for some $p \in \mathbf{P}$ and $i \notin I(p)$.

Proof. For $p \in \mathbf{P}$ we write

$$S(p) = \{1, \ldots, d\} \smallsetminus I(p) \qquad \text{and} \qquad S^*(p) = \{i \in S(p) \mid |\xi_i|_p \neq |\eta_i|_p\};$$

we remark right away that $\eta_i \neq 0$ for every $i \in S(p) \smallsetminus S^*(p)$, because otherwise $\xi_i = \eta_i = 0$ and $\varphi_{\mathbb{Q}} - \psi_{\mathbb{Q}}$ would have rank less than d contrary to Lemma 1.2. We set

$$b = \prod_{p \in \mathbf{P}} \prod_{i \in S^*(p)} \max\{|\xi_i|_p, |\eta_i|_p\}, \qquad \eta = \prod_{p \in \mathbf{P}} \prod_{i \in S(p) \smallsetminus S^*(p)} |\eta_i|_p$$

and, for $n \in \mathbb{N}$,

$$g(n) = b^n \cdot \eta^n \cdot \prod_{i=1}^{d} |\xi_i^n - \eta_i^n|_\infty \qquad \text{and} \qquad f(n) = \prod_{p \in \mathbf{P}} \prod_{i \in S(p) \smallsetminus S^*(p)} \left| (\xi_i \eta_i^{-1})^n - 1 \right|_p.$$

From formula (3.1) in Proposition 3.4 and using the ultrametric property, we deduce that, for every $n \in \mathbb{N}$,

$$
\begin{aligned}
R(\varphi^n, \psi^n) &= \prod_{i=1}^{d} |\xi_i^n - \eta_i^n|_\infty \cdot \prod_{p \in \mathbf{P}} \prod_{i \in S(p)} |\xi_i^n - \eta_i^n|_p \\
&= \prod_{i=1}^{d} |\xi_i^n - \eta_i^n|_\infty \cdot \prod_{p \in \mathbf{P}} \prod_{i \in S^*(p)} \max\{|\xi_i|_p^n, |\eta_i|_p^n\} \cdot \prod_{p \in \mathbf{P}} \prod_{i \in S(p) \smallsetminus S^*(p)} |\xi_i^n - \eta_i^n|_p \\
&= \prod_{i=1}^{d} |\xi_i^n - \eta_i^n|_\infty \cdot b^n \cdot \eta^n \cdot \prod_{p \in \mathbf{P}} \prod_{i \in S(p) \smallsetminus S^*(p)} \left| (\xi_i \eta_i^{-1})^n - 1 \right|_p \\
&= g(n) f(n).
\end{aligned}
$$

Now we open up the absolute values in the product $P(n) = \prod_{i=1}^{d} |\xi_i^n - \eta_i^n|_\infty$. Complex eigenvalues ξ_i in the spectrum of $\varphi_{\mathbb{Q}}$, respectively η_i in the spectrum of $\psi_{\mathbb{Q}}$, appear in pairs with their complex conjugate $\bar{\xi}_i$, respectively $\bar{\eta}_i$. Moreover, such pairs can be lined up with one another in a simultaneous triangularisation. If ξ_i and η_i are both real eigenvalues of $\varphi_{\mathbb{Q}}$ and $\psi_{\mathbb{Q}}$ then exactly as in Example 1.3 above we have $|\xi_i^n - \eta_i^n|_\infty = \delta_{1,i}^n - \delta_{2,i}^n$, where $\delta_{1,i} = \max\{|\xi_i|_\infty, |\eta_i|_\infty\}$ and $\delta_{2,i} = \frac{\xi_i \cdot \eta_i}{\delta_{1,i}}$.

Hence we can expand the product $P(n)$, using an appropriate symmetric polynomial, to obtain an expression of the form

$$(4.1) \qquad\qquad g(n) = \sum_{j \in J} c_j w_j^n,$$

where J is a finite index set, $c_j \in \{-1, 1\}$ and $\{w_j \mid j \in J\} \subseteq \mathbb{C} \setminus \{0\}$. Consequently, the coincidence Reidemeister zeta function can be written as

$$Z_{\varphi,\psi}(s) = \exp\left(\sum_{j \in J} c_j \sum_{n=1}^{\infty} \frac{f(n)(w_j s)^n}{n}\right).$$

If $S(p) = S^*(p)$ for all $p \in \mathbf{P}$, then $f(n) = 1$ for all $n \in \mathbb{N}$ and it follows that $Z_{\varphi,\psi}(s) = \prod_{j \in J}(1 - w_j s)^{-c_j}$ is a rational function.

Now suppose that $S(p) \neq S^*(p)$ for some $p \in \mathbf{P}$. By Lemma 4.1, it suffices to show that

$$Z_{\varphi,\psi}^*(s) = \sum_{j \in J} c_j \sum_{n=1}^{\infty} f(n)(w_j s)^n$$

has a natural boundary at its radius of convergence. Moreover, from [1, Lem. 2 and Lem. 17] (compare Lemma 4.2) we have $\limsup_{n\to\infty} f(n)^{1/n} = 1$. Hence, for each $j \in J$, the series

$$\sum_{n=1}^{\infty} f(n)(w_j s)^n$$

has radius of convergence $|w_j|_\infty^{-1}$.

As $|\xi_i|_\infty \neq |\eta_i|_\infty$ for $1 \leq i \leq d$, there is a dominant term w_m in the expansion (4.1) for which

$$|w_m|_\infty = b \cdot \eta \cdot \prod_{i=1}^{d} \max\{|\xi_i|_\infty, |\eta_i|_\infty\}$$

and $|w_m|_\infty > |w_j|_\infty$ for all $j \in J \setminus \{k\}$. Thus it suffices to show that $\sum_{n=1}^{\infty} f(n)(w_m s)^n$ has its circle of convergence as a natural boundary. This is the case, because $\sum_{n=1}^{\infty} f(n)s^n$ has the unit circle as a natural boundary by Lemma 4.2. $\qquad\square$

We remark that the special case of Theorem 4.3, where $\psi = \mathrm{id}_G$ and φ is an automorphism of the abelian group G has been discussed in [11, Thm. 5.8], via methods from commutative algebra as in [1]. The following example of a coincidence Reidemeister zeta function with natural boundary comes from a 3-adic extension of the circle doubling map and is essentially known; see [5].

Example 4.4. Consider the endomorphisms $\varphi\colon g \mapsto 6g$ and $\psi\colon g \mapsto 3g$ of the abelian group $\mathbb{Z}[\frac{1}{3}]$, written additively. A straightforward calculation, in line with Lemma 3.4, shows that, for $n \in \mathbb{N}$,

$$R(\varphi^n, \psi^n) = |\mathrm{Coker}(\varphi^n - \psi^n)| = |6^n - 3^n|_\infty \cdot |6^n - 3^n|_3 = |2^n - 1|_\infty \cdot |2^n - 1|_3.$$

Hence, the coincidence Reidemeister zeta function is

$$Z_{\varphi,\psi}(s) = \exp\left(\sum_{n=1}^{\infty} \frac{|2^n - 1|_\infty \cdot |2^n - 1|_3}{n} s^n\right),$$

This zeta function was studied in [5]. Their calculations show that the modulus of the coincidence Reidemeister zeta function satisfies

$$|Z_{\varphi,\psi}(s)|_\infty = \left|\frac{1-s}{1-2s}\right|_\infty \cdot \left|\frac{1-(2s)^2}{1-s^2}\right|_\infty^{\frac{1}{2}} \cdot \left|\frac{1-s^2}{1-(2s)^2}\right|_\infty^{\frac{1}{6}} \cdot \prod_{j=1}^{\infty} \left|\frac{1-(2s)^{2\cdot3^j}}{1-s^{2\cdot3^j}}\right|_\infty^{1/(2\cdot9^j)}.$$

It follows that the series defining the zeta function has zeros at all points of the form $\frac{1}{2}e^{2\pi i/3^r}$, $r \geq 1$, whence $|s| = \frac{1}{2}$ is a natural boundary for the coincidence Reidemeister zeta function $Z_{\varphi,\psi}(s)$.

In the next example we look at a coincidence Reidemeister zeta function for a tame pair of endomorphisms of $G \cong \mathbb{Z}^2$ that cannot be simultaneously triangularised; the example shows that possible a Pólya–Carlson dichotomy in this generality needs to allow for new outcomes.

Example 4.5. Consider the automorphisms $\varphi, \psi \colon \mathbb{Z}^2 \to \mathbb{Z}^2$ that are given by

$$(x, y)\varphi = (x, y) \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = (x, x + y), \quad (x, y)\psi = (x, y) \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} = (x + y, y).$$

Using Lemma 1.2, it is easy to check that, for $n \in \mathbb{N}$,

$$R(\varphi^n, \psi^n) = |\mathrm{Coker}(\varphi^n - \psi^n)| = n^2.$$

Hence, the coincidence Reidemeister zeta function is

$$Z_{\varphi,\psi}(s) = \exp\left(\sum_{n=1}^{\infty} n s^n\right) = \exp\left(\frac{s}{(1-s)^2}\right).$$

Clearly, $Z_{\varphi,\psi}(s)$ is neither a rational function nor does the zeta function have a natural boundary at its radius of convergence.

Finally we generalise Theorem 4.3 to deduce the main result that was stated in the introduction.

Proof of Theorem 1.4. Assertion (1) follows from Proposition 2.1, and (2) follows from Proposition 3.4. Thus it remains to prove (3), and we follow closely the proof of Theorem 4.3.

For $1 \le k \le c$ and $p \in \mathbf{P}$ we write $S_k(p) = \{1, \ldots, d_k\} \smallsetminus I_k(p)$ and $S_k^*(p) = \{i \in S_k(p) \mid |\xi_{k,i}|_p \ne |\eta_{k,i}|_p\}$. We set

$$b = \prod_{k=1}^{c} \prod_{p \in \mathbf{P}_k} \prod_{i \in S_k^*(p)} \max\{|\xi_{k,i}|_p, |\eta_{k,i}|_p\}, \qquad \eta = \prod_{k=1}^{c} \prod_{p \in \mathbf{P}_k} \prod_{i \in S_k(p) \smallsetminus S_k^*(p)} |\eta_{k,i}|_p,$$

and as in the proof of Theorem 4.3 we deduce from (1) and (2) that, for $n \in \mathbb{N}$,

$$R(\varphi^n, \psi^n) = g(n) \cdot f(n),$$

where

$$g(n) = b^n \cdot \eta^n \cdot \prod_{k=1}^{c} \prod_{i=1}^{d_k} |\xi_{k,i}^n - \eta_{k,i}^n|_\infty \quad \text{and} \quad f(n) = \prod_{k=1}^{c} \prod_{p \in \mathbf{P}_k} \prod_{i \in S_k(p) \smallsetminus S_k^*(p)} \left|(\xi_{k,i}\eta_{k,i}^{-1})^n - 1\right|_p.$$

As in the proof of Theorem 4.3 we open up the absolute values in the product $P(n) = \prod_{k=1}^{c} \prod_{i=1}^{d_k} |\xi_{k,i}^n - \eta_{k,i}^n|_\infty$ to arrive at an expression of the form

$$g(n) = \sum_{j \in J} c_j w_j^n,$$

where J is a finite index set, $c_j \in \{-1, 1\}$ and $\{w_j \mid j \in J\} \subseteq \mathbb{C} \smallsetminus \{0\}$. Consequently, the coincidence Reidemeister zeta function can be written as

$$Z_{\varphi,\psi}(s) = \exp\left(\sum_{j \in J} c_j \sum_{n=1}^{\infty} \frac{f(n)(w_j s)^n}{n}\right).$$

and we conclude the argument along the same line as in the proof of Theorem 4.3. $\square$

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